Finite element techniques for the evaluation of energy flow parameters

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ABSTRACT

In applications of Statistical Energy Analysis (SEA) to complex engineering structures, procedures for the calculation of the SEA parameters are frequently unavailable. This note discusses two Finite Element techniques for identification of energy flow parameters: a waveguide and a modal approach.

1 Introduction

Statistical Energy Analysis (SEA) [1-4] has a long history in building acoustics and since the large NASA projects in the early sixties it has been recognised as a worthwhile method. However, it is only recently that it has become generally available through convenient, commercial software. The solutions of the SEA equations for power balance in structures are normally undemanding. Instead, demands on the software are routines for the evaluation of parameters needed for devising these equations. The procedures are generally based on asymptotic formulations for high frequencies and large structures supporting diffuse wave fields, available for a number of generic elements such as beams, thin-walled plates and shells and air volumes. Elements can be coupled at points, along (straight) lines and (plane) surfaces, where the coupling coefficients are normally found from "wave-approach" calculations. Some software has extended these formulations; yet, a major obstacle for the application of SEA to engineering structures is the lack of routines for the evaluation of SEA parameters.

The rationale for Finite Element (FE) evaluations of SEA parameters is simply the lack of explicit routines for complex elements. Although the FEM cannot predict the response of the complete structure, it can evaluate the modal data of substructures and the coupling of two substructures. This is of particular interest when elements support only a few resonances within the considered frequency bands, since then the wave fields cannot be diffuse [5], whereas the calculation burden with the FEM is reduced. Alternatively, two-dimensional (2-D) FE can generate dispersion relations that describe wave motion in general waveguides. The derived wave solutions may form the basis for SEA element formulations. This note discusses the evaluation of SEA parameters based upon such a ‘modal approach’ and a ‘waveguide approach’ FEA.
2 SEA formulation

The SEA formulation of the power balance in built-up structures will be briefly reported, as a background to the evaluation of its parameters, in Section 3 and 4. The starting point in an SEA is to subdivide the investigated structure into subsystems. The power balance equation for subsystem $i$ is given by

$$p_{in}^{(i)} = p_{dis}^{(i)} + \sum_{j \neq i} p_{\text{coup}}^{i,j},$$

where $p_{in}^{(i)}$ is the input power to the subsystem from external sources (e.g., mechanical or turbulent boundary layer excitation), $p_{dis}^{(i)}$ is the power dissipated through damping and $p_{\text{coup}}^{i,j}$ is the net power transmitted from subsystem $i$ to a neighbouring subsystem $j$ through mechanical coupling. In equation (1), steady state vibration is assumed and power is time averaged. For commonly used damping models, the dissipated power is written in the form

$$p_{dis}^{(i)} = \eta_i \omega E_i$$

where $\eta_i$ is the damping loss factor, $E_i$ is the time averaged energy stored in the subsystem and $\omega$ is the (angular) frequency. Equations (1) and (2) follow from basic physical principles; however, an expression of the coupling power is needed, and the form adopted represents the main SEA hypothesis of coupling power proportionality. It is assumed a priori that

$$p_{\text{coup}}^{i,j} = C^{i,j} \left( \frac{E_j}{n_j} - \frac{E_j}{n_j} \right)$$

where $n_j$ is the modal density (i.e. the number of modes per unit angular frequency band) and where the dimensionless parameter $C^{i,j}$ is the ‘conductivity’. (In standard SEA notation $C^{i,j} = \omega n_i \eta_{i,j}$, where $n_{i,j}$ is the coupling loss factor.) Equation (3) states that the net energy flow is proportional to the difference in ‘energy per mode’ in connected subsystems. The hypothesis is motivated by analogy with asymptotic high frequency results in structure borne sound analysis [6]. It has been demonstrated for two coupled oscillators (modes in connected structures) [7] and for an isolated structural mode exchanging energy with a diffuse sound field [8]. However, it is only approximately valid for multi-modal structures [9].

If the hypothesis (3) is adopted, it is convenient to introduce, as dependent variables, the ‘modal energies’, $E_{m,i}$ (the energy per mode times analysis bandwidth) sometimes referred as ‘modal power potentials’ [2], given by

$$E_{m,i} = E_i / n_i$$

and accordingly, equation (2) may be written

$$p_{dis}^{(i)} = M_i E_{m,i}^i, \quad M_i = \eta_i \omega n_i$$
The dimensionless modal overlap, $M$, describes dissipation of energy in an SEA. It is a measure of the probability for resonance in an element since the average spacing between resonances is $\delta \omega = 1/n$ and the ‘size’ of a resonance is of the order of its 3-dB bandwidth, $\eta \omega$. It is also significant in discussions of response amplification at resonance [10], spatial decay in waveguides [11], coupling strength [12-15] and statistical variability of response [16].

