Licentiate Thesis

Predictions of Effective Models in Neutrino Physics

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Abstract

Experiments on neutrino oscillations have confirmed that neutrinos have small, but non-zero masses, and that the interacting neutrino states do not have definite masses, but are mixtures of such states. The seesaw models make up a group of popular models describing the small neutrino masses and the corresponding mixing. In these models, new, heavy fields are introduced and the neutrino masses are suppressed by the ratio between the electroweak scale and the large masses of the new fields. Usually, the new fields introduced have masses far above the electroweak scale, outside the reach of any foreseeable experiments, making these versions of seesaw models essentially untestable. However, there are also so-called low-scale seesaw models, where the new particles have masses above the electroweak scale, but within the reach of future experiments, such as the LHC.

In quantum field theories, quantum corrections generally introduce an energy-scale dependence on all their parameters, described by the renormalization group equations. In this thesis, the energy-scale dependence of the neutrino parameters in two low-scale seesaw models, the low-scale type I and inverse seesaw models, are considered.

Also, the question of whether the neutrinos are Majorana particles, i.e., their own antiparticles, has not been decided experimentally. Future experiments on neutrinoless double beta decay could confirm the Majorana nature of neutrinos. However, there could also be additional contributions to the decay, which are not directly related to neutrino masses. We have investigated the possible future bounds on the strength of such additional contributions to neutrinoless double beta decay, depending on the outcome of ongoing and planned experiments related to neutrino masses.

Keywords: Neutrino mass, lepton mixing, Majorana neutrinos, effective field theory, Weinberg operator, seesaw models, low-scale seesaw models, inverse seesaw, right-handed neutrinos, renormalization group, threshold effects, neutrinoless double beta decay.

Preface

This thesis is divided into two parts. Part I is an introduction to the subjects that form the basis for the scientific papers, while Part II consists of the three papers included in the thesis and listed below.

Part I of the thesis is organized as follows. In Chapter 1, a general introduction to the subject of particle physics is given. Chapter 2 deals with the standard model of particle physics and some simple extensions of it, with emphasis put on neutrino masses and lepton mixing. Chapter 3 gives an overview of the seesaw models, treating in some detail the type I and inverse versions. Chapter 4 introduces the concepts of regularization and renormalization in quantum field theories and discusses renormalization group equations in seesaw models. In Chapter 5, the process of neutrinoless double beta decay and its connection to neutrino masses is briefly reviewed, while Chapter 6 is a short summary of the results and conclusions found in the papers of Part II. Finally, in Appendix A, all the renormalization group equations of the type I seesaw model are given.

Note that Part II of the thesis should not be considered as merely an appendix, but as being part of the main text of the thesis. The papers include discussion and interpretation of the result presented in them. Since simple repetition of this material seems unnecessary, the reader is referred to the papers themselves for the results and the discussion, except for a short summary in Chapter 6. The background material presented in the first five chapters contains both a more broad introduction of the considered topics, as well as a more detailed and technical description of the models and methods considered in the papers. Hence, although there is necessarily some overlap with the corresponding sections in the papers, the more detailed discussion should be of help to the reader unfamiliar with those topics.

List of papers included in this thesis

[1] J. Bergström, M. Malinský, T. Ohlsson, and H. Zhang
Renormalization group running of neutrino parameters in the inverse seesaw model
Physical Review **D81**, 116006 (2010)
`arXiv:1004.4628`

[2] J. Bergström, T. Ohlsson, and H. Zhang
Threshold effects on renormalization group running of neutrino parameters in the low-scale seesaw model
Physics Letters **B698**, 297 (2011)
arXiv:1009.2762

[3] J. Bergström, A. Merle, and T. Ohlsson
Constraining new physics with a positive or negative signal of neutrino-less double beta decay
Journal of High Energy Physics **05**, 122 (2011)
arXiv:1103.3015

List of papers not included in this thesis

[4] J. Bergström and T. Ohlsson
Unparticle self-interactions at the Large Hadron Collider
Physical Review **D80**, 115014 (2009)
arXiv:0909.2213

The contributions of the author of the thesis to the papers

Besides discussing methods, results, and conclusions of all the papers together with the other authors, the main contributions to the articles are

[1] I did a substantial part of the numerical computations, produced many of the plots, and did some of the analytical computations. I revised the manuscript and wrote some parts of it.

[2] I did all the analytical computations and wrote the corresponding sections of the manuscript. The contents of the manuscript and its revisions were decided upon together with the other authors.

[3] I did many of the numerical computations as well as the few analytical calculations which were involved. I wrote some parts of the manuscript and revised it.

Notation and Conventions

The metric tensor on Minkowski space that will be used is

$$(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1) \tag{1}$$

Dimensionful quantities will be expressed in units of $\hbar$ and c. Thus, one can effectively put $\hbar = c = 1$. As a result, both time and length are expressed in units of inverse mass,

$$[t] = T = M^{-1}, \quad [l] = L = M^{-1}.$$

Also, the Einstein summation convention is employed, meaning that repeated indices are summed over, unless otherwise stated.

Erratum

In paper I [1], there are factors of v^2, where v is the vacuum expectation value of the Higgs field, missing in Eq. (32). It should read

$$\begin{aligned} \text{“}\, m_\nu\big|_{M_{i-1}} &\simeq bv^2 \left[\kappa + Y_\nu M_{\mathrm{R}}^{-1} M_S (M_{\mathrm{R}}^T)^{-1} Y_\nu^T\right]\big|_{M_i} + (a-b)v^2\kappa\big|_{M_i} \\ &= bm_\nu\big|_{M_i} + \Delta v^2\kappa\big|_{M_i}\,,\ \text{”}. \end{aligned}$$

Acknowledgments

I would like to thank my supervisor, Prof. Tommy Ohlsson, for his advice and the collaboration which have resulted in the papers included in this thesis, and for making sure that our projects have been completed in good time. Thanks also to He Zhang for the collaboration that led to the papers in Refs. [1] and [2], and for all that I have learned from him during our discussions. Thanks to Alexander Merle for the collaboration leading to the paper in Ref. [3], and for reading this thesis and giving advice on how to improve it. Thanks to all the people I have shared an office with, including Michal Malinský, Henrik Melbéus, Sofia Sivertsson, Martin Heinze, Alexander Ludkiewicz, Martin Sundin, Jonas de Woul, Pedram Hekmati, and to all the other people in the departent, for nice company and interesting discussions.

Special thanks to the people who have had a very big influence on me, given me endless support, and who have made it at all possible for me to complete my thesis, even though they do not understand very much about physics. They are my Mamma and my Pappa, and my grandparents. Also, I want to thank my sister Elsa, my aunts Sabine and Maria, Majsan, Khalid, Jakob, and Olle for many nice weekends and holidays. Thanks also to all my friends for all the good times we have spent together.

Finally, I very much want to thank my Natasha for making me smile every day, for cooking vaary tasty foodie, and for always supporting me.

Johannes Bergström
Stockholm, June 2011

Contents

Part I

Introduction and background material

Chapter 1

Introduction

It is an amazing fact that Nature supplies us with interesting physical phenomena on all accessible scales: from the large size and age of the Universe to the very small distances and time scales associated with heavy elementary particles. Physics, in its most general sense, is the study of the constituents of Nature and their properties.

Since physics is a science, its practitioners should follow the *scientific method*. In its most basic form, it specifies the relation between experiment and theory and how a theory is supposed to be validated or falsified. For a theory to be scientific it must be *falsifiable*, i.e., it must be possible to conduct experiments which disagree with the predictions of the theory. The falsifiability should exist in practise and not only in principle.

The goal of physics is to describe the various phenomena and properties of physical objects with the help of theories or models, which are generally specified and analyzed using mathematics. For a theory to be a valid theory, its predictions should agree with the experimental data collected to date. Also, it should be able to make *new* predictions which can be compared to future experimental data. It is important, however, to realize that the validity of a theory is defined only within a certain range or set of phenomena. A theory can be perfectly valid within one range but not within others. For example, non-relativistic classical mechanics is perfectly valid when all objects have small velocities compared to the speed of light, but not when they are comparable to it. Inherent in the above definition of a theory is the fact that a theory can never be *prooved* in a rigorous or tautological sense as theorems of mathematics can. Instead, it is based on certain assumptions or postulates, the validity of which can only be supported by the agreement of the theory's predictions with experiments.

Elementary particle physics is the study the most fundamental building blocks of the Universe, of which all other objects are composed. The elementary particles are the particles for which there exists no evidence of substructure. Thus, the property of being elementary is not fixed, and particles once thought to be elementary could turn out not to be so in the future. Since, generally, high energies are needed in order to study the elemetary particles, the field also goes under the name of high-energy physics. These highly energetic particles can be created in man-made particle accelerators, but also in natural environments in the Universe, such as stars, galaxies, and supernovae. A very good way to test theories of particle physics is to build machines, particle accelerators, that collide particles together and then observing what comes flying out in what directions and with what energies. One such is the Large Hadron Collider (LHC), which has been built in a circular tunnel 27 kilometers in circumference beneath the French-Swiss border near Geneva, Switzerland. It has at the time of the writing of this thesis been taking data for some time, and when the it becomes fully operational, it will perform proton-proton collision at a center of mass energy of $\sqrt{s} = 14$ TeV.

To be able to do physics at all scales, one needs to use different appropriate descriptions of Nature in different circumstances. The corresponding theory is then an *effective theory*, which needs to capture all the relevant physics, but also needs to disregard all the irrelevant physics. For example, when studying the ballistics of golf balls, one should not have to take into account neither the radius of the Milky Way, nor the mass the mass of the top quark.

More precisely, the common idea is that if there are parameters which are either very large or very small compared to the physical quantities one is interested in, one should set the small parameters to zero and the large to infinity. Hopefully, this will lead to a simpler theory, which can then be used to perform calculation with reasonably good accuracy. If one then wants to improve the accuracy of these calculations, one can include the effects of the large and small parameters by treating them as perturbations about this simple initial analysis.

Another very important concept in science is *Occam's razor*, which in its most basic form states that a theory which makes fewer assumptions is to be preferred over one that makes more. In other words, when choosing between two descriptions of a set of phenomena, one should choose the simpler one over the more complex one. However, when comparing two theories it might not always be clear which of them is simpler, although usually theories having more free parameters can be considered as more complex. Bayesian model selection is a rigorous method to determine which of two models is to be prefered, where the complexity of a model is automatically taken into account. In this approach, the simpler theory will be seleted even if it fits the data somewhat worse. Only if the more complex model fits the data significantly better, will it be prefered [5].

It is in no way self-evident that physical theories should be formulable using mathematics. However, this seems to be an empirical fact. This made Eugene Wigner make his famous comment on "the unreasonable effectiveness of mathematics in the natural sciences"[6]:

> "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for

better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."

Today, the established theory of the Universe on its most fundamental level is the *standard model* (SM) of particle physics. It describes all known fundamental particles and how they interact with each other, except for the gravitational interaction. It has been tested to great precision in a very large amount of experiments and has been found to be a good description of fundamental particles and their interactions at energies probed so far [7]. Since its formulation in the 1970's, it has (almost) remained unmodified. During its life, it has made a vast number of predictions which have later been confirmed by experiments. This includes the existence of new particles such as the Z- and W-bosons, the top quark, and the tau neutrino. The only part of the standard model yet to be confirmed is the existence of the Higgs boson, which is related to the mechanism of generating the masses of the particles in the SM.

Gravity is not included in the SM, but is instead treated separately, usually using the general theory of relativity. Note the standard model is a *quantum* theory, while general relativity is inherently classical. Although it would be pleasant to have the SM and gravity unified in a full quantum theory, most such attempt lack testability, which is due to the fact that they only make unique predictions for processes at energies much higher than will ever be possible to study. Also, cosmological obervations indicate that there exist large amounts of massive partices in the Universe that have not been detected. Since none of the particles in the SM can consitute this *dark matter*, one expects that there are new particles waiting to be discovered in the future.

Chapter 2

The standard model of particle physics and slightly beyond

The standard model (SM) of particle physics is the currently accepted theoretical framework for the description of the elementary particles and their interactions. It has been tested to great precision in a very large amount of experiments and has been found to be a good description of fundamental particles and their interactions at energies probed so far [7].

In this chapter an introduction to the SM is given. Emphasis is put on those aspects of the SM which are most relevant for the topics dealt with later in the thesis, i.e., the lepton sector in general and neutrino masses and lepton mixing in particular. First, the concept of a quantum field theory is introduced, followed by a review of the construction of the SM. Then, general fermion mass terms and the principles of effective quantum field theories are reviewed. Quark and lepton masses and mixing are treated, followed by the experimental consequences of massive neutrinos and, finally, the discussion also goes slightly beyond the SM by including right-handed neutrinos. For reviews and deeper treatments of the SM, see, e.g., Refs. [8–12].

2.1 Quantum field theory

A classical field is a function associating some quantity to each point of space-time, and is an object with an infinite number of dynamical degrees of freedom. The SM is a *quantum field theory* (QFT), and as such it deals with the quantum mechanics of fields. Basically, this means that the classical fields are quantized, i.e., are promoted to operators. A classical field theory can be specified by a *Lagrangian* *density* $\mathcal{L}$ (usually just called the Lagrangian), which is a function of the collection of fields $\Phi = \Phi(x)$, i.e., $\mathcal{L} = \mathcal{L}(\Phi(x))$. The *action* functional is given by

$$S[\Phi] = \int \mathcal{L}\, \mathrm{d}^D x \tag{2.1}$$

and gives the dynamics of the fields through the *Euler–Lagrange* equations of motion.

The QFTs we will study will basically also be defined by a Lagrangian. However, in QFT, one is not interested in the values of the fields themselves, which are not well-defined, but instead other quantities such as correlation functions and S-matrix elements. From these one can then calculate observable quantities such as cross-sections and decay rates of particles associated with the fields.

Symmetries and symmetry arguments have played and still play an important role in physics in general, and in QFT in particular. One important class of symmetries are *space-time symmetries*, which are symmetries involving the space-time coordinates. The QFTs we will consider will all be relativistic QFTs, meaning that the Lorentz group is a symmetry group of the theory. This implies that the fields we consider have to transform under some representation of the Lorentz group. The lowest dimensional representations correspond to the most commonly used types of fields,

- A *scalar field* has spin 0,
- A *spinor field* has spin 1/2,
- A *vector field* has spin 1.

Another kind of symmetries are *internal symmetries*, which are symmetries only involving the dynamical degrees of freedom (here the fields), and not the space-time coordinates. A very important and useful class of such symmetries are the *gauge symmetries*, which will be the main principle behind the construction of the SM. Finally, the theories we consider will be *local*, which basically means that the Lagrangian is a local expression is the fields, i.e., that it only depends on the fields at a single space-time point.

The terms in the Lagrangian are usually classified as either

- A *kinetic term*, which is quadratic in a single field and involves derivatives,
- A *mass term*, which is quadratic in a single field and does not involve derivatives, or
- An *interaction term*, which involves more than two fields.

A constant term in the Lagrangian would essentially correspond to an energy density of the vacuum or a cosmological constant. Since this term is usually irrelevant for particle physics, it will not be discussed any further. Finally, there could also be terms linear in a field (only for a scalar which is also a singlet under all other symmetries), implying that the minimum of the classical Hamiltonian is not at zero field value. Since these fields do not appear in the models we consider, neither this will be further mentioned.

