Master of Science Thesis

Phenomenology of Hyperbolic Large Extra Dimensions for Hadron Colliders

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Abstract

The subject of extra dimensions has experienced a renewed interest in recent years. Arkani-Hamed, Dimopoulos, and Dvali (ADD) have pointed out that it is possible that there exist extra dimensions that are as large as micrometer sized, if the Standard Model fields are restricted to a four-dimensional brane. In the ADD model, only the graviton is able to probe the extra dimensions. The main motivation for this model is that it could solve the hierarchy problem between the electroweak scale and the Planck scale by lowering the Planck scale to 1 TeV. However, in the ADD model, the radius of the extra dimensions is large, giving a new hierarchy problem between this radius and the electroweak scale. In addition, there are significant constraints on the model coming from astrophysics.

To improve on the ADD model, it is possible to consider a model with curved extra dimensions. An interesting scenario is provided by extra dimensions of hyperbolic geometry. In this case, it is possible to bring the Planck scale down to 1 TeV without the need of a large radius. Also, the constraints that are important for the ADD model can be completely avoided.

The most efficient probe of large extra dimensions is particle physics. In particular, it is possible to study their phenomenology in high-energy particle accelerators such as the Large Hadron Collider, which will be completed in 2008. The phenomenology of the ADD model has been extensively studied. In this thesis, we consider the phenomenology of a model where the internal space is a hyperbolic disc. We obtain the Kaluza-Klein spectrum approximately and study the Kaluza-Klein modes. The results are cross sections for production of a graviton together with a photon or a hadronic jet, which are the most important reactions for LHC physics.

Key words: large extra dimensions, ADD model, hyperbolic extra dimensions, LHC.
Preface

The subject of extra dimensions has its origins in the beginning of the twentieth century. In recent years, the subject has experienced a renewed interest, due to the proposal that extra dimensions could be as large as sub-millimeter sized, if gravity is the only field that is allowed to propagate through them. In this case, it may be possible to probe these extra dimensions in the next generation of particle accelerators.

In principle, models of extra dimensions of this kind could be tested by searching for deviations from Newton’s inverse-square law of gravity. However, because of the weakness of gravity at small scales, it is hard to perform any such experiments. A more profitable alternative is to study the consequences of extra dimensions in elementary particle physics. This leads us to study high-energy physics at particle accelerators.

The main motivation behind models of large extra dimensions is that they provide an elegant solution to the hierarchy problem between the electroweak scale and the Planck scale. In these models, the weakness of gravity is described by the fact that it is diluted in the higher-dimensional spacetime. This means that the Planck scale is only an effective scale, which we observe at low energies. In truth, there is another fundamental mass scale for gravity, which could be as low as the electroweak scale. However, if the extra dimensions are assumed to be flat, then, in order to have such a small fundamental mass scale, the radius of these dimensions has to be very large. This leads to a new hierarchy problem between the radius and the electroweak scale. Hence, this solution to the hierarchy problem is not completely satisfactory. In addition, there are constraints on the model, the most important coming from astroparticle physics. For the cases of one and two extra dimensions, these constraints are significant.

Because of these problems, some modifications of the model have been proposed. Of particular interest is the possibility that the geometry of the extra dimensions could by hyperbolic. In this case, a small fundamental mass scale could be obtained with a small radius. Also, the constraints that are important for a flat internal space completely vanish in this case. However, in models with a hyperbolic internal space, it is not possible to perform all the calculations analytically. Instead, some approximations have to be made. To the knowledge of the author, no concrete calculations of phenomenological predictions have been performed for these models.
The aim of this thesis is to provide concrete results for a model incorporating hyperbolic geometry. More specifically, we calculate particle collision cross sections that are relevant for the Large Hadron Collider (LHC) at CERN, Geneva, Switzerland.

Overview of the thesis

The thesis is structured as follows. In Ch. 1, we give a general introduction to theoretical physics and particle physics. We mention some of the problems faced by particle physicists, and how these can be solved in models of extra dimensions.

In Ch. 2, we consider the mathematical formulation of the Standard Model of particle physics. Then, we introduce universal extra dimensions, as these provide the conceptually most simple example of extra dimensions. We describe the concept of Kaluza-Klein decomposition, which provides an alternative, four-dimensional picture of extra dimensions. In this picture, any fields defined in the higher-dimensional spacetime appear as infinite so-called Kaluza-Klein towers of states of different masses in four dimensions. We also describe how to relate the parameters of the higher-dimensional theory to the effective four-dimensional picture.

In Ch. 3, we introduce large extra dimensions and the ADD model. In this model, gravity is the only force that can probe the extra dimensions. Because of this, we briefly review general relativity and its formulation in the setting of extra dimensions. To make any predictions for the LHC, we also need to know how gravity couples to Standard Model particles. In order to obtain a description of these couplings, we consider an effective quantum theory of gravity. Then, we perform the Kaluza-Klein expansion of the graviton, and describe how the Kaluza-Klein modes couple to Standard Model particles.

In Ch. 4, we introduce a new model, based on an internal space of hyperbolic geometry. We describe this geometry and its consequences. Then, we consider the Kaluza-Klein decomposition of the graviton. In the hyperbolic model, it is not possible to obtain this analytically, which means that we have to resort to some approximations. The remaining part of the chapter is devoted to collecting information about the Kaluza-Klein spectrum and modes, and finally, to consider the couplings of these modes to the Standard Model fields.

Finally, in Ch. 5, we consider the phenomenology of large extra dimensions that is relevant for the LHC. We begin by describing some general collider phenomenology. Then, we consider the phenomenology of the ADD model, and briefly outline the calculations of cross section relevant for the LHC. Finally, we modify these results to the hyperbolic model, and present a number of different cross sections for this model.

Notation and conventions

Throughout this thesis, we employ the Einstein summation convention, meaning that we implicitly sum over repeated indices (one covariant and one contravariant),
unless otherwise stated.

Ordinary four-dimensional indices will be denoted by lower-case Greek letters, extra-dimensional indices will be denoted by lower-case Roman letters and indices of the full higher-dimensional spacetime will be denoted by upper-case Roman letters.

We will denote any field defined in higher-dimensional spacetime by a tilde, in order to distinguish it from its four-dimensional counterpart, i.e., the field $\tilde{\phi}$ is the generalization of the field $\phi$.

The sign convention for the Minkowski metric is

$$ (g_{\mu\nu}) = \text{diag}(1, -1, -1, -1). \quad (1) $$

We will employ natural units, setting $c = \hbar = 1$. However, we will not put $G = 1$, since in the higher-dimensional picture, this is not a fundamental constant, but only a low-energy approximation.

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Chapter 1

Introduction

Nature provides us with an abundance of complex phenomena, which have always fascinated mankind. The task of physics is to study different such phenomena in a systematic way, trying to find general principles and patterns, and putting these into the form of mathematical models. *A priori*, there is nothing that says that Nature can be described in such simple terms. Yet, this seems to be the case, at least approximately. Throughout the history of physics, physicists have managed to make many accurate predictions on natural phenomena, ranging from the behavior of fluids to the history of the Universe. One of the reasons for this success is that the subject of physics is restricted in the kinds of questions that it attempts to answer. The aim of physics is to *describe* Nature, not to *explain* it; at least not on a fundamental level. Questions concerning why Nature behaves the way it does cannot be answered by physics; they belong to the subject of philosophy.

Another reason is the combination between theory and experiment. For a long time, physics has made its progress through a successful interplay between theoretical considerations and experimental measurements. While theoretical models are suggested by observed patterns in Nature, they often turn out to take on a life of their own, and make new predictions which can be verified by experiments.

This thesis is concerned with the theoretical aspects of physics. The aim of this branch of the subject is to construct mathematical models and theories of Nature. The single most important criterion for any such model, regardless of which subfield of physics it belongs to, is that it agrees with experimental observations. Also, a good model has to make well-defined predictions that can actually be compared to the results of experiments. The model should be able to reproduce old results as well as to predict new ones.

It is important to understand that, in physics, it is not possible to rigorously prove that a theory is correct. At some point, one has to make some assumptions, or *postulates*, which are motivated only by the fact that they agree with empirical observations. These postulates are in no way “holy”—if they turn out to contradict empirical observations they have to be modified in order to resolve the situation.
Chapter 1. Introduction

This should make it clear that in physics there is no such thing as a final theory of any given phenomenon. It is simply meaningless to state that a theory is “correct”. No matter how many observations are made, we can never be sure that there is no single experiment that would contradict the theory. As the range of parameters that experiments can probe is expanded, completely new effects may appear. These have to be taken into account in any model that seeks to describe the phenomenon in this expanded region. This does not mean that the old model, which was successful in describing the phenomenon in the more restricted range, is wrong; it simply has a restricted range of applicability, but inside that range it is still a good model. In this sense, any model comes with a range of validity, though what that range may be is often far from obvious. Accordingly, any new theory seeking to extend an old one must be able reproduce the successful results of the old model.

One of the most important and long-standing goals of physics is to obtain a description of matter on its most fundamental level. For a long time, it was believed that the atom was the fundamental building block of Nature. This view was changed at the end of the nineteenth century, beginning with the discovery of the electron. It was established that the atom consists of a small nucleus of protons and neutrons, surrounded by a cloud of electrons. Today, the electron is still considered to be an elementary particle, while the nucleons, i.e., the protons and the neutrons, consist of even smaller building blocks, which are the quarks.