Equation (3) defines the conductivity and may be the basis for its evaluation. However, standard wave approach procedures consider the coupling of semi-infinite elements based upon the observation that for ‘weakly coupled’ elements that are ‘large enough’, the conductivity does not depend on the precise shape or size of the connected elements but only on their wave characteristics. Thus, the conductivity may be given by

$$C^{i,j} = \left( \frac{P^{i,j}_{\text{coupled}}}{E_{n,j}} \right) c^{i,j} |M_{s} \rightarrow 0^\ast. \tag{6}$$

In applications of SEA, the conservation of energy is for each substructure formulated using the notation in equations (1-5). The resulting equations are assembled in matrix form and solved. There are no numerical difficulties involved in this, since the system matrix is real, symmetric and positive definite. A major difficulty in an SEA is instead the evaluation of the parameters in (1-5). These are here denoted the SEA parameters and are: the modal overlap, $M$; the modal density, $n$; the conductivity, $C$ and the input power, $P_{in}$. In the next sections, the attention is drawn to two FE techniques for the evaluation of the SEA parameters.

## 3 Waveguide approach

An early reference on the waveguide approach for the evaluation of energy flow parameters is Heckl’s study of plate-beam-plate coupling [16]. It considers structures that are simply supported along the waveguide and sinusoids describe the cross-sectional motion, whereas travelling waves describe motion along the plates. The transmission factors, for each cross-sectional mode, are evaluated and subsequently averaged, upon the assumption of a ‘wide’ waveguide.

Current SEA procedures directly assume coupling of infinite width and diffuse wave field. Such procedures for plates coupled directly [18] or through reinforcing beams [19] and shell strips [20] are the backbone of SEA software. The procedures require the dispersion relations, relating frequency and wave heading to wave number; however, these are frequently unavailable or are treated in an over simplified manner. Examples of structures that can cause difficulties are as follows: rib-stiffened plates for medium frequencies when neither ‘smearred-out’ theory [21] nor SEA sub-structuring [22] is valid; sandwich and honeycomb structures and extruded aluminium profiles, found in ‘space frame’ concept cars and in the walls of lightweight trains, for example. Now, the FEM can formulate the dispersion relations for such structures. The application of which, and the subsequent evaluation of SEA parameters, is discussed below.
3.1 Formulation of waves in uniaxial waveguides

The homogenous equations describing harmonic motion in waveguides with uniform cross sectional properties may generally be written as

\[ \sum_{n=0} a_n \frac{\partial^n}{\partial x^n} U - \omega^2 M U = 0, \]  

(7)

\[ u(x,t) = \text{Re} \left( U(x) e^{-i \alpha t} \right), \]  

(8)

where \( M \) is the mass matrix and \( K_n \) are ‘stiffness’ matrices and the entries of the vector \( u \) contains the response variables. Examples of systems obeying equation (7) are Timoshenko beams and thin-walled plates that are simply supported along the waveguide

\[ K_2 = \begin{bmatrix} -EI_y & 0 \\ 0 & -GA_k \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & GA_k \\ GA_k & 0 \end{bmatrix}, \quad K_0 = \begin{bmatrix} GA_k & 0 \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \rho I_y & 0 \\ 0 & \rho A \end{bmatrix}, \quad u = \begin{bmatrix} \phi \\ w \end{bmatrix}, \]  

(9)

and thin-walled plates that are simply supported along the waveguide

\[ K_4 = D; \quad K_2 = -2 \alpha_p^2 D; \quad K_0 = \alpha_p^4 D; \quad M = m; \]  

\[ \alpha_p = p \pi / L_y; \quad u = w_p; \quad w(x,y) = \sum_p w_p(x) \sin(\alpha_p y) \]  

(10)

where \( E \) is Young’s modulus, \( I_y \) is the area moment of inertia, \( G \) is the shear modulus, \( A \) is the cross sectional area, \( k \) is the shear correction factor, \( \rho \) is the density, \( L_y \) is the plate width, \( D \) is the bending rigidity, \( m \) is the mass per unit area, \( w \) is the transverse displacement and \( \phi \) is the cross sectional rotation. Other, more complex, examples are thin-walled Vlasov-Timoshenko beams [23-24] and fluid-filled pipes [25].

Now, even more complex waveguides can be handled by FE methods. For such an analysis, the starting point is a variational formulation, based on, e.g., Hamilton’s principle, for the structure. Two-dimensional FE shape functions describe the motion’s \( y \)- and \( z \)-dependence where, without loss of generality, it is assumed that the waveguide is aligned with the \( x \)- axis. It follows that ‘nodal’ displacements are functions of \( x \). The response variables are given by expressions of the form

\[ w(x,y,z) = \Theta(y,z)^T u(x) \]  

(11)

where the entries of the vector \( u \) are the nodal displacements and those of \( \Theta \) are 2-D FE shape functions, which generally are non-zero only within one finite element. The functions (11) are inserted into the functional, whereupon the corresponding Euler-Lagrange equations are of form of (7).
Possibly, this mode of analysis first appeared in reference [26]. Studies that are more recent describe wave propagation in laminated composite plates [27], thin-walled beams [28], railway track [29-30], rib-stiffened panels [31], anisotropic shells and beams [32-33], fluid-filled pipes [34] and a railway car structure [35].

