Anyons in $(1+1)$ dimensions and the deformed Calogero-Sutherland model

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Master of Science Thesis

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Stockholm, Sweden 2011

Typeset in $\LaTeX$

Examensarbete inom ämnet fysik för avläggande av civilingenjörsexamen
Graduation thesis on the subject Physics for the degree of Master of Science in Engineering

TRITA-FYS 2011:16
ISSN 0280-316X
ISRN KTH/FYS/--11:16--SE

Printed in Sweden by Universitetsservice US AB, Stockholm 2011

Abstract

This thesis deals with a conformal field theoretical treatment of abelian anyons in $(1+1)$-dimensions and their relation to the integrable Calogero-Sutherland models. We generalize previous work relating anyons to the Calogero-Sutherland model by showing that the correlation function of the anyon field operators corresponds to the eigenfunctions of the deformed Calogero-Sutherland model. Our results suggest a physical application of the deformed Calogero-Sutherland model in the context of the fractional quantum Hall effect (FQHE).

A key aspect for this work is the introduction of the dual anyon field operators, which obey a natural generalization of the canonical anti-commutation relation.

Key words: FQHE, Anyons, Integrable many-body system,
Deformed Calogero-Sutherland model

Sammanfattning

Denna avhandling behandlar abelska anyoner, med hjälp av konform fältteori, i $(1 + 1)$-dimensioner och deras relation till de fullt lösbara Calogero-Sutherland modellerna. Vi visar att anyon fältoperatorernas korrelationsfunktion motsvarar egenfunktionerna av den deformerade Calogero-Sutherland modellen, vilket är en generalisering av tidigare arbete kring anyonernas relation till Calogero-Sutherland modellen. Våra resultat tyder på en fysisk tillämpning av den deformerade Calogero-Sutherland modellen i samband med den fraktionerade kvanthalleffekten (FQHE).

En viktig aspekt för detta arbete är införandet av dual anyon operatorn, som uppfyller en naturlig generalisering av den kanoniska anti-kommuteringsrelation.

Contents

Introduction

> “ The motivation for a physicist to study 1-dimensional problems is best illustrated by the story of the man who, returning home late at night after an alcoholic evening, was scanning the ground for his key under a lamppost; he knew, to be sure, that he had dropped it somewhere else, but only under the lamppost was there enough light to conduct a proper search.” -F. Calogero [1]

Much has changed since this statement was published in 1970. One dimensional systems are, today, a reality for those dealing with experimental condensed matter physics and have produced some very interesting physical results, such as carbon nanotubes.

The 1-dimensional problems of particular interest are those described by exactly solvable models (integrable systems). Integrable systems play an important role as they provide an exact solutions that offers insight into properties which might not be gotten from approximations. Integrable models are also used as known solutions from which physicist start using approximative methods, such as perturbation theory. Integrable models are, therefore, essential as they provide a theoretical basis for our physical reality or, at least, act as a starting points for known unknowns. In this thesis certain integrable models play an important role, namely the original Calogero-Sutherland (CS) model [2] [3] and the generalization of the Calogero-Sutherland model known as the deformed Calogero-Sutherland (dCS) model [4] [5] [6].

A motivation for this work is the fractional quantum Hall effect (FQHE) discovered in 1982 by Tsui *et al.* [7]. The FQHE corresponds to plateaus in the Hall conductivity of two-dimensional electron systems subject to very low temperatures and extremely high magnetic fields. For the integer QHE the plateau region is quantized into positive integer, ν_{fill}, which is commonly referred to as the filling factor,[1] multiples of the inverse von Klitzing constant ($\mathrm{R}_{\mathrm{K}}^{-1} = (h/e^2)^{-1}$) and is explained as the manifestation of a single particle phenomena (see *e.g.* [8] and the references therein).

[1] In the bulk of the thesis the parameter ν will represent the statistical parameter of the anyons and should not be confused with the filling factor ν_{fill}.

In the FQHE the quantum number ν_{fill} can take the value of a positive rational number instead of a positive integer. This was observed only in samples with high mobility and it was concluded that the effect must be related to the electron-electron interaction, rather than being a single electron phenomena. A successful approximation for the groundstate wavefunction was done by R. Laughlin in 1983 [9], which corresponded to a filling factor of one over a odd positive integer ($\nu_{\text{fill}} = 1/(2m+1)$). The wavefunction described a circular droplet of condensed two-dimensional electron gas into an incompressible liquid known as a quantum Hall (QH) liquid. The Laughlin wavefunction also incorporated that each electron has precisely $2m$ pinned magnetic vortices attached to it. Simulations done by Laughlin, in the same paper, showed that the wavefunction corresponded to the observations of Tsui *et al.* for $\nu_{\text{fill}} = 1/3$.

A similar treatment was made by Haldane and Halperin (HH) [10] [11] for filling fractions not explained by the Laughlin wavefunction as the superposition of quasi-particles and quasi-holes creating "daughter" states from the existing "parent" states, starting from the Laughlin $\nu_{\text{fill}} = 1/3$ state, put on a spherical surface with a magnetic monopole at origo.

An alternative approach was given by Jain [12] for the $\nu_{\text{fill}} = p/(2mp \pm 1)$ plateaus, where m and p are positive integers, by the construction of composite fermions, *i.e.* fermions bound to a flux tube, and explained the FQHE as a manifestation of the integer QHE with composite fermions. The method presented by Jain had the advantage of being simpler than the HH approach, while simultaneously predicting the stability of filling factors and thereby new fractions that were bound to appear. The method does not explain all the observed QH fractions as it predicts fractions with odd denominators only.

Topological field theory (the celebrated Chern-Simons theory [13]) indicates that the low-energy excitations of the boundary states play a crucial role in the FQHE [14] [15]. By restricting the Laughlin wavefunction to the boundary, the Calogero-Sutherland groundstate is obtained with a shift in the center of mass (CoM) energy (c.f. Section 3.1.2). It was later discovered that the particles where quasi-particles exhibiting not just bosonic and fermionic statistics, but a new type of particle statistics which interpolates between bosonic and fermionic [16]. The quasi-particles were named anyons [17],[2] since they can gain any phase shift during exchange. It is known that there is a direct connection between the Calogero-Sutherland type models and anyons in $1+1$ dimensions [18] [19] and the FQHE [20]. A recent renewed high level of interest in this topic is sparked by the existence of non-abelian anyonic, topological states found in the QH states at $\nu_{\text{fill}} = 5/2$, which can be used in the creation of topological quantum computers [21].

This thesis is a generalization of the work by E. Langmann and A. Carey [19], which introduced the anyon field operators using the boson-fermion correspondence in $(1+1)$-dimensions (references on earlier work in this direction are discussed in [19]). They also showed the explicit construction of an self-adjoint anyonic

[2] Not to be confused with anions.

operator $\mathcal{H}^{\nu,3}$ corresponding to the second quantization of the Calogero-Sutherland Hamiltonian (c.f. Section 3.2). This thesis will continue by introducing a new type of anyonic field operator, corresponding to an dual anyon. Further on, one can show that the correlation functions of these operators, combined with previous anyon operators, correspond to solutions of the dCS model. We will also show that the self-adjoint operator $\mathcal{H}^{\nu,3}$, introduced in [19], will give the dCS differential operator when acting upon a vector created by the many-body anyon and dual anyon field operators. This allows to naturally interpret the $\mathcal{H}^{\nu,3}$ operator as the second quantization of the dCS differential operator.

Outline of thesis

At first, an informal summary of the results is provided, avoiding the technicalities of the main text.

Chapter 1 will introduce the basic concepts and the mathematical objects used in this thesis and their relations. It is recommended that readers unfamiliar with the subject skim through Chapter 1 briefly before reading the rest.

In Chapter 2, the boson-fermion correspondence will be discussed and extended in order to construct anyon field operators in $(1+1)$-dimensions. Finally, the dual anyon field operator is introduced, and we show that the anyon and dual anyon field operators obey a generalization of the canonical anti-commutation relation (CAR), and that they have other interesting algebraic relations.

Chapter 3 gives a brief introduction to the Calogero-Sutherland (CS) and the deformed Calogero-Sutherland (dCS) models. We then explicitly show the construction of a self-adjoint operator corresponding to the second quantization of the dCS differential operator and its relation to the anyon operators. We also show that the expectation value of these operators give the CS Hamiltonian and the CS groundstate wavefunction (up to a constant factor and a center of mass (CoM) shift).

In Chapter 4 a new eigenvector of the second quantized dCS differential operator will be constructed by using the anyon and dual anyon field operators. Then we show that the expectation value of the $\mathcal{H}^{\nu,3}$ operator and this new eigenvector corresponds to the dCS differential operator and the deformed groundstate eigenfunction, up to a constant and a CoM shift. Using this result, a general method for constructing eigenfunctions of the dCS differential operator is obtained.

Throughout the thesis, all proofs that require mathematical calculations are put into appendices B-E. Appendix A contains a summery of the most common notations used throughout this thesis.

Summary of results

In this Section a brief and compact summary is given that does not include the technicalities of the main text of this thesis. The result of the summary is obtained from the results in the main text for the formal distributions in a suitable distributional sense. See Appendix A for details on the notations used.

Summary

The Calogero-Sutherland model is a quantum mechanical model describing $N \in \mathbb{N}$ indistinguishable particles on a circle with circumference $L > 0$, which we denote by S_L, via a translationally invariant, inverse quadratic potential. The Calogero-Sutherland model is defined by the Hamiltonian

$$H_N := -\sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + 2\lambda(\lambda - 1) \sum_{k'<k} V(x_k - x_{k'}) \tag{1}$$

where

$$V(r) := \frac{\pi^2}{L^2} \frac{1}{\sin^2\left(\frac{\pi}{L}(r)\right)}$$

and $x_k \in S_L$ is such that $x_k > x_{k'}$ for all $k > k'$.

The CS model has the groundstate (GS) wavefunction

$$\psi_0(\mathbf{x}) := \prod_{k'<k} \left(\sin\left(\frac{\pi}{L}(x_k - x_{k'})\right)\right)^{\lambda} \tag{2}$$

where $\psi_0(\mathbf{x}) \in L^2(S_L^N)$ for $\lambda \geq -\frac{1}{2}$, and corresponds to an eigenvalue of

$$E_0 := \frac{\pi^2 \lambda^2 N \left(N^2 - 1\right)}{3L^2}$$

The CS model has exact eigenfunctions of the form

$$\Psi_{\mathbf{n}}(\mathbf{x}) = \mathcal{P}_{\mathbf{n},\lambda}(\mathbf{w}) \psi_0(\mathbf{x})$$

where $\mathcal{P}_{\mathbf{n},\lambda}(\mathbf{w})$ are polynomials which are invariant under permutations of particle coordinates, labeled by partitions $\mathbf{n} := (n_1, n_2, \ldots, n_N) \in \mathbb{N}^N$, and $w_k := e^{\frac{2\pi}{L} i x_k}$.

The deformed Calogero-Sutherland (dCS) model is defined by the differential operator

$$\tilde{H}_{N,M} := -\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2} + \lambda\sum_{j=1}^{M}\frac{\partial^2}{\partial y_j^2} + 2\,(1-\lambda)\sum_{k=1}^{N}\sum_{j=1}^{M}V(x_k - y_j) + 2\lambda\,(\lambda-1)\sum_{k'<k}V(x_k - x_{k'}) + \frac{2\,(\lambda-1)}{\lambda}\sum_{j'<j}V(y_j - y_{j'}) \quad (3)$$

where $y_j \in S_L$ for all $j = 1, 2, \ldots, M$ such that $y_j > y_{j'}$ for $j > j'$, and $x_k \neq y_j$ for all j, k.

The dCS differential operator has exact eigenfunctions which are similar to the exact eigenfunctions of the CS Hamiltonian. The eigenfunction corresponding to $\psi_0(\mathbf{x})$ is denoted by $\tilde{\psi}_0(\mathbf{y},\mathbf{x})$, which we refer to as the deformed groundstate (dGS) eigenfunction, and is defined as

$$\tilde{\psi}_0(\mathbf{y},\mathbf{x}) := \prod_{k'<k}\sin^{\lambda}\left(\frac{\pi}{L}(x_k - x_{k'})\right)\prod_{j'<j}\sin^{\frac{1}{\lambda}}\left(\frac{\pi}{L}(y_j - y_{j'})\right)\prod_{k=1}^{N}\prod_{j=1}^{M}\frac{1}{\sin\left(\frac{\pi}{L}(x_k - y_j)\right)}$$

The eigenvalue of the dCS differential operator corresponding to $\tilde{\psi}_0(\mathbf{y},\mathbf{x})$ is

$$\tilde{E}_0 := \frac{\pi^2}{3L^2}\frac{(N\lambda - M)\,(N\lambda - \lambda - M)\,(N\lambda + \lambda - M) - M\left(\lambda^2 - 1\right)}{\lambda}$$

Abelian anyons are quantum particles which are characterized by their particle exchange relation where they can obtain any phase shift. Anyon quantum field operators would be characterized by the relation

$$\phi_1(x)\phi_2(y) = e^{\pm i\vartheta}\phi_2(y)\phi_1(x) \quad \text{for } x \lessgtr y,\ x, y \in S_L\ , x \neq y$$

with a phase $\vartheta \in \mathbb{R}$. Thus the anyons become bosonic if $\vartheta = 2\pi n$ or fermionic if $\vartheta = \pi\,(2n+1)$ where $n \in \mathbb{Z}$.

Starting from the boson-fermion correspondence in $(1+1)$ dimensions, there is a generalization which allows us to construct field operators satisfying the exchange relation for abelian anyons [19]. The formal anyon field operators are denoted by $\phi^{\nu}(x)$, where ν is the statistical parameter[3] of the anyon field operators, and obey[4]

$$\begin{aligned} \phi^{\nu}(x) &= \phi^{-\nu}(x)^* \\ \phi^{\nu}(x)\phi^{\nu'}(y) &= e^{-i\pi\nu\nu'\operatorname{sgn}(x-y)}\phi^{\nu'}(y)\phi^{\nu}(x) \quad , \ x \neq y,\ x, y \in S_L \end{aligned}$$

where $\pi\nu\nu' \in \mathbb{R}$. Our construction of the anyon field operators require that the statistical parameter of the anyon field operator is an integer times an arbitrary

[3] The statistical parameter ν is not the filling factor for the fractional quantum Hall effect.

[4] We will denote complex conjugation and Hermitian conjugate by *

constant $\nu_0 \in \mathbb{R} \setminus \{0\}$ (c.f. Section 2.2).

By generalizing the construction of the fermion W-algebra to anyons, we construct an operator valued generating function for the anyon differential operators (c.f. Appendix D.3) and thereby construct the anyon differential operators up to second order and their relations (c.f. Section 3.2).

The second quantization of the dCS differential operator is denoted by $\mathcal{H}^{\nu,3}$ (given by Eq. (3.20)) and is a self-adjoint operator obeying the highest weight condition

$$\mathcal{H}^{\nu,3}\Omega = 0 \quad , \forall \, \nu$$

where Ω denotes the vacuum vector in the Fock space $\mathcal{F}$ (c.f. Section 1.1).

The anyon field operators and the $\mathcal{H}^{\nu,3}$ operator obey the commutation relation

$$\left[\mathcal{H}^{\nu,3}, \phi^{\nu}(x)\right]\Omega = -\frac{\partial^2}{\partial x^2}\phi^{\nu}(x)\Omega$$

The commutation relation holds for all ν as long as the $\mathcal{H}^{\nu,3}$ operator and anyon field operator have the same statistical parameter.

The zeroth order anyon differential operator is denoted by $\mathcal{H}^{\nu,1}$ and obeys

$$\begin{aligned} \mathcal{H}^{\nu,1}\Omega &= 0 \\ \left[\mathcal{H}^{\nu,1}, \phi^{\pm\nu}(x)\right] &= \pm\phi^{\pm\nu}(x) \end{aligned}$$

The $\mathcal{H}^{\nu,3}$ operator can be interpreted as the charge operator for anyons with statistical parameter ν.

Let $\Phi^{\nu}(\mathbf{x})$ denote the product of $N \in \mathbb{N}$ anyon field operators, *i.e.*

$$\Phi^{\nu}(\mathbf{x}) := \phi^{\nu}(x_1)\phi^{\nu}(x_2)\cdots\phi^{\nu}(x_N)$$

where $\mathbf{x} := (x_1, x_2, \ldots, x_N) \in S_L^N$ is such that $x_k > x_{k'}$ for all $k > k'$.

The $\mathcal{H}^{\nu,3}$ operator and the many-body anyon field operator $\Phi^{\nu}(\mathbf{x})$ obey the commutation relation

$$\left[\mathcal{H}^{\nu,3}, \Phi^{\nu}(\mathbf{x})\right]\Omega = \left(-\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2} + 2\nu^2\left(\nu^2-1\right)\sum_{k'<k}V(x_k - x_{k'})\right)\Phi^{\nu}(\mathbf{x})\Omega \qquad (4)$$

Comparing Eq. (4) to Eq. (1), we see that the commutation relation yields the CS Hamiltonian for $\lambda = \nu^2$. The $\mathcal{H}^{\nu,3}$ operator can therefore be interpreted as the second quantization of the CS Hamiltonian.

Let η be an eigenvector of the $\mathcal{H}^{\nu,3}$ operator, belonging to a dense, invariant domain, with eigenvalue $\mathcal{E}$. Then the function $F_\eta(\mathbf{x})$, defined as

$$F_\eta(\mathbf{x}) := \langle \eta, \Phi^{\nu}(\mathbf{x})\Omega\rangle$$

is an eigenfunction of the CS Hamiltonian with eigenvalue $\mathcal{E}$, since

$$\left\langle \eta, \mathcal{H}^{\nu,3}\Phi^{\nu}(\mathbf{x})\Omega\right\rangle = H_N\left\langle \eta, \Phi^{\nu}(\mathbf{x})\Omega\right\rangle$$

We show in Section 3.3 that there exist a eigenvector η_{CS}, which corresponds to the GS wavefunction of the CS model, up to a constant and with a center of mass

(CoM) shift (c.f. Corollary 3.3.6).

We introduce $\phi^{-\frac{1}{\nu}}(y)$ (c.f. Eq. (2.19)) and postulate that the $\phi^{-\frac{1}{\nu}}$ operator corresponds to the dual anyon field operator to an anyon field operator with statistical parameter ν. The dual anyon field operator is a well-defined field operator if and only if $\frac{1}{\nu\nu_0} \in \mathbb{Z}$, where ν_0 is the same constant used in the construction of the anyon field operators. The anyon and the dual anyon field operators obey (c.f. Eqs. (2.20) and (2.22))

$$\left\{\phi^{\nu}(x), \phi^{-\frac{1}{\nu}}(y)\right\} = L\delta(x-y)\phi^{\left(\nu-\frac{1}{\nu}\right)}(x)$$

where $\delta(x)$ is the Dirac delta and the factor of L is due to the fact that the anyon operators are dimensionless.

The second quantization of the CS Hamiltonian has an interesting duality for the operators with statistical parameter ν and $-1/\nu$ given by

$$\mathcal{H}^{\nu,3} = -\nu^2\mathcal{H}^{-\frac{1}{\nu},3} + \frac{\pi}{3L^2}\frac{\left(\nu^4-1\right)}{\nu^2}\mathcal{H}^{-\frac{1}{\nu},1}$$

So it is possible to create eigenvectors of the $\mathcal{H}^{\nu,3}$ operator by using the dual anyon operator.

These two relations support the notion that the dual operator for an anyon field operator ϕ^{ν} has to be $\phi^{-\frac{1}{\nu}}$ rather than the conjugate $\phi^{-\nu}$.

Let $\varphi^{\nu}(\mathbf{y},\mathbf{x})$ denote the product of $M \in \mathbb{N}$ dual anyon field operators and $N \in \mathbb{N}$ anyon field operators, *i.e.*

$$\varphi^{\nu}(\mathbf{y},\mathbf{x}) := \Phi^{-\frac{1}{\nu}}(\mathbf{y})\Phi^{\nu}(\mathbf{x})$$

where $\mathbf{y} := (y_1, y_2, \ldots, y_M) \in S_L^M$ is such that $y_j > y_{j'}$ for all $j > j'$ and $x_k \neq y_j$ for all k, j.

The operator $\varphi^{\nu}(\mathbf{y},\mathbf{x})$ and the $\mathcal{H}^{\nu,3}$ operator obey the commutation relation (c.f. Theorem 4.1.2)

$$\left[\mathcal{H}^{\nu,3}, \varphi^{\nu}(\mathbf{y},\mathbf{x})\right]\Omega = \left(\tilde{H}_{N,M} + \frac{\pi^2}{3L^2}\frac{\nu^4-1}{\nu^2}M\right)\varphi^{\nu}(\mathbf{y},\mathbf{x})\Omega \tag{5}$$

where $\tilde{H}_{N,M}$ is the dCS differential operator for $\lambda = \nu^2$.

Equation (5) allows the construction of eigenfunctions of the dCS differential operator (c.f. Theorem 4.2.1), which is as follows.

Let $\tilde{\eta}$ be an eigenvector of the $\mathcal{H}^{\nu,3}$ operator, with eigenvalue $\tilde{\mathcal{E}}$, and belong to an dense, invariant domain. Then there exists a function $\tilde{F}_{\tilde{\eta}}(\mathbf{y},\mathbf{x})$ defined as

$$\tilde{F}_{\tilde{\eta}}(\mathbf{y},\mathbf{x}) := \langle\tilde{\eta}, \varphi^{\nu}(\mathbf{y},\mathbf{x})\Omega\rangle$$

which[5] is an eigenfunction of the dCS differential operator with eigenvalue

$$\tilde{H}_{N,M}\tilde{F}_{\tilde{\eta}}(\mathbf{y},\mathbf{x}) = \left(\tilde{\mathcal{E}} - \frac{\pi^2}{3L^2}\frac{\nu^4-1}{\nu^2}M\right)\tilde{F}_{\tilde{\eta}}(\mathbf{y},\mathbf{x})$$

Creating $\tilde{\eta}$ vector corresponding to the exact solutions of the dCS model is outside the scope of this thesis. We restrict ourselves to the simplest possible eigenvector, $\tilde{\eta}_0$, for now.

The Fock space inner product of a vector created by the $\varphi^{\nu}(\mathbf{y},\mathbf{x})$ field operator and the $\tilde{\eta}_0$ vector is denoted by

$$\langle \tilde{\eta}_0, \varphi^{\nu}(\mathbf{y},\mathbf{x})\Omega\rangle = \tilde{F}_0(\mathbf{y},\mathbf{x})$$

where $\tilde{F}_0(\mathbf{y},\mathbf{x})$ is given by (c.f. Section 4.2 or Corollary 4.2.2)

$$\tilde{F}_0(\mathbf{y},\mathbf{x}) := \kappa\ \tilde{\psi}_0(\mathbf{y},\mathbf{x})\ e^{-i\frac{\pi(N\nu^2-M)}{L\nu^2}\left(\nu^2\sum_{k=1}^{N}x_k - \sum_{j=1}^{M}y_j\right)}$$

where

$$\tilde{\psi}_0(\mathbf{y},\mathbf{x}) :=$$
$$\prod_{j'<j}\sin^{\frac{1}{\nu^2}}\left(\frac{\pi}{L}(y_j-y_{j'})\right)\prod_{k'<k}\sin^{\nu^2}\left(\frac{\pi}{L}(x_k-x_{k'})\right)\prod_{k=1}^{N}\prod_{j=1}^{M}\frac{1}{\sin\left(\frac{\pi}{L}(x_k-y_j)\right)}$$

and $\kappa \in \mathbb{C}$ is a constant. The correlation function equals the dGS eigenfunction of the dCS differential operator, up to a constant and a CoM shift, for $\lambda = \nu^2$. We show that this is indeed the case where the eigenvalue corresponds to the eigenvalue of the dGS eigenfunction with a CoM shift contribution (c.f. Section 4.2).

[5]It is assumed that the $\tilde{\eta}$ vector is not orthogonal to the vector created by the many-body operator $\varphi^{\nu}(\mathbf{y},\mathbf{z})$.

Chapter 1

Preliminaries

There are several important notations that have to be explained before getting into the subject. In this Chapter, we introduce the general concepts used throughout this work and their relations. Section 1.4 is on formal distributions and follows Chapter 2 of [22], while the rest follows Refs. [19] and [23].

1.1 Introductory explanations

We denote $\mathcal{H}$ as the 1-particle Hilbert space, where $\mathcal{H}$ is given as the direct sum of two infinite dimensional subspaces, *i.e.*

$$\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$$

where $\mathcal{H}_\pm = P_\pm \mathcal{H}$ and $P_\pm$ are projection operators, obeying

$$P_\pm^2 = P_\pm^* = P_\pm \quad , \; P_+ + P_- = I$$

where I is the identity operator. We also denote the Hilbert space inner product as $(\; , \;)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$.

The fermion Fock space is defined as

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{S}_- \mathcal{H}^{\otimes n} \tag{1.1}$$

where $\mathcal{S}_-$ is the anti-symmetrizing operator and $\mathcal{H}^{\otimes n}$ is the n-fold tensor product of $\mathcal{H}$ with $\mathcal{S}_- \mathcal{H}^{\otimes 0} := \mathbb{C}$. As a shorthand, we use $\mathcal{F}$ to denote the fermion Fock space $\mathcal{F}(\mathcal{H})$.

Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis in $\mathcal{H}$ such that

$$P_- e_n = \begin{cases} e_n & , \text{ if } n < 0 \\ 0 & , \text{ otherwise} \end{cases}$$

We define a representation of the fermion field algebra[1] $\psi_n^* := \psi^*(e_n)$ and $\psi_n := (\psi_n^*)^*$ obeying the *canonical anti-commutation relations* (CAR)

$$\{\psi_n \ , \ \psi_m\} = \{\psi_n^* \ , \psi_m^*\} = 0$$
$$\{\psi_n \ , \ \psi_m^*\} = \delta_{n,m} \tag{1.2}$$

where $\{\ ,\ \}$ is the anti-commutator bracket (c.f. Appendix A).

We consider the quasi-free representation of the fermion field algebra associated with our Hilbert space $\mathcal{H}$ and where the negative states are filled, *i.e.*

$$\psi_n\Omega = \psi_{-n-1}^*\Omega = 0 \quad , \text{ for all } n \geq 0$$

where we denote the vacuum vector in the Fock space as $\Omega \in \mathcal{F}$.

It is, sometimes, preferential to use a basis independent characterization of the fermion operators. Let $f \in \mathcal{H}$ and define $\psi^*(f) := \sum_{n\in\mathbb{Z}} f_n\psi_n^*$, $\psi(f) := (\psi^*(f))^*$, and where $f_n := (f, e_n)_{\mathcal{H}}$. Equation (1.2) implies

$$\{\psi(f), \psi(g)\} = \{\psi^*(f), \psi^*(g)\} = 0$$
$$\{\psi(f), \psi^*(g)\} = (f, g)_{\mathcal{H}}\, I \quad \forall\ f, g \in \mathcal{H}$$

where the scalar product $(f, g)_{\mathcal{H}}$ is linear in g.

