Investigation of nanophotonic devices based on transformation optics

Transforming reflective optical devices

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INVESTIGATION OF NANOPHOTONIC DEVICES
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Abstract

Transformation optics (TO), which provides an elegant way of molding the flow of light to one’s wishes, has become one of the most popular photonics research areas during the last few years. Owing to stringent material parameters of transformation media, TO is in general not favourable for designing practical applications. The recent proposal of carpet cloak, a device that optically hides an anomaly on an otherwise flat reflective surface, simplifies material requirements due to the relaxed boundary condition on the cloak’s reflective border, thus providing the prospect of realization at optical wavelength. In light of this approach, this thesis introduces a general procedure for transforming reflective optical devices, including in particular focal mirrors and diffraction gratings. The curved or zigzagged surfaces of such devices are flattened through a smooth coordinate mapping which makes convenient use of the loose boundary conditions on the reflective surface. The resulting devices are transformation media without extreme material parameters. For two-dimensional structures, it is even possible to attain an approximate dielectric-only implementation when considering only transverse-electric or transverse-magnetic incidence. The flattened reflective devices are finally adapted to operate in a transmission mode, creating focal lenses and transmissive diffraction gratings. It is illustrated through full-wave simulation that the performance of these transformation optical devices — under the right circumstances also for the dielectric only implementations — surpasses their traditional equivalents.
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Chapter 1

Introduction

1.1 Background on transformation optics

The ambition of man to control the flow of light goes back to ancient times. The oldest optical device in the history of mankind must be the mirror, remains of ancient mirrors date back to about 8000 years ago [1]. Theories on the propagation of light, necessary for designing and understanding optical devices, have also developed over a long time span. It was for example in the first century AD that Hero of Alexandria described how light rays propagating in uniform media follow the path of shortest length [2]. The modern version of this principle was developed in the 17th century by Pierre de Fermat: light travels along a path of extremal optical distance [3]. This knowledge is still one of the main contributors to understanding the operation of modern optical devices. When a full description of wave propagation is needed, we need to advance 200 years in time in order meet James Clerk Maxwell. It are his famous equations [4] that form the cornerstone of modern optical theory including the subject of this thesis, transformation optics.

Although Maxwell’s equations had earlier been investigated in differential geometries [5], the actual emergence of transformation optics occurred in 2006. That year, two independent papers, by the hand of Ulf Leonhardt [6] and Sir John B. Pendry [7], were published discussing optical invisibility devices through coordinate transformations. Leonhardt gave a thorough review of conformal cloaking based on the Helmholtz equation [8], creating two dimensional devices having an isotropic and dielectric material profile [9]. Pendry approached the subject through differential geometry, electromagnetic waves are space transformed by a device with an anisotropic dielectric and magnetic response. A few months after publication of these papers, transformation optics really emerged as a hot topic as — based upon Pendry’s principle — David Schurig et al. demonstrated an invisibility cloak operating at microwave frequencies [10].
Over the last few years, many publications concerning transformation optics have seen the light. There are lenses \cite{11} (operating beyond the diffraction limit \cite{12}), power collectors \cite{13, 14}, polarisation splitters \cite{15, 16} and rotators \cite{17, 18}, electromagnetic wormholes \cite{19} and black holes \cite{20} and many more. Invisibility cloaking, the topic where the transformation optics mania began with, remains however the most popular field of research. This is not surprising as humankind has always been fascinated by this illusion. Documented examples go back to for example ancient Greece and Herodotus’ Histories, recounting how Perseus used the invisibility helmet of Hades in order to escape Medusa’s lair \cite{21}. The reader can certainly think of plenty examples of invisibility devices in contemporary fiction. In reality, it is even difficult (impossible) to achieve free space invisibility cloaking at one wavelength. One needs either extreme material parameters \cite{22} or metamaterial magnetic response, obtained by using resonating and thus lossy structures \cite{23}.

Carpet cloaking \cite{24}, this is the act of concealing an irregularity in otherwise perfectly plane reflective surface, serves as a refugee centre for invisibility cloaks. The less stringent material parameters make this type of cloak very popular \cite{25–27}.

1.2 Outline of this thesis

The remainder of this thesis follows a constructive course. Chapter 2 provides the mathematical description of differential geometry necessary to understand the concept transformation optics. This concept is presented in chapter 3 where it is seen how electromagnetic space is transformed through altered material parameters. The function of chapter 4 is to give an overview of some fundamental devices and makes the reader familiar with the math in the meanwhile. Chapter 5 constitutes the main dish of this thesis; the concept carpet cloaking is reversed and functional non-flat mirrors are transformed into planar devices, making use of the non-restrictive sliding boundary conditions at the surface. The transformed devices are finally extended in order to operate in transmission. Chapter 6 at last gives an endnote on the results of chapters 4 and 5 together with a glance on possible future work.
Chapter 2

Basics of differential geometry

A fundamental theory closely related to transformation optics is Fermat’s principle. Within a material with refractive index \( n(r) \), light rays follow paths of extremal optical distance. This principle explains interesting phenomena such as mirages and can moreover aid in describing practical devices such as lenses. When it comes to designing devices that voluntarily control the flow of light, Fermat’s principle does not suffice because it only describes isotropic dielectric media and gives no direct creational aid.

Transformation optics needs the full-blown power of Maxwell’s equations. As optical devices are created through a transformation of space, we need to know how these equations govern light propagation in this transformed space. This chapter acts as an introduction to differential geometry and provides the founding math of the following chapters. It is a mere compilation of some important expressions from standard differential geometry theory \[28\] and largely follows the structure of the excellent work by Leonhardt and Philbin \[29\]. The inclusion of the following theory is though accounted for by the insight it provides in later solutions and it makes this work self sustainable in addition.

2.1 The metric tensor

Consider the distance \( L \) between two points \( A \) and \( B \) along a trajectory \( \zeta \) in an arbitrary three-dimensional Euclidean space \( \{x^i, i = 1, 2, 3\} \). This distance should not depend on its coordinate system, so an identical path length will be found in a second space \( \{x'^i, i' = 1, 2, 3\} \) which is a bijective differentiable mapping from the first one. The metric tensor provides us with local information on the distance between two infinitesimal close points in these metric spaces, it is thus a necessity for calculating general distances \( L \).

Although the following theory is valid for an arbitrary pseudo-Euclidean space, per example a Minkowski space including time geometry \[29\], this thesis only handles three-dimensional
Chapter 2. Basics of differential geometry

Euclidean spaces which allows us to drop the explicit numbering of the different dimensions \( x^i \). Our reference system is a Cartesian space \( x^i = \{x, y, z\} \) for which the square distance between two adjacent points is known to be given by

\[
\begin{align*}
 ds^2 &= dx^2 + dy^2 + dz^2 = \delta_{ij} dx^i dx^j. 
\end{align*}
\]

\( \delta_{ij} \) is Kronecker’s delta and the compact Einstein notation has been introduced:

\[
\begin{align*}
 A^i B_i &= \sum_i A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3.
\end{align*}
\]

Expression 2.1 can be generalized by defining the symmetric metric tensor \( g_{ij} \) which enables calculation of distance in non-Cartesian systems

\[
\begin{align*}
 ds^2 &= g_{ij} dx^i dx^j = g_{i'j'} dx^{i'} dx^{j'}.
\end{align*}
\]

Making use of the transformation matrix \( \Lambda_i^{i'} \)

\[
\begin{align*}
 dx^i &= \frac{\partial x^i}{\partial x^{i'}} dx^{i'} = \Lambda_i^{i'} dx^{i'},
\end{align*}
\]

we find the relation between \( g_{ij} \) and its transformed brother \( g_{i'j'} \) to be

\[
\begin{align*}
 g_{i'j'} &= \Lambda_{i'}^{i} \Lambda_{j'}^{j} g_{ij}.
\end{align*}
\]

It is interesting to remember this relation in matrix form where \( G \) represents the metric tensor and \( \Lambda \) is the matrix representation of the transformation tensor \( \Lambda_i^{i'} \). With the first index in tensor notation corresponding to the rows in matrix notation, a convention used throughout this whole thesis, the metric transformation looks like

\[
\begin{align*}
 G' = \Lambda^T \cdot G \cdot \Lambda.
\end{align*}
\]

Finally we define \( g \) \( (g') \) as being the determinant of \( G \) \( (G') \).

2.2 Vectors, one-forms and general tensors

The elements \( dx^i \) composing infinitesimal line segments are in essence vectors. An arbitrary vector \( V \) thus transformed according to the same transformation principle 2.4. As in each coordinate system \( x^i \) a vector consists of components \( V^i \) along base vectors \( e_i \), we can write

\[
\begin{align*}
 V &= V^i e_i = \Lambda_{i'}^{i} V^{i'} e_i (= V^{i'} e_{i'}). 
\end{align*}
\]

This expression shows how the transformation defines coordinate bases \( e_{i'} \):
\[ e_i = \lambda^i_{i'} e_{i'} \]  

It is important to keep in mind that throughout this work coordinate bases will be used, these are generally neither orthogonal nor unitary. When later on expressions in e.g. cylindrical coordinates are used, they might differ from what the reader is used to since most textbooks choose to express themselves by using orthonormal base vectors.

Returning to the vector identity of the components \( dx^i \), the length of a vector \( V \) can be written as

\[ |V|^2 = V \cdot V = g_{ij} V^i V^j = g_{i'j'} V^{i'} V^{j'} . \]  

The invariance of this equation is explained by the inverse transformation characteristics of the metric tensor and the vector components. The Einstein notation proves to be very convenient since the index position indicates how objects will transform and it invites us to define a new entity: the covariant vector or one-form. The one-form is defined as a vector with lowered indices \( V_i \) and simplifies the appearance of metric independent products such as (2.9).

\[ V_i = g_{ij} V^j , \]  

\[ U \cdot V = g_{ij} U^i V^j = U_i V^i = U^{i'} V_{i'} = U_{i'} V^{i'} , \]  

\[ g^{ij} g_{jk} = \delta^i_k . \]  

The last expression, using an index raised Kronecker delta, defines the inverse metric tensor \( g^{ij} \) and allows to transform one-forms back to regular vectors.

