

Threshold effects on renormalization group running of neutrino parameters in the low-scale seesaw model

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Abstract

We show that, in the low-scale type-I seesaw model, renormalization group running of neutrino parameters may lead to significant modifications of the leptonic mixing angles in view of so-called seesaw threshold effects. Especially, we derive analytical formulas for radiative corrections to neutrino parameters in crossing the different seesaw thresholds, and show that there may exist enhancement factors efficiently boosting the renormalization group running of the leptonic mixing angles. We find that, as a result of the seesaw threshold corrections to the leptonic mixing angles, various flavor symmetric mixing patterns (e.g., bi-maximal and tri-bimaximal mixing patterns) can be easily accommodated at relatively low energy scales, which is well within the reach of running and forthcoming experiments (e.g., the LHC).

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I. INTRODUCTION

Experiments on neutrino oscillations have opened up a new window in searching for new physics beyond the Standard Model (SM) during the past decade. Since neutrinos are massless particles within the renormalizable SM, one often extends the SM particle content in order to accommodate massive neutrinos. Among various theories giving rise to neutrino masses, the seesaw mechanism attracts a lot of attention in virtue of its natural and simple description of tiny neutrino masses. In the conventional type-I seesaw model [1–4], three right-handed neutrinos are introduced with super-heavy Majorana masses far away from the electroweak scale $\Lambda_{\text{EW}} = \mathcal{O}(100)$ GeV. Small neutrino masses are strongly suppressed by the ratio between the electroweak scale and the large mass of the right-handed neutrinos, which on the other hand, leaves the theory lacking in experimental testability. However, there are alternatives that allow us to realize the seesaw mechanism at an experimentally accessible level, e.g., the TeV scale. One popular way to lower the seesaw scale is to introduce additional singlet fermions, which have the same masses as the right-handed neutrinos but different CP parity [5]. They can be combined with the right-handed neutrinos to form four-component Dirac fields, while the lepton number is broken by a small Majorana mass insertion. In such a scenario, the masses of the light neutrinos are strongly suppressed by the small Majorana mass insertion instead of the seesaw scale, while the non-unitarity effects in the lepton flavor mixing can also be boosted to an observable level [6]. Other possibilities like structural cancellation [7–10] or the minimal flavor seesaw mechanism [11, 12] may be employed to construct low-scale seesaw models.

In principle, the current neutrino parameters are observed via low-energy neutrino oscillation experiments. On the other hand, the seesaw-induced neutrino mass operator is usually given at the seesaw scale. Therefore, neutrino parameters are subject to radiative corrections, i.e., they are modified by renormalization group (RG) running effects. Typically, at energy scales lower than the seesaw threshold (i.e., the mass scale of the corresponding seesaw particle), the RG running behavior of neutrino masses and leptonic mixing parameters should be described in an effective theory, which is essentially the same for different seesaw models. However, at energy scales higher than the seesaw threshold, full seesaw theories have to be considered, and the interplay between the heavy and light sectors could make the RG running effects particularly different compared to those in the effective the-

ory. However, in the spirit of some grand unified theories (GUTs), a unified description of fermion masses and flavor mixing depends on the lepton flavor structure at the GUT scale, which inevitably requires the RG running between seesaw particle thresholds and above. The full sets of renormalization group equations (RGEs) in the type-I [1–4], type-II [13–18] and type-III [19] seesaw models have been derived, both in the SM, and in the Minimal Supersymmetric Standard Model (MSSM) [20–26]. The general feature of the running parameters have also been intensively studied in the literature, and it has been shown that there could be sizable radiative corrections to the leptonic mixing parameters at superhigh energy scales (see, e.g., Refs. [27, 28] and references therein). In particular, certain flavor symmetric mixing patterns can be achieved at the GUT scale indicating that there might exist some flavor symmetries similar to the gauge symmetry (see, e.g., Ref. [29] and references therein).

Since the RG evolution together with the threshold effects of the heavy seesaw particles may result in visible corrections to the leptonic mixing parameters, there is a need to look into the RG running effects on the leptonic mixing parameters in the TeV seesaw model. In this work, we will explore in detail the RG evolution of neutrino masses and leptonic mixing parameters in the TeV type-I seesaw model. In particular, we will show that the threshold effects play a crucial role in the RG running of neutrino parameters, and some phenomenologically interesting flavor symmetric mixing patterns may be feasible even at an observable energy scale.

The remaining part of this work is organized as follows. In Sec. II, we first present general RGEs of the neutrino parameters in the type-I seesaw model; in particular, the matching conditions in crossing the seesaw thresholds. Then, the threshold corrections to the neutrino parameters are discussed in detail in Sec. III. Based on the analytical results, we illustrate a phenomenologically interesting numerical example in Sec. IV, in which the bi-maximal leptonic mixing pattern at TeV scale is shown to be compatible with experimental data when the seesaw threshold corrections are properly taken into account. Finally, a brief summary and our conclusions are given in Sec. V. In addition, there are two appendices including the detailed analytical treatment of the threshold effects.

II. RGES IN THE TYPE-I SEESAW MODEL

The simplest type-I seesaw model is constructed by extending the SM particle content with three right-handed neutrinos $\nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})$ together with a Majorana mass term of right-handed neutrinos. The mass part of the lepton sector Lagrangian reads

$$-\mathcal{L} = \overline{\ell}_L \phi Y_e e_R + \overline{\ell}_L \tilde{\phi} Y_\nu \nu_R + \frac{1}{2} \overline{\nu}_R^c M_R \nu_R + \text{h.c.}, \quad (1)$$

where ϕ is the SM Higgs boson, while ℓ_L and e_R denote lepton doublets and right-handed charged leptons, respectively. Here Y_e and Y_ν are the corresponding Yukawa coupling matrices with M_R being the Majorana mass matrix of the right-handed neutrinos. Without loss of generality, one can always perform a basis transformation and work in the basis in which M_R is diagonal, i.e., $M_R = \text{diag}(M_1, M_2, M_3)$. At energy scales below the right-handed neutrino masses, the heavy right-handed neutrinos should be integrated out from the theory and the masses of the light neutrinos are effectively given by a non-renormalizable dimension-five operator

$$-\mathcal{L}_\nu^{d=5} = \frac{1}{2} (\overline{\ell}_L \phi) \cdot \kappa \cdot (\phi^T \ell_L^c) + \text{h.c.}, \quad (2)$$

where the effective coupling matrix κ can be obtained from the type-I seesaw formula

$$\kappa = Y_\nu M_R^{-1} Y_\nu^T, \quad (3)$$

and the Majorana mass matrix of the left-handed neutrinos is

$$m_\nu \equiv \kappa v^2, \quad (4)$$

with $v \simeq 174$ GeV being the vacuum expectation value of the Higgs field.