To exemplify the approach, Figure 1 display a generic rib-stiffened aircraft (ship or railway car) panel. The corresponding FE model, shown in Figure 2, uses standard cubic shape functions for the flexural motion and linear functions for the in-plane motion, as detailed in [31]. This coarse FE model should, for a 1 mm thick panel of aluminium, perform well up to 50 Hz while accurate analysis up to 5 kHz requires no more than 100 nodes.

Details of the FE modelling procedures are found in the references. The discussion in what follows concerns the usefulness of the formulation (7) for the evaluation of energy flow parameters.

3.1.1 Wave solutions

The set of equations (7) has constant coefficients so their solutions are exponential functions. Consequently, the ‘nodal’ displacements are of the form $U = \Phi e^{ikx}$ and upon inserting this into the equation, the dispersion relations are given by

$$\left[ K(ik) - \omega^2 M \right] \Phi = 0; \quad K = \sum_{n=0} K_n (ik)^n. \quad (12)$$

For a given frequency $\omega$, this is an eigenvalue problem for the wavenumbers $k$, which is solved by standard methods, see e.g. [31]. Similarly, for a given real valued $k$ this is an even more convenient, standard, eigenvalue problem for $\omega$. The choice between these two is a matter of viewpoint. For undamped motion at a given frequency, the solutions to (12), are of any of three forms: a) real $k$ corresponding to wave solutions; b) imaginary $k$ corresponding to exponentially decaying nearfield solutions and c) complex $k$ corresponding to ‘standing decaying waves’. These latter come in complex conjugate pairs and are oscillating and decaying. Figure 3 shows the real valued wavenumbers for the panel in Figure 1.
3.2 Modal density and group velocity

For structures without damping, the matrices $\mathbf{K}$ and $\mathbf{M}$ are real. Consequently, if $k = k(\omega)$ is a real valued eigenvalue to (12) and $\Phi$ is the corresponding eigenfunction, then $-k$ is also an eigenvalue with eigenfunction $\Phi^*$. One of the solutions to the equations of motion (7) is therefore given by

$$\mathbf{U} = a(\Phi e^{ikx} + \Phi^* e^{-ikx}) = b \left( \text{Re}(\Phi) \cos(kx) + i \text{Im}(\Phi) \sin(kx) \right),$$

(13)

where $a$ and $b$ are constants.

To satisfy general boundary conditions to a waveguide all evanescent nearfield solutions to (12) are needed. One particular boundary condition, however, is fulfilled for the propagating wave solutions, without any interaction between them. It follows that the eigenmodes of a waveguide of length $L$, obeying the 'convenient' boundary condition, are of the form of (13), when $k$ is given by

$$k = k_p = p \pi / L,$$

(14)

where $p$ is a positive integer. The eigen frequencies, $\omega_{p,r}$, are given by the standard eigenvalue problem that follows from (12) when $k = k_p$, whereas the eigenvectors $\Phi_{p,r}$ and the trigonometric functions in (13) specify the corresponding eigenmodes $\Psi_{p,r}$. The orthogonality of two eigenmodes $\Psi_{p,r}$ and $\Psi_{q,s}$ for $p \neq q$ follows from the properties of the
trigonometric functions and for \( p = q, r \neq s \) it follows from the orthogonality of the set of eigenvectors \( \{\Phi_{p,r}\} \).

For elements that are many wavelengths long, the modal density is asymptotically independent of the boundary condition. Hence, it follows from (14), the asymptotic modal density, for any of the branches of the dispersion relations, is given by

\[
n(\omega) = \frac{\partial N}{\partial \omega} = \frac{\partial (k L / \pi)}{\partial \omega} = \frac{L}{\pi} \frac{\partial k}{\partial \omega}, \tag{15}\]

If the wave numbers \( k = k(\omega) \) have continuous derivatives, this is equally expressed

\[
n(\omega) = \frac{L}{\pi c_g}; \quad c_g = \frac{\partial \omega}{\partial k}, \tag{16}\]

where \( c_g \) is the group velocity for the considered wave type. This is the speed by which wave energy propagates along the waveguide.