The currently accepted description of matter on its most fundamental level is the Standard Model (SM) of particle physics. The SM is one of the most successful theories ever in the history of physics. Richard Feynman has pointed out that results from quantum electrodynamics (QED), which is a part of the SM, agree with experimental data to a precision that is equivalent to measuring the distance between Los Angeles and New York to a precision of the width of a human hair. The SM describes all the known particles and interactions of Nature except for gravity. It was developed in the 1970s and has successfully passed a number of important tests since then. Many of the particles found in Nature were theoretically predicted by the SM before they were experimentally discovered. The last of the matter particles of the SM to be found was the tau neutrino, in the year 2000. Today, the only missing piece of experimental evidence is that of the existence of the so-called Higgs boson, which is predicted by the SM to be responsible for the generation of the masses of all massive particles. The search for the Higgs boson is one of the main motivations for the presently ongoing construction of the Large Hadron Collider (LHC) at CERN. It is expected that, with the improved energy reach of the LHC compared to existing colliders, it will be possible to detect the Higgs boson.

All forces of Nature can be described in terms of four fundamental interactions: the electromagnetic, the weak, the strong, and the gravitational. The first three of these are included in the SM. One of the most interesting predictions of the SM is that, at high energies, the electromagnetic and weak forces are unified in a single entity, known as the electroweak force. However, at energies below the electroweak scale $M_{EW} \approx 246$ GeV, this electroweak symmetry is broken, due to the Higgs
mechanism. It is likewise expected that the electroweak and strong forces will be unified at an energy scale somewhere around the order of $10^{15}$ GeV, in a Grand Unified Theory (GUT). Taking this idea even further, an ultimate theory of Nature should unify all the known forces, including gravity, in a Theory of Everything (TOE). However, since gravity is not even included in the SM, it is less clear how such a theory could be constructed.

As successful as the SM has proven to be, it is equally well-known that it suffers from a number of shortcomings. Most obviously, the SM does not include gravity. Naively, it may seem like a straightforward task to incorporate gravity into the SM, but it turns out that it is not. The successful way of treating the other fundamental forces does not work for gravity. In technical terms, the result of treating gravity in the same way is that the theory does not become renormalizable. This means that calculations give rise to infinities which cannot be avoided.

The reason that we can use the SM although it does not include gravity is that, at the energy scales that experiments can probe today, the gravitational force is tremendously weak compared to the other fundamental forces. The reason that gravity is important on macroscopic scales is that the strong and weak forces have very short ranges, and the electromagnetic force loses much of its importance because of the average charge neutrality of most macroscopic objects. Whenever the other fundamental forces become important, gravity is completely negligible, except at very high energies. This complicates the construction of a quantum theory of gravity—we cannot make any experiments where both gravity and the SM interactions are relevant, and hence, it is not possible to test such a theory.

The fact that gravity is such a weak force compared to the other fundamental forces actually presents a grave problem in itself. The electroweak scale, at which the electromagnetic and weak forces are unified, is of the order of 100 GeV, and this value is essentially given by the mass of the Higgs boson. The relevant mass scale for gravity, on the other hand, is the so-called Planck mass, which is of the order of $10^{19}$ GeV. Thus, the mass of the Higgs boson differs from the Planck mass by about 17 orders of magnitude. In classical physics, this would not be a problem. However, in a quantum theory, the bare mass of the Higgs boson receives additional contributions, which are theoretically predicted to be of the order of the largest energy scale of the theory, i.e., the Planck mass. The physical mass of the Higgs boson, which is what could be observed, is the sum of the bare mass and the quantum corrections. For this physical mass to be smaller than the Planck mass by 17 orders of magnitude, the bare mass and the quantum corrections must be equal in magnitude to a precision of 17 digits. Assuming that these two quantities are not related, this means that such a value for the Higgs mass is highly improbable. To obtain such a value, the bare mass would have to be fine-tuned to a very specific value. A physical theory suffering from such a fine-tuning problem is often referred to as being unnatural. This particular fine-tuning problem is known as the hierarchy problem, and it is one of the greatest mysteries of modern physics. In this thesis, we will consider a solution to the hierarchy problem in the setting of extra dimensions.
Another problem of the SM is related to cosmology. The observed baryonic matter content in the Universe only corresponds to about 4% of that expected from cosmological observations [1]. The most popular solution to this problem is to assume that there exists some unknown contribution to the matter content in the Universe. Since we have not seen this matter, we conclude that it does not interact electromagnetically, and hence, it has been termed Dark Matter (DM). Unfortunately, there is no particle in the SM that could serve as a viable DM candidate. This means that, unless the idea of DM is wrong, something is missing in the SM. For more information on DM, see e.g. Ref. [2].

A third problem is related to the phenomenon of neutrino oscillations, which is now a well-established empirical fact [3–5]. The problem is that for neutrino oscillations to be possible neutrinos have to be massive, which they are not in the SM. One could treat neutrino masses in the same way as any other particle masses, using the Higgs mechanism. However, this leads to the problem of describing why the neutrinos are so much lighter than any of the other massive particles in the SM.

The SM contains 19 parameters, whose values are not predicted by the theory, but have to be obtained from experimental measurements. Theoretically, it is desirable to reduce this number of free parameters. There are also questions like why there are exactly three generations of quarks. It is hoped that a more fundamental theory will provide answers to these questions.

At present, there is an abundance of different possible models seeking to solve one or more of the problems of the SM. These are often collectively referred to as beyond-the-SM models. Among the most important of these are different models of so-called supersymmetry\footnote{Extending the SM to the minimal supersymmetric SM (MSSM) has the disadvantage that we would have more than a hundred free parameters instead of 19.} [6] and models including additional spatial dimensions. The main problem of particle physics today is that there is a need of new experimental data, which could help to settle the question of whether any of these models accurately describe Nature.

To experimentally study physics on small distance scales, very high energy densities are needed. One of the most important methods used to obtain such high-energetic states is to accelerate particles to large velocities, collide them, and observe what comes out of the reaction. During the twentieth and twenty-first centuries, several particle colliders have been constructed for this purpose. The most important ones include the Large Electron Positron collider (LEP) at CERN, which served to confirm many of the predictions of the SM, and the Tevatron at Fermilab, Batavia, Illinois, USA, where the discovery of the top quark was announced in 1995. In the year 2000, the LEP was taken out of service in order to build a new collider, the LHC. When completed, the LHC will be the most powerful particle accelerator ever built. At present, it is scheduled to become operative in 2008. This means that particle physicists may finally obtain the new experimental input needed to probe physics beyond the electroweak scale and put the many existing theoretical models to test.
In the meantime, it is an important task of particle physicists to work out the experimental consequences of the different models, so that their predictions can be compared to LHC experimental data. This field relating theory to experimental predictions is known as phenomenology. In this thesis, we work out the phenomenology of a model incorporating two extra spatial dimensions. In particular, we focus our attention on predictions that are relevant for the LHC.

Many of the problems faced by particle physicists today have solutions in the context of models that include additional spatial dimensions. Historically, such models were first considered in the beginning of the twentieth century by Theodor Kaluza [7] and Oskar Klein [8]. The theory that they developed is now known as Kaluza-Klein (KK) theory. The motivation for this theory was to obtain a unified description of gravity and electromagnetism, which at that time were the only known fundamental forces. The way to achieve this in KK theory is to extend spacetime with an additional spatial dimension and postulate that the equations of general relativity hold in this extended spacetime. This can be shown to be equivalent to general relativity plus electromagnetism in four-dimensional spacetime.

A common question that people ask when first hearing about the idea of extra dimensions is: could they exist? The answer is that they could, but empirical measurements put significant constraints on their properties. In particular, any additional dimensions have to be different from the ordinary three dimensions by being very small. Klein suggested that this could be achieved if the extra dimensions are closed, for example like a circle. By comparing theoretical predictions of models with extra dimensions to experimental data, which have so far revealed no signs of extra dimensions, their size can be constrained. The reason that they have not been observed is that the energy that is available at today’s experiments is too small to be able to resolve them. From a phenomenological point of view, this presents a problem, since it means that we are unable to perform any experiments on these dimensions, unless the experimental energy level available is increased considerably. But as we shall see, more recent developments indicate that there are possible scenarios in which the required energy scale is small enough that it will be available in the near future.

In this thesis, we will not go deeper into the original KK theory. Instead, we will investigate some related models. At the end of the twentieth century, it was pointed out that there is a possibility that there exists extra dimensions, but that not all particles are allowed to propagate through them. Instead, some particles are restricted to move on one or more lower-dimensional so-called branes. In particular, it has been proposed that the SM particles are confined to these branes, while gravity is free to propagate throughout the full higher-dimensional spacetime. This leads to a very elegant solution of the hierarchy problem. In this model, the reason that gravity is so weak compared to the other fundamental forces is that it is diluted in the extra dimensions, while the other forces are confined to the four-dimensional brane. Another interesting consequence of this possibility is that the extra dimensions could be quite large, up to the order of micrometers.
For this reason, this idea is usually referred to as models of large extra dimensions. Such models will be considered in some detail in Ch. 3.

The possibility of having large extra dimensions also means that the Planck scale, $M_{\text{Pl}}$, the energy scale at which gravity becomes strong, may be much lower than expected. In the most extreme case, it could be small enough to become available at the LHC. In this case, the LHC would be able to probe the realm of quantum gravity, a most exciting prospect.

Models including extra dimensions could also provide solutions to other problems of the SM. As examples, we mention the problem of DM [9,10] and the problem of the smallness of neutrino masses [11,12]. However, in this thesis, we will not pay much attention to these problems.

If there exists additional spatial dimensions beyond the three that we know of, this would of course not only affect the field of particle physics, but would fundamentally alter the way we view Nature. The special role played by particle physics is that it requires high energies to be studied, and hence, it is the area of physics where extra dimensions would have the greatest impact. Turned around, it means that particle physics is the area that is most well-suited for studying the possibility of extra dimensions. This will be further clarified in this thesis, since we will consider the implications of extra dimensions for high-energy collider phenomenology.
Chapter 2

Physics in extra dimensions

To give an introduction to extra dimensions and to some methods used to treat them, the simplest setting is that of universal extra dimensions (UEDs). In these models, there is no brane, and all particles in Nature are free to propagate through the full higher-dimensional spacetime.

