Pricing Bermudan options -
A nonparametric estimation approach *

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Abstract

A nonparametric alternative to the Longstaff-Schwartz estimation of conditional expectations is suggested for pricing of Bermudan options. The method is based on regularization of a least-squares minimization, with a Tikhonov-type smoothing put on the partial differential equation which characterizes the underlying price processes. This approach can hence be viewed as a combination of the Monte Carlo method and the PDE method for the estimation of conditional expectations. The estimation method turns out to be robust with regard to the size of the smoothing parameter.

Keywords: Optimal stopping, regularization, nonparametric estimation, conditional expectations, Bermudan options, Snell envelope.

1 Introduction

The pricing of Bermudan contracts has been deeply studied in the literature (c.f. [3], [11] and [1]). In most cases, the problem may only be solved by numerical computations. Which type of numerical computation to use, depends on the structure of the contract. If the contract involved is based on a few assets, say up to three or four, then pricing based on certain kinds of discretized schemes for the underlying assets, e.g. binomial trees, trinomial trees or partial differential equations, works well. However, as the number of dimensions increase, this approach fails, due to what is known as the curse of dimensionality. This simply means that the number of computations needed

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to price a contract grows too fast with respect to the number of dimensions, for the method to be feasible. Thus, to price contracts in higher dimensions, one needs to resort to other means.

In 2001, a new approach was provided by Longsta and Schwartz in [11]. The method was based on simulations, where they used projection onto a set of pre-determined functions, to estimate the value of not exercising the contract at each possible stopping time and in each simulation. In other words, they provided a way to estimate the expected value of not exercising, conditioned on the value of the underlying processes, and used this estimation to give a stopping strategy for each simulation. The result is a method of pricing Bermudan contracts by simulation, which did not suffer the curse of dimensionality and was fast in terms of number of computations needed. Moreover, given a good choice of projection functions, the prices were very accurate.

What remains is to find a “good” choice of the projection functions. Even though a simple second degree polynomial is sufficient for many contracts, given certain levels of accuracy, the search for nonparametric estimates may provide a viable alternative.

In the literature one can find other means of estimating the value of continuation for use in simulation approaches. Some of these may also be classified as regression approaches. For example, Carriere [5] uses regression with splines and Tsitsiklis and Roy [13] uses a linear combination of Hilbert space basis functions to approximate a value function. There are also a number of improvements and developments of these approaches in the literature.

Kloedko and Schoenmakers published a paper ([10]) where another approach was introduced. The approach is based on policy iteration and is shown to give good results for high-dimensional problems. However, due to the use of sub-simulations in the method, the calculation times may be too large for some problems. Further developments of this method are made in [2], where the authors improve both the accuracy and the efficiency of the method so that accurate prices could be obtained for Bermudan derivatives quickly and with minimal handcrafting.

This paper takes as starting point the regression approaches, most notably the method proposed by Longstaff and Schwartz, and formulates a nonparametric estimation method. The focus is to try to eliminate the handcrafting needed for pricing high-dimensional Bermudan contracts. Due to the ill-posed nature of the introduced approach (c.f. [8]), a smoothing operator is used, which uses known dynamics of the conditional expectation to project the solution onto a reduced space where the true solution is included. The necessary proof of convergence is presented, independently of the particular choice of regularization function, as well as regularization parameter. This estimator is illustrated in some examples, where the partial differential operator, which characterizes the underlying model, is used as regularization
operator. The intuition behind this is that the partial differential equation should equal zero, for any solution to the pricing problem. Thus, given that the discretization gives some error, it is reasonable to demand that the norm of this discretized equation should at least be small.

A drawback of the method is presented in the way the smoothing operator is calculated. Since the smoothing operator takes the form of a PDE the calculations means, as it is now, introducing the use of a grid. As such, the curse of dimensionality is likely to still pose a problem in higher dimensions. Hence, the problem of pricing Bermudans in higher dimensions is not fully solved by the method. But nevertheless, the method is a first take on a novel approach towards solving that problem, and it does seem like further development can be done. In fact, as of the time of this writing, some success has been made in relaxing the use of the grid. But the results are still too preliminary to be able to draw any conclusions on extending the method to higher dimensions.

The paper is organized as follows. In Section 2 preliminaries for the pricing methodology is presented. In Section 3, an overview of the algorithm used to price the Bermudan contracts is given, assuming that a way of estimating the conditional expectations is known. A short summary of the projection method by Longstaff and Schwartz is also presented. Examples of results are provided in the case of second degree polynomial as parametrization, in the basic Black-Scholes setting. In Section 4, a nonparametric method of estimating these conditional expectations is proposed, based on regularization, and proof of convergence is provided. In Section 5, the results of the nonparametric estimation are compared with the results obtained by using the second degree polynomial. In Section 6, the results presented in this paper are discussed.

2 Preliminaries

Consider the probability space \( \{\Omega, \mathcal{F}, Q\} \) where a \( d\)-dimensional continuous time Markov process \( \{S_t\}_{t \geq 0} \) lives. On this probability space, define the filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \); the augmentation of the filtration generated by \( S \). Under these conditions, \( \mathcal{F} \) satisfies the usual conditions (c.f. [9] page 10), and the process \( S \) will of course be measurable with respect to this filtration.

Let \( C \) be a financial contract which depends on \( S \). \( C \) is Bermudan, i.e. the owner of the contract has the right to exercise the contract at one of the pre-determined set of times \( 0 \leq t_1 < t_2 < \ldots < t_{N-1} < t_N = T \). When the contract is exercised, the owner receives the payment \( \Phi(S_t) \), where \( t \) is the chosen exercise time.

2.1 Asset dynamics

Let \( S^{(1)} \) (the first element in \( S \)) be the money market account process, i.e.
\[
\begin{align*}
S_0^{(1)} &= 1, \\
\frac{dS_t^{(1)}}{S_t^{(1)}} &= r_t S_t^{(1)} dt, \quad t > 0,
\end{align*}
\]

where, \( r_t \) is the, possibly stochastic, continuously compounded risk-free rate at time \( t \). We assume that the market is complete and arbitrage free (c.f. [3] page 177-180), and that \( \mathbb{Q} \) is the unique martingale measure with respect to the numeraire \( S^{(1)} \).

### 2.2 Stopping times

Define \( \mathcal{T} \) as the set of stopping times taking values in the set \( \{t_k\}_{k=1}^N \), i.e. the set of stopping times which are feasible strategies for exercising the contract \( \mathcal{C} \). The price of \( \mathcal{C} \) at time \( t \) is

\[
V_t = S_t^{(1)} \text{ess sup}_{\tau \in \mathcal{T}} \mathbb{E}^\mathbb{Q}_F \left[ \frac{\Phi(S_\tau)}{S_\tau^{(1)}} \right],
\]

by the optional sampling theorem (c.f. [9] page 19).