### 3.2.1 Evaluation of the modal density

The modal density and the group velocity may be calculated from the difference of wave numbers at two adjacent frequencies. This requires the identification of two solutions to the dispersion relations. In light of the complexity of dispersion curves, such as the one in Figure 3, this is not straightforward when one branch approaches another. Some linear algebra illustrates an immediate solution. First, evaluate the derivative of equation (12) with respect to frequency

\[
\frac{\partial}{\partial \omega} \left( \left[ K(ik) - \omega^2 M \right] \Phi \right) =
\]

\[
= \left[ \frac{\partial K(ik)}{\partial k} \frac{\partial k}{\partial \omega} - 2\omega M \right] \Phi + \left[ K(ik) - \omega^2 M \right] \frac{\partial \Phi}{\partial \omega} = 0 \tag{17}\]

Then, consider the right-eigenvector, \( \Phi_r \), to (12), which is a solution to

\[
\Phi_r^T \left[ K(ik) - \omega^2 M \right] = \left[ K(ik) - \omega^2 M \right]^T \Phi_r^T = 0. \tag{18}\]

For isotropic structures, one may as in reference [28, 33] transform equation (12) to a real valued symmetric eigenvalue problem, in which case, the left and right eigenvectors are equal. In other cases, the right eigenvector is efficiently found by inverse iteration [36, Section 5.8].

Finally, equation (17) is multiplied from the left by \( \Phi_r^T \) and by virtue of equation (18), we have that
\[ \Phi^T_R \ast \left[ \frac{\partial K(ik)}{\partial k} \frac{\partial k}{\partial \omega} - 2\omega M \right] \ast \Phi = 0. \]  \hspace{1cm} (19)

This is, since \( \partial k/\partial \omega \) is a scalar, equally written as

\[ \frac{\partial k}{\partial \omega} = \frac{2\omega \Phi^T_R \ast M \ast \Phi}{\Phi^T_R \ast K' \ast \Phi}, \quad K' = \frac{\partial K(ik)}{\partial k}. \]  \hspace{1cm} (20)

An SEA, say of a cylindrical structure, may consider as an SEA element each branch of the dispersion relations [37], a group of branches [38] or all branches [39]. In either case, the modal density is calculated from (20) by matrix algebra of the kind that computers love.

### 3.3 Input Power

The equations of motion for a waveguide that is excited by a force, which is distributed on the cross-section but concentrated at \( x = x_0 \), are, in analogy to (7), given by

\[ (1 - i\eta_c) \sum_{n=0} K_n \frac{\partial}{\partial x^n} U - \omega^2 (1 + i\eta_v) M U = F \delta(x - x_0) \]  \hspace{1cm} (21)

where \( F \) is the generalised force vector and \( \eta_c \) and \( \eta_v \) are loss factors. The response is given by a modal analysis, based on the modes discussed in Section 3.2. From this follows the input power, for random \( x_0 \),

\[ P_{in} = \text{Re} \left[-i \omega \Phi^H \Phi \right] = \sum_{p,r} \text{Re} \left\{ \frac{-i \omega \Phi^H \Phi^T p, r \Phi^T_{r,p} F}{(1 - i\eta_c) \omega^2_{p, r} - (1 + i\eta_v) \omega^2} \Phi^T_{r,p} M \Phi^T_{p, r} \right\}. \]  \hspace{1cm} (22)

where upper index "\(^H\)" denotes complex conjugate and transpose. The frequency averaged input power is calculated as in reference [6, Sect. 4.IV.c]

\[ \langle P_{in} \rangle = \sum_{r} \frac{\pi}{2} n_r \frac{\Phi^H \Phi^T_{r,p} F}{\Phi^T_{r,p} M \Phi^T_{p, r}} \]  \hspace{1cm} (23)

where index \( p \) now indicates the frequency dependence of the eigenvectors and \( n_r \) is the modal density for branch \( r \).

Reference [35] (presented at Novem 2000) calculates the input power from a point force on a transverse beam in the railway car structure in Figure 4. The result agrees with measurements, for a wide frequency region.
3.4 Vibration Conductivity for waveguides

Consider two end-coupled elements, where the response in the first, directly excited, element is but for the possible presence of nearfield terms at the junction, given by (13). The energy transmission at the connection is described by the transmission factor $\tau$ that applies for the corresponding semi-infinite elements. Thus, if the power $P_{inc}$ is incident on the junction then the transmitted power, $P_r$, is given by

$$P_r = \tau P_{inc} = \tau e_{inc}, \quad (24)$$

where $e_{inc}$ is the energy density of the incoming wave. In a standing wave field, c.f. (13), the energy density is in general not constant (except for a rod). Instead, an average energy density is considered. This average is calculated either: a) by averaging in the far field over exactly half a wavelength; b) by averaging over a distance that is much longer than the wavelength, yet shorter than the distance required for a substantial damping decay, which is of the order of $1/\eta k$, or, c) as a statistical expectation when the incoming and reflected waves are assumed incoherent. For all three cases, the average energy density, $e_{av}$, is given by the sum of the energy density in the incoming wave and the reflected wave, so that for a reverberant system

$$e_{av} = \frac{e_{av}}{1 + (1 - \tau)} = \frac{n E_{av}/L}{2 - \tau}, \quad (25)$$
where $L$ is the element length. Upon this basis, the conductivity is from equation (6) and (16) given by

$$C^{ij} = \left[ \frac{P_{x \alpha}^{ij}}{E_m^{x \alpha}} \right]_{x^i y^j \rightarrow 0} = c_k^i \frac{\pi}{2 - \tau} \frac{n_i E_m^{x \alpha}}{L} \frac{1}{E_m^{x \alpha}} = \frac{\tau}{\pi}.$$

(26)

This is the ‘travelling wave estimate’ of the conductivity. The simplicity of the expression (26) for the conductivity is a strong argument for using modal energy as variable in vibrational energy analysis (“Ockham’s razor”, see Encyclopædia Britannica Online).