We begin this chapter by giving a more technical introduction to the SM. In particular, we will describe how the SM is constructed as a so-called gauge theory. Then, we introduce UEDs, in the simplest possible setting of a single extra dimension with the topology of a circle. We describe how to rewrite the higher-dimensional action on a form that gives an equivalent four-dimensional picture of the higher-dimensional spacetime. This is performed with a so-called KK decomposition of the fields. Finally, we mention how to relate the parameters of the higher-dimensional and four-dimensional pictures, and consider some important consequences of these relations.

2.1 The Standard Model of particle physics

The SM of particle physics is the currently accepted description of physics on its most fundamental level, in terms of elementary particles.

According to the SM, the particle content in Nature can be divided into three categories. These are the matter particles, which are spin 1/2 fermions, the force carriers, which are spin 1 bosons, and the Higgs boson, which is a scalar. The Higgs boson is responsible for giving the particles their masses. It has not yet been experimentally detected.

The matter particles can be further divided into two groups. They are the quarks

\[
\begin{pmatrix}
  u \\
  d
\end{pmatrix}
\begin{pmatrix}
  c \\
  s
\end{pmatrix}
\begin{pmatrix}
  t \\
  b
\end{pmatrix},
\]  

(2.1)
and the leptons

\[
\begin{pmatrix}
  e \\
  \nu_e \\
  \mu \\
  \nu_\mu \\
  \tau \\
  \nu_\tau
\end{pmatrix}.
\] (2.2)

All of these particles also have corresponding antiparticles, except possibly the neutrinos, which could be their own antiparticles. The two groups are divided into three generations, as indicated in Eqs. (2.1) and (2.2). In the lepton sector, each generation consists of a charged lepton (the electron, the muon, and the tau) and their corresponding neutrinos.

The force carriers mediate the interactions among the elementary particles. The SM includes three fundamental interactions—the electromagnetic, the strong, and the weak force. Hence, there are three kinds of force carriers; the photon for electromagnetism, the gluons (of which there are eight) for the strong force, and the intermediate vector bosons \(W^+, W^-, Z^0\) for the weak force. The photon and the gluons are massless, while the intermediate vector bosons, together with the massive matter particles, acquire masses through the Higgs mechanism. Gravity is supposed to be mediated by a massless spin-2 boson, known as the graviton.

The mathematical language that describes the SM is quantum field theory. In this language, all particles correspond to excitations of different fields, of which there is one for every species of particles. Modern field theories are described in terms of the Lagrangian formalism. The central quantity in such a theory is the action, \(S\) which can be written as an integral over the Lagrangian density (or simply Lagrangian, for short), \(\mathcal{L}\)

\[S = \int d^4x \mathcal{L}.\] (2.3)

The Lagrangian is generally a function of the fields, their derivatives, and the spacetime coordinates. The classical equations of motion for \(\mathcal{L}\) are obtained from the principle of stationary action, \(dS = 0\), leading to the Euler-Lagrange equations

\[\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.\] (2.4)

In a quantized theory, the fields are replaced by operators, and this leads to the interpretation of their excitations as particles. For more information on this, see e.g. Ref. [13].

The SM is a gauge theory based on the symmetry group \(SU(3) \otimes SU(2) \otimes U(1)\). It is constructed in the following way. The starting point is the Dirac Lagrangian for a massless spinor field \(\psi\),

\[\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi.\] (2.5)

This Lagrangian is invariant under the gauge transformations

\[\psi \rightarrow V \psi,\]
\[\bar{\psi} \rightarrow V^\dagger \bar{\psi},\] (2.6)

where \(V = \exp (i \alpha^a t^a)\). The quantities \(t^a\) are the generators of the symmetry group. The parameters \(\alpha^a\) are independent of the spacetime coordinates, i.e., the gauge
transformation is global. Now, the Lagrangian is also required to be invariant under local gauge transformations, with the $\alpha^a$ depending on the spacetime coordinates. In order to achieve this, a set of so-called gauge fields, $A^a_\mu$, has to be introduced, having the transformation property

$$A^a_\mu \rightarrow V(x) \left( A^a_\mu + \frac{i}{g} \partial_\mu \right) V^\dagger(x).$$

Using these gauge fields, the Lagrangian is made invariant under local gauge transformations by replacing the derivative $\partial_\mu$ by the covariant derivative, $D_\mu \equiv \partial_\mu - igA^a_\mu t^a$. The number of gauge fields is equal to the number of generators.

Through the term $g\bar{\psi}A^a_\mu t^a \psi$ in the Lagrangian, the gauge fields couple to the matter field $\psi$. In fact, the gauge fields are exactly the force mediating particles, and $g$ is the corresponding coupling constant. It is also possible to construct an invariant term in the gauge fields, which is of the form

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},$$

where the field strength tensor $F^a_{\mu\nu}$ is defined as

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu.$$  

The quantities $f^{abc}$ are the structure constants of the symmetry group, which are defined by the commutation relations $[t^a, t^b] = f^{abc} t^c$.

The first example of a gauge theory of this kind was quantum electrodynamics (QED), which is an Abelian gauge theory. This means that the generators of the symmetry group, which in this case is $U(1)$, commute. In fact, $U(1)$ has only a single generator, and is thus trivially Abelian. In QED, $A_\mu$ is the photon field, $\psi$ is the electron field, and $\bar{\psi}$ is the positron field. The coupling constant is the elementary electric charge, $e$.

Inspired by QED, Yang and Mills generalized these ideas to non-Abelian gauge theories, which are now known as Yang-Mills theories [14]. The most important example of such a non-Abelian gauge theory in the context of particle physics is quantum chromodynamics (QCD), describing color charged particles, i.e., the quarks and the gluons. The gauge group for QCD is $SU(3)$.

For an Abelian gauge theory, the last term in Eq. (2.10) is zero. In this case, the kinetic term (2.9) includes only terms of second order in the gauge fields, which means that the gauge fields do not interact among themselves. A non-Abelian gauge theory, on the other hand, includes terms of third and fourth order in the kinetic term, which means that the gauge fields are self-interacting. This is the case for QCD, where the gluon fields interact among themselves.

So far, all the fields that have been considered have been massless. Including an explicit mass term of the form $\bar{\psi} m \psi$ in the Lagrangian breaks the gauge symmetry, which cannot be allowed. In the SM, the masses of the particles are dynamically generated through the Higgs mechanism. However, this will not be important in this thesis, and hence we will not describe it any further.
2.2 Universal extra dimensions

One of the possible ways to extend physics beyond the SM is to introduce one or more additional spatial dimensions. Depending on the exact model considered, extra dimensions could provide mechanisms describing possible solutions to one or more of the problems of the SM. Extra dimensions arise naturally in string theory, where ten spacetime dimensions are needed in order for the theory to be free of anomalies.

The simplest example of extra dimensions is the class of models known as universal extra dimensions (UEDs). In these models, it is postulated that there exists one or more additional spatial dimensions, perpendicular to the ordinary three dimensions. The extra-dimensional space is called the internal space. Particle physics experiments reaching energy scales of the order of 100 GeV have revealed no trace of extra dimensions. This means that any model including extra dimensions must reduce to conventional four-dimensional spacetime in the low-energy limit. The simplest way to accomplish this is by assuming that the extra dimensions are compact and very small. The topology of the internal space could, for example, be that of a circle with a small radius. If the extra dimensions are small, high energy is needed in order to resolve them, which explains why they have not been observed. By using experimental data, the size of the extra dimensions can be constrained.

2.3 Kaluza-Klein decomposition

Fundamental to any model of extra dimensions is the idea of KK decomposition of a field. This idea provides an equivalent four-dimensional view of the extra dimensions. As an example, we consider a complex scalar field \( \tilde{\phi} \) of mass \( m \) in a model with one additional spatial dimension. The internal space is assumed to have the topology of a circle with radius \( R \). It is assumed that the action describing the field is the natural generalization of the corresponding four-dimensional field

\[
\tilde{S} = \int d^4x \int_{0}^{2\pi R} dy \left( \partial_M \tilde{\phi}^* \partial^M \tilde{\phi} - m^2 \tilde{\phi}^* \tilde{\phi} \right).
\]

Any single-valued field defined on a circle has to be periodic with period \( 2\pi R \). This means that it can be expanded in a Fourier series

\[
\tilde{\phi}(x^\mu, y) = \frac{1}{(2\pi R)^{1/2}} \sum_{n=-\infty}^{\infty} \phi^{(n)}(x^\mu) e^{i n y R}.
\]

Inserting this expansion into Eq. (2.11) and using the orthonormality of the basis functions to integrate out the fifth dimension, we obtain

\[
\tilde{S} = \sum_{n=-\infty}^{\infty} \int d^4x \left[ \partial_\mu \phi^{(n)*} \partial^\mu \phi^{(n)} - \left( m^2 + \frac{n^2}{R^2} \right) \phi^{(n)*} \phi^{(n)} \right].
\]
This is an action for an infinite number of four-dimensional complex scalar fields $\phi^{(n)}$ with masses $\sqrt{m^2 + n^2/R^2}$. Note that there is no reference to the fifth dimension left in Eq. (2.13). What has been done is to reformulate the problem in an equivalent four-dimensional form. In this picture, the momentum of the field along the fifth dimension is interpreted as a mass. This could also be observed by considering the energy-momentum relation $E^2 = p^2 + m^2 = p_4^2 + p_5^2 + m^2$, where $p_4$ is the projection of the momentum onto the familiar four dimensions and $p_5$ is its projection onto the fifth dimension. Since the fifth dimension is compact, $p_5$ is quantized and restricted to a set of discrete values. The five-dimensional field is represented by a countably infinite number of four-dimensional fields. This set of fields is known as a KK tower. The individual fields are known as the KK modes of the higher-dimensional field $\tilde{\phi}$ and the set of masses is known as the KK spectrum. Note that the $n = 0$ mode has the same mass as the conventional four-dimensional field with mass $m$. In the higher-dimensional picture, this corresponds to a field with no momentum in the fifth dimension. From the expansion (2.12), it is also clear that this zero mode is independent of the extra-dimensional coordinate. This mode is interpreted as the conventional four-dimensional field $\phi$. This means that the theory effectively reduces to a four-dimensional one for energies smaller than the mass of the first excited KK mode.

