On the Performance of Optimal Input Signals for Frequency Response Estimation

Bo Wahlberg, Håkan Hjalmarsson and Petre Stoica

***Abstract*—We consider the problem of minimum-variance excitation design for frequency response estimation based on Finite Impulse Response (FIR) and Output Error (OE) models. The objective is to minimize the power of the input signal to be used in the system identification experiment subject to a model accuracy constraint. For FIR and OE models this leads to a finite dimensional Semi-Definite Programming optimization problem. We study, in detail, how to apply this approach to the estimation of the frequency response at a given frequency, ω. The first case concerns minimizing the asymptotic variance of the estimated frequency response based on a FIR model estimate. We compare the optimal input signal with a sinusoidal signal with frequency ω that gives the same model accuracy, and show that the input power can, at best, be reduced by a factor of two when using the optimal input signal. Conditions are given under which the sinusoidal signal is optimal, and it is shown that this is a common case for higher order FIR models. Next, we study FIR model based estimation of the absolute value and phase of the frequency response at a given frequency, ω. We derive the corresponding optimal input signals and compare their performances with that of a sinusoidal input signal with frequency ω. The relative reduction of input power when using the optimal solution is at best a factor of two. Finally, we discuss how to extend the FIR results to OE system identification by using an input parametrization proposed by Stoica and Söderström (1982).**

I. Introduction

Consider a scalar discrete time asymptotically stable time-invariant linear dynamical system with impulse response sequence $\{g_k\}$, input signal sequence $\{u(t)\}$, output signal sequence $\{y(t)\}$, and additive zero mean white noise $\{e(t)\}$ with variance λ_e. The input to output relation is given by

$$y(t)=\sum_{k=0}^{\infty} g_k u(t-k)+e(t). \tag{1}$$

The corresponding frequency response function is defined by

$$G(e^{i\omega})=\sum_{k=0}^{\infty} g_k e^{-i\omega k}, \quad \omega\in(-\pi,\pi], \quad i=\sqrt{-1}, \tag{2}$$

and is a most important tool in for example filter design and feedback control systems.

Frequency response function estimation is a fundamental problem in both spectral estimation and system identification. We will study parametric methods, such as Prediction Error Model (PEM), for frequency response estimation, see *e.g.* [12] and [7]. Assume θ to be the model parameter vector, θ_o is the "true" system parameter vector, and let $\hat{\theta}$ be a PEM estimate of θ_o based on N measured input output observations, $\{u(t),y(t),\, t=1,\ldots,N\}$. By $G_o(e^{i\omega})=G(e^{i\omega},\theta_o)$ and $\hat{G}(e^{i\omega})=G(e^{i\omega},\hat{\theta})$ we denote the "true" frequency response function and the corresponding estimate. Under standard quasi-stationary assumptions the normalized estimation error variance converges to the asymptotic (large N) error variance $\mathrm{AsVar}\{\hat{G}(e^{i\omega})\}$[1] , *i.e.*,

$$\mathrm{E}\{|\sqrt{N}[\hat{G}(e^{i\omega})-G_o(e^{i\omega})]|^2\}\to \mathrm{AsVar}\{\hat{G}(e^{i\omega})\}, N\to\infty. \tag{3}$$

If it is possible to choose the input signal excitation in the data collection experiment, an interesting approach is to determine the input signal that minimizes the uncertainty $\mathrm{AsVar}\{\hat{G}(e^{i\omega})\}$ at a given frequency. This makes sense only if we constrain, for example, the power (variance) of the input signal $\mathrm{E}\{u^2(t)\}\le\gamma$, as otherwise the error can be made arbitrarily small. We will formulate this optimization problem as a finite dimensional Semi-Definite Program, and by analyzing its solution characterize trade-offs involved in optimal input design. A main result is that a sinusoidal input signal with frequency ω often is minimum power optimal, but that there are cases when another signal is up to a factor of two more power efficient.

The problem of finding optimal input signals for parameter estimation in dynamic systems is a classical topic. An excellent survey of the field up to 1974 is given in [8]. There has recently been a renewed interest in the optimal input design problem, and a survey of the current state of the art is given in [4]. We will use the least costly identification framework introduced in [1]. The paper [9] has recently showed the equivalence of least costly and traditional experiment design. Simple first order FIR examples illustrating the theory to be presented can be found in [15]. The paper [10] studies the minimum amount of input energy required to estimate a FIR system within a given accuracy as a function of the model complexity.

The outline of this paper is as follows. Section II contains a variance analysis of estimated frequency response functions based on FIR models. The corresponding minimum-variance input design problem is studied in Section III, while Section IV contains a brief discussion on how to extend the FIR results to Output Error (OE) models. A numerical example is given in Section V, and Section VI concludes the paper.

II. Frequency Response Variance Expressions

We will consider identification of n:th order Finite Impulse Response (FIR) systems, for which $g_k=0,\ k>n$. The FIR

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B. Wahlberg and H. Hjalmarsson are with the Automatic Control Lab and ACCESS, School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden. (e-mail: bo.wahlberg@ee.kth.se, hakan.hjalmarsson@ee.kth.se.)

P. Stoica is with the Department of Information Technology, Uppsala University, SE-75 105 Uppsala, Sweden. (e-mail: ps@it.uu.se).