A general tensor is defined as an operator that transforms according to definition 2.4 and whose indices can be lowered by applying the metric tensor or elevated using the inverse metric tensor.

### 2.3 Vector products

Calculating the vector product in Cartesian coordinates is a relatively complicated operation. In order to reproduce its anti-symmetricity and the cyclic nature of its base vectors, the permutation symbol \([ijk]\)

\[ [ijk] = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases} \]  

(2.13)
can be used to define the Levi-Civita tensor $\epsilon^{ijk}$. Given the identity $\epsilon^{ijk} = [ijk]$ in right-handed Cartesian coordinate systems $x^i$, it is found that for a general system $x^i$

$$
\epsilon^{i'j'k'} = \Gamma'_{i'}^i \Gamma'_{j'}^j \Gamma'_{k'}^k [ijk] = \frac{[i'j'k']}{\det \Lambda} = \pm \sqrt{g}.
$$

The Levi-Civita tensor of an arbitrary coordinate system $x^{i'}$ can hence be deduced from a right-handed Cartesian reference frame $x^i$ using its transformation matrix $\Lambda'$ as defined by 2.4. In the more convenient latter expression using the determinant of the metric tensor, the plus (minus) sign applies to right-handed (left-handed) base $e_i$.

Vector products can now be written as

$$
(U \times V)^i = \epsilon^{ijk} U_j V_k, \quad (U \times V)_i = \epsilon_{ijk} U^j V^k
$$

which introduces the complementary form of the Levi-Civita tensor:

$$
\epsilon_{ijk} = g_{il} g_{jm} g_{kn} \epsilon^{lmn} = \pm \sqrt{g} [ijk].
$$

### 2.4 Partial and covariant derivatives

Consider the scalar field $\Psi$ of which it is possible to take the partial derivative along $x^i$. Introducing a helpful and compact notation, we write

$$
\frac{\partial}{\partial x^i} \Psi = \Psi_{,i}.
$$

It is clear that, in Cartesian coordinates, these components constitute the gradient vector $\nabla \Psi$. From the lower index notation, you might however already have guessed that these derivatives transform as a one-form. The gradient vector in an arbitrary coordinate system can thus be derived by raising the one-form’s index:

$$
(\nabla \Psi)^i = g^{ij} \Psi_{,j}.
$$

When taking the derivative of a vector field $V$, one must be more thoughtful. Since both scalar fields $V^i$ and their base vectors $e_i$ are in general place dependent, the Leibniz rule must be applied:

$$
\frac{\partial}{\partial x^i} V = \frac{\partial V^j}{\partial x^i} e_j + V^j \frac{\partial e_j}{\partial x^i} = \frac{\partial V^j}{\partial x^i} e_j + V^j \Gamma^k_{ji} e_k,
$$

in which the Christoffel symbols $\Gamma^k_{ji}$ are defined. Without elaborating the properties of these symbols, be aware that they are not tensors and thus do not transform like tensors. An economical way of calculating these symbols is given by
\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}) \]  
(2.20)

from which one can directly see symmetry in the \( jk \)-coordinates. As has been done for the partial derivative, we now introduce a compact notation for the covariant derivative as

\[ V^j_{;i} = V^j_{,i} + \Gamma^j_{ki} V^k, \quad \frac{\partial}{\partial x^i} V = V^j_{;i} e_j. \]  
(2.21)

An important endnote of this paragraph is that the introduced derivatives can be applied on general tensors resulting in a new tensor. A useful resulting property is the ability to raise the index of the covariant differential operator,

\[ V^i_{;j} = g^{jk} V^i_{;k}. \]  
(2.22)

### 2.5 Divergence, curl and Laplacian

The previously introduced notations allow for an economical way of calculating and writing the operators defining wave propagation. The divergence for instance can be directly deduced from (2.21):

\[ \nabla \circ V = V^i_{;i} = V^i_{,i} + \Gamma^i_{ji} V^j. \]  
(2.23)

With the help of (2.20) and some algebra this formula can be greatly simplified to

\[ \nabla \circ V = \frac{1}{\sqrt{g}} (\sqrt{g} V^i)_{;i}, \]  
(2.24)

where \( g \) has previously been defined as the determinant of the metric tensor. The curl can straightforwardly be calculated with help of the Levi-Civita tensor:

\[ \left( \nabla \times V \right)^i = \epsilon^{ijk} V_{kj} = \frac{\pm [ijk]}{g} V_{kj}. \]  
(2.25)

Since in the covariant expression above the terms holding Christoffel symbols cancel out, it is possible to use partial derivatives. Combining expression (2.18) with the divergence formula, we finally obtain the last equation of this chapter, the Laplacian

\[ \nabla^2 \Psi = (\nabla \Psi)^i_{;i} = (g^{ij} \Psi_{;j})_{;i} = \frac{1}{\sqrt{g}} (\sqrt{g} g^{ij} \Psi_{;j})_{;i}. \]  
(2.26)

Overlooking this chapter, we see that the metric tensor appears to be an utmost important tool allowing direct calculation of the latest introduced operators. These expressions, together with the knowledge on how to transform vectors and their one-forms will allow deduction and understanding of all future formulas.
Chapter 2. Basics of differential geometry
Chapter 3

Transforming space through differential geometry

Using the previously introduced knowledge on differential geometries, this chapter shows how to describe Maxwell’s equations in an arbitrary differential geometry. Starting from their easiest form, describing wave propagation in electromagnetic free space for Cartesian coordinate systems, a correspondence between geometries and electromagnetic media will unravel itself. Subsequently a general way of calculating the dielectric and magnetic properties of these so-called transformation media will be presented. The transformation of free space is generalized to transforming non-empty space after which this chapter concludes with some notes on material parameters and their simplification.

Although visual representations would certainly clarify some of the concepts presented in this chapter, one will have to browse to the next chapter in order to find some. As images can easily be misinterpreted if one has no proper insight in the math, the focus is still put on the latter. Possible unclear parts of the theory will become comprehensible during the elaboration of the first transformation optical devices.

3.1 Maxwell’s equations in differential geometries

The behaviour of electromagnetic waves is fully described by Maxwell’s equations

\[
\begin{align*}
\nabla \cdot D &= \rho_f, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \\
\nabla \cdot B &= 0, \quad \nabla \times H = J_f + \frac{\partial D}{\partial t},
\end{align*}
\]

where \(\rho_f\) and \(J_f\) are the free charge density and free current density respectively. The electric and magnetic fields are related through the constitutive relations
Chapter 3. Transforming space through differential geometry

\[ D = \epsilon_0 \epsilon E \ , \ B = \mu_0 \mu H . \] (3.2)

Here the second rank tensors \( \epsilon \) and \( \mu \) are defined relatively to their free space values \( \epsilon_0 \) and \( \mu_0 \), a convention followed throughout the rest of this thesis. Note furthermore that the speed of light in vacuum is defined by this permittivity and permeability as \( \epsilon_0 \mu_0 = c^{-2} \). Making use of equations 2.24 and 2.25, we can easily write out Maxwell’s equations for an arbitrary differential geometry. In a space without free charges and currents, the following relations hold:

\[
\begin{align*}
(\sqrt{g} D^i)_{,i} &= 0 , \quad [ijk] E_{k,j} = \frac{\partial B^i}{\partial t} , \\
(\sqrt{g} B^i)_{,i} &= 0 , \quad [ijk] H_{k,j} = -\frac{\partial D^i}{\partial t} .
\end{align*}
\] (3.3)

Writing all field components in their one-form and furthermore evaluating this equation in free space, we straightforwardly find

\[
\begin{align*}
(\epsilon_0 \sqrt{g} g^{ij} E_i)_{,i} &= 0 , \quad [ijk] E_{k,j} = -\frac{\partial(\pm \mu_0 \sqrt{g} g^{ij} H_j)}{\partial t} , \\
(\mu_0 \sqrt{g} g^{ij} H_i)_{,i} &= 0 , \quad [ijk] H_{k,j} = \frac{\partial(\pm \epsilon_0 \sqrt{g} g^{ij} E_j)}{\partial t} .
\end{align*}
\] (3.4)

It is interesting to compare this expression with Maxwell’s equations in Cartesian space, which can be found by evaluating 3.3 in its metric \( g_{ij} = \delta_{ij} \):

\[
\begin{align*}
D^i_{,i} &= (\epsilon_0 \epsilon^{ij} E_j)_{,i} = 0 , \quad [ijk] E_{k,j} = -\frac{\partial(\mu_0 \mu^{ij} H_j)}{\partial t} , \\
B^i_{,i} &= (\mu_0 \mu^{ij} H_j)_{,i} = 0 , \quad [ijk] H_{k,j} = \frac{\partial(\epsilon_0 \epsilon^{ij} E_j)}{\partial t} .
\end{align*}
\] (3.5)

It appears that a Cartesian physical space with dielectric and magnetic response \( \epsilon^{ij} \) and \( \mu^{ij} \) given by

\[
\epsilon_T^{ij} = \mu_T^{ij} = \sqrt{g} g^{ij}
\] (3.6)

is governed by identically the same equations as an electromagnetic free space with metric \( g_{ij} \). This discovery is one of the fundaments of transformation optics: when playing with math and interpreting 3.4 as a right-handed Cartesian space with material parameters 3.6 the electromagnetic space seems to transform according to the metric tensor \( g_{ij} \). A transformation medium is thus defined as any medium that can be traced back to a coordinate transformation of a right-handed Cartesian space. Finally note that \( \epsilon^{ij} \) equals \( \mu^{ij} \) making the transformation medium impedance-matched. This is a logical necessity since a differentiable coordinate transformation may nowhere introduce scattering. Also observe the anisotropic nature of the material parameters as the metric tensor is only limited to symmetricity.
3.2 Transformation media expressed in arbitrary differential geometries