The above derivation was performed for a complex scalar field in one additional dimension, but similar results hold for any field in any number of dimensions. For more examples, see e.g. Ref. [15].

### 2.4 Higher-dimensional coupling constants

Assuming that four-dimensional spacetime is a low-energy approximation of some higher-dimensional space implies that the coupling constants that are observed at low energies are only effective ones, which are replaced by some more fundamental constants at higher energies. It is straightforward to obtain a relation between these constants. Consider as an example a self-interacting scalar field in the model that was described in the previous section. The interaction term is given by

$$\tilde{S} = \int d^4x \int_0^{2\pi R} dy \tilde{\lambda} \phi^4. \tag{2.14}$$

For low energies, the only available mode of the KK tower is the zero mode $\phi^{(0)}$. Inserting the KK expansion (2.12) into Eq. (2.14), neglecting the modes with $n > 0$ and integrating out the extra dimension, we obtain

$$\tilde{S} = 2\pi R \int d^4x \tilde{\lambda} \left( \frac{\phi^{(0)}}{(2\pi R)^{1/2}} \right)^4 = \frac{1}{2\pi R} \int d^4x \tilde{\lambda} \left( \phi^{(0)} \right)^4. \tag{2.15}$$

This is to be compared with the conventional four-dimensional action

$$S = \int d^4x \frac{\lambda \phi^4}{4!}. \tag{2.16}$$
Since the four-dimensional field $\phi$ is to be identified with the zero mode $\phi^{(0)}$ of the KK expansion, we obtain the relation
\[ \tilde{\lambda} = 2\pi R \lambda. \] (2.17)

One of the most important results of this equation is that, since the four-dimensional coupling constant $\lambda$ is dimensionless, the fundamental coupling constant $\tilde{\lambda}$ has negative mass dimension. This means that the higher-dimensional model is not renormalizable (see e.g. Ref. [13], Ch. 10). For high energies, the model breaks down and gives divergent results, which cannot be removed by any renormalization process. Because of this, the model cannot be regarded as a fundamental theory of Nature, but only as another low-energy approximation, which has to be replaced by a more fundamental theory at higher energies.

This does not mean that the model is useless. What we are dealing with is an effective theory. This concept is related to models which are not necessarily valid for any energy scales, but describe different phenomena only in restricted ranges. An effective theory is a low-energy approximation of a more fundamental theory, called the UV-completion of the effective theory. This UV-completion may or may not be known. For an effective theory, a cutoff scale $\Lambda$ is introduced, being the highest energy for which the theory is valid.

A classical example of an effective theory in particle physics is Fermi’s theory of weak interactions [16, 17], which does not include the intermediate vector bosons. The cutoff scale of this theory is the mass scale of these gauge bosons, which is of the order of 100 GeV. For higher energies, Fermi’s effective theory has to be replaced by its UV-completion, which is the GWS model of electroweak interactions [18–20].

Since experiments are not yet even on energy levels large enough to probe extra dimensions, it is hard to speculate on what the UV-completion of a theory of extra dimensions could be. Nevertheless, some attempts have been made. One possibility is that the UV-completion is provided by string theory. Another example is the idea known as dimensional deconstruction. In such models, spacetime is four-dimensional for high energies. As the energy is lowered, the extra dimensions are dynamically generated through a phase transition similar to the Higgs mechanism of the SM. For more information on this subject, see e.g. Refs. [21, 22].
Chapter 3

Large extra dimensions—the ADD model

One of the more popular models of extra dimensions is the ADD model [23], named after its originators Arkani-Hamed, Dimopoulos, and Dvali. This model differs from UEDs by assuming that there exists a four-dimensional brane in the higher-dimensional spacetime. The fields of the SM are confined to this brane, while gravity is free to propagate through the full spacetime. The brane is identified with the familiar four-dimensional spacetime. This approach to extra dimensions provides a solution to the hierarchy problem. However, this solution is not as satisfactory as it first appears to be, due to some problems which will be discussed in this chapter.

We also mention that there are other models of extra dimensions that include branes. One of the most important is the so-called Randall-Sundrum (RS) model [24, 25]. In this model, two branes are introduced, and the SM fields are confined to one of them. However, this model will not be considered further in this thesis.

In this chapter, we study the ADD model. We do this for two reasons. First, this model provides an introduction to the subject of large extra dimensions in a relatively simple context. Second, the hyperbolic disc model that we will consider later is in many respects a straightforward generalization of the ADD model. For this reason, most calculations in this chapter are provided in some detail. For the hyperbolic disc model, the calculations are completely analogous. This means that, when dealing with that model, we will be able to rely on the general principles developed in this chapter.

We start by discussing the geometry of the ADD model. Then, we continue to study the dynamics of the higher-dimensional graviton in this model. In particular, we investigate the KK modes of the graviton and their couplings to the SM fields. Finally, we discuss some drawbacks of the ADD model.
3.1 The geometry of the ADD model

In the introduction to extra dimensions in the previous chapter, it was assumed that the extra dimensions were like the familiar four dimensions in every respect except for being compact. In particular, it was assumed that any field could propagate through the full spacetime. However, there is also an interesting possibility that some of the fields might be restricted to a lower-dimensional manifold. This idea was introduced by Arkani-Hamed, Dimopoulos, and Dvali in 1998 and the resulting model is known as the ADD model [23]. It turns out that this idea gives rise to some interesting new consequences.

In the ADD model, it is postulated that the SM fields are confined to a four-dimensional manifold, known as a brane (in analogy with a two-dimensional membrane). This brane is to be identified with the familiar four-dimensional spacetime. The only field having dynamical degrees of freedom in the extra dimensions is the graviton, and possibly also exotic fields carrying no SM gauge charges. Indeed, since the metric tensor describes the structure of spacetime, it is hard to imagine a model where it would not exist in all of spacetime. Exotic fields will not be considered in this thesis.

The brane and the mechanism that confines the SM fields to it could have a number of different origins. One possibility is demonstrated in the original article by Arkani-Hamed, Dimopoulos, and Dvali [23]. There are also other possible mechanisms, e.g., models inspired by string theory [26].

Like UEDs, the ADD model is an effective model, which is only valid up to some cutoff scale. This cutoff scale will be discussed later. This means that, for energies lower than this cutoff scale, we do not need to consider the nature of the brane or the confining mechanism. In a more fundamental theory, the brane should also be treated as a dynamical field, with a finite thickness in the extra dimensions. However, as long as this thickness is smaller than the inverse of the cutoff scale of the effective model, it can be neglected. In this thesis, the brane will be considered to be completely static, with no dynamical degrees of freedom. A model including a dynamical brane has been treated in e.g., Ref. [27].

The topology of spacetime in the ADD model is a direct product of ordinary four-dimensional spacetime and the internal space, i.e., of the form $M^4 \otimes T^d$. Four-dimensional spacetime is assumed to be described by the Minkowski metric of special relativity. In the ADD model, the extra dimensions are assumed to be flat and compactified on a torus. For simplicity, it will be assumed that all the extra dimensions have the same physical radius $R$. The number of extra dimensions will be denoted by $d$ and the total number of dimensions by $N = 4 + d$. The metric tensor for this $N$-dimensional spacetime is a generalization of the four-dimensional Minkowski metric

$$ (\tilde{g}_{MN}) \equiv (\tilde{\eta}_{MN}) = \text{diag}(1, -1, -1, -1, \ldots, -1). \quad (3.1) $$
No attempt will be made to motivate the existence of such a geometry. The ADD model is viewed as a relatively simple and illustrative example.

3.2 General relativity

Since the graviton is the only field that can probe the extra dimensions in the ADD model, it is important to study how it behaves in a higher-dimensional spacetime. For this reason, we give a brief review of general relativity, which will later be generalized to the setting of the ADD model.

For practical purposes, the most important result of general relativity is Einstein’s field equations

\[ G_{\mu\nu} - \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \] \hspace{1cm} (3.2)

Here, \( \Lambda \) is a cosmological constant, \( R_{\mu\nu} \) is the Ricci tensor, \( R \) is the Ricci scalar, \( g_{\mu\nu} \) is the metric tensor, \( G \) is Newton’s gravitational constant, and \( T_{\mu\nu} \) is the energy-momentum tensor of the system under consideration. For the definition of these and other related quantities, see e.g. Ref. [28].

We will make much use of the fact that Einstein’s equations can be obtained by applying the principle of stationary action, \( d\mathcal{S} = 0 \), to the Einstein-Hilbert action \[ S = \bar{M}_{\text{Pl}}^2 \int \sqrt{|g|} d^4x \left( R - 2\Lambda \right) + \int \sqrt{|g|} d^4x \mathcal{L}_{\text{matter}}, \] \hspace{1cm} (3.3)

which is to be considered as a function of the metric tensor \( g_{\mu\nu} \) and its first order derivatives \( \partial_{\lambda} g_{\mu\nu} \). Here, \( \mathcal{L}_{\text{matter}} \) is the Lagrangian describing the matter and energy content of the system and \( \bar{M}_{\text{Pl}} \) is the reduced Planck mass.