1.1.1 Second quantization maps

Definition 1.1.1. An operator A is *Hilbert-Schmidt* (HS) if and only if

$$\mathrm{Tr}_{\mathcal{H}}(A^*A) < \infty$$

where $\mathrm{Tr}_{\mathcal{H}}$ is the Hilbert space trace.

The set of all HS operators is denoted as $\mathcal{B}_2(\mathcal{H})$. For any operator A on $\mathcal{H}$, there is a decomposition given by

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \quad A_{\epsilon\epsilon'} := P_\epsilon A P_{\epsilon'} \ , \ \epsilon, \epsilon' = \pm$$

Definition 1.1.2. An operator U on $\mathcal{H}$ is said to obey the *Hilbert-Schmidt condition* if $U_{\pm\mp} \in \mathcal{B}_2(\mathcal{H})$.

[1]Note that * is used to represent both complex conjugation and Hilbert space adjoint throughout this work.

Let $\mathcal{U}$ denote the set of all unitary operators on $\mathcal{H}$ that obey the HS condition. For any $U \in \mathcal{U}$, there is an unitary operator[2] on the Fock space $\mathcal{F}$ denoted by $\Gamma(U)$ and defined, up to a phase, by

$$\Gamma(U)\psi^*(f)\Gamma(U)^* = \psi^*(Uf) \quad , \forall\, U \in \mathcal{U},\ f \in \mathcal{H}$$

$$\Gamma(U)^* = \Gamma(U^*)$$

This implies that

$$\Gamma(U_1)\Gamma(U_2) = \sigma(U_1, U_2)\Gamma(U_1 U_2) \quad \text{for all } U_1, U_2 \in \mathcal{U} \tag{1.3}$$

where σ is a non-trivial[3] $U(1)$-valued two-cocycle associated with $\mathcal{U}$, such that the product in Eq. (1.3) is associative.

Let $\mathfrak{g}$ denote the set of all bounded operators acting on the Hilbert space $\mathcal{H}$, and satisfying the HS condition. For an operator $A \in \mathfrak{g}$, define $d\Gamma(A)$ as the quasi-free 2nd quantized operator obeying the following conditions

$$[d\Gamma(A), \psi^*(f)] = \psi^*(Af) \quad , \forall\, f \in \mathcal{H}$$

$$\langle d\Gamma(A)\rangle = 0$$

where $\langle\ \rangle$ denotes the vacuum expectation value (c.f. Appendix A). Since $A \to d\Gamma(A)$ is linear, there is a natural extension to linear combinations of operators. So for $A = A_1 + iA_2 \in \mathfrak{g}$, we have $d\Gamma(A) = d\Gamma(A_1 + iA_2) = d\Gamma(A_1) + id\Gamma(A_2)$. Although A is a bounded operator, the operator $d\Gamma(A)$ is unbounded in general, and therefore needs to be defined on a dense subspace $\mathcal{D} \subset \mathcal{H}$ [23]. The treatment of the operator $d\Gamma(A)$ can also be extended to certain unbounded operators A [24].

It is also known that

$$[d\Gamma(A)\ , d\Gamma(B)] := d\Gamma([A, B]) + iS(A, B) \tag{1.4}$$

where $[\ ,\ ]$ denotes the commutator (c.f. Appendix A). The term $iS(A, B)$ is referred to as the Schwinger term in the physics literature. Our use of the quasi-free representation (c.f. Section 1.1) requires us to normal-order the second quantization map, $d\Gamma(A)$, and the Schwinger term arises due to the normal-ordering (this will be discussed in Section 1.2.1). The Schwinger term is given by[4]

$$iS(A, B) := \sum_{n\in\mathbb{Z}} (e_n, P_- A P_+ B P_- e_n)_{\mathcal{H}} - (e_n, P_- B P_+ A P_- e_n)_{\mathcal{H}} \tag{1.5}$$

---

[2] Also known as an "implementer"

[3] Such that there is no transformation $U_j \to \mathfrak{U}_j$ such that $\Gamma(U) \to b(U)\Gamma(U)$, where $b(U)$ is a $U(1)$-valued function, which gives $\sigma(\mathfrak{U}_1, \mathfrak{U}_2) = 1$. The two-cocycle is non-trivial only if $\mathcal{H}_+$ and $\mathcal{H}_-$ both are infinite dimensional.

[4] This is just one of the possible ways to calculate the Schwinger term. Another way is given in Ref. [19]

Let $S_L = [-L/2, L/2]$ denote a circle with circumference L.
The Hilbert space is then set to $L^2(S_L)$ throughout the rest of the work. We can then define an orthonormal basis for $L^2(S_L)$ as follows,

$$e_n(x) := \frac{1}{\sqrt{2\pi}}\; e^{\frac{2\pi}{L} i\left(n+\frac{1}{2}\right)x} \quad , n \in \mathbb{Z}, x \in S_L \tag{1.6}$$

Fourier transformation is given by

$$f_n = \frac{1}{\sqrt{2\pi}} \int_{S_L} dx\; f(x) e^{-i\frac{2\pi}{L}\left(n+\frac{1}{2}\right)x} \quad , n \in \mathbb{Z}$$

where

$$f = \frac{2\pi}{L} \sum_{n\in\mathbb{Z}} f_n e_n$$

The projection operators $P_\pm$ are then given by

$$P_\pm\; e_n = \theta\left(\pm\left(n+\frac{1}{2}\right)\right)\; e_n \quad , n \;\in\; \mathbb{Z} \tag{1.7}$$

where θ is the Heaviside step function (c.f. Appendix A).

Let $\mathcal{G} = \mathrm{Map}(S_L, U(1))$ be the set of smooth maps from the circle to $U(1)$. Any $\varphi \in \mathcal{G}$ can be written as $\varphi = e^{if}$ where f is a smooth map[5] $S_L \to \mathbb{C}$.

In general, f is assumed to be of the form

$$f(x) = \frac{2\pi}{L}\omega x + \tilde{P}_+\alpha(x) + \tilde{P}_-\alpha(x) + \tilde{P}_0\alpha(x) \tag{1.8}$$

where $\omega := \frac{1}{2\pi}\left(f(\frac{L}{2}) - f(\frac{-L}{2})\right)$ is the winding number of φ. We also have $\alpha(x)$, which is a periodic smooth map, given as

$$\alpha(x) := \sum_{n\in\mathbb{Z}} \alpha_n e^{\frac{2\pi}{L} inx}$$

where

$$\alpha_n := \int\limits_{S_L} dx\; \alpha(x) e^{-\frac{2\pi}{L} inx}$$

We also introduced $\tilde{P}_\pm$ and $\tilde{P}_0$ as projection operator, obeying

$$\tilde{P}_\epsilon^2 = \tilde{P}_\epsilon^* = \tilde{P}_\epsilon \quad , \epsilon = +, -, 0$$
$$\tilde{P}_+ + \tilde{P}_- + \tilde{P}_0 = I$$

such that

$$\tilde{P}_+\alpha(x) = \sum_{n>0} \alpha_n e^{\frac{2\pi}{L} inx}\;,\;\; \tilde{P}_-\alpha(x) = \sum_{n<0} \alpha_n e^{\frac{2\pi}{L} inx}\;,\;\; \tilde{P}_0\alpha(x) = \alpha_0$$

[5] f should be a smooth map to $\mathbb{R}$ but can be extended to $\mathbb{C}$, due to linearity of the sum operator.

For all $\varphi \in \mathcal{G}$, we have

$$\Gamma(\varphi) = e^{id\Gamma(f)}$$

where

$$[d\Gamma(f_1), d\Gamma(f_2)] = iS(f_1, f_2)$$

The 2-cocycle becomes

$$\sigma(U(f_1), U(f_2)) = e^{-iS(f_1,f_2)/2}$$

by using Eq. (1.32).

The choice of phase for the implementer is such that

$$\left\langle \Gamma(e^{if}) \right\rangle = 0 \quad \text{if } \omega \neq 0$$

1.2 Normal ordering

An important mathematical result used for calculations in many-body systems, is the boson-fermion correspondence (bosonization) in (1 + 1)-dimensions. Normal ordering is then required in order to compensate for the infinites that arise from using the quasi-free representation. An important aspect of normal-ordering is that all normal-ordered operator products are well-behaved and have a well-defined operator product expansion.

1.2.1 Fermion normal-ordering

A fermion normal-ordered operator has a groundstate represented by the vacuum vector Ω, which gives that the vacuum expectation value (VEV) of a fermion normal-ordered product is always zero. We include the definition of fermion normal-ordering only for bilinear forms of the fermion creation and annihilation operators.

We denote a *fermion normal-ordered* operator by : : and define the sum operator as

$$d\Gamma(A) \equiv : \left(\sum_{n,m\in\mathbb{Z}} (A)_{n,m} \, \psi_n^* \psi_m \right) := \sum_{n,m\in\mathbb{Z}} (A)_{n,m} : \psi_n^* \ \psi_m : \tag{1.9}$$

where $(A)_{n,m} := (e_n, Ae_m)_{\mathcal{H}}$ are the matrix element of the operator A. The fermion normal-ordered form of the bilinear fermion creation and annihilation operators is given by

$$: \psi_n^* \psi_m := \begin{cases} -\psi_m \psi_n^* & \text{, if } \ n = m < 0 \\ \psi_n^* \psi_m & \text{, otherwise} \end{cases} = \psi_n^* \psi_m - \theta \left(-n - \frac{1}{2} \right) \delta_{n,m} \tag{1.10}$$

for all $n, m \in \mathbb{Z}$. So in the special case where the operator A is the identity operator I (where $(I)_{n,m} = \delta_{n,m}$), Eq. (1.9) becomes

$$\sum_{n\in\mathbb{Z}} : \psi_n^* \psi_n : = \sum_{n=0}^{\infty} \psi_n^* \psi_n - \psi_{-n-1} \psi_{-n-1}^* \tag{1.11}$$

Equations (1.9) and (1.10), together with

$$\begin{aligned} :\psi_n\psi_m: &= \psi_n\psi_m \\ :\psi_n^*\psi_m^*: &= \psi_n^*\psi_m^* \end{aligned}$$

for all $n, m \in \mathbb{Z}$, define fermion normal-ordering for bilinear forms .

1.2.2 Boson normal-ordering

Boson normal-ordering of operators is denoted by ${}^*_*\ {}^*_*$ and is defined by

$${}^*_* R^\omega e^{\sum\limits_{n\in\mathbb{Z}} c_n\rho_n} {}^*_* = {}^*_* e^{\sum\limits_{n\in\mathbb{Z}} c_n\rho_n} R^\omega {}^*_* = e^{\sum\limits_{n<0} c_n\rho_n} e^{\frac{1}{2}c_0\rho_0} R^\omega e^{\frac{1}{2}c_0\rho_0} e^{\sum\limits_{n>0} c_n\rho_n}$$

$${}^*_* AB {}^*_* = {}^*_* \left({}^*_* A {}^*_*\right)\left({}^*_* B {}^*_*\right) {}^*_*$$

where $\omega \in \mathbb{Z}$, A, B are some operators, c_n are constants, and the operators R and ρ_n are introduced in Section 1.3. Boson normal-ordering will be used in order to remove divergences from the operators $\Gamma(\varphi)$, introduced in the previous Section. Note that a boson normal-ordered product of operators is not necessarily a fermion normal-ordered product, since a boson normal-ordered operator could have a non-zero VEV. Examples of boson normal-ordering are also found in Appendix B and throughout this work.

1.3 Many-body operators

1.3.1 Operators of interest

Let s_n, $n \in \mathbb{Z}$, denote the shift operator such that $s_n e_m = e_{m-n}, m \in \mathbb{Z}$. The matrix elements of the shift operator are given by $(s_n)_{n',m} = \delta_{n',m-n}$, so that the many-body equivalent becomes

$$\rho_n := d\Gamma(s_n) = \sum_{m\in\mathbb{Z}} :\psi^*_{m-n}\psi_m: \tag{1.12}$$

The operators ρ_n, for $n \neq 0$, are commonly referred to as the oscillation operators, and obey

$$\rho_n\Omega = 0 \quad , \forall\ n \geq 0 \tag{1.13}$$
$$\rho_n^* = \rho_{-n}$$

The oscillation operators obeys the commutation relation

$$[\rho_n, \rho_m] = n\delta_{n,-m} \quad , \ \forall n, m \in \mathbb{Z} \tag{1.14}$$

Since $[s_n, s_m] = 0$, the commutation relation is non-trivial only due to the Schwinger term (see Appendix D.1 for details and proofs). Due to their commutation relation, the oscillation operators can be interpreted as bosonic operators.[6]

[6]One can create bosonic operators by rescaling the shift operators by $1/\sqrt{n}$, see Eq. (C.2).

A special case of the oscillation operators is the self-adjoint operator Q defined as

$$Q := \rho_0 = d\Gamma(I) \tag{1.15}$$

This operator counts the number of particles minus the number of holes ($Q = N_{\text{particles}} - N_{\text{holes}}$) and is equal to the operator in Eq. (1.11). The particles have a positive charge[7] 1 and the holes have charge -1. So the operator Q represent a measurement of the total charge of the system. By definition, the charge of the vacuum is zero. Inserting $n = 0$ in equation (1.14) gives

$$[Q, \rho_m] = 0 \quad \forall\, m \in \mathbb{Z} \tag{1.16}$$

Thus ρ_n does not change the total particle number of the system.

One other important operator is the "ladder operators", defined as

$$R := \Gamma(s_{-1}) \tag{1.17}$$

which, as one can prove, obeys

$$R^{-w} \rho_n R^w = \rho_n + \delta_{n,0}\; wI \quad ,\forall\, n, w \in \mathbb{Z} \tag{1.18}$$

The unitary operator R raises the total particle number by exactly one, something that no bosonic operator can do due to Eq. (1.16). These operators, commonly known as the *Klein factors*, fills the upper-most empty state of the system, while the inverse (Hermitian conjugate) would empty the upper-most filled state.

The precise mathematical construction of the ladder operators are outside the scope of this thesis. For more detail and explicit construction of the Klein factors see [25]. Due to their properties, the Klein operators are only well defined when raised to a power which is an integer.

1.3.2 Formal field operators

The fermion field operators are defined as

$$\psi^*(x) := \frac{1}{\sqrt{L}} \sum_{n \in \mathbb{Z}} \psi_n^*\; e^{-i\frac{2\pi}{L}(n+\frac{1}{2})x} \quad , \; \psi(x) := \frac{1}{\sqrt{L}} \sum_{n \in \mathbb{Z}} \psi_n\; e^{i\frac{2\pi}{L}(n+\frac{1}{2})x}$$

in position space, $x \in S_L$. They obey the formal CAR.

$$\{\psi(x), \psi(y)\} = \{\psi^*(x), \psi^*(y)\} = 0$$

$$\{\psi^*(x), \psi(y)\} = \delta(x - y)$$

for all $x, y \in S_L$ and where $\delta(x)$ is the Dirac delta-function.

[7] The charge unit is set to 1 throughout this work

In position space, the oscillation operator are given by

$$\rho(x) := \frac{1}{L}\sum_{n\in\mathbb{Z}} \rho_n e^{i\frac{2\pi}{L}nx} \quad , \; x \in S_L \tag{1.19}$$

and obey

$$\rho(x)^* = \rho(x)$$
$$[\rho(x), \rho(y)] = \frac{1}{2\pi i}\; \partial_x \delta(x-y) = -\frac{1}{2\pi i}\partial_y \delta(x-y) \tag{1.20}$$

where $\partial_x := \frac{\partial}{\partial x}$.

By using Eq. (1.12), the operator in Eq. (1.19) can be expressed as

$$\rho(x) = :\psi^*(x)\psi(x): \tag{1.21}$$

and is therefore interpreted as the fermion density operator.

1.3.3 Operator regularization

Note that the formal field operators presented in Section 1.3.2 are operator valued distributions and should be treated carefully. One way to do that would be to ”smear” out the operators with a well defined test-function in order to have a well defined operator on $\mathcal{F}$.

Another useful method to deal with this problem is to include a regularization in the Fourier transformation, *e.g.*

$$\rho_\varepsilon(x) := \sum_{n\in\mathbb{Z}} \rho_n \; e^{\frac{2\pi}{L}inx} e^{-\frac{2\pi}{L}|n|\varepsilon}$$

which is a well defined operator for $\varepsilon > 0$. Taking the limit $\varepsilon \downarrow 0$ would then yield the operator-valued distributions again. This technique simplifies the operator product expansions (OPE) as it also removes short-distance divergences. This method will, however, only be used in Appendix D.2.

1.4 Formal distributions

In Section 1.3.3 it was made clear that one should be careful when dealing with operator-valued distributions. But the regularization introduced, while making things more well-defined, will complicate some of the calculations in Chapters 3 and 4, by adding terms that vanish when taking the limits ($\varepsilon \downarrow 0$).

So for this work, the regularization term will be changed from $e^{-\varepsilon|n|}$ to $e^{\;\varepsilon n}$, $\varepsilon > 0$. At first, this seems like an ill-advised choice since the sum over positive integers, for the real space representation, will grow exponentially. It will not alter the calculations as all operators and OPE's will be applied to the vacuum from the left, such that the exponentially growing part of the field operators will be truncated after a finite number of terms.

But now these object are not well-defined operators and the use of them is a delicate matter. So this Section will try to explain the methods to make calculations with such objects well-defined and we follow Chapter 2 in [22].

1.4.1 Preliminaries

Definition 1.4.1. A *formal distribution*, in the indeterminates $z, w, \dots$, is defined as a series of the form

$$\sum_{n,m,\dots\in\mathbb{Z}} a_{n,m,\dots} z^n w^m \cdots$$

where $a_{n,m,\dots}$ are elements of a vector space V over $\mathbb{C}$ and the indeterminates $z, w, \dots$ have values in V. The formal distribution forms a vector space denoted by $V\left[\left[z, z^{-1}, w, w^{-1}, \dots\right]\right]$

Definition 1.4.2. For a formal distribution $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$, the *residue* is defined as

$$\mathrm{Res}_z f(z) = f_{-1} \tag{1.22}$$

This implies

$$\mathrm{Res}_z \partial_z f(z) = 0$$

$$\mathrm{Res}_z \left(\partial_z f(z)\right) g(z) = -\mathrm{Res}_z f(z) \partial_z g(z)$$

where $f(z), g(z)$ are formal distributions and ∂_z denotes differentiation in the indeterminate of the formal distribution. The last equation is the usual integration by parts, provided that the operator product fg is defined.

Let $\mathbb{C}\left[z, z^{-1}\right]$ denote the algebra of Laurent polynomials. Then for a formal distribution $f(z) \in V\left[\left[z, z^{-1}\right]\right]$, there is a pairing $V\left[\left[z, z^{-1}\right]\right] \times \mathbb{C}\left[z, z^{-1}\right] \to V$. So the Laurent polynomials should be viewed as the test-functions for the formal distributions and $\mathrm{Res}_z f(z)\phi(z)$ is well-defined for any $f \in V\left[\left[z, z^{-1}\right]\right]$ and $\phi \in \mathbb{C}\left[z, z^{-1}\right]$.

1.4.2 Formal delta-function

Definition 1.4.3. The *formal delta-function*, denoted by $\delta(z-w)$, is a formal distribution in the indeterminates z and w defined as

$$\delta(z-w) := \frac{1}{z} \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \tag{1.23}$$

Assume that $F(z_1, z_2)$ is a function of two complex variables z_1 and z_2 with poles at $z_1 = 0$, $z_2 = 0$ and $|z_1| = |z_2|$. Then we denote the power series expansion of $F(z_1, z_2)$ in the domain $|z_1| > |z_2|$ as $\mathrm{i}_{z_1,z_2} F$.

The formal delta-function can expanded into different domains as

$$\delta(z-w) = \mathrm{i}_{z,w}\frac{1}{(z-w)} - \mathrm{i}_{w,z}\frac{1}{(z-w)} \tag{1.24}$$

where

$$\begin{aligned} \mathrm{i}_{z,w}\frac{1}{z-w} &= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n \\ \mathrm{i}_{w,z}\frac{1}{z-w} &= -\frac{1}{z}\sum_{n=-1}^{-\infty}\left(\frac{w}{z}\right)^n \end{aligned}$$

So for all $j \in \mathbb{N}_0$, we have

$$\partial_w^{(j)}\delta(z-w) := \mathrm{i}_{z,w}\frac{1}{(z-w)^{j+1}} - \mathrm{i}_{w,z}\frac{1}{(z-w)^{j+1}} \tag{1.25}$$

where $A^{(j)} := A^j/j!$ for any operator A.

There are also some important properties of the formal delta-function that are useful later on.

Proposition 1.4.4. *1. For any formal distribution* $f(z) \in V\left[\left[z, z^{-1}\right]\right]$

$$\mathrm{Res}_z f(z)\delta(z-w) = f(w) \tag{1.26}$$

and the product $f(z)\delta(z-w)$ *is well-defined.*

2.

$$f(z)\delta(z-w) = f(w)\delta(z-w) \tag{1.27}$$

3.

$$\delta(z-w) = \delta(w-z) \tag{1.28}$$

4. For all $j \in \mathbb{N}_0$

$$\partial_z^{(j)}\delta(z-w) = (-1)^j\partial_w^{(j)}\delta(z-w) \tag{1.29}$$

Proof. See Appendix E.1.1 □

We define the formal sign function as

$$\operatorname{sign}(z-w) := -\frac{i}{\pi}\left(\ln\left(\frac{z}{|z|}\right) - \ln\left(\frac{w}{|w|}\right) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\left(\frac{z}{w}\right)^n\right) \tag{1.30}$$

which satisfies

$$\begin{aligned} \operatorname{sign}(z-w) &= -\operatorname{sign}(w-z) \\ \frac{2\pi i}{L} z\partial_z \operatorname{sign}(z-w) &= \frac{2z}{L}\delta(z-w) \end{aligned}$$

For indeterminates $z \propto e^{i\frac{2\pi}{L}y}$ and $w \propto e^{i\frac{2\pi}{L}x}$, $x, y \in S_L$, of a formal distribution, the integral over the variable y can be changed to

$$\int_{S_L} \frac{dy}{L}(\cdot) = \oint\limits_{\gamma} \frac{dz}{2\pi i}\frac{1}{z}(\cdot) = \operatorname{Res}_z \frac{1}{z}(\cdot)$$

So the formal distribution $\frac{z}{L}\delta(z-w)$ is the equivalence of the Dirac delta function on the circle S_L since[8]

$$\int_{S_L} \frac{dy}{L} z\delta(z-w) = \oint\limits_{\gamma} \frac{dz}{2\pi i}\frac{1}{z} z\delta(z-w) = \operatorname{Res}_z \delta(z-w) = \int_{S_L} dy\ \delta(y-x)$$

Most of the calculations done using formal distributions where checked by using the regularized operators (c.f. Section 1.3.3). There were no relevant deviation. Interested readers are encouraged to check for themselves.

1.5 Baker-Campbell-Hausdorff formulas

For the convenience of the reader, we collect various Baker-Campbell-Hausdorff (BCH) type formulas that will be used repeatedly.

Let A and B be two arbitrary operators and $h(A)$ some function of A.

Lemma 1.5.1. *For $C := [A, B]$ satisfying $[A, C] = [B, C] = 0$, the following holds*

1.

$$e^{-B} A e^{B} = A + C$$

which gives that

$$\left[A, e^{B}\right] = C\ e^{B} \tag{1.31}$$

[8]There is an abuse of notation as both the formal delta function and the Dirac delta are symbolized by a δ.

2.

$$e^{-B}h(A)e^{B} = h(A+C) \tag{1.32}$$

3.

$$e^{A}e^{B} = e^{C/2}e^{A+B} = e^{C}e^{B}e^{A} \tag{1.33}$$

Proof. See Appendix E.1.2 □

Lemma 1.5.2. *For any operators A and B satisfying $[A,B] = CB$ and $[A,C] = [B,C] = 0$, then $h(A)B = Bh(A+C)$.*

Proof. Taylor expanding $h(A)$ and using that $A^nB = B(A+C)^n$ since $AB = B(A+C)$. □

The special case of $h(A) = e^A$ is an important relation for this work, and gives

$$\begin{aligned} e^{A}B &= Be^{A+C} \\ Be^{A} &= e^{A-C}B \end{aligned} \tag{1.34}$$

Chapter 2

Anyons in $(1+1)$ dimensions

Abelian anyons are defined by their exchange relation where they obtain any phase shift.

Abelian anyon quantum field operators should therefore satisfy the relation

$$\phi_1(x)\phi_2(y) = e^{i\vartheta \mathrm{sgn}(x-y)}\phi_2(y)\phi_1(x) \quad , \; x \neq y, \; x, y \in S_L, \text{ and } \vartheta \in \mathbb{R} \tag{2.1}$$

where "sgn" is the sign function. It is clear from the relation above that, if ϑ is an even multiplet of π, the anyons become bosons, while if ϑ is an odd integer times π, the anyons exhibit fermionic properties. As mentioned in the introduction, there is a interesting relation between anyons and the FQHE which serves as a motivation for this work.

This Chapter will give a brief introduction to the boson-fermion correspondence in $(1+1)$ dimensions and the extension which allow us to construct $(1+1)$ dimensional[1] anyon field operators. Then continue by introducing the corresponding dual anyon field operator and show some very interesting properties.

2.1 Boson-fermion correspondence

In $(1+1)$ dimensions, there exist relations between fermions and bosons. One of these relations was given by Eq. (1.21). Another is the following.

[1]Normally when talking about anyons, they are referring to the quasi-particles in $(2+1)$ dimensions. We therefore stress that the anyon operators that we are using are in (1+1) dimensions only.

Lemma 2.1.1. *All vectors in the fermion Fock space $\mathcal{F}$ can be represented as linear combinations of*

$$B(\{m_n\}_{n=1}^{\infty}, \omega) = \prod_{n=1}^{\infty} \frac{1}{\sqrt{m_n!}} \left(\frac{\rho_{-n}}{\sqrt{n}}\right)^{m_n} R^{\omega}\Omega \quad , m_n \in \mathbb{N}_0,\ \omega \in \mathbb{Z},\ \sum_n m_n < \infty \tag{2.2}$$

or of

$$F(\{m_k, n_k\}_{k=0}^{\infty}) = \prod_{k=0}^{\infty} (\psi_k^*)^{m_k} (\psi_{-k-1})^{n_k}\Omega\ , m_k, n_k \in \{0,1\}\ ,\ \sum_k m_k + n_k < \infty \tag{2.3}$$

Both form a complete orthonormal basis that can be used to span the entire Fock space.

Proof. See Appendix C.1. □

Lemma 2.1.1 allows every operator in the fermion Fock space $\mathcal{F}$ to be expressed in terms of the Klein factors and oscillation operators.

The indeterminates of the formal distributions, w and z, that are used throughout this work are defined as

$$w := e^{\frac{2\pi}{L} ix} e^{\frac{2\pi}{L}\varepsilon} \tag{2.4}$$

$$z := e^{\frac{2\pi}{L} iy} e^{\frac{2\pi}{L}\varepsilon'} \tag{2.5}$$

where $x, y \in S_L$ and $\varepsilon, \varepsilon'$ are constants greater than zero.