The previous paragraph shows how to define a transformation medium in right-handed Cartesian space. For some transformations however, it is advantageous to use another metric $\gamma_{ij}$ for the physical space. Once again we start from the free space Maxwell’s equations in a coordinate system with metric tensor $g_{ij}$, given by (3.4). Instead of interpreting these equations as a transformation of a right-handed Cartesian coordinate system, we now use $\gamma_{ij}$ as a reference coordinate system:

\[
\begin{align*}
(\sqrt{\gamma} D^i),i &= 0 \quad [ijk] E_{k,j} = -\frac{\partial (\pm \sqrt{\gamma} B^i)}{\partial t}, \\
(\sqrt{\gamma} B^i),i &= 0 \quad [ijk] E_{k,j} = \frac{\partial (\pm \sqrt{\gamma} D^i)}{\partial t}.
\end{align*}
\]

(3.7)

The interpretation that the transformed electromagnetic space with metric $g_{ij}$ is actually positioned in a physical space with metric $\gamma_{ij}$ directly leads towards the transformation medium

\[
\epsilon_{ij}^{T} = \mu_{ij}^{T} = \pm \frac{\sqrt{\epsilon}}{\sqrt{\gamma}} g_{ij}^{T} = \pm \frac{\sqrt{\epsilon}}{\sqrt{\gamma}} \Lambda_{i'}^{i'} \Lambda_{j'}^{j'} g_{ij}^{T}
\]

(3.8)

with the plus-sign holding in case of equal handedness for electromagnetic and physical space.

From calculational point of view, it is interesting to develop (3.8) in a handy matrix notation. For this purpose, we define $g_{ij}'$ as the original empty electromagnetic space in the coordinate system $x_i'$. The transformation $\Lambda_{i'}^{i'}$ maps these coordinates to the transformed coordinates $x_i$ with its corresponding metric $g_{ij}$. These transformed coordinates represent physical space which is characterized by the metric $\gamma_{ij}$. With $G'$, $\Gamma$ and $\Lambda$ the matrix representations of $g_{ij}'$, $\Gamma_{ij}$ and $\Lambda_{i'}^{i'}$ respectively, the relative permittivity and permeability can be calculated as

\[
\epsilon_T = \mu_T = \frac{\sqrt{\det G'}}{\sqrt{\det \Gamma}} \Lambda \cdot G'^{-1} \cdot \Lambda^T \frac{1}{\det \Lambda}.
\]

(3.9)

3.3 Non-empty space transformations

The expressions for $\epsilon_T$ and $\mu_T$ derived so far apply on transformation media. It is thus in principle possible to trace these material parameters back to coordinate transformation originating from Cartesian free space. Nothing forbids us however to perform a coordinate transformation on a non-empty space. For this purpose we explicitly rewrite equation (3.3) in function of its material parameters and find
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\[(\sqrt{g} \epsilon_0 \epsilon^{ij} E_j)_i = 0 \quad , \quad [ijk] E_{k,j} = -\frac{\partial(\pm \sqrt{g} \mu_0 \epsilon^{ij} H_j)}{\partial t},\]

\[(\sqrt{g} \mu_0 \epsilon^{ij} H_j)_i = 0 \quad , \quad [ijk] H_{k,j} = \frac{\partial(\pm \sqrt{g} \epsilon_0 \epsilon^{ij} E_j)}{\partial t} .\]

(3.10)

Comparison of this expression with its free space equivalent [3.4] shows that the instances \(\epsilon^{ij}\) and \(\mu^{ij}\) replace the terms \(g^{ij}\). Interpreting 3.10 as situated in a space with metric \(\gamma_{ij}\) therefore gives a likewise relation between 3.8 and our new material parameters

\[\epsilon_T^{ij} = \pm \frac{\sqrt{g}}{\sqrt{\gamma}} \epsilon^{ij}, \quad \mu_T^{ij} = \pm \frac{\sqrt{g}}{\sqrt{\gamma}} \mu^{ij}.\]

(3.11)

With \(\epsilon'\) and \(\mu'\) the matrix representations of the original material parameters \(\epsilon^{ij'}\) and \(\mu^{ij'}\), the latest expressions transforms into the following matrix equivalent:

\[\epsilon_T = \frac{\sqrt{\det G}}{\sqrt{\det \Gamma}} \Lambda \cdot \epsilon' \cdot \Lambda^T, \quad \mu_T = \frac{\sqrt{\det G}}{\sqrt{\det \Gamma}} \Lambda \cdot \mu' \cdot \Lambda^T.\]

(3.12)

By now it should also have become apparent that the material parameters are metric dependent and that for free space \(\epsilon^{ij}\) and \(\mu^{ij}\) are given by the inverse metric \(g^{ij}\). Inspired by this equality as well as by the metric invariant scalar product, it makes sense to look into \(\epsilon^{ij}_j\) and \(\mu^{ij}_j\). These expressions produce normalized eigenvalues, therefore giving \(\delta^{ij}\) in free space. Normalized permittivity and permeability tensors give more insight in the physical picture and will thus be included in final results, also avoiding ostensible conflicts with other sources on this topic.

As every material expression is given by a metric dependent upper case tensor, \(\epsilon^{ij}\) and \(\mu^{ij}\) in coordinate system \(x^i\) are easily transformed to another coordinate system \(x'^i\) by the general tensor transformation rule

\[\epsilon'^{ij'} = \Lambda^{i'}_i \Lambda^{j'}_j \epsilon^{ij}\]

(3.13a)

or

\[\epsilon'' = \Lambda_c \cdot \epsilon \cdot \Lambda_c,\]

(3.13b)

where \(\Lambda_c\) is the matrix equivalent of the conversion operator \(\Lambda^{''}_i\). This expression allows for calculation of original material parameters \(\epsilon^{ij'}\) (\(\mu^{ij'}\)) and makes it possible to evaluate the transformed \(\epsilon_T^{ij}\) (\(\mu_T^{ij}\)) in for instance a Cartesian coordinate system.

### 3.4 Simplification of material parameters

Earlier in this chapter, it has been shown that transformation media are described by equal permittivity and permeability tensors, characterized by six location dependent parameters.
Even if a transformation is carefully chosen, the resulting device will in general be impossible to fabricate with modern technology. The necessity of magnetic responses poses the biggest problem due to lack of utilizable natural materials, especially in the optical spectrum. Magnetic optical metamaterials are difficult to fabricate due to the sub-wavelength dimension of structures, not to mention high absorption losses. Another difficulty with transformation optical devices is that a significant local deformation leads towards materials parameters approaching zero and infinity. It has for instance been shown that every line transformed cloak, such as a cylindrical invisibility device, requires infinite responses along its inner boundary [30].

A first simplifying step is to perform a transformation which is identical along one spatial axis. When talking about such a two dimensional device, this thesis will always consider invariance along the Cartesian $z$-direction. In such a device, the number of space dependent parameters reduces from six to four since the two shear parameters with one component in the $z$-direction reduce to zero. Material requirements can be further reduced by only considering transverse-electric (TE) or transverse-magnetic (TM) waves. In the first case, there is only an electric field in the $z$-direction and a magnetic field in the plane of propagation — the permittivity components in this plane are arbitrary as is the permeability along the $z$-axis. The analogue holds for the TM-case. A design strategy for obtaining non-magnetic material parameters for TM-applications will be integrated in a later part of this thesis. The TE-case is already strongly documented, a brief description of its properties will finish this chapter.

### 3.4.1 Conformal transformations

It would be interesting to find a transformation for which the permittivity and permeability in the plane of propagation reduce to unity. Resulting apparatuses operating in the TE-regime could thus be manufactured using simple isotropic dielectric media. Using equation 3.9 the material parameters for a Cartesian transformation of free space from $x^i$ to $x'^i$ look as follows:

$$
\epsilon_T = \mu_T = \frac{1}{\frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'} \frac{\partial y}{\partial x'}} \begin{bmatrix}
(\frac{\partial x}{\partial x'})^2 + (\frac{\partial x}{\partial y'})^2 & \frac{\partial x}{\partial x'} \frac{\partial y}{\partial x'} & \frac{\partial x}{\partial y} \frac{\partial y}{\partial y'} & 0 \\
\frac{\partial x}{\partial x'} \frac{\partial y}{\partial x'} & (\frac{\partial y}{\partial y'})^2 & (\frac{\partial y}{\partial x'})^2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \epsilon^{zz} & 0
\end{bmatrix}
$$

(3.14)

where $\epsilon^{zz}$ equals $\mu^{zz}$. This last identity is valid for transformations of the form

$$
\frac{\partial x}{\partial x'} = \frac{\partial y}{\partial y'} \quad \& \quad \frac{\partial x}{\partial y'} = -\frac{\partial y}{\partial x'},
$$

(3.15)

embodies conformal transformations. A visualization of this type of transformation is given by an isotropic scaling and rotation of square elements $dx' \times dy'$ to new square elements
dx × dy. If two lines intersect orthogonally in the original space, their intersection in the transformed space will thus also be orthogonal. Looking back at equations 3.14 and 3.15, it appears that all waves propagating in the xy-plane perceive the same refractive index

\[ n = \sqrt{\epsilon z} = \frac{1}{\sqrt{(\frac{\partial x}{\partial x'})^2 + (\frac{\partial x}{\partial y'})^2}}. \] (3.16)

In the Helmholtz approximation of the wave equation, conformal transformations hence result in isotropic dielectric media functioning for arbitrary polarizations. With \( k_0 \) being the free space wavenumber, the Helmholtz equation in empty space \( x' \) has the form

\[ (\nabla^2 + k_0^2) \Psi = (4 \frac{\partial}{\partial w'} \frac{\partial}{\partial w'^*} + \frac{\partial}{\partial w'^*} \frac{\partial}{\partial w'} + n^2 k_0^2) \Psi = 0. \] (3.17)

This equation introduces the complex number \( w' = x' + i y' \) and its complex conjugate \( w'^* \). Basic knowledge of complex numbers suffices to verify the first equality in 3.17. In the transformed plane \( w = x + i y \), Helmholtz’s equation looks as follows:

\[ (4 \frac{\partial}{\partial w} \frac{\partial}{\partial w'^*} + n^2 k_0^2) \Psi = (4 \frac{\partial w'}{\partial w} \frac{\partial w'^*}{\partial w'^*} \frac{\partial w'}{\partial w'^*} \frac{\partial w'^*}{\partial w'} + n^2 k_0^2) \Psi = 0. \] (3.18)

This derivation is correct if the transformation is a conformal one, for which 3.15 translates to the simple

\[ \frac{\partial w}{\partial w'^*} = 0. \] (3.19)

Identifying \( n \) from equations 3.17 and 3.18, it appears to be equal to \( |\frac{\partial w'}{\partial w}| \) which identifies with the expected 3.16.