Because of the seesaw thresholds, the RG running of neutrino parameters needs to be treated separately. At energies above the seesaw thresholds, the full theory should be considered and the relevant beta-functions are given by [20–23]

$$16\pi^2 \mu \frac{dY_e}{d\mu} = (\alpha_e + C_e^e H_e + C_e^\nu H_\nu) Y_e, \quad (5)$$

$$16\pi^2 \mu \frac{dY_\nu}{d\mu} = (\alpha_\nu + C_\nu^e H_e + C_\nu^\nu H_\nu) Y_\nu, \quad (6)$$

$$16\pi^2 \mu \frac{dM_R}{d\mu} = C_R M_R (Y_\nu^\dagger Y_\nu) + C_R (Y_\nu^\dagger Y_\nu)^T M_R, \quad (7)$$

where $H_f = Y_f Y_f^\dagger$ for $f = e, \nu, u, d$ and the coefficients $(C_e^e, C_e^\nu, C_\nu^e, C_\nu^\nu, C_R) = (3/2, -3/2, -3/2, 3/2, 1)$ in the SM. The coefficient α_ν is flavor blind and reads

$$\alpha_\nu = \text{tr}(3H_u + 3H_d + H_e + H_\nu) - \left(\frac{9}{20}g_1^2 + \frac{9}{4}g_2^2 \right). \quad (8)$$

If we make use of m_ν at energy scales above the seesaw threshold, we can derive from Eqs. (3) and (5)-(7)

$$\frac{dm_\nu}{dt} \equiv \dot{m}_\nu = 2\alpha_\nu m_\nu + (C_\nu^e H_e + C_m H_\nu) m_\nu + m_\nu (C_\nu^e H_e + C_m H_\nu)^T, \quad (9)$$

with $C_m = 1/2$. Here, for simplicity, we have defined $t = \ln \mu / (16\pi^2)$.

At energies below the seesaw thresholds, i.e., in the effective theory, neutrino masses are attached to the dimension-five operator, and the RGE of κ reads

$$\dot{\kappa} = \alpha_\kappa \kappa + (C_\nu^e H_e) \kappa + \kappa (C_\nu^e H_e)^T, \quad (10)$$

where

$$\alpha_\kappa = 2\text{tr}(3H_u + 3H_d + H_e) + \lambda - 3g_2^2, \quad (11)$$

with λ denoting the SM Higgs self-coupling constant.

In crossing the seesaw thresholds, one should ensure that the full and effective theories give identical predictions for physical quantities at low energy scales, and therefore, the physical parameters of both theories have to be related to each other. In the case of the neutrino mass matrix, this means relations between the effective coupling matrix κ and the parameters Y_ν and M_R of the full theory. This is technically called *matching* between the full and effective theories. For the simplest case, if the mass spectrum of the heavy singlets is degenerate, namely $M_1 = M_2 = M_3 = M_0$, one can simply make use of the tree-level matching condition at the scale $\mu = M_0$

$$\kappa|_{M_0} = Y_\nu M_R^{-1} Y_\nu^T|_{M_0}. \quad (12)$$

In the most general case with non-degenerate heavy singlets, i.e., $M_1 < M_2 < M_3$, the situation becomes more complicated and the heavy singlets have to be sequentially decoupled from the theory [30]. For instance, at energy scales between the n -th and $(n-1)$ -th thresholds, the heavy singlets are partially integrated out, leaving only a $3 \times (n-1)$ sub-matrix in Y_ν , which is non-vanishing in the basis, where the heavy singlet mass matrix is

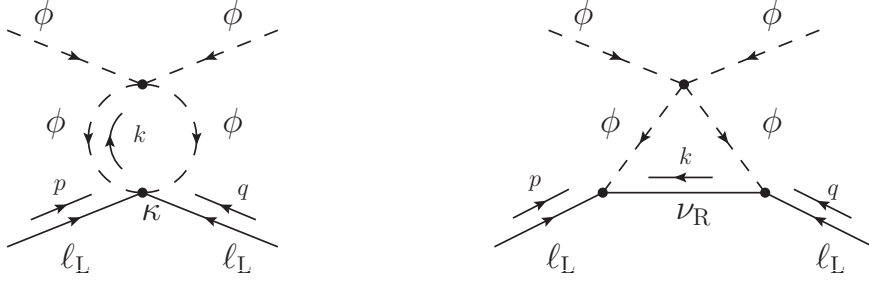


FIG. 1: Feynman diagrams of the one-loop λ corrections to the neutrino mass operator in the effective theory (left) and in the full theory (right), where p, q and k are the corresponding momenta.

diagonal. The decoupling of the n -th heavy singlet leads to the appearance of an effective dimension-five operator similar to that in Eq. (2), and the effective neutrino mass matrix below M_n is described by two parts

$$m_\nu^{(n)} = v^2 \left[\kappa^{(n)} + Y_\nu^{(n)} \left(M_R^{(n)} \right)^{-1} Y_\nu^{(n)T} \right], \quad (13)$$

where (n) labels the quantities relevant for the effective theory between the n -th and $(n-1)$ -th thresholds. In the SM, the RGEs for the two terms above have different coefficients for the gauge coupling and Higgs self-coupling contributions, which can be traced back to the decoupling of the right-handed neutrinos from the full theory. For instance, we show in Fig. 1 a comparison between the λ corrections in the effective theory (left diagram) and in the full theory (right diagram). The loop integral of the left diagram, i.e.,

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_\phi^2} \frac{1}{(k - p - q)^2 - m_\phi^2}$$

is UV divergent, whereas the loop integral of the right diagram, i.e.,

$$\int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + M_n}{k^2 - M_n^2} \frac{1}{(k + p)^2 - m_\phi^2} \frac{1}{(q - k)^2 - m_\phi^2}$$

is UV finite. Therefore, there is no λ correction to the neutrino mass operator in the full theory, while λ enters the beta-function of κ in the effective theory. See also Ref. [31] for a detailed discussion. We remark that, if the relevant seesaw threshold is above the soft SUSY-breaking scale, such a mismatch is absent in the MSSM due to the supersymmetric structure of the MSSM Higgs and gauge sectors. Therefore, this feature may result in significant RG running effects only in the SM, in particular when the mass spectrum of the heavy neutrinos is quite hierarchical.

III. THRESHOLD EFFECTS IN RGEs

We continue to determine the threshold corrections to the neutrino parameters. According to Eq. (13) the neutrino mass matrix between the seesaw thresholds consists of two parts κ and $Y_\nu M_R^{-1} Y_\nu^T$. The beta-functions of these two parts have different coefficients in the terms proportional to the gauge couplings and λ [31]. In the framework of the SM, flavor non-trivial parts in the beta-functions of κ and Y_ν are dominated by H_ν and H_e , which are essentially negligible in the RG running. For example, in a basis where H_e is diagonal, the largest entry of H_e is proportional to the square of the tau Yukawa coupling, i.e., y_τ^2 , which is at the order of 10^{-4} and too small to be significant in the RG running. As for H_ν , stringent constraints from the unitarity of the leptonic mixing matrix indicate that their contributions in the RG running are not comparable to the flavor blind parts, i.e., the gauge couplings and λ . Hence, we neglect the flavor non-trivial parts in the beta-functions in the following analytical analysis.