Proposition 2.1.2. *The fermion operators can be written as*

$$\begin{aligned} \psi(z) &:= \frac{1}{\sqrt{L}} e^{i\frac{\pi}{L}yQ} R^{-1} e^{i\frac{\pi}{L}yQ} e^{\sum\limits_{n<0}\frac{1}{n}\rho_n z^n} e^{\sum\limits_{n>0}\frac{1}{n}\rho_n z^n} \\ \psi^*(z) &:= \frac{1}{\sqrt{L}} e^{-i\frac{\pi}{L}yQ} R e^{-i\frac{\pi}{L}yQ} e^{-\sum\limits_{n<0}\frac{1}{n}\rho_n z^n} e^{-\sum\limits_{n>0}\frac{1}{n}\rho_n z^n} \end{aligned} \tag{2.6}$$

Proof. See Appendix E.2.1 □

The normalization factor $\frac{1}{\sqrt{L}}$ ensures that the operators obey the formal CAR and that they have the correct dimensions.

Introducing the regularized map from the circle to the real axis $\tilde{f}_z(s_{-1}) \in \mathrm{Map}(S_L, \mathbb{C})$, where s_{-1} is the shift operator for $n = -1$, as

$$\tilde{f}_z(s_{-1}) := -i\ln\left(\frac{s_{-1}}{|s_{-1}|}\right) + i\ln\left(\frac{z}{|z|}\right) + \alpha_z^-(s_{-1}) + \alpha_z^+(s_{-1}) \tag{2.7}$$

where $\alpha^{\pm} \in C^{\infty}(S_L, \mathbb{C})$ is a smooth map from the circle onto the complex plane, defined as

$$\alpha_z^{\pm}(s_{-1}) := \pm i \sum_{n=1}^{\infty} \frac{1}{n} \left(z^{\pm n}\right)\ s_{\pm n} \tag{2.8}$$

such that $\tilde{P}_{\pm}\rho(z) = -\frac{i}{L} z\partial_z d\Gamma(\alpha_z^{\pm})$, for $\rho(z) := \frac{1}{L}\sum\limits_{n\in\mathbb{Z}} \rho_n z^n$.

It is proven in [19] that the maps $\tilde{f}_z$ and $\alpha_z^{\pm}$ obey

$$1) \qquad iS(\alpha_z^+, \alpha_w^-) = i\alpha_z^-(w) \tag{2.9}$$

$$2) \qquad iS(\tilde{f}_z, \tilde{f}_w) = -iS(\tilde{f}_w, \tilde{f}_z) = i\tilde{f}_w(z) \tag{2.10}$$

Proposition 2.1.2 and Eqs. (2.7)-(2.10) and gives

$$\begin{aligned} \psi(z) &= \frac{1}{\sqrt{L}} \, {}^*_*\Gamma(e^{-i\tilde{f}_z})^*_* \\ \psi^*(z) &= \frac{1}{\sqrt{L}} \, {}^*_*\Gamma(e^{i\tilde{f}_z})^*_* \end{aligned}$$

$$\psi^*(w)\psi(z) = e^{-i\pi \ \mathrm{sign}(w-z)}\psi(z)\psi^*(w) \quad , \ w \neq z \tag{2.11}$$

where the sign function is given by

$$\mathrm{sign}(w-z) = \frac{1}{\pi}\tilde{f}_z(w) \tag{2.12}$$

Using equation (2.11) yields[2] the formal CAR. For greater details, see [26] or [27].

2.2 Boson-anyon correspondence

The unitary operators ${}^*_*\Gamma(e^{i\tilde{f}})^*_*$ can be extended as ${}^*_*\Gamma(e^{i\tilde{f}})^*_* \to {}^*_*\Gamma(e^{im\tilde{f}})^*_*$. These operators are commonly known as *vertex operators* and obey the exchange relation

$${}^*_*\Gamma(e^{im_1\tilde{f}_w})^*_* \ {}^*_*\Gamma(e^{im_2\tilde{f}_z})^*_* = e^{i\pi m_1 m_2 \ \mathrm{sign}(w-z)} \ {}^*_*\Gamma(e^{im_2\tilde{f}_z})^*_* \ {}^*_*\Gamma(e^{im_1\tilde{f}_w})^*_*$$

which is fermionic if m_1m_2 is an odd integer and bosonic if m_1m_2 is even. The transformation is only well-defined iff $m_1, m_2 \in \mathbb{Z}$, due to the restrictions from the Klein factors on the vertex operators. This does not meet the requirement for abelian anyons, where m_1m_2 should be any real number.

It can be shown that the 2-cocycle of Eq. (1.8) is invariant under rescaling such[3] as $\omega \to \frac{\omega}{\lambda}, \tilde{P}_0\alpha(x) \to \lambda\tilde{P}_0\alpha(x)$ for all $\lambda \neq 0$ [19]. We introduce the rescaling parameter $\nu_0 (\in \mathbb{R} \setminus \{0\})$ and transform $\tilde{f}$ such that

$$\tilde{f}_z(s_{-1}) \to f_z(s_{-1}) := -\frac{i}{\nu_0}\ln(s_{-1}) + i\nu_0 \ln\left(\frac{z}{|z|}\right) + \alpha_z^+(s_{-1}) + \alpha_z^-(s_{-1}) \tag{2.13}$$

obeying

$$iS(f_w, f_z) = iS(\tilde{f}_w, \tilde{f}_z) = i\pi \ \mathrm{sign}(w-z)$$

---

[2] It is actually the vacuum expectation value, of the anti-commutation, that yields the CAR

[3] ω is the winding number (c.f. Eq. (1.8)) and not to be confused with an indeterminate of a formal distribution

Definition 2.2.1. The *anyon field operators* are defined as

$$\phi^{\nu}(z) := {}^*_*\Gamma(e^{i\nu f_z})^*_* = \left(\phi^{-\nu}\left((z^*)^{-1}\right)\right)^* \tag{2.14}$$

which is a well-defined operator for all *statistical parameter* $\nu \in \mathbb{R}$, if and only if $\frac{\nu}{\nu_0} \in \mathbb{Z}$. The anyon field operators obey the exchange relation

$$\phi^{\nu}(w)\phi^{\nu'}(z) = e^{-i\pi\nu\nu' \operatorname{sign}(w-z)}\phi^{\nu'}(z)\phi^{\nu}(w) \quad , \nu\nu' \in \mathbb{R} \tag{2.15}$$

which satisfies the condition for anyons (Eq. (2.1)). Anyons with the same statistical parameter are referred to as the same *species*.

The anyon operators can be expressed as

$$\phi^{\nu}(z) = e^{-i\frac{\pi}{L}\nu\nu_0 yQ} R^{\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0 yQ} e^{-\nu\sum\limits_{n<0}\frac{1}{n}\rho_n z^n} e^{-\nu\sum\limits_{n>0}\frac{1}{n}\rho_n z^n} \tag{2.16}$$

It should be noted that the anyon operators become the fermion operators for $\nu_0 = |\nu| = 1$, *i.e.*

$$\begin{aligned} \psi(z) &= \frac{1}{\sqrt{L}}\,\phi^{-1}(z) \\ \psi^*(z) &= \frac{1}{\sqrt{L}}\,\phi^{1}(z) \end{aligned}$$

Remark 2.2.2. This is an important relation that can be used as a useful check, throughout this work, since all operators, calculations and relations should revert to fermion operators and well known fermionic relation by setting $\nu_0 = |\nu| = 1$.

Remark 2.2.3. If $|\nu\nu'|$ (c.f. Eq. (2.15)) is an odd number, greater than one, the anyon operators become field operators for quasi particles known as composite fermions. The anyon field operators are rescaled fermion field operator if and only if $\nu_0 = |\nu| = |\nu'| = 1$.

Lemma 2.2.4. *The anyon operators obey the following commutation relations*

1.

$$[Q, \phi^{\nu}(z)] = \frac{\nu}{\nu_0}\phi^{\nu}(z) \tag{2.17}$$

2.

$$[\rho(w), \phi^{\nu}(z)] = \frac{\nu}{L}\left(z\delta(z-w) + \frac{1}{\nu_0} - 1\right)\phi^{\nu}(z) \tag{2.18}$$

Proof. See Appendix E.2.2 □

2.3 Dual anyon operators

The result collected in the previous Section are well known (c.f. [19]). This Section will introduce the corresponding dual anyon field operator and its relation to the anyon field operators.

Postulate 2.3.1. *The operators $\phi^{-\frac{1}{\nu}}, \nu \in \mathbb{R} \setminus \{0\}$, correspond to the dual anyon field operator of species ν and is defined as*

$$\phi^{-\frac{1}{\nu}}(z) := \,{}^*_*\Gamma(e^{-\frac{i}{\nu}f_z})^*_* = \; e^{i\frac{\pi}{L}\frac{\nu_0}{\nu}yQ}R^{-\frac{1}{\nu\nu_0}}e^{i\frac{\pi}{L}\frac{\nu_0}{\nu}yQ}e^{\frac{1}{\nu}\sum\limits_{n<0}\frac{1}{n}\rho_n z^n}e^{\frac{1}{\nu}\sum\limits_{n>0}\frac{1}{n}\rho_n z^n} \tag{2.19}$$

and is a well-defined operator iff $\frac{1}{\nu\nu_0} \in \mathbb{Z}$. The dual anyon field operators satisfy the exchange relation for abelian anyon operators, with a phase shift of $\vartheta = \frac{\pi}{\nu\nu'} \in \mathbb{R}$ (c.f. Eq. (2.1)).

It is later shown that there is a natural extension between the anyon second order differential operator of species ν and $-1/\nu$ (c.f. Section 3.2.2) and that vectors created by using anyon and dual anyon operators are eigenvectors of these operators with different eigenvalues (c.f. Section 4.1).

Proposition 2.3.2. *The anyon and dual anyon operators obey the CAR*

$$\left\{\phi^{\nu}(w), \phi^{-\frac{1}{\nu}}(z)\right\} = z\delta(z-w)\; {}^*_*\phi^{\nu}(w)\phi^{-\frac{1}{\nu}}(z)^*_* \tag{2.20}$$

for all ν.

Proof. See Appendix E.2.3 □

This is very similar to the anti-commutation relation that the bosonized fermion field operators obey.

Remark 2.3.3. Proposition 1.4.4 *2* yields

$$\delta(z-w)^*_*\phi^{\nu}(w)\phi^{-\frac{1}{\nu}}(z)^*_* = \delta(z-w)^*_*\phi^{\nu}(z)\phi^{-\frac{1}{\nu}}(z)^*_* = \delta(z-w)\phi^{(\nu-\frac{1}{\nu})}(z) \tag{2.21}$$

which gives

$$\left\{\phi^{\nu}(w), \phi^{-\frac{1}{\nu}}(z)\right\} = z\delta(z-w)\phi^{(\nu-\frac{1}{\nu})}(z) \tag{2.22}$$

It is natural to prescribe the conjugate of the anyon field operators ($\phi^{-\nu}$) as the corresponding dual anyon operators. This prescription does not satisfy the relation for a dual particle. Comparing Eq. (2.20) with the relation between the anyon operator and its conjugate, which obeys a non-canonical commutation relations for the special cases of ν^2 being a positive even number or anti-commutation relations for the special cases where ν^2 is an odd positive integer. These relation is given by

$$\left[\phi^\nu(w), \phi^{-\nu}(z)\right] = \,{}^*_*\phi^\nu(w)\phi^{-\nu}(z)^*_*\left(e^{\frac{1}{2}\left|\ln\left(\left|\frac{w}{z}\right|\right)\right|}\left(\sqrt{wz}\right)\right)^{\nu^2}\partial_w^{\left(\nu^2-1\right)}\delta(z-w) \quad (2.23)$$

if $\nu^2 \in 2\mathbb{N}$ and

$$\left\{\phi^\nu(w), \phi^{-\nu}(z)\right\} = \,{}^*_*\phi^\nu(w)\phi^{-\nu}(z)^*_*\left(e^{\frac{1}{2}\left|\ln\left(\left|\frac{w}{z}\right|\right)\right|}\left(\sqrt{wz}\right)\right)^{\nu^2}\partial_w^{\left(\nu^2-1\right)}\delta(z-w) \quad (2.24)$$

if $\nu^2 \in 2\mathbb{N}_0+1$. There is nothing, in this work, that would indicates that ν^2 should satisfy any of these special cases.

Define $\mathbf{w} := (w_1, w_2, \ldots, w_N)$ where

$$w_k := e^{\frac{2\pi}{L}ix_k}e^{\frac{2\pi}{L}\varepsilon_k} \quad (2.25)$$

such that $w_k \neq w_{k'}$, for all $k \neq k'$.

Let $\Phi^\nu(\mathbf{w})$ denote the product of $N \in \mathbb{N}$ anyon field operators with the same statistical parameter ν, *i.e.*

$$\Phi^\nu(\mathbf{w}) := \phi^\nu(w_1)\phi^\nu(w_2)\cdots\phi^\nu(w_N) \quad (2.26)$$

The boson normal-ordered form of the operator $\Phi^\nu(\mathbf{w})$ is given by

$$\,{}^*_*\Phi^\nu(\mathbf{w})^*_* = e^{-i\frac{\pi}{L}\nu\nu_0\left(\sum\limits_{k=1}^N x_k\right)Q}R^{N\frac{\nu}{\nu_0}}e^{-i\frac{\pi}{L}\nu\nu_0\left(\sum\limits_{k=1}^N x_k\right)Q}\times \\ e^{-\nu\sum\limits_{k=1}^N\sum\limits_{n<0}\frac{1}{n}\rho_n w_k^n}e^{-\nu\sum\limits_{k=1}^N\sum\limits_{n>0}\frac{1}{n}\rho_n w_k^n} \quad (2.27)$$

such that the vacuum expectation value (VEV) of the normal-ordered product is

$$\langle{}^*_*\Phi^\nu(\mathbf{w})^*_*\rangle = \delta_{N\frac{\nu}{\nu_0},0}$$

Lemma 2.3.4. *The boson normal-ordering of* $\Phi^\nu(\mathbf{w})$ *is given by*

$$\Phi^\nu(\mathbf{w}) = \mathcal{J}^\nu(\mathbf{w})\,{}^*_*\Phi^\nu(\mathbf{w})^*_* \quad (2.28)$$

where

$$\mathcal{J}^\nu(\mathbf{w}) := \prod_{1\leq k'<k\leq N}\mathrm{i}_{w_k,w_{k'}}\left(b(w_k,w_{k'})\right)^{\nu^2}$$

and

$$b(w,z) := \left|\frac{z}{w}\right|^{\frac{1}{2}}\frac{w-z}{\sqrt{wz}}$$

Proof. See Appendix B.2.1 □

Let $\Phi(\{w_k,\nu_k\}_{k=1}^N)$ denote the product of $N\,(\in\mathbb{N})$ anyon field operators of different species, *i.e.*

$$\Phi(\{w_k,\nu_k\}_{k=1}^N) := \phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2)\cdots\phi^{\nu_N}(w_N)$$

where $\frac{\nu_k}{\nu_0}\in\mathbb{Z}\ \forall k$ and $w_k\neq w_{k'}$ for all $k\neq k'$. The boson normal-ordered form of the operator $\Phi(\{w_k,\nu_k\}_{k=1}^N)$ is given by

$$\,^*_*\Phi(\{w_k,\nu_k\}_{k=1}^N)^*_* = e^{-i\frac{\pi}{L}\nu_0\left(\sum_{k=1}^N \nu_k x_k\right)Q} R^{\frac{1}{\nu_0}\sum_{k=1}^N \nu_k} e^{-i\frac{\pi}{L}\nu_0\left(\sum_{k=1}^N \nu_k x_k\right)Q}\times$$
$$e^{-\sum_{k=1}^N\sum_{n<0}\frac{\nu_k}{n}\rho_n w_k^n} e^{-\sum_{k=1}^N\sum_{n>0}\frac{\nu_k}{n}\rho_n w_k^n}$$

such that the VEV becomes

$$\left\langle \,^*_*\Phi(\{w_k,\nu_k\}_{k=1}^N)^*_*\right\rangle = \delta_{\frac{1}{\nu_0}\sum_{k=1}^N\nu_k,0}$$

Lemma 2.3.5. *Boson normal-ordering the operator* $\Phi(\{w_k,\nu_k\}_{k=1}^N)$ *gives*

$$\Phi(\{w_k,\nu_k\}_{k=1}^N) = \mathcal{I}(\{w_k,\nu_k\}_{k=1}^N)\ \,^*_*\Phi(\{w_k,\nu_k\}_{k=1}^N)^*_* \tag{2.29}$$

where

$$\mathcal{I}(\{w_k,\nu_k\}_{k=1}^N) := \prod_{1\le k<k'\le N} \mathrm{i}_{w_{k'},w_k}\left(b(w_{k'},w_k)\right)^{\nu_k\nu_{k'}}$$

and $b(w_{k'},w_k)$ *is defined in Lemma* 2.3.4.

Proof. See Appendix B.2.2 □

Define $\mathbf{z} := (z_1,z_2,\ldots,z_M)$ where

$$z_j := e^{\frac{2\pi}{L}iy_j}e^{\frac{2\pi}{L}\varepsilon'_j} \tag{2.30}$$

such that $z_j\neq z_{j'}$ for all $j\neq j'$ and $z_j\neq w_k\ \forall j,k$.

Let $\varphi^\nu(\mathbf{z},\mathbf{w})$ denote the product of $M\in\mathbb{N}$ dual anyon operators and $N\in\mathbb{N}$ anyon operators, *i.e.*

$$\varphi^\nu(\mathbf{z},\mathbf{w}) := \Phi^{-\frac{1}{\nu}}(\mathbf{z})\Phi^\nu(\mathbf{w}) \tag{2.31}$$

where $\Phi^\nu(\mathbf{w})$ is the same as in Eq. (2.26). The boson normal-ordered form of the operator $\varphi^\nu(\mathbf{z},\mathbf{w})$ is given by

$$\,^*_*\varphi^\nu(\mathbf{z},\mathbf{w})^*_* =$$
$$e^{-i\frac{\pi}{L}\nu_0\left(\nu\sum_{k=1}^N x_k-\frac{1}{\nu}\sum_{j=1}^M y_j\right)Q} R^{N\frac{\nu}{\nu_0}-M\frac{1}{\nu\nu_0}} e^{-i\frac{\pi}{L}\nu_0\left(\nu\sum_{k=1}^N x_k-\frac{1}{\nu}\sum_{j=1}^M y_j\right)Q}$$
$$\times\, e^{-\sum_{n<0}\frac{1}{n}\rho_n\left(\nu\sum_{k=1}^N w_k^n-\frac{1}{\nu}\sum_{j=1}^M z_j^n\right)} e^{-\sum_{n>0}\frac{1}{n}\rho_n\left(\nu\sum_{k=1}^N w_k^n-\frac{1}{\nu}\sum_{j=1}^M z_j^n\right)}$$

such that the VEV is

$$\langle \,^*_*\varphi^\nu(\mathbf{z},\mathbf{w})^*_*\rangle = \delta_{N\frac{\nu}{\nu_0}-M\frac{1}{\nu\nu_0},0}$$

Lemma 2.3.6. *Boson normal-ordering the operator* $\varphi^\nu(\mathbf{z},\mathbf{w})$ *yields*

$$\varphi^\nu(\mathbf{z},\mathbf{w}) = \mathcal{J}^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{J}^\nu(\mathbf{w}) \prod_{j=1}^{M}\prod_{k=1}^{N} i_{w_k,z_j}\frac{1}{b(w_k,z_j)}\ {}^*_*\varphi^\nu(\mathbf{z},\mathbf{w})^*_* \tag{2.32}$$

Proof. This comes from boson normal-ordering the two different operators $\Phi^\nu(\mathbf{w})$ and $\Phi^{-\frac{1}{\nu}}(\mathbf{z})$ separately by using Lemma 2.3.4. Then use Lemma 2.3.5 for the special case of two boson normal-ordered operators. □

Remark 2.3.7. If the operator product was instead given by $\tilde{\varphi}^\nu(\mathbf{z},\mathbf{w}) := \Phi^\nu(\mathbf{w})\Phi^{-\frac{1}{\nu}}(\mathbf{z})$ the boson normal-ordering of the field operator would be given by

$$\tilde{\varphi}^\nu(\mathbf{z},\mathbf{w}) = \mathcal{J}^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{J}^\nu(\mathbf{w}) \prod_{k=1}^{N}\prod_{j=1}^{M} \mathrm{i}_{z_j,w_k}\frac{1}{b(z_j,w_k)}{}^*_*\tilde{\varphi}^\nu(\mathbf{z},\mathbf{w})^*_*$$

where ${}^*_*\tilde{\varphi}^\nu(\mathbf{z},\mathbf{w})^*_* = {}^*_*\varphi^\nu(\mathbf{z},\mathbf{w})^*_*$.

Chapter 3

Anyons and the CS model

The first part of this Chapter will give a short description of the integrable Calogero-Sutherland (CS) models expressed in terms of the indeterminates of formal distributions. We will then show that the Laughlin wavefunction reduces to the CS groundstate, up to a constant factor and with a center of mass (CoM) shift, when restricted to the boundaries of a circular quantum Hall droplet. We will also give a brief introduction to a generalization of the CS model which is referred to as the deformed Calogero-Sutherland (dCS) model [4] [5].

In the second part of this Chapter we will introduce a self-adjoint operator $\mathcal{H}^{\nu,3}$ by generalizing the construction of the fermion W-algebra for anyon field operators. This operator $\mathcal{H}^{\nu,3}$ was previously constructed in [19] using a somewhat different method. We will also show some interesting relations of the $\mathcal{H}^{\nu,3}$ operator.

In the final Section we show that the expectation value of the $\mathcal{H}^{\nu,3}$ operator and the product of $N \in \mathbb{N}$ anyon field operators yields the CS groundstate, with a CoM shift, and the CS Hamiltonian.

3.1 The Calogero-Sutherland type model

The CS model describes an N-body system on a circle of length L interacting with a translationally invariant centrifugal (inverse quadratic) potential. The model was first proposed by F. Calogero in [2] which also included a harmonic (quadratic) potential and fully solved for the three-body case in [28] and for arbitrary N in [1]. The model was extended to translationally invariant systems by B. Sutherland in [3], and solved fully in [29]. The CS model was originally expressed in terms of variables on the circle S_L, we find it convenient to re-express them in this Section in terms of the indeterminates of a formal distribution, w_k and z_j (c.f. Eqs. (2.25) and (2.30)).

3.1.1 The CS model

The Calogero-Sutherland model is a quantum many-body model of $N(\in \mathbb{N})$ indistinguishable particles interacting via a translationally invariant two-body potential. The model has the groundstate (GS) wavefunction

$$\psi_0(\mathbf{w}) = \prod_{k'<k} \left(\frac{1}{2i} \frac{w_k - w_{k'}}{\sqrt{w_k w_{k'}}} \right)^{\lambda} \tag{3.1}$$

where $\mathbf{w} = (w_1, \ldots, w_N)$ such that $w_k \neq w_{k'}$ for all $k \neq k'$. The GS wavefunction corresponds to the CS Hamiltonian defined as

$$H_N := \frac{4\pi^2}{L^2} \sum_{k=1}^{N} (w_k \partial_{w_k})^2 + g \frac{4\pi^2}{L^2} \sum_{k'<k} \tilde{V}(w_k, w_{k'}) \tag{3.2}$$

with the two-body potential

$$\tilde{V}(w_k, w_{k'}) := -\frac{w_k w_{k'}}{(w_k - w_{k'})^2} \tag{3.3}$$

and the coupling constant $g = 2\lambda(\lambda - 1)$, with the constraint $g > -\frac{1}{4}$ in order to avoid an energy spectrum that is unbounded from below. The GS wavefunction corresponds to an eigenvalue of

$$E_0 = \frac{\pi^2 \lambda^2 N \left(N^2 - 1\right)}{3L^2} \tag{3.4}$$

The CS Hamiltonian has exact eigenfunctions of the form

$$\Psi_{\mathbf{n}}(\mathbf{w}) = \mathcal{P}_{\mathbf{n},\lambda}(\mathbf{w}) \psi_0(\mathbf{w}) \tag{3.5}$$

with $\mathbf{n} = (n_1, n_2, \ldots, n_N) \in \mathbb{N}^N$, and $\mathcal{P}_{\mathbf{n},\lambda}(\mathbf{w})$ are symmetric[1] polynomials.

The model describes fermions for λ an odd integer and hard sphere bosons when λ is an even integer. The special case of $\lambda = 1$ would give a model describing N free fermions on a circle with periodic boundary conditions.

When λ is any real number, the GS wavefunction gains a phase shift of $e^{\pm i\pi\lambda}$ during particle position exchange.

3.1.2 Relation to the Laughlin state

The Laughlin state [9] is given by[2]

$$\Psi(m; \mathrm{z}_1, \ldots, \mathrm{z}_N) = e^{-\sum_{l=1}^{N} |\mathrm{z}_l|^2} \prod_{l<l'} (\mathrm{z}_{l'} - \mathrm{z}_l)^{2m+1} \quad , \; m \in \mathbb{N}_0 \tag{3.6}$$

[1] w.r.t. particle position permutation

[2] The magnetic length l_B is set to 1

where[3] $2m+1$ corresponds to one over the filling factor and $\mathrm{z}_l := \frac{1}{2}(X_l + iY_l)$ is a dimensionless complex number representing the position of the l-th electron, (X_l, Y_l), in the 2-dimensional quantum Hall (QH) droplet.

The low energy degrees of freedom of a QH liquid are at the boundary, due to the incompressibility of the QH liquid. For electrons at the edge of a circular QH droplet with radius r we have

$$\mathrm{z}_l = \frac{1}{2} r e^{i\vartheta_l} \quad , \vartheta_l \in S_{2\pi}$$

Inserting this ansatz into the Laughlin state yields

$$\begin{aligned}\Psi(m;\vartheta_1,\ldots,\vartheta_N) &= e^{-\frac{1}{4}r^2\sum_{l=1}^{N}1} \prod_{l<l'} \left(\frac{r}{2}\right)^{2m+1} \left(e^{i\vartheta_{l'}} - e^{i\vartheta_l}\right)^{2m+1} \\ &= (ir)^{N(2m+1)\frac{N-1}{2}} e^{-\frac{1}{4}r^2 N} \prod_{l<l'} e^{i(2m+1)\frac{\vartheta_{l'}+\vartheta_l}{2}} \left(\sin\left(\frac{\vartheta_{l'}-\vartheta_l}{2}\right)\right)^{2m+1} \qquad (3.7)\end{aligned}$$

Comparing Eqs. (3.7) and (3.1) (or rather Eq. (2)) shows that the Laughlin wavefunction for the electrons at the edge of the QH droplet corresponds to the CS groundstate, up to a constant[4] and a CoM shift, for $L = 2\pi$ and $\lambda = 2m+1$. This suggests that the CS groundstate wavefunction describes the edge degrees of freedom of a QH liquid.