Conformal mapping looks very interesting at first sight but design of practical devices is severely limited by the definition 3.19. Firstly, conformal maps preserving free space at infinity go hand in hand with singularities resulting in unrealizable refractive index profiles. Secondly, when designing a medium for practical applications, usually defined by boundary conditions, a conformal map will in general not exist.

Quasi-conformal maps \([31]\), introducing negligible anisotropic scaling factors, give the engineer some breathing room but their practical use is still limited. The isotropic carpet cloak, able to hide a small bump or pit of an otherwise planar reflector, is good example of this transformation and its limits \([32, 33]\). As this device is an invisibility cloak, it requires identity transformations at its outer surface. The resulting device must consequently be much bigger than the hidden anomaly for the induced anisotropy to be negligible. Because of their
attractive isotropic profiles, we will look into the performance of some TE-devices in chapter 5. Note though that this type of devices yields only good results under appropriately boundary conditions [34].
Chapter 3. Transforming space through differential geometry
Chapter 4

Fundamental transformations and their applications

With an overview of some basic devices based on a relatively simple transformation, this chapter illustrates the power of transformation optics. A simple wave compressing slab will serve as an appetizer as its straightforward properties will ease familiarization with the mathematical concepts. Next is a very similar transformation leading up to a more interesting application, namely a polarization splitter. The invisibility cloak, often called the ultimate optical illusion and probably the most intriguing device that transformation optics provides, is up next. The final topics presented in this chapter are a lossless and reflectionless waveguide bend and a beam collimator. All handled problems will be presented in a two-dimensional manner, that is with an identity mapping along the $z$-axis. They can however easily extended to a full space transformation.

Many other transformations can be investigated but this chapter only aims at clearly elaborating the most fundamental and practical ones. The transformation optical creation of lenses [35] — which might be considered to be the most fundamental optical devices after mirrors — is missing in this chapter as this matter will be extensively handled in the next one.

4.1 Unidirectional space compressing slab

In this section, transformation optics will be used to compress a slab region of space. A practical application of this type of transformation would involve compression of an optical device inside the slab, making it more compact [36,37]. Only interested in the theoretical concepts, the following example will compress free space. Consider thus the following transformation on the Cartesian free space $x' = \{x', y', z'\}$:
The original metric \( g_{ij}' \), equal to \( \delta_{ij}' \), will be transformed to \( g_{ij} \) according to transformation rule 2.5. Making the interpretation that the electromagnetic coordinates \( x^i \) are laying in regular Cartesian physical space, equation 3.6 determines the material parameters performing the given transformation:

\[
\epsilon_T = \mu_T = \begin{bmatrix}
1/2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}.
\] (4.2)

Consider now an slab of this material, perpendicular to the \( x \)-axis, surrounded by air. If the slab has thickness \( L \), this setup corresponds with local compression of free space over a distance \( L' = 2L \). Consequently the distance along the \( x \)-axis between points at opposite sides of the slab is decreased by \( L' - L \) which equals \( L \) for the given compression ratio. This transformation is illustrated by figure 4.1(a) and the material parameters 4.2 are for the TE-case confirmed through simulation (see figure 4.1(b)). All simulations in this thesis were performed using the commercially available software packet COMSOL Multiphysics 4.1.

\[ x = x'/2 \quad , \quad y = y' \quad & \quad z = z'. \] (4.1)
wave propagating in the $yz$-plane is nowhere compressed and should thus have a free space wavelength.

4.2 Waveguide polarization splitter

Consider a slab waveguide in the $x'y'$-plane with $x'$ the direction of wave propagation. As $x'$ goes from zero to $L$, the waveguide will be linearly shifted over a distance $d$ in the $y'$-direction. In the region $0 \leq x' \leq L$ the transformation

$$x = x', \quad y = y' + \frac{x'}{L} d \quad \& \quad z = z'$$

applies. The areas left and right of this region are given by a simple identity transformation and a shift of $y'$ over $d$ respectively. With unit permeability for the waveguide and its surroundings, the same procedure as for the slab compression is used to find the transformed permeability tensor

$$\mu_T = \begin{bmatrix} 1 + \left( \frac{d}{L} \right)^2 & \frac{d}{L} & 0 \\ \frac{d}{L} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (4.4)

In order to calculate the permittivity, the more general approach 3.12 needs to be used. Starting from an isotropic permittivity $\epsilon(y')$, the transformed tensor is however given by the simple $\epsilon_T = \epsilon(y') \mu_T$.

If the waveguide carries a TE-mode with a $z'$-polarized electric field and magnetic fields in the $x'y'$-plane, this mode is only susceptible for $\epsilon^{zz'}$ and the $\mu^{ij'}$-components in the $x'y'$-field. Analogously, a TM-mode will be subject to the opposite material components. A polarization splitter is now constructed by shifting the TE material parameters over a distance $d_{TE}$ which is different from $d_{TM}$, the distance over which the TM-mode will be shifted.

Note that if a silicon waveguide surrounded by air serves as input waveguide, separated output waveguides are not just two silicon waveguides. The permeability equals unity over the whole domain $x > L$ but something strange has happened with the original permittivity tensor. The polarization splitting transformation translates the $z'$-component of $\epsilon(y')$ over a distance $d_{TE}$ and the other components over $d_{TM}$. The TE output waveguide is thus defined by the permittivity profile $\epsilon^{zz} = \epsilon(y' - d_{TE})$ with all other guide components equal to unity. The TM waveguide is analogously defined by $\epsilon^{zz} = \epsilon^{yy} = \epsilon(y' - d_{TM})$ and unity $\epsilon^{zz}$ at the position of this waveguide. The exit material parameters can in practice be simplified to two copies of the original waveguide if mode coupling between those is negligible. Figure 4.2 shows simulational results of a polarization splitter and illustrates the effect of the mentioned simplification. It
also appears that, for well confined modes, it suffices to transform the material parameters for the silicon midsection only.

\[ n_{Si} = 3.45 \]

Figure 4.2: Simulational results of a polarisation splitter for a silicon input waveguide with a width of 200 nm surrounded by air. Depending on the investigated mode, the input waveguide is excited by a uniform electric or magnetic field and the corresponding transverse field is plotted. (a) shows a splitter without transformation. Transforming only the silicon splitter, good results are obtained for the TE-mode (b) while the less confined TM-mode (c) has poor performance. The TM-mode requires a full transformation of both splitter and exit waveguides (d).

Simulation parameters: \( \lambda_0 = 1550 \text{ nm} \), \( L = 600 \text{ nm} \), \( d_{TE} = 300 \text{ nm} \), \( d_{TM} = -300 \text{ nm} \).

4.3 Invisibility cloaking

There is no doubt that invisibility cloaking is the most discussed topic within transformation optics. An overview of transformations would therefore be incomplete without mentioning the mother of all transformation optical devices: the cylindrical invisibility cloak [10]. Consider an electromagnetic empty space in cylindrical coordinates \( x' = \{ r', \theta', z' \} \) having the following
well known relation with an empty Cartesian space \( x'' = \{ x''', y''', z'' \} \):

\[
x'' = r' \cos(\theta') \quad , \quad y'' = r' \sin(\theta') \quad \& \quad z'' = z.
\] (4.5)

The metric tensor of \( x'' \) is derived to be unitary with exception of \( g_{\theta'\theta'} = r'^2 \). Consider now a strictly monotonic and differentiable radial transformation \( r(r') \) from \( x'' \) to \( x' = \{ r, \theta, z \} \) with the following properties: \( r(0) = a, \ r(b) = b \) and \( r(r') = r' \) for \( r' > b \). This \( z \)- and \( \theta \)-invariant transformation will compress the volume \( r' \leq b \) to \( a \leq r \leq b \) allowing no electromagnetic field within the inner area \( r < a \). When the transformed electromagnetic space \( x' \) is interpreted to have the physical metric \( \gamma_{ij} \) corresponding with a regular cylindrical coordinate system, equation 3.11 determines the material parameters of this cylindrical cloak:

\[
\epsilon_T = \mu_T = \begin{bmatrix} r' R & 0 & 0 \\ 0 & \frac{1}{r' R} & 0 \\ 0 & 0 & \frac{r'}{R} \end{bmatrix} \tag{4.6}
\]

with \( R \) defined as the derivative of \( r \) with respect to \( r' \). Note hereby that most literature presents normalized values giving the tangential component \( \epsilon_{\theta\theta} = \mu_{\theta\theta} = \frac{r}{r' R} \).