[1] A common but, less formal, notation is $\mathrm{Var}\{\hat{G}(e^{i\omega})\}=\mathrm{AsVar}\{\hat{G}(e^{i\omega})\}/N$.

model is defined by

$$y(t) = g_0 u(t) + g_1 u(t-1) + \ldots + g_n u(t-n) = \varphi^T(t)\mathbf{g}, \quad (4)$$

$$\varphi(t) = [u(t), u(t-1), \ldots, u(t-n)]^T, \ \mathbf{g} = [g_0, g_1, \ldots, g_n]^T. \quad (5)$$

This is a linear regression model and the PEM estimate of $\mathbf{g}$ can be found by the least squares method. The statistical properties of the least squares estimate $\hat{\mathbf{g}}$ are well known, see *e.g.* [7]. We will use $\mathbf{g}_o$ to denote the true system parameters and $\Delta\mathbf{g} = \hat{\mathbf{g}} - \mathbf{g}_o$ for the corresponding estimation error. The asymptotic (large N) covariance matrix, $\lim_{N\to\infty} \mathrm{Cov}\{\sqrt{N}\Delta\mathbf{g}\} = \mathrm{AsCov}\{\hat{\mathbf{g}}\}$, of the least squares estimate equals

$$\mathrm{AsCov}\{\hat{\mathbf{g}}\} = \mathbf{P}, \quad \mathbf{P} = \lambda_e \mathbf{R}^{-1}, \quad (6)$$

$$\mathbf{R}_{jk} = \bar{\mathrm{E}}\{u(t+1-j)u(t+1-k)\}, \ j,k = 1\ldots n+1, \quad (7)$$

where $\bar{\mathrm{E}}$ denotes quasi-stationary expectation as defined in [7]. For a stationary zero mean stochastic process this equals ordinary expectation and $r_\tau = \mathrm{E}\{u(t)u(t-\tau)\}$ is the covariance function (auto-covariance sequence). Note that $\mathbf{R} = \bar{\mathrm{E}}\{\varphi(t)\varphi^T(t)\}$ is a $(n+1)\times(n+1)$ symmetric Toeplitz covariance matrix with first column $[r_0, r_1, \ldots, r_n]^T$. We use the prefix As for results that are asymptotic in the number of data points, N. We will study functions of the estimated impulse response vector, and, in particular, the frequency response function, (2),

$$G(e^{i\omega}) = \Gamma^*(e^{i\omega})\mathbf{g}, \quad \Gamma(e^{i\omega}) = [1, e^{i\omega}, \ldots, e^{in\omega}]^T, \quad (8)$$

where the super-index $*$ denotes conjugate transpose. We will use the notations $\hat{G}(e^{i\omega}) = \Gamma^*(e^{i\omega})\hat{\mathbf{g}}$ and $G_o(e^{i\omega}) = \Gamma^*(e^{i\omega})\mathbf{g}_o$ for the estimated and the true frequency response function, respectively.

The analysis will be based on first order Taylor series expansions and is therefore valid in the small error regime, that is for large number of data N. We use the definition $\mathrm{Cov}\{Z\} = \mathrm{E}\{ZZ^*\}$ for zero mean complex vector-valued random variables. We will use the notation $\Delta F(G(e^{i\omega})) = F(\hat{G}(e^{i\omega})) - F(G_o(e^{i\omega}))$ for functions $F(\cdot)$ of the frequency response. First we will derive the variance of $\Delta G(e^{i\omega}) = \hat{G}(e^{i\omega}) - G_o(e^{i\omega}) = \Gamma^*(e^{i\omega})\Delta\mathbf{g}$, where $\Delta\mathbf{g} = \hat{\mathbf{g}} - \mathbf{g}_o$. Using (6), the asymptotic error variance of the frequency response estimate is

$$\mathrm{AsVar}\{\hat{G}(e^{i\omega})\} = \Gamma^*(e^{i\omega})\mathbf{P}\Gamma(e^{i\omega}). \quad (9)$$

For the real part and the imaginary part of $G(e^{i\omega})$ we have

$$\mathbf{S}(e^{i\omega}) = \begin{bmatrix} \mathrm{Re}\left\{G(e^{i\omega})\right\} \\ \mathrm{Im}\left\{G(e^{i\omega})\right\} \end{bmatrix} = \mathbf{V}(e^{i\omega})^T\mathbf{g}, \quad (10)$$

$$\mathbf{V}(e^{i\omega}) = \left[\mathrm{Re}\{\Gamma(e^{i\omega})\}, \ -\mathrm{Im}\{\Gamma(e^{i\omega})\}\right]. \quad (11)$$

Two other important representation of $G(e^{i\omega})$ are

$$\mathbf{T}(e^{i\omega}) = \begin{bmatrix} \mathrm{Abs}\left\{G(e^{i\omega})\right\} \\ \mathrm{Arg}\left\{G(e^{i\omega})\right\} \end{bmatrix}, \ \mathbf{Q}(e^{i\omega}) = \begin{bmatrix} \log|G(e^{i\omega})| \\ \mathrm{Arg}\left\{G(e^{i\omega})\right\} \end{bmatrix}, \quad (12)$$

where Abs and Arg denote the absolute value, $|G(e^{i\omega})|$, and the argument (phase) of $G(e^{i\omega})$, respectively. Recall that $\log G(e^{i\omega}) = \log|G(e^{i\omega})| + i\mathrm{Arg}\left\{G(e^{i\omega})\right\}$. For a complex number $z = x + iy = r[\cos(\phi) + i\sin(\phi)]$, a first order perturbation analysis gives

$$\begin{bmatrix} \Delta r \\ \Delta\phi \end{bmatrix} = \frac{1}{r^2}\begin{bmatrix} rx & ry \\ -y & x \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \quad (13)$$