3.1.3 The deformed Calogero-Sutherland model

The deformed Calogero-Sutherland (dCS) model is defined by the differential operator

$$\begin{aligned}\tilde{H}_{N,M} := \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2 - \frac{4\pi^2}{L^2}\lambda\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2 \\ + \frac{4\pi^2}{L^2}2(\lambda-1)\left(\frac{1}{\lambda}\sum_{j'<j}\tilde{V}(z_j,z_{j'}) + \lambda\sum_{k'<k}\tilde{V}(w_k,w_{k'})\right) \\ + \frac{4\pi^2}{L^2}2(1-\lambda)\sum_{j=1}^{M}\sum_{k=1}^{N}\tilde{V}(w_k,z_j) \qquad (3.8)\end{aligned}$$

where the two-body potential is given by Eq. (3.3).

[3]Note that the z_l used in Laughlin's wavefunction are not indeterminates of formal distributions.

[4]Note that none of the wavefunction have been normalized

The dCS model has exact eigenfunctions similar to Eq. (3.5), with ψ_0 replaced by $\tilde{\psi}_0(\mathbf{z},\mathbf{w})$, which we refer to as the deformed groundstate (dGS) eigenfunction, defined as

$$\tilde{\psi}_0(\mathbf{z},\mathbf{w}) := \prod_{j'<j}\left(\frac{1}{2i}\frac{z_j - z_{j'}}{\sqrt{z_j z_{j'}}}\right)^{\lambda^{-1}} \prod_{k'<k}\left(\frac{1}{2i}\frac{w_k - w_{k'}}{\sqrt{w_k w_{k'}}}\right)^{\lambda} \prod_{j=1}^{M}\prod_{k=1}^{N}\left(\frac{1}{2i}\frac{w_k - z_j}{\sqrt{w_k z_j}}\right)^{-1} \tag{3.9}$$

which corresponds to an eigenvalue

$$\tilde{E}_0 := \frac{\pi^2}{3L^2}\frac{(N\lambda - M)(N\lambda - \lambda - M)(N\lambda + \lambda - M) - M\left(\lambda^2 - 1\right)}{\lambda} \tag{3.10}$$

of the dCS differential operator.

Remark 3.1.1. Due to the sign in front of the z-derivatives, the differential operators $\tilde{H}_{N,M}$ cannot represent a physical system for $\lambda > 0$ and the dGS eigenfunction is not square integrable. So the dCS model, unlike the original CS model, cannot be considered as a quantum mechanical model.

3.2 The $\mathcal{H}^{\nu,s}$ operators

As shown in Appendix D.1, there exists fermion operators such that

$$\left[W^s, \psi^{(*)}(w)\right] = \left(\frac{2\pi}{L}\right)^{s-1} (w\partial_w)^{s-1}\,\psi^{(*)}(w)$$

which can also be expressed in terms of the fermion density operators (c.f. Appendix D.2)

Since the anyon field operators are a generalization of the fermion field operators there should exist a generalization of the fermion W^s operators. It turns out that this is not as simple since there arises a need for correction terms for higher order derivatives. We construct the generalization of the W^s operators for $s = 1,2,3$, which was introduced in [19], using a different method.

3.2.1 Construction of the anyon differential operators

It is not trivial to construct the second quantization of the anyon differential operators. The first additional terms manifests for the second derivative (c.f. Appendix D.3.2). We restrict ourselves to the $s \leq 3$ operators (higher order derivatives are interesting but outside the scope of this thesis).

There exists a generalization of the fermion W^s operators for anyons with statistical parameter ν and is denoted by $W^{\nu,s}$.

Lemma 3.2.1. *The $W^{\nu,s}$ operators, for $s = 1, 2$, obey*

$$[W^{\nu,s}, \phi^\nu(w)] = \nu^{2-s}\left(\frac{2\pi}{L}\right)^{s-1}(w\partial_w)^{s-1}\phi^\nu(w) \quad s = 1, 2 \tag{3.11}$$

For $s > 2$ operators additional terms arise.

Proof. see Appendix D.3.2 □

For the first two cases, $s = 1, 2$, it is a minor challenge to construct the desired operators for the anyon field operators.

Lemma 3.2.2. *The $\mathcal{H}^{\nu,s}$, $s = 1, 2$, operators are defined as*

$$\mathcal{H}^{\nu,s} := \frac{1}{\nu^{2-s}}W^{\nu,s} \tag{3.12}$$

and obey

$$\mathcal{H}^{\nu,s}\Omega = 0 \tag{3.13}$$

$$[\mathcal{H}^{\nu,s}, \phi^\nu(w)] = \left(\frac{2\pi}{L}w\partial_w\right)^{s-1}\phi^\nu(w) \tag{3.14}$$

for all ν.

Proof. This can be seen from Eq. (3.11) in Lemma 3.2.3 and Section D.3. □

It is natural to interpret the $\mathcal{H}^{\nu,1}$ operator as the charge operator for anyons with statistical parameter ν. Equations (3.11) and (D.26) gives that the first order differential operator is not actually dependent on the statistical parameter ν (only ν_0), but is will still be written as $\mathcal{H}^{\nu,2}$ in order to have a unified notation.

The operator of interest is for $s = 3$.

Lemma 3.2.3. *The $W^{\nu,3}$ operator and the anyon field operator obey*

$$\begin{aligned}\left[W^{\nu,3}, \phi^\nu(w)\right] &= \frac{1}{\nu}\frac{4\pi^2}{L^2}(w\partial_w)^2\phi^\nu(w) \\ &- \frac{4\pi^2}{L}\left(\nu^2 - 1\right){}^*_*(w\partial_w\rho(w))\,\phi^\nu(w){}^*_* + \frac{\pi^2\left(\nu^2-1\right)\left(\nu^2-2\right)}{3\nu L^2}\phi^\nu(w)\end{aligned} \tag{3.15}$$

Proof. see Appendix D.3.2 □

There are two terms that need to be eliminated. This is partially done by the introduction of the $\mathcal{C}$ operator.

Lemma 3.2.4. *The operator*

$$\mathcal{C} := \frac{4\pi^2}{L^2} \sum_{n>0} {}^*_* n \rho_{-n} \rho_n {}^*_* \tag{3.16}$$

obeys

$$\mathcal{C} R^\omega \Omega = 0 \quad , \ \forall \omega \in \mathbb{Z} \tag{3.17}$$

$$\{\mathcal{C}, \phi^\nu(w)\} = 2 {}^*_* \mathcal{C} \phi^\nu(w) {}^*_* - \frac{4\pi^2}{L} \nu {}^*_* \left(w\partial_w \rho(w)\right) \phi^\nu(w) {}^*_* \tag{3.18}$$

Proof. See Appendix E.3.1 □

The commutation relation can then be written as

$$\begin{aligned}
[\mathcal{C}, \phi^\nu(w)] &= \{\mathcal{C}, \phi^\nu(w)\} - 2\phi^\nu(w)\mathcal{C} \\
&= -\frac{4\pi^2}{L} \nu {}^*_* \left(w\partial_w \rho(w)\right) \phi^\nu(w) {}^*_* + 2\left({}^*_* \mathcal{C} \phi^\nu(w) {}^*_* - \phi^\nu(w)\mathcal{C}\right)
\end{aligned} \tag{3.19}$$

The remarkable property of the $\mathcal{C}$ operator (Eq. (3.17)) allows us to construct the desired operator as long as it is applied to a vector $R^\omega\Omega$ for any $\omega \in \mathbb{Z}$.

Define the $\mathcal{H}^{\nu,3}$ operator as

$$\mathcal{H}^{\nu,3} := \nu W^{\nu,3} - \left(\nu^2 - 1\right)\mathcal{C} - \frac{\pi^2}{3L^2}\left(\nu^2 - 1\right)\left(\nu^2 - 2\right)\mathcal{H}^{\nu,1} \tag{3.20}$$

Lemma 3.2.5. *The $\mathcal{H}^{\nu,3}$ operator satisfies*

$$\mathcal{H}^{\nu,3}\Omega = 0 \tag{3.21}$$

$$\left[\mathcal{H}^{\nu,3}, \phi^\nu(w)\right] = \frac{4\pi^2}{L^2}\left(w\partial_w\right)^2 \phi^\nu(w) - 2\left(\nu^2 - 1\right)\left({}^*_* \mathcal{C}\phi^\nu(w) {}^*_* - \phi^\nu(w)\mathcal{C}\right) \tag{3.22}$$

Proof. This relation is a simple consequence of Lemmas 3.2.2, 3.2.3, and 3.2.4. □

Lemma 3.2.4 yields

$$\left[\mathcal{H}^{\nu,3}, \phi^\nu(w)\right] R^\omega\Omega = \frac{4\pi^2}{L^2}\left(w\partial_w\right)^2 \phi^\nu(w) R^\omega \Omega \quad , \ \forall \omega \in \mathbb{Z} \tag{3.23}$$

3.2.2 The duality of species ν and $-1/\nu$

As mentioned in Section 2.3, the $\mathcal{H}^{\nu,3}$ operator is closely related to the $\mathcal{H}^{-\frac{1}{\nu},3}$ operator. To be precise:

Lemma 3.2.6. *There is a duality between the $\mathcal{H}^{\nu,3}$ operator and the $\mathcal{H}^{-\frac{1}{\nu},3}$ operator given by*

$$\mathcal{H}^{\nu,3} = -\nu^2 \mathcal{H}^{-\frac{1}{\nu},3} + \frac{\pi^2}{3L^2}\frac{\left(\nu^4 - 1\right)}{\nu^2}\mathcal{H}^{-\frac{1}{\nu},1} \tag{3.24}$$

Proof. This relation is a consequence of Eqs. (3.20), (D.27) . □

It is therefore possible to create eigenvectors of the $\mathcal{H}^{\nu,3}$ operator using the dual anyon field operators.

It is also seen that the anyon charge operator also obeys a similar duality with

$$\mathcal{H}^{\nu,1} = -\frac{1}{\nu^2}\mathcal{H}^{-\frac{1}{\nu},1} \tag{3.25}$$

The duality concept also holds for the $\mathcal{H}^{\nu,2}$ operator as well but it is simple since the operator is independent of the statistical parameter,[5] *i.e.*

$$\mathcal{H}^{\nu,2} = \mathcal{H}^{-\frac{1}{\nu},2} \tag{3.26}$$

Remark 3.2.7. It is possible to construct new operators that obey the same duality relation as the $\mathcal{H}^{\nu,2}$ operator (c.f. Eq. (3.26)): We introduce the $\tilde{\mathcal{H}}^{\nu,s}$ operators, $s = 1, 2, 3$, which obey

$$\tilde{\mathcal{H}}^{\nu,s} = \tilde{\mathcal{H}}^{-\frac{1}{\nu},s} \quad , \ s = 1, 2, 3$$

The $\tilde{\mathcal{H}}^{\nu,s}$ operators, where ν_0 remains independent of the statistical parameter ν, are given by

$$\begin{aligned}
\tilde{\mathcal{H}}^{\nu,1} &:= \nu\mathcal{H}^{\nu,1} \\
\tilde{\mathcal{H}}^{\nu,2} &:= \mathcal{H}^{\nu,2} \\
\tilde{\mathcal{H}}^{\nu,3} &:= \frac{1}{\nu}\mathcal{H}^{\nu,3} + \frac{\pi^2}{3L^2}h(\nu)\tilde{\mathcal{H}}^{\nu,1}
\end{aligned} \tag{3.27}$$

where $h(\nu)$ is any function satisfying

$$h(\nu) - h\left(-\nu^{-1}\right) = \frac{\nu^4 - 1}{\nu^2} \quad , \ e.g., \ \ h(\nu) = \nu^2$$

The second part on the r.h.s. of Eq. (3.27) is unimportant and can be removed by adding a constant shift to the eigenvalues of the $\tilde{\mathcal{H}}^{\nu,3}$ operator.

Remark 3.2.8. For two arbitrary statistical parameters ν_j and ν_k, satisfying $\frac{\nu_j}{\nu_0} \in \mathbb{Z} \ni \frac{\nu_k}{\nu_0}$, the relation of the second order differential operator is given by

$$\mathcal{H}^{\nu_j,3} = \frac{\nu_j}{\nu_k}\mathcal{H}^{\nu_k,3} - \frac{(\nu_j\nu_k + 1)(\nu_j - \nu_k)}{\nu_k}\mathcal{C} - \frac{\pi^2}{3L^2}\nu_j\nu_k\left(\nu_j^2 - \nu_k^2\right)\mathcal{H}^{\nu_k,1} \tag{3.28}$$

where

$$\mathcal{H}^{\nu_j,1} = \frac{\nu_k}{\nu_j}\mathcal{H}^{\nu_k,1}$$

and

$$\mathcal{H}^{\nu_j,2} = \mathcal{H}^{\nu_k,2}$$

as long as ν_0 remains unchanged.

---

[5] The operators depend on ν_0, which could depend on ν, but is to remain "fixed" for this thesis

Remark 3.2.9. The $\tilde{\mathcal{H}}^{\nu,3}$ operator in Eq. (3.27) with $h(\nu) = \nu^2$ has an interesting relation for two arbitrary statistical parameters ν_j, ν_k, defined as in Remark 3.2.8, given by

$$\tilde{\mathcal{H}}^{\nu_j,3} = \tilde{\mathcal{H}}^{\nu_k,3} + \frac{(\nu_k\nu_j + 1)(\nu_k - \nu_j)}{\nu_j\nu_k}\mathcal{C}$$

3.2.3 The groundstate eigenvector

It is later shown that there exists certain eigenvectors of the $\mathcal{H}^{\nu,3}$ operator that are of particular interest for the CS or dCS models (c.f. Corollary 3.3.5 or Theorem 4.2.1). The construction and implementation of all of these eigenvectors is outside the scope of this thesis, and we restrict ourselves to only the simplest eigenvectors.

Lemma 3.2.10. *The vector*

$$\eta_0 := R^{\omega_{\mathcal{H}}}\Omega \quad , \ \omega_{\mathcal{H}} \in \mathbb{Z} \tag{3.29}$$

is an eigenvector of the $\mathcal{H}^{\nu,3}$ *operator and corresponds to an eigenvalue of* $\mathcal{H}^{\nu,3}$*,*

$$\mathcal{H}^{\nu,3}\eta_0 = \mathcal{E}_0\eta_0 \quad , \quad \mathcal{E}_0 := \frac{\pi^2}{3L^2}\left(4\nu\nu_0^3\omega_{\mathcal{H}}^3 - \nu^3\nu_0\omega_{\mathcal{H}}\right) \tag{3.30}$$

Proof. Using Eqs. (3.20) and (3.17) gives that the $\mathcal{C}$ operator vanishes when applied to a vector such as η_0 for all $\omega_{\mathcal{H}} \in \mathbb{Z}$. Using Eq. (D.27) for the $W^{\nu,3}$ operator and the fact that the $\breve{W}^s$ commute with the Klein factors and annihilate the vacuum implies that only the charge operator remain. Thus, Eq. (3.30) can be rewritten as

$$\mathcal{H}^{\nu,3}\eta_0 = \nu\left(\frac{4\pi^2\nu_0^3}{3L^2}Q^3 + \frac{\pi^2\nu_0\left(2-3\nu^2\right)}{3L^2\nu^2}Q\right)\eta_0 + \frac{\pi^2\nu_0}{3L^2\nu}\left(\nu^2-1\right)\left(\nu^2-2\right)Q\eta_0$$

where

$$Q\eta_0 = \omega_{\mathcal{H}}\eta_0$$

which gives Eq. (3.30). □

The parameter $\omega_{\mathcal{H}}$ is determined such that

$$\langle\eta_0, \Psi\rangle \neq 0 \tag{3.31}$$

where $\Psi \in \mathcal{F}$ is an eigenvector of the $\mathcal{H}^{\nu,3}$ operator. The relation given by Eq. (3.31) will be seen in Sections 3.3 and 4.1 where Ψ will correspond to an vector created by using the anyon and dual anyon field operators.

The two cases that are of interest for this thesis are the vectors η_{CS} and η_{dCS} defined as

$$\eta_{\mathrm{CS}} := R^{\left(N\frac{\nu}{\nu_0}\right)}\Omega \quad , \quad N \in \mathbb{N} \tag{3.32}$$

$$\eta_{\mathrm{dCS}} := R^{\left(N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}\right)}\Omega \quad , \quad N, M \in \mathbb{N} \tag{3.33}$$

where η_{CS} and η_{dCS} corresponds to the CS and dCS model respectively. We refer to these eigenvectors as the groundstate eigenvectors and the reason will become clear in Corollary 3.3.6 and Corollary 4.2.2.

See [19] for the construction of an η eigenvector corresponding to the exact eigenfunctions of the CS model.

Using Eq. (3.32) for the groundstate eigenvalue in Eq. (3.30) does not give the exact groundstate eigenvalue of the CS model (c.f. Eq. (3.4)). This is due to a CoM shift that arises from using the groundstate vectors as will be seen more clearly in Sections 3.3 and 4.2.

3.3 Anyons relation to the CS model

Definition 3.3.1. Let A and B be two arbitrary operators where $B = \mathcal{N}_*^* B_*^*$. The $*$ - operation is defined as

$$A * B := \mathcal{N}_*^* A B_*^*$$

For this Section, we define $\Phi^\nu(\mathbf{w})$ as the product of N anyon operator with the same statistical parameter ν, *i.e.*

$$\Phi^\nu(\mathbf{w}) := \phi^\nu(w_1)\cdots\phi^\nu(w_N)$$

with $\mathbf{w} = (w_1, \ldots, w_N)$ where $w_k \neq w_{k'}$ for all $k \neq k'$ and such that $1 < |w_1| < |w_2| < \ldots < |w_N| < \infty$

Lemma 3.3.2. *The operators $\Phi^\nu(\mathbf{w})$ and $W^{\nu,3}$ obey the commutation relation*

$$\begin{aligned}\nu\left[W^{\nu,3}, \Phi^\nu(\mathbf{w})\right] = {} & \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\Phi^\nu(\mathbf{w}) \\ & - \frac{4\pi^2}{L}\nu\left(\nu^2-1\right)\sum_{k=1}^{N}\left(w_k\partial_{w_k}\rho(w_k)\right) * \Phi^\nu(\mathbf{w}) \\ & - \frac{4\pi^2}{L^2}2\nu^2\left(\nu^2-1\right)\sum_{k'<k}\frac{w_k w_{k'}}{\left(w_k-w_{k'}\right)^2}\Phi^\nu(\mathbf{w}) \\ & + \frac{\pi^2}{3L^2}\left(\nu^2-1\right)\left(\nu^2-2\right)N\Phi^\nu(\mathbf{w}) \end{aligned} \tag{3.34}$$

This relation hold for all ν.

Proof. See Appendix E.3.2 □

Lemma 3.3.3. *The $\mathcal{C}$ operator, defined in Lemma 3.2.4, and the operator $\Phi^\nu(\mathbf{w})$ obey the anti-commutation relation*

$$\{\mathcal{C}, \Phi^\nu(\mathbf{w})\} = 2\mathcal{C} * \Phi^\nu(\mathbf{w}) - \frac{4\pi^2}{L^2}\nu\sum_{k=1}^{N}\left(w_k\partial_{w_k}\rho(w_k)\right) * \Phi^\nu(\mathbf{w}) \tag{3.35}$$

Proof. See Appendix E.3.3 □

Proposition 3.3.4. *The $\mathcal{H}^{\nu,3}$ operator and the N-body anyon operator $\Phi^{\nu}(\mathbf{w})$ obey the commutation relation*

$$\begin{aligned}\left[\mathcal{H}^{\nu,3},\Phi^{\nu}(\mathbf{w})\right] = &\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\Phi^{\nu}(\mathbf{w}) \\ &-\frac{4\pi^2}{L^2}\,2\nu^2\left(\nu^2-1\right)\sum_{k'<k}\frac{w_k w_{k'}}{\left(w_k-w_{k'}\right)^2}\,\Phi^{\nu}(\mathbf{w}) \\ &-2\left(\nu^2-1\right)\left(\mathcal{C}*\Phi^{\nu}(\mathbf{w})-\Phi^{\nu}(\mathbf{w})\mathcal{C}\right)\end{aligned} \quad (3.36)$$

Proof. This can be obtained by using Lemmas 3.3.2 and 3.3.3 and that $\left[\mathcal{H}^{\nu,1},\Phi^{\nu}(\mathbf{w})\right] = N\Phi^{\nu}(\mathbf{w})$. □

The last two terms in Eq. (3.36) vanish if applied to a vector $R^{\omega}\Omega$, $\omega\in\mathbb{Z}$, due to the property of the $\mathcal{C}$ operator (c.f. Lemma 3.2.4), *i.e.*

$$\begin{aligned}\left[\mathcal{H}^{\nu,3},\Phi^{\nu}(\mathbf{w})\right]R^{\omega}\Omega = &\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\Phi^{\nu}(\mathbf{w})R^{\omega}\Omega \\ &-\frac{4\pi^2}{L^2}\,2\nu^2\left(\nu^2-1\right)\sum_{k'<k}\frac{w_k w_{k'}}{\left(w_k-w_{k'}\right)^2}\,\Phi^{\nu}(\mathbf{w})R^{\omega}\Omega\end{aligned} \quad (3.37)$$

Comparing Eqs. (3.2) and (3.37) gives that

$$\left[\mathcal{H}^{\nu,3},\Phi^{\nu}(\mathbf{w})\right]R^{\omega}\Omega = H_N\Phi^{\nu}(\mathbf{w})R^{\omega}\Omega$$

where H_N is the CS Hamiltonian with coupling constant $g = 2\nu^2\left(\nu^2-1\right)$.

Corollary 3.3.5. *Let η be an eigenvector of the $\mathcal{H}^{\nu,3}$ operator belonging to a dense, invariant domain and satisfying*

$$\mathcal{H}^{\nu,3}\eta = \mathcal{E}\eta$$

Then the following function

$$F_{\eta}(\mathbf{w}) := \langle\eta,\Phi^{\nu}(\mathbf{w})\Omega\rangle$$

is an eigenfunction of the CS Hamiltonian, given in Eq. (3.2) for $\lambda=\nu^2$, with eigenvalue $\mathcal{E}$.

Proof. This is a simple consequence of Eq. (3.21) and Proposition 3.3.4. □

Corollary 3.3.6. *The function $F_0(\mathbf{w})$, defined as*

$$F_0(\mathbf{w}) := \langle\eta_{CS},\Phi^{\nu}(\mathbf{w})\Omega\rangle = \mathcal{J}^{\nu}(\mathbf{w})e^{-i\frac{\pi}{L}\nu^2N\sum_{k=1}^{N}x_k} \quad (3.38)$$

where $\mathcal{J}^{\nu}(\mathbf{w})$ is given in Lemma 2.3.4, is an eigenfunction of the CS Hamiltonian. The correlation function $F_0(\mathbf{w})$ corresponds to the GS wavefunction of the CS model *for* $\lambda = \nu^2$*, up to a constant factor and with a CoM shift, with eigenvalue* $\mathcal{E}_{CS}$ *given by*

$$\mathcal{E}_{CS} = \frac{\pi^2}{3L^2}\left(4N^3\nu^4 - N\nu^4\right) = E_0 + \frac{\pi^2}{L^2}\nu^4 N^3 \tag{3.39}$$

where E_0 *is the CS groundstate energy given by Eq.* (3.4) *for* $\lambda = \nu^2$*.The additional term on the r.h.s. of Eq.* (3.39) *is exactly the contribution from the CoM shift in* $F_0(\mathbf{w})$*.*

Proof. See Appendix E.3.4. □

Chapter 4

Anyons and the dCS model

In this Chapter we show the relation of the anyon field operators and the dual anyon field operators to the deformed Calogero-Sutherland (dCS) model by generalizing the computations in Section 3.3.

Section 4.1 will show that the commutator of the $\mathcal{H}^{\nu,3}$ operator with $M \in \mathbb{N}$ dual anyon field operator and $N \in \mathbb{N}$ anyon field operators yields the dCS differential operator.

In Section 4.2 a general method for constructing eigenfunctions to the dCS differential operator is introduced. Using this method, it will be shown that the expectation value of a vector created by using the anyon and dual anyon field operators with the η_{dCS} vector (c.f. Eq. (3.33)) yields the deformed groundstate (dGS) eigenfunction of the dCS differential operator.

4.1 Anyons and the dCS differential operator

Define $\varphi^\nu(\mathbf{z},\mathbf{w})$ as the product of M dual anyon field operators and N anyon field operators, *i.e.*

$$\varphi^\nu(\mathbf{z},\mathbf{w}) := \Phi^{-\frac{1}{\nu}}(\mathbf{z})\Phi^\nu(\mathbf{w})$$

where $\mathbf{z} := (z_1,\ldots,z_M)$ such that $1 < |z_1| < \ldots < |z_M| < \infty$, $\mathbf{w} := (w_1,\ldots,w_N)$ such that $1 < |w_1| < \ldots < |w_N| < \infty$, and $|w_k| > \max(|z_j|)$ for all k.

Remark 4.1.1. This added requirement depends on the positioning of the many-body operators. For a product $\tilde{\varphi}^\nu(\mathbf{z},\mathbf{w}) := \Phi^\nu(\mathbf{w})\Phi^{-\frac{1}{\nu}}(\mathbf{z})$ the added constraint would have been $|z_j| > \max(|w_k|)$ for all j.

