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by

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Analytical solutions are presented for the input mobilities of an isotropic infinite plate with no fluid load. An infinitely long, massless beam is assumed to excite one of the faces of the plate at a fairly low frequency. The plate is described by means of the general equations of motion for three-dimensional elastic bodies, which leads to solutions for the plate that beside bending, longitudinal, and shear waves also incorporate low-frequency Lamb and Love modes. It is shown that the expression for the mobility due to a shear wave excited by an in-line force is similar to that of the mobility due to the longitudinal wave excited by a horizontal force; the only difference is that the shear wave speed is inserted instead of the longitudinal wave speed. It is also shown that, in the case of excitation with a vertical or a horizontal force, the mobilities with respect to the longitudinal waves are much smaller than those connected with the bending waves. An interesting conclusion is that the influence of the Lamb and Love modes is, in general, less important than in the case of point excitation.

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INTRODUCTION

The problem under consideration is illustrated in Fig. 1. An infinite elastic plate is excited on its upper face by, in turn, each of the three line forces shown in the figure. These force distributions are characterized as "vertical," "horizontal," and "in-line." There is also a corresponding fourth mode of excitation, the rocking moment shown in Fig. 1(d), but this case has been treated before (Ljunggren, 1987a). The excitation is harmonic and the frequency is comparatively low. The forces are applied to the plate by means of a massless beam, the width of which is small compared with the wavelength of the bending wave of the plate; this condition also implies that the width of the beam is small compared with the wavelengths of the other propagating plate waves.

The motion of the beam is expressed in the form of a mobility. The mobilities used in this paper are defined as the complex ratio of a velocity (or rotational velocity) and a related force (or moment) per unit length.

Line mobilities are of general interest for the study of interaction between a mechanical (vibrational) source and a plate in such cases where the contact area can be regarded as one or several straight lines. Line mobilities can also be useful for estimating the sound propagation at junctions between different structural members. Moreover, it is thought that information on line mobilities can be useful for the study of low-frequency vibrations of structures, which consist of plates connected to each other at right angles. This is especially the case for the part of the mobility that is due to the "local reaction" (a reactive field built up of low-frequency Lamb and/or Love modes), as the expressions for infinitely long lines can often be used as approximations for finite structures.

Parts of the problem have been treated before. Results obtained from the classic thin-plate solutions are presented in the well-known book *Structure-Borne Sound*, by Cremer

FIG. 1. The four modes of line excitation: (a) vertical force, (b) horizontal force, (c) in-line force, and (d) rocking moment.

et al. (1973). Thus the mobility with respect to the bending waves excited by a vertical force is given by

$$M_B^V = (1 - j) / (4\rho\delta c_B). \quad (1)$$

Here, ρ is the density, δ is the thickness of the plate, and c_B is the phase velocity of the propagating bending wave. The expression is valid for a time dependence $\exp(j\omega t)$. The mobility with respect to a horizontal force and the ensuing longitudinal plate wave can be obtained from the material presented in the same book as

$$M_L^H = 1 / (2\rho\delta c_L), \quad (2)$$

where c_L is the propagation speed of the (quasi-)longitudinal wave.

A three-dimensional approach was used by Heckl (1981) for the case of excitation with a vertical line force. Heckl used a Fourier transform technique (similar to that used in the present paper) and evaluated the mobility by means of a numerical integration. Some of Heckl's results will be compared with those obtained in the present paper, see Sec. IV C.

In the present work, the motion of the plate is obtained from the general differential equations of motion for three-dimensional elastic bodies. The boundary conditions are established from prescribed stresses on the surfaces of the plate. A solution for the region of the plate outside the beam is readily obtained in the form of integrals in a complex wavenumber plane. The integrals are made unambiguous by the introduction of a loss factor. The mobilities are finally calculated from the displacements of the plate at the edge of the beam.

In order to check the solutions, comparisons are made with some approximate expressions. These are derived under the assumption that the line sources can be regarded as superpositions of point sources. The expressions for the line mobilities are, in this way, obtained as normalized integrals of the fields due to the point sources.

I. A TWO-DIMENSIONAL PLATE SOLUTION

A coordinate system is introduced with the plate in the xy plane and with the boundaries of the plate as $z = 0$ and $z = \delta$. The direction of the z axis is defined as vertical and, hence, the plate is in a horizontal position.

The plate is assumed to be of uniform thickness and infinite extent. The material of the plate is homogeneous, isotropic, and linearly elastic. It is described by Lamé's elastic constants λ and μ and by the density ρ . In some cases, Young's modulus E and Poisson's ratio ν are also used in order to obtain more compact expressions and to show the connections to the usual thin-plate expressions. The relationship between the two sets of constants is well known and can, for the present case, be written as

$$E = 2\mu(3\lambda + 2\mu) / (\lambda + \mu), \quad \nu = \lambda / (2\lambda + 2\mu). \quad (3)$$

Solutions are derived for the motion of the plate when the excitation has a space and time dependence of the form $\exp(-jk_x x + j\omega t)$, where k_x is the wavenumber and ω the radian frequency of the excitation, and t is the time. The excitation will, in turn, have the form of a horizontal force distribution (in the x direction), of a transversal force distribution (in the y direction), and of a vertical force distribution (in the z direction).

The general equations of motion can according to Southwell (1941) be written as [the time factor $\exp(j\omega t)$ is omitted here and in the remainder of the paper]

$$\begin{aligned} (\lambda + 2\mu) \left(\frac{\partial \Delta}{\partial x} \right) + 2\mu \left(\frac{\partial \omega_y}{\partial z} - \frac{\partial \omega_z}{\partial y} \right) &= -\rho\omega^2 u, \\ (\lambda + 2\mu) \left(\frac{\partial \Delta}{\partial y} \right) + 2\mu \left(\frac{\partial \omega_z}{\partial x} - \frac{\partial \omega_x}{\partial z} \right) &= -\rho\omega^2 v, \quad (4) \\ (\lambda + 2\mu) \left(\frac{\partial \Delta}{\partial z} \right) + 2\mu \left(\frac{\partial \omega_x}{\partial y} - \frac{\partial \omega_y}{\partial x} \right) &= -\rho\omega^2 w, \end{aligned}$$

where u , v , and w are the displacements in the x , y , and z directions, respectively. The dilation Δ and the rotations ω_x , ω_y , and ω_z are related to the displacements by the following expressions:

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \quad (5)$$

$$2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad 2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Three components of the stress tensor are needed: σ_z , τ_{zx} , and τ_{zy} . These can be obtained from the dilation and the displacements by the following general relations (see Southwell, 1941):

$$\begin{aligned} \sigma_z &= \lambda \Delta + 2\mu \left(\frac{\partial w}{\partial z} \right), \\ \tau_{zx} &= \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \\ \tau_{zy} &= \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right). \end{aligned} \quad (6)$$

The directions of the stresses are shown in Fig. 2.

A. Horizontal excitation

If the plate is assumed to be excited on one side only, the solution for the displacement can be split up in two parts, one of which is antisymmetric with respect to the plane $z = \delta/2$ and the other symmetric (Ljunggren, 1983). If, on the other hand, the excitation is antisymmetric or symmetric in the same sense, the solution will consist of an antisymmetric or symmetric part only. The more usual case of excitation on

FIG. 2. Directions of stresses.

one side can then be regarded as a superposition of the anti-symmetric and symmetric cases. The restriction to antisymmetric or symmetric excitation facilitates the mathematical manipulations in a decisive way in this case, and will, consequently, be used here.