This implies

$$\Delta\mathbf{T}(e^{i\omega}) \approx \mathbf{X}^T(e^{i\omega})\Delta\mathbf{S}(e^{i\omega}) = \mathbf{W}^T(e^{i\omega})\Delta\mathbf{g}, \quad (14)$$

$$\mathbf{X}(e^{i\omega}) = \frac{1}{|G_o(e^{i\omega})|^2}\times \begin{bmatrix} |G_o(e^{i\omega})|\mathrm{Re}\{G_o(e^{i\omega})\} & -\mathrm{Im}\{G_o(e^{i\omega})\} \\ |G_o(e^{i\omega})|\mathrm{Im}\{G_o(e^{i\omega})\} & \mathrm{Re}\{G_o(e^{i\omega})\} \end{bmatrix}, \quad (15)$$

$$\mathbf{W}(e^{i\omega}) = \frac{1}{|G_o(e^{i\omega})|^2}\times \left[|G_o(e^{i\omega})|\mathrm{Re}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}, \ -\mathrm{Im}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]. \quad (16)$$

Expressions (6), (10) and (14) imply the following estimation error covariance matrix expressions

$$\mathrm{AsCov}\left\{\begin{bmatrix} \mathrm{Re}\left\{\hat{G}(e^{i\omega})\right\} \\ \mathrm{Im}\left\{\hat{G}(e^{i\omega})\right\} \end{bmatrix}\right\} = \mathbf{V}^T(e^{i\omega})\mathbf{P}\mathbf{V}(e^{i\omega}), \quad (17)$$

$$\mathrm{AsCov}\left\{\begin{bmatrix} \mathrm{Abs}\left\{\hat{G}(e^{i\omega})\right\} \\ \mathrm{Arg}\left\{\hat{G}(e^{i\omega})\right\} \end{bmatrix}\right\} = \mathbf{W}^T(e^{i\omega})\mathbf{P}\mathbf{W}(e^{i\omega}), \quad (18)$$

where $\mathbf{V}(e^{i\omega})$ and $\mathbf{W}(e^{i\omega})$ are defined by (11) and (16), respectively. Since $\Delta\log|G(e^{i\omega})| \approx \Delta|G(e^{i\omega})|/|G(e^{i\omega})|$, the asymptotic covariance of $\mathbf{Q}(e^{i\omega})$, defined by (12), is obtained by scaling the first column of $\mathbf{W}$ by $|G(e^{i\omega})|$,

$$\mathrm{AsCov}\left\{\begin{bmatrix} \log|\hat{G}(e^{i\omega})| \\ \mathrm{Arg}\left\{\hat{G}(e^{i\omega})\right\} \end{bmatrix}\right\} = \mathbf{Y}^T(e^{i\omega})\mathbf{P}\mathbf{Y}(e^{i\omega}), \quad (19)$$

$$\mathbf{Y}(e^{i\omega}) = \frac{1}{|G_o(e^{i\omega})|^2}\times \left[\mathrm{Re}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}, \ -\mathrm{Im}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]. \quad (20)$$

The diagonal relations in (18) equal

$$\mathrm{AsVar}\{|\hat{G}(e^{i\omega})|\} = \frac{1}{|G_o(e^{i\omega})|^2}\times \left[\mathrm{Re}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]^T\mathbf{P}\left[\mathrm{Re}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right], \quad (21)$$

$$\mathrm{AsVar}\left\{\mathrm{Arg}\left\{\hat{G}(e^{i\omega})\right\}\right\} = \frac{1}{|G_o(e^{i\omega})|^4}\times \left[\mathrm{Im}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]^T\mathbf{P}\left[\mathrm{Im}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]. \quad (22)$$

If $y = x^2$ then $\Delta y \approx 2x\Delta x$, which together with (21) give

$$\mathrm{AsVar}\{|\hat{G}(e^{i\omega})|^2\} = 4\left[\mathrm{Re}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]^T\mathbf{P}\left[\mathrm{Re}\{G_o(e^{i\omega})\Gamma(e^{i\omega})\}\right]. \quad (23)$$

III. Optimal Input Design for Frequency Response Estimation

In this section we will study optimal input design for FIR model based frequency response estimation. We will consider quantities that are approximate linear functions of the impulse coefficients $\eta = \mathbf{c}^*\mathbf{g}$, $\mathbf{c} = [c_0, c_1, \ldots, c_n]^T$. The special case of frequency response estimation corresponds to $\mathbf{c} = \Gamma(\omega)$, defined by (8). In Section II we have derived expressions for $\mathbf{c}$ for different decompositions of $G(e^{i\omega})$, and showed

$\text{AsCov}\{\hat{\eta}\} = \lambda_e \mathbf{c}^* \mathbf{R}^{-1} \mathbf{c}$, where $\mathbf{R}$ is the symmetric Toeplitz matrix with first column $\mathbf{r} = [r_0, r_1, \dots r_n]^T$. Later on in this section, we allow the matrix $\mathbf{R}$ to be rank deficient as long as the asymptotic variance of the least squares estimate of η is finite. This is the case if $\mathbf{c}$ belongs to the range space of $\mathbf{R}$. Then $\text{AsVar}\{\hat{\eta}\} = \lambda_e \mathbf{c}^* \mathbf{R}^\dagger \mathbf{c}$, where $\dagger$ denotes any pseudo inverse, *c.f.* [11].

The sequence $\{r_0, r_1, \dots r_n\}$ is the initial part of a covariance sequence of a stationary stochastic process if the corresponding symmetric Toeplitz matrix with this sequence as its first column is positive definite, $\mathbf{R} > 0$. By using the less restrictive condition $\mathbf{R} \geq 0$ we allow for rank deficient solutions. In this case the Carathéodory parametrization of a covariance matrix, see [12], can be used to find a time realization of the input signal $u(t)$. This will involve sinusoidal signals.