Theorem 4.1.2. *The $\mathcal{H}^{\nu,3}$ operator and the operator $\varphi^\nu(\mathbf{z},\mathbf{w})$ obey the commutation relation*

$$\begin{aligned}\left[\mathcal{H}^{\nu,3},\varphi^\nu(\mathbf{z},\mathbf{w})\right] &= \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\varphi^\nu(\mathbf{z},\mathbf{w}) - \frac{4\pi^2}{L^2}\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2\varphi^\nu(\mathbf{z},\mathbf{w})\\ &-\frac{4\pi^2}{L^2}2\left(\nu^2-1\right)\left(\nu^2\sum_{k'<k}\frac{w_kw_{k'}}{\left(w_k-w_{k'}\right)^2}\varphi^\nu(\mathbf{z},\mathbf{w})+\frac{1}{\nu^2}\sum_{j'<j}\frac{z_jz_{j'}}{\left(z_j-z_{j'}\right)^2}\varphi^\nu(\mathbf{z},\mathbf{w})\right)\\ &-\frac{4\pi^2}{L^2}2\left(1-\nu^2\right)\sum_{k=1}^{N}\sum_{j=1}^{M}\frac{w_kz_j}{\left(w_k-z_j\right)^2}\varphi^\nu(\mathbf{z},\mathbf{w})+\frac{\pi^2}{3L^2}\frac{\left(\nu^4-1\right)}{\nu^2}M\,\varphi^\nu(\mathbf{z},\mathbf{w})\\ &-2\left(\nu^2-1\right)\left(\mathcal{C}*\varphi^\nu(\mathbf{z},\mathbf{w})-\varphi^\nu(\mathbf{z},\mathbf{w})\mathcal{C}\right)\end{aligned}\tag{4.1}$$

Proof. See Appendix E.4.1. □

The last two terms in Eq (4.1) vanish if the commutation is applied to a vector $R^\omega\Omega$, for any $\omega\in\mathbb{Z}$. This gives that

$$\begin{aligned}\left[\mathcal{H}^{\nu,3},\varphi^\nu(\mathbf{z},\mathbf{w})\right]R^\omega\Omega &= \Bigg(\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2-\frac{4\pi^2}{L^2}\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2\\ &-\frac{4\pi^2}{L^2}2\nu^2\left(\nu^2-1\right)\sum_{k'<k}\frac{w_kw_{k'}}{\left(w_k-w_{k'}\right)^2}-\frac{4\pi^2}{L^2}\frac{2}{\nu^2}\left(\nu^2-1\right)\sum_{j'<j}\frac{z_jz_{j'}}{\left(z_j-z_{j'}\right)^2}\\ &-\frac{4\pi^2}{L^2}2\left(1-\nu^2\right)\sum_{k=1}^{N}\sum_{j=1}^{M}\frac{w_kz_j}{\left(w_k-z_j\right)^2}+\frac{\pi^2}{3L^2}\frac{\left(\nu^4-1\right)}{\nu^2}M\Bigg)\varphi^\nu(\mathbf{z},\mathbf{w})R^\omega\Omega\end{aligned}\tag{4.2}$$

Comparing Eqs. (3.8) and (4.2) gives

$$\left[\mathcal{H}^{\nu,3},\varphi^\nu(\mathbf{z},\mathbf{w})\right]R^\omega\Omega=\left(\tilde{H}_{N,M}+\frac{\pi^2}{3L^2}\frac{\left(\nu^4-1\right)}{\nu^2}M\right)\varphi^\nu(\mathbf{z},\mathbf{w})R^\omega\Omega\tag{4.3}$$

where $\tilde{H}_{N,M}$ is the dCS differential operator given in Eq (3.8) for $\lambda=\nu^2$.

4.2 Anyons and the dCS eigenfunctions

Theorem 4.1.2 allows us to construct eigenfunctions of the dCS differential operator.

Theorem 4.2.1. *Let $\tilde{\eta}$ be an eigenvector of the $\mathcal{H}^{\nu,3}$ operators, belonging to a dense, invariant domain, and satisfying*

$$\mathcal{H}^{\nu,3}\tilde{\eta} = \tilde{\mathcal{E}}\tilde{\eta}$$

Then the function $\tilde{F}_{\tilde{\eta}}(\mathbf{z},\mathbf{w})$ defined as

$$\tilde{F}_{\tilde{\eta}}(\mathbf{z},\mathbf{w}) := \langle \tilde{\eta}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega\rangle$$

which is well-defined and non-trivial, is an eigenfunction of the dCS differential operator with eigenvalue

$$\tilde{\mathcal{E}} - \frac{\pi^2}{3L^2}\frac{\nu^4-1}{\nu^2}M$$

Proof. See Appendix E.4.2 □

The construction of the exact solutions to the dCS model is outside the scope of this thesis. We restrict ourselves to the simplest eigenvector η_{dCS} and construct the deformed groundstate (dGS) eigenfunction.

Corollary 4.2.2. *The function $\tilde{F}_0(\mathbf{z},\mathbf{w})$, defined as*

$$\tilde{F}_0(\mathbf{z},\mathbf{w}) := \langle \eta_{dCS}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega\rangle$$

where

$$\langle \eta_{dCS}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega\rangle = \mathcal{J}^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{J}^{\nu}(\mathbf{w})e^{-i\frac{\pi}{L}\nu^{-2}\left(N\nu^2-M\right)\left(\nu^2\sum_{k=1}^{N}x_k-\sum_{j=1}^{M}y_j\right)}\prod_{k=1}^{N}\prod_{j=1}^{M}\sqrt{\left|\frac{w_k}{z_j}\right|\frac{\sqrt{z_jw_k}}{w_k-z_j}} \tag{4.4}$$

is en eigenfunction of the dCS differential operator. The correlation function $\tilde{F}_0(\mathbf{z},\mathbf{w})$ corresponds to the dGS eigenfunction, up to a constant factor and a CoM shift, since

$$\tilde{H}_{N,M}\tilde{F}_0(\mathbf{z},\mathbf{w}) = \left(\tilde{E}_0 + \frac{\pi^2}{L^2}\frac{\left(N\nu^2-M\right)^3}{\nu^2}\right)\tilde{F}_0(\mathbf{z},\mathbf{w}) \tag{4.5}$$

where $\tilde{E}_0$ is given by Eq (3.10), for $\lambda = \nu^2$, and the added term in Eq. (4.5) is exactly the contribution from the CoM shift in $\tilde{F}_0(\mathbf{z},\mathbf{w})$.

Proof. See Appendix E.4.3 □

Chapter 5

Conclusions

We have generalized the results in [19] to the deformed Calogero-Sutherland model.

We have shown that there exists a duality between anyons with statistical parameter ν and $-\frac{1}{\nu}$ and that they obey a natural generalization of the formal canonical anti-commutation relation. We have also shown that this duality exists for the $\mathcal{H}^{\nu,3}$ operator. This would suggest that the corresponding dual operator for an anyon with statistical parameter ν is an anyon field operator with statistical parameter $-\frac{1}{\nu}$ rather than the Hermitian conjugate. We would like to stress that it is not possible to conclude which of the field operators that correspond to the quasi-particle or quasi-hole field operators based on our work. This question is left open for further inquiry.

We have also shown that the expectation value of a vector created by using anyon and dual anyon field operators and the $\mathcal{H}^{\nu,3}$ operator corresponds to the deformed Calogero-Sutherland (dCS) differential operator and deformed groundstate (dGS) eigenfunction when using one of the simplest eigenvectors of the second quantized CS Hamiltonian. It should, therefore, be possible to construct the exact eigenfunctions of the dCS model by using an η vector similar to the one proposed in [19] for the exact solutions to the CS model.

While the dCS model is a most natural generalization of the CS model from a mathematical point of view, it does not allow for a simple quantum mechanics interpretation. The dCS differential operator cannot directly be called a Hamiltonian since the sign in front of the y differentiation would suggest that the dual particles have a mass of $-\frac{1}{2\nu^2}$ (in natural units). The "Hamiltonian" would correspond to particles with negative mass for $\nu^2 > 0$, which is unphysical.

The second difficulty lies with the dGS eigenfunction. The eigenfunction of the dCS differential operator is not square integrable for any value of ν and thus does not have the same norm as a wavefunction.

Our work suggests that the function $\tilde{F}_0$ can be interpreted as the N anyon and M dual anyon correlation function. The interpretation of the correlation function is outside the scope of this thesis and warrants further investigation. There exists no physical application of the dCS model up to now, as far as we know.

The only restriction on the statistical parameter ν of anyon field operator was that there exists a constant ν_0 such that the statistical parameter is an integer times ν_0. With the introduction of the dual anyon field operator we also get that the statistical parameter ν should be one over an integer when multiplied with the same constant ν_0, *i.e.*

$$\frac{\nu}{\nu_0} = p \in \mathbb{Z} \quad \text{and} \quad \frac{1}{\nu\nu_0} = q \in \mathbb{Z}$$

Combining these two requirements result in

$$\nu^2 = \frac{p}{q} \in \mathbb{Q} \quad \text{and} \quad \frac{1}{\nu_0^2} = qp \in \mathbb{Z}$$

Sections 3.1.2 and 3.3 shows that correlation function of N anyon field operators yields the Laughlin wavefunction at the boundary of a circular quantum Hall droplet, up to a constant and a CoM shift, which corresponds to a filling factor of one over ν^2 which could have any value. The dual anyon operators restrict the values of ν^2 to only rational numbers. This indicates that the dual anyons are also needed in order to construct a full theoretical formalism of the fractional quantum Hall effect (FQHE).

We hope that this provides a starting point of a physical interpretation of the dCS model in the context of the FQHE.

Acknowledgements

I would like to thank my supervisor Prof. Edwin Langmann for giving me the chance to work on this very interesting project and for his guidance and support. My sincerest gratitude to Jonas de Woul for helping me with the introduction to bosonization, explaining some of the advanced mathematics needed for this project, and helping me with LaTeX. I would also like to thank Martin Sundin for helping me with some of the advanced mathematics. Many thanks to the rest of the Mathematical Physics group and the Theoretical Physics department for the interesting conversations during lunches and Friday afternoon coffee.

Appendix A

Summary of notations

- The fundamental constants are set to[1] $\hbar = c = e = 1$ through the work.
- $\mathbb{N}$ denotes the set of all natural numbers (positive integers).
  $\mathbb{N} := \{1, 2, 3, \ldots\}$.
- $\mathbb{N}_0$ denotes the set of all natural numbers including zero.
  $\mathbb{N}_0 := \{0, 1, 2, 3 \ldots\}$.
- $\mathbb{Z}$ denotes the set of all integers.
  $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- S_L denotes a circle of length L.
- $\delta_{n,m}$ is the Kronecker delta defined as $\delta_{n,m} = \begin{cases} 1 & \text{, if } n = m \\ 0 & \text{, otherwise} \end{cases}$
- $\theta(x)$ is the Heaviside step function defined as $\theta(x) = \begin{cases} 1 & \text{, if } x > 0 \\ 0 & \text{, if } x < 0 \end{cases}$ and not specified for $x = 0$.
- $\text{sgn}(x)$ is the sign function defined as $\text{sgn}(x) = \begin{cases} +1 & \text{, if } x > 0 \\ -1 & \text{, if } x < 0 \end{cases}$ and not specified for $x = 0$.
- $[\ ,\ ]$ is the commutation brackets defined as $[A, B] = AB - BA$.
- $\{\ ,\}$ is the anti-commutation brackets defined as $\{A, B\} = AB + BA$.
- $(\ ,\ )_{\mathcal{H}}$ denotes the Hilbert space inner product $(\ ,\ )_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$.

[1] $\hbar$ is Planck constant divided by 2π, c is the speed of light and e is the elementary charge.

- $\mathrm{Tr}_{\mathcal{H}}$ denotes the trace of our Hilbert space defined as $\mathrm{Tr}_{\mathcal{H}}(A) := \sum_{n\in\mathbb{Z}} (e_n, Ae_n)_{\mathcal{H}}$ for any operator A.
- $\langle \ldots \rangle$ denotes the vacuum expectation value defined as $\langle A\rangle = \langle \Omega, A\Omega\rangle$ for any operator A.
- $\psi_n^{(*)}$ is the fermion annihilation (creation) operator, obeying the CAR.
- $d\Gamma$ is the fermion normal-ordered, 2'nd quantization operator defined as in Eq. (1.9).
- Q is the charge operator, defined as in Eq. (1.15).
- ρ_n , for $n \neq 0$, are the oscillation operator and are defined as in Eq. (1.12).
- $R^{\pm 1}$ are the Klein factors, which change the particle number of the system by ± 1.
- $\varepsilon \downarrow 0$ denotes the limit $\lim_{\varepsilon\to 0^+}$.
- $: \ldots :$ denotes fermion normal-ordering of the operator, or operator product, "$\ldots$" defined in Section 1.2.1.
- ${}^*_*\ldots{}^*_*$ denotes boson normal-ordering of the operator, or operator product "$\ldots$" as described in Section 1.2.2.
- $\delta(z-w)$ is the formal delta function given by Eq. (1.23).
- $\mathrm{sign}(z-w)$ is the formal sign function given by Eq. (1.30).
- w is an indeterminate of a formal distribution given by Eq. (2.4)
- w_k, where $k = 1, \ldots, N$, are indeterminates of a formal distribution, such that $w_k \neq w_{k'}$ for all $k \neq k'$
- z is an indeterminate of a formal distribution given by Eq. (2.5)
- z_j, where $j = 1, \ldots, M$, are indeterminates of a formal distribution, such that $z_j \neq z_{j'}$ for all $j \neq j'$
- ϕ^{ν} denotes the anyon operator of species ν defined as in Eq. (2.16).
- $\phi^{-\frac{1}{\nu}}$ denotes the dual anyon operator of species ν, defined as in Eq. (2.19).
- The product notation $\prod_{l'<l}$ is short for $\prod_{1\leq l'<l\leq \max(l)} = \prod_{l'=1}^{\max(l)-1}\prod_{l=l'+1}^{\max(l)} = \prod_{l=2}^{\max(l)}\prod_{l'=1}^{l-1}$, where $l', l = 1, 2, \ldots, \max(l)$.
- The sum notation $\sum_{l'<l}$ is short for $\sum_{l'=1}^{\max(l)-1}\sum_{l=l'+1}^{\max(l)} = \sum_{l=2}^{\max(l)}\sum_{l'=1}^{l-1}$, where $l', l = 1, 2, \ldots, \max(l)$.

Appendix B

Boson normal-ordering relations

In this Appendix, we show some of the normal-ordering[1] relations that are needed.

B.1 Normal-ordering products of Q's and R's

It might not be clear but operators consisting of powers of the charge operator and the Klein factors can also be normal-ordered. This is useful when evaluating the derivatives of the bosonized operators. The most used example is given by the normal-ordering of the Klein factors and exponentials of the charge operator.

Postulate B.1.1. *The normal-ordering of exponentials of the charge operator and the Klein factors are given by*

$$ {}^*_* R^n e^{i\alpha Q} {}^*_* = e^{i\alpha Q/2} R^n e^{i\alpha Q/2} \tag{B.1} $$

for any $n \in \mathbb{Z}, \alpha \in \mathbb{C}$.

Lemma B.1.2. *The normal-ordered product of Klein factors and the charge operator, raised to an arbitrary number, is given by*

$$ {}^*_* Q^m R^n {}^*_* := \frac{1}{2^m} \sum_{n'=0}^{m} \binom{m}{n'} Q^{n'} R^n Q^{m-n'} \quad , \forall\ n \in \mathbb{Z}, m \in \mathbb{N} \tag{B.2} $$

[1] In this part, normal-ordering is always meant as boson normal-ordering.

Proof. Taking equation (B.1) and Taylor expanding with regard to α around 0 gives that

$$\begin{aligned} &{}^*_*R^n + i\alpha Q R^n - \frac{\alpha^2}{2}Q^2R^n - \frac{i\alpha^3}{6}Q^3R^n + \mathcal{O}(\alpha^4){}^*_* = \\ &\quad \left(1 + i\frac{\alpha}{2}Q - \frac{\alpha^2}{8}Q^2 - \frac{i\alpha^3}{48}Q^3\right) R^n \left(1 + i\frac{\alpha}{2}Q - \frac{\alpha^2}{8}Q^2 - \frac{i\alpha^3}{48}Q^3\right) + \mathcal{O}(\alpha^4) \end{aligned} \tag{B.3}$$

which yields

$$\begin{aligned} {}^*_*QR^n{}^*_* &= \frac{1}{2}\left(QR^n + R^nQ\right) \\ {}^*_*Q^2R^n{}^*_* &= \frac{1}{4}\left(R^nQ^2 + 2QR^nQ + Q^2R^n\right) \\ {}^*_*Q^3R^n{}^*_* &= \frac{1}{8}\left(R^nQ^3 + 3QR^nQ^2 + 3Q^2R^nQ + Q^3R^n\right) \end{aligned}$$

Generalizing the results for Q to the power of an arbitrary number gives equation (B.2). □

B.2 Normal-ordering products of anyon operators of arbitrary number

The products of N anyon operators has a very interesting correlation function which corresponds to the ground-state of the Calogero-Sutherland (CS) model (c.f. Section 3.1.1). This is most easily seen if the product is normal-ordered. There are also similar result for a multi-species product of anyon field operators.

We define $\mathbf{w} = (w_1, w_2, \dots, w_N)$ such that $w_k \neq w_{k'}$ $\forall k \neq k'$ and w_k is given by Eq. (2.25).

B.2.1 Product with the same species. Proof of Lemma 2.3.4

The proof is given by first normal-ordering the products of 2 and 3 anyon operators and then generalizing to a product of N operators.

2 anyon operators

The product of two anyon field operators can be written as

$$\begin{aligned} \phi^\nu(w_1)\phi^\nu(w_2) = &\left(e^{-i\frac{\pi}{L}\nu\nu_0 x_1 Q} R^{\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0 x_1 Q} e^{-i\frac{\pi}{L}\nu\nu_0 x_2 Q} R^{\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0 x_2 Q}\right)_{(1)} \times \\ &\left(e^{-\nu\sum\limits_{n<0}\frac{1}{n}\rho_n w_1^n} e^{-\nu\sum\limits_{n>0}\frac{1}{n}\rho_n w_1^n} e^{-\nu\sum\limits_{m<0}\frac{1}{m}\rho_m w_2^m} e^{-\nu\sum\limits_{m>0}\frac{1}{m}\rho_m w_2^m}\right)_{(2)} \end{aligned} \tag{B.4}$$

Since the two parts of Eq. (B.4) $\big((\ldots)_{(1)}$ and $(\ldots)_{(2)}\big)$ commute, they can be normal-ordered separately. Normal-ordering the first part by using Eq. (1.34) gives

$$(\ldots)_{(1)} = e^{-i\frac{\pi}{L}\nu^2(x_1-x_2)} {}^*_*(\ldots)_{(1)}{}^*_*$$

where

$${}^*_*(\ldots)_{(1)}{}^*_* = e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2)Q} R^{\,2\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2)Q}$$

The second part can be normal-ordered by using equations (1.14) and (1.32), which gives that

$$(\ldots)_{(2)} = e^{\left[-\nu\sum_{n>0}\frac{1}{n}\rho_n w_1^n, -\nu\sum_{m<0}\frac{1}{m}\rho_m w_2^m\right]} {}^*_*(\ldots)_{(2)}{}^*_*$$

where

$${}^*_*(\ldots)_{(2)}{}^*_* = e^{-\nu\sum_{n<0}\frac{1}{n}\rho_n w_1^n} e^{-\nu\sum_{m<0}\frac{1}{m}\rho_m w_2^m} e^{-\nu\sum_{n>0}\frac{1}{n}\rho_n w_1^n} e^{-\nu\sum_{m>0}\frac{1}{m}\rho_m w_2^m}$$

Summarizing the results gives that

$$\begin{aligned}\phi^\nu(w_1)\phi^\nu(w_2) &= {}^*_*\phi^\nu(w_1)\phi^\nu(w_2){}^*_*\; e^{i\frac{\pi}{L}\nu^2(x_2-x_1)} e^{-\nu^2\sum_{n>0}\frac{1}{n}w_1^n w_2^{-n}} \\ &= {}^*_*\phi^\nu(w_1)\phi^\nu(w_2){}^*_*\; \mathrm{i}_{w_2,w_1}\left|\frac{w_1}{w_2}\right|^{\frac{\nu^2}{2}}\left(\frac{w_1-w_2}{\sqrt{w_1w_2}}\right)^{\nu^2}\end{aligned}$$

So the final result becomes

$$\phi^\nu(w_1)\phi^\nu(w_2) = \mathcal{J}^\nu(w_1,w_2){}^*_*\phi^\nu(w_1)\phi^\nu(w_2){}^*_* \tag{B.5}$$

where $\mathcal{J}^\nu$ is defined in Lemma 2.3.4.

Product of 3 anyon operators

Using equation (B.5) the product 3 anyon field operators can be written as

$$\phi^\nu(w_1)\phi^\nu(w_2)\phi^\nu(w_3) = \mathcal{J}^\nu(w_1,w_2){}^*_*\phi^\nu(w_1)\phi^\nu(w_2){}^*_*\phi^\nu(w_3) \tag{B.6}$$

where

$$\begin{aligned}{}^*_*\phi^\nu(w_1)\phi^\nu(w_2){}^*_*\phi^\nu(w_3) = \Big(&e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2)Q} R^{\,2\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2)Q} \\ &\times e^{-i\frac{\pi}{L}\nu\nu_0x_3Q} R^{\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0x_3Q}\Big)_{(1)} \times \\ \Big(e^{-\nu\sum_{n<0}\frac{1}{n}\rho_n(w_1^n+w_2^n)} &e^{-\nu\sum_{n>0}\frac{1}{n}\rho_n(w_1^n+w_2^n)} e^{-\nu\sum_{m<0}\frac{1}{m}\rho_m w_3^m} e^{-\nu\sum_{m>0}\frac{1}{m}\rho_m w_3^m}\Big)_{(2)}\end{aligned}$$

The two parts are again treated separately giving

$$(\ldots)_{(1)} = e^{-i\frac{\pi}{L}\nu^2(x_1+x_2)} e^{i\frac{\pi}{L}2\nu^2x_3} {}^*_*(\ldots)_{(1)}{}^*_*$$

$$(\dots)_{(2)} = e^{\left[-\nu \sum\limits_{n>0} \frac{1}{n}\rho_n (w_1^n + w_2^n), -\nu \sum\limits_{m<0} \frac{1}{m}\rho_m w_3^m\right]} {}^*_*(\dots)_{(2)}{}^*_*$$

where

$${}^*_*(\dots)_{(1)}{}^*_* = e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2+x_3)Q} R^{\,3\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2+x_3)Q}$$

$${}^*_*(\dots)_{(2)}{}^*_* = e^{-\nu \sum\limits_{n<0} \frac{1}{n}\rho_n (w_1^n + w_2^n + w_3^n)} e^{-\nu \sum\limits_{n>0} \frac{1}{n}\rho_n (w_1^n + w_2^n + w_3^n)}$$

Inserting the equations into Eq. (B.6) gives

$$\phi^\nu(w_1)\phi^\nu(w_2)\phi^\nu(w_3) = {}^*_*\phi^\nu(w_1)\phi^\nu(w_2)\phi^\nu(w_3){}^*_* \times \mathcal{J}^\nu(w_1, w_2) e^{-i\frac{\pi}{L}\nu^2(x_1+x_2-2x_3)} e^{-\nu^2 \sum\limits_{n>0} \frac{1}{n}\left(w_1^n w_3^{-n} + w_2^n w_3^{-n}\right)}$$

where

$$\begin{aligned} &e^{-i\frac{\pi}{L}\nu^2(x_1+x_2-2x_3)} e^{-\nu^2 \sum\limits_{n>0} \frac{1}{n}\left(w_1^n w_3^{-n} + w_2^n w_3^{-n}\right)} \\ &= \left(e^{-i\frac{\pi}{L}\nu^2(x_1-x_3)} e^{-\nu^2 \sum\limits_{n>0} \frac{1}{n} w_1^n w_3^{-n}}\right) \left(e^{-i\frac{\pi}{L}\nu^2(x_2-x_3)} e^{-\nu^2 \sum\limits_{n>0} \frac{1}{n} w_2^n w_3^{-n}}\right) \\ &= \left(\mathrm{i}_{w_3,w_1} \left|\frac{w_1}{w_3}\right|^{\frac{\nu^2}{2}} \left(\frac{w_3 - w_1}{\sqrt{w_3 w_1}}\right)^{\nu^2}\right) \left(\mathrm{i}_{w_3,w_2} \left|\frac{w_2}{w_3}\right|^{\frac{\nu^2}{2}} \left(\frac{w_3 - w_2}{\sqrt{w_3 w_2}}\right)^{\nu^2}\right) \end{aligned}$$

which gives that

$$\mathcal{J}^\nu(w_1, w_2) e^{-i\frac{\pi}{L}\nu^2(x_1+x_2-2x_3)} e^{-\nu^2 \sum\limits_{n>0} \frac{1}{n}\left(w_1^n w_3^{-n} + w_2^n w_3^{-n}\right)} = \mathcal{J}^\nu(w_1, w_2, w_3)$$

Summarizing the result yields

$$\phi^\nu(w_1)\phi^\nu(w_2)\phi^\nu(w_3) = \mathcal{J}^\nu(w_1, w_2, w_3) {}^*_*\phi^\nu(w_1)\phi^\nu(w_2)\phi^\nu(w_3){}^*_* \tag{B.7}$$

where

$${}^*_*\phi^\nu(w_1)\phi^\nu(w_2)\phi^\nu(w_3){}^*_* = e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2+x_3)Q} R^{\,3\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0(x_1+x_2+x_3)Q} \times e^{-\nu \sum\limits_{n<0} \frac{1}{n}\rho_n (w_1^n + w_2^n + w_3^n)} e^{-\nu \sum\limits_{n>0} \frac{1}{n}\rho_n (w_1^n + w_2^n + w_3^n)}$$

Product of N anyon operators

Generalizing the result for any $N(\in \mathbb{N})$. Equation (2.26) can be re-written as

$$\Phi^\nu(\mathbf{w}) = \mathcal{J}^\nu(\{w_k\}_{k=1}^{N-1}) {}^*_*\Phi^\nu(\{w_k\}_{k=1}^{N-1}){}^*_* \, \phi^\nu(w_N) \tag{B.8}$$

where

$$
{}^*_*\Phi^\nu(\{w_k\}_{k=1}^{N-1})^*_*\ \phi^\nu(w_N) =
$$
$$
\left(e^{-i\frac{\pi}{L}\nu\nu_0\left(\sum\limits_{k=1}^{N-1} x_k\right)Q} R^{(N-1)\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0\left(\sum\limits_{k=1}^{N-1} x_k\right)Q} e^{-i\frac{\pi}{L}\nu\nu_0 x_N Q} R^{\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0 x_N Q} \right)_{(1)}
$$
$$
\times \left(e^{-\nu\sum\limits_{k=1}^{N-1}\sum\limits_{n<0}\frac{1}{n}\rho_n w_k^n} e^{-\nu\sum\limits_{k=1}^{N-1}\sum\limits_{n>0}\frac{1}{n}\rho_n w_k^n} e^{-\nu\sum\limits_{m<0}\frac{1}{m}\rho_m w_N^m} e^{-\nu\sum\limits_{m>0}\frac{1}{m}\rho_m w_N^m} \right)_{(2)} \tag{B.9}
$$

Using

$$
(\dots)_{(1)} = e^{-i\frac{\pi}{L}\nu^2\left(\sum\limits_{k=1}^{N-1} x_k\right)} e^{i\frac{\pi}{L}\nu^2(N-1)x_N}\, {}^*_*(\dots)_{(1)}{}^*_*
$$
$$
(\dots)_{(2)} = e^{\left[-\nu\sum\limits_{k=1}^{N-1}\sum\limits_{n>0}\frac{1}{n}\rho_n w_k^n, -\nu\sum\limits_{m<0}\frac{1}{m}\rho_m w_N^m\right]}\, {}^*_*(\dots)_{(2)}{}^*_*
$$

gives that

$$
{}^*_*\Phi^\nu(\{w_k\}_{k=1}^{N-1})^*_*\ \phi^\nu(w_N) = \left(\prod_{k'=1}^{N-1} \mathrm{i}_{w_N,w_{k'}} \left(\left|\frac{w_{k'}}{w_N}\right|^{\frac{1}{2}} \frac{w_N - w_{k'}}{\sqrt{w_N w_{k'}}} \right)^{\nu^2}\right) {}^*_*\Phi^\nu(\mathbf{w})^*_*
$$

where ${}^*_*\Phi^\nu(\mathbf{w})^*_*$ is defined in Eq. (2.27). This proves Lemma 2.3.4 since

$$
\mathcal{J}^\nu(\{w_k\}_{k=1}^{N-1}) \prod_{k'=1}^{N-1} \mathrm{i}_{w_N,w_{k'}} \left(\left|\frac{w_{k'}}{w_N}\right|^{\frac{1}{2}} \frac{w_N - w_{k'}}{\sqrt{w_N w_{k'}}} \right)^{\nu^2} =
$$
$$
\left(\prod_{1\le k'<k\le N-1} \mathrm{i}_{w_k,w_{k'}} \left(\left|\frac{w_{k'}}{w_k}\right|^{\frac{1}{2}} \frac{w_k - w_{k'}}{\sqrt{w_{k'} w_k}} \right)^{\nu^2}\right) \prod_{k'=1}^{N-1} \mathrm{i}_{w_N,w_{k'}} \left(\left|\frac{w_{k'}}{w_N}\right|^{\frac{1}{2}} \frac{w_N - w_{k'}}{\sqrt{w_N w_{k'}}} \right)^{\nu^2}
$$
$$
= \mathcal{J}^\nu(\mathbf{w})
$$

B.2.2 Products of different species. Proof of Lemma 2.3.5

The treatment in Section B.2.1 can also be used to normal-order products of anyon field operators with individual statistical parameters ν_k, where $\nu_k/\nu_0 \in \mathbb{Z}\ \forall k$.