2.2 Basic structure of the standard model

The SM is a gauge theory, and as such its form is dictated by the principle of gauge invariance. A gauge theory is defined by specifying the gauge group, the fermion and scalar particle content, and their representations. The gauge group for the SM is given by

$$G_{\mathrm{SM}} = SU(3)_{\mathrm{C}} \otimes SU(2)_{\mathrm{L}} \otimes U(1)_{\mathrm{Y}}, \tag{2.2}$$

which is a twelve-dimensional *Lie group*. Here $SU(3)_{\mathrm{C}}$ is the eight-dimensional gauge group of Quantum Chromodynamics (QCD), where the subscript stands for "color", which the corresponding quantum number is called. The group $SU(2)_{\mathrm{L}} \otimes U(1)_{\mathrm{Y}}$ is the four-dimensional gauge group of the Glashow–Weinberg–Salam model of weak interactions [13–15]. As will be described later, only the left-handed fermions are charged under the $SU(2)_{\mathrm{L}}$ subgroup, and hence the subscript "L". The symbol "Y" represents the weak hypercharge. We will now proceed to describe the particles of the SM and their interactions.

2.2.1 The gauge bosons

The part of the SM Lagrangian containing the kinetic terms as well as the self-interactions of the gauge fields is determined by gauge invariance and is given by

$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \tag{2.3}$$

where $a \in \{1, 2, \ldots, 8\}$, $i \in \{1, 2, 3\}$, and the field strength tensors are given in terms of the gauge fields as

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2.4}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \varepsilon^{ijk} W^j_\mu W^k_\nu, \tag{2.5}$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_3 f^{abc} A^b_\mu A^c_\nu. \tag{2.6}$$

Here, (g_2, ε^{ijk}) and (g_3, f^{abc}) are the coupling and structure constants of $SU(2)_{\mathrm{L}}$ and $SU(3)_{\mathrm{C}}$, respectively. Note that gauge invariance excludes the possibility of a mass term for the gauge fields, and thus, the gauge bosons are massless. This is a problem, since some gauge bosons, i.e., the W-bosons and the Z-boson, are observed to be massive [7]. To incorporate massive gauge bosons, the gauge symmetry has to be broken in some way. This can, for example, be done through spontaneous symmetry breaking, in which case the fundamental Lagrangian, but not the vacuum, respects the symmetry.

2.2.2 The fermions

The next step in the construction of the SM is the introduction of the fermions and the specification of their charges. The fermions of the SM come in two groups, called *quarks* and *leptons*, which in turn come in three *generations* each. First, for any Lorentz vector Q^μ, define

$$\not{Q} \equiv \gamma^\mu Q_\mu. \tag{2.7}$$

Then, given the representations of the fermion ψ, the kinetic term and the interactions with the gauge bosons are determined by the requirement of gauge invariance and are given by

$$\mathcal{L}_\psi = \mathrm{i}\overline{\psi}\not{D}\psi. \tag{2.8}$$

Here $\overline{\psi} = \psi^\dagger\gamma^0$,

$$D_\mu = \partial_\mu - \mathrm{i}g_1 B_\mu Y - \mathrm{i}g_2 W^i_\mu \tau^i - \mathrm{i}g_3 G^a_\mu t^a \tag{2.9}$$

is the covariant derivative, g_1 is the coupling constant of $U(1)_\mathrm{Y}$, Y the hypercharge of ψ, the τ^i's the representation matrices of $SU(2)_\mathrm{L}$, and the t^a's the representation matrices of $SU(3)_\mathrm{C}$. Note that the hypercharge and representation matrices depend on which fermion is being considered, and that, if ψ is a singlet under some subgroup, then the generator of that group is zero when acting on ψ.

The fermion fields in the SM all have definite *chirality*, meaning that they transform under two different representations of the Lorentz group. The two different kinds of chirality are *left-handed* or *right-handed*, as denoted by the subscripts "L" and "R". The quark fields are organized as

$$q_{\mathrm{L}i} = \begin{pmatrix} u_{\mathrm{L}i} \\ d_{\mathrm{L}i} \end{pmatrix},\ u_{\mathrm{R}i},\ d_{\mathrm{R}i},$$

where $i \in \{1,2,3\}$ is the generation index. They are all in the fundamental representation of $SU(3)_\mathrm{C}$, while the left-handed $q_{\mathrm{L}i}$'s are doublets and the right-handed $u_{\mathrm{R}i}$'s and $d_{\mathrm{R}i}$'s are singlets of $SU(2)_\mathrm{L}$. The lepton fields are all singlets of $SU(3)_\mathrm{C}$ and organized as

$$\ell_{\mathrm{L}i} = \begin{pmatrix} \nu_{\mathrm{L}i} \\ e_{\mathrm{L}i} \end{pmatrix},\ e_{\mathrm{R}i},$$

where the $\ell_{\mathrm{L}i}$'s are doublets and the $e_{\mathrm{R}i}$'s are singlets of $SU(2)_\mathrm{L}$. For both quarks and leptons, the names assigned to the components of the doublets correspond to the names of the fields which appear in the Lagrangian after the electroweak symmetry has been broken.

In order to restore the symmetry between the quark and lepton fields, one can also introduce the right-handed neutrinos $\nu_{\mathrm{R}i}$ in the list. However, they would be total singlets of the SM gauge group and are not needed to describe existing experimental data, and should thus be excluded in a minimal model.[1] The hypercharges

[1] The other right-handed fermions are seen directly in the interactions with the gauge bosons, since they are not gauge singlets. Also, they are required for describing the masses of these fermions.

of all the fermions are given in such a way that the correct electric charges are assigned after the spontaneous breaking of the gauge symmetry.

The rest of this thesis will mostly be concerned with the electroweak sector of the SM, while QCD will not be discussed in detail. The electroweak interactions can largely be studied separately from QCD, since $SU(3)_\mathrm{C}$ remains unbroken and there is no mixing between the gauge fields of $SU(3)_\mathrm{C}$ and $SU(2)_\mathrm{L} \otimes U(1)_\mathrm{Y}$.

This concludes the introduction of the basic structure of the SM. However, in order to give a good description of experimental data, the gauge symmetry of the SM needs to be broken. Before the description of this breaking, a short summary of the different kinds of possible mass terms for fermions will be given.

2.3 Fermion mass terms

There are in general two types of mass terms for a fermion ψ that can be constructed, both giving the same kinematical masses. The first one is called a *Dirac mass term*, and has the form

$$-\mathcal{L}_\mathrm{Dirac} = m\overline{\psi}\psi. \tag{2.10}$$

However, the chiral fields included in the SM satisfy

$$\overline{\psi_\mathrm{L/R}}\,\psi_\mathrm{L/R} = 0 \tag{2.11}$$

due to the definition of chirality and $\overline{\psi}$, and thus terms on the form of Eq. (2.10) vanish for all the fields of the SM. One could try to remedy this by defining a new field

$$\chi \equiv \psi_\mathrm{L}^{(1)} + \psi_\mathrm{R}^{(2)}, \tag{2.12}$$

but in the SM, the left-handed and right-handed fields transform under different representations of $SU(2)_\mathrm{L}$, and thus the resulting mass term,

$$m\overline{\chi}\chi = m\left(\overline{\psi_\mathrm{L}^{(1)}}\psi_\mathrm{R}^{(2)} + \overline{\psi_\mathrm{R}^{(2)}}\psi_\mathrm{L}^{(1)}\right), \tag{2.13}$$

will not be gauge invariant.

To construct the second type of fermion mass term, called a *Majorana mass term*, one first would need to introduce the *charge conjugation operator* as

$$\hat{C} : \psi \to \psi^c = C\overline{\psi}^T, \tag{2.14}$$

where the matrix C satisfies

$$C^\dagger = C^T = C^{-1} = -C. \tag{2.15}$$

The Majorana mass term is then given by

$$-\mathcal{L}_\mathrm{Majorana} = \frac{1}{2}m\overline{\psi^c}\psi + \mathrm{H.c.}, \tag{2.16}$$

where "H.c." denotes the Hermitian conjugate, and m can always be made real and positive by redefining the phase of ψ. However, this kind of mass term is also not gauge invariant, unless ψ is a gauge singlet. Any Abelian charges of ψ will be broken by two units. If the fermions ψ is chiral, as in the SM, the mass term can be rewritten as

$$-\mathcal{L}_{\text{Majorana}} = \frac{1}{2} m \overline{\xi} \xi, \tag{2.17}$$

where $\xi \equiv \psi + \psi^c$ is called a *Majorana field*, since it obeys $\xi^c = \xi$, called the *Majorana condition*. After field quantization, the Majorana condition on the field ξ will imply the equality of the particle and antiparticle states. Since a Majorana field has only half the independent components of the Dirac field, a theory with the former is simpler and more economical than one with the latter.

In conclusion, none of the fermion fields in the *unbroken* SM can have a mass term, and thus, all SM fermions are massless. The only possible exception is the right-handed neutrino, which is a gauge singlet and can hence have a Majorana mass term. This is a problem, since the fermions existing in Nature are observed to be massive.[2]

2.4 The scalar sector and the Higgs mechanism

In order to make the model described above consistent with experiments, one needs to introduce some mechanism to break the SM gauge symmetry in such a way to give the fermions and three of the gauge bosons masses. In the SM, this is achieved through the *Higgs mechanism* [16–21]. It is implemented by introducing one complex scalar $SU(2)_{\text{L}}$ doublet ϕ, called the *Higgs field*, which is described by the Lagrangian

$$\mathcal{L}_{\text{scalar}} = |D_\mu \phi|^2 - V(\phi), \tag{2.18}$$

where the *scalar potential* is given by

$$V(\phi) = -\mu^2 |\phi|^2 + \frac{\lambda}{4} |\phi|^4. \tag{2.19}$$

If $\mu^2 > 0$, the minimum of the potential will not be at $\phi = 0$, but instead where

$$v \equiv |\phi| = \sqrt{\frac{2\mu^2}{\lambda}}, \tag{2.20}$$

which is called the *vacuum expectation value* (VEV) and experimentally determined to have a value of approximately 174 GeV. Under standard conventions, the vacuum is such that

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{2.21}$$

[2]The exceptions are the neutrinos, the masses of which have not been measured directly. However, the evidence for neutrino oscillations, to be discussed later, requires that they have small, but non-zero masses.

breaking electroweak gauge invariance. As it turns out, this generates mass terms for the electroweak gauge bosons such that there are three massive gauge fields:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp \mathrm{i}W_\mu^2), \quad \text{with masses } m_W = g_2\frac{v}{\sqrt{2}}, \tag{2.22}$$

$$Z_\mu = \frac{1}{\sqrt{g_2^2+g_1^2}}(g_2 W_\mu^3 - g_1 B_\mu), \quad \text{with mass } m_Z = \sqrt{g_2^2+g_1^2}\frac{v}{\sqrt{2}}, \tag{2.23}$$

and

$$A_\mu = \frac{1}{\sqrt{g_2^2+g_1^2}}(g_1 W_\mu^3 + g_2 B_\mu), \quad \text{with mass } m_A = 0. \tag{2.24}$$

The fields $W_\mu^\pm$, Z_μ, and A_μ are identified as the fields associated with the W-bosons, the Z-boson, and the photon, respectively.

The Higgs mechanism accomplishes the breaking

$$SU(2)_\mathrm{L} \otimes U(1)_\mathrm{Y} \to U(1)_\mathrm{QED}, \tag{2.25}$$

where $U(1)_\mathrm{QED}$ is the gauge group of Quantum Electrodynamics (QED). The quantity Q is the fermion electric charge quantum number, i.e., the electric charge of a given fermion in units of the proton charge e, given as

$$Q = T^3 + Y, \qquad e = \frac{g_1 g_2}{\sqrt{g_2^2+g_1^2}}, \tag{2.26}$$

where T^3 is the third component of the $SU(2)_\mathrm{L}$ weak isospin. These assignment give the usual QED couplings of the fermions to the photon field, while the interactions with the W-bosons, i.e., the *charged-current* interactions, are given by[3]

$$\mathcal{L}_\mathrm{cc} = \frac{g_2}{\sqrt{2}} W_\mu^+ \overline{u_\mathrm{L}}\gamma^\mu d_\mathrm{L} + \frac{g_2}{\sqrt{2}} W_\mu^+ \overline{\nu_\mathrm{L}}\gamma^\mu e_\mathrm{L} + \text{H.c.} \tag{2.27}$$

The introduction of a scalar field also opens up the possibility of further interactions with fermions through *Yukawa interactions*, having the form

$$-\mathcal{L}_\mathrm{Yuk} = \overline{\ell_\mathrm{L}}\phi Y_e e_\mathrm{R} + \overline{q_\mathrm{L}}\phi Y_d d_\mathrm{R} + \overline{q_\mathrm{L}}\tilde{\phi} Y_d d_\mathrm{R} + \text{H.c.} \tag{2.28}$$

Here $\tilde{\phi} = \mathrm{i}\tau_2\phi^*$, where τ_2 is the second Pauli matrix, each fermion field is a vector consisting of the corresponding field from each generation, and Y_f for $f = e, u, d$ are Yukawa coupling matrices. When the Higgs field acquires its VEV, Dirac mass terms

$$-\mathcal{L}_\mathrm{mass} = \overline{e_\mathrm{L}} M_e e_\mathrm{R} + \overline{u_\mathrm{L}} M_u u_\mathrm{R} + \overline{d_\mathrm{L}} M_d d_\mathrm{R} + \text{H.c.}, \tag{2.29}$$

are generated. Here the mass matrices

$$M_f = Y_f v \tag{2.30}$$

for $f = e, u, d$ are arbitrary complex 3×3 matrices, and as such are in general not diagonal. In this case, the *flavor eigenstates*, which are the states participating

[3]There will also be interactions with the Z-boson, the neutral current interactions, which will, however, not be discussed any further.

in the weak interactions, are not the same as the *mass eigenstates*, which are the states which propagate with definite masses. For n fermion generations, one would expect that each of the matrices Y_f contains n^2 complex , or $2n^2$ real, parameters. However, not all these parameters are physical, a point which will be discussed later.

Finally, it should be noted that one real degree of freedom of the Higgs fields is left as a physical field after the breaking of electroweak symmetry. The quantum of this field is called the *Higgs boson* and is the only particle in the SM yet to be experimentally confirmed. One of the main goals of the Large Hadron Collider is to study the mechanism of electroweak symmetry breaking, and, if the Higgs mechanism is an accurate description, find the Higgs boson.

2.5 Effective field theory

So far, only terms in the Lagrangian with a small number of fields have been considered. From the kinetic term of a field, one can calculate the mass dimension of the field. This is because, in natural units, the action in Eq. (2.1) is dimensionless. Denote the mass dimension of X as $[X]$. Then, if space-time is D-dimensional, the Lagrangian has to have the mass dimension D, since $[\,\mathrm{d}^D x] = -D$. Since $[\partial_\mu] = 1$, one obtains for the case $D = 4$

$$[\phi] = [A^\mu] = 1, \tag{2.31}$$

$$[\psi] = \frac{3}{2}. \tag{2.32}$$

From this, the mass dimensions of the constants multiplying all other terms in the Lagrangian can then be determined by the fact that the total mass dimension is 4.