The force distribution of the excitation is assumed to contain no vertical or transversal components. Apart from this excitation, no account is taken of the influence of the surrounding medium. Thus the boundary conditions can be taken as

$$\sigma_z = \tau_{zy} = 0 \quad (z = 0, z = \delta), \quad (7)$$

$$\tau_{zx} = -p_x \quad (z = 0), \quad (8)$$

$$\tau_{zx} = \pm p_x \quad (z = \delta), \quad (9)$$

where $p_x = \hat{p}_x \exp(-jk_x x)$ is the external force distribution. The plus sign in Eq. (9) is valid for symmetric excitation and the minus sign is for the opposite case.

A solution for the displacements can be obtained in the following way. To start with, all the field variables are assumed to have the same x and y dependence as the external force distribution, viz, $\exp(-jk_x x)$. If the expressions for the dilatation and the rotations, Eqs. (5), are now inserted in the general equations of motion, Eqs. (4), the result becomes

$$\begin{aligned} \Delta &= (\rho\omega^2)^{-1} \left[(\lambda + 2\mu) \left(\frac{\partial^2 \Delta}{\partial x^2} \right) + (\lambda + 2\mu) \left(\frac{\partial^2 \Delta}{\partial z^2} \right) \right], \\ 2\omega_x &= \left(\frac{2\mu}{\rho\omega^2} \right) \left[-jk_x \left(\frac{\partial \omega_z}{\partial z} \right) - \frac{\partial^2 \omega_x}{\partial z^2} \right], \\ 2\omega_y &= \left(\frac{2\mu}{\rho\omega^2} \right) \left(k_x^2 \omega_y - \frac{\partial^2 \omega_y}{\partial z^2} \right), \\ 2\omega_z &= \left(\frac{2\mu}{\rho\omega^2} \right) \left[k_x^2 \omega_z - jk_x \left(\frac{\partial \omega_x}{\partial z} \right) \right]. \end{aligned} \quad (10)$$

The first and the third of these equations can be solved directly,

$$\Delta = [A^+ \exp(-\alpha z) + A^- \exp(+\alpha z)] \exp(-jk_x x), \quad (11)$$

$$\omega_y = [B^+ \exp(-\beta z) + B^- \exp(+\beta z)] \exp(-jk_x x), \quad (12)$$

where

$$\alpha^2 = k_x^2 - h^2, \quad \beta^2 = k_x^2 - k^2, \quad (13)$$

and where h and k are the wavenumbers of the P and S waves, respectively,

$$h^2 = \rho\omega^2/(\lambda + 2\mu), \quad k^2 = \rho\omega^2/\mu. \quad (14)$$

The remaining rotations can be solved by combining the second and the fourth of Eqs. (12). This gives

$$\omega_z = [C^+ \exp(-\beta z) + C^- \exp(+\beta z)] \exp(-jk_x x), \quad (15)$$

and a similar expression for ω_x . However, there is no need to use both of the rotations ω_x and ω_z with the present type of excitation, and so ω_x is dropped.

A solution for the displacement of the P wave can now be obtained by insertion of the solution for the dilatation in Eq. (4) while keeping the rotations equal to zero. This procedure gives

$$\begin{aligned} u_1 &= (jk_x/h^2) [A^+ \exp(-\alpha z) + A^- \exp(+\alpha z)] \\ &\quad \times \exp(-jk_x x), \\ v_1 &= 0, \\ v_1 &= (\alpha/h^2) [A^+ \exp(-\alpha z) - A^- \exp(+\alpha z)] \\ &\quad \times \exp(-jk_x x). \end{aligned} \quad (16)$$

The solution for the SV wave is obtained in a similar manner by insertion of the expression for ω_y in Eqs. (4),

$$\begin{aligned} u_2 &= (2\beta/k^2) [B^+ \exp(-\beta z) - B^- \exp(+\beta z)] \\ &\quad \times \exp(-jk_x x), \\ v_2 &= 0, \\ w_2 &= (-j2k_x/k^2) [B^+ \exp(-\beta z) + B^- \exp(+\beta z)] \\ &\quad \times \exp(-jk_x x). \end{aligned} \quad (17)$$

The displacement of the SH wave is finally obtained as

$$\begin{aligned} u_3 &= 0, \\ v_3 &= (j2k_x/k^2) [C^+ \exp(-\beta z) + C^- \exp(+\beta z)] \\ &\quad \times \exp(-jk_x x), \\ w_3 &= 0. \end{aligned} \quad (18)$$

The stress components given by Eqs. (6) can now be expressed in the following way with the help of the expressions for the displacements:

$$\begin{aligned} \sigma_z &= \{ (\lambda - 2\mu\alpha^2/h^2) [A^+ \exp(-\alpha z) + A^- \exp(+\alpha z)] \\ &\quad + (j4k_x\mu\beta/k^2) [B^+ \exp(-\beta z) - B^- \exp(+\beta z)] \} \exp(-jk_x x), \\ \tau_{zx} &= \mu \{ (-j2k_x\alpha/h^2) [A^+ \exp(-\alpha z) - A^- \exp(+\alpha z)] \\ &\quad - [2(k_x^2 + \beta^2)/k^2] [B^+ \exp(-\beta z) + B^- \exp(+\beta z)] \} \exp(-jk_x x), \\ \tau_{zy} &= (-j2\mu k_x/k^2) [C^+ \exp(-\beta z) - C^- \exp(+\beta z)] \exp(-jk_x x). \end{aligned} \quad (19)$$

Insertion of these expressions for displacements and stresses into the boundary conditions, Eqs. (7)–(9), gives the following set of equations:

$$\begin{aligned} (\lambda/\mu - 2\alpha^2/h^2) (A^+ + A^-) &+ (j4k_x\beta/k^2) (B^+ - B^-) = 0, \\ (\lambda/\mu - 2\alpha^2/h^2) [A^+ \exp(-\alpha\delta) + A^- \exp(\alpha\delta)] &+ (j4k_x\beta/k^2) [B^+ \exp(-\beta\delta) - B^- \exp(\beta\delta)] = 0, \\ \mu \{ (-j2k_x\alpha/h^2) (A^+ - A^-) - [2(k_x^2 + \beta^2)/k^2] \} &\times (B^+ + B^-) = -\hat{p}_x, \\ \mu \{ (-j2k_x\alpha/h^2) [A^+ \exp(-\alpha\delta) - A^- \exp(\alpha\delta)] &- [2(k_x^2 + \beta^2)/k^2] [B^+ \exp(-\beta\delta) + B^- \exp(\beta\delta)] \} = \pm \hat{p}_x, \\ (-j2\beta k_x/k^2) (C^+ - C^-) &= 0, \\ (-j2\beta k_x/k^2) [C^+ \exp(-\beta\delta) - C^- \exp(\beta\delta)] &= 0. \end{aligned} \quad (20)$$

The solution for the antisymmetric case can be written as

$$\begin{aligned} A^+ &= \hat{p}_x (j\beta k_x h^2/\mu) [\exp(\alpha\gamma) \sinh(\beta\gamma)/D_A], \\ A^- &= -\hat{p}_x (j\beta k_x h^2/\mu) [\exp(-\alpha\gamma) \sinh(\beta\gamma)/D_A], \\ B^+ &= \hat{p}_x [k^2(k_x^2 + \beta^2)/4\mu] [\exp(\beta\gamma) \sinh(\alpha\gamma)/D_A], \\ B^- &= \hat{p}_x [k^2(k_x^2 + \beta^2)/4\mu] [\exp(-\beta\gamma) \sinh(\alpha\gamma)/D_A], \\ C^+ &= C^- = 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} D_A &= (k_x^2 + \beta^2)^2 \sinh(\alpha\gamma) \cosh(\beta\gamma) \\ &\quad - 4\alpha\beta k_x^2 \cosh(\alpha\gamma) \sinh(\beta\gamma), \end{aligned} \quad (22)$$

and where γ is half the thickness of the plate $\gamma = \delta/2$.