The proof for Lemma 2.3.5 is very similar to the proof for Lemma 2.3.4, except changing the statistic parameter to individual ones. It should not be necessary to prove it entirely. The method for a product of two anyon field operator, with different statistical parameters, is useful for calculating other relations, e.g. calculating (anti-)commutation relations, so it will be shown explicitly.

Normal-ordering a product of two anyon operators with different species

The product is given by

$$\begin{aligned}\phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2) &= \\ &\left(e^{-i\frac{\pi}{L}\nu_0\nu_1 x_1 Q} R^{\frac{\nu_1}{\nu_0}} e^{-i\frac{\pi}{L}\nu_0\nu_1 x_1 Q} e^{-i\frac{\pi}{L}\nu_0\nu_2 x_2 Q} R^{\frac{\nu_2}{\nu_0}} e^{-i\frac{\pi}{L}\nu_0\nu_2 x_2 Q}\right)_{(1)} \\ &\times \left(e^{-\nu_1 \sum\limits_{n<0}\frac{1}{n}\rho_n w_1^n} e^{-\nu_1 \sum\limits_{n>0}\frac{1}{n}\rho_n w_1^n} e^{-\nu_2 \sum\limits_{m<0}\frac{1}{m}\rho_m w_2^m} e^{-\nu_2 \sum\limits_{m>0}\frac{1}{m}\rho_m w_2^m}\right)_{(2)}\end{aligned} \tag{B.10}$$

Normal-ordering the parts separately by

$$(\ldots)_{(1)} = e^{-i\frac{\pi}{L}\nu_1\nu_2 x_1} e^{i\frac{\pi}{L}\nu_1\nu_2 x_2} {}^*_*(\ldots)_{(1)}{}^*_*$$

$$(\ldots)_{(2)} = e^{\left[-\nu_1 \sum\limits_{n>0}\frac{1}{n}\rho_n w_1^n, -\nu_2 \sum\limits_{m<0}\frac{1}{m}\rho_m w_2^m\right]} {}^*_*(\ldots)_{(2)}{}^*_*$$

which gives that

$$\begin{aligned}\phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2) &= {}^*_*\phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2)^*_* \left(e^{-i\frac{\pi}{L}(x_1-x_2)} e^{-\sum\limits_{n>0}\frac{1}{n}w_1^n w_2^{-n}}\right)^{\nu_1\nu_2} \\ &= {}^*_*\phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2)^*_* \; \mathrm{i}_{w_2,w_1}\left(\left|\frac{w_1}{w_2}\right|^{\frac{1}{2}} \frac{w_1-w_2}{\sqrt{w_1 w_2}}\right)^{\nu_1\nu_2}\end{aligned}$$

where

$$\begin{aligned}{}^*_*\phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2)^*_* = e^{-i\frac{\pi}{L}\nu_0(\nu_1 x_1+\nu_2 x_2)Q} R^{\frac{\nu_1+\nu_2}{\nu_0}} e^{-i\frac{\pi}{L}\nu_0(\nu_1 x_1+\nu_2 x_2)Q} \times \\ e^{-\sum\limits_{n<0}\frac{1}{n}\rho_n(\nu_1 w_1^n+\nu_2 w_2^n)} e^{-\sum\limits_{n>0}\frac{1}{n}\rho_n(\nu_1 w_1^n+\nu_2 w_2^n)}\end{aligned}$$

such that

$$\langle {}^*_*\phi^{\nu_1}(w_1)\phi^{\nu_2}(w_2)^*_*\rangle = \delta_{\frac{\nu_1+\nu_2}{\nu_0},0}$$

Appendix C

Fock space equality

C.1 Boson completeness

As stated in Lemma 2.1.1, the basis vectors of the fermion Fock space $\mathcal{F}$ can be represented in terms of the boson operators and the Klein factors. The Fock spaces spanned by the basis vectors

$$
\begin{aligned}
B(\{m_n\}_{n=1}^{\infty}, \omega) &= \prod_{n=1}^{\infty} \frac{\left(a_n^{\dagger}\right)^{m_n}}{\sqrt{m_n!}} R^{\omega} \Omega \\
&\text{or} \\
F(\{m_k, n_k\}_{k=0}^{\infty}) &= \prod_{k=0}^{\infty} \left(\psi_k^*\right)^{m_k} \left(\psi_{-k-1}\right)^{n_k} \Omega
\end{aligned}
\tag{C.1}
$$

where

$$
a_n^{\dagger} := \frac{\rho_{-n}}{\sqrt{n}} \tag{C.2}
$$

$m_n \in \mathbb{N}_0$, $m_k, n_k \in \{0, 1\}$, under the requirement that $\sum_n m_n < \infty$, $\sum_k m_k + n_k < \infty$, are equal.

Since the boson operators can be expressed in terms of the fermion operators (c.f. (1.12)), the Fock space spanned by the bosonic vectors $B(\{m_n\}_{n=n}^{\infty}, \omega)$ is at least a subspace of the fermion Fock space spanned by the vectors in equation (2.3). In order to show that the two Fock spaces are isomorphic, one can calculate the partition function $\mathcal{Z}$. The partition function are equal if every state in the Fock space spanned by the $F(\{m_k, n_k\}_{k=0}^{\infty})$ also occurs in the Fock space spanned by the basis $B(\{m_n\}_{n=1}^{\infty}, \omega)$.

The partition function is defined as[1]

$$\mathcal{Z} := \mathrm{Tr}(e^{-\beta H}) \tag{C.3}$$

where H is the Hamiltonian given by

$$H = \begin{cases} H_B = \frac{\pi}{L} Q^2 + \frac{2\pi}{L} \sum_{n=1}^{\infty} n \, {}^*_* a_n^\dagger a_n {}^*_* & \text{, Boson case} \\ H_F = \frac{2\pi}{L} \sum_{k \in \mathbb{Z}} \left(k + \frac{1}{2}\right) : \psi_k^* \psi_k : & \text{, Fermion case} \end{cases} \tag{C.4}$$

and $\beta > 0$.

The eigenvalues of the Hamiltonian are

$$H_B B(\{m_n\}, \omega) = \left(\frac{\pi}{L} \omega^2 + \frac{2\pi}{L} \sum_{n=1}^{\infty} n m_n \right) B(\{m_n\}, \omega) \tag{C.5}$$

$$H_F F(\{m_k, n_k\}) = \left(\frac{2\pi}{L} \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) (m_k + n_k) \right) F(\{m_k, n_k\}) \tag{C.6}$$

Using equation (C.5), the boson partition function becomes

$$\mathcal{Z}_B = \sum_{\omega \in \mathbb{Z}} e^{-\frac{\pi\beta}{L}\omega^2} \prod_{n=1}^{\infty} \sum_{m_n=0}^{\infty} e^{-\frac{2\pi}{L}\beta n m_n}$$

where

$$\sum_{m_n=0}^{\infty} e^{-\frac{2\pi}{L}\beta n m_n} = \frac{1}{1 - e^{-\frac{2\pi}{L}\beta n}}$$

and by using the *Jacobi triple identity* (see *e.g.* Chapter 5 of [22])

$$\sum_{m \in \mathbb{Z}} x^{m^2} y^{2m} = \prod_{m=1}^{\infty} \left(1 - x^{2m}\right) \left(1 + x^{2m-1} y^2\right) \left(1 + x^{2m-1} y^{-2}\right) \tag{C.7}$$

which gives

$$\mathcal{Z}_B = \prod_{m=1}^{\infty} \left(1 - x^{2m}\right) \left(1 + x^{2m-1}\right)^2 \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n}} = \prod_{m=1}^{\infty} \left(1 + x^{2m-1}\right)^2 \tag{C.8}$$

where we use $x = e^{-\frac{\pi\beta}{L}}$ and $y = 1$.

[1]The chemical potential μ is set to 0 for simplicity. The result is the same for non-zero μ.

Using Eq. (C.6), the fermion partition function becomes

$$\mathcal{Z}_F = \prod_{k=0}^{\infty} \sum_{m_k, n_k=0}^{1} e^{-\frac{2\pi\beta}{L}\left(k+\frac{1}{2}\right)(m_k+n_k)} \tag{C.9}$$

using

$$\sum_{m_k, n_k=0}^{1} e^{-\frac{2\pi\beta}{L}\left(k+\frac{1}{2}\right)(m_k+n_k)} = 1 + 2e^{-\frac{2\pi\beta}{L}\left(k+\frac{1}{2}\right)} + e^{-2\frac{2\pi\beta}{L}\left(k+\frac{1}{2}\right)}$$
$$= \left(1 + e^{-\frac{2\pi\beta}{L}(k+1/2)}\right)^2 = \left(1 + x^{2k+1}\right)^2$$

and

$$\prod_{k=0}^{\infty} \left(1 + x^{2k+1}\right)^2 = \prod_{n'=1}^{\infty} \left(1 + x^{2(n'-1)+1}\right)^2 = \prod_{n'=1}^{\infty} \left(1 + x^{2n'-1}\right)^2 \quad , n' = k+1$$

Inserting these relations into Eq. (C.9) and comparing to Eq. (C.8) shows that

$$\mathcal{Z}_F = \prod_{n'=1}^{\infty} \left(1 + x^{2n'-1}\right)^2 = \mathcal{Z}_B$$

Appendix D

The W-current

D.1 The $W_{1+\infty}$ algebra

Define

$$w_n^s := e^{-i\frac{\pi}{L}nx}\hat{p}^{s-1}e^{-i\frac{\pi}{L}nx} \quad , n \in \mathbb{Z} \; , s \in \mathbb{N} \tag{D.1}$$

where $x \in S_L$ and $\hat{p} := -i\frac{\partial}{\partial x}$. We can construct an operator valued generating function (OVGF) by

$$w_n(a) := \sum_{s=1}^{\infty} \frac{(ia)^{s-1}}{(s-1)!} w_n^s = e^{-i\frac{\pi}{L}nx}e^{ia\hat{p}}e^{-i\frac{\pi}{L}nx} = e^{-i\frac{\pi}{L}na}e^{-i\frac{2\pi}{L}nx}e^{ia\hat{p}} \tag{D.2}$$

where a is a free parameter and equation (1.32) was used to obtain the final result.

The $W_{1+\infty}$ algebra, which is the W_∞ algebra with a central extension, is then generated by the OVGF

$$W_n(a) := d\Gamma(w_n(a)) \tag{D.3}$$

satisfying

$$W_n(a)\Omega = 0 \quad \forall n \geq 0$$

The representation of the $W_{1+\infty}$ algebra is given by

$$[W_n(a), W_m(b)] = d\Gamma\left([w_n(a), w_m(b)]\right) + c \cdot iS(w_n(a), w_m(b)) \tag{D.4}$$

where c is the central element satisfying

$$[W_n, c] = 0 \quad c^* = c$$

Using Eq. (1.5), the Schwinger term is given by

$$iS(w_n(a), w_m(b)) = \sum_{k \in \mathbb{Z}} \big((e_k, P_- w_n(a) P_+ w_m(b) P_- e_k) \\ - (e_k, P_- w_m(b) P_+ w_n(a) P_- e_k)\big) \tag{D.5}$$

where $e_k = e_k(x)$ is the basis vector in $L^2(S_L)$, defined as in Eq. (1.6), and obey the following relations

$$\begin{aligned} P_\pm e_k &= \theta\left(\pm\left(k+\frac{1}{2}\right)\right) e_k \\ e^{-i\frac{2\pi}{L}nx} e_k &= e_{k-n} \\ e^{ia\hat{p}} e_k(x) = e_k(x+a) &= e^{\frac{2\pi}{L}i\left(k+\frac{1}{2}\right)a} e_k(x) \end{aligned}$$

for all $k \in \mathbb{Z}$ and where $\theta(x)$ is the Heaviside step function (c.f. Appendix A).

This gives that

$$P_+ w_m(b) P_- e_k = \begin{cases} \theta(-k-\frac{1}{2})\theta(k-m+\frac{1}{2}) e^{i\frac{2\pi}{L}\left(k-\frac{m}{2}+\frac{1}{2}\right)b} e_{k-m} & \text{, if } m<0 \\ 0 & \text{, otherwise} \end{cases}$$

Similarly $P_- w_n(a) e_{k-m}$ is only non-zero if $n \geq -m > 0$. Using that the basis vectors are orthonormal, i.e. $(e_k, e_{k-n-m}) = \delta_{n+m,0}$, yields

$$\sum_{k\in\mathbb{Z}} (e_k, P_- w_n(a) P_+ w_m(b) P_- e_k) = \begin{cases} \sum_{k=-n}^{-1} e^{i\frac{2\pi}{L}\left(k+\frac{n}{2}+\frac{1}{2}\right)\cdot(a+b)} \delta_{n,-m} & \text{, if } m<0 \\ 0 & \text{, otherwise} \end{cases}$$

$$= \begin{cases} \delta_{n,-m} \cdot \frac{\sin\left(\frac{\pi}{L}n(a+b)\right)}{\sin\left(\frac{\pi}{L}(a+b)\right)} & \text{, if } m<0 \\ 0 & \text{otherwise} \end{cases}$$

Similar treatment is done for the second part of Eq. (D.5) which yields

$$\sum_{k\in\mathbb{Z}} (e_k, P_- w_m(b) P_+ w_n(a) P_- e_k) = \begin{cases} \sum_{k=-m}^{-1} e^{i\frac{2\pi}{L}\left(k+\frac{m}{2}+\frac{1}{2}\right)\cdot(a+b)} \delta_{m,-n} & \text{, if } m>0 \\ 0 & \text{, otherwise} \end{cases}$$

$$= \begin{cases} -\delta_{m,-n} \frac{\sin\left(\frac{\pi}{L}n(a+b)\right)}{\sin\left(\frac{\pi}{L}(a+b)\right)} & \text{, if } m>0 \\ 0 & \text{, otherwise} \end{cases}$$

Combining these results gives

$$iS(w_n(a), w_m(b)) = \delta_{n,-m} \frac{\sin\left(\frac{\pi}{L}n(a+b)\right)}{\sin\left(\frac{\pi}{L}(a+b)\right)} \quad ,\forall n,m \in \mathbb{Z} \tag{D.6}$$

By using equation (1.32), it is fairly straightforward to compute the commutation relation of the generating function

$$\begin{aligned} w_n(a)w_m(b) &= e^{-i\frac{\pi}{L}na}e^{-i\frac{2\pi}{L}nx}e^{ia\hat{p}}e^{-i\frac{\pi}{L}mb}e^{-i\frac{2\pi}{L}mx}e^{ib\hat{p}} \\ &= e^{-i\frac{\pi}{L}(am-bn)}w_{n+m}(a+b) \end{aligned}$$

and

$$\begin{aligned} w_m(b)w_n(a) &= e^{-i\frac{\pi}{L}mb}e^{-i\frac{2\pi}{L}mx}e^{ib\hat{p}}e^{-i\frac{\pi}{L}na}e^{-i\frac{2\pi}{L}nx}e^{ia\hat{p}} \\ &= e^{i\frac{\pi}{L}(am-bn)}w_{n+m}(a+b) \end{aligned}$$

which yields

$$[w_n(a), w_m(b)] = 2i\sin\left(\frac{\pi}{L}(bn-am)\right)w_{n+m}(a+b)$$

Collecting the results yields that

$$[W_n(a), W_m(b)] = 2i\sin\left(\frac{\pi}{L}(bn-am)\right)W_{n+m}(a+b) + c\cdot\delta_{n,-m}\frac{\sin\left(\frac{\pi}{L}n(a+b)\right)}{\sin\left(\frac{\pi}{L}(a+b)\right)} \tag{D.7}$$

where

$$W_{n+m}(a+b) := d\Gamma\left(w_{n+m}(a+b)\right)$$

Of particular interest are the oscillation operators and the generators of the Virasoro algebra, which are contained in the OVGF for $c = 1$.

The oscillation operator can be written as

$$\rho_n := d\Gamma(w_n^1) = \lim_{a\downarrow 0} W_n(a) \tag{D.8}$$

which gives that

$$[\rho_n, \rho_m] = \lim_{a\downarrow 0}\lim_{b\downarrow 0}[W_n(a), W_m(b)] = n\delta_{n,-m} \tag{D.9}$$

Define

$$L_n := d\Gamma(w_n^2) = \lim_{a\downarrow 0}\left(-i\partial_a W_n(a)\right) \tag{D.10}$$

which satisfies

$$[L_n, L_m] = \lim_{a,b\downarrow 0} -\partial_a\partial_b\left[W_n(a), W_m(b)\right] = (n-m)\,L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12} \tag{D.11}$$

Eq. (D.11) is known as the Virasoro algebra and the operator L_n are then the generators of the Virasoro algebra.

Note that the Hamiltonian in $(1+1)$ dimensions is given by $H = L_0$ and Eq. (D.7) can be used to calculate the several important commutation relations.

D.2 The Sugawara construction

In this Section, the real space representation of the operators contained in Eq. (D.3) are constructed for $s \leq 3$.

Consider the point-split product of regularized fermion field operators, expressed in terms of bosonic operators as

$$\begin{aligned}\psi_\varepsilon^*(x)\psi_\varepsilon(x-a) = &\frac{1}{L} e^{-i\frac{\pi}{L}xQ} R e^{-i\frac{\pi}{L}xQ} e^{i\frac{\pi}{L}(x-a)Q} R^{-1} e^{i\frac{\pi}{L}(x-a)Q} \\ &e^{-\sum\limits_{n<0}\frac{1}{n}\rho_n e^{i\frac{2\pi}{L}nx} e^{-\frac{2\pi}{L}|n|\varepsilon}} e^{-\sum\limits_{n<0}\frac{1}{n}\rho_n e^{i\frac{2\pi}{L}nx} e^{-\frac{2\pi}{L}|n|\varepsilon}} \times \\ &\times e^{\sum\limits_{m<0}\frac{1}{m}\rho_m e^{i\frac{2\pi}{L}m(x-a)} e^{-\frac{2\pi}{L}|m|\varepsilon}} e^{\sum\limits_{m>0}\frac{1}{m}\rho_m e^{i\frac{2\pi}{L}m(x-a)} e^{-\frac{2\pi}{L}|m|\varepsilon}}\end{aligned} \tag{D.12}$$

Making the point-split product into a boson normal-ordered form, by using the BCH formulas (equations (1.33) and (1.34)), yields

$$\psi_\varepsilon^*(x)\psi_\varepsilon(x-a) = e^{i\frac{\pi}{L}a} e^{\sum\limits_{n>0}\frac{1}{n}e^{i\frac{2\pi}{L}na}e^{-\frac{4\pi}{L}|n|\varepsilon}} {}^*_*\psi_\varepsilon^*(x)\psi_\varepsilon(x-a)^*_*$$

where

$$\begin{aligned}&{}^*_*\psi_\varepsilon^*(x)\psi_\varepsilon(x-a)^*_* = \\ &\quad\frac{1}{L} e^{i\frac{2\pi}{L}aQ} e^{\sum\limits_{n<0}\frac{1}{n}\rho_n e^{i\frac{2\pi}{L}nx} e^{-\frac{2\pi}{L}|n|\varepsilon}\left(e^{-i\frac{2\pi}{L}na}-1\right)} e^{\sum\limits_{n>0}\frac{1}{n}\rho_n e^{i\frac{2\pi}{L}nx} e^{-\frac{2\pi}{L}|n|\varepsilon}\left(e^{-i\frac{2\pi}{L}na}-1\right)}\end{aligned}$$

It is now possible to fermion normal-ordered the point-split product. Using

$$\langle {}^*_*\psi_\varepsilon^*(x)\psi_\varepsilon(x-a)^*_* \rangle = \frac{1}{L}$$

gives

$$\begin{aligned}:\psi_\varepsilon^*(x)\psi_\varepsilon(x-a): &= e^{i\frac{\pi}{L}a} e^{\sum\limits_{n>0}\frac{1}{n}e^{i\frac{2\pi}{L}na}e^{-\frac{4\pi}{L}|n|\varepsilon}} \left({}^*_*\psi_\varepsilon^*(x)\psi_\varepsilon(x-a)^*_* - \frac{I}{L}\right) \\ &= \frac{e^{i\frac{\pi}{L}a}}{1-e^{i\frac{2\pi}{L}(a+2i\varepsilon)}}\left({}^*_*\psi_\varepsilon^*(x)\psi_\varepsilon(x-a)^*_* - \frac{I}{L}\right)\end{aligned} \tag{D.13}$$

Since a fermion normal-ordered product is a well defined operator product, it is possible to remove the regularization parameter, *i.e.* taking $\varepsilon \downarrow 0$

$$:\psi^*(x)\psi(x-a): = \frac{e^{i\frac{\pi}{L}a}}{1-e^{i\frac{2\pi}{L}a}}\left({}^*_*\psi^*(x)\psi(x-a)^*_* - \frac{I}{L}\right) \tag{D.14}$$

Taylor expanding the r.h.s. of Eq. (D.14) gives that[1]

$$:\psi^*(x)\psi(x-a): = \rho(x) - ia\left(\pi {}^*_*\rho(x)^2{}^*_* + \frac{1}{2i}{}^*_*\partial_x\rho(x){}^*_*\right)$$
$$-\frac{a^2}{2}\left(\frac{4\pi^2}{3}{}^*_*\rho(x)^3{}^*_* - \frac{1}{3}{}^*_*\partial_x^2\rho(x){}^*_* - 2\pi i{}^*_*\rho(x)\partial_x\rho(x){}^*_* - \frac{\pi^2}{6L^2}\rho(x)\right) + \mathcal{O}(a^3) \quad \text{(D.15)}$$

higher order terms are neglected since they are of little interest for this thesis.

The fermion normal-ordered point-split product can also be written as

$$:\psi^*(x)\psi(x-a): = :\psi^*(x)e^{-ia\hat{p}}\psi(x): = :\psi^*(x)\left(\sum_{n=0}^{\infty}\frac{(-ia)^n}{n!}\hat{p}^n\right)\psi(x): \quad \text{(D.16)}$$

Equations (D.15) and (D.16) gives

$$:\psi^*(x)\psi(x): = \rho(x) \quad \text{(D.17)}$$

Equation (D.17) gives exactly the same relation as equation (1.21). This relation is commonly referred to as the *Kronig identity* in the literatures.

The relation of interest (c.f. Eq. (C.4)) is

$$:\psi^*(x)\,\hat{p}\,\psi(x): = \pi{}^*_*\rho(x)^2{}^*_* + \frac{1}{2}{}^*_*\hat{p}\ \rho(x){}^*_* \quad \text{(D.18)}$$

The l.h.s. of Eq. (D.18) is the position space representation for the generators of the Virasoro algebra. And

$$:\psi^*(x)\,\hat{p}^2\,\psi(x): = \frac{4\pi^2}{3}{}^*_*\rho(x)^3{}^*_* + \frac{1}{3}{}^*_*\hat{p}^2\rho(x){}^*_* + 2\pi{}^*_*\rho(x)\,\hat{p}\,\rho(x){}^*_* - \frac{\pi^2}{6L^2}\rho(x)$$

D.3 The anyon W-current

We define

$$\mathcal{W}^\nu(z;a) := C^\nu(a)\ :\phi^\nu(ze^{i\frac{2\pi}{L}a})\phi^{-\nu}(z): \quad \text{(D.19)}$$

where $C^\nu(a)$ is a normalization constant which plays an important role for higher order corrections, as will be seen. Rewriting Eq. (D.19) in the bosonized form gives

$$\mathcal{W}^\nu(z;a) =$$
$$\mathcal{N}^\nu(a)\left(e^{-i\frac{2\pi}{L}\nu\nu_0 aQ}e^{-\nu\sum\limits_{n<0}\frac{1}{n}\rho_n z^n\left(e^{i\frac{2\pi}{L}na}-1\right)}e^{-\nu\sum\limits_{n>0}\frac{1}{n}\rho_n z^n\left(e^{i\frac{2\pi}{L}na}-1\right)} - I\right) \quad \text{(D.20)}$$

where

$$\mathcal{N}^\nu(a) := \frac{i}{2L\nu^2\cos^{\nu^2}\left(\frac{\pi a}{L}\right)\tan\left(\frac{\pi a}{L}\right)} \quad \text{(D.21)}$$

[1] Differentiation is a linear operation and therefore preserves normal-ordering

The operator valued generating function for the anyon currents can then be defined as

$$\mathcal{W}^{\nu}(a) := \sum_{s=1}^{\infty} \frac{(-ia)^{s-1}\,\nu^{s-2}}{(s-1)!} W^{\nu,s} = L\mathrm{Res}_z \frac{1}{z}\mathcal{W}^{\nu}(z;a) \qquad \text{(D.22)}$$

Lemma D.3.1. *The operator $\mathcal{W}^{\nu}(a)$ is well defined on a dense subspace of $\mathcal{F}$ and leaves it invariant. The highest weight condition*

$$\mathcal{W}^{\nu}(a)\Omega = 0 \quad \forall\nu \in \mathbb{R}\setminus\{0\}, \forall a \in \mathbb{R}$$

applies to the $W^{\nu,s}$ operators as well by extension of Eq. (D.22)*.*

Proof. See Ref. [19] □

Using these generating functions, it is possible to calculate the analog of the fermion W-algebra for the anyons by calculating the commutation relation between the generating function and an anyon operator (see Section D.3.2).