The value process of the contract \( \mathcal{C} \) is given via the following backward recursive formula known as the Snell envelope (c.f. [3] page 78)

\[
\begin{align*}
V_{N} & = \Phi(S_N) \\
V_{k} & = \max \left\{ \Phi(S_k), S_k^{(1)} \mathbb{E}^\mathbb{Q}_F \left[ \frac{V_{k+1}}{S_{k+1}^{(1)}} \right] \right\}, \quad 1 \leq k \leq N - 1
\end{align*}
\]

This formulation gives a good intuitive understanding of the pricing problem. It is a dynamic programming problem, where at each time \( t_k \), the value of the contract at that specific time is the largest of the value of exercising, and the expected value of the contract at time \( t_{k+1} \), which can be interpreted as the value of not exercising the contract at time \( t_k \). In addition, an explicit form of the optimal stopping time is given (see e.g. [3] page 78)

\[
\hat{\tau} = \inf \left\{ t \in \{t_k\}_{k=1}^N : V_t = \Phi(S_t) \right\}.
\]

The intuition is that the contract is exercised the first time the value of exercising is equal to the value of the contract, i.e. when there is no excess value in not exercising. Let \( V_{t,\tau} \) be the value of the contract \( \mathcal{C} \) at time \( t \), provided that the contract is exercised at a stopping time \( \tau \in \mathcal{T} \). It is shown in [3] (page 78) that \( \hat{\tau} \in \mathcal{T} \) and that it solves the optimal stopping problem in (2), so that

\[
V_0 = \mathbb{E}^\mathbb{Q} \left[ \frac{\Phi(S_{\hat{\tau}})}{S_{\hat{\tau}}^{(1)}} \right].
\]
In addition, by the optimality of \( \hat{\tau} \) in (2), it holds that,

\[ V_{0,\tau} \leq V_{0,\hat{\tau}}, \quad \forall \tau \in T. \quad (5) \]

This means that any used stopping strategy will give a price which is lower than the optimal price, namely the price which is found by using the optimal stopping strategy.

### 2.3 Differential operators

Assume that the underlying process is of the form

\[ dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t, \]

where \( \{W_t\}_{t \geq 0} \) is an \( r \)-dimensional Brownian motion under \( Q \), \( \sigma(t, x) \in \mathbb{R}^{d \times r} \) and \( b(t, x) \in \mathbb{R}^d \), for \( t > 0, \ x \in \mathbb{R}^d \). Then the infinitesimal generator of \( S \) is the operator \( A \) defined such that, for any \( h \in C^2(\mathbb{R}^d) \) (i.e. the set of twice continuously differentiable functions on \( \mathbb{R}^d \)),

\[ (Ah)(x) = \sum_{i=1}^d b_t \frac{\partial h}{\partial x_i} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij}(x) \frac{\partial^2 h}{\partial x_i \partial x_j}, \]

where \( a_{ij}(x) = \sum_{k=1}^r \sigma_{ik}(x)\sigma_{kj}(x) \). For further reference, see [9] page 281.

The discounted price process \( \frac{V_t}{S_t^{(1)}} \) is a \( Q \)-martingale, thus, the function \( g(t, x) \) which satisfies that

\[ g(t, S_t) = \frac{V_t}{S_t^{(1)}} = \mathbb{E}^Q \left[ \frac{V_u}{S_u^{(1)}} \bigg| \mathcal{F}_t \right], \quad u \geq t, \]

would also satisfy

\[ (A + \partial_t)g(t, x) = 0, \quad (6) \]

which is a smoothness condition put on the price process. This property allows us to restrict the class of prices to those which satisfies (6) and for which \( g(t, x) \in C^{1,2} \).

### 3 Pricing

Let \( X : \Omega \to \mathbb{R}^d \) be any random vector, \( Y : \Omega \to \mathbb{R} \) be any square integrable random variable and let \( D = \{x_j, y_j\}_{j=1}^M \) be a set of observed outcomes of \( X \) and \( Y \). Assume a method of estimating \( f(X) = \mathbb{E}[Y|X] \) from the data \( D \), and denote this estimator by \( \hat{f}_D \). Then \( \hat{f}_D \) can be used in the algorithm to price the contract \( C \). The algorithm, and the way that \( \hat{f}_D \) is used in it, is described below.
Let \( s := \{s_{j,k}\}_{j=1,k=0}^{j=M,k=N} \) denote a set of \( M \) simulated trajectories of \( S \) evaluated at each time \( \{t_k\}_{k=1}^N \), so that \( s_{j,k} \) represent the value of the process in simulation \( j \), at time \( t_k \) and \( s_{j,0} = s_0 \) is a constant starting point.

Let \( \tau_j \) represent the stopping time for trajectory \( s_j \). At time \( t_N = T \), no other choice is given but to exercise the contract, so for each simulation, set \( \tau_j = N \). Take one step backwards in time, to \( t_{N-1} \). A decision must be made, namely which of the simulations that should be stopped at this time. The optimal stopping rule (4), states that the simulation should be stopped if the value of exercising is equal to the expected value of not doing so. That is, the decision is made based on the information

\[
S_{t_{N-1}}^{(1)} \frac{\Phi(S_{t_N})}{S_{t_N}} \mathbb{E}^{Q} \left[ F_{t_{N-1}} \right] = \mathbb{E}^{Q} \left[ S_{t_{N-1}}^{(1)} \frac{\Phi(S_{t_N})}{S_{t_N}} \mathbb{E}^{Q} \left[ F_{t_{N-1}} \right] \right] = \mathbb{E}^{Q} \left[ S_{t_{N-1}}^{(1)} \frac{\Phi(S_{t_N})}{S_{t_N}} S_{t_{N-1}} \right],
\]

by the Markov property.

The estimator \( \hat{f}_D \) can be used in the following way: For each \( k = N-1, \ldots, 1 \), set

\[
D_k = \left\{ s_{j,k}, s_{j,k}^{(1)} \Phi(s_{j,\tau_j}) \frac{s_{j,\tau_j}^{(1)}}{s_{j,\tau_j}} \right\}_{j=1}^M,
\]

i.e. \( X \) and \( Y \) is defined to be \( X = S_{t_k} \) and \( Y = S_{t_k}^{(1)} \frac{\Phi(S_{\tau_j})}{s_{\tau_j}^{(1)}} \), and for each \( j \) that satisfies

\[
\hat{f}_{D_k}(s_{j,k}) < \Phi(s_{j,k}),
\]

set \( \tau_j = k \), and leave all other \( \tau_j \) unchanged.