A linearly compressed cloak is illustrated in figure 4.3. Note that the diverging material parameters at the inner boundary are inherent to the line transformed cloak; this problem does not present itself for point transformed cloaks [30].

### 4.4 Sharply bent waveguide

Waveguide bends are necessary for photonic integration, sharper bends allowing for compacter integrated devices. As bends become sharper though, more power will be lost through radiation and reflection. Transformation optics to the rescue! [38,39] A slab waveguide which guides waves in the \( y' \)-direction is positioned in the right half plane and the upcoming transformation will disregard the left half plane. A uniform bend is introduced by fixing the space \( y' \leq 0 \) and mapping \( y' = L \) to the \( y' \)-axis. The remainder of the waveguide will then guide along the \( x' \)-axis. Note that the \( x' \)-value of the left interface between the silicon waveguide and its cladding defines the inner bend radius \( r_{in} \). In physical space, only the first quadrant will contain transformed material parameters which are defined by the transformation

\[
r = x' \quad , \quad \theta = \frac{y'}{L} \frac{\pi}{2} = \frac{y'}{R} \quad \& \quad z = z'.
\] (4.7)

Physical space is conveniently described by a cylindrical metric and the distance \( R = \frac{2L}{\pi} \) has been introduced. One can predict that material parameters along \( x' = R \) will remain intact
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Figure 4.3: Demonstration of the cylindrical invisibility cloak. The material parameters $\varepsilon_{zz}$ and $\varepsilon_{xx}$ are shown in (a) and (b) respectively. Excitation of the structure by a plane wave is simulated in (c) while (d) plots $E_z$ for an out of plane line current.

Simulation parameters: $\lambda_0 = 1550\,\text{nm}$, $a = 1\,\mu\text{m}$, $b = 2\,\mu\text{m}$. 
as this surface is free of local compression along any direction. Using equation 3.9, it is found that

\[
\mu_T = \begin{bmatrix}
\frac{R}{r} & 0 & 0 \\
0 & \frac{1}{rR} & 0 \\
0 & 0 & \frac{R}{r}
\end{bmatrix},
\]

which confirm this statement — \( \epsilon_T \) again being the product of \( \mu_T \) and the original relative permittivity. Note that truncation of the diverging material parameters for \( r \) approaching zero is not very disturbing if inversely transformed truncated values have little effect on the guided mode in the \( x' \)-space. One can analogously use untransformed silica in the third quadrant of physical space, where the material parameters should actually be derived from compressing the left half of electromagnetic space. Good performance can already be achieved by using transformed material parameters for the silicon waveguide while leaving all silica untransformed; this is also illustrated through simulation in figure 4.4.

### 4.5 Beam expander and collimator

The beam expander, or beam collimator in reverse use, will not be elaborated with the same detail as the previous devices but completeness requires its presence. A collimator designed for wave propagation in the \( x' \)-direction gradually compresses the \( y' \)-coordinate as the wave advances through the device and is finally cut off to free space. A typical device would look like a cylinder with radius \( R \) and height \( H \) with the circular surfaces forming the input and output plane. Incoming (laser) beams would be compressed or expanded by a factor \( R/r \) and off-centre beams would also undergo a lateral shift in space.

When two identical collimators are put together so that their compressed regions face each other, an input beam would exit the device as if it would have propagated through free space. This setup can be used to guide light through a small aperture as demonstrated in figure 4.5 through a 2D-simulation.
Figure 4.4: Reduction of bend loss for a 200 nm wide silicon slab waveguide in silica \(n_{Silica} = 1.53\). The bottom guide is excited with a uniform electric field. (a) depicts the \(E_z\)-field for an untransformed waveguide, in (b) only the guide itself contains transformed material parameters while the transformation is applied on the whole first quadrant in (c) (d) compares the power \(P(x)\) exiting the output guide to the power \(P_{str}\) that would exit the equivalent straight guide. As the input guide is not excited with a guided mode, ratios greater than one are possible.

Simulation parameters: \(\lambda_0 = 1550\, \text{nm},\ r_{in} = 200\, \text{nm},\ R = 300\, \text{nm}\).
Figure 4.5: Space compression through narrow slit in golden reflector using a beam collimator and expander. The central rectangle, $2H$ high and $2L$ wide, compresses electromagnetic space through an opening of height $2h$. An $E_z$-field with a Gaussian profile (beam width $H/2$) excites the left device boundary. As no transformation is applied in (a), a low power cylinder-like wavefront fills the right half space. (b) shows that using the device, a higher power Gaussian beam gets through the barrier. Legends for the plotted $E_z$-field are not provided as the colour saturation clearly illustrates the low and high power transmission for (a) and (b) respectively.

Simulation parameters: $\lambda_0 = 1550\,\text{nm}$, $L = H = 2\,\mu\text{m}$, $h = 250\,\text{nm}$.
Chapter 4. Fundamental transformations and their applications
Chapter 5

Reflection based transformation optical devices

The invariance of Maxwell’s equations for different geometries serves as foundation for different research directions. Thorough investigation of the previously described or essentially similar devices together with the study on transformations leading to manufacturable material parameters is the main dish however. Remarkable hereby is that none of these devices — with exception of the carpet cloak — have a reflective nature, which led up to the creation of this rather novel chapter.

The probable reason why reflective devices have not been proposed so far, together with why they should be looked into, is given in the first subsection. The repeatedly mentioned carpet cloak will then serve as an introductory example of a device operating in reflection. More functional applications, namely focal mirrors and directive gratings, will be described next. Having introduced a couple of reflective devices, the next treated subject is on how to compute transformations which can be implemented by dielectric-only material for the TE-mode. Finally it will be demonstrated that space can be unfolded in order to obtain transmissive devices with the same characteristics as their reflective counterparts.

5.1 Space transformations for reflective devices

When it comes to practical applications, transmissive optical devices have a general advantage over reflective ones. An optical communication fibre per example is an inherent transmissive device, the ideal fibre carries an output which is identical to the input signal [40]. The invisibility cloak serves a similar purpose: the cloak guides an electromagnetic field around an eventual obstacle in order to achieve wave propagation resembling undisturbed free space.

When one tries to achieve a function which is not inherently transmissive, reflective devices are
as a general rule less practical than transmissive ones as the latter readily separate input from output signals. The most striking evidence is the popularity of lenses over focal reflectors, while the latter possess all practical advantages apart from difficulty in separation of input and output. When using a space transformation to alter the shape of a reflector, one loses its main advantage — that is what one could call the trouble free environment apart from the reflector itself — but can on the other hand simplify the shape of the reflector itself.

Moving on to the math, consider light propagation in the electromagnetic space $x^i$ described by a Cartesian metric. Working in two dimensions with $z$-invariance, an infinite curve $\zeta'$ defines the boundary between air in one half space and highly reflective material on the other side. It is helpful to define another curve $\eta'$ which lies in free space but has its start- and endpoint (which may be positioned at infinity) on the interface $\zeta'$, $\Omega'$ denotes the enclosed surface. The electromagnetic space $x^i$, with figure 5.1(a) serving as an illustration, will now be transformed to $x^i$ through a transformation with some restrictions. Free space is identity mapped with exception of the area $\Omega'$ which can be transformed arbitrarily, this transformation should of course not interfere with the previous identity mapping. Note that these rules already define the modified material parameters of free space and the reflector at their boundary. We will now assume that the transformation does not induce extreme compression at this boundary; this allows us to disregard the transformation for the reflector itself and use its original permittivity and permeability values. Figure 5.1(b) exemplifies the transformation rules by performing an arbitrary transformation on the earlier illustrated electromagnetic space.

![Figure 5.1](image)

**Figure 5.1:*** Example of a reflective transformation from free space (a) to device (b). This device, situated between the identity transformed boundary $\eta$ (green) and the reflector $\zeta$ (blue), performs the same electromagnetic function as free space with the electromagnetic mirror position $\zeta'$ (red). The transformation within the device can be freely chosen with the limit of identity transformation on its air boundary $\eta$. 
The described transformation results in the following electromagnetic behaviour: waves enter the device at the boundary \( \eta' = \eta \), propagate through \( \Omega \) and reflect on the relocated boundary \( \zeta \). After exiting the device, reflected waves will resemble free space propagation with reflection on the original mirror \( \zeta' \).

An important property of the reflective transformation is the loose boundary condition on the new interface \( \zeta \). It is for instance possible to define a map \( x'^i \) to \( x^i \) which is non-identical on this interface but for which its physical appearance does not change. As we are only interested in \( \zeta' \) and \( \zeta \), the physical shape of the reflecting interface, this transformation has an extra degree of freedom when compared with transmissive devices. Wise use of this so called sliding boundary condition will result in less stringent material parameters.

### 5.2 Carpet cloaking

Consider a flat reflecting surface containing a local irregularity. If an electromagnetic wave reflects on this well or bump, it will obviously be scattered. The carpet cloak, sometimes also called ground plane cloak, is a device that manipulates incident waves in order to hide the irregularity. The most obvious solution for this problem is a regular cylindrical or spherical cloak cut in half. Incident light will be guided around the inside bell and reflect on the flat contact surface between cloak and ground plane [41]. If light is not guided away from the irregularity but only reshaped as to mask the irregularity, one can however obtain a device with more relaxed material parameters. A lot of research has been performed on the carpet cloak because of its relatively simple material requirements [42]. For smooth irregularities, it is even possible to perform a quasi-conformal transformation resulting in an isotropic dielectric material. The resulting devices needs to be much bigger than the hidden irregularity though for this simplification to be justified [43].