For the RG running of the neutrino parameters from one seesaw threshold M_n to the nearby one M_{n-1} , the two parts of the neutrino mass matrix are basically re-scaled to $a\kappa$ and $bY_\nu M_R^{-1} Y_\nu^T$, with $a \neq b$. Explicitly, the mass matrix of the light neutrinos at $\mu = M_{n-1}$ approximates to

$$\begin{aligned} m_\nu|_{M_{n-1}} &\simeq b v^2 (\kappa + Y_\nu M_R^{-1} Y_\nu^T)|_{M_n} + (a - b) v^2 \kappa|_{M_n} \\ &= b \left[m_\nu|_{M_n} + (ab^{-1} - 1) v^2 \kappa|_{M_n} \right] = b \left(m_\nu|_{M_n} + \varepsilon v^2 \kappa|_{M_n} \right), \end{aligned} \quad (14)$$

where

$$a = \exp \left(\int_{\frac{1}{16\pi^2} \ln M_n}^{\frac{1}{16\pi^2} \ln M_{n-1}} \alpha_\kappa dt \right) \simeq \left(\frac{M_{n-1}}{M_n} \right)^{\frac{\alpha_\kappa}{16\pi^2}}, \quad (15)$$

$$b = \exp \left(\int_{\frac{1}{16\pi^2} \ln M_n}^{\frac{1}{16\pi^2} \ln M_{n-1}} 2\alpha_\nu dt \right) \simeq \left(\frac{M_{n-1}}{M_n} \right)^{\frac{2\alpha_\nu}{16\pi^2}}, \quad (16)$$

$$\varepsilon = \exp \left[\int_{\frac{1}{16\pi^2} \ln M_n}^{\frac{1}{16\pi^2} \ln M_{n-1}} (\alpha_\kappa - 2\alpha_\nu) dt \right] - 1 \simeq \left(\frac{M_{n-1}}{M_n} \right)^{\frac{\alpha_\kappa - 2\alpha_\nu}{16\pi^2}} - 1. \quad (17)$$

In the limit $M_{n-1} = M_n$, one can easily observe that the second term of Eq. (14) disappears.

In the low-scale seesaw model, when the evolution of the neutrino mass matrix crosses two thresholds (e.g., from M_n to M_{n-1}), the threshold corrections, i.e., the second term in Eq. (14), provides corrections to the neutrino mass matrix. In comparison to the RG running

corrections, the threshold effects are much more significant in changing the value of the neutrino parameters. In order to see this point, we recall that when the neutrino parameters are evaluated below the seesaw scale, i.e., in the effective theory, the only flavor non-trivial corrections in the beta-functions are related to the charged lepton Yukawa coupling Y_e . As we mentioned before, the largest entry in Y_e is constrained by the tau mass, and can only be at percentage level. Therefore, no visible RG running effects can be gained in the effective theory. At energies above the seesaw scale, Y_ν contributes to the RG running of the neutrino mass matrix as shown in Eq. (9). In the ordinary seesaw models with GUT scale right-handed neutrino masses, Y_ν can be sizable, e.g., at order unity. However, in the low-scale seesaw model, Y_ν should be relatively small in order to keep the stability of the masses of the light neutrinos. Although there exist mechanisms that could stabilize neutrino masses without the requirement of a tiny Y_ν , there is still very strong restrictions from the unitarity of the leptonic mixing matrix. For example, if one of the right-handed neutrinos is located around the electroweak scale Λ_{EW} , the unitarity of the leptonic mixing matrix sets a general upper bound $H_\nu < 10^{-3}$ unless severe fine-tuning is involved. Therefore, large RG running effects on the mixing parameters can hardly be obtained above the seesaw scale. In the case of threshold corrections, the situation becomes quite different, since in Eq. (14) the flavor non-trivial corrections are proportional to the gauge couplings and λ , which do not suffer from the unitarity constraints and can be of order unity. In particular, if the light neutrino mass spectrum is highly degenerate, sizable corrections to the neutrino parameters could be naturally obtained.

In order to investigate the threshold effects on the neutrino parameters, we present series expansions of the leptonic mixing angles and the light neutrino masses. In particular, we make use of the so-called Casas–Ibarra parametrization [32], in which Y_ν is determined by

$$Y_\nu = U \sqrt{D_\kappa} O \sqrt{M_R}, \quad (18)$$

in a basis where M_R is diagonal. Here, U is the leptonic mixing matrix D_κ denotes the eigenvalue matrix of κ , while $O = R_{23}(\vartheta_1)R_{13}(\vartheta_2)R_{12}(\vartheta_3)$ with $R_{ij}(\vartheta_k)$ being the elementary rotations in the 23, 13, and 12 planes, respectively. Different from the leptonic mixing angles, ϑ_i are in general complex.¹ For simplicity, we will assume CP conservation in this in-

¹ See Ref. [33] for a discussion on the role of the matrix O in the RG evolution of the neutrino parameters in the MSSM.

vestigation (except from in the appendices), i.e., O is a real orthogonal matrix parametrized by three real angles. A detailed derivation with CP-violating effects can be found in Appendix A. The neutrino mass matrix at the scale $\mu = M_n$ can be diagonalized by using a unitary transformation, viz.

$$m_\nu|_{M_n} = U \text{diag}(m_1, m_2, m_3) U^T, \quad (19)$$

with m_i ($i = 1, 2, 3$) being the light neutrino masses. Then, at the scale $\mu = M_{n-1}$, the first-order correction to the leptonic mixing matrix is calculated by

$$\tilde{U}_{\alpha j} = U_{\alpha j} + \varepsilon v^2 \sum_{k \neq j} \frac{(U^T \kappa U)_{kj}}{m_j - m_k} U_{\alpha k} = U_{\alpha j} + \varepsilon \sum_{k \neq j} \frac{\sqrt{m_j m_k}}{m_j - m_k} O_{kn} O_{jn} U_{\alpha k} \quad (20)$$

and the masses of the light neutrinos are given by

$$\tilde{m}_i = b m_i + \varepsilon v^2 (U^T \kappa U)_{ii} = (b + \varepsilon O_{in}^2) m_i. \quad (21)$$

Here n denotes the radiative corrections between the n -th and $(n-1)$ -th thresholds. See Appendix A for the detailed derivation of Eqs. (20) and (21).

Inserting the parametrizations of U and O into the above equations, one can easily obtain the threshold corrections to the leptonic mixing parameters. For example, evolving the RGEs from M_3 to M_2 , the leptonic mixing angles are modified to

$$\tilde{\theta}_{13} \simeq \theta_{13} + \varepsilon c_1 c_2 \sqrt{m_3} \left(\frac{s_1 c_2 s_{12} \sqrt{m_2}}{m_3 - m_2} + \frac{s_2 c_{12} \sqrt{m_1}}{m_3 - m_1} \right), \quad (22)$$

$$\tilde{\theta}_{12} \simeq \theta_{12} + \varepsilon s_1 s_2 c_2 \frac{\sqrt{m_1 m_2}}{m_2 - m_1}, \quad (23)$$

$$\tilde{\theta}_{23} \simeq \theta_{23} + \varepsilon c_1 c_2 \sqrt{m_3} \left(-s_2 s_{12} \frac{\sqrt{m_1}}{m_3 - m_1} + s_1 c_2 c_{12} \frac{\sqrt{m_2}}{m_3 - m_2} \right), \quad (24)$$

where $s_i = \sin \vartheta_i$ and $c_i = \cos \vartheta_i$, and the terms proportional to εs_{13} have been ignored. The light neutrino masses at the scale $\mu = M_2$ are approximately given by

$$\tilde{m}_1 = (b + \varepsilon s_2^2) m_1, \quad (25)$$

$$\tilde{m}_2 = (b + \varepsilon s_1^2 c_2^2) m_2, \quad (26)$$

$$\tilde{m}_3 = (b + \varepsilon c_1^2 c_2^2) m_3. \quad (27)$$

We refer the reader to Appendix B for the complete analytical derivation of the threshold corrections to the leptonic mixing angles and the light neutrino masses.