A. Optimal Input Problems

Consider the optimization problem[2]

$$\text{P1}: \quad \min_{\mathbf{r}} \mathbf{c}^* \mathbf{R}^{-1} \mathbf{c}, \quad \text{s.t.} \quad r_0 \leq \gamma, \quad \mathbf{R} \geq 0. \tag{24}$$

The objective is to minimize the (normalized) variance of $\hat{\eta}$ subject to a bound on the power (variance) of the input signal. The optimal solution will satisfy $r_0 = \gamma$. As shown in [9], P1 is equivalent to the *Least Costly* identification optimization problem, [1],

$$\text{P2}: \quad \min_{\mathbf{r}} r_0, \quad \text{s.t.} \quad \mathbf{c}^* \mathbf{R}^{-1} \mathbf{c} \leq 1, \quad \mathbf{R} \geq 0. \tag{25}$$

Optimization problem P2 aims to find the minimum power input signal for which the normalized asymptotic variance of $\hat{\eta} = \mathbf{c}^* \hat{\mathbf{g}}$ is smaller than or equal to one. It is possible to rescale P2 to allow for other variance value constraints. Also scaling the solution of P2 to obtain $r_0 = \gamma$, gives the solution to problem P1. Using Schur complements arguments, [2], it is possible to show that Problem P2, (25), is equivalent to the optimization problem

$$\text{P3}: \quad \min_{\mathbf{r}} r_0, \quad \text{s.t.} \quad \mathbf{R} \geq \mathbf{c}\mathbf{c}^*. \tag{26}$$

Notice that the constraint in (26) implies $\mathbf{R} \geq 0$, and allows for rank deficient solutions. Problem P3 is a Semi-Definite Program (SDP), since the constraint in (26) is a linear matrix inequality in the covariance sequence r_k, $k = 0, \dots, n$. The analytic solution to P3 is derived in Appendix A for $n = 1$ (the 2×2 matrix case). We are not aware of a general analytic solution of P3 for $n \geq 2$, but it is a convex optimization problem (a Semi-Definite Program/SDP) for which excellent numerical methods exist.

The structure of P3 can be used to derive necessary conditions for optimality. A covariance sequence vector $\mathbf{r}^\star$ is called *optimal* if it has the smallest objective value r_0 among all vectors that satisfy the constraint. The matrix $\mathbf{R} - \mathbf{c}\mathbf{c}^*$ is positive semi-definite if and only if all its principal minors are positive semi-definite, see [3]. When $\mathbf{c}$ has complex valued elements we have the following lower bound

$$r_0^\star \geq \max_{j,k=0,\dots,n} \frac{|c_j|^2 + |c_k|^2 + \sqrt{(|c_j|^2 - |c_k|^2)^2 + 4[\text{Im}(c_j c_k^*)]^2}}{2}. \tag{27}$$

This bound follows from Proposition 2 in Appendix A. Any $\mathbf{r}$ satisfying the constraint $\mathbf{R} - \mathbf{c}\mathbf{c}^* \geq 0$ gives an upper bound for the optimal r_0. If it is possible to find a vector for which the upper bound is equal to the lower bound above, the corresponding signal is optimal.

[2]For matrices $X \geq Y$ means that $[X - Y]$ is a positive semi-definite matrix.

B. Sinusoidal Input Signals

We will compare the optimal solution of (26) to the case when a sinusoidal signal $u(t) = C\cos(\bar{\omega}t)$ is used. The quasi-stationary covariance function of this sinusoidal signal, see [7], equals

$$r_0 = \begin{cases} C^2, & \bar{\omega} = 0, \pi \\ \frac{C^2}{2}, & \text{otherwise} \end{cases}, \quad r_k = r_0 \cos(\bar{\omega}k), \quad k \geq 1. \tag{28}$$

The corresponding Toeplitz covariance matrix for the sinusoidal signal is

$$\begin{aligned} \mathbf{R} &= r_0 \text{Re}\{\Gamma(e^{i\bar{\omega}}) \Gamma^*(e^{i\bar{\omega}})\} \\ &= \frac{r_0}{2} [\Gamma(e^{i\bar{\omega}}) \Gamma^*(e^{i\bar{\omega}}) + \Gamma(e^{-i\bar{\omega}}) \Gamma^*(e^{-i\bar{\omega}})], \end{aligned} \tag{29}$$

The rank of the matrix $\mathbf{R}$ is one for $\bar{\omega} = 0, \pi$ and otherwise two. Therefore it is only possible to estimate a function of the frequency response $G(e^{i\bar{\omega}})$ at $\omega = \bar{\omega}$. We need to use a pseudo-inverse of $\mathbf{R}$ in the frequency response variance expression (9),

$$\text{AsVar}\{\hat{G}(e^{i\omega})\} = \frac{\lambda_e}{r_0} \Gamma^*(e^{i\omega}) \left[\text{Re}\{\Gamma(e^{i\omega}) \Gamma^*(e^{i\omega})\} \right]^\dagger \Gamma(e^{i\omega}). \tag{30}$$

Now $\text{AsVar}\{\hat{G}(e^{-i\omega})\} = \text{AsVar}\{\hat{G}(e^{i\omega})\}$ since the impulse response is real valued, and hence

$$\begin{aligned} &\text{AsVar}\{\hat{G}(e^{i\omega})\} = \\ &\quad \tfrac{\lambda_e}{r_0} \tfrac{1}{2} \text{Trace} \Big\{ \Gamma^*(e^{i\omega}) \left[\text{Re}\{\Gamma(e^{i\omega}) \Gamma^*(e^{i\omega})\} \right]^\dagger \Gamma(e^{i\omega}) \\ &\quad + \Gamma^*(e^{-i\omega}) \left[\text{Re}\{\Gamma(e^{-i\omega}) \Gamma^*(e^{-i\omega})\} \right]^\dagger \Gamma(e^{-i\omega}) \Big\} = \\ &\tfrac{\lambda_e}{r_0} \text{Trace} \Big\{ \left[\text{Re}\{\Gamma(e^{i\omega}) \Gamma^*(e^{i\omega})\} \right]^\dagger \left[\text{Re}\{\Gamma(e^{i\omega}) \Gamma^*(e^{i\omega})\} \right] \Big\} \\ &\quad = \begin{cases} \dfrac{\lambda_e}{r_0}, & r_0 = C^2, \quad \omega = 0, \pi, \\ \dfrac{2\lambda_e}{r_0}, & r_0 = \dfrac{C^2}{2}, \quad \text{otherwise} \end{cases}. \end{aligned} \tag{31}$$