The corresponding solution for the symmetric case is

$$\begin{aligned} A^+ &= \hat{p}_x (j\beta k_x h^2/\mu) [\exp(\alpha\gamma) \cosh(\beta\gamma)/D_S], \\ A^- &= \hat{p}_x [j\beta k_x h^2/\mu] [\exp(-\alpha\gamma) \cosh(\beta\gamma)/D_S], \\ B^+ &= \hat{p}_x [k^2(k_x^2 + \beta^2)/4\mu] [\exp(\beta\gamma) \cosh(\alpha\gamma)/D_S], \\ B^- &= -\hat{p}_x [k^2(k_x^2 + \beta^2)/4\mu] [\exp(-\beta\gamma) \\ &\quad \times \cosh(\alpha\gamma)/D_S], \\ C^+ &= C^- = 0, \\ D_S &= (k_x^2 + \beta^2)^2 \cosh(\alpha\gamma) \sinh(\beta\gamma) \\ &\quad - 4\alpha\beta k_x^2 \sinh(\alpha\gamma) \cosh(\beta\gamma). \end{aligned} \quad (23)$$

Insertion gives the displacements on the upper face as $[u = \hat{u} \exp(-jk_x x)$, etc.]

$$\begin{aligned} \hat{u}_A &= \hat{p}_x 2Y_A^{Hx}/j\omega, \quad \hat{u}_S = \hat{p}_x 2Y_S^{Hx}/j\omega, \\ \hat{w}_A &= \hat{p}_x 2Y_A^{Hz}/j\omega, \quad \hat{w}_S = \hat{p}_x 2Y_S^{Hz}/j\omega, \end{aligned} \quad (24)$$

where the indices A and S denote antisymmetric and symmetric motion, respectively, and where

$$\begin{aligned} Y_A^{Hx} &= -[j\omega\beta k^2(1+\nu)/E] / [(k_x^2 + \beta^2)^2 \coth(\beta\gamma) \\ &\quad - 4\alpha\beta k_x^2 \coth(\alpha\gamma)], \\ Y_S^{Hx} &= -[j\omega\beta k^2(1+\nu)/E] / [(k_x^2 + \beta^2)^2 \tanh(\beta\gamma) \\ &\quad - 4\alpha\beta k_x^2 \tanh(\alpha\gamma)], \\ Y_A^{Hz} &= [\omega k_x(1+\nu)/E] [(k_x^2 + \beta^2) \tanh(\alpha\gamma) \\ &\quad - 2\alpha\beta \tanh(\beta\gamma)] / [(k_x^2 + \beta^2)^2 \tanh(\alpha\gamma) \\ &\quad - 4\alpha\beta k_x^2 \tanh(\beta\gamma)], \\ Y_S^{Hz} &= [\omega k_x(1+\nu)/E] [(k_x^2 + \beta^2) \coth(\alpha\gamma) \\ &\quad - 2\alpha\beta \coth(\beta\gamma)] / [(k_x^2 + \beta^2)^2 \coth(\alpha\gamma) \\ &\quad - 4\alpha\beta k_x^2 \coth(\beta\gamma)], \end{aligned} \quad (25)$$

and where Y_A^{Hx} , etc., denote the "admittances" of the plate (these admittances can be regarded as generalizations of the "spectral input admittance" introduced by Smith, 1976).

The displacements for the case of single-sided excitation can now be obtained as

$$\begin{aligned} \hat{u} &= (\hat{u}_A + \hat{u}_S)/2 = (\hat{p}_x/j\omega) (Y_A^{Hx} + Y_S^{Hx}), \\ \hat{w} &= (\hat{w}_A + \hat{w}_S)/2 = (\hat{p}_x/j\omega) (Y_A^{Hz} + Y_S^{Hz}). \end{aligned} \quad (26)$$

B. Transversal excitation

This case is quite easy to solve and so there is no reason to treat the symmetric and antisymmetric cases separately. Thus the boundary conditions are here taken as

$$\begin{aligned} \sigma_z &= 0 \quad (z=0, z=\delta), \\ \tau_{zx} &= 0 \quad (z=0, z=\delta), \\ \tau_{zy} &= -p_y \quad (z=0), \quad \tau_{zy} = 0 \quad (z=\delta), \end{aligned} \quad (27)$$

which gives the solution for the displacement on the upper face as (a detailed derivation can be found in Ljunggren, 1987b)

$$\begin{aligned} \hat{v} &= \hat{p}_y Y^T/j\omega, \\ Y^T &= [j\omega \cosh(\beta\delta)] / [G\beta \sinh(\beta\delta)], \end{aligned} \quad (28)$$

where G is the shear modulus

$$G = \mu = E/[2(1+\nu)]. \quad (29)$$

C. Vertical excitation

A solution for this case can be obtained in the same way as the two cases above. With an exciting force distribution p_z in the vertical direction on the upper face, the solution can, in the present notation, be written as

$$\begin{aligned} \hat{u} &= (\hat{p}_z/j\omega) (Y_A^{Vx} + Y_S^{Vx}), \\ \hat{w} &= (\hat{p}_z/j\omega) (Y_A^{Vz} + Y_S^{Vz}), \\ Y_A^{Vx} &= Y_A^{Hz}, \quad Y_S^{Vx} = Y_S^{Hz}, \\ Y_A^{Vz} &= -[j\omega\alpha k^2(1+\nu)/E] / [(k_x^2 + \beta^2)^2 \tanh(\alpha\gamma) \\ &\quad - 4\alpha\beta k_x^2 \tanh(\beta\gamma)], \\ Y_S^{Vz} &= -[j\omega\alpha k^2(1+\nu)/E] [(k_x^2 + \beta^2)^2 \coth(\alpha\gamma) \\ &\quad - 4\alpha\beta k_x^2 \coth(\beta\gamma)]. \end{aligned} \quad (30) \quad (31) \quad (32)$$

This result has previously been obtained by Heckl (1981), but in a different way. Related expressions in a somewhat different form have also been presented by Bycroft (1956) and Paul and Muthiyalu (1969).

II. LOW-FREQUENCY SOLUTIONS FOR A LINE SOURCE

A. The stress distributions due to the beam

The boundary conditions are, in the present case, established from prescribed stresses on the faces of the plate. Alternatively, a prescription of a certain motion of the beam could have been used. This latter approach leads to inhomogeneous boundary conditions, however, and the ensuing heavy mathematical work does not seem worth while in the present case.

In the cited paper on moment excitation (Ljunggren, 1987a), a certain distribution of the normal stress was specified:

$$p_z = 2Mx/[\pi b^2(b^2 - x^2)^{1/2}], \quad (|x| < b), \quad (33)$$

where M is the moment applied per unit length and $2b$ is the width of the beam. The reason for this choice was that this particular distribution occurs in the static case of excitation of a semi-infinite solid with an infinitely stiff beam (Awojobi and Grootenhuys, 1965), and, hence, it could be expected to

give a reasonably close approach to the real distribution in the case of the plate, at least if the beam is narrow compared with the thickness of the plate and if the frequency is low.

Distributions of this type were also used in two related papers on point excitation (Ljunggren, 1983, 1984).