This procedure gives a stopping strategy for each simulation, which can be used to calculate the price as

\[
V_0 \approx \frac{1}{M} \sum_{j=1}^{M} \Phi(s_{j,\tau_j}) \frac{s_{j,\tau_j}^{(1)}}{s_{j,\tau_j}} ,
\]

for a good estimator \( \hat{f}_D \) and large \( M \), since

\[
V_0 = \mathbb{E}^{Q} \left[ \frac{\Phi(S_{\tau})}{S_{\tau}} \right] = \lim_{M \to \infty} \frac{1}{M} \sum_{j=1}^{M} \frac{\Phi(s_{j,\tau})}{s_{j,\tau}^{(1)}} \quad a.s.,
\]

by the law of large numbers.
Another pricing method would be to iterate backwards in the same way, with the difference of using

\[
D_k = \left\{ s_{j,k}, \frac{s_j^{(1)} V_{t+1}}{s_{j,k}^{(1)}}, \frac{V_{t+1}}{s_{j,k+1}} \right\}_{j=1}^M
\]

with the advantage of not having to keep track of the stopping times \( \tau_j \). This method would be to just recursively calculate the price \( V_t \), for each \( k \). The problem with this approach is that if the estimator is not consistent, then errors in \( V_t \) would accumulate, giving larger error in the final result.

What remains for the pricing procedure to be complete, is to determine a good estimator \( \hat{f}_D \). First, a short review of the approach suggested by Longstaff and Schwartz for that estimator \( \hat{f}_D \) is presented, and then, two examples of pricing using this method will be given, to point out the strengths and weaknesses of the assumptions about this specific choice of \( \hat{f}_D \).

### 3.1 The Longstaff-Schwartz approach

This approach, described in [11], suggests the estimator \( \hat{f}_D \) to be the projection of \( f \) onto a finite set of vectors \( \{\Psi_n\}_{n=1}^K \), belonging to \( L^2(\Omega, \mathcal{F}, \mathbb{Q}) \). It is shown that the best unbiased linear estimator on this form, given the data \( D \), is

\[
\hat{f}_D(x) = \sum_{n=1}^K \hat{c}_n \Psi_n(x),
\]

where,

\[
\{\hat{c}_n\}_{n=1}^K = \arg\min_{\{c_n\}_{n=1}^K} \sum_{j=1}^M \left( y_j - \sum_{n=1}^K c_n \Psi_n(x_j) \right)^2
\]

is a set of real constants. A simple example of the use of this estimator is given below.

### 3.1.1 Example 1: Bermudan put

Let \( S_t \in \mathbb{R}^2 \) for \( 0 < t < T = 1 \), and let

\[
\begin{cases}
S_0^{(1)} = 1, \\
\frac{dS_t^{(1)}}{dt} = 0.1S_t^{(1)} dt, & t > 0
\end{cases}
\quad
\begin{cases}
S_0^{(2)} = 100, \\
\frac{dS_t^{(2)}}{dt} = 0.1S_t^{(2)} dt + 0.3S_t^{(2)} dW_t, & t > 0
\end{cases}
\]

where \( \{W_t\}_{t>0} \) is a one-dimensional Brownian motion, and define

\[
\Phi(x) = \max\{100 - x^{(2)}, 0\}.
\]
The projection vectors used are the polynomials

\[ \Psi_n(x) = (x^{(2)})^{n-1}, \quad n = 1, 2, 3, \quad x = \{x^{(1)}, x^{(2)}\}, \]

and \( M = 10000 \) simulations. Ten time steps are used, i.e.

\[ \{t_k\}_{k=1}^{10} = \left\{ \frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, 1 \right\}. \]

Consider the time \( t_9 \). Since there is only one step to maturity, the Black-Scholes formula implies

\[
S_{t_9}^{(1)} \mathbb{E}^{Q} \left[ \frac{\Phi(S_T)}{S_T^{(1)}} \right] | S_{t_9} = 100 \exp^{-0.1(T-t_9)} N(-d_2) - S_{t_9}^{(2)} N(-d_1),
\]

where \( N \) denote the cumulative normal distribution, and

\[
\begin{cases}
    d_1 = \left( \frac{\mu + \frac{1}{2} \sigma^2 (T-t_9)}{0.3 \sqrt{T-t_9}} \right), \\
    d_2 = d_1 - 0.3 \sqrt{T-t_9}.
\end{cases}
\]

Since the analytic solution is known, the conditional expectation may be compared with the estimator \( \hat{f}_{D_9} \), where

\[ D_9 = \left\{ s_{j,9}, s_{j,9}^{(1)} \frac{\Phi(s_{j,N})}{\Phi^{(1)}(s_{j,N})} \right\}_{j=1}^{M}. \]

This comparison is illustrated in Figure 1, where it can be seen that \( \hat{f}_{D_9} \), drawn with thick line, does not agree very well with the conditional expectation, drawn with a solid line. However, the algorithm only uses \( \hat{f}_{D_9} \) to decide whether or not to exercise at time \( t_9 \). The contract should be exercised if \( \hat{f}_{D_9} \) is lower than the value of exercising. The figure shows that the conditional expectation and \( \hat{f}_{D_9} \) is on the same side of the value of exercising almost along the whole axis. That is, the right decision whether to exercise or not at time \( t_9 \), is likely to be made by using \( \hat{f}_{D_9} \).

### 3.1.2 Bermudan digital

Consider the implications of using the same projection vectors, but with the following payoff function:

\[ \Phi(x) = 100 I_A(x^{(2)}) + 1000 \max\{100 - x^{(2)}, 0\}, \]

where \( A = [160, \infty) \), and \( I_A(x) \) denotes the indicator function. All other parameters are as in the previous example.
Figure 1: Bermudan put. The solid line with dots is $f_{D_2}$ where a second degree polynomial is used, the line plotted with dashes is the value of exercising and the solid line is the conditional expectation. All values are discounted to $t_0$. The second figure a magnification of the first figure.
Figure 2: Bermudan digital: The solid line with dots is $\hat{f}_{D_9}$, where a second degree polynomial is used, the line plotted with dashes is the value of exercising and the solid line is the conditional expectation. All values are discounted to $t_9$.

In this example the Black-Scholes formula cannot be used to compare with $\hat{f}_{D_9}$. However, since the Cox-Ross-Rubinstein binomial model converges to the geometric Brownian motion when the number of time steps tends to infinity (see [6]), the values obtained with the binomial model can be used. The result for $t_9$ is shown in Figure 2.

From the figure it can be deduced that $\hat{f}_{D_9}$ still fits poorly to the conditional expectation. In addition, there are large regions where using $\hat{f}_{D_9}$ to determine when to exercise the contract will result in the wrong decision. This means that it can be expected that the price of the contract is less accurate than in the previous example.

4 Nonparametric estimation

As seen in the two examples in the previous section, the use of a basic second degree polynomial as parametrization of $\hat{f}_D$ can be useful, although good performance cannot be guaranteed. Other choices of vectors could give better results, but this means that the method will have to be tailored to each type of contract to ensure good results. In addition, either prior information about the shape of the conditional expectations must be given, or some trial and error approach must be used. Thus, it might be a good idea to search for nonparametric alternatives of $\hat{f}_D$.