Here we have used the fact that $\text{Trace}\{\mathbf{R}^\dagger \mathbf{R}\} = \text{rank}\{\mathbf{R}\}$ and that the rank of $\mathbf{R}$ for the sinusoidal case is one or two, depending on the frequency. Expression (31) is identical to asymptotic variance of the Empirical Transfer Function Estimate (ETFE), see [7]. The same expression is also obtained in [5] for periodic input signals with $\mathbf{R}$ having full rank. The derivation is, however, different.

Next consider (23) with $\bar{\omega} = \omega$,

$$\begin{aligned}\text{AsVar}\{|\hat{G}(e^{i\omega})|^2\} = \\ \tfrac{4\lambda_e}{r_0}\mathbf{g}_o^T[\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}][\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}]^\dagger \times \\ [\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}]^T\mathbf{g}_o = \\ \tfrac{4\lambda_e}{r_0}\mathbf{g}_o^T[\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}]\mathbf{g}_o = \tfrac{4\lambda_e}{r_0}|G_o(e^{i\omega})|^2. \end{aligned} \quad (32)$$

Hence, we also have a simple and intuitive variance expression for $\text{AsVar}\{|\hat{G}(e^{i\omega})|^2\}$, where the variance is proportional to the squared amplitude, $|G_o(e^{i\omega})|^2$.

For the amplitude we have the variance expression, *c.f.* (21):

$$\text{AsVar}\{|\hat{G}(e^{i\omega})|\} = \frac{\lambda_e}{r_0}, \quad (33)$$

that is half the variance of the frequency response estimate at $\omega \neq 0, \pi$, *c.f.* (31).

Finally, for the phase we have, *c.f.* (22):

$$\text{AsVar}\{\text{Arg}\{\hat{G}(e^{i\omega})\}\} = \frac{\lambda_e}{r_0|G_o(e^{i\omega})|^2}, \quad \omega \neq 0, \pi. \quad (34)$$

For $\omega = 0$ or π the variance is zero since the phase can only be 0 or π. Also notice that the phase is not well defined for zeros on the unit circle, that is if $G_o(e^{i\omega}) = 0$.

C. Scaling

Recall the previous scaling by the number of data N, $\text{Var}\{\hat{G}(e^{i\omega})\} \approx \text{AsVar}\{\hat{G}(e^{i\omega})\}/N$. To simplify the analysis we will also scale the input optimization problem by setting $\lambda_e = 1$, *i.e.* using unit noise variance. The *optimal* solution for the scaled problem will be denoted by $\{r_k^\star\}$. The unscaled optimal solution is $r_k^{\text{opt}} = [\lambda_e/N]r_k^\star$. The scaling affects the asymptotic variance expressions derived in the previous sub-section for sinusoidal signals. We will use the over-bar notation $\{\bar{r}_k\}$ for covariance sequence of a sinusoidal signal with frequency ω that gives a specified estimation error variance equal to one. For example, the scaled sinusoidal solution corresponding to $\bar{r}_0 = 1$, $\omega = 0, \pi$ and $\bar{r}_0 = 2$, $\omega \neq 0, \pi$, gives $\text{AsVar}\{\hat{G}(e^{i\omega})\} = 1$, *c.f.* (31).

D. Frequency Response

For the complex valued representation of the frequency response $G(e^{i\omega}) = \Gamma^*(e^{i\omega})\mathbf{g}$, we have $\mathbf{c} = \Gamma(e^{i\omega})$. The lower bound (27) with $j = 1$ then implies that

$$r_0^\star \geq 1 + |\sin(\omega k)|, \quad k = 1, \ldots, n. \quad (35)$$

This lower bound, which is achievable for $n = 1$, should be compared to the sinusoidal upper bound

$$r_0^\star \leq \bar{r}_0 = \begin{cases} 1, & \omega = 0, \pi \\ 2, & \text{otherwise} \end{cases}, \quad (36)$$

obtained by using a sinusoidal signal with frequency $\bar{\omega} = \omega$. We directly see that the lower bound equals the upper bound for $\omega = 0, \pi$ and $\omega k = \pi/2 + m\pi$, $m = 0, \ldots, (k-1)$. This gives the following optimality result:

Proposition 1: At the frequencies $\omega = 0, \pi$ and

$$\omega = \frac{\pi}{2}\frac{(1+2m)}{k}, \quad k = 1, \ldots, n, \quad m = 0, \ldots, (k-1), \quad (37)$$

a sinusoidal signal with frequency $\bar{\omega} = \omega$ and power $\bar{r}_0 = 1$, $\omega = 0, \pi$ and $\bar{r}_0 = 2$, $\omega \neq 0, \pi$, is a solution to the optimal input problem P3, (26), for frequency response estimation with FIR models of order n.