The corresponding distributions in the case of excitation with a line force in the horizontal, in-line, or vertical direction are

$$p_x = N / [\pi(b^2 - x^2)^{1/2}], \quad (34)$$

$$p_y = Q / [\pi(b^2 - x^2)^{1/2}], \quad (35)$$

$$p_z = F / [\pi(b^2 - x^2)^{1/2}], \quad (36)$$

where N , Q , and F are the horizontal, in-line, and vertical forces per unit length of the beam. These expressions are valid for the area directly under the beam, $|x| < b$; otherwise $p_x = p_y = p_z = 0$.

This type of distribution is attractive since the Fourier transform is very compact:

$$\check{p}_x = \int_{-b}^b \frac{N}{[\pi(b^2 - x^2)^{1/2}]} \exp(jk_x x) dx = NJ_0(k_x b). \quad (37)$$

It should be noted, however, that this particular distribution cannot be said to appear in the case of a semi-infinite solid at low frequencies, as the corresponding mobilities in these cases are infinitely large, see, e.g., Adams and Bogy (1976). It will, nevertheless, be used in this work as a first approximation; some other distributions will be discussed later on.

B. Some preliminary remarks on the solutions

The solution for the free part of the plate, $x \gg b$ and $x \ll -b$, can now be obtained by means of an inverse Fourier transform. Taking the case of a vertical force as an example, the vertical amplitude can be written as

$$w = \left(\frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \left(\frac{p_z}{j\omega} \right) (Y_A^{Vz} + Y_S^{Vz}) \exp(-jk_x x) dk_x. \quad (38)$$

This integral can be evaluated analytically by means of the residue theorem. The location of the poles in the complex k_x plane is illustrated in Fig. 3 for the case of low frequencies. Part (a) of the figure is applicable to integrands containing admittances of the antisymmetric kind, part (b) for integrands that are symmetric in the same sense, and part (c) for integrands describing pure in-plane motion. All the poles are of first order and there is no need for any branch cuts as the integrands are even in α and β . It has been shown (Ljunggren, 1987a, 1988) that it is possible to derive approximations of the admittances for different regions in the (real) k_x/ω plane. This technique can also be used in the present case. Thus, in the vicinity of the bending wave poles, use is made of the "thin-plate" conditions,

$$h\gamma \ll 1, \quad k\gamma \ll 1, \quad k_x \gamma \ll 1. \quad (39)$$

With this approach, the antisymmetric admittance turns into an admittance for bending waves,

$$Y_B^{Vz} = [j\omega 12(1 - \nu^2)] / [E\delta^3(k_x^4 - k_B^4)]. \quad (40)$$

Here, k_B is the wavenumber of the free bending wave, which, in the present notation, can be written as

$$k_B = [6k^2(1 - \nu)/\delta^2]^{1/4}. \quad (41)$$

The expression for the displacement due to the bending wave now takes a well-known form:

$$\hat{w}_B = \hat{p}_z [12(1 - \nu^2)] / [E\delta^3(k_x^4 - k_B^4)], \quad (42)$$

see, e.g., Cremer *et al.* (1973).

If the thin-plate conditions are applied to the symmetric admittance, an admittance for the longitudinal plate waves is obtained:

$$Y_L^{Vz} = [j\omega\delta(1 - \nu^2)(k_x^2 - k_L^2)] / [4E(k_x^2 - k_L^2)], \quad (43)$$

$$k_L^2 = k^2(1 - \nu)/2. \quad (44)$$

Near the poles of the Lamb waves, a couple of "low-frequency approximations" can be used:

FIG. 3. Location of poles in the complex wavenumber plane: (a) antisymmetric motion, (b) symmetric motion, and (c) pure in-plane motion.

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$$\alpha\gamma = k_x\gamma - h^2\gamma/2k_x, \quad \beta\gamma = k_x\gamma - k^2\gamma/2k_x. \quad (45)$$

This approach leads to (Ljunggren, 1987a)

$$\begin{aligned} Y_{ALR}^{V_z} &= [j\omega(1-\nu^2)/E][2\cosh^2(k_x\gamma)]/ \\ &\quad \{k_x[\sinh(2k_x\gamma) - 2k_x\gamma]\}, \\ Y_{SLR}^{V_z} &= [j\omega(1-\nu^2)/E][2\sinh^2(k_x\gamma)]/ \\ &\quad \{k_x[\sinh(2k_x\gamma) + 2k_x\gamma]\}, \end{aligned} \quad (46)$$

for the antisymmetric and symmetric admittance, respectively.

The integration contour can be taken as the real axis together with an arc in the lower half of the complex k_x plane. It is then evident that the integrals for the bending and longitudinal waves are ambiguous, since some of the poles are situated on the real axis. The reason for this anomaly is that no losses have been introduced in the model. If a complex Young's modulus E with a loss factor η is introduced in the usual way,

$$\bar{E} = E(1 + j\eta), \quad (47)$$

the poles on the real axis to the right of the origin are moved below the axis and those to the left of the origin above the axis.

The poles of the expressions for the Lamb waves are given by the characteristic equations for very low frequencies (Sherwood, 1958; Tamm and Weis, 1959):

$$\sinh(2k_x\gamma) - 2k_x\gamma = 0, \quad \sinh(2k_x\gamma) + 2k_x\gamma = 0, \quad (48)$$

which both have an infinite number of roots. The roots are here denoted ξ_{Ap} and ξ_{Sp} , respectively, where A indicates antisymmetric motion, S indicates symmetric motion, and p is the number of the root, $p = 1, \dots, \infty$.

It is seen from Fig. 3 that the Lamb poles are located symmetrically with respect to the imaginary axis. It has been pointed out by Tamm and Weis (1959) that a physical plate mode of this type must consist of a pair of waves with wavenumbers positioned in this way. The mode can then be interpreted as a standing and exponentially decaying wave. The fact that the Lamb poles appear in pairs is important for the mobilities too. It is evident that the part of the mobility due to the Lamb modes can be written in the form of two infinite sums, one of which is due to the poles in the third quadrant and the other to the poles in the fourth quadrant. It has been shown (Ljunggren, 1983) that each term of the sum from the third quadrant can be added to a corresponding term in the sum from the fourth quadrant and that the sum of these two terms constitutes a part of the mobility which is always imaginary. Thus the contribution from all the Lamb waves is imaginary. This is only to be expected, of course, as the low-frequency Lamb waves carry no energy in the steady state. This also shows that the reactive field due to the Lamb waves must be confined to a region near the excitation area.

In the case of in-line excitation, a propagating shear wave will be excited together with an infinite number of low-frequency Love waves. If the "thin-plate" conditions are introduced, the admittance with respect to the shear wave can be simplified to

$$Y_S^T = j\omega/[G\delta(k_x^2 - k^2)]. \quad (49)$$

Instead of a special low-frequency approximation, the original formulation of the admittance must be used. If it is written in the form

$$Y_{LR}^T = j\omega/[Gk_x \tanh(k_x\delta)], \quad (50)$$

it is clearly seen that the wavenumbers of the free Love waves are given by

$$\xi_n = \pm jn\pi/\delta. \quad (51)$$

As the wavenumbers of the low-frequency Love waves are imaginary, the corresponding waves are exponentially decaying in about the same manner as the field associated with the imaginary bending wavenumber. Furthermore, the contribution to the mobility from each Love wave pole can be shown to be imaginary.