The Planck mass, \( M_{\text{Pl}} \), is defined as \( M_{\text{Pl}} \equiv \frac{1}{\sqrt{G}} \approx 1.2 \cdot 10^{19} \text{ GeV} \). For energies larger than the Planck mass, gravity is expected to become as strong as the SM interactions. In this region, gravity can no longer be treated classically, but has to be replaced by some quantized theory. The reduced Planck mass is introduced for convenience, in order to simplify some equations, and is defined as \( \bar{M}_{\text{Pl}} \equiv M_{\text{Pl}} / \sqrt{8\pi} \approx 2.4 \cdot 10^{18} \text{ GeV} \).

In general relativity, the definition of the energy-momentum tensor in terms of the matter Lagrangian is given by

\[ T^{\mu\nu} = \frac{\delta \left( \sqrt{|g|} \mathcal{L}_{\text{matter}} \right)}{\delta g_{\mu\nu}}. \] \hspace{1cm} (3.4)

From now on, rather than Einstein’s field equations (3.2), the principle of stationary action, applied to the action (3.3), is taken to be the fundamental principle of general relativity. This point of view makes the generalization to higher dimensions straightforward.
3.2.1 Gravitational waves and the graviton

On a fundamental level, gravity, just like the SM interactions, is expected to be mediated by a gauge boson, which is known as the graviton. The graviton is a spin-2 field, in contrast to the SM gauge fields which have spin 1.

The graviton is treated by considering small perturbations of the metric. This is performed by making a first-order expansion in $\frac{1}{M_{Pl}}$ about some background metric $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{h_{\mu\nu}}{M_{Pl}}. \quad (3.5)$$

The field $h_{\mu\nu}$ is identified as the graviton, and it is assumed that $|h_{\mu\nu}|/M_{Pl} \ll |\bar{g}_{\mu\nu}|$.

The background metric is most commonly chosen to be the Minkowski metric, but other choices are also possible [29]. In this chapter, we assume it to be the Minkowski metric.

The reason for the factor $\frac{1}{M_{Pl}}$ is that the graviton in four spacetime dimensions has mass dimension one. This can be observed by expressing the Einstein-Hilbert Lagrangian in terms of $h_{\mu\nu}$, keeping only terms up to second order in $\frac{1}{M_{Pl}}$. This gives rise to a kinetic term $\partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu}$. If the graviton has mass dimension one, then this term has mass dimension four, which makes the action dimensionless, as required. The relevant mass scale for gravity is the reduced Planck scale, $\bar{M}_{Pl}$, and it is this quantity that has to be used to normalize the field $h_{\mu\nu}$.

In order to obtain the equations of motion for the graviton, we expand the vacuum field equations to first order in $\frac{1}{M_{Pl}}$. In this way, the field equations are linearized. To this order, the indices of $h_{\mu\nu}$ can be raised and lowered using the background Minkowski metric, since $h^\mu_\nu = \eta^\mu_\lambda h_\lambda^\nu + \mathcal{O}(\frac{1}{M_{Pl}^2})$.

Since $\partial_\lambda \eta_{\mu\nu} = 0$, the Christoffel symbols are

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} M_{Pl}^{-1} (\partial_\lambda h^\mu_\nu + \partial_\nu h^\lambda_\mu - \partial_\mu h^\lambda_{\nu}) \quad (3.6)$$

The Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2} M_{Pl}^{-1} \left( \partial_\lambda \partial_\mu h^\lambda_\nu + \partial_\nu \partial_\mu h^\lambda_\mu - \partial_\lambda \partial_\nu h^\lambda_\mu + \partial_\mu \partial_\nu h \right), \quad (3.7)$$

where we have used the notation $h \equiv h^\mu_\nu$. By contracting the indices of the Ricci tensor, we obtain the Ricci scalar

$$R = M_{Pl}^{-1} \left( \partial_\lambda \partial_\sigma h^{\lambda\sigma} - \partial_\lambda \partial^\lambda h \right). \quad (3.8)$$

Putting these expressions together, the Einstein tensor is

$$G_{\mu\nu} = \frac{1}{2} M_{Pl}^{-1} \left( \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\lambda \partial_\sigma h^{\lambda\sigma} - \partial_\lambda \partial^\lambda h_{\mu\nu} - \partial_\mu \partial_\nu h - \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \partial^\lambda h \right). \quad (3.9)$$

The vacuum equations of motion are $G_{\mu\nu} = 0$. 

3.3. Quantum theory of gravitation

In order to obtain physical results, we have to choose a specific gauge for the graviton. One possible choice is the harmonic gauge \[29\], giving the condition

\[
\partial_{\mu} h^{\mu\nu} = \frac{1}{2} \partial^\nu h. \tag{3.10}
\]

Using this condition, the Einstein tensor is simplified as follows:

\[
G_{\mu\nu} = \frac{1}{2} M_{Pl}^{-1} \left[ \partial_{\mu} \left( \partial_{\lambda} h_{\nu}^{\lambda} - \frac{1}{2} \partial_{\nu} h \right) + \partial_{\nu} \left( \partial_{\lambda} h_{\mu}^{\lambda} - \frac{1}{2} \partial_{\mu} h \right) - \partial_{\lambda} \partial^{\lambda} h + \eta_{\mu \nu} \partial_{\lambda} \left( \partial_{\sigma} h^{\lambda \sigma} - \frac{1}{2} \partial^{\lambda} h \right) + \frac{1}{2} \partial^{\lambda} h \right].
\]

\[= -\frac{1}{2} \partial^{\lambda} \partial^{\lambda} \left( h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h \right). \tag{3.11}\]

Now, the equations of motion are

\[
\partial_{\lambda} \partial^{\lambda} \left( h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h \right) = 0. \tag{3.12}
\]

Taking the trace of this equation, we obtain

\[
0 = \partial_{\lambda} \partial^{\lambda} (h - 2h) = -\partial_{\lambda} \partial^{\lambda} h, \tag{3.13}
\]

which results in the final form of the equations of motion

\[
\Box h_{\mu \nu} = 0, \tag{3.14}
\]

where \(\Box \equiv \partial_{\mu} \partial^{\mu}\) is the d’Alembert operator. This is the three-dimensional wave equation for a particle traveling with the speed of light. Thus, the linearized field equations have wave solutions.

3.3 Quantum theory of gravitation

We have now considered some basic aspects of general relativity. In chapter 5, we will consider the experimental signatures of extra dimensions in particle colliders. This means that we also need to know how the graviton couples to the SM fields. For this purpose, we need a quantum theory of gravity.

Like the SM, general relativity can be seen as a gauge theory. For the SM, the gauge transformations act on internal spaces and have no relation to spacetime transformations. For general relativity, on the other hand, the gauge transformations are general coordinate transformations, which act on spacetime. The Einstein-Hilbert Lagrangian (3.3) is invariant under these transformations. Naively, it seems straightforward to quantize general relativity in the same way as the SM is quantized. However, this path leads to serious problems. The reason is that
the resulting theory is not renormalizable. However, we can still use the theory, provided that we consider it to be an effective theory.

We now describe the coupling of gravity to QED, starting with the ordinary QED Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (3.15)$$

The action for a field theory in flat space can be written as

$$S = \int d^4x \, \mathcal{L}. \quad (3.16)$$

For a curved spacetime, the volume element $d^4x$ is changed to $\sqrt{|g|} d^4x$, where $g$ is the determinant of the metric tensor. Hence, the Lagrangian is multiplied by the factor $\sqrt{|g|}$.

In order for the theory to be invariant under the general coordinate transformations of general relativity, the gauge field of gravity, i.e., the graviton, has to be added to the covariant derivative. This presents a subtle problem, related to the fact that there exist no spinor representations of the general coordinate transformations. The solution to this problem is to make a change of reference frame in order to relate the general coordinate transformations to Lorentz transformations, which do have spinor representations. The transformation matrices are the vierbein fields $e^a_\mu(x)$, where the lower index refers to general coordinate transformations and the upper index refers to Lorentz transformations. For an excellent discussion on this problem and on quantum gravity in general, see Ref. [30]. The end result is the Lagrangian

$$\mathcal{L} = \sqrt{|g|} \left( i\bar{\psi} \gamma^a D_a \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (3.17)$$

where

$$D_a = e^b_\mu \left( \partial_\mu - ieQ A_\mu + \frac{1}{2} \sigma^{bc} e^c_\nu \partial_\mu e_{b\nu} \right). \quad (3.18)$$

Here, $eQ$ is the coupling constant, where the elementary charge $e$ has been factored out, and $\sigma^{ab} = (\gamma^a \gamma^b - \gamma^b \gamma^a)/4$.

The coupling of gravity to QCD is completely analogous and will not be described further in this thesis. The results can be found in e.g. Ref. [31].

Finally, the coupling of the graviton to the SM fields is obtained by expanding the Lagrangian (3.17) to first order in $\bar{M}_p^{-1}$. We obtain

$$\sqrt{|g|} \mathcal{L} = \sqrt{|g|} \mathcal{L} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} + \bar{M}_p^{-1} \frac{\delta \left( \sqrt{|g|} \mathcal{L} \right)}{\delta g_{\mu\nu}} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} h_{\mu\nu} + \mathcal{O} (\bar{M}_p^{-2})$$

$$= \mathcal{L} + \bar{M}_p^{-1} T^{\mu\nu} h_{\mu\nu} + \mathcal{O} (\bar{M}_p^{-2}), \quad (3.19)$$

where the definition of the energy-momentum tensor (3.4) has been used.
Hence, to first order in \( \hat{M}_{\text{Pl}}^{-1} \), the part of the action coupling gravity to the SM fields is

\[
S_{\text{int}} = \frac{1}{\hat{M}_{\text{Pl}}} \int d^4x T^\mu_\nu h_{\mu\nu}.
\] (3.20)

The coupling between gravity and the SM fields is suppressed by the small factor \( \hat{M}_{\text{Pl}}^{-1} \), which is the reason why such high energies would be needed in order to experimentally test conventional quantum gravity.