It is now possible to further classify the interaction terms according to the mass dimension of the corresponding coupling constant. Field theory textbooks usually argue that a QFT should be "renormalizable", meaning that all divergences appearing should be possible to cancel with a finite number of counter terms. One can show that this is equivalent to having coupling constants with only non-negative mass dimensions, or equivalently, that the combinations of the fields in all terms in the Lagrangian have total mass dimension not greater than the space-time dimensionality. Otherwise, one needs an infinite number of counter terms, and hence, an infinite number of unknown parameters, resulting in loss of predictive power of the theory.

An *effective* field theory Lagrangian, on the other hand, contains an infinite number of terms

$$\mathcal{L}_{\mathrm{EFT}} = \mathcal{L}_D + \mathcal{L}_{D+D_1} + \mathcal{L}_{D+D_2} + \cdots , \tag{2.33}$$

where $\mathcal{L}_D$ is the renormalizable Lagrangian, $\mathcal{L}_{D+D_i}$ contains terms of dimension $D + D_i$, and $0 < D_1 < D_2 < \cdots$. Although there is an infinite number of terms in $\mathcal{L}_{\mathrm{EFT}}$, one still has *approximate* predictive power. The coupling constants in $\mathcal{L}_{D+D'}$

have the form $g\Lambda^{-D'}$, where g is dimensionless and Λ is some energy scale. The amplitude resulting from this interaction will then be proportional to $g(E/\Lambda)^{D'}$, and potentially suppressed additionally by loop factors. Thus, one can perform computations for processes at some scale $E < \Lambda$ with an error of order $g(E/\Lambda)^{D'}$ if one keeps terms up to $\mathcal{L}_{D+D'}$ in $\mathcal{L}_{\mathrm{EFT}}$. Thus, an effective field theory is just as useful as a renormalizable one, as long as one is satisfied with a certain finite accuracy of the predictions.[4] This also means that the leading contributions for a given process at low energies are induced by the operators of lowest dimensionality.

Given a renormalizable field theory involving a heavy field of mass M, one can integrate out the heavy field from the generating functional to produce an effective theory with an effective Lagrangian below M, consisting of a tower of effective operators. For example, in QED, one can integrate out the electron field to produce an effective Lagrangian, the Euler–Heisenberg Lagrangian

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{a}{m_e^4}\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{b}{m_e^4}F^{\mu\nu}F_{\nu\sigma}F^{\sigma\rho}F_{\rho\mu} + \mathcal{O}\left(\frac{F^6}{m_e^8}\right), \quad (2.34)$$

where the dimensionless constants a and b can be found explicitly in terms of the electromagnetic coupling constant. However, even if one has no idea of what the high-energy theory is, one can still write down this unique Lagrangian (with unknown a and b, treated as free parameters) by simply imposing Lorentz, gauge, charge conjugation, and parity invariance. In other words, the only effect of the high-energy theory is to give explicit (and possibly correlated, as functions of the high-energy parameters) values of the coupling constants in the low-energy theory.

Also, note that perturbative renormalization of effective operators can be performed in the same way as for those usually called "renormalizable", as long as one chooses the renormalization scheme wisely and works to a given order in E/Λ. In other words, to a given order in E/Λ, the effective theory contains only a finite number of operators, and working to a given accuracy, the effective theory behaves for all practical purposes like a renormalizable quantum field theory: only a finite number of counter terms are needed to reabsorb the divergences [22]. For deeper treatments of effective field theory, see Refs. [22–25].

In conclusion, the Lagrangian of the SM can actually be considered to contain terms of arbitrary dimensionality, of which the usual renormalizable SM Lagrangian is the lowest order low-energy approximation. The allowed terms are given by the requirements of gauge and Lorentz invariance and any other assumed symmetries. A dimension-five operator, which gives the lowest-order contribution to neutrino masses, will be discussed in Section 2.7.1.

[4] The accuracy is in practise almost always finite anyway if one is using perturbation theory.

2.6 Quark masses and mixing

The mass terms for quarks in Eq. (2.29), i.e.,

$$-\mathcal{L}_{\text{q-mass}} = \overline{u_{\mathrm{L}}} M_u u_{\mathrm{R}} + \overline{d_{\mathrm{L}}} M_d d_{\mathrm{R}} + \text{H.c.}, \tag{2.35}$$

couple different quark flavors to each other, i.e., they mix the quark flavors. To find the mass eigenstate fields, i.e., the fields of which the excitations are propagating states, define a new basis of the quark fields by

$$u_{\mathrm{L}} = U_{\mathrm{L}} u'_{\mathrm{L}}, \quad u_{\mathrm{R}} = U_{\mathrm{R}} u'_{\mathrm{R}}, \quad d_{\mathrm{L}} = V_{\mathrm{L}} d'_{\mathrm{L}}, \quad d_{\mathrm{R}} = V_{\mathrm{R}} d'_{\mathrm{R}}, \tag{2.36}$$

where $U_{\mathrm{L}}, U_{\mathrm{R}}, V_{\mathrm{L}},$, and V_{R} are some unitary 3×3 matrices. The kinetic terms of the quark fields are still diagonal in this new basis. Then, choose the unitary matrices such that

$$U_{\mathrm{L}}^{\dagger} M_u U_{\mathrm{R}} = D_u, \quad V_{\mathrm{L}}^{\dagger} M_d V_{\mathrm{R}} = D_d, \tag{2.37}$$

where D_u and D_d are real, positive, and diagonal. This choice of unitary matrices is possible for any complex matrices M_u and M_d. Thus, the fields

$$u'_i = u'_{\mathrm{L}i} + u'_{\mathrm{R}i}, \quad d'_i = d'_{\mathrm{L}i} + d'_{\mathrm{R}i} \tag{2.38}$$

are Dirac mass eigenstate fields with masses $m_{u,i} = (D_u)_{ii}$ and $m_{d,i} = (D_d)_{ii}$, respectively.

However, the interactions of the quarks with the gauge bosons originating from Eq. (2.27) will not be diagonal anymore, but instead be given by

$$\begin{aligned} \mathcal{L}_{Wud} &= \frac{g_2}{\sqrt{2}} W_\mu^+ \overline{u_{\mathrm{L}}} \gamma^\mu d_{\mathrm{L}} + \text{H.c.} \\ &= \frac{g_2}{\sqrt{2}} W_\mu^+ \overline{u'_{\mathrm{L}}} U_{\mathrm{L}}^{\dagger} V_{\mathrm{L}} \gamma^\mu d'_{\mathrm{L}} + \text{H.c.} \\ &= \frac{g_2}{\sqrt{2}} W_\mu^+ \overline{u'_{\mathrm{L}}} U_{\mathrm{CKM}} \gamma^\mu d'_{\mathrm{L}} + \text{H.c.}, \end{aligned} \tag{2.39}$$

where the unitary matrix $U_{\mathrm{CKM}} = U_{\mathrm{L}}^{\dagger} V_{\mathrm{L}}$ is the *Cabibbo-Kobayashi-Maskawa* (CKM) or *quark mixing* matrix [26, 27].

A general unitary $n \times n$ matrix has n^2 real parameters of which $n(n-1)/2$ are mixing angles and $n(n+1)/2$ are phases. However, by rephasing the left-handed quark fields, one can remove $(2n-1)$ phases of the CKM matrix. If one then rephases the right-handed quark fields in the same way, the Lagrangian will be left invariant. This means that these phases of the quark fields are not observable, and that neither are the removed phases from the CKM matrix. To conclude, the number of physical parameters for n quark generations are $2n$ masses, $n(n-1)/2$ angles and $(n-1)(n-2)/2$ phases. The total number of physical parameters of the quark sector is thus (n^2+1), which is to be compared to the naive expectation of $2n^2$ for each Yukawa matrix, i.e., $4n^2$ in total.

The CKM matrix can be parametrized in many ways, but the standard parametrization is, for three generations, given by

$$\begin{aligned} U_{\mathrm{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13}e^{-\mathrm{i}\Delta} \\ 0 & 1 & 0 \\ -S_{13}e^{\mathrm{i}\Delta} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-\mathrm{i}\Delta} \\ -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{\mathrm{i}\Delta} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{\mathrm{i}\Delta} & S_{23}C_{13} \\ S_{12}S_{23} - C_{12}C_{23}S_{13}e^{\mathrm{i}\Delta} & -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{\mathrm{i}\Delta} & C_{23}C_{13} \end{pmatrix}, \end{aligned} \tag{2.40}$$

where $C_{ij} = \cos\Theta_{ij}$ and $S_{ij} = \sin\Theta_{ij}$, Θ_{12}, Θ_{23}, and Θ_{13} are the quark mixing angles, and Δ is the CP-violating phase. The values of the quark mixing parameters have been inferred from experiments and the mixing angles have been found to be relatively small [7].

2.7 Lepton masses and mixing

The mass term for the charged leptons in Eq. (2.29), i.e.,

$$-\mathcal{L}_{\text{e-mass}} = \overline{e_L} M_e e_{\mathrm{R}} + \text{H.c.}, \tag{2.41}$$

can be diagonalized in the same way as the down quark mass term by defining

$$e_{\mathrm{L}} = V_{\mathrm{L}} e'_{\mathrm{L}}, \quad e_{\mathrm{R}} = V_{\mathrm{R}} e'_{\mathrm{R}}, \tag{2.42}$$

where V_{L} and V_{R} are unitary matrices such that

$$V_{\mathrm{L}}^\dagger M_e V_{\mathrm{R}} = D_e, \tag{2.43}$$

where D_e is real, positive, and diagonal. Then,

$$e'_i = e'_{\mathrm{L}i} + e'_{\mathrm{R}i} \tag{2.44}$$

are Dirac fields with masses $m_{e,i} = (D_e)_{ii}$. Also note that V_{L} diagonalizes $M_e M_e^\dagger$ and V_{R} diagonalizes $M_e^\dagger M_e$, i.e.,

$$V_{\mathrm{L}}^\dagger \left(M_e M_e^\dagger\right) V_{\mathrm{L}} = V_{\mathrm{R}}^\dagger \left(M_e^\dagger M_e\right) V_{\mathrm{R}} = D_e^2, \tag{2.45}$$

and that similar relations hold for the quark mass matrices M_u and M_d. If neutrinos would be massless, i.e., have no mass terms, one could define the rotated neutrino fields by

$$\nu_{\mathrm{L}} = V_{\mathrm{L}} \nu'_{\mathrm{L}}, \tag{2.46}$$

in which case the charged current interaction in Eq. (2.27) would still be diagonal.

2.7.1 Neutrino masses without right-handed neutrinos

In the SM, the left-handed neutrinos do not obtain their masses through Yukawa interactions as the other fermions do. This is because there is no need for the introduction of right-handed neutrinos to describe experimental data, and hence, in the spirit of simplicity, there are no such fields in the SM to which the neutrinos could couple.

The minimal, most simple, and most economical way to describe neutrino masses in the SM is based on an effective operator of dimension five (cf. Section 2.5), sometimes called the *Weinberg operator* [28]. It is given by

$$-\mathcal{L}_\nu^{d=5} = \frac{1}{2}\left(\overline{\ell_{\rm L}}\phi\right)\kappa\left(\phi^T\ell_{\rm L}^c\right) + \text{H.c.}, \tag{2.47}$$

and is the only dimension-five operator allowed by the SM symmetries. Here, κ is a complex 3×3 matrix having the dimension of an inverse mass. However, using the anticommutativity of the fermion fields, one can show that only the symmetric part of κ is physically relevant, i.e., κ can always be chosen to be symmetric. However, since the Higgs field acquires a VEV as in Eq. (2.21), the term in Eq. (2.47) will lead to a Majorana mass term for the light neutrinos,

$$-\mathcal{L}_{\text{Maj,L}} = \frac{1}{2}\overline{\nu_{\rm L}}M_{\rm L}\nu_{\rm L}^c + \text{H.c.}, \tag{2.48}$$

with $M_{\rm L} = v^2\kappa$. Just as the mass matrices considered previously, $M_{\rm L}$ is in general not diagonal. One can redefine the neutrino fields as

$$\nu_{\rm L} = U_{\rm L}\nu'_{\rm L}, \tag{2.49}$$

where $U_{\rm L}$ is chosen such that

$$U_{\rm L}^\dagger M_{\rm L} U_{\rm L}^* = D_{\rm L}, \tag{2.50}$$

with $D_{\rm L}$ real, positive, and diagonal. This is always possible for symmetric $M_{\rm L}$, a well-known theorem in linear algebra. Equations (2.42) and (2.49) imply that the interaction in Eq. (2.27) takes the form

$$\begin{aligned}
\mathcal{L}_{W\nu e} &= \frac{g_2}{\sqrt{2}}W_\mu^+\overline{\nu_{\rm L}}\gamma^\mu e_{\rm L} + \text{H.c.} \\
&= \frac{g_2}{\sqrt{2}}W_\mu^+\overline{\nu'_{\rm L}}U_{\rm L}^\dagger V_{\rm L}\gamma^\mu e'_{\rm L} + \text{H.c.} \\
&= \frac{g_2}{\sqrt{2}}W_\mu^+\overline{\nu'_{\rm L}}U^\dagger\gamma^\mu e'_{\rm L} + \text{H.c.},
\end{aligned} \tag{2.51}$$

where $U = V_{\rm L}^\dagger U_{\rm L}$ is the *lepton mixing matrix*, also referred to as the *Pontecorvo-Maki-Nakagawa-Sakata* (PMNS) matrix [29–31]. The difference to the quark sector is that the Majorana mass term is not invariant under rephasings of the mass eigenstate fields. Thus, the phases of the Majorana neutrino fields are physical and cannot be removed from the lepton mixing matrix. It follows that there are $(n-1)$ additional physical phases for n generations. The lepton mixing matrix is usually parametrized as

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{diag}\left(e^{i\rho}, e^{i\sigma}, 1\right) \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\rho} & & \\ & e^{\mathrm{i}\sigma} & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13}e^{\mathrm{i}\rho} & s_{12}c_{13}e^{\mathrm{i}\sigma} & s_{13}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23}e^{\mathrm{i}\rho} - c_{12}s_{23}s_{13}e^{\mathrm{i}(\delta+\rho)} & c_{12}c_{23}e^{\mathrm{i}\sigma} - s_{12}s_{23}s_{13}e^{\mathrm{i}(\delta+\sigma)} & s_{23}c_{13} \\ s_{12}s_{23}e^{\mathrm{i}\rho} - c_{12}c_{23}s_{13}e^{\mathrm{i}(\delta+\rho)} & -c_{12}s_{23}e^{\mathrm{i}\sigma} - s_{12}c_{23}s_{13}e^{\mathrm{i}(\delta+\rho)} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \tag{2.52}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, θ_{12}, θ_{23}, and θ_{13} are the lepton mixing angles, δ the CP-violating Dirac phase, and σ and ρ CP-violating Majorana phases. However, note that there exists different parametrizations, differing in the convention for the CP-violating phases. Hence, excluding the well-measured charged lepton masses, there is a maximum of 9 parameters in the lepton sector of the SM, separated as 3 neutrino masses, 3 mixing angles, and 3 CP-violating phases.