D.3.1 Construction of the $W^{\nu,s}$ operator

Define

$$\tilde{\rho}(z) := \rho(z) - \frac{1-\nu_0}{L}Q$$

with which we obtain (c.f. Eqs. (D.19) and (2.14))

$$\left(e^{-i\frac{2\pi}{L}\nu\nu_0 aQ} e^{-\nu\sum_{n<0}\frac{1}{n}\rho_n z^n\left(e^{i\frac{2\pi}{L}na}-1\right)} e^{-\nu\sum_{n>0}\frac{1}{n}\rho_n z^n\left(e^{i\frac{2\pi}{L}na}-1\right)} - I \right) = \exp\left(-2\pi i\nu\sum_{k=1}^{\infty}\frac{a^k}{k!}\left(\frac{2\pi i}{L}z\partial_z\right)^{k-1}\tilde{\rho}(z)\right)$$

Introducing the operator $\breve{\rho}(z)$ defined as

$$\breve{\rho}(z) := \frac{1}{L}\sum_{n\neq 0}\rho_n z^n \qquad \text{(D.23)}$$

which is constructed such that it commutes with the Klein factors. We also define the self-adjoint $\breve{W}^s$, $s \in \mathbb{N}$, operators as

$$\breve{W}^s := \frac{L^{s-1}}{s}\,\mathrm{Res}_z \frac{1}{z} {}^*_*\left(\breve{\rho}(z)\right)^s{}^*_* \qquad \text{(D.24)}$$

which also commute with the Klein factors. The $\breve{W}^s$ operators obey

$$\breve{W}^s R^{\omega}\Omega = 0 \quad , \forall s \in \mathbb{N},\ \forall\omega \in \mathbb{Z}$$

These operators will serve as a more convenient way of expressing the other operators.

The fermion density operator can be written as

$$\rho(z) = \breve{\rho}(z) + \frac{1}{L}Q$$

which gives that

$$\tilde{\rho}(z) = \breve{\rho}(z) + \frac{\nu_0}{L}Q$$

The $W^{\nu,s}$ operators, which are a formal power series of $\mathcal{W}^{\nu}(y;a)$, can then be expressed as

$$W^{\nu,1} = \nu_0 Q \tag{D.25}$$

$$W^{\nu,2} = \frac{\pi}{L}\left(2\breve{W}^2 + \nu_0^2 Q^2\right) \tag{D.26}$$

$$W^{\nu,3} = \frac{4\pi^2}{L^2}\breve{W}^3 + \frac{8\pi^2\nu_0}{L^2}Q\breve{W}^2 + \frac{4\pi^2\nu_0^3}{3L^2}Q^3 + \frac{\pi^2\nu_0\left(2-3\nu^2\right)}{3\nu^2L^2}Q \tag{D.27}$$

Remark D.3.2. In the $W^{\nu,3}$ power series, there is a term $-\frac{4\pi^2}{\nu L}\,{}^*_*\tilde{\rho}(z)\; z\partial_z\tilde{\rho}(z)^*_*$ and it is not obvious right away that this term vanishes. Upon integration and taking the limit, that particular term becomes $\sum\limits_{n\neq 0} n\,{}^*_*\rho_{-n}\rho_n{}^*_* = \sum\limits_{n<0} n\rho_n\rho_{-n} + \sum\limits_{n>0} n\rho_{-n}\rho_n = \sum\limits_{n>0} n\rho_{-n}\rho_n - n\rho_{-n}\rho_n = 0$ regardless of the vector it is applied to.

D.3.2 The $W^{\nu,s}$ relations

The commutation relation between the generating function (Eq. (D.22)) and an anyon operator of the same species is given by taking equations (D.20) and (2.16)

$$[\mathcal{W}^{\nu}(a), \phi^{\nu}(w)] = L\ \mathrm{Res}_z\left[\mathcal{W}^{\nu}(z;a), \phi^{\nu}(w)\right] \tag{D.28}$$

Boson normal-ordering each term of the commutation separately, using the techniques in Appendix B, gives that

$$\mathcal{W}^{\nu}(z;a)\phi^{\nu}(w) = {}^*_*\mathcal{W}^{\nu}(z;a)\phi^{\nu}(w)^*_* e^{-i\frac{\pi}{L}\nu^2 a} e^{-\nu^2\sum\limits_{n=1}^{\infty}\frac{1}{n}z^n w^{-n}\left(e^{i\frac{2\pi}{L}na}-1\right)}$$

and

$$\phi^{\nu}(w)\mathcal{W}^{\nu}(z;a) = {}^*_*\mathcal{W}^{\nu}(z;a)\phi^{\nu}(w)^*_* e^{i\frac{\pi}{L}\nu^2 a} e^{\nu^2\sum\limits_{n=-1}^{-\infty}\frac{1}{n}z^n w^{-n}\left(e^{i\frac{2\pi}{L}na}-1\right)}$$

The normalization constant of ${}^*_*\mathcal{W}(z;a)\phi^{\nu}(a)^*_*$ expanded as a formal power series in terms of a would give $\mathcal{N}^{\nu}(a) \propto \frac{1}{a} + \mathcal{O}(a)$. Since the relation of interest are up to the second power of a, it is only necessary to expand the exponent to the third power. This gives that

$$\sum_n \frac{1}{n}\left(\frac{z}{w}\right)^n \left(e^{i\frac{2\pi}{L}na}-1\right) \approx \frac{2\pi i a}{L}\sum_n \left(\frac{z}{w}\right)^n + \frac{1}{2}\left(\frac{2\pi i a}{L}\right)^2 \sum_n n\left(\frac{z}{w}\right)^n + \frac{1}{6}\left(\frac{2\pi i a}{L}\right)^3 \sum_n n^2\left(\frac{z}{w}\right)^n + \mathcal{O}(a^4)$$

Using

$$\sum_n n\left(\frac{z}{w}\right)^n = z\partial_z \sum_n \left(\frac{z}{w}\right)^n$$
$$\sum_n n^2\left(\frac{z}{w}\right)^n = \left(z\partial_z + z^2\partial_z^2\right)\sum_n \left(\frac{z}{w}\right)^n$$

and Eqs. (1.24), (1.25) gives

$$[\mathcal{W}^\nu(z;a),\phi^\nu(w)] = {}^*_*\Gamma(e^{i\nu F})^*_* \left(c_0 + a\cdot c_1 + a^2\cdot c_2 + \mathcal{O}(a^3)\right) \tag{D.29}$$

where

$$\begin{aligned}\Gamma(e^{i\nu F}) &:= {}^*_*\mathcal{W}^\nu(z;a)\phi^\nu(w)^*_*/\mathcal{N}^\nu(a)\\ &= e^{-i\frac{\pi}{L}\nu\nu_0(x+a)Q} R^{\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0(x+a)Q}\times\\ &\quad e^{-\nu\sum_{n<0}\frac{1}{n}\rho_n\left(z^n\left(e^{\frac{2\pi}{L}ina}-1\right)+w^n\right)} e^{-\nu\sum_{n>0}\frac{1}{n}\rho_n\left(z^n\left(e^{\frac{2\pi}{L}ina}-1\right)+w^n\right)}\end{aligned} \tag{D.30}$$

and

$$c_0 := \frac{z}{L}\delta(z-w) \tag{D.31}$$

$$c_1 := -\frac{i\pi z}{L^2}\left(\nu^2-1\right)\partial_z\left(z\delta(z-w)\right) \tag{D.32}$$

$$c_2 := -\frac{\pi^2\left(\nu^2-1\right)\left(\nu^2-2\right)}{6L^3}\, z\,\left(2\partial_z^2 z^2\delta(z-w) - 2\partial_z z\delta(z-w) + \delta(z-w)\right) \tag{D.33}$$

Expanding the operator part of Eq. (D.28) gives that

$$\frac{1}{\nu}\left[W^{\nu,1},\phi^\nu(w)\right] = L\mathrm{Res}_z \frac{1}{z} c_0\, {}^*_*\Gamma(e^{i\nu F})^*_*\Big|_{a=0} \tag{D.34}$$

$$i\left[W^{\nu,2},\phi^\nu(w)\right] = L\mathrm{Res}_z\frac{1}{z}\left(c_0\left(\partial_a {}^*_*\Gamma(e^{i\nu F})^*_*\right)\Big|_{a=0} + c_1\, {}^*_*\Gamma(e^{i\nu F})^*_*\Big|_{a=0}\right) \tag{D.35}$$

$$\begin{aligned}-\left[\nu W^{\nu,3},\phi^\nu(w)\right] = L\mathrm{Res}_z\frac{1}{z}\Big(& c_0\left(\partial_a^2 {}^*_*\Gamma(e^{i\nu F})^*_*\right)\Big|_{a=0}\\ &+2c_1\left(\partial_a {}^*_*\Gamma(e^{i\nu F})^*_*\right)\Big|_{a=0} + 2c_2\, {}^*_*\Gamma(e^{i\nu F})^*_*\Big|_{a=0}\Big)\end{aligned} \tag{D.36}$$

The short distance limit and derivatives of the operator part of Eq. (D.30) is given by

$$ {}^{*}_{*}\Gamma(e^{i\nu F})^{*}_{*}\Big|_{a=0} = \phi^{\nu}(w) $$

$$ \partial_{a}{}^{*}_{*}\Gamma(e^{i\nu F})^{*}_{*} = -i\nu 2\pi^{*}_{*}\left(\rho(ze^{\frac{2\pi}{L}ia}) + \frac{\nu_0 - 1}{L}Q\right)\Gamma(e^{i\nu F})^{*}_{*} $$

which gives that

$$ \left(\partial_{a}{}^{*}_{*}\Gamma(e^{i\nu F})^{*}_{*}\right)\Big|_{a=0} = -2\pi i\nu^{*}_{*}\left(\rho(z) + \frac{\nu_0 - 1}{L}Q\right)\phi^{\nu}(w)^{*}_{*} $$

and

$$ \partial^{2}_{a}{}^{*}_{*}\Gamma(e^{i\nu F})^{*}_{*} = -4\pi^2\nu^{2}{}^{*}_{*}\left(\rho(ze^{\frac{2\pi}{L}ia}) + \frac{\nu_0 - 1}{L}Q\right)^2\Gamma(e^{i\nu F})^{*}_{*} \\ - 2\pi i\nu^{*}_{*}\rho'(ze^{\frac{2\pi}{L}ia})\Gamma(e^{i\nu F})^{*}_{*} $$

where $\rho'(ze^{\frac{2\pi}{L}ia}) = \partial_a\rho(ze^{\frac{2\pi}{L}ia}) = \frac{2\pi}{L}iz\partial_z\rho(ze^{\frac{2\pi}{L}ia})$. Which gives

$$ \left(\partial^{2}_{a}{}^{*}_{*}\Gamma(e^{i\nu F})^{*}_{*}\right)\Big|_{a=0} = -4\pi^2\nu^{2}{}^{*}_{*}\left(\rho(z) + \frac{\nu_0 - 1}{L}Q\right)^2\phi^{\nu}(w)^{*}_{*} \\ + \frac{4\pi^2}{L}\nu^{*}_{*}\left(z\partial_z\rho(z)\right)\phi^{\nu}(w)^{*}_{*} $$

Lemma B.1.2 is used in order to obtain the results.

Using Proposition 1.4.4 and partial integration gives

$$ \frac{1}{\nu}\left[W^{\nu,1}, \phi^{\nu}(w)\right] = \phi^{\nu}(w) \tag{D.37} $$

$$ i\left[W^{\nu,2}, \phi^{\nu}(w)\right] = -2\pi i\nu^{*}_{*}\left(\rho(w) + \frac{\nu_0 - 1}{L}Q\right)\phi^{\nu}(w)^{*}_{*} = \frac{2\pi i}{L}w\partial_w\phi^{\nu}(w) \tag{D.38} $$

where the second part of the integration, in Eq. (D.35), cancels out when doing partial integration.

The sought after commutation relation is

$$ -\left[\nu W^{\nu,3}, \phi^{\nu}(w)\right] = \\ \left(-4\pi^2\nu^{2}{}^{*}_{*}\left(\rho(w) + \frac{\nu_0 - 1}{L}Q\right)^2\phi^{\nu}(w)^{*}_{*} + \frac{4\pi^2}{L}\nu^{*}_{*}\left(w\partial_w\rho(w)\right)\phi^{\nu}(w)^{*}_{*}\right) \\ + \frac{4\pi^2}{L}\nu\left(\nu^2 - 1\right)^{*}_{*}\left(w\partial_w\rho(w)\right)\phi^{\nu}(w)^{*}_{*} - \frac{\pi^2\left(\nu^2 - 1\right)\left(\nu^2 - 2\right)}{3L^2}\phi^{\nu}(w) $$

which can be re-written as

$$\left[\nu W^{\nu,3}, \phi^{\nu}(w)\right] = \frac{4\pi^2}{L^2}\left(w\partial_w\right)^2\phi^{\nu}(w) - \frac{4\pi^2}{L}\nu\left(\nu^2-1\right){}^{*}_{*}\left(w\partial_w\rho(w)\right)\phi^{\nu}(w){}^{*}_{*} + \frac{\pi^2\left(\nu^2-1\right)\left(\nu^2-2\right)}{3L^2}\phi^{\nu}(w) \quad \text{(D.39)}$$

Summarizing the results gives that

$$[W^{\nu,s}, \phi^{\nu}] = \nu^{2-s} i^{1-s}\frac{\partial^{s-1}}{\partial x^{s-1}}\phi^{\nu}(w) \quad , s = 1, 2$$

But for higher order terms there is a need for correction terms in order to obtain a similar result.

Note that these relation hold as long as the operators and the anyon field operator have the same statistical parameter ν.

Appendix E

Additional proofs

E.1 Proofs for Chapter 1

E.1.1 Proof of Proposition 1.4.4

Proof. 1. Let $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{n\pm q}$, $q \in \mathbb{Z}$ be any formal distribution. Then

$$\text{Res}_z f(z)\delta(z-w) = \sum_{n,m\in\mathbb{Z}} f_n w^m \text{Res}_z z^{n-m-1\pm q} = \sum_{n\in\mathbb{Z}} f_n w^{n\pm q} = f(w)$$

2. *2* follows from *1*.

3.

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n = \sum_{n\in\mathbb{Z}} z^{-n}w^{n-1} = \sum_{n\in\mathbb{Z}} z^n w^{-n-1} = \delta(w-z)$$

4. Using

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n = \sum_{n\in\mathbb{Z}} z^{-n}w^{n+1}$$

it is clear that Proposition 1.4.4 *4* holds.

□

E.1.2 Proof of Lemma 1.5.1

Proof. 1. Create an operator-valued function $\mathcal{T}(s) = e^{-sB}Ae^{sB}$. Then

$$\frac{d\mathcal{T}}{ds} = e^{-sB}(-B)\cdot Ae^{sB} + e^{-sB}Ae^{sB}B = e^{-sB}[A,B]\,e^{sB} = C$$

and all higher order derivatives vanish since $[B, C] = 0$. Taylor expanding $\mathcal{T}(s)$ gives

$$\mathcal{T}(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!}\left(\frac{d^n\mathcal{T}}{ds^n}\Big|_{s=0}\right) = \mathcal{T}(0) + s\,[A,B]$$

Taking the limits $s = 1$ yields Eq. (1.31). This relation implies

$$\left[A, e^B\right] = Ae^B - e^BA = e^B\left(e^{-B}Ae^B - A\right) = e^BC$$

2. The relation is given by the Taylor expansion of $f(A)$ and using (1.31)

3. Create a new operator-valued function $\mathcal{Y}(s) := e^{sA}e^{sB}$, where

$$\frac{d\mathcal{Y}}{ds} = e^{sA}Ae^{sB} + e^{sA}e^{sB}B = \mathcal{Y}(s)\,(A + B + sC)$$

which gives that

$$\mathcal{Y}(s) = e^{s(A+B)}e^{s^2C/2}$$

and taking the limit $s = 1$ yields Eq. (1.33). The final part is obtained by using Eq. (1.32) in the special case of $f(A) = e^A$.

□

E.2 Proofs for Chapter 2

E.2.1 Proof of Proposition 2.1.2

Proof. The density operator was the following commutation relation with the fermion field operators, which can be checked by using Eq. (1.21) and the CAR,

$$\left[\rho(w), \psi^{(*)}(z)\right] = \underset{(+)}{-}\frac{z}{L}\delta(z-w)\psi^{(*)}(z)$$

where $\rho(w) := \frac{1}{L}\sum_{n\in\mathbb{Z}} \rho_n w^n \in V\left[\left[w, w^{-1}\right]\right]$. A comparison of the commutation relation to equations (1.20) and (1.31), would suggest that

$$\psi^{(*)}(z) \propto \exp\left(\underset{(-)}{+}L\int^{z} dz'\ \frac{1}{z'}\rho(z')\right)$$

The commutation relation between the charge operator and the fermion field operators is given by

$$\left[Q, \psi^{(*)}(x)\right] = \underset{(+)}{-}\psi^{(*)}(x)$$

Since it is known that the charge operator does not commute with the Klein factors, the commutation relation would suggest that $\psi \propto R^{-1}$, $\psi^* \propto R$.

The fermion operators could then be written as

$$\begin{aligned}\psi(z) &\propto R^{-1}\exp\left(L\int^{z} dz'\ \frac{1}{z'}\rho(z')\right)\\ \psi^*(z) &\propto R\ \exp\left(-L\int^{z} dz'\ \frac{1}{z'}\rho(z')\right)\end{aligned}$$

□

E.2.2 Proof of Lemma 2.2.4

Proof. 1. Equation (2.17) is seen by comparing equations (1.18) and (2.16) with Lemma 1.5.2.

2. The fermion density operator can be split into two parts such as $\rho(w) = \breve{\rho}(w) + \frac{Q}{L}$ where $\breve{\rho}$, given by equation (D.23), is such that it commutes with the Klein factors. Using Eq. (1.31) gives that

$$[\breve{\rho}(w), \phi^\nu(z)] = \left(\left[P_-\breve{\rho}(w), -\nu\sum_{m>0}\frac{\rho_m}{m}z^m\right] + \left[P_+\breve{\rho}(w), -\nu\sum_{m<0}\frac{\rho_m}{m}z^m\right]\right)\phi^\nu(z)$$

which gives

$$[\breve{\rho}(w), \phi^\nu(z)] = \frac{\nu}{L}\left(z\delta(z-w) - 1\right)\phi^\nu(z)$$

□

E.2.3 Proof of Proposition 2.3.2

Proof. Boson normal-ordering the products yields[1]

$$\begin{aligned}\phi^\nu(w)\phi^{-\frac{1}{\nu}}(z) =&\ e^{-\frac{1}{2}\ln\left|\frac{w}{z}\right|}\sqrt{\frac{w}{z}}\ e^{\sum\limits_{n>0}\frac{1}{n}w^n z^{-n}}\ {}^*_*\phi^\nu(w)\phi^{-\frac{1}{\nu}}(z)^*_*\\ &= \mathrm{i}_{z,w}\ e^{-\frac{1}{2}\ln\left|\frac{w}{z}\right|}\sqrt{zw}\ \frac{1}{z-w}\,{}^*_*\phi^\nu(w)\phi^{-\frac{1}{\nu}}(z)^*_*\end{aligned}$$

[1] For more details see Appendix B

and

$$\phi^{-\frac{1}{\nu}}(z)\phi^{\nu}(w) = e^{\frac{1}{2}\ln\left|\frac{w}{z}\right|}\sqrt{\frac{z}{w}}e^{\sum_{n>0}\frac{1}{n}w^{-n}z^{n}} \; {}^{*}_{*}\phi^{-\frac{1}{\nu}}(z)\phi^{\nu}(w)^{*}_{*}$$
$$= -\mathrm{i}_{w,z}e^{\frac{1}{2}\ln\left|\frac{w}{z}\right|}\sqrt{zw}\frac{1}{z-w} \; {}^{*}_{*}\phi^{-\frac{1}{\nu}}(z)\phi^{\nu}(w)^{*}_{*}$$

where

$$\mathrm{i}_{z,w} \quad \ln\left|\frac{w}{z}\right| < 0$$
$$\mathrm{i}_{w,z} \quad \ln\left|\frac{w}{z}\right| > 0$$

and

$${}^{*}_{*}\phi^{\nu}(w)\phi^{-\frac{1}{\nu}}(z)^{*}_{*} = {}^{*}_{*}\phi^{-\frac{1}{\nu}}(z)\phi^{\nu}(w)^{*}_{*}$$

So the anti-commutation relation becomes

$$\left\{\phi^{\nu}(w), \phi^{-\frac{1}{\nu}}(z)\right\} = e^{\frac{1}{2}\left|\ln\left|\frac{w}{z}\right|\right|}\sqrt{zw}\left(\mathrm{i}_{z,w}\frac{1}{z-w} - \mathrm{i}_{w,z}\frac{1}{z-w}\right) \; {}^{*}_{*}\phi^{\nu}(w)\phi^{-\frac{1}{\nu}}(z)^{*}_{*}$$
$$= e^{\frac{1}{2}\left|\ln\left|\frac{w}{z}\right|\right|}\sqrt{zw}\;\delta(z-w)\; {}^{*}_{*}\phi^{\nu}(w)\phi^{-\frac{1}{\nu}}(z)^{*}_{*} = z\delta(z-w)\; {}^{*}_{*}\phi^{\nu}(w)\phi^{-\frac{1}{\nu}}(z)^{*}_{*} \quad \text{(E.1)}$$

where Proposition 1.4.4 *2* is used in order to obtain the final result. □

E.3 Proofs for Chapter 3

E.3.1 Proof of Lemma 3.2.4

Proof. Using Eqs. (2.16), (1.31) and that the $\mathcal{C}$ operator commutes with the charge operators and the Klein factors yields that

$$\mathcal{C}\phi^{\nu}(w) = \frac{4\pi^2}{L^2}\sum_{n>0} n\rho_{-n}\phi^{\nu}(w)\rho_n + \frac{4\pi^2}{L^2}\sum_{n>0} n\rho_{-n}\left[\rho_n, -\nu\sum_{m<0}\frac{1}{m}\rho_m w^m\right]\phi^{\nu}(w)$$
$$= {}^{*}_{*}\mathcal{C}\phi^{\nu}(w)^{*}_{*} + \frac{4\pi^2}{L^2}\nu\sum_{n>0} n\rho_{-n}w^{-n}\phi^{\nu}(w)$$

and

$$\phi^{\nu}(w)\mathcal{C} = \frac{4\pi^2}{L^2}\sum_{n>0} n\rho_{-n}\phi^{\nu}(w)\rho_n + \frac{4\pi^2}{L^2}\sum_{n>0} n\phi^{\nu}(w)\left[-\nu\sum_{m>0}\frac{1}{m}\rho_m w^m, \rho_{-n}\right]\rho_n$$
$$= {}^{*}_{*}\phi^{\nu}(w)\mathcal{C}^{*}_{*} - \frac{4\pi^2}{L^2}\nu\phi^{\nu}(w)\sum_{n>0} n\rho_n w^n$$

Combining these two gives that

$$\{\mathcal{C}, \phi^\nu(w)\} = 2 \,{}^*_*\mathcal{C}\phi^\nu(w)^*_* + \frac{4\pi^2}{L^2}\nu \sum_{n>0} n\rho_{-n}w^{-n}\phi^\nu(w) - n\phi^\nu(w)\rho_n w^n$$

Using

$$w\partial_w \rho(w) = \frac{1}{L}\sum_{n\in\mathbb{Z}} n\rho_n w^n = \frac{1}{L}\sum_{n>0}\left(n\rho_n w^n - n\rho_{-n}w^{-n}\right)$$

yields that

$$\{\mathcal{C}, \phi^\nu(w)\} = 2 \,{}^*_*\mathcal{C}\phi^\nu(w)^*_* - \frac{4\pi^2}{L}\nu \,{}^*_*(w\partial_w\rho(w))\,\phi^\nu(w)^*_*$$

□

E.3.2 Proof of Lemma 3.3.2

Proof. The commutation relation can be re-written as

$$\begin{aligned}\left[\nu W^{\nu,3}, \Phi^\nu(\mathbf{w})\right] = &\left[\nu W^{\nu,3}, \phi^\nu(w_1)\right]\phi^\nu(w_2)\cdots\phi^\nu(w_N)\\ &+ \phi^\nu(w_1)\left[\nu W^{\nu,3}, \phi^\nu(w_2)\right]\phi^\nu(w_3)\cdots\phi^\nu(w_N) + \ldots\\ &+ \phi^\nu(w_1)\cdots\phi^\nu(w_{N-1})\left[\nu W^{\nu,3}, \phi^\nu(w_N)\right]\end{aligned}$$

Using Eq. (D.39) gives

$$\begin{aligned}\left[\nu W^{\nu,3}, \Phi^\nu(\mathbf{w})\right] = &\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\Phi^\nu(\mathbf{w}) - \frac{4\pi^2}{L}\nu\left(\nu^2-1\right)(\ldots)_1\\ &+ \frac{\pi^2}{3L^2}\left(\nu^2-1\right)\left(\nu^2-2\right)N\Phi^\nu(\mathbf{w})\end{aligned}$$

where

$$\begin{aligned}(\ldots)_1 = \Big(&{}^*_*\left(w_1\partial_{w_1}\rho(w_1)\right)\phi^\nu(w_1)^*_*\phi^\nu(w_2)\cdots\phi^\nu(w_N)\\ &+\phi^\nu(w_1)^*_*\left(w_2\partial_{w_2}\rho(w_2)\right)\phi^\nu(w_2)^*_*\phi^\nu(w_3)\cdots\phi^\nu(w_N) + \ldots\\ &+\phi^\nu(w_1)\cdots\phi^\nu(w_{N-1})^*_*\left(w_N\partial_{w_N}\rho(w_N)\right)\phi^\nu(w_N)^*_*\Big)\end{aligned} \tag{E.2}$$

Eq. (E.2) expressed in terms of the oscillation operators is

$$\begin{aligned}(\ldots)_1 = \frac{1}{L}\sum_{n<0} n\rho_n w_1^n \Phi^\nu(\mathbf{w}) + \frac{1}{L}\phi^\nu(w_1)\sum_{n>0} n w_1^n \rho_n \phi^\nu(w_2)\cdots\phi^\nu(w_N) \\ + \frac{1}{L}\phi^\nu(w_1)\sum_{n<0} n\rho_n w_2^n \phi^\nu(w_2)\phi^\nu(w_3)\cdots\phi^\nu(w_N) \\ + \frac{1}{L}\phi^\nu(w_1)\phi^\nu(w_2)\sum_{n>0} n\rho_n w_2^n \phi^\nu(w_3)\cdots\phi^\nu(w_N) + \ldots \\ + \phi^\nu(w_1)\cdots\phi^\nu(w_{N-1})\sum_{n<0} n\rho_n w_N^n \phi^\nu(w_N) + \Phi^\nu(\mathbf{w})\sum_{n>0} n\rho_n w_N^n\end{aligned}$$

Using Eq. (1.31) gives

$$\begin{aligned}(\ldots)_1 = \sum_{k=1}^{N} (w_k \partial_{w_k}\rho(w_k)) * \Phi^\nu(\mathbf{w}) \\ + \left(\frac{1}{L}\sum_{k'=1}^{N-1}\sum_{k=k'+1}^{N}\sum_{n>0} n w_{k'}^n \left[\rho_n, -\nu\sum_{m<0}\frac{1}{m}\rho_m w_k^m\right]\right)\Phi^\nu(\mathbf{w}) \\ + \left(\frac{1}{L}\sum_{k'=2}^{N}\sum_{k=1}^{k'-1}\sum_{n<0} n w_{k'}^n \left[-\nu\sum_{m>0}\frac{1}{m}\rho_m w_k^m, \rho_n\right]\right)\Phi^\nu(\mathbf{w})\end{aligned}$$

which becomes

$$(\ldots)_1 = \sum_{k=1}^{N} (w_k \partial_{w_k}\rho(w_k)) * \Phi^\nu(\mathbf{w}) + \frac{2\nu}{L}\sum_{k'<k}\frac{w_k w_{k'}}{(w_k - w_{k'})^2}\Phi^\nu(\mathbf{w})$$

Collecting the results yields Eq. (3.34). □

E.3.3 Proof of Lemma 3.3.3

Proof.