C. Solution for the case of excitation with a horizontal force

The formal solutions for the displacements u and w due to a horizontal line force can, for the free part of the plate, be written as

$$\begin{aligned} u &= \frac{N}{j\omega 2\pi} \int_{-\infty}^{\infty} J_0(k_x b) (Y_A^{Hx} + Y_S^{Hx}) \\ &\quad \times \exp(-jk_x x) dk_x, \end{aligned} \quad (52)$$

$$\begin{aligned} w &= \frac{N}{j\omega 2\pi} \int_{-\infty}^{\infty} J_0(k_x b) (Y_A^{Hz} + Y_S^{Hz}) \\ &\quad \times \exp(-jk_x x) dk_x. \end{aligned} \quad (53)$$

The approximate expressions for the admittances near the bending wave poles take the form

$$Y_A^{Hx} = [j\omega 3(1-\nu^2)/E\delta](k_x^2 - k^2)/(k_x^4 - k_B^4), \quad (54)$$

$$\begin{aligned} Y_A^{Hz} &= [\omega(1+\nu)/Ek^2][k_x^3 k^2(1-\nu/2) - k_x k_B^4]/ \\ &\quad (k_x^4 - k_B^4). \end{aligned} \quad (55)$$

The displacements of the bending wave can now, for $x \gg b$ and small values of b , be obtained as

$$\begin{aligned} u_B &= -[jN 3(1-\nu^2)/(4Ek_B\delta)][(1 - k^2/k_B^2) \\ &\quad \times \exp(-jk_B x) + j(1 + k^2/k_B^2)\exp(-k_B x)], \end{aligned} \quad (56)$$

$$\begin{aligned} w_B &= -[N(1+\nu)/4E][(1 - \nu/2 - k_B^2/k^2) \\ &\quad \times \exp(-jk_B x) + (1 - \nu/2 + k_B^2/k^2) \\ &\quad \times \exp(-k_B x)]. \end{aligned} \quad (57)$$

The corresponding expressions for the admittances with respect to the longitudinal plate wave can be simplified to

$$Y_L^{Hx} = [j\omega(1-\nu^2)/E\delta]/(k_x^2 - k_L^2), \quad (58)$$

$$Y_L^{Hz} = -[\omega k_x(1-\nu^2)/2Ek^2](2h^2 - k^2)/(k_x^2 - k_L^2), \quad (59)$$

which gives the displacement of the longitudinal wave as ($x \gg b$)

$$u_L = -[jN(1-\nu^2)/(2Ek_L\delta)]\exp(-jk_L x), \quad (60)$$

$$w_L = -[N\nu(1+\nu)/4E]\exp(-jk_L x). \quad (61)$$

The corresponding solution for $x \leq -b$ can be obtained in the same way as

$$u_B = -[j3N(1-\nu^2)/(4Ek_B\delta)] \left[(1-k^2/k_B^2) \times \exp(jk_Bx) + j(1+k^2/k_B^2)\exp(k_Bx) \right], \quad (62)$$

$$w_B = \left(\frac{N(1+\nu)}{4E} \right) \left[\left(\frac{1-\nu}{2} - \frac{k_B^2}{k^2} \right) \exp(jk_Bx) + \left(\frac{1-\nu}{2} + \frac{k_B^2}{k^2} \right) \exp(k_Bx) \right], \quad (63)$$

$$u_L = -[jN(1-\nu^2)/(2Ek_L\delta)] \exp(jk_Lx), \quad (64)$$

$$w_L = [N\omega(1+\nu)/4E] \exp(jk_Lx). \quad (65)$$

It is seen that the horizontal displacements of the two solutions for the two regions [Eqs. (56) and (62); (60) and (64)] tend to the same value for $x \rightarrow 0$. This is not the case, however, for the vertical displacements. This disagreement is due to a rocking motion of the beam. If the beam is slender compared to the wavelengths, it is possible to define an angle ϕ of the rocking motion of the beam as

$$\phi = [(w)_{x=b} - (w)_{x=-b}]/2b. \quad (66)$$

Insertion gives

$$\phi_B = 3N(1-\nu^2)(1-j)/(2Ek_B\delta^2), \quad (67)$$

$$\phi_L = -N\nu(1+\nu)/(4Eb). \quad (68)$$

The expression for the angle ϕ_B is derived here under the assumption that the width of the beam is small (compared with the wavelength of the bending wave) but not too small (compared with the thickness of the plate). The necessity for such an assumption is only to be expected in view of the results by Smith (1976) and Smith and Dym (1978) for the related case of excitation with a rocking moment.

The expressions for the horizontal displacements and the angle of the rocking motion can be used to derive the following expressions for the input mobilities:

$$M_B^H \equiv j\omega(u_B)_{x=b}/N \\ = 3\omega(1-\nu^2) \frac{(1-k^2/k_B^2 + j + jk^2/k_B^2)}{(4Ek_B\delta)}, \quad (69)$$

$$M_L^H \equiv j\omega(u_L)_{x=b}/N \\ = \omega(1-\nu^2)/(2Ek_L\delta) = 1/(2\rho c_L\delta) \quad (70)$$

$$M_B^{H\phi} \equiv j\omega\phi_B/N = 3\omega(1-\nu^2)(1+j)/(2Ek_B\delta^2), \quad (71)$$

$$M_L^{H\phi} \equiv j\omega\phi_L/N = -j\omega\nu(1+\nu)/(4Eb). \quad (72)$$

Not only the expressions for $M_B^{H\phi}$ and $M_L^{H\phi}$ but also those for the other two mobilities have here been derived under the assumption of a small width of the beam.

For low frequencies, where $k_B^2 \ll k^2$, it is possible to simplify the expression for M_B^H to

$$M_B^H = 3\omega(1-\nu^2)(1+j)/(4Ek_B\delta). \quad (73)$$

This expression can also be derived from the solution for a rocking moment M by taking $F = M/\gamma$.

Equations (71) and (72) show that

$$M_B^H/M_B^{H\phi} = \delta/2. \quad (74)$$

The underlying reason for this relationship can be explained with the help of Fig. 4. It is seen from this figure that

$$\phi = \frac{u}{\gamma} = \frac{2u}{\delta}, \quad \text{or} \quad \frac{u}{\phi} = \frac{\delta}{2}, \quad (75)$$

which corresponds to the relation given by Eq. (74).

The remaining part of the solution, the local reaction, can be evaluated by using the following approximations of the admittances:

$$Y_{LRA}^{Hx} = [j\omega(1-\nu^2)/E] [2 \sinh^2(k_x\gamma)] / \{k_x [\sinh(2k_x\gamma) - 2k_x\gamma]\}, \quad (76)$$

$$Y_{LRS}^{Hx} = [j\omega(1-\nu^2)/E] [2 \cosh^2(k_x\gamma)] / \{k_x [\sinh(2k_x\gamma) + 2k_x\gamma]\}, \quad (77)$$

$$Y_{LRA}^{Hz} = [-\omega(1+\nu)/2E] [(1-2\nu)\sinh(2k_x\gamma) + 2k_x\gamma] / \{k_x [\sinh(2k_x\gamma) - 2k_x\gamma]\}, \quad (78)$$

$$Y_{LRS}^{Hz} = [-\omega(1+\nu)/2E] [(1-2\nu)\sinh(2k_x\gamma) - 2k_x\gamma] / \{k_x [\sinh(2k_x\gamma) + 2k_x\gamma]\}. \quad (79)$$