In conclusion, the SM can incorporate massive neutrinos, while also indicating that they should be light, a reflection of the fact that the first tree-level mass term has a dimension equal to five and not less. Whatever high-energy theory one can come up with, it always reduces to the SM with the Weinberg operator at low energies (unless it is forbidden by some exact symmetry of the high-energy theory, or if the neutrinos have a Dirac mass term, cf. Sec. 2.7.2). If it does not (to a good approximation), it has been ruled out by experiments. However, writing κ as

$$\kappa = \frac{\tilde{\kappa}}{\Lambda_\nu}, \tag{2.53}$$

with $\tilde{\kappa}$ dimensionless and Λ_ν some energy scale, we have that, since $v \simeq 174$ GeV,[5]

$$\Lambda_\nu \simeq \frac{v^2}{m_\nu}\tilde{\kappa} \simeq 3 \cdot 10^{13}\ \mathrm{GeV}\left[\frac{\mathrm{eV}}{m_\nu}\right]\tilde{\kappa}. \tag{2.54}$$

Since experiments indicate that the neutrino mass scale m_ν is of the order of 1 eV or smaller, this will imply that the scale Λ_ν will be very high (unless $\tilde{\kappa}$ is very small), out of reach of any foreseeable experiments. The scale Λ_ν could be within the reach of future experiments, if either $\tilde{\kappa}$ is very small, which can be natural,

[5] Since we are dealing with matrices, the eigenvalues of which are the physical masses, the individual components of M_{L} could be much larger than the eigenvalues if there are large cancellations.

or if for some reason the Weinberg operator is forbidden and neutrino masses are instead the result of even higher-dimensional operators [32].

Finally, we note that this operator necessarily introduces one more mass scale into the theory, above which the effective description ceases to be valid. Hence, one can actually say with great certainty that at some high-energy scale, some kind of new degrees of freedom should start to make themselves apparent. This is the reason that the evidence of neutrino masses is usually referred to as evidence for "physics beyond the SM". Of course, one expects the SM to not be a good description at arbitrarily large energy scales, and the evidence of new physics to first appear as the effect of effective operators. Not only is the Weinberg operator the only dimension-five operator allowed by the SM symmetries, it is also the only higher-dimensional operator which is required to have a non-zero coefficient, i.e., the only one for which a positive value has been inferred.

2.7.2 Neutrino masses with right-handed neutrinos

In order to restore the symmetry in the particle content of the SM, one can choose to extend it by adding 3 right-handed neutrinos $\nu_{\mathrm{R}i}$[6], often also denoted by $N_{\mathrm{R}i}$ or just N_i. Then, a new set of Yukawa couplings are allowed,

$$-\mathcal{L}_{\mathrm{Yuk},\nu} = \overline{\ell_{\mathrm{L}}}\tilde{\phi}Y_\nu \nu_{\mathrm{R}} + \mathrm{H.c.}, \tag{2.55}$$

which after electroweak symmetry breaking yields a Dirac-type mass terms as

$$-\mathcal{L}_{\mathrm{Dirac},\nu} = \overline{\nu_{\mathrm{L}}}M_{\mathrm{D}}\nu_{\mathrm{R}} + \mathrm{H.c.}, \tag{2.56}$$

with $M_{\mathrm{D}} = Y_\nu v$. However, since the right-handed neutrinos are total singlets under the SM gauge group, they can have Majorana masses on the form

$$-\mathcal{L}_{\mathrm{Maj,R}} = \frac{1}{2}\overline{\nu_{\mathrm{R}}^c}M_{\mathrm{R}}\nu_{\mathrm{R}} + \mathrm{H.c.}, \tag{2.57}$$

with M_{R} symmetric. Without loss of generality, one can always perform a basis transformation on the right-handed neutrino fields and work in the basis in which M_{R} is real, positive, and diagonal, i.e., $M_{\mathrm{R}} = \mathrm{diag}(M_1, M_2, M_3)$. Thus, the full Lagrangian describing the masses in the neutrino sector is given by

$$-\mathcal{L}_{\nu\text{-mass}} = \frac{1}{2}\overline{\nu_{\mathrm{L}}}M_{\mathrm{L}}\nu_{\mathrm{L}}^c + \overline{\nu_{\mathrm{L}}}M_{\mathrm{D}}\nu_{\mathrm{R}} + \frac{1}{2}\overline{\nu_{\mathrm{R}}^c}M_{\mathrm{R}}\nu_{\mathrm{R}} + \mathrm{H.c.} = \frac{1}{2}\overline{\Psi}\mathcal{M}_\nu\Psi^c + \mathrm{H.c.}, \tag{2.58}$$

where

$$\Psi = \begin{pmatrix} \nu_{\mathrm{L}} \\ \nu_{\mathrm{R}}^c \end{pmatrix}, \quad \mathcal{M}_\nu = \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix}. \tag{2.59}$$

Thus, it now has the form of a Majorana mass term for the field Ψ, with a symmetric 6×6 Majorana mass matrix $\mathcal{M}_\nu$. Diagonalization of this matrix leads in general

[6] Although there are models with different numbers of $\nu_{\mathrm{R}i}$'s, we stick to 3 in this section for simplicity and symmetry reasons.

to 6 Majorana mass eigenstates, each of which is a linear superposition of the left- and right-handed neutrinos, and vice versa.

An often considered special case is the case $M_{\mathrm{L}} = M_{\mathrm{R}} = 0$, in which case the resulting 6 Majorana fields can be combined into 3 Dirac fields. In this case, one again has the freedom to rephase the neutrino fields, just as in the quark sector. Thus, the form of the lepton mixing matrix is given by Eq. (2.52), but without the last matrix containing the Majorana phases, in analogy with the CKM matrix in Eq. (2.40). However, for this to be the case, the M_{R} term has to be forbidden by some additional exact symmetry. A new gauge symmetry will not work, since after this symmetry is broken, the Majorana mass term for the right-handed neutrino will in general be generated. In this respect, the neutrino sector of the SM is fundamentally different from the charged lepton and quark sectors. Also, the neutrino Yukawa couplings have to be very small, of the order of 10^{-11}. Although this is technically natural, unless the right-handed neutrinos have some additional kind of interaction, the right-handed neutrinos will in practise be undetectable.

Another special case, which as been studied intensively in the literature, is the case $M_{\mathrm{R}} \gg M_{\mathrm{D}}$, which will be further discussed in Ch. 3. This is possible, since M_{R} is not related to the electroweak symmetry breaking, while M_{D} is determined by the Higgs VEV. On the other hand, a commonly used naturalness criterion states that a number can be naturally small if setting it to zero increases the symmetry of the Lagrangian. Since without a Majorana mass for the right-handed neutrino, the $U(1)$ of total lepton number is a symmetry of the Lagrangian, a small M_{R} is also natural.

2.7.3 Experimental consequences of massive neutrinos

In a charged-current interaction, the left-handed component of the mass eigenstates e'_i will be produced. Defining this as a charged lepton, and the neutrino produced in association with it as a flavor eigenstate, U relates the neutrino flavor eigenstates $|\nu_\alpha\rangle$, and the mass eigenstates $|\nu_i\rangle$ as

$$|\nu_\alpha\rangle = U^*_{\alpha i}|\nu_i\rangle. \tag{2.60}$$

This will lead to *neutrino oscillations*, in which a neutrino of flavor α, produced in a charged-current interaction, can, after propagating a certain distance, be detected as a neutrino of a generally different flavor β [33–35]. Since the time evolutions of the mass eigenstates are simply given by multiplication of the exponential $\exp(-\mathrm{i}E_i t)$, the amplitude for this transition is

$$\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right) = \langle\nu_\beta|U^*_{\alpha i}e^{-\mathrm{i}E_i t}|\nu_i\rangle = \langle\nu_j|U_{\beta j}U^*_{\alpha i}e^{-\mathrm{i}E_i t}|\nu_i\rangle = U_{\beta i}\left(U^\dagger\right)_{i\alpha} e^{-\mathrm{i}E_i t}, \tag{2.61}$$

giving the transition probability as $P\left(\nu_\alpha \to \nu_\beta\right) = |\mathcal{A}\left(\nu_\alpha \to \nu_\beta\right)|^2$. As it turns out, neutrino oscillations are not sensitive to all the parameters in the neutrino sector, which are 7 for Dirac neutrinos and 9 for Majorana neutrinos. The sensitivity is

Parameter	Best-fit value & 1σ ranges
s_{12}^2	$0.312^{+0.017}_{-0.015}$
s_{23}^2	0.51 ± 0.06
s_{13}^2	$0.010^{+0.009}_{-0.006}$
Δm_{21}^2 [10^{-5} eV^2]	$7.59^{+0.20}_{-0.18}$
Δm_{31}^2 [10^{-3} eV^2]	2.45 ± 0.09

Table 2.1. The current best-fit values and 1σ ranges for the neutrino oscillation parameters, where the mass ordering is assumed to be normal [49].

restricted to 2 independent mass-squared differences $\Delta m_{31}^2 \equiv m_3^2 - m_1^2$, $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$, the 3 mixing angles, and the CP-violating Dirac phase δ. Neutrino oscillations have been verified by experiments on solar [36–39], atmospheric [40], and artificially produced neutrinos [41–45], and ranges for all parameters but the phase δ have been inferred, all these results being consistent with the three-neutrino mixing scheme. However, there are some experimental results which are not compatible with the other experiments and a three-neutrino mixing scheme, the most important ones being the old results of the LSND experiment [46, 47], which recently gained more support from the the MiniBooNE experiment [48]. The implications of these results are currently unclear.

Since the sign of Δm_{31}^2 is not known, neither is the ordering of the masses. The neutrino masses are said to have either normal or inverted ordering, depending on whether $m_1 < m_2 < m_3$ or $m_3 < m_1 < m_2$. The masses can also be quasi-degenerate, in which case $m_1 \simeq m_2 \simeq m_3$. Although neutrino oscillations are sensitive to the mass ordering, there is as of today no firm evidence for one or the other. The current best-fit values and 1σ ranges for the neutrino oscillation parameters from a global fit of neutrino oscillation data are given in Tab. 2.1, where the mass ordering is assumed to be normal [49]. For assumed inverted ordering, the values will change slightly, the main difference being in the best-fit value of Δm_{31}^2, changing to $-2.34 \cdot 10^{-3}$ eV^2. Also, there is a slight preference for nonzero θ_{13}, with $\theta_{13} = 0$ excluded at around 1.8σ, depending, however, on the data used and other assumptions. The determination of θ_{13} is expected to be improved significantly in the near future by the Double Chooz experiment [50–52]. To summarize, although neutrino oscillation experiments give a large amount of information about the neutrinos and the relevant parameters, they are insensitive to the absolute neutrino mass scale, cannot distinguish between Dirac and Majorana neutrinos, and give no information on the values of any possible Majorana phases.

However, the absolute neutrino mass scale can be determined by experiments of a different nature. For example, by studying the energy spectra of electrons emitted in beta decays of certain isotopes, one can constrain the effective kinematical electron-neutrino mass m_β, given by

$$m_\beta^2 = \sum_{i=1}^{3} |U_{ei}|^2 \, m_i^2 = m_1^2 c_{12}^2 c_{13}^2 + m_2^2 s_{12}^2 c_{13}^2 + m_3^2 s_{13}^2. \tag{2.62}$$

The best current upper limits on m_β are approximately 2.5 eV [53,54]. Near-future experiments such as MARE [55] and KATRIN [56, 57] are aiming to improve on this bound. The latter, in the case of vanishing m_β, is expected to put a 90 % confidence upper bound of approximately 0.2 eV [58]. For $m_\beta = 0.2$ eV, a zero value can be excluded at slightly over 2σ, while for $m_\beta = 0.35$, the exclusion would be around 5σ.

The absolute neutrino mass scale can also be probed by cosmological observations. The effective sum $\Sigma = m_1 + m_2 + m_3$ of neutrino masses can be inferred from measurements of the cosmic microwave background radiation when combined with results from other observations, such as of high-redshift galaxies, baryon acoustic oscillations, and type Ia supernovae [59, 60]. This has been performed by the WMAP experiment [61] and will be improved on by Planck [62].

Finally, the mass scale can, in the case of Majorana neutrinos, be measured in neutrinoless double beta ($0\nu\beta\beta$) decay experiments, which is sensitive to the effective mass $|m_{ee}|$, which is a function of all the parameters in the neutrino sector, except θ_{23} and the Dirac CP-violating phase. Thus, in principle, the values of all the remaining 3 parameters which oscillations experiments are insensitive to can be probed. Neutrinoless double beta decay will be discussed in more detail in Ch. 5.

Chapter 3

The seesaw models

The *seesaw models* are a group of models involving new, heavy degrees of freedom such that, in the low-energy theory where the heavy fields are integrated out, the effective operator in Eq. (2.47) is generated, resulting in a Majorana mass matrix for the light neutrinos. In other words, they are simple extensions of the SM such that, at low energy, the SM with the Weinberg operator is recovered and the matrix κ can be given in terms of the parameters of the high-energy theory. The Weinberg operator is usually generated at tree-level, but can also appear due to radiative corrections [63]. For the tree-level case, there are three main type of models, depending on which type of fields generate the Weinberg operator:

- The *Type I* seesaw models [64–67], where a number of fermionic SM singlets, basically right-handed neutrinos, are introduced,

- The *Type II* seesaw models [68–73], where scalar $SU(2)_{\mathrm{L}}$ triplets are introduced,

- The *Type III* seesaw models [74], where fermionic $SU(2)_{\mathrm{L}}$ triplets are introduced.

In general, there is nothing that prevents more than one of these sets of fields to be present simultaneously, giving combinations of seesaw models.

Usually, the new fields introduced have masses far above the electroweak scale, outside the reach of any foreseeable experiments, making these versions of seesaw models essentially untestable.[1] However, there are also seesaw models where the new particles have masses above the electroweak scale, but within the reach of future experiments such as the LHC, so-called *low-scale seesaw models*. For potential collider signatures of such models, see Refs. [82, 83] and references therein.

[1] They could affect processes at very high energies. For example, they could generate the baryon asymmetry of the Universe through *leptogenesis* [75–81].

In this chapter, the type I seesaw model as well as its variation the *inverse seesaw model* will be discussed in more detail. Both of these models can be constructed such that the new particles have masses at low energy scales, e.g., at the TeV scale, making them, in principle, testable in future experiments. The reader is referred to the references for more details on the other types of seesaw models. The type I seesaw model was studied in paper I of this thesis [1] and the inverse seesaw model in paper II [2].

3.1 The type I seesaw model

The type I seesaw model is the most studied of the seesaw models, and is basically a special case of the model introduced in Sec. 2.7.2, i.e., the particle content of the SM is extended with three right-handed neutrino fields $\nu_{\mathrm{R}i}$ with a Majorana mass matrix M_{R}, which has eigenvalues above the electroweak scale. Also, it is usually assumed that there are no other contributions to the masses of the light neutrinos. In this case, at energies below M_{R}, the Weinberg operator with

$$\kappa = Y_\nu M_{\mathrm{R}}^{-1} Y_\nu^T \tag{3.1}$$

is generated at tree-level (after phase redefinitions of the neutrino fields).[2] This can be represented diagrammatically as in Fig. 3.1. Thus, after electroweak symmetry breaking, there is a Majorana mass term with mass matrix

$$M_{\mathrm{L}} = v^2 \kappa = v^2 \cdot Y_\nu M_{\mathrm{R}}^{-1} Y_\nu^T = F M_{\mathrm{R}} F^T, \tag{3.2}$$

with $F = v Y_\nu M_{\mathrm{R}}^{-1}$. It is thus suppressed by a factor of $Y_\nu v M_{\mathrm{R}}^{-1}$ with respect to the electroweak scale.