What happens if a nonparametric alternative is used? For the data $D^M =$
\[ \{x_j, y_j\}_{j=1}^M, \text{consider the use of} \]
\[ \hat{f}_{DM}(x_j) = \hat{c}_j, \quad 1 \leq j \leq M, \]
where
\[ \{\hat{c}_j\}_{j=1}^M = \arg\min_{\{c_j\}_{j=1}^M} \sum_{j=1}^M (y_j - c_j)^2. \]

This problem is ill-posed according to [8] page 31, since it does not depend continuously on the input data. Instead, for any data, the solution is \( c = y \). In other words, the regression becomes overfitted.

The problem of finding \( \{\hat{c}_j\}_{j=1}^M \) can be written as
\[ \min_c \|y - c\|^2, \]
where \( y = (y_1, \ldots, y_M)^T, \ c = (c_1, \ldots, c_M)^T \) and where \( \|\cdot\| \) denotes the Euclidean norm.

To resolve the issue of ill-posedness a regularization is introduced, often referred to as Tikhonov regularization (c.f. [8] page 171). The Tikhonov regularization penalizes large deviations of the candidate solutions from some a priori chosen characteristics, where the penalty is expressed as the norm of the candidate solutions transformed by an operator \( B \) added to the problem setting:
\[ \min_c \|y - c\|^2 + \alpha \|Bc\|^2, \]
for some real scalar \( \alpha > 0 \), and some matrix \( B \). The matrix \( B \) can e.g. be chosen as to represent a discretization of the second derivative of the solution. This particular penalty, used e.g. in the Hodrick-Prescott filter (c.f. [7]), decrease the set of possible solutions to the ones having “enough smoothness” with respect to the discretized second derivative. The smoothing parameter \( \alpha \) is used to control the smoothness of the solution.

Knowing that the conditional expectation not only must have a second derivative, but also must satisfy (6), the natural restriction to our set of estimations \( \hat{f}_{DM}^M \) is to the ones satisfying
\[ (\mathcal{A} + \partial_t) \left( \frac{\hat{f}_{DM}(x)}{x^{(1)}} \right) = 0. \]

Assuming the use of some discretization of this partial differential equation (PDE), there will be some error. So instead of demanding that the PDE is exactly 0, the norm should be small, i.e. the discretized PDE can be used as a penalty function. To solve the discretized PDE, equidistant points in space are needed. But for our use of the estimator, it needs to be defined for
all simulated values of the underlying, i.e. the estimator needs to be defined on the whole space axis. A way of handling this is to create an equidistant discretization of the space and for each simulation value of the underlying, use the estimator value of the closest discretization point, with respect to some metric. This is called the Voronoi tessellation and is defined in detail below.

It should be noted that this tessellation will also have a regularizing effect, where a larger number of discretization points and thereby smaller tessellation regions will result in less regularization. Thus, the finer grid chosen, the larger value of the smoothing parameter $\alpha$ is needed, in order to put more weight on the regularization by the PDE.

4.1 Choosing $\hat{f}$

Let $X : \Omega \to \mathbb{R}^d$ be a random vector, let $Y : \Omega \to \mathbb{R}$ be a square integrable random variable and let $\Pi$ be a grid in the range of $X$, denoted $\mathcal{R}(X)$, i.e. $\Pi$ is a set of points in $\mathcal{R}(X)$ with cardinality $|\Pi|$. Let $c_1, c_0 \in \mathbb{R}^{||\Pi||}$ be two vectors of constants with the same size as $\Pi$, and let $G(\Pi, c_0, c_1) \in \mathbb{R}^{||\Pi||}$ be some function of $\Pi$, $c_1$ and $c_0$. Furthermore, let $\{A_n\}_{n=1}^{||\Pi||}$ be the Voronoi tessellation of $\mathcal{R}(X)$ generated by $\Pi$:

$$A_n = \{x \in \mathcal{R}(X) : \|x - x_n\| \leq \|x - x_m\|, x_m \in \Pi, \forall m\},$$

for $x_n \in \Pi$, where $\|\cdot\|$ is the Euclidean norm.

For a given set of random data $D^M = \{X_j, Y_j\}_{j=1}^{M}$, where $X, X_1, X_2, \ldots$ are i.i.d. and $Y, Y_1, Y_2, \ldots$ are i.i.d., define the estimator $\hat{f}_{D^M}(x; \Pi, c_1, G, \alpha)$ to be

$$\hat{f}_{D^M}(x; \Pi, c_1, G, \alpha) = \sum_{n=1}^{||\Pi||} \hat{c}_{0,n} I_{A_n}(x),$$

where

$$\hat{c}_0 = \arg\min_{c_0} \left[ \left\| \mathbf{Y} - \sum_{n=1}^{||\Pi||} c_{0,n} I_{A_n}(\mathbf{X}) \right\|^2 + \alpha \left\| G(\Pi, c_1, c_0) \right\|^2 \right], \quad (8)$$

$\mathbf{Y} = (Y_1, \ldots, Y_M)^T$, $\mathbf{X} = (X_1, \ldots, X_M)^T$ and $I_{A_n}(\mathbf{x}) = (I_{A_n}(x_1), \ldots, I_{A_n}(x_M))^T$.

In the method proposed here, the same pricing algorithm as in section 3 is used. But at each time $t_k$, the estimator

$$\hat{f}_{D^M_k}(x; \Pi_k, V_{t_{k+1}}(\Pi_k), F_A, \alpha),$$
is used, where set $D^M_k$ is set as before, and where $F_A$ is a one-step forward finite difference discretization of (6) on $\Omega_k$. Thus, $F_A$ determines the relation between

$$V_{k+1}(\Omega_k) \in \mathbb{R}^{||\Omega_k||}$$

and

$$V_k(\Omega_k) \in \mathbb{R}^{||\Omega_k||}$$

which is the result of a discretization of the operator $(A + \partial_t)$. Here, $\Omega_k$ is some grid on $\mathbb{R}^d$ defined for each $t_k$, which should be chosen so that the probability that $S_{t_k}$ takes values outside of $\Omega_k$ is small, but also so that for each $n$ there is at least some set of simulation values $s_{j_n,k}$, where $j_n \in [1, \ldots, M]$, so that $s_{j_n,k} \in A_n$. How this grid should be chosen is left for the user to decide, together with the problem of choosing a good smoothing parameter $\alpha$.

Though it has not been studied in this paper, there are methods of finding $\alpha$. One way is to do an analysis of the problem, and make an a priori guess about the value, another way is to use some data-driven method. More on this can be found in [8].