### 3.4 General relativity in higher dimensions

General relativity is straightforward to generalize to the higher-dimensional spacetime of the ADD model. The Einstein-Hilbert action is assumed to be of the same form as in four dimensions

\[
\tilde{S} \propto \int \sqrt{|\tilde{g}|} d^N x \left( \tilde{R} - 2\tilde{\Lambda} \right).
\] (3.21)

In the higher-dimensional theory, a new fundamental mass scale \( \hat{M}_* \), analogous to the four-dimensional \( \hat{M}_{\text{Pl}} \), is introduced. The value of this new parameter is not predicted by the model, but has to be inferred from empirical observations. However, as the four-dimensional reduced Planck mass \( \hat{M}_{\text{Pl}} \) is the low-energy approximation of \( \hat{M}_* \), there exists a simple relation between these two quantities. This is considered in Sec. 3.4.1.

The complete expression for the action is obtained by multiplying the expression (3.21) by an appropriate power of \( \hat{M}_* \), which we now find by dimensional analysis. In natural units, the action is a dimensionless quantity. It is easy to show that the Ricci scalar has mass dimension two in any number of spatial dimensions, by using its definition in terms of the metric tensor. The volume element \( d^N x \) has mass dimension \(-N\). This means that the proportionality constant in Eq. (3.21) must have mass dimension \( N - 2 = d + 2 \). This means that the complete expression for the action is

\[
\tilde{S} = \hat{M}_*^{d+2} \int \sqrt{\tilde{g}} d^N x \left( \tilde{R} - 2\tilde{\Lambda} \right).
\] (3.22)

In Ref. [28], the derivation of Einstein’s equations from the action (3.22) is carried out. However, it is assumed that all the dimensions are infinite, and that the metric perturbation vanishes at infinity, which causes a surface term to vanish. In the ADD model, the additional dimensions are not infinite. However, going through the derivation, it is easy to convince oneself that the periodic boundary conditions of the metric insure that the surface term vanishes also in this case. Hence, the equations of motion in the higher-dimensional spacetime are

\[
\tilde{R}_{MN} - \frac{1}{2} \tilde{g}_{MN} \tilde{R} = \frac{1}{\hat{M}_*^{d+2}} \tilde{T}_{MN},
\] (3.23)

where we have assumed that the cosmological constant vanishes.
3.4.1 Matching parameters

As in the previous chapter, it is useful to relate the new parameters that have been introduced to the effective four-dimensional ones. To first order in $M_{\text{Pl}}^{-1}$, $\sqrt{|g|} = \sqrt{|\tilde{g}|}$ and $\tilde{R} = R$. Assuming that the energy is low enough so that the Ricci tensor is independent of the extra-dimensional coordinates and integrating out the extra dimensions in Eq. (3.21), we obtain the effective four-dimensional action

$$\tilde{S}_4 = \bar{M}_{\text{Pl}}^2 V_d \int \sqrt{|\tilde{g}|} d^4 x R,$$

(3.24)

Here, $V_d = (2\pi R)^d$ is the volume of the internal space. This expression is to be compared with the four-dimensional action

$$S = \bar{M}_{\text{Pl}}^2 \int \sqrt{|g|} d^4 x R.$$

(3.25)

The relation between the Planck scales is given by

$$\bar{M}_{\text{Pl}}^2 = \bar{M}_{\ast}^{d+2} V_d = \bar{M}_{\ast}^{d+2} (2\pi R)^d = \bar{M}_{\ast}^{d+2} R^d,$$

(3.26)

where in the last step we have introduced the quantity $M_{\ast} \equiv (2\pi)^d/(2^{d+1}) \bar{M}_{\ast}$, following Ref. [31].

One of the most interesting possibilities of the ADD model is that, by choosing the volume of the internal space to be large enough, the fundamental scale for gravity, $M_{\ast}$, could actually be as low as 1 TeV, which is of the same order of magnitude as the electroweak scale, $M_{\text{EW}}$. This would solve the hierarchy problem in a very elegant way, by eliminating the large difference between the two scales. To determine how large the radius of the internal space would have to be in order to achieve this, we solve for $R$ in Eq. (3.26), to obtain

$$R = \frac{\bar{M}_{\text{Pl}}^{2/d}}{\bar{M}_{\ast}^{(d+2)/d}} \approx 10^{32/d} \text{ TeV}^{-1} \approx 2 \cdot 10^{32+19d} \text{ m}.$$

(3.27)

For $d = 1, 2, 3$ and 4, we obtain $R \approx 2 \cdot 10^{13}$ m, 2 mm, 10 nm, and 20 pm, respectively. The case $d = 1$ is ruled out by experiments since it would violate Newton’s law of gravity on solar system scales. However, at the time when the ADD model was first considered, the cases $d > 2$ were all allowed, since gravity had only been probed down to distances of fractions of a millimeter. However, today gravity has been measured at smaller distances, and Newton’s law has been established at least down to distances of 55 \( \mu \text{m} \) [32]. This means that the case $d = 2$ is ruled out for $M_{\ast} = 1$ TeV.

In principle, the mass scale $M_{\ast}$ could be chosen to be even smaller than 1 TeV. However, $M_{\ast}$ is the scale at which quantum gravity effects should appear. Since no such effects have been observed in experiments reaching energy levels up to the order of the electroweak scale [33], this is experimentally ruled out.
3.5 Linearized equations of motion

In order to obtain the equations of motion for the higher-dimensional graviton, we repeat the procedure of Sec. 3.2.1. The metric is expanded about the background metric (3.1)

\[ \tilde{g}_{MN} = \tilde{\eta}_{MN} + \frac{\tilde{h}_{MN}}{\bar{M}^{d/2 + 1}}, \]

where \( |\tilde{h}_{MN}/\bar{M}^{d/2 + 1}| \ll |\tilde{\eta}_{MN}| \). The reason for the factor \( \bar{M}^{d/2 - 1} \) is completely analogous to the factor \( \bar{M}_{Pl}^{-1} \) in Sec. 3.2.1. Here, the fundamental energy scale for gravity is \( \bar{M} \) instead of \( \bar{M}_{Pl} \). Since Einstein’s field equations are of the same form as in four dimensions, the calculations are completely analogous. The only difference is the factor \( \bar{M}^{1} \). However, for the vacuum equations, this only contributes with an overall factor which cancels out. Thus, the equations of motion are

\[ \partial_M \partial_N \tilde{h}_{MN} = (\Box + \partial_n \partial^n) \tilde{h}_{MN} = 0. \]

3.6 Kaluza-Klein spectrum

We now perform a KK decomposition of the metric perturbation \( \tilde{h}_{MN} \). We introduce the Hilbert space \( L^2(T^d) \) of square-integrable functions on the \( d \)-dimensional torus with inner product

\[ \langle f, g \rangle = \int_{T^d} d^d x f(x) g(x). \]

We wish to express any reference to the extra dimensions in a four-dimensional form. Apart from the \( y \) dependence of the metric perturbation, the only such reference is in the kinetic term along the extra dimensions,

\[ \partial_n \partial^n \tilde{h}_{MN} = - (\partial_5^2 + \ldots + \partial_N^2) \tilde{h}_{MN}. \]

This is the Laplace operator in the flat internal space, acting on the metric perturbation. To bring out the massive graviton states, we solve the eigenvalue equation of this operator

\[ \partial_n \partial^n \psi = m^2 \psi. \]

The solutions of this equation, subject to the boundary condition that \( \psi \) is periodic under translations \( y \to y + 2\pi R \), are the harmonic functions

\[ \psi_n(y) = \frac{1}{(2\pi R)^{d/2}} e^{i n \cdot y}, \quad n \in \mathbb{Z}^d, \]

where we have used the short-hand notation \( n = (n_1, n_2, \ldots, n_d) \) and \( n \cdot y = n_1 y_1 + n_2 y_2 + \ldots + n_d y_d \). The normalization will be discussed later. These eigenfunctions
form a complete orthonormal set on the space $L^2(T^d)$. We expand the perturbation $\tilde{h}_{MN}$ along the extra dimension, using this set of basis functions

$$\tilde{h}_{MN}(x,y) = \frac{1}{(2\pi R)^{d/2}} \sum_n h^{(n)}_{MN}(x)e^{in\cdot y},$$  \hspace{1cm} (3.34)$$

where it is implicit that the sum is taken over all integers. Since the metric perturbation $\tilde{h}_{MN}$ is real, we require that $h^{(-n)}_{MN} = h^{(n)*}_{MN}$. Inserting the expansion into Eq. (3.14), we obtain

$$\sum_n \frac{1}{(2\pi R)^{d/2}} e^{in\cdot y} \left( \Box + \frac{|n|^2}{R^2} \right) h^{(n)}_{MN} = 0,$$  \hspace{1cm} (3.35)$$

or, since the harmonic functions are orthogonal, and hence, linearly independent

$$\left( \Box + \frac{|n|^2}{R^2} \right) h^{(n)}_{MN} = 0, \hspace{1cm} n \in \mathbb{Z}^d.$$  \hspace{1cm} (3.36)$$

For fixed $n$, this is the four-dimensional Klein-Gordon equation for a particle of mass $m = |n|/R$. Just as for the scalar field that was considered in Sec. 2.3, the higher-dimensional graviton $\tilde{h}_{MN}$ is represented by a countably infinite number of four-dimensional massive gravitons. In addition, the KK tower includes a massless mode that is independent of the extra dimensions, corresponding to the conventional four-dimensional graviton.