The next operator generated in the tower of effective interactions, relevant for neutrinos, is the dimension-six operator [84–87]

$$\mathcal{L}_\nu^{d=6} = \left(\overline{\ell_{\mathrm{L}}}\tilde{\phi}\right) C \mathrm{i}\partial\!\!\!/ \left(\tilde{\phi}^\dagger \ell_{\mathrm{L}}\right), \tag{3.3}$$

where the coefficient matrix is given, at leading order, by

$$C = \left(Y_\nu M_{\mathrm{R}}^{-1}\right)\left(Y_\nu M_{\mathrm{R}}^{-1}\right)^\dagger . \tag{3.4}$$

After electroweak symmetry breaking, this dimension-six operator leads to corrections to the kinetic terms for the light neutrinos. In order to keep the neutrino

[2] This is accurate for energies E below all the eigenvalues of M_{R}. For energies between two eigenvalues of M_{R}, only the right-handed neutrinos with masses above E should be integrated out. This will be discussed in more detail in Sec. 4.3.

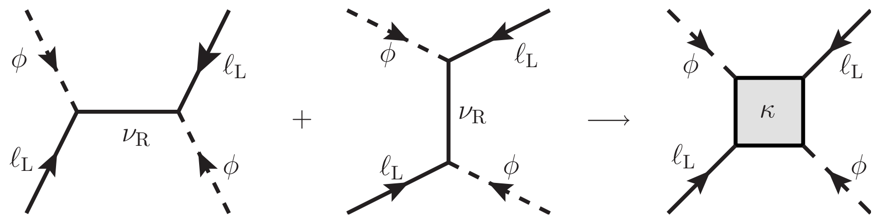

Figure 3.1. The generation of the Weinberg operator in the type I seesaw model.

kinetic energy canonically normalized, one has to rescale the neutrino fields, resulting in a non-unitary matrix relating the flavor and mass eigenstates, given by

$$N = \left(1 - \frac{v^2}{2} C\right) U = \left(1 - \frac{F F^\dagger}{2}\right) U, \tag{3.5}$$

where U diagonalizes the light neutrino mass matrix. For $|F| \gtrsim \mathcal{O}(0.1)$, non-negligible non-unitarity effects could be visible in the near detector of a future neutrino factory [88–92], and there are further constraints coming from the universality tests of weak interactions, rare leptonic decays, the invisible Z width, and neutrino oscillation data [93]. However, note that such a large value of F is incompatible with Eq. (3.2) unless severe fine tuning is involved so that large cancellations occur.

The model can also be analyzed by keeping the right-handed fields in the theory, breaking electroweak symmetry spontaneously, and then approximately block diagonalizing the resulting full mass matrix, given in Eq. (2.59) with $M_{\rm L} = 0$. This can be done using

$$U \simeq \begin{pmatrix} \mathbb{1} & \rho \\ -\rho^\dagger & \mathbb{1} \end{pmatrix}, \tag{3.6}$$

with $\rho = M_{\rm D} M_{\rm R}^{-1}$, giving

$$U^T \mathcal{M}_\nu U \simeq \begin{pmatrix} -M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T & 0 \\ 0 & M_{\rm R} \end{pmatrix} = \mathcal{D}_\nu \tag{3.7}$$

to lowest order in $M_{\rm D} M_{\rm R}^{-1}$. The upper left 3×3 block is then a Majorana mass matrix for the fields $R = \nu_{\rm L} + \rho \nu_{\rm R}^{\rm c}$, containing a small part of the gauge singlet right-handed neutrinos, proportional to $\rho \ll 1$. Also, the heavy neutrino mass eigenstate fields, which are mainly composed of the gauge singlet right-handed fields, also contain a small component of left-handed neutrino fields. Note that the matrix which enters into the lepton mixing matrix is the matrix which diagonalizes the upper left 3×3 block of the full mass matrix $\mathcal{M}_\nu$. However, this matrix will not necessarily be unitary, as it is only a part of the full unitary 6×6 matrix which diagonalizes the full mass matrix. This is how the non-unitarity enters in this way of looking at the model, which is to be compared with the effects of the dimension-six operator in Eq. (3.3).

In paper II of this thesis [2], some properties of the *low-scale* type I seesaw model were considered, in which the right-handed neutrinos have masses close to the electroweak scale. In the ordinary seesaw models with right-handed neutrino masses far above the electroweak scale, Y_ν can be sizable, e.g., at order unity. In the low-scale seesaw model, Y_ν should be relatively small in order to maintain the stability of the masses of the light neutrinos. However, there exist mechanisms that could stabilize neutrino masses without the requirement of a tiny Y_ν. For example, additional suppression could enter through a small lepton number violating contribution (as in the inverse seesaw model, cf. Sec. 3.2). Also, the neutrino masses could be generated radiatively, in which case the additional suppression is guaranteed by loop integrals [63]. Finally, neutrino masses could be forbidden at $d = 5$, but appear from effective operators of higher dimension [32]. In these cases, there will still be restrictions on Y_ν from the unitarity of the leptonic mixing matrix.

To fully specify the type-I seesaw model one needs to specify 18 parameters, in addition to the ones in the SM [94]. This can be done in different ways.[3] For example, in the *top-down* parametrization, the model is considered at high-energy scales, where the right-handed neutrinos are propagating degrees of freedom. As mentioned before, one can always choose a $\nu_{\rm R}$ basis where the mass matrix $M_{\rm R}$ is diagonal, with positive and real eigenvalues, i.e., $M_{\rm R} = D_{\rm R}$. The remaining neutrino Yukawa matrix Y_ν is an arbitrary complex matrix, from which 3 phases can be removed by phase redefinitions of the $\ell_{{\rm L}i}$'s, giving 15 additional parameters. Another useful and popular parametrization, more natural and relevant for low-energy physics, is the *Casas–Ibarra* parametrization [95]. First, it uses the real and diagonal matrices $D_{\rm R}$, $D_\kappa = D_{\rm L}/v^2$, and the leptonic mixing matrix U, containing a total of 12 parameters. The remaining 6 parameters are encoded in the matrix

$$O \equiv D_\kappa^{-1/2} U^\dagger Y_\nu M_{\rm R}^{-1/2}. \tag{3.8}$$

If the relation in Eq. (3.1) is to hold, O has to be a complex orthogonal matrix, which means that it can be written in the form $O = R_{23}(\vartheta_1)R_{13}(\vartheta_2)R_{12}(\vartheta_3)$ with $R_{ij}(\vartheta_k)$ being the elementary rotations in the 23, 13, and 12 planes, respectively. Different from the quark or lepton mixing angles, ϑ_i are in general complex.

3.2 The inverse seesaw model

The inverse seesaw model [96] is an extension of the type I seesaw model, in which the smallness of the neutrino masses is protected by a small amount of lepton number breaking instead of suppression by a very large mass scale. It contains three extra fermionic SM gauge singlets S_i, coupled to the right-handed neutrinos in a lepton-number conserving way, while the ordinary right-handed neutrino Majorana

[3]One can always choose the $\ell_{\rm L}$ basis such that $Y_e^\dagger Y_e = D_e$, containing three parameters.

mass term is forbidden by some additional symmetry. It is only through symmetric mass matrix $M_{\rm S}$ in the Majorana mass term $\overline{S^c} M_{\rm S} S$ that the lepton number is broken, and $M_{\rm S}$ can thus be naturally small. The relevant part of the Lagrangian is then, in the flavor basis,

$$-\mathcal{L}_{\rm IS} = \overline{\ell_{\rm L}} \tilde{\phi} Y_\nu \nu_{\rm R} + \overline{S^c} M_{\rm R} \nu_{\rm R} + \frac{1}{2} \overline{S^c} M_{\rm S} S + \text{H.c.} \tag{3.9}$$

Here, the fields $\nu_{{\rm R}i}$ and S_i are not mass eigenstates, but instead the Majorana mass matrix in the basis $\{\nu_{\rm R}, S\}$ is

$$\mathcal{M}_{\rm IS} = \begin{pmatrix} 0 & M_{\rm R} \\ M_{\rm R}^T & M_{\rm S} \end{pmatrix}. \tag{3.10}$$

For $M_{\rm S} \ll M_{\rm R}$, the right-handed neutrinos and the extra singlets S_i are, to lowest order, maximally admixed into three pairs of heavy Majorana neutrinos with opposite CP parities and essentially identical masses, with a splitting of the order of $M_{\rm S}$, and can as such be regarded as components of three heavy pseudo-Dirac neutrinos.

Integrating out these heavy fields yields the Weinberg operator with

$$\kappa = \left(Y_\nu M_{\rm R}^{-1}\right) M_{\rm S} \left(Y_\nu M_{\rm R}^{-1}\right)^T \tag{3.11}$$

at tree-level, which, after electroweak symmetry breaking as usual, yields a Majorana mass matrix for the light neutrinos as

$$M_{\rm L} = F M_{\rm S} F^T, \tag{3.12}$$

where $F = v Y_\nu M_{\rm R}^{-1}$. This is to be compared with Eq. (3.2) for the type I seesaw model. The diagrammatical representation is still given by the diagrams in Fig. 3.1, but with all the 6 heavy mass eigenstate fields appearing as intermediate states.

In spite of the underlying physics responsible, the particle content of the inverse seesaw model is essentially the same as that of the type-I seesaw model, but with six right-handed neutrinos. Thus, one can in principle treat the heavy singlets S_i as three additional right-handed neutrinos, possessing vanishing Yukawa couplings with the lepton doublets. It is also worth comparing the type I and inverse seesaw models with the discussion of the Weinberg operator in Sec. 2.7.1. In the seesaw models, the cutoff scale Λ_ν in Eq. (2.53) can essentially be identified with $M_{\rm R}$, which is generally above the electroweak scale. However, the dimensionless $\tilde{\kappa}$ in Eqs. (2.53) and (2.54) then have the order of magnitudes

$$\tilde{\kappa} = \begin{cases} \mathcal{O}(Y_\nu^2) & \text{type I seesaw,} \\ \mathcal{O}(Y_\nu^2 M_{\rm S} M_R^{-1}) & \text{inverse seesaw.} \end{cases} \tag{3.13}$$

Thus, $\tilde{\kappa}$ can be strongly suppressed by the potentially very small ratio $M_{\rm S} M_R^{-1}$ in the inverse seesaw model.

Finally, note that, in the inverse seesaw model, the correct light neutrino masses can be obtained even for $F = \mathcal{O}(1)$, i.e., for the new heavy fields around the electroweak scale and with large Yukawa couplings Y_ν, and that the non-unitarity effects are, as in the type I seesaw model, given by Eq. (3.5). Thus, as opposed to the ordinary type I seesaw model, large non-unitarity effects are possible in the inverse seesaw model.

Chapter 4

Renormalization group running

This chapter is a description of the concept of renormalization group (RG) running. First, the need for regularization and renormalization is described using a simple example. Then, the motivations for studying the RG running in the SM and seesaw models as well as the methods for solving the resulting RG equations are reviewed. The decoupling of the right-handed neutrinos and the use of effective theory is explained. Finally, the proper description of the running between the masses of the heavy particles is described and how this can lead to the so-called threshold effects in the running of the neutrino parameters.

4.1 The main idea

Calculations of quantum corrections, represented by loops in Feynman diagrams, to physical quantities (such as cross sections, decay rates, and particle masses), as well as unphysical ones (such as correlation functions), often yield divergent results. This implies that the calculated corrections are not uniquely defined, and as a result, neither are the predictions of the theory.

The standard way to deal with this issue is to implement a two-step procedure. First, one has to *regularize* the divergence by modifying the theory in some way. This is performed by introducing some parameter ϵ, such that the modified prediction is a well-defined function of ϵ and the original, divergent result is reobtained in the limit $\epsilon \to 0$. Then, one has to *renormalize* the theory by redefining its parameters, such that the prediction becomes finite in the $\epsilon \to 0$ limit. For this to be the case, the original parameters and fields appearing in the Lagrangian, the so-called *bare* parameters and fields, must formally diverge as $\epsilon \to 0$. In order to make these concepts more easy to grasp, a simple example will be used as an illustration.

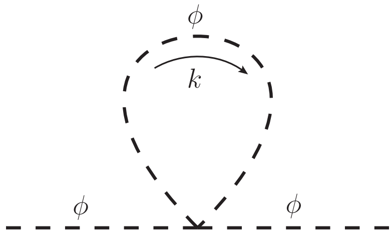

Figure 4.1. Self-energy diagram of a scalar field ϕ. Here, k is the loop momentum.

Consider the QFT with only a single real scalar field ϕ with mass m and quartic self-coupling λ. The one-loop self-energy diagram is given in Fig. 4.1, the value of which is

$$\mathrm{i}\Sigma_\phi = \frac{\lambda}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{4.1}$$

where k is the loop momentum. One way to evaluate these kinds of integrals is to perform a *Wick rotation* by changing variables to

$$k^0 \equiv \mathrm{i}k_{\mathrm{E}}^4, \quad \boldsymbol{k} \equiv \boldsymbol{k}_{\mathrm{E}}, \tag{4.2}$$

which implies that the Lorentz inner product is given by

$$k^2 = (k^0)^2 - \boldsymbol{k}^2 = -(k_{\mathrm{E}}^4)^2 - \boldsymbol{k}_{\mathrm{E}}^2 = -k_{\mathrm{E}}^2, \tag{4.3}$$

where $k_{\mathrm{E}}^2 = (k_{\mathrm{E}}^4)^2 + \boldsymbol{k}_{\mathrm{E}}^2$ is just the ordinary inner product in four-dimensional Euclidean space. Rerouting the integral in the complex plane and then going to spherical coordinates, $\mathrm{i}\Sigma_\phi$ can be calculated as

$$\mathrm{i}\Sigma_\phi = -\mathrm{i}\frac{\lambda}{2(2\pi)^4}\int \mathrm{d}\Omega \int_0^\infty \mathrm{d}k_{\mathrm{E}}\frac{k_{\mathrm{E}}^3}{k_{\mathrm{E}}^2 + m^2}, \tag{4.4}$$

which is divergent in the region of large k_{E}.

It is now time to regularize this integral, which in general can be done in a number of different ways. The simplest way, and arguably the physically most intuitive, is to use an ultraviolet cutoff. Simply cut off the integral at some large energy scale $k_{\mathrm{E}} = \Lambda$, i.e., only integrate up to Λ instead of ∞, giving

$$\Sigma_\phi = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right]. \tag{4.5}$$

The parameter ϵ can then, for example, be chosen as $\epsilon = \Lambda^{-1}$. Note that Λ is not the (fixed) energy scale up to which your theory is valid, but an arbitrary regularization scale. This regularization method has the disadvantage that is breaks gauge invariance, and is thus not suitable for gauge theories.