A direct way to prove this result is as follows: Note that $\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\} = \Gamma(e^{i\omega})\Gamma^*(e^{i\omega})$, $\omega = 0, \pi$. For $\omega \neq 0, \pi$, we have that $\bar{\mathbf{R}} = 2\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}$ implies that $\bar{\mathbf{R}} - \Gamma(e^{i\omega})\Gamma^*(e^{i\omega}) = \Gamma(e^{-i\omega})\Gamma^*(e^{-i\omega}) \geq 0$. The diagonal elements of the admissible Toeplitz matrix $\bar{\mathbf{R}}$ equals 2, and hence $r_0^\star \leq 2$. Next study the $1, (k+1)$ principal minor of $\mathbf{R} - \Gamma(e^{i\omega})\Gamma^*(e^{i\omega})$ at frequencies satisfying $\omega k = \pi/2$ (the cases $\omega k = \pi/2 + m\pi$, $m = 1, \ldots, (k-1)$ are similar)

$$\begin{bmatrix} r_0 & r_k \\ r_k & r_0 \end{bmatrix} - \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \geq 0. \quad (38)$$

This matrix is positive semi-definite if $r_0 \geq 1$ and $(r_0 - 1)^2 \geq r_k^2 + 1$, which implies $r_0^\star \geq 2$. Hence, the optimal solution is $r_0^\star = 2$.

Proposition 1 implies that when n increases, $r_0^\star = 2$ for more and more frequencies (their number increases as n^2) on a grid centered at $\pi/2$. The reason is $\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})$ becomes more and more restrictive for higher dimensions, while the sinusoidal solution is independent of n.

Let us compare the optimal solution with the white noise input case $\mathbf{R} = r_0\mathbf{I}$. Since the maximum eigenvalue of $\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})$ equals $n + 1$, we need an input power $r_0 \geq (n+1)$ to satisfy $r_0\mathbf{I} \geq \Gamma(e^{i\omega})\Gamma^*(e^{i\omega})$. For $n > 1$, this is substantially larger than what is needed for the optimal solution, which is in the interval $[1, 2]$.

E. Real and Imaginary / Log Gain and Phase

Next, we will now study

$$\begin{bmatrix} \text{Re}\left\{G(e^{i\omega})\right\} \\ \text{Im}\left\{G(e^{i\omega})\right\} \end{bmatrix}, \quad (39)$$

for which the asymptotic covariance expression is given by (17). Here, see (11),

$$\mathbf{c} = \left[\text{Re}\{\Gamma(e^{i\omega})\}, \ -\text{Im}\{\Gamma(e^{i\omega})\}\right]. \quad (40)$$

Problems P2, (25), and P3, (26), also make sense when $\mathbf{c}$, as in this case, is a matrix. Here $\mathbf{c}\mathbf{c}^* = \text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}$ is a symmetric Toeplitz matrix, which implies that the optimal solution to P3 (26) equals $r_k^\star = \cos(\omega k)$. This choice satisfies the equality $\mathbf{R} = \mathbf{c}\mathbf{c}^T$. Hence for this case a sinusoidal signal with frequency $\bar{\omega} = \omega$ and power $\bar{r}_0 = 1$ is an optimal solution to P3.

A similar result holds for

$$\begin{bmatrix} \log|G(e^{i\omega})| \\ \text{Arg}\left\{G(e^{i\omega})\right\} \end{bmatrix}, \quad (41)$$

where, see (20),

$$\mathbf{c}\mathbf{c}^* = \frac{1}{|G_o(e^{i\omega})|^2}\text{Re}\{\Gamma(e^{i\omega})\Gamma^*(e^{i\omega})\}, \quad (42)$$

and a sinusoidal signal with frequency $\bar{\omega} = \omega$ and power $\bar{r}_0 = 1/|G_o(e^{i\omega})|^2$ is an optimal solution to P3.

The results in this sub-section show that the corresponding two representations of the complex valued frequency response function are well balanced. In the next two subsections, we will study input signals that further reduce AsVar$\{|\hat{G}(e^{i\omega}|\}$, but at the price of increasing AsVar$\{\text{Arg}\left\{\hat{G}(e^{i\omega}\right\}\}$, and vice versa.

F. Frequency Gain

Next, we analyze P3 (26) for the amplitude frequency function estimate $|\hat{G}(e^{i\omega})|^2$, where

$$\mathbf{c} = 2\text{Re}\left\{G_o(e^{i\omega})\Gamma(e^{i\omega})\right\}, \tag{43}$$

the lower bound (27) implies

$$r_0^\star \geq 4\left[\text{Re}\left\{G_o(e^{i\omega})e^{i\omega k}\right\}\right]^2, \quad k = 0,\dots,n. \tag{44}$$

This bound is tight for $n = 1$, *i.e.*

$$r_0^\star = \max_{k=1,2}\{4\left[\text{Re}\left\{G_o(e^{i\omega})e^{i\omega k}\right\}\right]^2\}. \tag{45}$$

The scaled sinusoidal signal upper bound (32) is

$$r_0^\star \leq \bar{r}_0 = 4|G_o(e^{i\omega})e^{i\omega k}|^2 = 4\left[\text{Re}\left\{G_o(e^{i\omega})e^{i\omega k}\right\}\right]^2 + 4\left[\text{Im}\left\{G_o(e^{i\omega})e^{i\omega k}\right\}\right]^2, \quad k = 0,\dots,n. \tag{46}$$

In polar form, let

$$G_o(e^{i\omega}) = |G_o(e^{i\omega})|e^{i\phi^o}, \quad \phi^o = \text{Arg}\{G_o(e^{i\omega})\} \quad \Rightarrow \quad \text{Re}\left\{G_o(e^{i\omega})e^{i\omega k}\right\} = |G_o(e^{i\omega})|\cos(\omega k + \phi^o). \tag{47}$$

Then the difference between the upper and lower bound is

$$4|G_o(e^{i\omega})|^2 \min_{k=0,\dots,n} \sin^2(\omega k + \phi^o). \tag{48}$$

We directly see that if $\omega k + \phi^o = m\pi$ for a $k \in [0,\dots,n]$, where m is an integer, the upper bound and lower bound coincide and hence a sinusoidal signal with frequency ω is an optimal solution to P3 (26). This is a more restrictive result than for the frequency function given by Proposition 1. The numerical result in the next section confirm that a sinusoidal solution indeed is often optimal for the gain estimation problem.