The definitions from the previous section can be used to describe a generalized strategy for designing a carpet cloak. We start out from a perfect surface \( \zeta' \) which tends to be flat in literature but it can have an arbitrary shape. This surface contains an small irregularity resulting in a surface \( \zeta \) which is almost identical to the original one. The outer perimeter of the cloak \( \eta \) is drawn over the irregularity defining the device \( \Omega \). The cloak parameters follow from an arbitrary differentiable transformation that maps the undisturbed electromagnetic space \( \Omega' \) to the cloak region \( \Omega \). This cloaked region is confined between the identity mapped free space (up to \( \eta \)) and the irregularity containing surface \( \zeta \). As there exists a lot of literature regarding the carpet cloak, the carpet cloak will not be further discussed. Interested readers are invited to look into the referred publications.
5.3 Focal mirrors

While the carpet cloak hides reflector deformation, this section will investigate the opposite transformation. We design the shape of the reflector $\zeta'$ surrounded by air to have focal properties in electromagnetic space. A transformation will then be implemented that transforms this curved reflector to a straight mirror $\zeta$. We will thus design a device that, when placed against a straight mirror, mimics a function performing curved mirror surrounded by empty space. The most evident advantage of this transformation is the simple geometry of mirror $\zeta$. Another benefit is the potential relative compactness of the device if $\zeta$ lays in between $\zeta'$ and $\zeta$. Last but not least is the prospect of adapting the device to work in transmission and thus form a practically perfect lens, this aspect will be handled in the last section of this chapter.

We will now determine the shape of a $z'$-invariant focal reflector $\zeta'$ going through the origin of the Cartesian electromagnetic space $x^i$. The focal points $F'_1$ and $F'_2$ are chosen to lay on the negative $x'$-axis, located at distances $f'_1$ and $f'_2$ away from the origin respectively (note that the prime annotation can be dropped if the focal points lay outside of the device). Light will only propagate through air, so the condition for constructive interference at $F'_2$ for waves originating from $F'_1$, on their way reflected by the mirror, is a simple geometrical problem as illustrated in figure 5.2(a). If the points on the mirror $P'$ are given by coordinates $(x'_\zeta, y'_\zeta)$ and with $O'$ denoting the origin, one finds

$$\|F'_1 P'\| + \|P' F'_2\| = \|F'_1 O'\| + \|O' F'_2\|.$$  \hspace{1cm} (5.1)

With some simple algebra, this relation translates to the simple boundary expression

$$y'_\zeta^2 = - \frac{4x'_\zeta f'_1 f'_2 (x'_\zeta + f'_1 + f'_2)}{(f'_1 + f'_2)^2},$$  \hspace{1cm} (5.2)

defining an elliptical shape for $\zeta'$. A very common reflector, for instance used for long distance communication, has one of its focal points at infinity. This reflector is geometrically illustrated in figure 5.2(b) and its expression can be easily derived from 5.2 by taking the limit to minus infinity for $f'_1$ while we replace $f'_2$ with $p'$:

$$y'_\zeta^2 = - 4x'_\zeta p'.$$  \hspace{1cm} (5.3)

When transforming the focal mirror $\zeta'$ to a straight one, there is no theoretical difference between the elliptical (circular for $f'_1 = f'_2$) and parabolic case. Because of its mathematical simplicity however, we will now transform the parabolic mirror. In the electromagnetic space $x^i$, the parabolic mirror profile is limited by $x'_{\min}$ and the corresponding positive $y'$-value $y'_{\max}$. For $y'$-values greater than $y'_{\max}$ or smaller than $-y'_{\max}$, $x'_\zeta$ remains $x'_{\min}$. Figure 5.3(a) illustrates the described device by means of a wave simulation. The mirror $\zeta$ is defined by
Figure 5.2: Drawing of a focal reflector with two focal points \( f'_1 \) and \( f'_2 \) (a) or with one focal point \( p' \) and the other laying at infinity (b). For each point on the reflector (red curve), the sum of the distances between the focal points an the mirror (solid blue lines) is equal to a certain constant. For the parabolic mirror (b), the vertical line represents an equal phase front for a source at infinity and serves as a reference line for the distance calculation.

\[ x_\zeta = x'_{\text{min}} \] and the device \( \Omega \) is chosen to be a rectangle of thickness \( t \) and height \( 2y'_{\text{max}} \) placed in front of the flattened part of the mirror. We now chose to perform the most simple transformation from \( \Omega' \) to \( \Omega \), namely a linear compression of the \( x' \)-coordinate and identity transformations for \( y' \) and \( z' \). The definition \( l = x'_{\text{min}} - t \), this is the \( x' \)-coordinate of the left boundary \( \eta' \), helps us to write the transformation in the transparent form

\[ \frac{x - l}{t} = \frac{x' - l}{t} - \frac{y'^2}{4p'^2} - l \quad , \quad y = y' \quad \text{&} \quad z = z' . \quad (5.4) \]

Starting transformation matrix \( \Lambda \), the material parameters of the device \( \Omega \) can now be easily calculated using equation 3.9:

\[
\Lambda = \begin{bmatrix}
\frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
\frac{t}{x'^2} & \frac{t y'}{2p'} & \frac{x' - l}{(x'^2 - l)^2} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \equiv \begin{bmatrix}
\alpha & \beta & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} ,
\quad (5.5)
\]

\[ \epsilon_T = \mu_T = \frac{\Lambda \cdot \Lambda^T}{\det \Lambda} = \frac{1}{\alpha} \begin{bmatrix}
\alpha^2 + \beta^2 & \beta & 0 \\
\beta & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} . \quad (5.6) \]

Figure 5.3(b) shows how the calculated device performs the same function as the original curved mirror. This simulation also demonstrates the justifiability of using untransformed reflector material parameters for this non-extreme transformation. The reflector \( \zeta \) consists of the same highly reflective metal as \( \zeta' \) without any noticeable impact on the device’s performance. Note that the focal point lays closer to the flattened mirror than the focal distance.
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$p'$ suggests. The definition a new focal distance $p \doteq p' + x'_{\min}$, this is the distance between the focal point and the flattened mirror, takes away potential confusion when describing the transformed device without reference to the original parabolic mirror $\zeta'$.

![Simulation of the upper half of a parabolic mirror (a) and its transformed variant (b).](image)

**Figure 5.3:** Simulation of the upper half of a parabolic mirror (a) and its transformed variant (b).

The normalized electric field is plotted for a normally incident plane wave with unit amplitude ($E_z$).

Simulation parameters: $\lambda_0 = 1550$ nm, $p' = 8$ µm, $x'_{\min} = -2$ µm, $t = 1$ µm.

### 5.4 Diffraction gratings

Next to the focal mirror, the surface diffraction grating is another very common reflective device. This wavelength dependant device is for instance used in spectrometers [44]. In the $z'$-invariant electromagnetic space $x'^\prime$, the grating is defined by an arbitrary mirror profile $\zeta'$ that repeats itself in the $y'$-direction with minimum period $\Delta$. The angle $\theta_m$ for the $m$th diffraction order is given by the relation $\sin(\theta_m) = \frac{m \lambda_0}{\Delta}$, the zeroth-order angle corresponding with reflection on a flat mirror. The reflected power density in function of angle for plane wave illumination finally is the product of the response of one period and the number of lightened periods [45]. A first grating we will transform is the blazed one, which directs light towards the first diffraction order. The blazed profile $\zeta'$ is defined by one period $0 \leq y'_\zeta < \Delta$ for which

$$
(x'_\zeta, y'_\zeta) = \begin{cases} 
    x'_\zeta = d \left( \frac{\Delta}{\Delta} - 1 \right), & 0 < y'_\zeta < \Delta \\
    -d \leq x'_\zeta < 0, & y'_\zeta = 0
\end{cases},
$$

(5.7)

wherein $d$ equals half a wavelength. The resulting device is illustrated trough simulation in figure 5.4(a). Analogous to the parabolic mirror, this mirror can be transformed to the
minimum $x'$-value giving $x_\zeta = -d$ for the straight mirror. The material parameters for a slab of thickness $t$, resulting from a linear compression of the $x'$- to the $x$-coordinate, will not be included in this thesis. They can easily be found by applying the procedure used in equations [5.4] to [5.6]. Note that this transformation is problematic at the boundary between different periods $\Delta$, illustrated by figure [5.4(b)]. The transformation will pull the infinitesimal close points $P'_1 = (-d - t/2, +\delta y)$ and $P'_2 = (-d - t/2, -\delta y)$ away from each other. Note that the performance of the transformed device decreases with increasing $t$ as the room for incorrect boundary interaction increases. Figure [5.4(c)] illustrates that for the relatively small $t$ simulated, the transformed blazed grating has a performance similar to the original one.

A second interesting grating has its period $\Delta$ defined by an isosceles triangle (sawtooth) with its base on the $y'$-axis. With the height of the triangle again defined as $d = \frac{\lambda_0}{2}$, this grating maximizes reflection to both the first and the minus first diffraction order. The material parameters for the transformed slab can again be calculated by the recipe of equations [5.4] to [5.6]. Within the slab region $|y| \leq \frac{\Delta}{2}$ and with $l$ equal to $-d - t$, $\epsilon_T$ and $\mu_T$ are now defined by

$$\frac{x - l}{t} = \frac{x' - l}{|y'| \frac{2d}{\Delta} + t} , \quad y = y' \quad \& \quad z = z' \quad (5.8a)$$

and

$$\alpha = \frac{1}{|y'| \frac{2d}{\Delta t} + 1}, \quad \beta = \frac{-2d}{\Delta t} \left( |y'| \frac{2d}{\Delta t} + 1 \right)^2 . \quad (5.8b)$$

Unlike the blazed one, this transformed grating performs exactly the same function as its original counterpart, the mapping introduces no discontinuities. A simulation of the transformed sawtooth grating is shown in figure [5.4(d)].