Note that the mass differences in the denominator of Eqs. (22)-(24) could strongly enhance the threshold corrections. We give a rough estimate of the enhancement effects. Unless in some fine-tuning limits, κv^2 should generally be the same order of magnitude as the neutrino masses, i.e., $\kappa v^2 \sim m_\nu$. Taking into account the neutrino mass-squared differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$ measured by neutrino oscillation experiments, one can calculate that, if $m_1 \sim 0.05$ eV, $\kappa v^2/(m_1 - m_2) = \mathcal{O}(100)$ holds no matter whether the mass hierarchy is normal or inverted. This is a very distinctive feature of the RG running of the leptonic mixing angles in the seesaw model, since such enhancement factors do not exist in the quark sector. Due to the threshold effects, large corrections to the leptonic mixing angles can be achieved even at a very low energy scale, which provides us with a possibility of testing the threshold effects experimentally. On the other hand, for flavor symmetric theories constructed at low energy scales, e.g., the TeV scale, the seesaw threshold effects should be properly taken into account although the RG running happens in a relatively short distance.

Compared to the threshold corrections to the leptonic mixing angles, there is however no enhancement factor in the RG evolution of the neutrino masses. In the case of CP violation, the running CP-violation phases are also affected by the threshold effects. However, since there is no experimental information on leptonic CP violation, we will not present further analysis of the running CP-violating phases. In what follows, we will illustrate the threshold effects on the leptonic mixing angles and concentrate on the possibility of attaining flavor symmetric mixing patterns at a low energy scale.

IV. NUMERICAL ANALYSIS OF THRESHOLD EFFECTS IN RGES

In our numerical analysis, we adopt the values of the neutrino mass-squared differences Δm_{ij}^2 and the leptonic mixing angles θ_{ij} in the standard parametrization [34] from a global-fit of current experimental data in Ref. [35] that we assume to be given at the energy scale $\mu = M_Z$, where M_Z denotes the mass of the Z boson. The values of the quark and charged-lepton masses as well as the gauge couplings are taken from Ref. [36]. Furthermore, we take a representative value of the Higgs mass to be $m_H = 140$ GeV. For an illustrative purpose, the right-handed neutrino masses are chosen as $(M_1, M_2, M_3) = (200 \text{ GeV}, 500 \text{ GeV}, 1000 \text{ GeV})$. It is worth mentioning that our choice of the right-handed neutrino masses corresponds to $\varepsilon \sim -6 \times 10^{-3}$ between the scales M_3 and M_2 and $\varepsilon \sim -8 \times 10^{-3}$ between the scales M_2

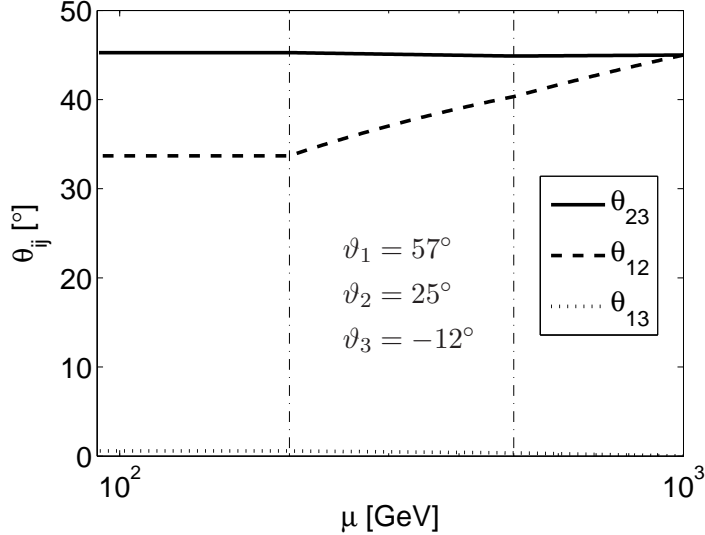


FIG. 2: The RG evolution of the leptonic mixing angles θ_{23} , θ_{12} , and θ_{13} with solid, dashed, and dotted curves, respectively. The threshold of the right-handed neutrinos are marked by dashed-dotted vertical lines. At the scale $M_3 = 1$ TeV, the bi-maximal mixing pattern, i.e., $\theta_{12} = \theta_{23} = \pi/4$ together with $\theta_{13} = 0$, is assumed, while the corresponding rotation angles in O are given in the plot. In addition, the light neutrino mass spectrum is taken to be normal hierarchy with $m_1(M_3) = 0.05$ eV. Because of the lacking of experimental evidence of leptonic CP violation, we assume all CP-violating phases to be zero in our calculation.

and M_1 , cf. Eq. (17).

In Fig. 2, we show one example of the threshold effects on the leptonic mixing angles. The rotation angles in O at the scale $\mu = M_3$ have been labeled in the plot. In addition, we use the normal mass hierarchy, i.e., $m_1 < m_2 < m_3$ together with $m_1(M_3) = 0.05$ eV, which translates into $m_1 \simeq 0.047$ eV at the M_Z scale. One observes that, with the threshold effects included, the exact bi-maximal mixing pattern (i.e., $\theta_{12} = \theta_{23} = 45^\circ$ and $\theta_{13} = 0$) at a very low energy scale could be compatible with experimental measurements. In particular, θ_{12} decreases by about ten degrees from M_3 to M_1 . This can be compared to a similar study of obtaining the bi-maximal mixing pattern at the GUT scale [37]. Such a significant effect can be understood from our analytical results. For instance, in the RG running between M_3 and M_2 , we have $s_1 s_2 c_2 = \sin(57^\circ) \sin(25^\circ) \cos(25^\circ) \simeq 0.3$, $\varepsilon \simeq -6 \cdot 10^{-3}$ as well as $\sqrt{m_1 m_2} / (m_2 - m_1) \simeq 43$ (at $\mu = M_3$), which leads to a correction $\theta'_{12} \simeq -4.7^\circ$ according to Eq. (23) [or Eq. (B9)]. Continuing the RG running between M_2 and M_1 [see Eq. (B24)],

we obtain a total correction $\theta'_{12} \simeq -9.2^\circ$ between M_3 and M_1 , which agrees approximately with the numerical value that is given by $\theta'_{12} \simeq -12.5^\circ$. In general, varying the value of the smallest light neutrino mass as well as changing the values of the Casas–Ibarra angles, we observe that the analytical results, which are only accurate for small corrections, are in very good agreement with the numerics.

Furthermore, a non-trivial θ_{13} can be expected at the M_Z scale because of the threshold corrections, although we have assumed $\theta_{13} = 0$ at high energy scales. Apart from the bi-maximal mixing pattern, other attractive flavor symmetric structures, e.g., the tri-bimaximal mixing pattern (i.e., $\theta_{12} \simeq 35.3^\circ$, $\theta_{23} = 45^\circ$ and $\theta_{13} = 0$) [38–40] can also be easily obtained through a proper choice of the rotation angles ϑ_i (see also Refs. [41–49] for discussions on possible RG corrections to the tri-bimaximal mixing pattern in other frameworks).