Another regularization method is the *Pauli-Villars* regularization, in which a smooth cutoff is introduced in the propagator by making the replacement

$$\frac{\mathrm{i}}{k^2 - m^2} \to \frac{\mathrm{i}}{k^2 - m^2} - \frac{\mathrm{i}}{k^2 - M^2} = \frac{\mathrm{i}(m^2 - M^2)}{(k^2 - m^2)(k^2 - M^2)}, \tag{4.6}$$

which can also be viewed as the introduction of a new fictitious particle with a large mass M and wrong overall sign of the propagator. The parameter ϵ can be taken to be M^{-1}. However, this method becomes too complicated for less simple theories, such as the SM.

Lattice regularization implies replacing the space-time continuum by a lattice with finite spacing $\epsilon = l$, removing modes of the field with momenta larger than l^{-1}. This regularization is automatically present in numerical non-perturbative calculations, but less suitable for analytical calculations using perturbation theory, since this regulator breaks Lorentz invariance.

Finally, the most widely used regularization method, which preserves Lorentz and gauge invariance, but perhaps the most non-intuitive one, is *dimensional regularization*. Here, the number of space-time dimensions is altered to $d = 4 - \epsilon$. Of course, one has to make sure that the mathematical framework one is using is properly generalized to arbitrary d. For example, the mass dimensions of the fields have now changed, so that Eqs. (2.31) and (2.32) are generalized to

$$[\phi] = [A^\mu] = \frac{d-2}{2}, \tag{4.7}$$

$$[\psi] = \frac{d-1}{2}. \tag{4.8}$$

In general, all the mass parameters still have the dimension of a mass, but all the other coupling constants need to be redefined in order to keep their mass dimensions (they are typically dimensionless). For example, in the SM and related theories, one has to make the replacements

$$\lambda \to \lambda_0 = \mu^\epsilon \lambda, \tag{4.9}$$

$$g_i \to g_{i0} = \mu^{\frac{\epsilon}{2}} g_i \quad \text{for } i \in \{1,2,3\}, \tag{4.10}$$

$$Y_f \to Y_{f0} = \mu^{\frac{\epsilon}{2}} Y_f \quad \text{for } f \in \{u,d,\nu,e\}. \tag{4.11}$$

Here, μ is an arbitrary energy scale, called the *renormalization scale*, and the subscript "0" denotes the bare quantities which have mass dimensions, while the couplings without this subscript denote the renormalized couplings. These relations between the bare and renormalized couplings are only the lowest order results, while loop corrections will modify the relations to include the renormalization constants.

Continuing the example, by Wick rotating and using standard formulae [8], the scalar self-energy can then be calculated as

$$
\begin{aligned}
\mathrm{i}\Sigma_\phi &= \mu^\epsilon \frac{\lambda}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^4} \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon} \\
&= -\mathrm{i}\frac{\lambda}{32\pi^2} (4\pi)^{\frac{\epsilon}{2}} \Gamma\left(-1 + \frac{\epsilon}{2}\right) m^2 \left(\frac{\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \\
&= \mathrm{i}\frac{\lambda}{32\pi^2} m^2 \left[\frac{2}{\epsilon} + \log\frac{\mu^2}{m^2} + \log 4\pi + 1 - \gamma_\mathrm{E} + \mathcal{O}(\epsilon)\right],
\end{aligned}
\tag{4.12}
$$

where the expression in middle line has been expanded in ϵ to yield the last line, and $\gamma_\mathrm{E} \simeq -0.5572$ is Euler's constant. It is now clear in which way the integral diverges in the limit $\epsilon \to 0$, and how the introduction of the scale μ ensures that the argument of the logarithm is dimensionless.

After having regularized a loop diagram, it is time for the step of renormalization. In general, loop corrections to two-point functions, as the one in the example, yield corrections to the corresponding field's mass and wave function normalization. If the corresponding correction is divergent in the limit $\epsilon \to 0$, the mass and wave function renormalizations also have to be divergent in this limit, leaving the renormalized masses and fields finite. Loop corrections to higher-order correlation functions require renormalization of the corresponding coupling constants. For example, a loop diagram with four external scalars requires renormalization of the coupling constant λ_0, yielding the renormalized coupling. There are in general a number of ways of renormalizing a QFT, the main groups of *renormalization schemes* being the *mass-dependent* and *mass-independent* schemes, meaning that the *counterterms* introduced to cancel the divergences are dependent and independent of the mass parameters of the theory, respectively.

The renormalized parameters are the ones one should relate to experiments, although they are not observables in the strict sense. For example, in perturbation theory, predictions for observables are expansions in the renormalized couplings, which are functions of the renormalization scale. Fixing a renormalization scale (usually of the order of the relevant energy in the process), the values of the coupling constants at that renormalization scale can be inferred from the experimental data. Note that the exact result for a physical observable should be independent of the renormalization scale (and more generally, the renormalization scheme), while individual terms in the perturbation expansion are not necessarily so. Thus, by choosing the renormalization scheme and scale wisely, i.e., in a way that the effective expansion parameter becomes small, one can optimize the expansion.

Writing m_0 for the bare mass of our example scalar field, the corrected mass squared will be $m_0^2 - \Sigma_\phi$.[1] Defining the counterterm δm^2 by $m_0^2 \equiv m^2 + \delta m^2$, the corrected propagator and the renormalized mass m will be finite if δm^2 is made

[1]In general, there will be a wave function renormalization as well, but in this example there is no need for this.

to diverge in such a way so that it cancels the divergence of Σ_ϕ exactly. There are different ways of accomplishing this, corresponding to different renormalization schemes, but the most widely used is the Minimal Subtraction (MS) scheme, or its modified version $\overline{\text{MS}}$. In MS, only the poles in ϵ are subtracted.[2] In our example, one takes

$$\delta m^2 = \frac{\lambda}{16\pi^2} m^2 \frac{1}{\epsilon}. \tag{4.13}$$

Now, by noting that all the bare parameters are independent of the renormalization scale μ, one can from Eq. (4.9) calculate that

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -\epsilon\lambda, \tag{4.14}$$

and hence

$$\mu \frac{\mathrm{d}m^2}{\mathrm{d}\mu} = \frac{\lambda}{16\pi^2} m^2, \tag{4.15}$$

and so the renormalized mass depends on the renormalization scale. Note that m^2 is renormalized multiplicatively, implying that if $m = 0$ at tree level, it will remain so after quantum effects are considered.

The above statement is generally true for fermion fields, but *not* for scalar fields. This is because the fermion mass terms generally break chiral symmetry, implying that it is natural to have them small, since the symmetry of the theory is then increased by setting the masses to zero. This is usually not the case for scalar masses, and so, using dimensional regularization, the quantum corrections are in general proportional to the mass of any particle running in the loop (but not to a high-energy cutoff scale). For example, in the type I seesaw model, there are corrections to the Higgs mass proportional to the Majorana masses of the right-handed neutrinos, which are generally much larger than the electroweak scale. Thus, it is not natural to have the SM Higgs mass at the electroweak scale, where it, however, has to be for phenomenological reasons. This often called the *hierarchy problem*. However, in the SM, where there are no right-handed neutrinos, all fermions are massless, and thus all quantum corrections to the Higgs mass are proportional to the mass itself, and thus smaller than the tree-level mass.[3] Thus, the question "What are the sizes of the quantum corrections to the Higgs mass?" will get different answers depending on in which theory the quantum corrections are calculated. Although this might seem bizarre, it is in fact quite natural, since in the SM, as an effective theory, all parameters, as determined by low-energy experiments, do not depend on any high-energy theory. However, if one has a high-energy theory, the Higgs mass will in general depend on any specific parameter of the high-energy theory, if all the other high-energy parameters are kept fixed. The hierarchy problem is

[2] In $\overline{\text{MS}}$, also the constant term $\ln 4\pi - \gamma_{\mathrm{E}}$ is subtracted, generally leading to a better convergence of the perturbation series.

[3] This is true even if one considers the broken SM with massive fermions, since all fermions have masses at the electroweak scale or below.

thus not really a problem in the SM, but rather a potential problem of high-energy extensions of the SM, in which particles with masses far above the electroweak scale appear. In these theories, obtaining a Higgs mass at the electroweak scale can involve fine-tuning between different parameters. This fine-tuning problem can be avoided if, for example, there are additional symmetries in the high-energy theory protecting the Higgs mass, or if the Higgs is composite.

4.2 Renormalization group running of lepton parameters in seesaw models

The example in the previous chapter can be generalized to more complicated theories, such as the SM and the seesaw models. The resulting *renormalization group equations* (RGEs) describe the dependence of the renormalized parameters on the renormalization scale μ, and constitute a system of coupled ordinary differential equations, one for each parameter. In mass-independent renormalization schemes, the RGEs have the general form

$$\mu\frac{\mathrm{d}P_i}{\mathrm{d}\mu} \equiv \beta_i\left(P_1,\ldots,P_n\right), \tag{4.16}$$

where P_i $(i = 1,\ldots,n)$ are the n parameters of the theory and the *beta functions* β_i do not depend explicitly on μ. The RGEs for the type I, type II, and type III seesaw models with both the SM and the *minimal supersymmetric standard model* (MSSM)[4] as underlying theories have been derived in the literature [100–106]. In this chapter, mainly the RGEs of the type I seesaw model will be discussed. They can be found in Appendix A, both for the extended SM and MSSM. The RGEs of both the SM and the inverse seesaw model can be obtained as special cases of the ones for the type I seesaw model.

Although it can be interesting to study the running of the gauge couplings and quark masses (or, rather, the quark Yukawa couplings), the main topic of this thesis is the RG evolution of the parameters in the lepton sector of the SM, and more specifically, the light neutrino masses and the lepton mixing parameters. In the case of the parameters outside the lepton sector of the SM, the RG running is usually studied because of the need to reconcile experimental measurements at different energies, which, without considering running, would not be compatible. For example, this is the case for the electromagnetic coupling constant. However, this is not the case in the lepton sector, since the current experimental uncertainties are generally much larger than the running effects, and because the lepton parameters have only been measured at relatively low energies so far. Instead, the reason to study them is that theoretical predictions of models beyond the SM, such as *grand unified theories* (GUTs), are valid at some high-energy scale, while experimental

[4]The MSSM is a extensively studied extension of the SM, where every SM particle has an additional partner having spin differing by one half. See, for example, Refs. [97–99] for reviews.

data are taken at low energies. Therefore, one has to take into account the running of the parameters between the high-energy (GUT) and low-energy (experimental) scales in order to compare the experimental results with the theoretical predictions. Extensions of the SM sometimes predict specific mixing patterns in the lepton sector, i.e., specific values for the lepton mixing matrix. Two such common symmetric mixing patters are the *bimaximal* mixing pattern [107–110] with $s_{12} = s_{23} = 1/\sqrt{2}$ and $s_{13} = 0$ [cf. Eq. (2.52)] and the *tri-bimaximal* mixing pattern [111–113] with $s_{12} = 1/\sqrt{3}$, $s_{23} = 1/\sqrt{2}$, and $s_{13} = 0$.

The RGE evolution of the neutrino masses and lepton mixing parameters can be determined through the evolution of the effective light neutrino mass matrix (in the effective or full theories) and the charged lepton Yukawa matrix, cf. Section 2.7.1. Often, the RG running of the effective light neutrino mass matrix and charged lepton Yukawa matrix are calculated numerically, after which the neutrino masses and lepton mixing parameters are determined by diagonalizing the mass matrix. However, one can also translate the full RGEs for the neutrino mass matrix into a system of differential equations for mixing angles, CP-violating phases, and light neutrino masses directly. The corresponding formulas have been discussed below the seesaw scale [114–116], as well as above the seesaw thresholds in the type I [117, 118], type II [104, 105], and type III [106] seesaw frameworks. Note that the usual diagonality assumption made on Y_e is not in general invariant under the RG running, and neither is the parametrization of the lepton mixing matrix in Eq. (2.52), which in general includes three additional unphysical phases which have to be rotated away in order to determine the physical mixing parameters.

In general, there are two different strategies for solving the RGEs. In the *top-down* approach, the initial conditions on the parameters are specified at a certain high-energy scale, often motivated by the flavor structure of a specific high-energy model. Once this is done, the running down to low energies and crossing the seesaw thresholds is relatively straightforward. In this approach, the main issue is the fact that only small regions of the parameter space of the full theory will lead to values of the low-energy parameters that are consistent with experiments, and this makes this approach difficult to implement in practice. In the *bottom-up* approach, on the other hand, the initial conditions on the parameters are specified at a low-energy scale, usually the electroweak scale. Hence, all the available experimental information is taken into account from the start. However, after running to higher energy scales, one reaches the seesaw threshold, where one has to match the effective and full theories. Then, since the number of parameters in the full theory is larger than in the effective one, one has to make additional assumptions on the parameters and flavor structure of the full theory.

The general features of the running of the lepton parameters have been studied in the literature, and it has been shown that there could be large radiative corrections to the lepton mixing parameters at super high-energy scales (see, e.g., Ref. [119] and references therein). In particular, certain flavor symmetric mixing patterns can be achieved at the GUT scale indicating that there might exist some flavor symmetries similar to the gauge symmetry (see, e.g., Ref. [120] and references therein).

4.3 Decoupling of right-handed neutrinos and threshold effects

The description using effective field theory plays an essential role in the study of the RG running of QFT parameters. The *Appelquist–Carazzone* theorem [121] states that the effect of heavy particles decouples at energies much smaller than their masses, and that they do not contribute to the beta functions at low energies. This can be seen explicitly if one uses a mass-dependent renormalization scheme, such as momentum space subtraction. However, this does not happen if one uses a mass-independent renormalization scheme, such as MS, since the beta functions are independent of masses. Generally, in perturbative calculations, one will obtain finite contributions to observables on the form

$$\log \frac{E^2}{\mu^2},$$

which is the reason why one generally should take $\mu \simeq E$ to minimize the effects of higher-order terms. However, one can also have potentially large logarithms,

$$\log \frac{M^2}{\mu^2},$$

where M is the mass of the heavy particle [24]. If $M \gg E$, these terms might destroy the perturbation expansion. In order to implement the decoupling in mass-independent schemes, one decouples the heavy particles "by hand" by integrating them out at the matching scale $\mu \simeq M$, and describing the RG running for $\mu < M$ using the effective theory.

This holds in general for all particles, and in particular for the ones in the SM, but here we will concentrate on the heavy neutrinos, which are assumed to have masses above the electroweak scale. In the previous discussion of the seesaw models, only the different regions of energy $E < M_{\mathrm{R}}$ and $E > M_{\mathrm{R}}$ were considered. However, the three right-handed neutrinos do in general not have the same masses, i.e., the masses can be non-degenerate. In that case, the heavy neutrinos have to be sequentially decoupled from the theory [122], leading to a series of effective field theories. Once again, it is worth to point out that perturbative renormalization of effective operators can be performed in the usual way, as long as one is satisfied with a finite accuracy and works to a given order in E/Λ.