5.5 Computing material friendly transformations

Up to now, the transformed material parameters have always been calculated by hand. Although this manual operation might improve one's insight in a transformation, there are a couple of serious drawbacks. For one, defining an analytic coordinate transformation becomes extremely cumbersome as geometries tend to become complicated. One could for instance define a region inside $\Omega'$ and close to $\eta$ that is not to be deformed by the transformation, thus leaving its material properties intact. This extra boundary condition will introduce either major mathematical complexity or wicked material parameters. These parameters themselves form the second reason for turning to computer aided transformations. In case of the parabolic mirror per example, the linear compression of the $x'$-coordinate invokes maximal lateral deformation on the symmetry axis and shear deformation builds up as $|y|$ increases.
Figure 5.4: Normal incident Gaussian beam with a beam width of 5 μm reflects on grating. The original blazed grating [(a)] is flattened, this results in a periodic medium of which $\epsilon^{xx}$ is shown in [(b)]. [(c)] shows a simulation of the flattened blazed grating and [(d)] displays the flattened sawtooth grating.

Simulation parameters: $\lambda_0 = 1550$ nm, $t = \lambda_0/2$, $\theta_m = 45^\circ$. 
Overall material parameter requirements would be lower if these deformations were smeared out over the entire transformation region.

It is common knowledge that nature often provides an elegant solution to mathematical problems, a statement also applying to the current issue. Let us thus describe spatial transformations with the most obvious useful physics theory: solid mechanics. Given a material’s elastic properties, this well-established theory is able to find the minimum elastic energy solution for the compression of $\Omega'$. One only needs to describe the position of the fixed boundary $\eta$ and the displacement of $\zeta'$. The most basic — and thus most appealing — way to describe a material’s solid properties is given by an isotropic and uniform medium with Young’s modulus $E$ and Poisson’s ratio $\nu$. If $E$ is uniform over $\Omega'$, the transformation will only depend on $\nu$. It is possible to reduce deformation in certain regions of $\Omega'$ by making $E$ place-dependent, this strategy makes it possible to solve the previously introduced problem of the (almost) incompressible region. All these statements can easily be derived from basic solid mechanics theory for which interested readers are referred to literature [46].

The following subsections provide all information necessary to readily design a dielectric only TM-device based on solid mechanics, finalized by its application on the parabolic mirror.

### 5.5.1 Obtaining material parameters through inverse transformation

The solid mechanics problem can be solved with the same software packet used for all frequency simulations in this thesis. COMSOL Multiphysics provides its spatial transformation solution in the untransformed space $x^i$ however, making a direct frequency simulation with the derived material properties impossible. A correct yet rather inelegant way of solving this problem would be to export the deformation matrix $\Lambda^i_{i'}(x^i e^i)$ together with the translation vector $u(x^i e^i)$ and import the modified matrix $\Lambda^i_{i'}(x^i e^i + u^i e^i)$. Note hereby that the displacement vector $u$ does not transform according to the rule $\Lambda^i_{i'}$, it is only a place dependent vector relating $x^i$ to $x'^i$.

A smarter solution to the material properties problem uses the inverse matrix transformation $x^i'(x^i)$. If a certain transformation $\Lambda^i_{i'} = \Lambda$ transforms free space to $\epsilon_T = \mu_T$, then the inverse transformation $\Lambda^i_{i} = \Lambda_{\text{inv}}$ must transform these back to free space. If one applies solid mechanics on the domain $\Omega$, transforming it to $\Omega'$, its solution

\[
\Lambda_{\text{inv}} = \begin{bmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\
\frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z}
\end{bmatrix}
\] (5.9)

can be readily used to find $\epsilon_T$ and $\mu_T$ in the frequency simulation domain $x^i$. Through
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Equation 3.12, $\epsilon_T$ is transformed back to its free space values $\epsilon_{fs}$:

$$\epsilon_{fs} = \frac{\Lambda_{inv} \cdot \epsilon_T \cdot \Lambda_{inv}^T}{\det \Lambda_{inv}} = 1$$  \hspace{1cm} (5.10)

Or

$$\epsilon_T(= \mu_T) = \det(\Lambda_{inv}) \Lambda_{inv}^{-1} \cdot \Lambda_{inv}^{-T} = \frac{\Lambda_{inv}^{-1} \cdot \Lambda_{inv}^{-T}}{\det \Lambda_{inv}^{-1}},$$  \hspace{1cm} (5.11)

A comforting result for parameters calculated through inverse transformation.

5.5.2 Dielectric only implementation of TM-devices

At the end of chapter 3, the transformation condition of non-magnetic media for $E_z$ polarized waves has been investigated. As only the $zz$-component of the permeability tensor is allowed to differ from one, the resulting conformal transformation and even the quasi-conformal severely restrict allowed operations. Let us thus now consider TM-devices and allow anisotropy in the $xy$-plane. Transformation limitations should now be less severe as only one permeability component needs to be equal to one.

The transformation condition can easily be derived from equation 3.14: the restriction $\mu_{zz} = 1$ translates to the transformation condition

$$\frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'} \frac{\partial y}{\partial y'} = 1.$$  \hspace{1cm} (5.12)

The remaining in plane of propagation $\epsilon^{ij}$ components, with $\epsilon^{zz}$ being arbitrary, follow from the same referred equation. We will now look into the meaning of condition 5.12 in order to obtain a user-friendly transformation rule. It is known that a conformal transformation of $x'$ resembles rotation and uniform stretching of its elements areas. We now expect to find a dual solution for TM-devices, meaning an allowance of shear deformation but a restriction on change in area. On intuition, we calculate the area of transformed elements. An infinitesimal square $dxe_x \times dy e_y$ will transform according to the rule $\Lambda'_{ij}$ into a parallelogram. As the area of a parallelogram is given by $\|v \times w\|$, the vectors $v$ and $w$ defining two nonparallel sides, it is found that the area of the transformed square is given by

$$\begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial x}{\partial x'} dx & \frac{\partial y}{\partial x'} dx & 0 \\ \frac{\partial x}{\partial y'} dy & \frac{\partial y}{\partial y'} dy & 0 \end{vmatrix} = \left| \frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} \right| dx dy = \left| \frac{\partial x}{\partial x'} \frac{\partial y}{\partial x'} \right| dx dy = dx dy.$$  \hspace{1cm} (5.13)

It appears that transformation rule 5.12 corresponds with an area preserving transformation. A solid mechanics plane strain approach with Poisson’s ratio $\nu = 1/2$, representing an incompressible medium, thus numerically solves for this type of (inverse) transformation [47]. Note
that an ideal solution will only exist for area preserving transformations with fully gliding boundary conditions. An approximate solution $\mu_{zz} \approx 1$ is found by solving the restricted problem (fixed boundary $\eta$, different areas for $\Omega$ and $\Omega'$) with a Poisson’s ratio $\nu \approx 1/2$. If the obtained solution approaches an ideal transformation, the permeability can be put one without serious impact on the device’s performance.

### 5.5.3 The dielectric only flattened parabolic mirror

COMSOL’s solid mechanics toolbox will now be used to create a dielectric only variant of the flattened parabolic mirror seen in section 5.3. As $x'$ and $x''$ have already been defined, we can directly proceed to describing the inverse transformation of $\Omega$. The $\eta$ boundary remains fixed, while the $x$-coordinate of $\zeta$ undergoes the positive displacement $u = -x'_\text{min} - \frac{y^2}{4p}$ for $|y| \leq y'_\text{max}$. $\zeta$ can displace freely along its $y$-coordinate as long as its contact points with $\eta$ remain fixed. The displacement solution, computed with $\nu$ close to $1/2$, leads to the permittivity and permeability tensors with $zz$-components as close to one as physically possible. One can guess that the outcome of this naive transformation deviates greatly from our purpose material. While the ideal displacement solution conserves the area of its elements, the total area $\Omega'$ is much bigger than $\Omega$, this leads to poor results. This problem is easily solved by a translation of the left boundary of $\eta$, together with all free space left of it, so that $\Omega$ and $\Omega'$ have an equal area. If we define $x' = x + \Delta x$ on the boundary $\eta$, conservation of area dictates that

$$2 \int_0^{y'_\text{max}} (x'_\text{max} - \frac{x'^2}{4p}) dy'_\zeta = \Delta x \cdot 2y'_\text{max}$$

which can be reduced to the simple expression

$$\Delta x = -\frac{2}{3} x''_\text{min} \cdot$$

(5.15)

The discontinuity of the redefined transformation at $y = \pm y'_\text{max}$ has no severe consequences for relatively small $t$. In case of larger device thicknesses, it might be wise however to make $\Delta x$ a multiple of the free space wavelength. We will solve for a relatively thin device, this makes it possible to relax the displacement requirements on the top and bottom edge of the boundary $\eta$: the $y$-coordinate remains fixed while free displacement is allowed for the $x$-coordinate. Figure 5.5 depicts the result of the computer generated transformation and the performance of the deduced device. This simulation also shows that the focal point in the $x'$-space has been shifted to the left by the translation over $\Delta x$: the distance $p$ between focal point and mirror is now given by $p = p' + x''_\text{min} + \Delta x = p' + x''_\text{min}/3$. As the material parameters diverge in the depicted top right corner of $\Omega$, the height of the transformed device has been reduced to 95% of its original size.
Figure 5.5: Inverse transformation of flattened parabolic mirror using transformation optics. (a) shows the displacement vector \((u, v)\) from \(x^i\) to \(x^i'\), these lead to the material parameters \(\varepsilon_T = \mu_T\) of which the \(xx\)-, \(yy\)- and \(zz\)-components are shown in (b). (c) depicts the normalized \(E\)-field for a unit amplitude incident TM-wave. The simplified non-magnetic device (d) has a similar performance since \(\mu_{zz}\) is relatively close to one.