The following results show that the estimator $\hat{f}_{DM}(x; \Pi, c_1, G, \alpha)$ converges to $f(x)$, where $f(X) = \mathbb{E}[Y \mid X]$, as the number of data points goes to infinity, and as the grid $\Pi$ increases. This holds for any choice of $G$, $c_1$ and $\alpha$.

**Proposition 1.** Assume that $\|G(\Omega, c_1, c_0)\|< \infty$ for any finite $\Pi$, $c_1$ and $c_0$. Then

$$\hat{f}_{DM}(x; \Pi, c_1, G, \alpha) \to \sum_{n=1}^{||\Omega||} \mathbb{E}[Y \mid X \in A_n] I_{A_n}(x) \quad a.s.,$$

as $M \to \infty$ for any $c_1$, $G$ and $\alpha$.

**Proof.** Let $c_1$, $G$ and $\alpha$ be arbitrary and define $N_n = \sum_{k=1}^{M} I_{A_n}(X_k)$, for $n = \{1, \ldots, ||\Omega||\}$. Since

$$\min_{c \in \mathbb{R}^{||\Omega||}} \frac{1}{M} \sum_{k=1}^{M} \left(Y_k - \sum_{n=1}^{||\Omega||} c_n I_{A_n}(X_k)\right)^2$$

is a minimum norm problem in $\mathbb{R}^{||\Omega||}$, the unique vector giving the minimal solution is $c^M$, where

$$c^M_n = \frac{1}{N_n} \sum_{k=1}^{M} Y_k I_{A_n}(X_k), \quad n = 1, \ldots, ||\Omega||.$$
Hence,

\[
\min_{c \in \mathbb{R}^{[\Pi]}} \frac{1}{M} \sum_{k=1}^{M} \left( Y_k - \sum_{n=1}^{[\Pi]} c_n I_{A_n}(X_k) \right)^2 = \frac{1}{M} \sum_{k=1}^{M} \left( Y_k - \sum_{n=1}^{[\Pi]} c_n^M I_{A_n}(X_k) \right)^2
\]

\[
= \frac{1}{M} \sum_{k=1}^{M} \left( Y_k - \sum_{n=1}^{[\Pi]} \frac{1}{N_n} \sum_{k=1}^{M} Y_k I_{A_n}(X_k) \right)^2.
\]

It follows that

\[
\left| \frac{1}{M} \sum_{k=1}^{M} \left( Y_k - \sum_{n=1}^{[\Pi]} c_n^M I_{A_n}(X_k) \right)^2 - \mathbb{E} \left( Y - \sum_{n=1}^{[\Pi]} \mathbb{E} [Y | X \in A_n] I_{A_n}(X) \right)^2 \right| \rightarrow 0 \text{ a.s.,}
\]

as \( M \to \infty \) since

\[
\frac{1}{M} \sum_{k=1}^{M} Y_k^2 \to \mathbb{E} Y^2 \text{ a.s.}
\]

\[
\frac{N_n}{M} \to P(X \in A_n), \ n \in \{1, \ldots, [\Pi]\} \text{ a.s.}
\]

\[
c_n^M = \frac{M}{N_n} \left( \frac{1}{M} \sum_{k=1}^{M} Y_k I_{A_n}(X_k) \right) \to \mathbb{E} [Y | X \in A_n] \text{ a.s.}
\]

as \( M \to \infty \). Thus, if

\[
S(c, M) := \frac{1}{M} \sum_{k=1}^{M} \left( Y_k - \sum_{n=1}^{[\Pi]} c_n I_{A_n}(X_k) \right)^2
\]

and

\[
E := \mathbb{E} \left( \sum_{n=1}^{[\Pi]} \mathbb{E} [Y | X \in A_n] I_{A_n}(X) \right)^2
\]

14
the above show that
\[ S(\hat{c}, M) \to E \quad \text{a.s.} \tag{9} \]
as \(M \to \infty\).

Let \( \hat{c} = (\hat{c}_1, \ldots, \hat{c}_{|\Pi|})^T \), where \( \hat{c}_n = \mathbb{E}[Y \mid X \in A_n] \). Then

\[ S(\hat{c}, M) \to E \quad \text{a.s.} \]

by the same reasoning as for \( c^M \). And since \( \|G(\Pi, c_1, c_0)\|^2 < \infty \) for any \( c_0 \), the constant \( \|G(\Pi, c_1, \hat{c})\|^2 \) is also finite. Thus, it follows that

\[ S(\hat{c}, M) + \frac{\|G(\Pi, c_1, \hat{c})\|^2}{M} \to E \quad \text{a.s.,} \tag{10} \]
as \( M \to \infty \).

Now, define \( \hat{c}^M \) as

\[ \hat{c}^M := \arg\min_{c \in \mathbb{R}^{|\Pi|}} S(c, M) + \frac{\|G(\Pi, c_1, c)\|^2}{M}. \]

Furthermore, define

\[
\mathcal{M}_{\epsilon,1} = \left\{ M \in \mathbb{N}^+ : S(\hat{c}^M, M) + \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} > E + \epsilon \right\}
\]
and
\[
\mathcal{M}_{\epsilon,2} = \left\{ M \in \mathbb{N}^+ : S(\hat{c}^M, M) + \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} < E - \epsilon \right\}
\]

Then, if it can be shown that both \( \mathcal{M}_{\epsilon,1} \) and \( \mathcal{M}_{\epsilon,2} \) are finite almost surely for any \( \epsilon > 0 \), it follows that

\[ S(\hat{c}^M, M) + \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} \to E \quad \text{a.s.,} \]
as \( M \to \infty \).

Assume that \( |\mathcal{M}_{\epsilon,1}| = \infty \). From (10) it follows that there almost surely exists an \( \hat{M} \in \mathcal{M}_{\epsilon,1} \) such that

\[ S(\hat{c}, M) + \frac{\|G(\Pi, c_1, \hat{c})\|^2}{M} < E + \frac{\epsilon}{2}, \quad M \geq \hat{M}. \]

But then, for any \( M \in \mathcal{M}_{\epsilon,1} \) such that \( M \geq \hat{M} \),

\[ E + \epsilon < S(\hat{c}^M, M) + \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} \leq S(\hat{c}, M) + \frac{\|G(\Pi, c_1, \hat{c})\|^2}{M} < E + \frac{\epsilon}{2}, \]
since \( \hat{c}^M \) is the minimizing set of constants. This is clearly a contradiction and hence it follows that \( \mathcal{M}_{\epsilon} \) is finite almost surely for any \( \epsilon > 0 \).