### 3.4.1 Example: Simply supported plate

Most procedures for the evaluation of SEA conductivity are based on the wave approach, where diffuse wave fields are assumed. An alternative, the waveguide approach, models each of the wave solutions to (12) as a 1-D SEA element [37]. This formulation is useful when vibration energies, group velocities or transmission factors are largely different for the various wave types. A third alternative is the modal approach where response is described by the subsystem modes. Normally, it assumes that all modes within a frequency band have equal energy. Lyon and DeJong state that this assumption is equivalent to the wave approach assumption that wave intensity is independent of wave heading [2, Section 10.2]. It is illustrative to demonstrate this statement, starting from the waveguide formulation.

Consider a thin-walled plate that is simply supported along the waveguide. Its free motion is described by equation (10), from which it follows that its wavenumber, modal density and group velocity are given by

$$k(\alpha_p) = \sqrt{k_{h}^2 - \alpha_p^2} = k_{h} \cos(\phi); \quad k_{h}^4 = \omega^2 m/D;$$

$$n(\alpha_p) = k_{h} L/(2\pi \omega \cos(\phi)); \quad c_g(\alpha_p) = L/\pi n(\alpha_p) = (2\omega/k_{h}) \cos(\phi).$$

(27)

At the plate end, it is connected to other structures with the energy flow described by a transmission factor, $\tau$, which depends on the wave heading or, equally, on the branch of the dispersion relations. The transmitted power is given by

$$P_{tr} = \sum_{p=1}^{p_{\max}} \tau(\alpha_p) P_m(\alpha_p) = \sum_{p=1}^{p_{\max}} c_g(\alpha_p) e(\alpha_p) \tau(\alpha_p)/(2 - \tau(\alpha_p)),$$

(28)

where $p_{\max} = \max(p: \alpha_p < k_{h})$ and $e(\alpha_p)$ is the energy density (per unit length). From equation (16) follows that, for a reverberant field and a ‘wide’ waveguide, the transmitted power is equally given by

$$P_{tr} = \frac{E_m}{\pi d \alpha} \sum_{p=1}^{p_{\max}} \frac{\tau(\alpha_p)}{2 - \tau(\alpha_p)} d\alpha \approx 2.$$ 

(29)

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1 Many researchers, see, e.g., [40, 52 Appendix C], don’t agree on the nominator but prefer to use a factor of 2 instead of $2 - \tau$. 

\[ E_m = \frac{L_y E_m}{\pi^2} \int_0^{\kappa} \frac{\tau(\alpha)}{2 - \tau(\alpha)} \, d\alpha = \frac{L_y k_p E_m}{\pi^2} \int_0^{\pi/2} \frac{\tau(\phi) \cos(\phi)}{2 - \tau(\phi)} \, d\phi , \]

where \( E_m \) is the modal energy, assumed equal for all modes, \( \alpha = n \pi / L_y = k_p \sin(\phi) \) and \( d\alpha = \pi / L_y \). From equation (6) and (29) follows the standard conductivity for line connections, valid for diffuse field [2, (10.2.6)]. Consequently, the similarity between the assumptions of diffuse wave field and modal equipartition is demonstrated for a simply supported panel.

### 3.5 Spectral Finite Element Method

The wave solutions to (12) can be used as shape functions in a new 3-D spectral FE formulation. A 20 node ‘finite’ element for the aircraft panel is indicated in Figure 5. It is based on shape functions of the form

\[ w(x, y, z) = \Theta(y, z) \Phi \text{diag}(e^{ikz}) A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]  

\[ (30) \]

where \( \Theta \) contains the 2-D shape function used in (11), the columns of \( \Phi \) and the entries of \( k \) are the eigenvectors and eigenvalues given by the dispersion relations (12). The vectors \( u_1 \) and \( u_2 \) are the nodal displacements and the matrix \( A \) is given by the boundary conditions that the element displacement is equal to the nodal displacements at the element ends. The dynamic stiffness matrix for an element having shape functions of the form of (30) may be evaluated by routines presented in [34].

The spectral FEM is similar to the direct dynamic stiffness method, e.g., [21]. A major advantage with the FE formulation is that a variational formulation provides a sound basis for approximate element formulations. Moreover, it is inherently simple to combine the spectral elements with standard finite elements for more complicated sub-structures and for detailed modelling of joints.