Simulation parameters: \(\lambda_0 = 1550\) nm, \(-x_{min}' = t = 2\) \(\mu\)m, \(p = 8\) \(\mu\)m, \(\nu = 0.495\).
5.6 Transmissive equivalents of reflective devices

Consider a transformed reflective device so that the mapped mirror $\zeta$ forms a flat surface. The electromagnetic influence of this mirror at its surface is simple: the Poynting vector of reflected radiation is the mirrored equivalent of the incident power, changing the sign of its perpendicular component. Apart from a uniform 180 degree phase shift, the reflected wave looks identical to the incoming one. We now define a new device by removing the reflector $\zeta$ and mirroring the material parameters $\epsilon_T = \mu_T$ around this plane $\zeta$ in order to fill the previously undefined space. It is clear that, up to the position $\zeta$, incident waves will behave identically in the original and new device. Afterwards, the new wave solution will consist of the mirror image around $\zeta$ of the originally reflected fields, be it without the phase shift. We have created a reflectionless device fulfilling the same function as its reflective brother but now in transmission.

The actual body of this thesis will now be concluded by some figures and notes on this latest finding. Different implementations of the transformed parabolic mirror are up first. Next to the already described full transformation and the TM-implementation, two other devices will we simulated. There is the TE-device, for which the transformation is calculated using the solid mechanical approach with $\nu \approx -1$ [47], and a regular dielectric lens serving as a reference. The profile and the refractive index of the dielectric lens are calculated using Fermat’s principle of extremal optical distance. A flat left interface at position $x = -d$, with $d$ the maximum thickness of the device, will not bend an incoming plane wave. Focus at point $(p,0)$ will obtained if the optical path length for all parallel incoming rays equals an identical distance $L$. The profile of the right lens facet $(x_l, y_l)$, giving thickness $d$ for $y = 0$ and zero thickness at the top of the lens (where $y = y_{\text{max}}$), is now given by

$$L = n d + p = \sqrt{h^2 + (p + d)^2} = n (d - x_l) + \sqrt{y_l^2 + (x_l + p)^2}$$  \hspace{1cm} (5.16)

which leads towards the following expressions for $n$ and $y_l$ in function of $x_l$:

$$n = \frac{\sqrt{h^2 + (p + d)^2} - p}{d},$$  \hspace{1cm} (5.17a)

$$y_l = \sqrt{(n x_l + p)^2 - (x_l - p)^2}.$$  \hspace{1cm} (5.17b)

Figure 5.6 depicts the simulations for the described devices. Their thicknesses have been increased with respect to the previous simulations, this decreases the errors for the TE- and TM-device caused by neglecting the slight inhomogeneity of the $\mu$-profile. The focal performance of the uniform dielectric lens, which is reformed to be symmetric around $x = x'_{\text{min}}$, also increases due to the lower refractive index $n$, decreasing reflections.
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Figure 5.6: Norm of electric field for unit amplitude plane wave incident on lens, top half plotted. (a) shows the result of a full transformation while (b) and (c) use simplified dielectric only material parameters, operating in TM and TE respectively. The ‘regular’ dielectric lens in figure (d) serves as a reference.

Simulation parameters: $\lambda_0 = 1550$ nm, $p' = 8$ µm, $x'_{\text{min}} = -2$ µm, $t = 6$ µm.
Comparing the different devices in figure 5.6, it is clear and unsurprising that a full transformation, including a magnetic response, attains best results. The TM- and the TE-lens operate with similar performance, apparently the applied transformation introduces a comparable error of unwanted shear deformation in the TE-case and area (de)compression for the TM-device. The uniform dielectric lens performs worst, having the lowest maximal field amplitude and the largest spot size. The spot size is explained for by the lens’s relative thickness, influencing the diffraction limit negatively. The focal strength is close to its maximum though, decreasing the thickness $d$ would reduce the spot size but increase reflections at the facets. Both the focal strength and the spot size have been improved by spreading the curvature of the lens over its two facets instead of keeping the left one straight.

Figure 5.7(a) plots the electric field cuts for the four devices and gives a clearer view on their spot sizes. The position of measurement is given by the rightmost black lines in figures 5.6 and corresponds with the constant $x$-value that contains the maximum field value. Due to the wave nature of light, together with the material simplification for the TE- and TM-case, these focal points lay further away from the lens than calculated. The focal positions were traced manually for each device individually. Figure 5.7(b) depicts the analogue for a plane wave incoming under a small angle $\theta$. All new focal points have different $x$- and $y$-coordinates but the graph centres their maximum field value, making comparison easy. Note finally that the transformation devices can handle relatively large incident wave angles while the relatively thick uniform dielectric lens can only handle small angles. Internal reflections become problematic as the direction of wave propagation near the lens’s surface tends to align with this surface.

![Electric field cuts for the devices in figure 5.6.](a) (b)

**Figure 5.7:** Electric field strength around focal point for the devices in figure 5.6. (a) matches with the there shown field while (b) corresponds with a wave incident under an angle $\theta = 10^\circ$.

Simulation results for the transmissive blazed - and sawtooth grating are shown in figure 5.8.
and \ref{fig:Benz:FarField1} respectively. The thickness of the transformation region $t$ (as defined in section \ref{sec:Benz:TotalThickness}) is chosen in a way that the uniform dielectric grating achieves good performance. Small thicknesses $d = 2t$ of this grating result in high reflection while too big $d$ again result in internal reflection issues. The profile of this uniform grating is closely related to the shape of the original reflective grating. The refractive index $n$ introduces an optical distance difference of $\lambda_0$ between rays travelling the distance $d$ in air and dielectric, it is thus readily found that $n = 1 + \lambda_0/d$.

![Figure 5.8](image)

**Figure 5.8:** Norm of electric field for unit amplitude plane wave incident on transmissive diffraction grating ($E_z$ with exception of (b): $E_y$). (a) shows the result of a full transformation while (b) and (c) use simplified dielectric only material parameters, operating in TM and TE respectively. The uniform dielectric grating in figure (d) serves as a reference. Simulation parameters: $\lambda_0 = 1550$ nm, $t = 3\lambda_0/2$, $\theta_m = 30^\circ$.

The performance of all grating implementations can be compared using figure \ref{fig:Benz:FarField2}. The electric field just right of the grating is used to calculate the field on a semicircle with a radius $r_{ff}$ much bigger than the total height of the grating. These far field values give a measure of directivity in function of angle. For both types of grating, the TE-device performs worst, this is due to the shear character of the performed transformation and the resulting simplification error. The best pick out of the remaining devices depends on the application. The full transformation gives the highest field value in the direction of the desired diffraction order $\theta_m$, the TM-transformed - and the uniform dielectric grating on the other hand provide better suppression of the other diffraction orders.
Figure 5.9: Norm of electric field for unit amplitude plane wave incident on transmissive diffraction grating ($E_z$ with exception of $E_y$). (a) shows the result of a full transformation while (b) and (c) use simplified dielectric only material parameters, operating in TM and TE respectively. The uniform dielectric grating in figure (d) serves as a reference. Only positive $y$-coordinates are plotted as these devices produce $x$-axis mirror symmetry. Simulation parameters: $\lambda_0 = 1550$ nm, $t = 3\lambda_0/2$, $\theta_m = 30^\circ$. 
Figure 5.10: Grating directivity for the blazed grating ((a)) and the sawtooth grating ((b)). For the devices in figures 5.8 and 5.9, the electric field strength at a distance $r_{ff} = 1 \text{ mm}$ is plotted in function of the angle with the $x$-axis.
Chapter 6

Conclusion and future work

This chapter summarizes the contributions of this thesis to the domain of transformation optics and indicates potential routes of related future research. Chapter 5 is basically the innovative part of this thesis, the previous chapters having an introductory nature, but the polarization splitter proposed in section 4.2 deserves further discussion. Literature only presents free space polarization splitters although silicon waveguide splitters, used in photonic integrated circuits, are very interesting from a fabrication point of view. If only the guide carrying the TM-mode is displaced (i.e. $d_{TE} = 0$), one obtains a dielectric-only polarization splitter which is anisotropic yet piecewise uniform. It should certainly be investigated how such a splitter can be (approximately) realized, per example using layered media citelayeredMedia.

Proceeding already to chapter 5, we see how this thesis introduces the idea of transforming functional reflective devices. While the carpet cloak — the only other reflective transformation optical device present in literature — hides irregularities on a flat mirror, we looked into the reverse transformation and flattened the reflecting surface of focal mirrors and diffraction gratings. The utility of the resulting devices looks limited at first sight but applications are certainly possible. Laboratory laser setups using non-flat reflectors can for instance be built using plane mirrors in combination with these flat transformation optical devices; the setup can be altered without ruining mirror alignment by a simple change of the mirror-adjacent device. The worth of transmissive equivalents for transformed focal mirrors and reflective diffraction gratings is self-evident; when physical implementation becomes possible and performance outranks costs, they can be used to outperform traditional lenses and transmissive gratings.

I finish this thesis with a note on the described process of material simplification given in section 5.5. The method of using solid mechanics to calculate coordinate transformations is not new, it has however not yet been applied to reflective devices — by which I mean the carpet cloak. In order to attain dielectric-only implementations, the area preserving Poisson’s
ratio for TM-devices inspired a redefinition of the boundary $\eta$ in order to obtain equal areas for $\Omega$ and $\Omega'$. The redefinition of $\eta$, which has not yet been reported before, has led to good results for all simulated TM-devices as well as TE-devices. This moreover suggests that the performance of the dielectric-only realization of the carpet cloak should improve as the lateral width of the cloak increases as this reduces the displacement of $\eta$ (a fairly agreeing quasi-conformal approach of this problem can be found in publication [43]). One can conclude that, as fabrication feasibility remains the main bottleneck of transformation optics, material simplification using quasi-conformal and area preserving maps needs thorough theoretical investigation.