G. Frequency Phase

Finally, we investigate the frequency phase function estimate Arg$\{\hat{G}(e^{i\omega})\}$, for which

$$\mathbf{c} = \frac{-1}{|G_o(e^{i\omega})|^2}\text{Im}\left\{G_o(e^{i\omega})\Gamma(e^{i\omega})\right\}, \tag{49}$$

and the result (27) gives

$$r_0^\star \geq \frac{1}{|G_o(e^{i\omega})|^4}\left[\text{Im}\left\{G_o(e^{-i\omega})e^{-i\omega k}\right\}\right]^2, \quad k = 0,\dots,n. \tag{50}$$

The upper bound given by a sinusoidal signal is, see (34),

$$r_0^\star \leq \tfrac{1}{|G_o(e^{i\omega})|^2} = \tfrac{1}{|G_o(e^{i\omega})|^4}\left[\left[\text{Re}\left\{G_o(e^{-i\omega})e^{-i\omega k}\right\}\right]^2 + \left[\text{Im}\left\{G_o(e^{-i\omega})e^{-i\omega k}\right\}\right]^2\right], \quad k = 0,\dots,n. \tag{51}$$

Using the polar form $G_o(e^{-i\omega}) = |G(e^{i\omega})|e^{-i\phi^o}$ the difference between the upper and lower bound is

$$\frac{1}{|G_o(e^{i\omega})|^2}\min_{k=0,\dots,n}\cos^2(\omega k + \phi^o). \tag{52}$$

Hence, a sinusoidal signal is optimal if there is an integer k such that

$$\omega k + \phi^o = \pm\frac{\pi}{2} + m2\pi, \quad m = 0,1,\dots \tag{53}$$

IV. A Remark on OE Models

A model of the form

$$y(t) = \frac{B(q)}{F(q)}u(t) + e(t),\ \theta = [b_1,\dots,b_n,f_1,\dots,f_n]^T, \tag{54}$$

$$B(q) = b_1q^{-1} + \ldots + b_nq^{-n},\ F(q) = 1 + f_1q^{-1} + \ldots + f_nq^{-n} \tag{55}$$

where q^{-1} is the delay operator, *i.e.* $q^{-1}u(t) = u(t-1)$, is called an Output Error (OE) model. The roots of $F(q)$ are assumed to be strictly inside the unit circle. The key result in [13] is that

$$\text{AsCov}\{\hat{\theta}\} = \lambda_e[\mathscr{S}(F_o,-B_o)\tilde{\mathbf{R}}_{2n}\mathscr{S}^T(F_o,-B_o)]^{-1}. \tag{56}$$

The matrix $\tilde{\mathbf{R}}_{2n}$ is the $2n \times 2n$ symmetric Toeplitz matrix with first row $[\tilde{r}_0,\tilde{r}_1,\dots,\tilde{r}_{2n-1}]$, where $\tilde{r}_k = \text{E}\{\tilde{u}(t)\tilde{u}(t-k)\}$ is the covariance sequence of the filtered input signal $\tilde{u}(t) = [1/F_o^2(q)]u(t)$. The matrix $\mathscr{S}(F_o,-B_o)$ is the Sylvester matrix associated with the "true" polynomials $-B_o(q)$ and $F_o(q)$, and has full rank if they have no common factors. In order to formulate the minimum variance optimal input signal design problem we have to relate the power of the original input signal $u(t)$ to the covariance sequence of $\tilde{u}(t)$,

$$\text{E}\{u^2(t)\} = \text{E}\{[F_o^2(q)\tilde{u}(t)]^2\} = \tilde{\mathbf{f}}_o^T\,\tilde{\mathbf{R}}_{2n+1}\,\tilde{\mathbf{f}}_o, \tag{57}$$

where $\tilde{\mathbf{f}}_o$ is the $n+1$ dimensional column vector constructed from the coefficients of $F_o^2(q)$. Note that we have to increase the dimension of the Toeplitz matrix to $(2n+1)\times(2n+1)$ to determine the variance of $u(t)$. This means that we need to optimize with respect to $\tilde{\mathbf{r}} = [\tilde{r}_0,\tilde{r}_1,\dots,\tilde{r}_{2n}]$. Assuming $\eta = \mathbf{c}^*\theta$, Problem P3 for OE models will have the form

$$\begin{aligned}\text{P3}:\quad & \min_{\tilde{\mathbf{r}}}\ \bar{f}_o^T\tilde{\mathbf{R}}_{2n+1}\bar{f}_o,\\ \text{s.t.}\quad & \tilde{\mathbf{R}}_{2n} \geq [\mathscr{S}(F_o,-B_o)^T]^{-1}\mathbf{c}\mathbf{c}^*[\mathscr{S}(F_o,-B_o)]^{-1},\\ & \tilde{\mathbf{R}}_{2n+1} \geq 0,\end{aligned} \tag{58}$$

which is a SDP in $\tilde{\mathbf{r}}$ with a slightly more complicated Toeplitz structure than for the FIR case. This observation is used in [14] to extend some of the FIR results presented in this paper to the OE case. The results are, however, less transparent than for FIR models.