Note that, although all the three mixing angles do not stay at their initial input values, the change of θ_{12} is generally larger than those of the other two mixing angles. In view of Eqs. (22)–(24), θ_{13} and θ_{23} could only be amplified by the inverse of $\Delta m_{32}^2 = m_3^2 - m_2^2$ or $\Delta m_{31}^2 = m_3^2 - m_1^2$, but not the inverse of $\Delta m_{21}^2 = m_2^2 - m_1^2$, since these angles are located in the third column of the standard parametrization. As we have shown in the previous section, the boosting effect of Δm_{21}^2 is much more significant than those of Δm_{32}^2 and Δm_{31}^2 . Therefore, θ_{12} receives a more significant RG correction. In principle, if the mass spectrum of the light neutrinos is highly degenerate, visible changes of all the leptonic mixing angles can be expected.

It should be mentioned that, in the supersymmetric framework, the RG running of leptonic mixing angles may also be enhanced by a large value of $\tan \beta$. However, it has been shown that, in the MSSM, θ_{12} generally tends to increase when running down from high energy scales [50, 51], which means that many flavor symmetric mixing patterns are unfavorable.

For completeness, we also show the RG running of the masses of the light neutrinos in Fig. 3. One reads directly from the figure that there is no significant threshold corrections to the neutrino masses, which is consistent with our analytical results. Note that we do not include CP-violating effects in our analysis. Now that they may affect the RG running behavior of mixing angles, a thorough study of the RG running CP-violating phases is worthy, but out of the scope of this work, and hence will be elaborated elsewhere.

As a result of the freedom in O , various RG running behaviors of the leptonic mixing

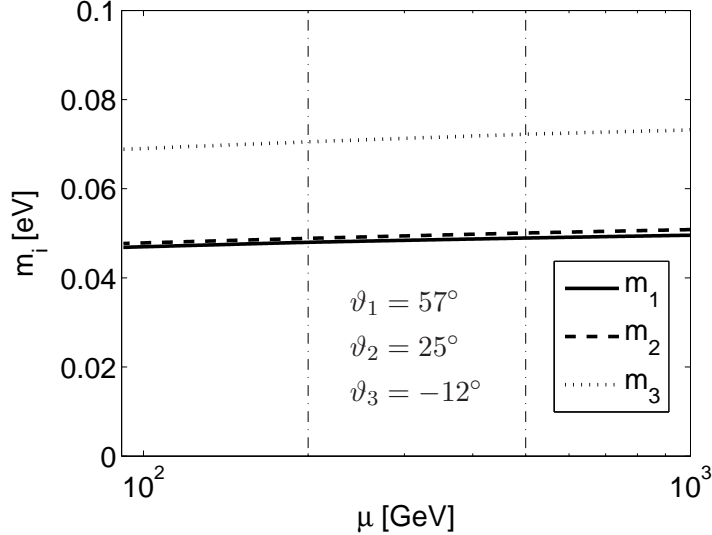


FIG. 3: The RG evolution of the masses of the light neutrinos m_1 , m_2 , and m_3 with solid, dashed, and dotted curves, respectively. The corresponding input parameters are the same as those in Fig. 2.

angles become viable for different choices of ϑ_i . Once a certain mixing pattern is fixed at a high energy scale, i.e. $\mu = M_3$, one can tune the rotation angles ϑ_i so as to achieve the experimentally measured data at low energy scales. Concretely, we show an example in Fig. 4, in which the bi-maximal mixing pattern of U is fixed at the scale $\mu = M_3$, while the parameter spaces of ϑ_i at 1σ , 2σ , and 3σ C.L. are shown in the plot. The colored areas correspond to the choice $m_1(M_3) = 0.05$ eV, while the curves $m_1(M_3) = 0.03$ eV. As we have estimated before, in the case of $m_1 = 0.05$ eV, significant corrections could be received for θ_{12} , and therefore, the bi-maximal mixing pattern can be easily fetched at higher energy scales. For the case of a smaller m_1 , the parameter spaces of ϑ_i shrink a lot, and a larger value of ϑ_1 , e.g., $\vartheta_1 \simeq 90^\circ$, is more favorable. In particular, $\vartheta_1 \simeq 0$ is ruled out, which can be seen from Eq. (23) where the threshold correction to θ_{12} is suppressed for small ϑ_1 .

We remark that, although we only consider the normal hierarchy of light neutrino masses in our numerical analysis, the main results are unchanged if the inverted mass hierarchy is assumed, i.e., $m_3 < m_1 < m_2$. The reason is simply that the RG running of θ_{12} is mainly affected by the inverse of Δm_{21}^2 , which does not flip its sign for different neutrino mass hierarchies. In the limit of a highly degenerate spectrum of light neutrino masses, i.e., $m_1 \simeq m_2 \simeq m_3$, sizable enhancements of RG running can also be acquired for both

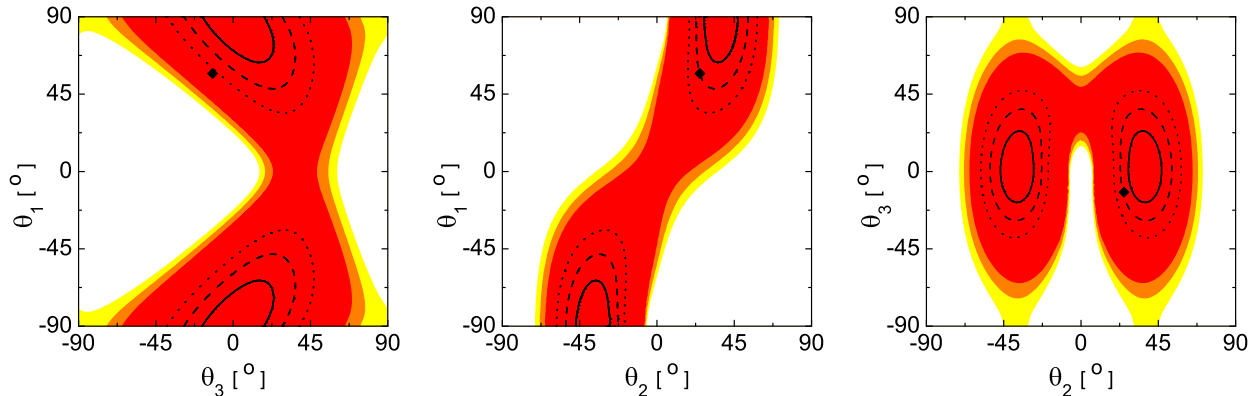


FIG. 4: Parameter spaces of ϑ_i relating the bi-maximal mixing pattern at the scale $\mu = M_3$ to the experimentally measured leptonic mixing angles at 1σ (red area, solid curve), 2σ (orange area, dashed curve), and 3σ (yellow area, dotted curve) C.L. The colored areas correspond to the choice $m_1(M_3) = 0.05$ eV, while the curves $m_1(M_3) = 0.03$ eV. The black diamonds represent the input values of ϑ_i in Figs. 2 and 3.