The length of a spectral element is limited only by geometrical discontinuities so the coupling of semi-infinite elements can be studied and transmission factors calculated. Reference [41] describes fluid-filled pipes with spectral FE and the flanges at their

![Figure 5. Three-dimensional FE model of aircraft panel.](image-url)
connection by standard FE. The calculated transmission factors are input data to an SEA model of a pipe structure, which is favourably compared to an accurate calculation [37].

A PhD project at KTH considers the railway car structure in Figure 4. The transverse beams and the long beams have thin-walled open cross-sections that are not described by beam theory above say 100 Hz [35]. The beams will be modelled by semi-infinite spectral elements with the coupling area modelled by standard finite elements, thus advancing the wave approach methodology for calculation of transmission factors.

4 Modal approach

4.1 A dedicated modal approach

Maxit and Guyader formulate the coupled equations of motion expressed by the substructures’ uncoupled modes [42-43]. They note that the direct interaction of substructures is on a mode-to-mode basis, commonly assumed in the modal approach to SEA, e.g. [2, Section 3.2]. Upon this basis, the coupling loss factors are identified by a superposition of Scharton and Lyon’s exact result for the frequency average coupling power between two oscillators [7]. SEA models are formulated where an element describes one substructure mode. Alternatively, an element describes the average modal energy in a substructure, based on the standard SEA assumption of modal equipartition.

Reference [43] presents several encouraging calculation examples that indicate the usefulness of this approach. The procedure seems valid if resonant vibrations dominate the response and, if the elements are not strongly coupled; the limits for this assumption still need to be established. Especially so, as Scharton and Lyon’s two-oscillator result is valid for arbitrary coupling strength, so ‘normal’ coupling strength criteria may seem too strict, when the modal overlap is less than unity.

4.2 An inverse method: the power injection method

The standard technique for identifying SEA parameters from measured or calculated response is the Power Injection Method (PIM) [44-46]. Using this method, a force is applied at a number of locations in a substructure, the input power is determined and the kinetic energy in each substructure is estimated. As in SEA, it is assumed that dissipation powers and coupling powers are proportional to vibration energy. Upon this basis, the input power to one substructure is related to dissipation and coupling powers in the entire structure. The scheme is repeated, applying force to all other substructures in turn; so the SEA parameters can be determined from a linear system of equations. The PIM, or equally the SEA equation for power balance, is the basis for identification of SEA parameters based on FE results, made, e.g., by Simmons [47], Fredö [48-49], Keane, Shankar and Beshara [50-51] and Mace and Shorter [52-53]. In some cases, the evaluation of the injected power is simplified for rain-on-the-roof excitation and most authors use modal substructuring for efficiency.

The PIM is based on the SEA hypothesis of coupling power proportionality, which may be only approximately valid. Generally, the observed response does not obey the SEA reciprocity relation [2, p. 87], so the results are not easily used for updating predictive SEA
models. Moreover, the equation system that need be solved is sometimes badly conditioned and therefore any errors in the model will be amplified, producing disappointing and perhaps non-physical results [49]. It seems that both the PIM and the validity of coupling power proportionality are favoured by weak coupling and modal overlap factors that are at least of the order of unity [54-55]. To improve the inverse procedures, various calculation schemes have been proposed [56-58].

### 4.3 An “exact” formulation

The present author believes it unwise to base an inverse method on an approximate formulation that may be ill conditioned. Consequently, reference [59] suggests that inverse SEA be based on Langley’s exact power balance relations [60]. These relations apply for vibrations in homogenous built-up structures that are proportionally damped and governed by linear equations and are excited by rain-on-the-roof. Without any further limitations, we have the following system of equations

$$C \hat{E} = P_m,$$  \hspace{1cm} (31)

where the entries of $P_m$ are the frequency averaged input powers within the frequency band $\Omega$. The symmetric matrix $C$ and the “vibration potential” $\hat{E}$ are given by

$$C = \pi^{-1} q Q^{-1} q; \quad \hat{E} = \pi r^{-1} q^{-1} E_k.$$  \hspace{1cm} (32)

The entries of $E_k$ are the substructure kinetic energies and those of the diagonal matrices $r$ and $q$ and the symmetric matrix $Q$ are given by

$$r_i = R_{ii}; \quad R_{ij} = \frac{1}{2\Omega Q_{ij}} \int_\Omega \int_{\gamma_i} \int_{\gamma_j} \rho_i(x) \omega^2 |G_{ij}(x,y,\omega)|^2 \, dx \, dy \, d\omega;$$

$$q_i = \frac{1}{\Omega} \text{Re} \left( \int_\Omega \int_{\gamma_i} -i \omega G_{ii}(x,x,\omega) \, dx \, d\omega \right);$$