V. Numerical Study

The performance index $J = (\bar{r}_0 - r_0^\star)/\bar{r}_0$ is used to compare the optimal solution $r_0^\star$ to the sinusoidal one $\bar{r}_0$. The frequency response estimation problem with $\mathbf{c} = \Gamma(e^{i\omega})$ defined by (8) is studied for models with $n = 1$ to 5. The optimal input problem (26) is solved using MATLAB/LMI LAB/YALMIP, [6], for $\omega_k = k\pi/360$, $k = 0,\dots,360$. The corresponding performance indices are given in Figure 1. As predicted by Proposition 1, a sinusoidal signal is optimal in the mid frequency range. For low and high frequencies, we can further reduce the input power by up to a factor of two by using an optimal input signal.

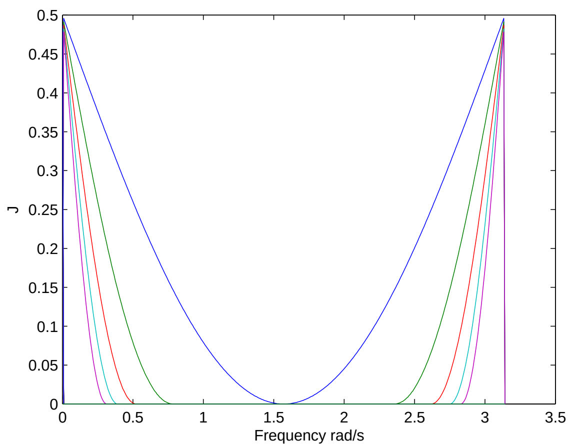

Fig. 1. The performance indices, $J(n)$, for frequency response estimation, $n = 1,..,5$. It is a monotonically decreasing function of n. The upper (blue) curve is for $n = 1$ and the lower curve (magenta) is for $n = 5$. Note that $J(n) = 0$ for $\omega = 0, \pi$.

Figure 1 also indicates that Proposition 1 only gives necessary conditions. A sinusoidal signal with frequency ω appears to be optimal in the whole interval $\omega \in [\pi/(2n), \pi - \pi/(2n)]$.

VI. Conclusion

A sinusoidal signal with frequency ω is the "natural choice" for estimating $G(e^{i\omega})$, but gives little information about other system properties. We have shown that for lower order systems up to fifty percent in input power reduction, compared to the sinusoidal signal, can be gained by using an optimal input signal. However for mid range frequencies and higher order systems a sinusoidal signal is minimum power optimal. One explanation is that when the model order increases structural information is of less importance and the optimal input just focuses on $G(e^{i\omega})$. We have also investigated what happens if we are mainly interested in $|G(e^{i\omega})|$ or $\text{Arg}\{G(e^{i\omega})\}$. The phase of $G_o(e^{i\omega})$ will then affect the optimal input signal.

An important problem in robust control applications is the estimation of the $\mathscr{H}_\infty$-norm of a linear dynamical system. This corresponds to estimating the maximum absolute value of the frequency response, $\max_\omega |G(e^{i\omega})|$. Iterative methods for input design for $\mathscr{H}_\infty$ estimation are discussed in [17]. In [16] the results presented herein are used to analyze performance and to derive optimal input signal for $\mathscr{H}_\infty$-norm identification. It is shown that a sinusoidal signal corresponding to the peak frequency always is minimum power optimal.

Appendix

A. Proof of (59)

Proposition 2: The optimal solution to P2, (25), and P3, (26), for $n = 1$ (the 2×2 matrix case), with $c = [c_0, c_1]^T$, is[3]

$$r_0^\star = \frac{|c_0|^2 + |c_1|^2 + \sqrt{(|c_0|^2 - |c_1|^2)^2 + 4[\text{Im}\{c_0 c_1^*\}]^2}}{2},$$
$$r_1^\star = \text{Re}\{c_0 c_1^*\}, \tag{59}$$

[3]We use the convention $\sqrt{x^2} = |x| \geq 0$.

which simplifies to $r_0^\star = \max\{c_0^2, c_1^2\}$, $r_1^\star = c_0 c_1$, for a real-valued vector $\mathbf{c}$.

Proof: The matrix $\mathbf{R} - \mathbf{c}\mathbf{c}^*$ is positive definite if the following conditions hold

$$(r_0 - |c_0|^2) \geq 0, \quad (r_0 - |c_1|^2) \geq 0, \tag{60}$$
$$(r_0 - |c_0|^2)(r_0 - |c_1|^2) - (r_1 - c_0 c_1^*)(r_1 - c_0^* c_1) \geq 0. \tag{61}$$

The second term of (61), $-(r_1 - c_0 c_1^*)(r_1 - c_0^* c_1) = -|r_1 - c_0 c_1^*|^2$, attains the minimum equal to $-[\text{Im}\{c_0 c_1^*\}]^2$ for $r_1 = r_1^\star = \text{Re}(c_0 c_1^*)$. This gives the least restrictive version of (61), $(r_0 - |c_0|^2)(r_0 - |c_1|^2) \geq [\text{Im}\{c_0 c_1^*\}]^2$. The second order equation in r_0, $(r_0 - |c_0|^2)(r_0 - |c_1|^2) = [\text{Im}\{c_0 c_1^*\}]^2$ has one solution that is larger than or equal to $\max\{|c_0|^2, |c_1|^2\}$ and one that is smaller than or equal to $\min\{|c_0|^2, |c_1|^2\}$. Only the larger solution will satisfy the constraints in (60). Solving the above second order equation shows that the covariance function of the optimal input signal is given by (59), which simplifies to $r_0^\star = \max\{c_0^2, c_1^2\}$, $r_1^\star = c_0 c_1$, for the real valued case.

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