θ_{13} and θ_{23} , of which the light neutrino mass hierarchies may play a key role in the RG running. Furthermore, in our numerical calculations, we have fixed the mass spectrum of the right-handed neutrinos, in which the interval between two such neutrinos is around a few hundred GeV. In general, a stronger hierarchy among the masses of the right-handed neutrinos can lead to more sizable RG corrections to the neutrino parameters, whereas a weakly hierarchical mass spectrum of the right-handed neutrinos would suppress the seesaw threshold effects.

V. SUMMARY AND CONCLUSIONS

In this work, we have studied the RG running of neutrino parameters in the low-scale type-I seesaw model. We have shown that, in the RG evolution of the leptonic mixing angles between the seesaw thresholds, significant radiative corrections could be obtained even for a short distance of RG running. In particular, we have presented analytical formulas for the RG corrections to the neutrino parameters in crossing the seesaw thresholds, which clearly indicate that sizable threshold corrections to the leptonic mixing matrix may occur due to the mismatch between different contributions to the mass matrix of the light neutrinos. In addition, we have demonstrated that, if the light neutrino mass spectrum is nearly degener-

ate, there exist large enhancement factors boosting the RG running of the leptonic mixing angles remarkably. A numerical example has also been given to show that, in the presence of low-scale right-handed neutrinos, the bi-maximal mixing pattern at TeV scale is fully compatible with the current measured leptonic mixing angles. This peculiar feature turns out to be very useful for the construction of a low energy scale flavor symmetric theory, and in particular, it provides us with the possibility of testing such a flavor symmetry at running and forthcoming experiments (such as the LHC). For the sake of keeping the seesaw scale within the reach of these experiments, the neutrino Yukawa coupling tends to be chosen very small in order to stabilize the mass scale of the light neutrinos. One might think that this prevents the low-scale type-I seesaw model from being testable, since the predominant interaction between the right-handed neutrinos and the SM fields is through the Yukawa coupling. However, this is not necessarily true in the presence of additional interactions. For example, if the right-handed neutrinos and other SM fields are charged under an additional gauge group, broken not far above the TeV scale, such a new gauge interaction could give rise to observable effects. Note that, although we mainly concentrate on the type-I seesaw model throughout this work, our main conclusions can be easily extended to various low-scale seesaw models. For example, in the type-I+II seesaw framework with three right-handed neutrinos and a heavy Higgs triplet field being introduced, both the right-handed neutrinos and the Higgs triplet may contribute to the masses of the light neutrinos, and the mismatch between different contributions would result in similar threshold corrections to the leptonic mixing angles.

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Appendix A: Derivation of the analytical formulas for the corrected leptonic mixing matrix and the light neutrino masses

In this appendix, we derive the first-order threshold corrections to the leptonic mixing matrix and the light neutrino masses. In general, the light neutrino mass matrix is complex and symmetric, i.e., $m_\nu = m_\nu^T$. In order to derive the series expansions, we write the perturbed mass matrix as

$$\tilde{m}_\nu = m_\nu + m'_\nu, \quad (\text{A1})$$

where m'_ν corresponds to a correction to m_ν . The corrected leptonic mixing matrix is given by $\tilde{U} = U + U'$ with U' being the correction of U . In analogy, one can define the real and diagonal matrix $\tilde{D} = D + D'$, i.e., $\tilde{D} = \text{diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$, and similarly for the matrices D and D' .

By definition, we have

$$\tilde{U}\tilde{D}\tilde{U}^T = \tilde{m}_\nu, \quad UDU^T = m_\nu, \quad (\text{A2})$$

and to lowest order in the corrections U' and D' , we find that

$$m'_\nu = \tilde{m}_\nu - m_\nu = \tilde{U}\tilde{D}\tilde{U}^T - UDU^T \simeq UDU'^T + U'DU^T + UD'U^T. \quad (\text{A3})$$

Multiplying with U^\dagger from the left and U^* from the right to the above equation, we obtain

$$U^\dagger m'_\nu U^* = DU'^T U^* + U^\dagger U' D + D'. \quad (\text{A4})$$

Now, we define an anti-Hermitian matrix T and a symmetric matrix Λ as

$$T = U^\dagger U', \quad (\text{A5})$$

$$\Lambda = U^\dagger m'_\nu U^*, \quad (\text{A6})$$

Here one can easily prove that $T^\dagger = -T$, and the diagonal elements of T are purely imaginary, i.e., $T_{ii} = -T_{ii}^*$. Using Eqs. (A4)-(A6), Λ can be computed as

$$\Lambda = -DT^* + TD + D'. \quad (\text{A7})$$

In order to obtain the corrections to the mass eigenvalues, we investigate the diagonal part of this matrix equation, which indicates that (no summation over repeated indices)

$$\Lambda_{ii} = -m_i T_{ii}^* + T_{ii} m_i + m'_i, \quad (\text{A8})$$

and the real part gives the corrections to the neutrino masses, i.e.,

$$m'_i = \text{Re}(\Lambda_{ii}). \quad (\text{A9})$$

In addition, the imaginary part is

$$\text{Im}(T_{ii}) = \frac{\text{Im}(\Lambda_{ii})}{2m_i}. \quad (\text{A10})$$

Next, to obtain the corrections to the leptonic mixing matrix, we study the non-diagonal part of Eq. (A7), which is

$$\Lambda_{ij} = -m_i T_{ij}^* + T_{ij} m_j, \quad i \neq j. \quad (\text{A11})$$

Now, the real and imaginary parts of Eq. (A11) yield

$$\text{Re}(T_{ij}) = \frac{\text{Re}(\Lambda_{ij})}{m_j - m_i}, \quad (\text{A12})$$

$$\text{Im}(T_{ij}) = \frac{\text{Im}(\Lambda_{ij})}{m_i + m_j}, \quad (\text{A13})$$

respectively. Therefore, all the elements of T can be obtained by using Eqs. (A10), (A12) and (A13), and U' can be calculated according to Eq. (A5).

Note that, if $m'_\nu = \xi m_\nu$, where ξ is a constant, then $\tilde{D} = (1 + \xi)D$ and $T = 0$, and thus, $\tilde{U} = U$. This means that, if m'_ν is proportional to m_ν , U will not be affected. As a special case, consider CP conservation, when the original mixing matrix U as well as the perturbation m'_ν are real. Then, $\Lambda^* = \Lambda$, and therefore, one has

$$m'_i = \Lambda_{ii}, \quad (\text{A14})$$

$$T_{ii} = 0, \quad (\text{A15})$$

$$T_{ij} = \frac{\Lambda_{ij}}{m_j - m_i}, \quad i \neq j. \quad (\text{A16})$$

As one application, we consider the threshold corrections in the main context. Below the scale $\mu = M_n$, one should take [cf. Eq. (14)]

$$m'_\nu = (b - 1)m_\nu + \varepsilon v^2 \kappa^{(n)}, \quad (\text{A17})$$

as the perturbation to m_ν . Using the matching condition at $\mu = M_n$, $\kappa^{(n)}$ can be expressed in terms of Y_ν as

$$\kappa^{(n)} = \frac{\mathcal{Y}_\nu^{(n)} \mathcal{Y}_\nu^{(n)T}}{M_n}, \quad (\text{A18})$$

where $\mathcal{Y}_\nu^{(n)}$ is the n -th column of Y_ν . Here we will make use of the Casas–Ibarra parametrization of Y_ν [cf., Eq. (18)], which allows us to express the corrections to U in terms of the rotation angles ϑ_i and the low-energy parameters. For a general U and Y_ν , one can calculate the matrix $\Lambda = \Lambda^{(n)}$, giving the contribution of $\kappa^{(n)}$ for μ below the threshold M_n ,²

$$\Lambda_{ij}^{(n)} = (U^\dagger m'_\nu U^*)_{ij} = (b-1)\delta_{ij}m_i + \varepsilon O_{in}O_{jn}\sqrt{m_i m_j}. \quad (\text{A19})$$