The displacements of the local reaction can now be obtained as ($x \geq b$)

$$u_{LR} = \left(\frac{N(1-\nu^2)}{E} \right) \left[\sum_p \text{Im} \left(J_0(\xi_{Ap}b) \frac{\exp(-j\xi_{Ap}x)}{(\xi_{Ap}\gamma)} \right) + \sum_p \text{Im} \left(J_0(\xi_{Sp}b) \frac{\exp(-j\xi_{Sp}x)}{(\xi_{Sp}\gamma)} \right) \right], \quad (80)$$

$$w_{LR} = \left(\frac{N(1-\nu^2)}{E} \right) \left[\sum_p \text{Re} \left(J_0(\xi_{Ap}b) \coth(\xi_{Ap}\gamma) \times \frac{\exp(-j\xi_{Ap}x)}{(\xi_{Ap}\gamma)} \right) + \sum_p \text{Re} \left(J_0(\xi_{Sp}b) \tanh(\xi_{Sp}\gamma) \frac{\exp(-j\xi_{Sp}x)}{(\xi_{Sp}\gamma)} \right) \right], \quad (81)$$

where ξ_{Ap} and ξ_{Sp} denote the roots of Eqs. (48) in the fourth quadrant. This gives the corresponding mobilities as

$$M_{LR}^H = [j\omega(1-\nu^2)/E] f_4(t), \quad (82)$$

$$M_{LR}^{H\phi} = [j\omega(1-\nu^2)/Eb] f_5(t), \quad (83)$$

where

FIG. 4. Excitation of a bending wave with a horizontal line force.

$$f_4(t) = \sum_p \operatorname{Im} \left(J_0(\xi_{Ap} b) \frac{\exp(-j\xi_{Ap} b)}{(\xi_{Ap} \gamma)} \right) + \sum_p \operatorname{Im} \left(J_0(\xi_{Sp} b) \frac{\exp(-j\xi_{Sp} b)}{(\xi_{Sp} \gamma)} \right), \quad (84)$$

$$f_5(t) = \sum_p \operatorname{Re} \left(J_0(\xi_{Ap} b) \coth(\xi_{Ap} \gamma) \frac{\exp(-j\xi_{Ap} b)}{(\xi_{Ap} \gamma)} \right) + \sum_p \operatorname{Re} \left(J_0(\xi_{Sp} b) \tanh(\xi_{Sp} \gamma) \frac{\exp(-j\xi_{Sp} b)}{(\xi_{Sp} \gamma)} \right), \quad (85)$$

$$t = b/\gamma = 2b/\delta. \quad (86)$$

The functions $f_4(t)$ and $f_5(t)$ have been calculated using the exact values of the roots of Eqs. (48). The graphs of the functions are shown in Fig. 5.

D. Solution for the case of excitation with an in-line force

There will be a displacement in the y direction only in this case. According to the discussion above, the displacement v can be written as

$$v = \left(\frac{1}{j\omega 2\pi} \right) \int_{-\infty}^{\infty} QJ_0(k_x b) Y^T \exp(-jk_x x) dk_x. \quad (87)$$

Upon using the expressions for the admittances given by Eqs. (49) and (50), the displacement of the shear wave can, for the region $x > b$, be written as

$$v_S = -(jQ/2Gk\delta) J_0(kb) \exp(-jkx), \quad (88)$$

and the displacement of the local reaction as

$$v_{LR} = \left(\frac{Q}{\pi G} \right) \sum_n I_0 \left(\frac{n\pi b}{\delta} \right) \left(\frac{1}{n} \right) \exp \left(\frac{-n\pi x}{b} \right). \quad (89)$$

For small values of b , the expressions for the corresponding mobilities become

$$M_S^{IL} = \omega/(2Gk\delta) = 1/(2\rho c_T \delta), \quad (90)$$

$$M_{LR}^{IL} = (j\omega/\pi G) f_6(t), \quad (91)$$

where

FIG. 5. The functions $f_4(t)$ and $f_5(t)$ describing the influence of b/γ on the local reactions due to a horizontal line force. —, $f_4(t)$; ---, $f_5(t)$.

$$f_6(t) = \sum_n I_0 \left(\frac{n\pi b}{\delta} \right) \left(\frac{1}{n} \right) \exp \left(\frac{-n\pi b}{\delta} \right). \quad (92)$$

The graph of the function $f_6(t)$ is shown in Fig. 6.

It is interesting to note that the expression for M_S^{IL} has the same form as that for M_L^H ; the only difference is that the propagation speed of the shear wave c_T is used in M_S^{IL} and the speed of the longitudinal wave c_L , in M_L^H .

E. Solution for the case of excitation with a vertical force

Only the vertical displacement w will be evaluated in this case. There will also be a small displacement in the x direction with the boundary conditions chosen here, but this displacement is not thought to be very interesting. The solution for the different parts of the vertical displacement can be obtained directly from the material presented above ($x > b$):

$$w_B = -\{j3F(1-\nu^2)/[E(k_B\delta)^3]\} [J_0(k_B b) \times \exp(-jk_B x) - jI_0(k_B b) \exp(-k_B x)], \quad (93)$$

$$w_L = -\{jF\delta(1+\nu)\nu^2 k_L/[8E(1-\nu)]\} \times J_0(k_L b) \exp(-jk_L x), \quad (94)$$

$$w_{LR} = \left(\frac{F(1-\nu^2)}{E} \right) \sum_p \operatorname{Im} \left(J_0(\xi_{Ap} b) \coth^2(\xi_{Ap} \gamma) \times \frac{\exp(-j\xi_{Ap} x)}{(\xi_{Ap} \gamma)} \right) + \left(\frac{F(1-\nu^2)}{E} \right) \times \sum_p \operatorname{Im} \left(J_0(\xi_{Sp} b) \tanh^2(\xi_{Sp} \gamma) \frac{\exp(-j\xi_{Sp} x)}{(\xi_{Sp} \gamma)} \right). \quad (95)$$

The mobilities can now be obtained in the same manner as before and, for the case of small values of b , be written as

$$M_B^V = 3\omega(1-\nu^2)(1-j)/[E(k_B\delta)^3] = (1-j)/(4\rho\delta c_B), \quad (96)$$

$$M_L^V = \omega(1+\nu)\nu^2 k_L \delta/[8E(1-\nu)], \quad (97)$$

$$M_{LR}^V = [j\omega(1-\nu^2)/E] f_7(t), \quad (98)$$

FIG. 6. The function $f_6(t)$ describing the influence of b/γ on the local reaction due to an in-line source.

$$f_7(t) = \sum_p \operatorname{Im} \left(J_0(\xi_{Ap} b) \coth^2(\xi_{Ap} \gamma) \frac{\exp(-j\xi_{Ap} b)}{(\xi_{Ap} \gamma)} \right) + \sum_p \operatorname{Im} \left(J_0(\xi_{Sp} b) \tanh^2(\xi_{Sp} \gamma) \times \frac{\exp(-j\xi_{Sp} b)}{(\xi_{Sp} \gamma)} \right). \quad (99)$$

The graph of the function $f_7(t)$ is given in Fig. 7.

III. DERIVATION OF LINE MOBILITIES FROM EXPRESSIONS FOR POINT-EXCITED FIELDS

A. Bending, longitudinal, and shear waves

A source in the form of a force applied along an infinite line can, as a first approximation, be regarded as an infinite number of point forces. Taking the case of vertical force as an example, the bending wave displacement w' due to a point force F' can be written as (Cremer *et al.*, 1973)

$$w' = (F'/j\omega) M_0 [H_0^{(2)}(k_B r) - H_0^{(2)}(-jk_B r)], \quad (100)$$

where M_0 is the classic point mobility, which, in the present notation, takes the following form:

$$M_0 = 3\omega(1 - \nu^2)/(2E\delta^3 k_B^2). \quad (101)$$

At a point on the line, the displacement due to the line force can be taken as

$$w = 2 \int_0^\infty w'(r) dr. \quad (102)$$

The integral of the Hankel function is well known. In the book by Abramowitz and Stegun (1965), it is expressed as

$$\int_0^\infty J_0(r) dr = 1, \quad \int_0^\infty Y_0(r) dr = 0. \quad (103)$$

This gives immediately

$$w = 2 \left(\frac{F'}{j\omega} \right) M_0 \left(\frac{1}{k_B} - \frac{1}{(-jk_B)} \right), \quad (104)$$

and the mobility becomes

FIG. 7. The function $f_7(t)$ describing the influence of b/γ on the local reaction due to a vertical line force.

$$M_B^\nu = 2M_0(1 - j)/k_B = 3\omega(1 - \nu^2)(1 - j)/[E(k_B \delta)^3], \quad (105)$$

in agreement with Eq. (96).