3.6.1 Normalization of the Kaluza-Klein modes

We will now discuss the normalization of the eigenfunctions (3.33). As solutions of the Eq. (3.32) they are only specified up to the normalization factor. This factor is determined by the constraint that the kinetic term in the Lagrangian for the KK modes $h^{(n)}_{\mu\nu}$ should be canonically normalized, i.e., of the form

$$L_{\text{kin}} = \frac{1}{2} \partial_\lambda h^{(n)}_{\mu\nu} \partial^\lambda h^{(n)*}_{\mu\nu}.$$  \hspace{1cm} (3.37)$$
Suppose that the Lagrangian for the higher-dimensional graviton $\tilde{h}^{MN}$ is canonically normalized. Then,

$$S_{\text{kin}} = \int d^N x \frac{1}{2} \partial_{\lambda} \tilde{h}^{\lambda MN} \partial^\lambda \tilde{h}^M_N$$

$$= \int d^N x \left( \frac{1}{2} \partial_{\lambda} \tilde{h}_{\mu \nu} \partial^\lambda \tilde{h}^{\mu \nu} + \ldots \right)$$

$$= \int d^4 x \int d^d y \left[ \frac{1}{2} \left( \sum_n \partial_{\lambda} h^{(n)}_{\mu \nu} \psi_n \right) \left( \sum_{n'} \partial^\lambda h^{(n')}_{\mu \nu} \psi_{n'} \right) + \ldots \right]$$

$$= \sum_n \sum_{n'} \int d^4 x \left( \frac{1}{2} \partial_{\lambda} h^{(n)}_{\mu \nu} \partial^\lambda h^{(n')}_{\mu \nu} \int d^d y \psi_n \psi_{n'} + \ldots \right)$$

$$= \sum_n \sum_{n'} \int d^4 x \left( \frac{1}{2} \partial_{\lambda} h^{(n)}_{\mu \nu} \partial^\lambda h^{(n')}_{\mu \nu} \delta^{(d)}(\|\psi_n\|)^2 + \ldots \right)$$

$$= \sum_n \int d^4 x \left( \|\psi_n\|^2 \frac{1}{2} \partial_{\lambda} h^{(n)}_{\mu \nu} \partial^\lambda h^{(n)}_{\mu \nu} + \ldots \right)$$

$$= \sum_n \int d^4 x \left( \|\psi_n\|^2 \frac{1}{2} \partial_{\lambda} h^{(n)}_{\mu \nu} \partial^\lambda h^{(n)}_{\mu \nu} + \ldots \right). \quad (3.38)$$

Comparing this expression to Eq. (3.37), this means that the proper normalization for the eigenmodes is $\|\psi_n\| = 1$, which agrees with Eq. (3.33).

### 3.7 Interaction Lagrangian

Now, we consider the coupling of the KK modes to the SM fields. For four space-time dimensions, the interaction term is given in Eq. (3.20), which is completely straightforward to generalize to higher-dimensional spacetime. We only need to compute the energy-momentum tensor. Since all the SM fields are confined to the brane, which is located at $y = y_0$, we have

$$\tilde{T}^{MN}(x, y) = \delta^M_{\mu} \delta^N_{\nu} T^{\mu \nu}(x) \delta^{(d)}(y - y_0), \quad (3.39)$$

where $T^{\mu \nu}$ is the ordinary SM energy-momentum tensor coupled to gravity and $\delta^{(d)}(y)$ is the Dirac distribution in $d$ dimensions.
In order to obtain the coupling between the SM fields and individual KK modes, we insert the expansion (3.34) into Eq. (3.20),

\[
\tilde{S}_{\text{int}} = \tilde{M}_{\text{Pl}}^{-1} \int d^4x \tilde{T}^{MN}(x,y) \tilde{h}_{MN}(x,y)
\]

\[
= \tilde{M}_{\text{Pl}}^{-1} \int d^4x T^{\mu\nu}(x) \tilde{h}_{\mu\nu}(x,y_0)
\]

\[
= M_{\text{Pl}}^{-1} \sum_n \int d^4x T^{\mu\nu}(x) \frac{1}{(2\pi R)^{d/2}} h^{(n)}_{\mu\nu}(x)e^{in \cdot y_0 R}
\]

\[
= \sum_n S^{(n)}_{\text{int}},
\]

where

\[
S^{(n)}_{\text{int}} = M_{\text{Pl}}^{-1} \int d^4x T^{\mu\nu}(x) \frac{1}{(2\pi R)^{d/2}} h^{(n)}_{\mu\nu}(x)e^{in \cdot y_0 R}.
\]

It will turn out that any physical quantity will only depend on the modulus squared of the quantity \(e^{in \cdot y_0 R}\). This means that we may set \(y_0 = 0\) without loss of generality, so that \(e^{in \cdot y_0 R} = 1\).

Using Eq. (3.26), the interaction terms can be written as

\[
S^{(n)}_{\text{int}} = \frac{1}{M_{\text{Pl}}} \int d^4x T^{\mu\nu}(x) h^{(n)}_{\mu\nu}(x).
\]

Hence, we observe that, just as in four-dimensional quantum gravity, Eq. (3.20), the coupling is suppressed by \(M_{\text{Pl}}^{-1}\). However, there are now a large number of KK modes of different masses, instead of a single, massless graviton. This means that we may still be able to obtain a measurable signal. The phenomenology of higher-dimensional gravity will be investigated in Ch. 5.

### 3.8 Problems with the model

We have stated that one of the appealing aspects of the ADD model is that it solves the hierarchy problem. But does it really? For \(d > 2\), the fundamental Planck scale can be lowered as far as to the electroweak scale, but at the same time a new length scale, \(R\), has been introduced. According to Eq. (3.27), the inverse of this scale is

\[
\frac{1}{R} = M_* \left( \frac{M_*}{M_{\text{Pl}}} \right)^{2/d} \approx 10^{-32/d} \text{ TeV} \ll M_*.
\]

Ideally, we would like this new energy scale to be of the same order of magnitude as the electroweak scale, but this is not possible in the ADD model. The new energy scale related to the radius of the extra dimensions is separated from the electroweak scale by 32/d orders of magnitude, which results in a large difference unless \(d\) is
large. What has been done is merely to translate the hierarchy problem—instead of a large hierarchy between the electroweak scale and the Planck scale we now have a large hierarchy between the electroweak scale and the large radius of the extra dimensions. This still needs a motivation, and hence, the hierarchy problem has not really been solved.

It is important to make sure that the implications of the model are not in serious conflict with other well-established physical phenomena. The smallness of the KK masses in the ADD model leads to some serious constraints on the fundamental mass scale, especially for $d = 2$. Many of these constraints are discussed in Ref. [34]. Here, we will only mention one of the most important examples, coming from astrophysics.

For $d = 2$, the mass of the lightest KK mode is approximately $10^{-4}$ eV. This means that in high-temperature systems, such as stars and supernovas, these modes could be produced in large numbers. Since the coupling of KK modes to matter is suppressed by $M_{Pl}^{-1}$, the KK modes interact very weakly with matter. This means that, once produced, almost all KK particles will avoid further interaction with matter, and will hence carry away energy. This could potentially alter the evolution of the system in a non-acceptable way. To obtain the strongest possible constraint, the hottest objects in the Universe have been studied. These are supernovas. The most well-studied supernova is SN1987A. The maximal temperature of this system is on the order of 50 MeV [35]. According to models of the supernova processes, most of the energy is carried away by neutrinos. This has also been confirmed by measurements of the neutrino flux from SN1987A. By requiring that no substantial fraction of the energy is carried away by KK gravitons, one obtains the bound $M_{*} > 30$ TeV for $d = 2$. For $d > 2$, it is still possible to have $M_{*} = 1$ TeV. For a more detailed analysis, see e.g. Ref. [35].
Chapter 4

Hyperbolic disc model

In the ADD model, the extra dimensions are assumed to be flat. In this chapter, a different model is considered, having curved extra dimensions. More specifically, the internal space has constant negative curvature. This means that the geometry is hyperbolic, rather than Euclidean. This leads to some interesting new possibilities.

We will first consider hyperbolic geometry and the geometry of the higher-dimensional space. Next, we will consider the KK tower of the graviton. In this model, it is not possible to solve for the spectrum analytically, which means that some approximations have to be made. Finally, we discuss some of the advantages and disadvantages of the model, comparing it with the ADD model.

4.1 Hyperbolic geometry

The flat internal space of the ADD model could be generalized to a curved space in an infinite number of ways. However, assuming that the internal space is homogeneous and isotropic reduces the number of choices significantly, leaving only three distinct choices. A homogeneous and isotropic space has constant curvature [36]. The only qualitative difference depends on whether this curvature is zero, positive or negative. A space of zero curvature is flat, while a space of constant positive curvature is the generalization of a sphere. A space with constant negative curvature is called hyperbolic, and is sometimes referred to as a pseudosphere. In this thesis, we consider a two-dimensional space of constant negative curvature. After having examined the geometry of this space in more detail, we will be able to motivate this choice.

There are a number of different ways to visualize a two-dimensional pseudosphere. For an overview of some of the different possibilities, see e.g. Ref. [37]. Unfortunately, unlike the ordinary two-dimensional sphere, it is not possible to embed the two-dimensional pseudosphere in three-dimensional Euclidean space. However, it is possible to embed it in a three-dimensional Lorentzian space with a metric
Chapter 4. Hyperbolic disc model

defined by the line element $ds^2 = dx_1^2 - dx_2^2 - dx_3^2$. Except for that fact, however, the treatment is closely analogous to that of the sphere.