Inserting the above equation into Eq. (A9), one directly arrives at the neutrino masses at the scale $\mu = M_{n-1}$, i.e.,

$$\tilde{m}_i = [b + \varepsilon \text{Re}(O_{in}^2)] m_i. \quad (\text{A20})$$

Also, the components of the matrix T can be calculated as

$$T_{ii}^{(n)} = \frac{i\varepsilon}{2} \text{Im}(O_{in}^2), \quad (\text{A21})$$

$$T_{ij}^{(n)} = \varepsilon \sqrt{m_i m_j} \left[\frac{\text{Re}(O_{in}O_{jn})}{m_j - m_i} + i \frac{\text{Im}(O_{in}O_{jn})}{m_i + m_j} \right], \quad i \neq j. \quad (\text{A22})$$

Note that O_{ij} may, in principle, have very large absolute values if they are complex. If O is real, or purely imaginary,³ the corrected mixing matrix at the lower scale $\mu = M_{n-1}$ reads

$$\tilde{U}_{\alpha j} = U_{\alpha j} + [UT^{(n)}]_{\alpha j} = U_{\alpha j} + \varepsilon \sum_{k \neq j} \frac{\sqrt{m_j m_k}}{m_j - m_k} O_{kn} O_{jn} U_{\alpha k}. \quad (\text{A23})$$

Appendix B: Corrections to the leptonic mixing angles and the light neutrino masses

In the following, we use the analytical results obtained in Appendix A to derive explicit expressions for the threshold corrections to the leptonic mixing angles and the light neutrino masses. Here, for simplicity, we assume CP is preserved, i.e., all the CP-violating phases are taken to be zero. Since the perturbation is linear in κ , one should, for μ between the n -th and $(n-1)$ -th thresholds, simply add the contributions from all the thresholds above μ .

² If all the right-handed neutrinos are located at the same energy scale, one would effectively obtain a summation over the index n , giving the result $\Lambda_{ij} = (b-1+\varepsilon)\delta_{ij}m_i = a\delta_{ij}m_i$, since $OO^T = 1$.

³ That is with “low-energy” but without “high-energy” CP-violation.

1. Threshold corrections between M_3 and M_2

a. Correction to θ_{13}

Using Eq. (A23) with $n = 3$, the corrected quantity to s_{13} is given by

$$\begin{aligned}
s'_{13} = \tilde{s}_{13} - s_{13} &= \tilde{U}_{e3} - U_{e3} = \sum_{k \neq 3} U_{ek} T_{k3}^{(3)} = \varepsilon \sum_{k \neq 3} \frac{\sqrt{m_3 m_k}}{m_3 - m_k} O_{k3} O_{33} U_{ek} \\
&= \varepsilon \frac{\sqrt{m_1 m_3} O_{13} O_{33}}{m_3 - m_1} c_{13} c_{12} + \varepsilon \frac{\sqrt{m_2 m_3} O_{23} O_{33}}{m_3 - m_1} c_{13} s_{12} \\
&= \varepsilon c_1 c_2 c_{13} \sqrt{m_3} \left(\frac{s_2 c_{12} \sqrt{m_1}}{m_3 - m_1} + \frac{s_1 c_2 s_{12} \sqrt{m_2}}{m_3 - m_2} \right). \tag{B1}
\end{aligned}$$

Taylor expanding this equation around $\tilde{s}_{13} = s_{13}$, one obtains

$$\begin{aligned}
\tilde{\theta}_{13} &= \arcsin(s_{13} + s'_{13}) = \theta_{13} + \frac{s'_{13}}{c_{13}} + \mathcal{O}[(s'_{13})^2] \\
&\simeq \theta_{13} + \varepsilon c_1 c_2 \sqrt{m_3} \left(\frac{s_1 c_2 s_{12} \sqrt{m_2}}{m_3 - m_2} + \frac{s_2 c_{12} \sqrt{m_1}}{m_3 - m_1} \right). \tag{B2}
\end{aligned}$$

In the nearly degenerate case, i.e., $m_1 \simeq m_2 \simeq m_3$, the above formula approximates to

$$\theta'_{13} = \tilde{\theta}_{13} - \theta_{13} \simeq \varepsilon \frac{c_1 c_2 (s_2 c_{12} + s_1 c_2 s_{12}) \sqrt{m_1 m_3}}{m_3 - m_1}. \tag{B3}$$

One can observe that the correction to θ_{13} is actually independent of s_{13} . Furthermore, note that this expression is proportional to $\sqrt{m_1 m_3}/(m_3 - m_1)$. Thus, the correction cannot be very large, since there is no $1/(m_2 - m_1)$ enhancement.

b. Correction to θ_{12}

Similarly, using Eq. (A23), the threshold correction to U_{e2} can be obtained as

$$\begin{aligned}
U'_{e2} &= \tilde{U}_{e2} - U_{e2} = \sum_{k \neq 2} U_{ek} T_{k2}^{(3)} = \varepsilon \sum_{k \neq 2} \frac{\sqrt{m_2 m_k}}{m_2 - m_k} O_{k3} O_{23} U_{ek} \\
&= \varepsilon s_1 c_2 \sqrt{m_2} \left(\frac{s_2 c_{12} c_{13} \sqrt{m_1}}{m_2 - m_1} - \frac{c_1 c_2 s_{13} \sqrt{m_3}}{m_3 - m_2} \right). \tag{B4}
\end{aligned}$$

To obtain the correction to s_{12} , we note that, in the standard parametrization,

$$\tilde{s}_{12} = \frac{\tilde{U}_{e2}}{\tilde{c}_{13}}. \tag{B5}$$

Expanding in the small correction s'_{13} , we have

$$\frac{1}{\tilde{c}_{13}} \simeq \frac{1}{c_{13}} \left(1 + \frac{s_{13}s'_{13}}{c_{13}^2} \right), \quad (\text{B6})$$

and therefore, we find that

$$\tilde{s}_{12} = (U_{e2} + U'_{e2}) \frac{1}{c_{13}} \left(1 + \frac{s_{13}s'_{13}}{c_{13}^2} \right) \simeq s_{12} + s_{12} \frac{s_{13}s'_{13}}{c_{13}^2} + \frac{U'_{e2}}{c_{13}}, \quad (\text{B7})$$

where the corrections s'_{13} and U'_{e2} have been calculated above. For $s_{13} \simeq 0$, one obtains

$$s'_{12} = \tilde{s}_{12} - s_{12} \simeq \varepsilon s_1 s_2 c_2 c_{12} \frac{\sqrt{m_1 m_2}}{m_2 - m_1}. \quad (\text{B8})$$