The case of a longitudinal wave due to a horizontal line force is more complicated. The field due to a point force N' consists not only of a longitudinal wave but also of a shear wave. The displacement η can, according to a previous paper (Ljunggren, 1984, Eq. 97), be written as (see Fig. 8 for definitions)

$$\eta_L = -[jN'(1 - \nu^2)/4E\delta](1/k_L r) H_1^{(2)}(k_L r), \quad (106)$$

$$\eta_S = -[jN'(1 + \nu)/2E\delta] \times [H_0^{(2)}(kr) - (1/kr) H_1^{(2)}(kr)]. \quad (107)$$

The mobility with respect to a horizontal line force is then obtained as

$$M_L^H \approx 2 \left(\frac{\omega(1 - \nu^2)}{4E\delta} \right) \int_\epsilon^\infty \left(\frac{1}{k_L r} \right) H_1^{(2)}(k_L r) dr + 2 \left(\frac{\omega(1 + \nu)}{2E\delta} \right) \int_\epsilon^\infty \left[H_0^{(2)}(kr) - \left(\frac{1}{kr} \right) H_1^{(2)}(kr) \right] dr, \quad (108)$$

where ϵ is a small length of the same order of magnitude as b . It can be shown that

$$\int_\epsilon^\infty \left[k_L^2 \left(\frac{1}{k_L r} \right) Y_1(k_L r) - k^2 \left(\frac{1}{kr} \right) Y_1(kr) \right] dr \approx 0. \quad (109)$$

Furthermore,

$$\int_0^\infty \left(\frac{1}{kr} \right) J_1(kr) dr = \frac{1}{k}, \quad (110)$$

see Abramowitz and Stegun (1965). Upon using these results, the line mobility becomes

$$M_L^H \approx \omega(1 + \nu)/(Ek\delta) + (k_L - k)/(2E\omega\rho\delta) = 1/(2\rho c_L \delta), \quad (111)$$

in agreement with the Cremer's classic expression, here Eq. (2).

It is possible to use this type of derivation for all the different parts of the force mobility due to the bending, longitudinal, and shear waves, see Table I.

B. Local reactions

In the case of "point" excitation of a plate with a vertical force F' , the displacement of the local reaction close to the

FIG. 8. Definition of the displacement η in the case of the horizontal point excitation.

TABLE I. Derivation of line mobilities from superpositions of point sources.

	Integral representation of mobilities	Evaluated expressions
Vertical force		
Bending wave	$M_b^V = 2 \int_0^\infty M_0 [H_0^{(2)}(k_b r) - H_0^{(2)}(-jk_b r)] dr$	$M_b^V = 2M_0(1-j)/k_b$ $= 3(1-\nu^2)(1-j)/[E(k_b \delta)^3]$
Longitudinal wave	$M_L^V = 2 \frac{\omega k^2 \nu^2 \gamma}{32\mu} \int_0^\infty H_0^{(2)}(k_L r) dr$	$M_L^V = 2\omega k^2 \nu^2 \gamma / (32\mu k_L)$ $= \omega(1+\nu)^2 k_L \delta / [8E(1-\nu)]$
Horizontal force		
Bending wave	$M_b^H = -2j\omega \frac{j3(1-\nu^2)}{8E\delta} \int_0^\infty \left(\frac{H_1^{(2)}(k_b r)}{k_b r} - \frac{(j2/\pi)K_1(k_b r)}{k_b r} \right) dr$	$M_b^H \approx 3\omega(1-\nu^2)(1+j)/(4Ek_b \delta)$
Longitudinal wave	$M_L^H = 2 \frac{\omega(1-\nu^2)}{4E\delta} \int_0^\infty \frac{1}{k_L r} H_1^{(2)}(k_L r) dr$ $+ 2 \frac{\omega(1+\nu)}{2E\delta} \int_0^\infty \left[H_1^{(2)}(kr) - (1/kr)H_1^{(2)}(kr) \right] dr$	$M_L^H \approx \omega(1+\nu)/Ek\delta + (k_L - k)/(2E\omega p\delta)$ $= 1/(2pc_L \delta)$
In-line force		
Shear wave	$M_s^H = 2 \frac{\omega(1-\nu^2)}{4E\delta} \int_0^\infty H_0^{(2)}(k_L r) dr$ $- (1/k_L r)H_1^{(2)}(k_L r) \int_0^\infty \frac{1}{kr} H_1^{(2)}(kr) dr$ $+ 2 \frac{\omega(1+\nu)}{2E\delta} \int_0^\infty \frac{1}{kr} H_1^{(2)}(kr) dr$	$M_s^H \approx \omega(1-\nu^2)/2Ek_L \delta + (k - k_L)/(2E\omega p\delta)$ $= \omega/(2Gk\delta) = 1/(2pc_\tau \delta)$

indenter and on the excited face can approximately be written as (Ljunggren, 1983)

$$\omega'_{LR} = (F'/j\omega)M_{LR}^F(2R/\pi r), \quad (112)$$

$$M_{LR}^F = j\omega(1-\nu^2)/2ER, \quad (113)$$

where M_{LR}^F is the point mobility for an indenter with radius R and where r is the distance from the point of observation to the center of the indenter. The expression is valid for

$$r \ll \gamma/\pi, \quad (114)$$

and was derived for the same type of stress distribution as that used for the line forces in the present paper.

When the distance r is larger than γ/π , the displacement is exponentially decreasing with r . Thus it should be possible to obtain at least an estimate of the corresponding line mobility by an integration up to this distance. If R and the lower integration limit are both put equal to b , this approach leads to

$$M_{LR}^V = [j\omega 2(1-\nu^2)/\pi E] \ln(\gamma/\pi b). \quad (115)$$

This is not the same result as that obtained in the other way, Eqs. (98) and (99). The difference is not thought to be too large in view of the rough approximations in the second case, however; a comparison between $f_j(t)$ and $(2/\pi)\ln(\gamma\pi/b)$ is shown in Fig. 9.

Solutions of the same type can be derived for the cases of excitation with horizontal and in-line force, too. As the displacement due to the corresponding point force is again of the type $\text{const} \cdot R/r$ for small values of r , it is seen that these line mobilities must exhibit the same type of behavior for small values of b/γ as in the case of vertical force. It should not be taken for granted, however, that the agreement

between the two solutions will be as convincing for the horizontal and in-line force as for the vertical force in the previous case. The reason for this is that the case of a vertical force is comparatively simple as the field due to the separate point forces as well as that due to the line source consist of Lamb modes only. In the case of a horizontal force, on the other hand, the field due to the line source still consists of Lamb modes only, whereas that due to a single point force consists of Love modes as well as Lamb modes (Ljunggren, 1984, Eq. (108)). This implies that not only the Lamb modes but also the Love modes will influence the interaction between the separate point sources in spite of the fact that the

FIG. 9. The function $f_j(t)$ calculated in two different ways. —, from the two-dimensional solution; ---, from superposition of point sources [Eq. (115)].

resulting field can be regarded as being built up of Lamb modes only.