When crossing the seesaw thresholds, one should make sure that the full and effective theories give identical predictions for physical quantities at low-energy scales, and therefore, the physical parameters of both theories have to be related to each other. In the case of the neutrino mass matrix, this means relations between the effective coupling matrix κ and the parameters Y_ν and M_R of the full theory. This is called *matching* the full and effective theories. For the simplest case, when the mass spectrum of the heavy singlets is degenerate, namely $M_1 = M_2 = M_3 = M_0$, one can simply make use of the tree-level matching condition at the scale $\mu = M_0$,

$$\kappa\big|_{M_0} = Y_\nu M_\mathrm{R}^{-1} Y_\nu^T\big|_{M_0}. \tag{4.17}$$

In the most general case with non-degenerate heavy singlets, i.e., $M_1 < M_2 < M_3$, the situation becomes more complicated. For μ between M_n and M_{n-1}, the heavy mass eigenstates $\left\{\nu_\mathrm{R}^n \dots \nu_\mathrm{R}^3\right\}$ are integrated out. In this effective theory, only a $3\times(n-1)$ sub-matrix of Y_ν remains, denoted by $Y_\nu^{(n)}$, as well as an $(n-1)\times(n-1)$ submatrix of M_R, denoted by $M_\mathrm{R}^{(n)}$. The decoupling of the n-th heavy singlet leads to the appearance of an effective dimension-five operator through the tree-level matching condition at $\mu = M_n$,

$$\kappa^{(n)}\big|_{M_n} = \kappa^{(n+1)}\big|_{M_n} + \frac{\mathcal{Y}_\nu^{(n)}\mathcal{Y}_\nu^{(n)T}}{M_n}\big|_{M_n}, \quad \text{for } n = 1,2,3, \tag{4.18}$$

where $\mathcal{Y}_\nu^{(n)}$ is the n-th column of Y_ν (i.e., the part which has been removed from Y_ν), and it is understood that $\kappa^{(4)} = 0$ is the effective operator in the full theory and $\kappa^{(1)} = \kappa$ is the effective operator with all the heavy fields decoupled. In between these scales, all the parameters are to be run using their respective RGEs. Note that the matching has to be done in a basis where the right-handed mass matrix is diagonal, since it is the eigenstate with a specific mass which is to be decoupled.

The renormalized effective neutrino Majorana mass matrix for μ below M_n is described by two parts as

$$m_\nu^{(n)} = v^2 \left[\kappa^{(n)} + Y_\nu^{(n)} \left(M_\mathrm{R}^{(n)}\right)^{-1} Y_\nu^{(n)T}\right], \tag{4.19}$$

where (n) labels the quantities relevant for the effective theory between the n-th and $(n-1)$-th thresholds. Both of these contributions run with the renormalization scale μ, and the running can be determined from Eqs. (A.1g), (A.1h), and (A.1j). As it turns out, the flavor non-diagonal parts of the running are the same in both the SM and the MSSM. However, the RGEs for the two terms have different flavor diagonal contributions, but only in the SM and not in the MSSM. In particular, the coefficients for the gauge coupling and Higgs self-coupling contributions are different. The flavor diagonal parts are α_κ and $2\alpha_\nu$, respectively, and they differ as

$$\alpha_\kappa - 2\alpha_\nu = \lambda + \frac{9}{10}g_1^2 + \frac{3}{2}g_2^2 \qquad \text{in the SM,} \tag{4.20}$$

$$\alpha_\kappa - 2\alpha_\mu = 0 \qquad \text{in the MSSM.} \tag{4.21}$$

Thus, the running of the two different parts contributing to the effective neutrino mass matrix in Eq. (4.19) has different gauge and Higgs self-coupling contributions. Since these couplings are in general rather large, there can be potentially

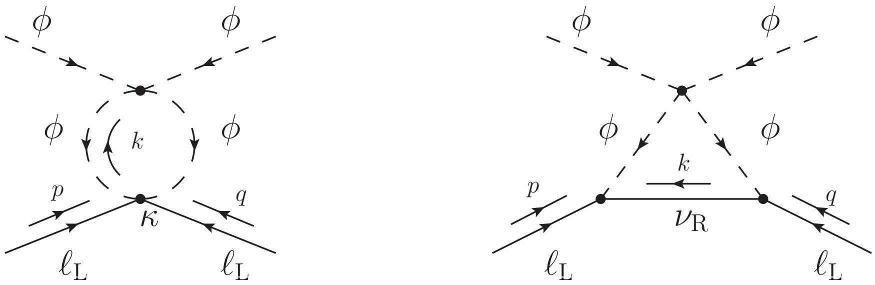

Figure 4.2. Feynman diagrams of the one-loop corrections to the neutrino mass matrix due to the Higgs self coupling in the effective theory (left) and in the full theory (right). Here, p and q are the external momenta and k is the loop momentum.

large running of the lepton mixing angles due to this "mismatch" between the two contributions. These effects are usually referred to as *threshold effects*.

As an example to visualize where this difference comes from, consider the corrections to the four-point functions relevant to the neutrino mass matrix from the Higgs self-coupling in Fig. 4.2. In the effective theory (left diagram), there is a contribution involving the loop integral

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2} \frac{1}{(k-p-q)^2 - m^2},$$

which is divergent, while the loop integral appearing in the full theory (right diagram) is given by

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\not{k} + M_n}{k^2 - M_n^2} \frac{1}{(k+p)^2 - m^2} \frac{1}{(q-k)^2 - m^2}$$

and is not divergent. Therefore, there are no corrections proportional to λ to the neutrino mass matrix in the full theory, while λ does enter the beta function of κ in the effective theory. See also Ref. [117] for a detailed discussion. If the relevant seesaw threshold is above the SUSY-breaking scale, such a mismatch is absent in the MSSM due to the supersymmetric structure of the MSSM Higgs and gauge sectors. Therefore, this may result in significant RG running effects only in the SM, but not in the MSSM.

In paper II of this thesis [2], the threshold effects on the RG running of the neutrino parameters in the type I seesaw model have been studied, while in paper I [1], the general RG running of the neutrino parameters in the inverse seesaw model has been investigated, including threshold effects.

Chapter 5

Neutrinoless double beta decay

In this chapter, the process of neutrinoless double beta decay will be described, and how it is related to neutrino masses. First, the process as mediated by light neutrino exchange will be discussed as well as the current experimental situation and the dependence on the nuclear physics. Second, the possibility of other contributions, parametrized by effective operators, will be described.

Beta decay is the decay of a nucleus accompanied by the emission of an electron or a positron. In *beta-minus* decay, an electron is emitted together with an electron antineutrino when a nucleus with mass number A (the number of nucleons) and atomic number Z (the number of protons) decays according to

$$(A, Z) \to (A, Z+1) + e^- + \overline{\nu_e}, \tag{5.1}$$

which on the level of the nucleons is essentially

$$n \to p + e^- + \overline{\nu_e}. \tag{5.2}$$

In *beta-plus* decay, a positron is emitted together with an electron neutrino in the process

$$p \to n + e^+ + \nu_e, \tag{5.3}$$

which, because of energy conservation, can only occur inside a nucleus. The final state positron can also be exchanged to an initial state electron, in which case the process is called *electron capture*. All these processes can be accurately described through the exchange of SM W-bosons, or, since the relevant energies are much lower than the W-boson mass, by using the standard four-fermion interaction between the proton, neutron, electron, and neutrino fields. (Or between the quark, electron, and neutrino fields in the quark-level description.) Historically, the properties of beta decay, specifically the apparent non-conservation of energy and angular momenta, was what led Wolfgang Pauli to suggest that there was an undetected neutral particle being emitted together with the electron.

In nuclei, where ordinary single beta decay is forbidden for kinematical reasons, *double beta decay* ($2\nu\beta\beta$) can be the dominant process. In this process, a nucleus with mass number A and atomic number Z decays through the emission of two electrons and two antineutrinos according to

$$(A, Z) \to (A, Z+2) + 2e^- + 2\overline{\nu_e}. \tag{5.4}$$

Double beta decay has been observed in around 10 nuclei [123], and the corresponding half-lives are very long, typically of the order of 10^{19} years or longer. This decay is essentially described as two "simultaneous" single beta decay processes, which is also the reason why the decay rates are so small.

5.1 Neutrinoless double beta decay through light neutrino exchange

If neutrinos are Majorana particles, it can be possible for some nuclei to undergo double beta decay *without* emission of neutrinos. Replacing the two external neutrinos with an internal line and working on the level of the quarks inside the nucleons, one obtains the diagram in Fig. 5.1, giving the process

$$(A, Z) \to (A, Z+2) + 2e^-. \tag{5.5}$$

In this decay process, lepton number is violated by two units. This will be referred to as the "standard" mechanism responsible for $0\nu\beta\beta$ [124,125]. Just as in the case of single beta decay, the internal momentum in the diagram is of the order of the typical energy transfer in the nucleus, and hence much smaller than the mass of the W-bosons. Thus, the quark-lepton interaction becomes point-like and can be described using the standard four-fermion interaction. However, due to the small neutrino masses, the light neutrino propagator will depend strongly on the energy transfer and can thus not be treated as point-like. In fact, due to the lightness of the neutrinos and the chirality structure of the charged current interaction vertices, the propagator of a Majorana mass eigenstate neutrino field with mass m_i will be [126]

$$P_{\rm L} \frac{\not{q} + m_i}{q^2 - m_i^2} P_{\rm L} = \frac{m_i}{q^2 - m_i^2} \simeq \frac{m_i}{q^2}, \tag{5.6}$$

where q is the transfered momentum.

In the calculation of the resulting decay rate Γ of a specific nucleus,[1] one can separate the dependence on the underlying particle physics and the nuclear physics by writing it as [124, 125]

$$\Gamma = G|\mathcal{M}^{0\nu}|^2|m_{ee}|^2. \tag{5.7}$$

Here, G is a known phase space factor, $\mathcal{M}^{0\nu}$ is the *nuclear matrix element* (NME),

[1]The decay rate is in general equal to the inverse half-life divided by $\ln 2$.

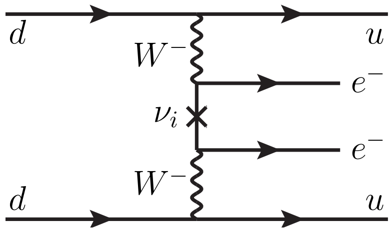

Figure 5.1. The leading order Feynman diagram for $0\nu\beta\beta$ through Majorana neutrino exchange.

containing all the dependence on the nuclear physics, and $|m_{ee}|$ is the effective neutrino mass given by

$$|m_{ee}| = \left|\sum_{i=1}^{3} U_{ei}^2 m_i\right| = \left|m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 e^{2i\alpha} + m_3 s_{13}^2 e^{2i\beta}\right|, \tag{5.8}$$

which is the magnitude of the ee-element of the neutrino mass matrix in the flavor basis. Here, α and β are the Majorana phases. This expression is given using a slightly different, but physically equivalent, parametrization of the lepton mixing matrix than what was used in Eq. (2.52). First, the neutrino fields have been given a common phase redefinition in order to make U_{e1} real. Then, the third neutrino mass eigenstate ν_3 has been given an additional phase redefinition so that the U_{e3} becomes independent of δ. The NME is given as the sum of two more basic matrix elements, the *Gamow–Teller* and *Fermi* type matrix elements as

$$\mathcal{M}^{0\nu} = \mathcal{M}_{\mathrm{GT}} - \frac{g_{\mathrm{V}}^2}{g_{\mathrm{A}}^2}\mathcal{M}_{\mathrm{F}}, \tag{5.9}$$

where g_{V}^2 and g_{A}^2 are two constants of order one. The matrix elements $\mathcal{M}_{\mathrm{GT}}$ and $\mathcal{M}_{\mathrm{F}}$ can be written as expectation values of certain operators between the initial and final nuclear states [124, 125]. However, since they are rather complicated and not needed for the discussion of the particle physics, they will not be discussed in detail.

In order to extract the values of the underlying particle physics parameters, one needs the values of the NMEs. The calculation of the matrix elements $\mathcal{M}_{\mathrm{GT}}$ and $\mathcal{M}_{\mathrm{F}}$ requires the knowledge of the wave functions of complicated nuclei and need to be calculated numerically using some nuclear physics model. This is a notoriously difficult task [127–129], and the corresponding uncertainties make the inference of the underlying parameters from experimental results uncertain.

In addition, a firm observation of $0\nu\beta\beta$ will actually always imply that neutrinos are Majorana particles. This is because any diagram leading to $0\nu\beta\beta$, regardless of its origin or form, can be extended by connecting the external electron lines with the quark lines by two W^-'s [130]. The resulting diagram will then be a four-loop diagram for a Majorana neutrino mass term. As of today, there is experimentally no other realistic way to determine if neutrinos are Dirac or Majorana particles. Also, as mentioned in Sec. 2.7.3, all the parameters which cannot be probed in oscillation experiments can, in principle, be so in $0\nu\beta\beta$ experiments. Hence, $0\nu\beta\beta$ experiments can give information on the absolute neutrino neutrino mass scale, and could in principle constrain the CP-violating Majorana phases. However, the large uncertainties in the NMEs propagate to the inference of the underlying parameters.

In order to detect this extremely rare process against the much larger background of ordinary double beta decay, one can make use of the fact that the energy spectra of the final state electron differ widely. In $2\nu\beta\beta$, the final state contains four particles in addition to the nucleus, out of which only two (the electrons) can realistically be detected, while the neutrinos carry their energy out of the detector. Thus, the energy of the electrons will be distributed in the region between 0 and the total energy being released, i.e., the *Q-value*. In contrast, due to the nucleus being very heavy compared to the electrons, most of the energy will be carried off by the electrons in $0\nu\beta\beta$, while being emitted almost back-to-back and monochromatically, and with the total energy equal to the Q-value. Using this, one can discriminate the signal ($0\nu\beta\beta$) from the background ($2\nu\beta\beta$).

Current lower limits at 90 % confidence level for the half-lives of different nuclei include $T_{1/2} > 1.9 \cdot 10^{25}$ years for ^{76}Ge [123], $T_{1/2} > 5.8 \cdot 10^{23}$ years for ^{100}Mo, $T_{1/2} > 2.1 \cdot 10^{23}$ years for ^{82}Se [131], and $T_{1/2} > 3.0 \cdot 10^{24}$ years for ^{130}Te [132]. Using calculated values of the corresponding NMEs, these limits can be translated into upper limits on $|m_{ee}|$. For example, Ref. [123] finds the upper limit $|m_{ee}| < 0.35$ eV at 90 % confidence level. There have also been claims of a measurement of neutrinoless double beta decay by a subgroup of the Heidelberg–Moscow collaboration [133–136], which would indicate that neutrinos are Majorana particles. However, the validity of these results is disputed, although the claimed decay rate cannot, at the moment, be excluded by other experiments [127]. Hence, a new generation of experiments are needed in order to either verify or rule out the above claim. Examples of present and future experiments include GERDA [137], EXO [138], Majorana [139], and MOON [140].

5.2 Other mechanisms of neutrinoless double beta decay

Although the standard mechanism for $0\nu\beta\beta$ is the exchange of light Majorana neutrinos, other mechanisms could very well appear in certain extensions of the SM, such as supersymmetric models and models with heavy neutrinos [141–144], as well as left-right symmetric models [145]. To discriminate between different mechanisms of $0\nu\beta\beta$, it will not be enough to detect $0\nu\beta\beta$ in a single isotope, but instead measurements for several different isotopes will be necessary [143, 146, 147].

Also, note that not only will the decay of different nuclei involve different NMEs, but so will also different decay mechanisms in the same nucleus.