$$Q_{ij} = \frac{1}{2\Omega} \int_\Omega \int_{\gamma_i} \int_{\gamma_j} \omega^2 |G_{ij}(x,y,\omega)|^2 \, dx \, dy \, d\omega,$$  \hspace{1cm} (33)

where $V_i$ is the ‘volume’ of substructure $i$ and $\rho_i$ is its density. The Green function $G_{ij}(x,y,\omega)$ relates the response of subsystem $i$ at location $x$ to a harmonic point load at location $y$ on subsystem $j$. Equation (31) relies on that an effective density may be attributed to each substructure, i.e., that $R_{ij} = R_{ii}$ for all $j$. This assumption is trivially fulfilled for uniform subsystems (Langley’s assumption [60]) but also if either of the SEA assumptions of equal modal energy and diffuse wave fields are fulfilled [59]. It should also be fulfilled in a dedicated modal approach where an SEA element describes the energy in only one mode. This application is yet unexplored.

Langley defines weak coupling by that a Green function attributed to an uncoupled substructure is, within a required accuracy, not modified when the substructure is
connected to the rest of the structure. Reference [15] presents a criterion by which this assumption is assessed. For weak coupling, defined in this manner, it follows that [60]

\[ q_i \approx \frac{\pi N_i}{2 \rho_i \Omega}, \quad \hat{E}_i \approx 2 E_{ij}/(N_i / \Omega) \]  \tag{34}

where \( N_i / \Omega \) is the modal density. For resonant vibration, the kinetic energy is equal to the strain energy and the vibration potential in equation (34) is equal to the modal energy times analysis bandwidth. It follows that, for weak coupling, equation (31) is an SEA model. Moreover, even if elements are strongly coupled and equation (34) is not valid, equation (31) still gives a valid description of the power balance, so it has a broader application than standard SEA procedures.

### 4.4 Application of FEA

The feasibility of FEA for the evaluation of energy balance parameters based on equation (31) is investigated [61-62]. Figure 6 shows the investigated structure. A standard FE package calculates, roughly, 400 of the uncoupled modes for each plate, expediently done on a PC. Upon the basis of the uncoupled modes, the coupled equations of motions are formulated, the Green function calculated and the power balance equation (31) formulated for 1/3-octave bands. The corresponding SEA model is identified from the entries of the matrix \( C \): the off-diagonal terms give the conductivity and the diagonal terms are the sum of the corresponding modal overlap and the conductivity.

Extensive parameter variations have been made [62] of which only some are discussed here. Figure 7 compares calculated modal overlap factors with those based on the given

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![Figure 6](image-url)  
**Figure 6.** Geometry of investigated plate structure.
loss factor and the asymptotic modal density for a plate [6, p. 489]; showing stable results even for strong coupling. Figure 8 compares three calculated conductivities with those found for an equivalent line connection, calculated by equation (29), and for six point connections, each calculated as in [2, Section 10.1], using the asymptotic mobility for an edge excited plate found in [6, p 317]. The line connection should apply at lower and the point connection at higher frequencies, when the distance between the connections is greater than half the wave length, which appears at roughly 200 Hz. Figure 8 shows that the precise shape and size of the top plate has only a minor influence on the conductivity, as is generally assumed in SEA. For strong coupling, the conductivity decreases with decreasing modal overlap factors [63-64]. It is claimed that coupling is strong when the factor $\gamma = 2 \frac{C_{ij}}{(\pi M_i M_j)}$ is greater than unity [12, 14-15]; a criterion met below 200 Hz for $\eta = 0.004$ and below 80 Hz for $\eta = 0.02$. It is disappointing that the asymptotic results in Figure 8 are, almost exactly, a factor of 2 lower than the FE results.

4.5 Future work

The FEM presented above shows promising results and the calculation burdens should not be overwhelmingly, if the SEA models are based on FEA of only directly coupled elements. (This is a general conclusion, since all vehicles, but submarines, have shell thickness of the order of $1\text{mm}$, with the length scale given by that of man, so the shell size is of the order of $1\text{m}^2$.) Hopefully, the methodology will soon be tested on an engineering structure. The work has provoked a number of research issues that are briefly discussed below.
Normally, SEA assumes conservative coupling, implying that all dissipation is within the elements. Often the connection of engineering structures introduces dissipation, which SEA models as increased element damping [2, Section 9.3]. This gives a non-local element formulation, since it depends on the properties of connected elements.

The derivation of the energy balance equation (31) is not based on the assumption of conservative coupling, so it should be useful also for dissipative coupling, as is briefly confirmed in reference [62]. The coupling damping loss factor, introduced by Keane and Beshara [51], is used for the derived SEA model. Its product with the modal density and frequency produces the ‘coupling damping factor’, which, in parallel to the conductivity, apparently depends only on the coupling mechanism and the wave characteristics of the connected elements, so a local element formulation seems possible [62]. This is not yet fully established.