In addition, Taylor expanding around $\tilde{s}_{12} = s_{12}$ gives

$$\theta'_{12} = \tilde{\theta}_{12} - \theta_{12} \simeq \varepsilon s_1 s_2 c_2 \frac{\sqrt{m_1 m_2}}{m_2 - m_1}, \quad (\text{B9})$$

where the right-hand side is independent of θ_{12} .

c. Correction to θ_{23}

Finally, using Eq. (A23), the threshold correction to $U_{\mu 3}$ is given by

$$\begin{aligned} U'_{\mu 3} &= \tilde{U}_{\mu 3} - U_{\mu 3} = \sum_{k \neq 3} U_{\mu k} T_{k3}^{(3)} = \varepsilon \sum_{k \neq 3} \frac{\sqrt{m_3 m_k}}{m_3 - m_k} O_{k3} O_{33} U_{\mu k} \\ &= \varepsilon c_1 c_2 \sqrt{m_3} \left(\frac{s_2 U_{\mu 1} \sqrt{m_1}}{m_3 - m_1} + \frac{s_1 c_2 U_{\mu 2} \sqrt{m_2}}{m_3 - m_2} \right). \end{aligned} \quad (\text{B10})$$

As for the corrections to s_{23} , we note that, in the standard parametrization,

$$\tilde{s}_{23} = \frac{\tilde{U}_{\mu 3}}{\tilde{c}_{13}}, \quad (\text{B11})$$

and thus, one obtains

$$\tilde{s}_{23} = (U_{\mu 3} + U'_{\mu 3}) \frac{1}{c_{13}} \left(1 + \frac{s_{13}s'_{13}}{c_{13}^2} \right) \simeq s_{23} + s_{23} \frac{s_{13}s'_{13}}{c_{13}^2} + \frac{U'_{\mu 3}}{c_{13}}, \quad (\text{B12})$$

where s'_{13} and $U'_{\mu 3}$ have been calculated above. For $s_{13} \simeq 0$, one obtains

$$s'_{23} = \tilde{s}_{23} - s_{23} \simeq \varepsilon c_1 c_2 c_{23} \sqrt{m_3} \left(-s_2 s_{12} \frac{\sqrt{m_1}}{m_3 - m_1} + s_1 c_2 c_{12} \frac{\sqrt{m_2}}{m_3 - m_2} \right), \quad (\text{B13})$$

giving that

$$\theta'_{23} = \tilde{\theta}_{23} - \theta_{23} \simeq \varepsilon c_1 c_2 \sqrt{m_3} \left(-s_2 s_{12} \frac{\sqrt{m_1}}{m_3 - m_1} + s_1 c_2 c_{12} \frac{\sqrt{m_2}}{m_3 - m_2} \right). \quad (\text{B14})$$

One finds that θ'_{23} is independent of θ_{23} . In the nearly degenerate limit, the above formula approximates to

$$\theta'_{23} = \varepsilon c_1 c_2 (s_1 c_2 c_{12} - s_2 s_{12}) \frac{\sqrt{m_1 m_3}}{m_3 - m_1}. \quad (\text{B15})$$

Similar to the corrections to θ_{13} , there is no strong enhancement factor.

d. Corrections to the light neutrino masses

According to Eq. (A20), the explicit expressions for the threshold corrections to the light neutrino masses are

$$m'_1 = \tilde{m}_1 - m_1 = [(b-1) + \varepsilon s_2^2] m_1, \quad (\text{B16})$$

$$m'_2 = \tilde{m}_2 - m_2 = [(b-1) + \varepsilon s_1^2 c_2^2] m_2, \quad (\text{B17})$$

$$m'_3 = \tilde{m}_3 - m_3 = [(b-1) + \varepsilon c_1^2 c_2^2] m_3. \quad (\text{B18})$$

2. Threshold corrections between M_2 and M_1

One may repeat the above analysis with $n = 2$, and calculate the threshold corrections to the neutrino parameters between M_2 and M_1 . Explicitly, for s_{13} , one obtains

$$s'_{13} = \varepsilon O_{32} c_{13} \sqrt{m_3} \left(\frac{c_2 s_3 c_{12} \sqrt{m_1}}{m_3 - m_1} + \frac{O_{22} s_{12} \sqrt{m_2}}{m_3 - m_2} \right), \quad (\text{B19})$$

and henceforth,

$$\theta'_{13} = \varepsilon O_{32} \sqrt{m_3} \left(\frac{c_2 s_3 c_{12} \sqrt{m_1}}{m_3 - m_1} + \frac{O_{22} s_{12} \sqrt{m_2}}{m_3 - m_2} \right), \quad (\text{B20})$$

where $O_{22} = c_1 c_3 - s_1 s_2 s_3$ and $O_{32} = -s_1 c_3 - c_1 s_2 s_3$. In the nearly degenerate limit, the above expression approximates to

$$\theta'_{13} \simeq \varepsilon O_{32} \frac{\sqrt{m_1 m_3}}{m_3 - m_1} (c_2 s_3 c_{12} + O_{22} s_{12}). \quad (\text{B21})$$

Then, for θ_{12} , one obtains

$$U'_{e2} = \varepsilon O_{22} \sqrt{m_2} \left(\frac{c_2 s_3 c_{12} c_{13} \sqrt{m_1}}{m_2 - m_1} - \frac{O_{32} s_{13} \sqrt{m_3}}{m_3 - m_2} \right). \quad (\text{B22})$$

Ignoring the small parameter s_{13} , we arrive at

$$s'_{12} \simeq \varepsilon c_2 s_3 (c_1 c_3 - s_1 s_2 s_3) c_{12} \frac{\sqrt{m_1 m_2}}{m_2 - m_1}, \quad (\text{B23})$$

which corresponds to

$$\theta'_{12} \simeq \varepsilon c_2 s_3 (c_1 c_3 - s_1 s_2 s_3) \frac{\sqrt{m_1 m_2}}{m_2 - m_1}. \quad (\text{B24})$$

Next, for θ_{23} , the correction to $U_{\mu 3}$ reads

$$U'_{\mu 3} = \varepsilon O_{32} \sqrt{m_3} \left(\frac{c_2 s_3 U_{\mu 1} \sqrt{m_1}}{m_3 - m_1} + \frac{O_{22} U_{\mu 2} \sqrt{m_2}}{m_3 - m_2} \right), \quad (\text{B25})$$

which, for $s_{13} \simeq 0$ and in the nearly degenerate case, gives

$$s'_{23} \simeq \varepsilon O_{32} (-c_2 s_3 s_{12} + O_{22} c_{12}) c_{23} \frac{\sqrt{m_1 m_3}}{m_3 - m_1}, \quad (\text{B26})$$

and

$$\theta'_{23} \simeq \varepsilon O_{32} (-c_2 s_3 s_{12} + O_{22} c_{12}) \frac{\sqrt{m_1 m_3}}{m_3 - m_1}. \quad (\text{B27})$$

Finally, the threshold corrections to the light neutrino masses are given by

$$m'_1 = [(b - 1) + \varepsilon c_2^2 s_3^2] m_1, \quad (\text{B28})$$

$$m'_2 = [(b - 1) + \varepsilon (c_1 c_3 - s_1 s_2 s_3)^2] m_2, \quad (\text{B29})$$

$$m'_3 = [(b - 1) + \varepsilon (s_1 c_3 + c_1 s_2 s_3)^2] m_3. \quad (\text{B30})$$

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