Instead of treating specific high-energy models, one can use effective field theory and look at the most general operators responsible for $0\nu\beta\beta$. First, one can alter the Lorentz and/or chirality structure of the effective four-fermion interaction, while keeping the neutrino propagator [126]. There is also the possibility that no neutrino exchange is involved in the new decay mechanism. If the virtual particles responsible for the decay are heavier than the typical nuclear energies, the whole process should be describable by a single point-like interaction in effective field theory. Since in this case there are four quark fields and two electron fields involved, the effective operator has to have a mass dimension equal to nine. The most general such Lagrangian is given by [148]

$$\mathcal{L}_{0\nu\beta\beta} = \frac{G_{\mathrm{F}}^2}{2} m_{\mathrm{p}}^{-1} \left(\epsilon_1 JJj + \epsilon_2 J^{\mu\nu} J_{\mu\nu} j + \epsilon_3 J^{\mu} J_{\mu} j + \epsilon_4 J^{\mu} J_{\mu\nu} j^{\nu} + \epsilon_5 J^{\mu} J j_{\mu}\right) + \text{H.c.}, \tag{5.10}$$

where J and j denote hadron and electron currents, respectively. The proportionality to the Fermi constant G_{F}^2 has been introduced, since this also appears in the standard mechanism, while the factor m_{p}^{-1} finally gives the coefficient the correct mass dimension.[2] The strengths of the different operators are parametrized by the (generally complex) dimensionless coefficients ϵ_i. Actually, there are many more operators in Eq. (5.10) then there seems to be at first sight. This is because different chirality structures are permitted for all the currents. The hadron currents in Eq. (5.10) are given by

$$J_{\mathrm{L,R}} = \overline{u}\left(1 \mp \gamma_5\right) d, \quad J^{\mu}_{\mathrm{L,R}} = \overline{u}\gamma^{\mu}\left(1 \mp \gamma_5\right) d, \quad J^{\mu\nu}_{\mathrm{L,R}} = \overline{u}\frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]\left(1 \mp \gamma_5\right) d, \tag{5.11}$$

and the electron ones by

$$j_{\mathrm{L,R}} = \overline{e}\left(1 \mp \gamma_5\right) e^c = 2\ \overline{e_{\mathrm{R,L}}} e^c_{\mathrm{R,L}}, \quad j^{\mu}_{\mathrm{L,R}} = \overline{e}\gamma^{\mu}\left(1 \mp \gamma_5\right) e^c = 2\ \overline{e_{\mathrm{L,R}}}\gamma^{\mu} e^c_{\mathrm{R,L}}. \tag{5.12}$$

Note that there are some more Lorentz invariant terms, which could have been added to Eq. (5.10), namely

$$\mathcal{L}'_{0\nu\beta\beta} = \frac{G_{\mathrm{F}}^2}{2} m_{\mathrm{p}}^{-1} \left(\epsilon_6 J^{\mu} J^{\nu} j_{\mu\nu} + \epsilon_7 J J^{\mu\nu} j_{\mu\nu} + \epsilon_8 J_{\mu\alpha} J^{\nu\alpha} j^{\mu}_{\nu}\right), \tag{5.13}$$

where the electron tensor currents are given by $j^{\mu\nu}_{\mathrm{L,R}} = \overline{e}\frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]\left(1 \mp \gamma_5\right) e^c$. However, one can show that all operators proportional to $\overline{e}\gamma^{\mu} e^c$, $\overline{e}\frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] e^c$, and $\overline{e}\gamma_5\frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] e^c$ vanish identically, since the electron fields anti-commute [149]. Thus, the terms in Eq. (5.13) all vanish and do not need to be considered.

The resulting half-life, including interferences, can be calculated from the interactions in the Lagrangian in Eq. (5.10) [148], and also interferences with the

[2] The choice of the proton mass is arbitrary, but in some sense natural since the proton appears as a final state, and since the typical energy transfer inside the nucleus is substantially smaller.

standard light neutrino exchange mechanism can be derived [3]. Note that, depending on the chiralities of the final state electrons, several of the interference terms will be suppressed, see Ref. [3] for a discussion.

Paper III of this thesis [3] deals with prospective constraints on the effective operators in Eq. (5.10) from future data on $0\nu\beta\beta$, in combination with data from single beta decay experiments and cosmological observations.

Chapter 6

Summary and conclusions

In Part I of this thesis, the theoretical background relevant for the scientific papers presented in Part II of the thesis has been dealt with. The standard model of particle physics has been briefly discussed, and emphasis has been put on the topics of neutrino masses and leptonic mixing. The seesaw models, in particular the type I and inverse seesaw models, have been reviewed, and so have the concepts of renormalization and renormalization group running. Finally, the process of neutrinoless double beta decay and how it can be triggered by exchange of Majorana neutrinos and by effective operators has been described. Throughout, the importance and benefits of using effective (quantum field) theories has been emphasized.

In Part II of the thesis, three scientific papers will be presented, which investigate the models and use the techniques introduced in Part I. In paper I [1], we have studied the renormalization group running of the lepton parameters in the inverse seesaw model. We have derived analytical formulas, describing the running of the neutrino parameters above the seesaw threshold in the SM and the MSSM. Also, a detailed numerical study of the RG running has been carried out. Because of the potentially large Yukawa couplings, significant running of the lepton mixing angles can be obtained. The running of the lepton mixing angles, in particular of θ_{12}, can be large if the mass spectrum of the light neutrinos is nearly degenerate. In addition, the effects of the seesaw thresholds are discussed. Some phenomenologically and theoretically interesting leptonic mixing patterns, the bimaximal and tri-bimaximal patterns, can be achieved at a high-energy scale once the RG running is taken into account. Finally, the RG evolution of the light neutrino masses and of the CP-violating phases has been studied.

In paper II [2], the renormalization group running of the neutrino parameters in a low-scale seesaw model with non-degenerate heavy neutrinos has been investigated. We have shown that significant radiative corrections can be obtained at low energies, and for a short distance of renormalization group running, as a result of threshold effects. Analytical formulas for the renormalization group corrections to the neutrino parameters in crossing the seesaw thresholds have been presented, indicating that the mismatch between different contributions to the mass matrix of the light neutrinos can lead to large corrections to the lepton mixing matrix. A numerical example has also been given to show that, in the presence of low-scale right-handed neutrinos, the bi-maximal mixing pattern at the TeV scale is fully compatible with the current measurements lepton mixing angles.

In paper III [3], we have investigated the possible future bounds on the strength of different short-range contributions to neutrinoless double beta decay. These bounds depend on the outcome of ongoing and planned experiments related to neutrino masses. Three scenarios, A, B, and C, are studied, corresponding to different combinations of experimental results. For two of the scenarios, we have determined the bounds on the coefficients ϵ_i of each point-like operator that could contribute to the decay, will for the remaining scenario, we obtain non-zero estimates. More accurate calculations of the nuclear matrix elements will improve the robustness of our results.

For more detailed conclusions, the reader is referred to corresponding papers.

Appendix A

Renormalization group equations in the type I seesaw model

In this appendix, the RGEs of the parameters of the type I seesaw model with the SM and the MSSM as underlying theories are given. The RGEs for the inverse seesaw model can be calculated as a special case of this model with six right-handed neutrinos.

A.1 SM with right-handed neutrinos

The renormalization group evolution of the parameters of the SM and the coefficient of the Weinberg operator are given by [100–103, 119, 120, 150]

$$16\pi^2\mu\frac{\mathrm{d}g_1}{\mathrm{d}\mu} = b_1 g_1^3, \tag{A.1a}$$

$$16\pi^2\mu\frac{\mathrm{d}g_2}{\mathrm{d}\mu} = b_2 g_2^3, \tag{A.1b}$$

$$16\pi^2\mu\frac{\mathrm{d}g_3}{\mathrm{d}\mu} = b_3 g_3^3, \tag{A.1c}$$

$$16\pi^2\mu\frac{\mathrm{d}Y_u}{\mathrm{d}\mu} = \left(\alpha_u + C_u^u H_u + C_u^d H_d\right) Y_u, \tag{A.1d}$$

$$16\pi^2\mu\frac{\mathrm{d}Y_d}{\mathrm{d}\mu} = \left(\alpha_d + C_d^u H_u + C_d^d H_d\right) Y_d, \tag{A.1e}$$

$$16\pi^2\mu\frac{\mathrm{d}Y_e}{\mathrm{d}\mu} = \left(\alpha_e + C_e^e H_e + C_e^\nu H_\nu^{(n)}\right) Y_e, \tag{A.1f}$$

$$16\pi^2\mu\frac{\mathrm{d}Y_\nu^{(n)}}{\mathrm{d}\mu} = \left(\alpha_\nu + C_\nu^e H_e + C_\nu^\nu H_\nu^{(n)}\right) Y_\nu^{(n)}, \tag{A.1g}$$

$$16\pi^2\mu\frac{\mathrm{d}M_\mathrm{R}^{(n)}}{\mathrm{d}\mu} = C_\mathrm{R} M_\mathrm{R}^{(n)} \left(Y_\nu^{(n)\dagger} Y_\nu^{(n)}\right) + C_\mathrm{R}\left(Y_\nu^{(n)\dagger} Y_\nu^{(n)}\right)^T M_\mathrm{R}^{(n)}, \tag{A.1h}$$

$$16\pi^2\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \alpha_\lambda^\lambda + \alpha_\lambda^g + \alpha_\lambda^Y, \tag{A.1i}$$

$$16\pi^2\mu\frac{\mathrm{d}\kappa^{(n)}}{\mathrm{d}\mu} = \alpha_\kappa \kappa^{(n)} + \left(C_\kappa^e H_e + C_\kappa^\nu H_\nu^{(n)}\right)\kappa^{(n)} + \kappa^{(n)}\left(C_\kappa^e H_e + C_\kappa^\nu H_\nu^{(n)}\right)^T, \tag{A.1j}$$

where $H_f = Y_f Y_f^\dagger$ for $f = e, \nu, u, d$, and (n) labels the quantities relevant for the effective theory between the n-th and $(n-1)$-th thresholds. The matching between the effective theories is described in the main text. GUT charge normalization for g_1 is used, which means that g_1 is related to the conventional SM coupling $\tilde{g}_1$ as $g_1^2 = \frac{5}{3}\tilde{g}_1^2$. The coefficients determining the evolution of the gauge couplings are

$$b_1 = \frac{41}{10}, \quad b_2 = -\frac{19}{16}, \quad b_3 = -7. \tag{A.2}$$

The beta functions for the Yukawa couplings each consist of a flavor diagonal part and a flavor non-diagonal part. The flavor diagonal parts are given by

$$\alpha_u = \mathrm{tr}\left(3H_u + 3H_d + H_e + H_\nu^{(n)}\right) - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2, \tag{A.3}$$

$$\alpha_d = \mathrm{tr}\left(3H_u + 3H_d + H_e + H_\nu^{(n)}\right) - \frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2, \tag{A.4}$$

$$\alpha_e = \mathrm{tr}\left(3H_u + 3H_d + H_e + H_\nu^{(n)}\right) - \frac{9}{4}g_1^2 - \frac{9}{4}g_2^2, \tag{A.5}$$

$$\alpha_\nu = \mathrm{tr}\left(3H_u + 3H_d + H_e + H_\nu^{(n)}\right) - \frac{9}{20}g_1^2 - \frac{9}{4}g_2^2, \tag{A.6}$$

$$\alpha_\kappa = 2\mathrm{tr}\left(3H_u + 3H_d + H_e + H_\nu^{(n)}\right) + \lambda - 3g_2^2, \tag{A.7}$$

while the coefficients determining the flavor non-diagonal parts are given by

$$C_u^u = C_d^d = C_e^e = C_\nu^\nu = \frac{3}{2}, \tag{A.8}$$

$$C_u^d = C_d^u = C_e^\nu = C_\nu^e = -\frac{3}{2}, \tag{A.9}$$

$$C_\mathrm{R} = 1, \tag{A.10}$$

$$C_\kappa^e = -\frac{3}{2}, \quad C_\kappa^\nu = \frac{1}{2}. \tag{A.11}$$

Finally, the RGE evolution of the Higgs self-coupling constant is determined by[1]

$$\alpha_\lambda^\lambda = 6\lambda^2 - 3\lambda\left(\frac{3}{5}g_1^2 + 3g_2^2\right) + \lambda\mathrm{tr}\left(3H_u + 3H_d + H_e + H_\nu^{(n)}\right), \tag{A.12}$$

$$\alpha_\lambda^g = 3g_2^4 + \frac{3}{2}\left(\frac{3}{5}g_1^2 + 3g_2^2\right)^2, \tag{A.13}$$

$$\alpha_\lambda^Y = -8\mathrm{tr}\left(3H_u^2 + 3H_d^2 + H_e^2 + \left(H_\nu^{(n)}\right)^2\right). \tag{A.14}$$

A.2 MSSM with right-handed neutrinos

If instead the MSSM is the underlying theory, the RGEs in Eqs. (A.1) (except for Eq. (A.1i), since the parameter λ is absent in the MSSM) still hold above the supersymmetry-breaking scale, but with different coefficients. Below the scale of supersymmetry-breaking, one recovers the SM as an effective theory, and the corresponding RGEs should be used. The coefficients determining the evolution of the gauge couplings are

$$b_1 = \frac{33}{5}, \quad b_2 = 1, \quad b_3 = -3. \tag{A.15}$$

The flavor diagonal terms read

$$\alpha_u = \mathrm{tr}\left(3H_u + H_\nu^{(n)}\right) - \frac{13}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2, \tag{A.16}$$

$$\alpha_d = \mathrm{tr}\left(3H_d + H_e\right) - \frac{7}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2, \tag{A.17}$$

$$\alpha_e = \mathrm{tr}\left(3H_d + H_e\right) - \frac{9}{5}g_1^2 - 3g_2^2, \tag{A.18}$$

$$\alpha_\nu = \mathrm{tr}\left(3H_u + H_\nu^{(n)}\right) - \frac{3}{5}g_1^2 - 3g_2^2, \tag{A.19}$$

$$\alpha_\kappa = 2\alpha_\nu, \tag{A.20}$$

while the flavor non-diagonal are determined by

$$C_u^u = C_d^d = C_e^e = C_\nu^\nu = 3, \tag{A.21}$$

$$C_u^d = C_d^u = C_e^\nu = C_\nu^e = 1, \tag{A.22}$$

$$C_\mathrm{R} = 2, \tag{A.23}$$

$$C_\kappa^e = C_\kappa^\nu = 1. \tag{A.24}$$

The fact that $\alpha_\kappa = 2\alpha_\nu$ leads to the absence of threshold effects in the MSSM, as discussed in the main text.

[1] The interaction term is $(\lambda/4)\,\phi^4$.

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Part II

Scientific papers

I

Paper I

J. Bergström, M. Malinský, T. Ohlsson, and H. Zhang
Renormalization group running of neutrino parameters in the inverse seesaw model
Physical Review **D81**, 116006 (2010)
arXiv:1004.4628

II

Paper II

J. Bergström, T. Ohlsson, and H. Zhang

Threshold effects on renormalization group running of neutrino parameters in the low-scale seesaw model

Physics Letters **B698**, 297 (2011)

arXiv:1009.2762

Paper III

J. Bergström, A. Merle, and T. Ohlsson

Constraining new physics with a positive or negative signal of neutrino-less double beta decay

Journal of High Energy Physics **05**, 122 (2011)
arXiv:1103.3015

III