Now assume that \( |\mathcal{M}_{\epsilon,2}| = \infty \). From (9) it follows that there almost surely exists an \( \hat{M} \in \mathcal{M}_{\epsilon,2} \) such that
\[
S(c^M, M) > E - \frac{\epsilon}{2}, \quad M \geq \hat{M}.
\]
But then, for any \( M \in \mathcal{M}_{\epsilon,2} \) such that \( M \geq \hat{M} \),
\[
E - \frac{\epsilon}{2} < S(c^M, M) \leq S(\hat{c}^M, M) \leq S(\hat{c}^M, M) + \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} < E - \epsilon,
\]
since \( \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} \geq 0 \) and \( c^M \) is a minimizing set of constants. This is also a contradiction and it thereby follows that \( \mathcal{M}_{\epsilon,2} \) is finite almost surely for any \( \epsilon > 0 \). Hence, by the projection theorem (see [12]),
\[
\min_{c \in \mathbb{R}^{[\Pi]}} S(c, M) + \frac{\|G(\Pi, c_1, c)\|^2}{M} = S(\hat{c}^M, M) + \frac{\|G(\Pi, c_1, \hat{c}^M)\|^2}{M} \to E \text{ a.s.}
\]
Therefore,
\[
\hat{c}^M \to E[Y | X \in A_n] \quad \text{a.s.} \tag{11}
\]
as \( M \to \infty \). Thus,
\[
\hat{f}_{DM}(x; \Pi, c_1, G, \alpha) = \sum_{n=1}^{[\Pi]} \hat{c}^M_n I_{A_n}(x) \to \sum_{n=1}^{[\Pi]} E[Y | X \in A_n] I_{A_n}(x) \quad \text{a.s.}
\]
as \( M \to \infty \).

The following result state that if the grid \( \Pi \) tend to infinity in a particular way, then the estimator converges to the conditional expectation. In the proof of this, the following definitions and lemma are needed.

Let \( \Pi \) be a grid in \( A \subseteq \mathbb{R}^d \) and let
\[
\{A_n^{[\Pi]}\}_{n=1}^{k}, \quad k \geq 1
\]
be the Voronoi tessellation on \( A \) which is generated by \( \Pi \). Define the outer sets to be the sets \( \{A_n\}_{n \in N_D} \), where
\[
N_D = \left\{ n \in (1, \ldots, [\Pi]) : \sup_{x,y \in A_n} ||x - y|| = \infty \right\}, \tag{12}
\]
and define the inner sets to be the sets \( \{ A_n \}_{n \in \mathbb{N}} \), where

\[
N_I = \left\{ n \in (1, \ldots, |\Pi|) : \sup_{x,y \in A_n} \|x - y\| < \infty \right\},
\]

(13)

and where \( \| \cdot \| \) denotes the Euclidean norm.

**Lemma 1.** Assume that \( f \) is twice differentiable, where \( f(X) = \mathbb{E}[Y|X] \). Furthermore, assume that for some \( a > 0 \), \( f(x) \) is locally Lipschitz on \( \{ x \in \mathbb{R}^d : \|x\| > a \} \). Then \( f \) is Lipschitz.

**Proof.** Since \( f \) is twice differentiable, \( \nabla f \) is continuous. Thus, in any compact set \( K \), \( \nabla f \) is bounded and \( f \) is therefore locally Lipschitz on \( K \). Set \( K = \{ x \in \mathbb{R}^d : \|x\| \leq a \} \), then \( f \) is locally Lipschitz on \( K \), and by assumption \( f \) is also locally Lipschitz on \( \{ x \in \mathbb{R}^d : \|x\| > a \} \). Hence, \( f \) is Lipschitz. \( \square \)

The assumptions in the above lemma are in our application quite reasonable. The twice differentiability is already assumed when talking about the infinitesimal generator, and the Lipschitz property for large \( X \) can be argued for in the following way: \( X \) represents the value of the underlying at some time \( t \), while \( Y \) represents the payoff received by using the stopping strategy which is pre-determined for all times greater than \( t \). Thus, for any payoff function which is linear for large enough values on the underlying (think e.g. about an option), the expected payoff will also be linear for large enough values \( X \). Therefore, there must exist some \( a > 0 \) such that \( f(x) \) is locally Lipschitz on \( \{ x \in \mathbb{R}^d : \|x\| > a \} \).

**Proposition 2.** Let

\[
\{ A_{n,k} \}_{n=1}^{\Pi_k}, \quad k \geq 1,
\]

be the Voronoi tessellations on \( R(X) \) generated by a sequence of grids \( \Pi_k \). Let \( N_{I,k} \) and \( N_{O,k} \) be the indices of the inner and outer sets respectively of tessellation \( k \), given by (12) and (13). Assume that

1. \( f \) is twice differentiable, where \( f(X) = \mathbb{E}[Y|X] \),
2. \( f \) is locally Lipschitz on \( \{ x \in \mathbb{R}^d : \|x\| > a \} \), for some \( a \),
3. \( P \left( X \in \bigcup_{n \in N_{O,k}} A_{n,k} \right) \to 0 \), as \( k \to \infty \),
4. \( \max_{n \in N_{I,k}} \sup_{x,y \in A_{n,k}} \|x - y\| \to 0 \), as \( k \to \infty \).

Then,

\[
\lim_{k \to \infty} \lim_{M \to \infty} \hat{f}_{DM}(X; \Pi_k, c_1, G, \alpha) = \mathbb{E}[Y|X]
\]

in \( L^2 \), for any \( c_1 \), \( G \) and \( \alpha \).
Proof. Let $B_{n,k} = \{ \omega : X(\omega) \in A_{n,k} \}$ and let $K_L$ denote the Lipschitz constant which exists by Lemma 1. From Proposition 1, and the disjoint property of the Voronoi tessellation (which holds except possibly on a null set),

$$
\begin{align*}
E \left[ \left( E[ Y | X ] - \hat{f}_{D^M} (X; \Pi_k, c_1, G, \alpha) \right)^2 \right] \\
= E \left[ \left( f(X) - \sum_{n=1}^{||\Pi_k||} E \left[ f(X) | X \in A_{n,k} \right] I_{A_{n,k}} (X) \right)^2 \right] \\
= \sum_{n=1}^{||\Pi_k||} E \left[ \left( f(X) - E \left[ f(X) | X \in A_{n,k} \right] \right)^2 I_{A_{n,k}} (X) \right] \\
= \sum_{n=1}^{||\Pi_k||} P(B_{n,k}) E \left[ \left( f(X) - E \left[ f(X) | B_{n,k} \right] \right)^2 \right] |B_{n,k}| \\
= \sum_{n \in N_{I,k}} P(B_{n,k}) E \left[ \left( f(X) - E \left[ f(X) | B_{n,k} \right] \right)^2 \right] |B_{n,k}| \\
\quad + \sum_{n \in N_{O,k}} P(B_{n,k}) E \left[ \left( f(X) - E \left[ f(X) | B_{n,k} \right] \right)^2 \right] |B_{n,k}| \\
\leq \sum_{n \in N_{I,k}} P(B_{n,k}) \sup_{x,y \in A_{n,k}} (f(x) - f(y))^2 \quad \quad + \sum_{n \in N_{O,k}} P(B_{n,k}) E \left[ f(X)^2 \right] |B_{n,k}| \\
\leq \sum_{n \in N_{I,k}} P(B_{n,k}) K_L^2 \sup_{x,y \in A_{n,k}} \|x - y\|^2 \quad \quad + \sum_{n \in N_{O,k}} E \left[ f(X)^2 I_{A_{n,k}} (X) \right] \\
\leq K_L^2 \max_{n \in N_{I,k}, x,y \in A_{n,k}} \|x - y\|^2 \quad + \quad E \left[ f(X)^2 I\{ X \in \cup_{n \in N_{O,k}} A_{n,k} \} \right] \\
\leq K_L^2 \epsilon^2 + \epsilon,
\end{align*}
$$

for large enough $k$, for any $\epsilon > 0$. \hfill $\square$

Corollary 1.