Remark E.3.1. The definition of boson normal-ordering and the definition of the $\mathcal{C}$ operator, given in Eq. (3.16) implies that that the anti-commutation relation of the $\mathcal{C}$ operator and the many-body anyon field operator $\Phi^\nu(\mathbf{w})$ is equivalent to the anti-commutation relation of the $\mathcal{C}$ operator and the boson normal-ordered form of the many-body anyon field operator, given in Eq. (2.27), due to Lemma 2.3.4.

Using Eq. (1.31) gives that

$$\mathcal{C}{}^*_*\Phi^\nu(\mathbf{w})^*_* = \mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_* + \frac{4\pi^2}{L^2}\sum_{n>0} n\rho_{-n}\left[\rho_n, -\nu\sum_{k=1}^N\sum_{m<0}\frac{1}{m}\rho_m w_k^m\right]\cdot {}^*_*\Phi^\nu(\mathbf{w})^*_*$$
$$= \mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_* - \frac{4\pi^2}{L^2}\nu\sum_{k=1}^N\sum_{m<0} m\rho_m w_k^m {}^*_*\Phi^\nu(\mathbf{w})^*_* \quad \text{(E.3)}$$

where $\mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_* = {}^*_*\mathcal{C}\Phi^\nu(\mathbf{w})^*_*$ and

$${}^*_*\Phi^\nu(\mathbf{w})^*_*\mathcal{C} = \mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_* + {}^*_*\Phi^\nu(\mathbf{w})^*_*\cdot\frac{4\pi^2}{L^2}\sum_{n>0} n\left[-\nu\sum_{k=1}^N\sum_{m>0}\frac{1}{m}\rho_m w_k^m, \rho_{-n}\right]\rho_n$$
$$= \mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_* - {}^*_*\Phi^\nu(\mathbf{w})^*_*\cdot\frac{4\pi^2}{L^2}\nu\sum_{k=1}^N\sum_{m>0} m\rho_m w_k^m \quad \text{(E.4)}$$

Combining Eqs. (E.3) and (E.4) gives that

$$\{\mathcal{C}, {}^*_*\Phi^\nu(\mathbf{w})^*_*\} = 2\mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_*$$
$$-\frac{4\pi^2}{L^2}\nu\sum_{k=1}^N\sum_{m<0} m\rho_m w_k^m {}^*_*\Phi^\nu(\mathbf{w})^*_* - {}^*_*\Phi^\nu(\mathbf{w})^*_*\cdot\frac{4\pi^2}{L^2}\nu\sum_{k=1}^N\sum_{m>0} m\rho_m w_k^m$$
$$= 2\mathcal{C} * {}^*_*\Phi^\nu(\mathbf{w})^*_* - \frac{4\pi^2}{L^2}\nu\sum_{k=1}^N (w_k\partial_{w_k}\rho(w_k)) * {}^*_*\Phi^\nu(\mathbf{w})^*_*$$

□

E.3.4 Proof of Corollary 3.3.6

Proof. Using Lemma 2.3.4 the correlation function can be rewritten as

$$F_0(\mathbf{w}) := \langle\eta_{\text{CS}}, \Phi^\nu(\mathbf{w})\Omega\rangle = \mathcal{J}^\nu(\mathbf{w})\,\langle\eta_{\text{CS}}, {}^*_*\Phi^\nu(\mathbf{w})^*_*\Omega\rangle$$

where

$$\langle\eta_{\text{CS}}, {}^*_*\Phi^\nu(\mathbf{w})^*_*\Omega\rangle =$$
$$\left\langle R^{-N\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0\sum_{k=1}^N x_k Q} R^{N\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0\sum_{k=1}^N x_k Q} e^{-\nu\sum_{k=1}^N\sum_{n<0}\frac{1}{n}\rho_n w_k^n} e^{-\nu\sum_{k=1}^N\sum_{n>0}\frac{1}{n}\rho_n w_k^n}\right\rangle$$
$$= \left\langle R^{-N\frac{\nu}{\nu_0}} e^{-i\frac{\pi}{L}\nu\nu_0\sum_{k=1}^N x_k Q} R^{N\frac{\nu}{\nu_0}}\right\rangle = e^{-i\frac{\pi}{L}\nu^2 N\sum_{k=1}^N x_k}$$

This result is obtained since the oscillation operator ρ_n commutes with the charge operator and the Klein factors for $n \neq 0$ and by using Eq. (1.13).

Comparing $F_0(\mathbf{w})$ and the GS wavefunction of the CS model (Eq. (3.1)) shows that the correlation function $F_0(\mathbf{w})$ can be written as

$$F_0(\mathbf{w}) = c\psi_0(\mathbf{w})g(\mathbf{w})$$

where $c \in \mathbb{C}$ is a constant and

$$g(\mathbf{w}) = e^{-i\frac{\pi}{L}\nu^2 N \sum\limits_{k=1}^{N} x_k} = \prod_{k=1}^{N} \left(\frac{w_k}{|w_k|}\right)^{-\frac{N\nu^2}{2}}$$

So the CS Hamiltonian (c.f. Eq. (3.2)) acting on the correlation function $F_0(\mathbf{w})$ gives

$$\begin{aligned} H_N F_0(\mathbf{w}) = c\left(H_N\psi_0(\mathbf{w})\right) g(\mathbf{w}) \\ + \frac{8\pi^2}{L^2} c \sum_{k'=1}^{N} \left(w_{k'}\partial_{w_{k'}}\psi_0(\mathbf{w})\right)\left(w_{k'}\partial_{w_{k'}} g(\mathbf{w})\right) \\ + \frac{4\pi^2}{L^2} c\psi_0(\mathbf{w}) \sum_{k'=1}^{N} \left(w_{k'}\partial_{w_{k'}}\right)^2 g(\mathbf{w}) \end{aligned} \quad \text{(E.5)}$$

where

$$H_N\psi_0(\mathbf{w}) = E_0\psi_0(\mathbf{w})$$

$$\sum_{k'=1}^{N} \left(w_{k'}\partial_{w_{k'}}\psi_0(\mathbf{w})\right)\left(w_{k'}\partial_{w_{k'}} \prod_{k=1}^{N}\left(\frac{w_k}{|w_k|}\right)^{-\frac{N\nu^2}{2}}\right) = 0$$

and

$$\sum_{k'=1}^{N} \left(w_{k'}\partial_{w_{k'}}\right)^2 g(\mathbf{w}) = g(\mathbf{w})\frac{\nu^4 N^2}{4}\sum_{k'=1}^{N} 1 = g(\mathbf{w})\frac{N^3\nu^4}{4}$$

which gives that

$$H_N F_0(\mathbf{w}) = E_0 F_0(\mathbf{w}) + \frac{4\pi^2}{L^2}\frac{N^3\nu^4}{4} F_0(\mathbf{w}) = \mathcal{E}_{\mathrm{CS}} F_0(\mathbf{w})$$

□

E.4 Proofs for Chapter 4

E.4.1 Proof of Theorem 4.1.2

Proof. The commutation can be re-written as

$$\begin{aligned} \left[\mathcal{H}^{\nu,3}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\right] &= \left[\mathcal{H}^{\nu,3}, \Phi^{-\frac{1}{\nu}}(\mathbf{z})\Phi^{\nu}(\mathbf{w})\right] \\ &= \left[\mathcal{H}^{\nu,3}, \Phi^{-\frac{1}{\nu}}(\mathbf{z})\right]\Phi^{\nu}(\mathbf{w}) + \Phi^{-\frac{1}{\nu}}(\mathbf{z})\left[\mathcal{H}^{\nu,3}, \Phi^{\nu}(\mathbf{w})\right] \end{aligned} \quad \text{(E.6)}$$

Proposition 3.3.4 gives that

$$\begin{aligned}\Phi^{-\frac{1}{\nu}}(\mathbf{z})\left[\mathcal{H}^{\nu,3},\Phi^{\nu}(\mathbf{w})\right] = \Phi^{-\frac{1}{\nu}}(\mathbf{z})\Big(&\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\Phi^{\nu}(\mathbf{w})\\ &-\frac{4\pi^2}{L^2}2\nu^2\left(\nu^2-1\right)\sum_{k'<k}\frac{w_k w_{k'}}{\left(w_k-w_{k'}\right)^2}\ \Phi^{\nu}(\mathbf{w})\\ &-2\left(\nu^2-1\right)\left(\mathcal{C}*\Phi^{\nu}(\mathbf{w})-\Phi^{\nu}(\mathbf{w})\mathcal{C}\right)\Big)\end{aligned}$$

which yields

$$\begin{aligned}\Phi^{-\frac{1}{\nu}}(\mathbf{z})\left[\mathcal{H}^{\nu,3},\Phi^{\nu}(\mathbf{w})\right] = &\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\varphi^{\nu}(\mathbf{z},\mathbf{w})\\ &-\frac{4\pi^2}{L^2}\sum_{k'<k}\frac{w_k w_{k'}}{\left(w_k-w_{k'}\right)^2}\ \varphi^{\nu}(\mathbf{z},\mathbf{w})\\ &-2\left(\nu^2-1\right)\left(\Phi^{-\frac{1}{\nu}}(\mathbf{z})\left(\mathcal{C}*\Phi^{\nu}(\mathbf{w})\right)-\varphi^{\nu}(\mathbf{z},\mathbf{w})\mathcal{C}\right)\end{aligned}$$

By using Lemma 3.2.6, the first commutation in Eq. (E.6) becomes

$$\begin{aligned}\left[\mathcal{H}^{\nu,3},\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right]\Phi^{\nu}(\mathbf{w}) = &-\nu^2\left[\mathcal{H}^{-\frac{1}{\nu},3},\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right]\Phi^{\nu}(\mathbf{w})\\ &+\frac{4\pi^2}{L^2}\frac{\left(\nu^4-1\right)}{\nu^2}\left[\mathcal{H}^{-\frac{1}{\nu},1},\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right]\Phi^{\nu}(\mathbf{w})\end{aligned}$$

where

$$\left[\mathcal{H}^{-\frac{1}{\nu},1},\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right] = M\Phi^{-\frac{1}{\nu}}(\mathbf{z})$$

and

$$\begin{aligned}\left[\mathcal{H}^{-\frac{1}{\nu},3},\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right] = &\frac{4\pi^2}{L^2}\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2\Phi^{-\frac{1}{\nu}}(\mathbf{z})\\ &-\frac{4\pi^2}{L^2}\frac{2}{\nu^2}\left(\frac{1}{\nu^2}-1\right)\sum_{j'<j}\frac{z_j z_{j'}}{\left(z_j-z_{j'}\right)^2}\Phi^{-\frac{1}{\nu}}(\mathbf{z})\\ &-2\left(\frac{1}{\nu^2}-1\right)\left(\mathcal{C}*\Phi^{-\frac{1}{\nu}}(\mathbf{z})-\Phi^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{C}\right)\end{aligned}$$

by using Proposition 3.3.4. This gives that

$$
\begin{aligned}
-\nu^2\left[\mathcal{H}^{-\frac{1}{\nu},3},\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right]\Phi^{\nu}(\mathbf{w}) &= -\frac{4\pi^2}{L^2}\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2\varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad -\frac{4\pi^2}{L^2}\frac{2}{\nu^2}\left(\nu^2-1\right)\sum_{j'<j}\frac{z_jz_{j'}}{\left(z_j-z_{j'}\right)^2}\varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad -2\left(\nu^2-1\right)\left(\left(\mathcal{C}*\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right)\Phi^{\nu}(\mathbf{w})-\Phi^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{C}\Phi^{\nu}(\mathbf{w})\right)
\end{aligned}
$$

Combing the results, so far, yields

$$
\begin{aligned}
\left[\mathcal{H}^{\nu,3},\varphi^{\nu}(\mathbf{z},\mathbf{w})\right] &= \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2\varphi^{\nu}(\mathbf{z},\mathbf{w})-\frac{4\pi^2}{L^2}\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2\varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad -\frac{4\pi^2}{L^2}\left(\nu^2-1\right)\left(2\nu^2\sum_{k'<k}\frac{w_kw_{k'}}{\left(w_k-w_{k'}\right)^2}+\frac{2}{\nu^2}\sum_{j'<j}\frac{z_jz_{j'}}{\left(z_j-z_{j'}\right)^2}\right)\varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad +\frac{4\pi^2}{L^2}M\frac{\left(\nu^4-1\right)}{\nu^2}\varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad -2\left(\nu^2-1\right)\left(\left(\mathcal{C}*\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right)\Phi^{\nu}(\mathbf{w})+\Phi^{-\frac{1}{\nu}}(\mathbf{z})\left(\mathcal{C}*\Phi^{\nu}(\mathbf{w})\right)\right. \\
&\qquad \left.-\Phi^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{C}\Phi^{\nu}(\mathbf{w})-\varphi^{\nu}(\mathbf{z},\mathbf{w})\mathcal{C}\right)
\end{aligned}
\tag{E.7}
$$

We use Remark E.3.1 when evaluating the final part of Eq. (E.7).

$$
\begin{aligned}
\left(\mathcal{C}*{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\right){}^*_*\Phi^{\nu}(\mathbf{w})^*_* &= \frac{4\pi^2}{L^2}\sum_{n>0}n\rho_{-n}{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\rho_n{}^*_*\Phi^{\nu}(\mathbf{w})^*_* \\
= \frac{4\pi^2}{L^2}\sum_{n>0}n\rho_{-n}{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_*\rho_n &+ \frac{4\pi^2}{L^2}\sum_{n>0}n\rho_{-n}{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\left[\rho_n,{}^*_*\Phi^{\nu}(\mathbf{w})^*_*\right] \\
&= \mathcal{C}*\left({}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_*\right) \\
&\quad +\frac{4\pi^2}{L^2}\sum_{n>0}n\rho_{-n}{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\left[\rho_n,-\nu\sum_{k=1}^{N}\sum_{m<0}\frac{1}{m}\rho_mw_k^m\right]{}^*_*\Phi^{\nu}(\mathbf{w})^*_* \\
&= \mathcal{C}*\left({}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_*\right)+\frac{4\pi^2}{L^2}\nu\sum_{k=1}^{N}n\rho_{-n}w_k^{-n}{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_*
\end{aligned}
$$

Un-normal-ordering the anyon many-body operators gives that

$$
\left(\mathcal{C}*\Phi^{-\frac{1}{\nu}}(\mathbf{z})\right)\Phi^{\nu}(\mathbf{w}) = \mathcal{C}*\varphi^{\nu}(\mathbf{z},\mathbf{w})-\frac{4\pi^2}{L^2}\nu\sum_{k=1}^{N}\sum_{n<0}n\rho_nw_k^n\varphi^{\nu}(\mathbf{z},\mathbf{w}) \tag{E.8}
$$

Using the exact same method gives that

$$\Phi^{-\frac{1}{\nu}}(\mathbf{z})\left(\mathcal{C} * \Phi^{\nu}(\mathbf{w})\right) = \mathcal{C} * \varphi^{\nu}(\mathbf{z},\mathbf{w}) + \varphi^{\nu}(\mathbf{z},\mathbf{w}) \cdot \frac{4\pi^2}{L^2}\frac{1}{\nu} \sum_{j=1}^{M}\sum_{n>0} n\rho_n z_j^n \tag{E.9}$$

and finally

$$\begin{aligned}
{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\mathcal{C}^*_*\Phi^{\nu}(\mathbf{w})^*_* &= \frac{4\pi^2}{L^2}\sum_{n>0} n^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\rho_{-n}\rho_n{}^*_*\Phi^{\nu}(\mathbf{w})^*_* \\
&= \frac{4\pi^2}{L^2}\sum_{n>0} n^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\rho_{-n}\left({}^*_*\Phi^{\nu}(\mathbf{w})^*_*\rho_n + [\rho_n, {}^*_*\Phi^{\nu}(\mathbf{w})^*_*]\right) \\
&= {}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\left(\mathcal{C} * {}^*_*\Phi^{\nu}(\mathbf{w})^*_*\right) + \frac{4\pi^2}{L^2}\nu\sum_{k=1}^{N}\sum_{n>0} n w_k^{-n}\ {}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\rho_{-n}{}^*_*\Phi^{\nu}(\mathbf{w})^*_*
\end{aligned}$$

Using Eq. (E.9) gives that

$$\begin{aligned}
{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*\mathcal{C}^*_*\Phi^{\nu}(\mathbf{w})^*_* &= \mathcal{C} * \left({}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_*\right) \\
&\quad + {}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_* \cdot \frac{4\pi^2}{L^2}\frac{1}{\nu} \sum_{j=1}^{M}\sum_{n>0} n\rho_n z_j^n \\
&\quad + \frac{4\pi^2}{L^2}\nu\sum_{k=1}^{N}\sum_{n>0} n\rho_{-n} w_k^{-n}\ {}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^{**}_{**}\Phi^{\nu}(\mathbf{w})^*_* \\
&\quad + \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\sum_{n>0} n w_k^{-n}\left[{}^*_*\Phi^{-\frac{1}{\nu}}(\mathbf{z})^*_*, \rho_{-n}\right]{}^*_*\Phi^{\nu}(\mathbf{w})^*_*
\end{aligned}$$

Which yields

$$\begin{aligned}
\Phi^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{C}\Phi^{\nu}(\mathbf{w}) &= \mathcal{C} * \varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad + \varphi^{\nu}(\mathbf{z},\mathbf{w}) \cdot \frac{4\pi^2}{L^2}\frac{1}{\nu}\sum_{j=1}^{M}\sum_{n>0} n\rho_n z_j^n - \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\sum_{n<0} n\rho_n w_k^n\ \varphi^{\nu}(\mathbf{z},\mathbf{w}) \\
&\quad + \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\sum_{j=1}^{M}\sum_{n>0} n z_j^n w_k^{-n}\ \varphi^{\nu}(\mathbf{z},\mathbf{w})
\end{aligned}$$

Collecting the results yields

$$\begin{aligned}
&\left(\mathcal{C} * \Phi^{-\frac{1}{\nu}}(\mathbf{z})\right)\Phi^{\nu}(\mathbf{w}) + \Phi^{-\frac{1}{\nu}}(\mathbf{z})\left(\mathcal{C} * \Phi^{\nu}(\mathbf{w})\right) - \Phi^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{C}\Phi^{\nu}(\mathbf{w}) = \\
&\qquad \mathcal{C} * \varphi^{\nu}(\mathbf{z},\mathbf{w}) - \frac{4\pi^2}{L^2}\sum_{k=1}^{N}\sum_{j=1}^{M}\sum_{n>0} n z_j^n w_k^{-n}\ \varphi^{\nu}(\mathbf{z},\mathbf{w})
\end{aligned}$$

where

$$\sum_{n>0} n z_j^n w_k^{-n} = \frac{w_k z_j}{(w_k - z_j)^2}$$

□

E.4.2 Proof of Theorem 4.2.1

Consider the Fock space inner product

$$\left\langle \tilde{\eta}, \mathcal{H}^{\nu,3} \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega \right\rangle = \left\langle \mathcal{H}^{\nu,3}\tilde{\eta}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega \right\rangle \tag{E.10}$$

Using Eq. (3.21) gives that the l.h.s. of Eq. (E.10) can be written as

$$\left\langle \tilde{\eta}, \mathcal{H}^{\nu,3} \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega \right\rangle = \left\langle \tilde{\eta}, \left[\mathcal{H}^{\nu,3}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\right]\Omega \right\rangle$$

Theorem 4.1.2 gives that the l.h.s. of Eq. (E.10) equals the dCS differential operator action on the correlation function $\tilde{F}_{\tilde{\eta}}(\mathbf{z},\mathbf{w})$ with an additional term (c.f. Eq. (4.3)), *i.e.*

$$\left\langle \tilde{\eta}, \mathcal{H}^{\nu,3} \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega \right\rangle = \left(\tilde{H}_{N,M} + \frac{\pi^2}{3L^2} \frac{\nu^4 - 1}{\nu^2} M \right) \langle \tilde{\eta}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega \rangle$$

The r.h.s. of Eq. (E.10) equals $\tilde{\mathcal{E}}\tilde{F}_{\tilde{\eta}}(\mathbf{z},\mathbf{w})$ since $\tilde{\eta}$ is an eigenvector of the $\mathcal{H}^{\nu,3}$ operator with eigenvalue $\tilde{\mathcal{E}}$.

E.4.3 Proof of Corollary 4.2.2

Proof. Lemma 2.3.6 and Eq. (3.33) gives that the inner product can be written as

$$\langle \eta_{\mathrm{dCS}}, \varphi^{\nu}(\mathbf{z},\mathbf{w})\Omega \rangle = \left\langle R^{-\left(N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}\right)} \varphi^{\nu}(\mathbf{z},\mathbf{w}) \right\rangle =$$

$$\mathcal{J}^{-\frac{1}{\nu}}(\mathbf{z})\mathcal{J}^{\nu}(\mathbf{w}) \prod_{k=1}^{N}\prod_{j=1}^{M} \left|\frac{w_k}{z_j}\right|^{\frac{1}{2}} \frac{\sqrt{w_k z_j}}{w_k - z_j} \left\langle R^{-\left(N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}\right)} {}^*_*\varphi^{\nu}(\mathbf{z},\mathbf{w})^*_* \right\rangle$$

where

$$\left\langle R^{-\left(N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}\right)} {}^*_*\varphi^{\nu}(\mathbf{z},\mathbf{w})^*_* \right\rangle =$$

$$\left\langle R^{-\left(N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}\right)} e^{-i\frac{\pi}{L}\nu_0\left(\nu\sum_{k=1}^{N} x_k - \frac{1}{\nu}\sum_{j=1}^{M} y_j\right)Q} R^{N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}} \right\rangle =$$

$$e^{-i\frac{\pi}{L}\nu_0\left(N\frac{\nu}{\nu_0} - M\frac{1}{\nu\nu_0}\right)\left(\nu\sum_{k=1}^{N} x_k - \frac{1}{\nu}\sum_{j=1}^{M} y_j\right)}$$

$$= e^{-i\frac{\pi}{L}\nu^{-2}\left(N\nu^2 - M\right)\left(\nu^2\sum_{k=1}^{N} x_k - \sum_{j=1}^{M} y_j\right)}$$

The correlation function $\tilde{F}_0(\mathbf{z},\mathbf{w})$ can be rewritten as

$$\tilde{F}_0(\mathbf{z},\mathbf{w}) = c\tilde{\psi}_0(\mathbf{z},\mathbf{w})g(\mathbf{z},\mathbf{w})$$

where c is a constant, $\tilde{\psi}_0(\mathbf{z},\mathbf{w})$ is the dGS eigenfunction, and

$$\begin{aligned} g(\mathbf{z},\mathbf{w}) &= e^{-i\frac{\pi}{L}\left(\frac{N\nu^2-M}{\nu^2}\right)\left(\nu^2\sum_{k=1}^{N}x_k-\sum_{j=1}^{M}y_j\right)} \\ &= \prod_{k=1}^{N}\prod_{j=1}^{M}\left(\frac{w_k}{|w_k|}\right)^{-\frac{N\nu^2-M}{2}}\left(\frac{z_j}{|z_j|}\right)^{\frac{N\nu^2-M}{2\nu^2}} \end{aligned}$$

The dCS differential operator acting on the correlation function is

$$\begin{aligned} \tilde{H}_{N,M}\tilde{F}_0(\mathbf{z},\mathbf{w}) = c\left(\tilde{H}_{N,M}\tilde{\psi}_0(\mathbf{z},\mathbf{w})\right)g(\mathbf{z},\mathbf{w}) \\ + \frac{8\pi^2}{L^2}c\sum_{k=1}^{N}\left(w_k\partial_{w_k}\tilde{\psi}_0(\mathbf{z},\mathbf{w})\right)\left(w_k\partial_{w_k}g(\mathbf{z},\mathbf{w})\right) \\ + c\tilde{\psi}_0(\mathbf{z},\mathbf{w})\frac{4\pi^2}{L^2}\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2 g(\mathbf{z},\mathbf{w}) - \frac{8\pi^2}{L^2}\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\tilde{\psi}_0(\mathbf{z},\mathbf{w})\right)\left(z_j\partial_{z_j}g(\mathbf{z},\mathbf{w})\right) \\ - c\tilde{\psi}_0(\mathbf{z},\mathbf{w})\frac{4\pi^2}{L^2}\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2 g(\mathbf{z},\mathbf{w}) \end{aligned}$$

where

$$\tilde{H}_{N,M}\tilde{\psi}_0(\mathbf{z},\mathbf{w}) = \tilde{E}_0\tilde{\psi}_0(\mathbf{z},\mathbf{w})$$

$$\begin{aligned} \sum_{k=1}^{N}\left(w_k\partial_{w_k}\tilde{\psi}_0(\mathbf{z},\mathbf{w})\right)\left(w_k\partial_{w_k}g(\mathbf{z},\mathbf{w})\right) \\ - \nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\tilde{\psi}_0(\mathbf{z},\mathbf{w})\right)\left(z_j\partial_{z_j}g(\mathbf{z},\mathbf{w})\right) = 0 \end{aligned}$$

and

$$\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2 g(\mathbf{z},\mathbf{w}) = g(\mathbf{z},\mathbf{w})\sum_{k=1}^{N}\frac{\left(N\nu^2-M\right)^2}{4} = g(\mathbf{z},\mathbf{w}) = \frac{N\left(N\nu^2-M\right)^2}{4}$$

$$\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2 g(\mathbf{z},\mathbf{w}) = g(\mathbf{z},\mathbf{w})\sum_{j=1}^{M}\left(\frac{N\nu^2-M}{2\nu^2}\right)^2 = g(\mathbf{z},\mathbf{w})\frac{M\left(N\nu^2-M\right)^2}{4\nu^4}$$

which gives

$$\sum_{k=1}^{N}\left(w_k\partial_{w_k}\right)^2 g(\mathbf{z},\mathbf{w})-\nu^2\sum_{j=1}^{M}\left(z_j\partial_{z_j}\right)^2 g(\mathbf{z},\mathbf{w})=\frac{\left(N\nu^2-M\right)^3}{4\nu^2}$$

□

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