4.5.1 Dissipative coupling

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4.5.2 Stochastic structures

The application of SEA requires some statistical ensemble for theoretical models. In principle, when the design parameters are random, FEA may determine the response statistics through Monte Carlo simulations. This is, however, very costly and the analysis in references [47-53, 61-62] consider only single deterministic structures. Fredø denotes this mode of calculation ‘SEAL’ (SEA Like) to emphasise the difference between prediction models for single structure and ensemble average response [49].

Figure 8. Conductivity, Square plate coupled to top plate; square, chopped plate $\eta = 0.02$; circle, triangular plate $\eta = 0.02$; diamond, triangular plate $\eta = 0.004$; dashed, analytical, point coupling; dotted, analytical, line coupling.
The most apparent effect of uncertainty is that the modal frequencies are random. The expected response spectrum is then flattened as if the size of the elements were increased (except for that there are no pronounced anti-resonances). It has been observed (see e.g., [12, 65-67]) that, if the uncertainties are large enough, the statistics of the response ‘saturates’. (The limit when saturation appears is sometimes referred to as ‘the acoustic limit’ [12].) In the saturated state, the expected value of the vibrational energy flow and its variance will be almost independent of both the size and the distribution of the uncertainties of the resonance frequencies. Scrutinising the derivation of equation (31), it is seen that this equation predicts ensemble averaged response, if the matrices $q$ and $Q$ are averaged similarly [15]. Hopefully, future research will investigate the criteria for saturation of these matrices and calculate the expected value and variance of the energy flow in the saturated state.

4.5.3 The mid-frequency problem

A major obstacle to SEA is that most engineering structures consist not only of parts that have many resonances and which are distinctly different ‘elements’ but also of stiff and strongly connected parts, most often designed to support and secure the structure. Often, the greatest responses are found in the flexible parts with the transmission of vibration in the supporting structure. An FE-analysis of such structures is not possible: because the computational burden due to the excessive number of degrees of freedom in the flexible parts and because standard FE procedures are not, yet, able to handle frequency dependant material parameters and boundary conditions. On the other hand, SEA does not apply for strongly coupled and stiff structures with few resonances. Consequently, high frequency prediction of noise and vibration in built-up structures, such as vehicles, requires a mixed approach.

One such mixed approach is ‘fuzzy’ structure theory [68-69]. In this theory, the ‘master’ structure is modelled deterministically. The influence of fuzzy attachments on the master structure is, upon a stochastic assumption, modelled as an increase of inertia and damping. Reference [70] brings fuzzy theory further by allowing full SEA modelling of ‘local’ substructures whereas deterministic FEA is used for the ‘global’ structure. It appears, however, as the chosen path does not allow analysis of systems where vibration energy is transmitted from one deterministic structure to another, through stochastic substructures. For random excitation, it is not possible to predict the response but only its energy (or equally, the magnitude of response). The same applies for random structure.

The power balance equation (31) requires the exact Green’s function for the entire structure, which, apparently, makes the formulation unnecessary and impractical. However, the formulation may be used as a starting point for deriving equations for the frequency averaged vibrational energy in structures where the response in parts is described with modal bases found from FE analysis and in parts is described, as in SEA, with the averaged modal response in members of statistically defined ensembles of similar elements.

First, the coupled equations of motion for stationary vibrations are formulated with the response in each element formally expressed in a local modal basis. The Green’s function is found upon inverting the system matrix. The analysis may be simplified upon the assumption that some of the substructures are ‘weakly coupled’ thus allowing the Green’s function to be calculated while parts of the inversion is made by a simple series expansion,
as in [71]. It should be noted: not only ‘soft’ joints, large blocking masses and large impedance shifts, favour ‘weak coupling’ but also large mode counts [15]. Consequently, the saving in computation effort, produced by weak coupling, corresponds to those parts of the structure that is best handled with SEA.

Secondly, using the Greens function, the substructure vibration energies are expressed as linear functions of the input powers. Here, it may be assumed that there are uncertainties in some of the elements that are of the order required to reach the saturated state. Upon this basis, the energy relations are simplified using the results discussed above, and, the desired formulation of the conservation of energy is reached. That is, a set of equations for the conservation of vibrational energy where parts of the structure are described ‘exactly’ (with modal bases found from FE analysis) and where parts of the structure have a condensed statistical description (based on asymptotic results for high frequency vibration in large structures with uncertainties) are obtained.

5 Conclusions

This note draws attention to two FE techniques for studies of vibrational energy flow. The first uses special 2-D finite elements for devising the dispersion relations for waveguides. Thereby, the versatility of the FEM can be put to use for studies of vibrational wave propagation in complex structures that otherwise would be inaccessible. The second technique uses the standard FEM for prediction of vibration response. The novelty here lies in the application of the power balance relations (31), ensuring a symmetric formulation. Moreover, this formulation does not rely on the assumption of coupling power proportionality, giving it wider scope for applications, compared to standard SEA procedures.

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