The case of an in-line force is similar to that of a horizontal force as the field around a point force consists of both Lamb and Love modes. The main difference is that the resulting field due to the line force is built up of Love modes instead of Lamb modes.

IV. DISCUSSION

A. Influence of the stress distribution

The results given in Sec. II show that the mobilities with respect to the bending, longitudinal, and shear waves depend neither on the width of the beam nor the stress distribution under the beam. The reason for this is of course that the width of the beam is assumed to be much smaller than the appropriate wavelengths. However, in the case of the Lamb and Love waves, the wavelengths tend to zero with increasing order number, and the width of the beam, however small, cannot be small compared with the wavelengths of all these waves. Not only the width of the beam but also the stress distribution must then be expected to have an influence on the magnitude of the local reaction.

In order to test the effect of different stress distributions, the local reaction due to a vertical line force has been calculated for two different distributions beside the one used before. These new distributions are defined as

$$p_u = F/2b \quad (\text{uniform}), \quad (116)$$

$$p_p = F2(b^2 - x^2)^{1/2}/\pi b^2 \quad (\text{parabolic}). \quad (117)$$

The three distributions are illustrated in Fig. 10 [the hyperbolic distribution shown in the figure is the distribution used above, see Eq. (36)]. The corresponding Fourier transforms for the new distributions are

$$\check{p}_u = F \sin(k_x b)/k_x b, \quad (118)$$

$$\check{p}_p = F2J_1(k_x b)/k_x b. \quad (119)$$

The displacements of the local reaction can be obtained by inspection from Eqs. (98) and (99). The new versions of the function $f_7(t)$ have been evaluated in the same way as the original one and the result is given in Fig. 11. As expected, the displacement of the local reaction becomes smaller when

FIG. 10. Illustration of the three stress distributions: —, hyperbolic; ---, uniform; - · -, parabolic.

FIG. 11. The function $f_7(t)$ calculated for different stress distributions. —, hyperbolic; ---, uniform; - · -, parabolic.

the distribution turns from hyperbolic to uniform and from uniform to parabolic, i.e., with decreasing stress amplitude at the rim of the beam.

B. The importance of the different parts of the solutions

In the case of excitation with a horizontal or vertical line force, the contributions from the bending waves are much more important for the mobilities than the contributions from the longitudinal plate waves. This is immediately recognized from the following expressions for the ratio of the two parts of the mobilities:

$$|M_L^H/M_B^H| = k_B \delta / (3k_L \delta), \quad (120)$$

$$|M_L^V/M_B^V| = \nu^2 k_L \delta (k_B \delta)^3 / [48(1 - \nu^2)]. \quad (121)$$

Since $k_B \ll k_L$ and since $k_B \delta < 1$, it is clear that

$$|M_L^H/M_B^H| \ll 1, \quad |M_L^V/M_B^V| \ll 1. \quad (122)$$

The importance of the contributions from the local reactions can be deduced in a similar manner from the following expressions:

FIG. 12. The function $f_3(t)$ describing the influence of b/γ on the local reaction due to a rocking moment.

TABLE II. The uncoupled line mobilities for infinite plates at low frequencies.

	Vertical force	Horizontal force	In-line force	Rocking moment
Bending wave	$1/[2\rho\delta c_b(1+j)]$	$3\omega(1-\nu^2)(1+j)/(4Ek_b\delta)$	...	$3\omega(1-\nu^2)(1+j)/(Ek_b\delta^3)$
Longitudinal wave	$\omega(1+\nu)\nu^2k_L\delta/[8E(1-\nu)]$	$1/(2\rho c_L\delta)$	...	$j\omega k_L^2\delta\nu^2(1+\nu)/[8bE(1-\nu)]$
Shear wave	...	...	$1/(2\rho c_T\delta)$	...
Local reaction, Lamb-type	$[j\omega(1-\nu^2)/E]f_7(t)$	$[j\omega(1-\nu^2)/E]f_4(t)$	...	$[j\omega 4(1-\nu^2)/(\pi Eb^2)]f_5(t)$
Local reaction, Love-type	...	...	$(j\omega/\pi G)f_6(t)$	...

$$|M_{LR}^H/M_B^H| = (2k_b\delta/3)f_4(t), \quad (123)$$

$$|M_{LR}^L/M_S^L| = (2k\delta/\pi)f_6(t), \quad (124)$$

$$|M_{LR}^V/M_B^V| = [(k_b\delta)^3/6]f_7(t). \quad (125)$$

The expressions are valid in the frequency range given by $k_b\delta < 1$. It is then clear that in the case of a horizontal or vertical force, the mobility due to the local reaction can equal that due to the bending wave only if $f_3(t) > 3/2$ and $f_7(t) > 6$, respectively. This corresponds to a value of b/γ less than 0.035 in the first case and a value much less than 0.01 in the second case.

A relationship between the wavenumber of the shear wave and that of the bending wave is given by Eq. (41). This expression shows that the limit for the validity of the bending wavenumbers, $k_b\delta = 1$, corresponds to a value of $k\delta = 0.49$ (with $\nu = 0.3$). Equation (124) then shows that the mobility due to the local reaction, in the case of in-line excitation at the frequency given by $k_b\delta = 1$, equals the mobility due to the shear wave if $f_6(t) \approx 3.21$, i.e., $b/\gamma \approx 0.05$.

These examples show that in the case of line force excitation, the local reactions must be expected to be small in most cases in practice, and in general not so important as the local reactions for point force sources, cf. the discussions in the cited papers on point mobilities (Ljunggren 1983, 1984). It should be observed, however, that the local reaction is fairly important in the case of a moment along an infinite line, as the ratio of the contributions from the local reaction upon that of the bending wave can be written as (from the material presented in the cited paper on moment excitation)

$$|M_{LR}^R/M_B^R| = (2k_b\delta^3/3\pi b^2)f_3(t), \quad (126)$$

where $f_3(t)$ is given in Fig. 12. Equation (126) shows that the local reaction can be important if the width of the beam is only slightly smaller than the thickness of the plate.

C. Comparison with previous work

In 1981, Heckl presented results for a vertical line force with a uniform stress distribution. The results were obtained by means of a numerical evaluation of an integral which corresponds to the right-hand side of the present Eq. (38) with $x = 0$. Due to the numerical integration used, the validity of the results are not restricted to low frequencies only as in the present case, and many of the conclusions that can be drawn concern slightly higher frequencies. Heckl shows, however, that there is change of signs of the imaginary part of the mobility and that this change occurs at a comparative-

ly low frequency. If, for example, $2b/\lambda_S = 0.01$ (λ_S = shear wavelength), the change of signs occurs at a frequency given by $k\gamma = 0.35$.

A corresponding change of signs can be found from the present material, too. Equations (96) and (98) show that this change should occur at the frequency given by

$$3\omega(1-\nu^2)/[E(k_b\delta)^3] = [\omega(1-\nu^2)/E]f_7(t). \quad (127)$$

With the given value of $2b/\lambda_S$, this equation is satisfied by $k\gamma = 0.52$. It is surprising that this value should be so much higher than the one obtained by Heckl, but it should be borne in mind that $k\gamma = 0.52$ corresponds to $k_b\delta = 1.5$, which means that the frequency is well above the usual limit for the validity of the bending wavenumbers.

D. A survey of the line mobilities

The expressions for the uncoupled line mobilities are summarized in Table II. The results for the case of the rocking moment are taken from a previous paper (Ljunggren, 1987a). The function $f_3(t)$, however, has been recalculated using the exact formulation of Eq. (39) in the cited paper instead of the approximate one of Eq. (41e).

Expression for the rocking motion due to a horizontal force can be found above, see Eqs. (71), (72), and (83).

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