$$
E \left[ \left( Y - \hat{f}_{D^M} (X; \Pi_k, c_1, G, \alpha) \right)^2 \right] \rightarrow E \left[ (Y - E[ Y | X ])^2 \right],
$$

as $k \rightarrow \infty$.

Proof. For any $Z \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$, denote $\|Z\|^2 = E \left[ Z^2 \right]$. Then, by the triangle inequality,

$$
\begin{align*}
\| Y - \hat{f}_{D^M} (X; \Pi_k, c_1, G, \alpha) \| \\
\leq \| Y - E[ Y | X ]\| + \| E[ Y | X ] - \hat{f}_{D^M} (X; \Pi_k, c_1, G, \alpha) \| \\
\rightarrow \| Y - E[ Y | X ]\| = \sqrt{E \left[ (Y - E[ Y | X ])^2 \right]},
\end{align*}
$$

as $k \rightarrow \infty$. \hfill $\square$
The last result of this section provides a closed form solution for the constants $\hat{c}_0$, given a particular form of the function $G$. For this define, for any vector $v \in \mathbb{R}^m$, and any $m \geq 1$, $\hat{v}$ as every element of $v$ except the first and the last one:

$$\hat{v} = (v_2, \ldots, v_{m-1})^T.$$  

**Proposition 3.** Assume that $G$ is of the form

$$G(\Pi, c_0, c_1) = a\hat{c}_1 + Bc_0,$$

where $a \in \mathbb{R}$ and $B \in \mathbb{R}^{(|\Pi|-2) \times |\Pi|}$. Then, the constants $\hat{c}_0$ defined in (8), are given by

$$\hat{c}_0 = (A^T A + \alpha B^T B)^{-1} (A^T y - \alpha a\hat{c}_1),$$

where $A_{j,n} = I_{A_n}(x_j)$, given that $A^T A$ has full rank.

**Proof.** Let $A_{j,n} = I_{A_n}(x_j)$, so that $A \in \mathbb{R}^{M \times |\Pi|}$. $\hat{c}_0$ is the set of constants that minimizes

$$\sum_{j=1}^{M} \left( y_j - \sum_{n=1}^{\|\Pi\|} A_{j,n}c_{0,n} \right)^2 + \alpha (a\hat{c}_1 + Bc_0)^T (a\hat{c}_1 + Bc_0)$$

$$= \frac{1}{2} (y - Ac_0)^T (y - Ac_0) + \frac{\alpha}{2} (a\hat{c}_1 + Bc_0)^T (a\hat{c}_1 + Bc_0).$$

The first order conditions give

$$0 = -A^T y + A^T Ac_0 + \alpha aB^T\hat{c}_1 + \alpha B^T Bc_0,$$

which gives that

$$\hat{c}_0 = (A^T A + \alpha B^T B)^{-1} (A^T y - \alpha a\hat{c}_1).$$

To establish convexity of the minimizing problem, assuring uniqueness of $\hat{c}_0$, $A^T A + \alpha B^T B$ needs to be positive definite, which is the case if enough simulations are made for at least one simulation to occupy each set $A_n$, which gives $A^T A$ full rank. □

5 Numerical examples

In the setting of Sections 3.1.1 and 3.1.2, let

$$\frac{V_t(S^{(2)}_t)}{S^{(1)}_t} = \frac{v(t, S^{(2)}_t)}{S^{(1)}_t} = g(t, S^{(1)}_t, S^{(2)}_t),$$

then $V_t$ satisfies
\[ 0 = \frac{\partial v}{\partial t}(t, y) - rv(t, y) + ry \frac{\partial v}{\partial y}(t, y) + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial y^2}(t, y) \]

which is commonly known as the Black-Scholes equation (c.f. [4] page 97).

To derive \( F_A \) from this, the grid must first be chosen. Let \( \Delta y = (y_{\text{max}} - y_{\text{min}}) / 200 \), where \( y_{\text{max}} = 2.1232 \) and where \( y_{\text{min}} = 0.5258 \). This particular choice asserts that

\[ P \left( \left\{ S_T^{(2)} > y_{\text{max}} \right\} \cup \left\{ S_T^{(2)} < y_{\text{min}} \right\} \right) = 0.01. \]

Another implication is that, for \( t_k < T \) there will be a higher probability that no simulations will fill the outer sets of the tessellation. However, the simplicity of having a constant grid justifies this choice for this simple example.

Therefore, let \( \Pi_k = \{ y_1, \ldots, y_{200} \} \) for all \( k \), where \( y_i = y_{\text{min}} + (i - 1) \Delta y \).

Finite difference approximations are used for the partial derivatives. Let \( v_{k,i} = v(t_k, y_i) \), then

\[
\begin{align*}
\frac{\partial v}{\partial t}(t_k, y_i) & \approx \frac{v_{k+1,i} - v_{k,i}}{\Delta t} \\
\frac{\partial v}{\partial y}(t_k, y_i) & \approx \frac{v_{k,i+1} - v_{k,i-1}}{2\Delta y} \\
\frac{\partial^2 v}{\partial y^2}(t_k, y_i) & \approx \frac{v_{k,i+1} - 2v_{k,i} + v_{k,i-1}}{(\Delta y)^2},
\end{align*}
\]

which yields

\[
0 = \frac{\partial v}{\partial t}(t_k, y_i) - rv(t_k, y_i) + ry_i \frac{\partial v}{\partial y}(t_k, y_i) + \frac{1}{2} \sigma^2 y_i^2 \frac{\partial^2 v}{\partial y^2}(t_k, y_i)
\approx \frac{v_{k+1,i} - v_{k,i}}{\Delta t} - rv_{k,i} + ry_i \frac{v_{k,i+1} - v_{k,i-1}}{2\Delta y} + \frac{1}{2} \sigma^2 y_i^2 \frac{v_{k,i+1} - 2v_{k,i} + v_{k,i-1}}{(\Delta y)^2}
= \frac{1}{\Delta t} (v_{k+1,i} - a_i v_{k,i-1} + b_i v_{k,i} + c_i v_{k,i+1}),
\]

where

\[
\begin{align*}
a_i &= \frac{1}{2} \left( \frac{\sigma y_i}{\Delta y} \right)^2 - \frac{ry_i}{2\Delta y} \\
b_i &= -\left( \frac{1}{\Delta t} + r + \frac{\sigma y_i}{\Delta y} \right)^2 \\
c_i &= \frac{1}{2} \left( \frac{\sigma y_i}{\Delta y} \right)^2 + \frac{ry_i}{2\Delta y},
\end{align*}
\]
for $i = 1, \ldots, 200$, and where $\Delta t = t_{k+1} - t_k$ for all $k$. Hence,

$$
\begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\approx \frac{1}{\Delta t}
\begin{bmatrix}
v_{k+1,2} \\
\vdots \\
v_{k+1,199}
\end{bmatrix} +
\begin{bmatrix}
a_2 & b_2 & c_2 & 0 & 0 & 0 & \ldots & 0 \\
0 & a_3 & b_3 & c_3 & 0 & 0 & \ldots & 0 \\
0 & 0 & a_4 & b_4 & c_4 & 0 & \ldots & 0 \\
\vdots \\
0 & \ldots & 0 & 0 & 0 & a_{199} & b_{199} & c_{199}
\end{bmatrix}
\begin{bmatrix}
v_{k,1} \\
\vdots \\
v_{k,200}
\end{bmatrix}
$$

$$
= \frac{1}{\Delta t} \hat{V}_{t_{k+1}}(\Pi_k) + BV_t(\Pi_k).
$$

This means that

$$
F_A(\Pi_k, c_0, c_1) = \frac{1}{\Delta t} \hat{c}_1 + Bc_0.
$$

Below, numerical results are given from the examples in sections 3.1.1 and 3.1.2, which is obtained by the use of $\hat{f}_{D_k}(x; \Pi_k, V_{t_{k+1}}(\Pi_k), F_A, \alpha)$, where $\Pi_k$ and $F_A$ is given by the above, and where $D_k$ is given as before.

### 5.1 Bermudan put

In Figure 3 a comparison is shown between $\hat{f}_{D_9}(x; \Pi_k, V_{t_{k+1}}(\Pi_k), F_A, \alpha)$ and the Black-Scholes price. The figure shows that $\hat{f}_{D_9}(x; \Pi_k, V_{t_{k+1}}(\Pi_k), F_A, \alpha)$ give a smooth fit to the true expectation. In addition, the prices below show good accuracy, for a wide range of penalty parameters, $\alpha$. The prices are also an improvement, compared to the price obtained with a second degree polynomial, for a wide range of $\alpha$:

<table>
<thead>
<tr>
<th>Method</th>
<th>$\alpha$</th>
<th>Price</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>-</td>
<td>8.225</td>
<td>-</td>
</tr>
<tr>
<td>Longstaff</td>
<td>-</td>
<td>7.933</td>
<td>3.5%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>0.001</td>
<td>8.1234</td>
<td>1.2%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>0.01</td>
<td>8.0718</td>
<td>1.9%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>0.1</td>
<td>8.0876</td>
<td>1.7%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>1</td>
<td>8.0692</td>
<td>1.9%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>10</td>
<td>8.0782</td>
<td>1.8%</td>
</tr>
</tbody>
</table>

### 5.2 Bermudan digital

In Figure 4, it is seen that $\hat{f}_{D_9}(x; \Pi_k, V_{t_{k+1}}(\Pi_k), F_A, \alpha)$ fits well to the true expectation in this example as well. Moreover, the results still show stable accuracy, but with a larger improvement, compared to the price obtained with a second degree polynomial:
Figure 3: Bermudan put: The solid line with dots is $\hat{f}_{D_9}(x; \Pi_k, V_{t_k+1}(\Pi_k), F_A, \alpha)$, the line plotted with dashes is the value of exercising, and the solid line is the true value of not exercising. All values are discounted to $t_9$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\alpha$</th>
<th>Price</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>-</td>
<td>92.69</td>
<td>-</td>
</tr>
<tr>
<td>Longstaff</td>
<td>-</td>
<td>84.41</td>
<td>8.9%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>0.001</td>
<td>91.19</td>
<td>1.62%</td>
</tr>
<tr>
<td>PDE-reg</td>
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<td>90.84</td>
<td>2.00%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>0.1</td>
<td>91.33</td>
<td>1.45%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>1</td>
<td>90.97</td>
<td>1.85%</td>
</tr>
<tr>
<td>PDE-reg</td>
<td>10</td>
<td>87.4</td>
<td>5.71%</td>
</tr>
</tbody>
</table>

6 Conclusions

The method suggested in this paper is an attempt to solve the discrete optimal stopping problem by combining the Monte Carlo method with the PDE method to estimate conditional expectations. This combination is done in the sense that the space of solutions is restricted from the whole of $L^2(\Omega, \mathcal{F}, \mathbb{Q})$, to the set of solutions that also satisfy the partial differential equation which characterizes the underlying process. To achieve this, the norm of the discretized PDE of the solutions is used as a penalty function.

It is shown that the Tikhonov regularized method gives stable and accurate results, for a wide range of values of the smoothing parameter. In addition, it gives better prices compared to the use of a second degree poly-
Figure 4: Bermudan digital: The solid line with dots is $\hat{f}_{D_0}(x; \Pi_k, V_{k+1}(\Pi_k), F_A, \alpha)$, the line plotted with dashes is the value of exercising, and the solid line is the true value of not exercising. All values are discounted to $t_0$.

nominal, in the examples provided here. The real gain is the nonparametric way that the method operates, combined with the fact that the estimation of the conditional expectations is much better on the whole space. To be specific, the method does not need insight on what projection vectors to use for each new contract. Granted, the regularized method requires insight on what $\alpha$ to use, but there are methods of doing this systematically (see [14] page 97).

The biggest drawback of the method presented in this paper is that the PDE is calculated on a grid. As such, the curse of dimensionality is likely to still pose a problem as the number of dimensions increase. Hence, as it is now, the method does not solve the problem that it was intended to solve. Nevertheless, it is a first take on a novel approach, and further developments of the method do seem possible. In fact, some attempts of relaxing the use of the grid have been done as of the time of this writing, with some success. However, the results are still too preliminary to be able to draw any conclusions on the performance in higher dimensions.

We note also that in terms of computation times, the classic Longstaff-Schwartz method is unmatched. So for the gain of not having parametrization in the Tikhonov regularization, the cost in terms of computation time is increased. Hence, an interesting question that remains unanswered is the behaviour of this increase in computation time for higher dimensions.
References