Just like for the sphere embedded in Euclidean space, the equation defining the pseudosphere in Lorentzian space is $\|x\| = R_c$, where $R_c$ is the curvature radius. For the sphere, $R_c$ is just its physical radius. Writing this equation out in terms of coordinates, we obtain

$$\|x\|^2 = x_1^2 - x_2^2 - x_3^2 = R_c^2. \quad (4.1)$$

This equation is identically satisfied with the parametrization

$$\begin{align*}
x_1 &= R_c \cosh(\tau), \\
x_2 &= R_c \sinh(\tau) \cos(\varphi), \\
x_3 &= R_c \sinh(\tau) \sin(\varphi),
\end{align*} \quad (4.2)$$

where $\tau \in [0, \infty)$ and $\varphi \in [0, 2\pi)$ are polar coordinates. In terms of these parameters, the line element is given by

$$ds^2 = -R_c^2 \left[ d\tau^2 + \sinh^2(\tau) \, d\varphi^2 \right]. \quad (4.3)$$

The physical distance from the origin to a point $(\tau, \varphi)$ is $R_c \tau$. One could also work directly with the physical radius by introducing the new coordinate $r \equiv R_c \tau$. In terms of the coordinates $(r, \varphi)$, the line element is

$$ds^2 = -dr^2 - v^{-2} \sinh^2(vr) \, d\varphi^2, \quad (4.4)$$

where the curvature $v \equiv R_c^{-1}$ has been introduced.

The metric for the full spacetime is defined by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - dr^2 - v^{-2} \sinh^2(vr) d\varphi^2, \quad (4.5)$$

where $g_{\mu\nu}$ is the Minkowski metric. Hence, we have

$$(g_{MN}) = \text{diag}[1, -1, -1, -1, -v^{-2} \sinh^2(vr)]. \quad (4.6)$$

### 4.1.1 Compactification

The hyperbolic space that we have considered so far is open, *i.e.*, infinite in extent. An infinite internal space would violate empirical observations, so the space has to be compactified. There are in principle two ways of achieving this. The first is to explicitly introduce a boundary, *i.e.*, cut out a piece of the space, like for example a disc. The other option is to consider a closed space without a boundary, similar to a sphere or a torus. Such a space can be constructed from the open hyperbolic space by “wrapping it up” around itself. Mathematically, such spaces are called quotient spaces, and are constructed by identifying certain sets of points of the space. In this way, one can, for example, construct the circle $S^1$ out of the real line $\mathbb{R}$ by identifying points that are separated by an integer number. Quotient spaces lead to
a generalization of periodic boundary conditions, since any single-valued function
defined on this quotient space has to take the same values for any two points that
are identified. For more information on quotient spaces, see e.g. Ref. [38]. For a
discussion on closed hyperbolic extra dimensions, see Ref. [39].

In this thesis, we only consider the first alternative, i.e., a space with a bound-
ary. More specifically, the internal space of the model is a compact two-dimensional
hyperbolic disc. This model has been considered in Ref. [40], but there, the calcu-
lations are performed by discretizing the disc.

An important aspect of the geometry of the model is where the four-dimensional
brane intersects the two-dimensional disc. In the ADD model, the internal space
is completely homogeneous, which means that any two locations of this point are
equivalent. This is reflected in the fact that the KK modes are of the form $e^{i m \cdot y}$,
having moduli equal to unity. The physical predictions of the theory only depend
on these moduli squared, and hence, they do not depend on the choice of the point
$y_0$. For the hyperbolic disc, this homogeneity is destroyed by the introduction of
the boundary of the disc. This has the result that the moduli of the KK states do
depend on the location of the brane in the radial direction. Translational invariance
in the angular direction is still maintained, which means that the moduli do not
depend on the angular coordinate. This will be investigated in more detail in
Sec. 4.4.

We now argue that a space of negative curvature is a good choice for our pur-
poses. We focus on the problem of the hierarchy between the radius of the ex-
tra dimensions and the electroweak scale, which was mentioned at the end of the
last chapter. The point to note is that in the relation between the Planck scales,
Eq. (3.26), it is not the radius, but rather the volume, of the internal space that is
important. Consider the area of a hyperbolic disc of radius $L$

$$A = \int dA = \int_0^L \int_0^{2\pi} \sqrt{|g|} dr d\phi = 4\pi v^{-2} \sinh^2 \left( \frac{vL}{2} \right).$$  (4.7)

For large $vL$, the area increases exponentially with the radius, and hence, it is pos-
sible to obtain a large volume from a small radius. For a disc of positive curvature,
on the other hand, the corresponding relation is

$$A = 4\pi v^{-2} \sin^2 \left( \frac{vL}{2} \right).$$  (4.8)

Since spaces of constant positive curvature are closed, this area has a maximum
value of $4\pi v^{-2}$. In addition, it increases even slower with the radius than for a flat
disc. Hence, at least in terms of this hierarchy problem, negative curvature is the
better choice. The relation among the three cases is shown in Fig. 4.1.

### 4.1.2 Solution of Einstein’s field equations

We have described the geometry of the hyperbolic disc, but we have not really
discussed whether a solution of Einstein’s field equations of the form (4.6) exists.
But what is meant by saying that a certain solution exists? Einstein’s field equations are \( G_{MN} = T_{MN}/M_{Pl}^2 \). The energy-momentum tensor on the right-hand side is uniquely determined by the left-hand side, which can be computed for any metric. Thus, mathematically speaking, any metric is a solution of Einstein’s field equations for some energy-momentum tensor. However, from a physical point of view, we want an interpretation of this energy-momentum tensor. Hence, what is meant by saying that a solution exists is that the field equations for this particular solution lead to a physically realistic energy-momentum tensor.

The energy-momentum tensor corresponding to the metric (4.6) is given by

\[
(T_{MN}) = \text{diag}(-v^2, v^2, v^2, v^2, 0, 0). \tag{4.9}
\]

It is constant, but it is not due to a standard cosmological constant, which is of the form \( T_{MN} = -\Lambda g_{MN} \).

A general energy-momentum tensor is symmetric, which means that there exists a frame in which it is diagonal. In this frame, the 00-component is usually interpreted as an energy density and the diagonal spatial components as stresses in the respective directions. For the hyperbolic disc model, the 00-component is negative definite, similar to the case of a cosmological constant.

We have not been able to find a satisfactory interpretation of the energy-momentum tensor (4.9). A discussion of this is given in Ref. [40]. Furthermore,
in Ref. [39], it is motivated that a solution of Einstein’s field equations exists in a similar model.

4.2 Matching parameters

The result for the relation between the fundamental mass scales and the volume of the internal space, Eq. (3.26), still holds for the hyperbolic disc model. To simplify the comparison with the results from the ADD model, we introduce the quantity $M_\ast \equiv (2\pi)^{1/2}M_*$. Using the expression for the volume of a hyperbolic disc, Eq. (4.7), we obtain

$$\frac{\sinh^2(vL/2)}{\pi v^2} = \frac{M_\ast^2}{M_p^4}.$$  \hspace{1cm} (4.10)

Solving for $L$, we find

$$L = 2v^{-1}\arcsinh\left(\frac{\pi v^2 M_\ast^2}{M_p^4}\right).$$  \hspace{1cm} (4.11)

Choosing e.g. $M_* = v = 1$ TeV gives $L \approx 150M_*^{-1}$. This is much closer to the fundamental scale than what is obtained in the ADD model, requiring only $\mathcal{O}(100)$ coefficients. In this sense, the hyperbolic disc model provides a more satisfactory solution to the hierarchy problem than the ADD model.

4.3 Linearized equations of motion

In the usual manner, we consider small perturbations about the background metric (4.6)

$$\bar{g}_{MN} \rightarrow \bar{g}_{MN} + \frac{1}{\bar{M}_4^2} \bar{h}_{MN},$$  \hspace{1cm} (4.12)

where $|\bar{h}_{MN}/\bar{M}_4^2| \ll |\bar{g}_{MN}|$ and $\bar{M}_4^{-2}$ is the proper normalization factor in six spacetime dimensions.

In the ADD model, the KK modes and masses are obtained as the solutions of the eigenvalue equation for the Laplace operator on the internal space, which is an $n$-dimensional torus. In the present model, the internal space is curved, being a hyperbolic manifold. The most natural way to generalize the analysis of the KK modes is to replace the Laplace operator by the corresponding operator on a curved manifold. For a manifold $M$, there is a unique invariant second order differential operator acting on scalar fields defined on $M$ [41]. It is known as the Laplace-Beltrami (LB) operator. The metric perturbation is a spin-2 field in the full higher-dimensional spacetime. However, we will only consider the 4-dimensional part $\bar{h}_{\mu\nu}$, which transforms as a scalar on the internal space. Therefore, the LB
Chapter 4. Hyperbolic disc model

The LB operator will be denoted by $\Delta_{LB}$. In a coordinate system $(x^n)$, its action on a scalar field is

$$\Delta_{LB}\psi = \frac{1}{\sqrt{|g|}} \partial_n \left( \sqrt{|g|} g^{nm} \partial_m \psi \right).$$ \hfill (4.13)

Note that for a flat space $g_{nm} = \text{const.}$, so that $\Delta_{LB}\psi = \partial_n \partial^n \psi$, which is the ordinary Laplace operator.

For a hyperbolic space with polar coordinates $(r, \varphi)$, we obtain, using the metric (4.6),

$$\Delta_{LB}\psi = -\frac{1}{\sinh(vr)} \frac{\partial}{\partial r} \left[ \sinh(vr) \frac{\partial \psi}{\partial r} \right] - \frac{v^2}{\sinh^2(vr)} \frac{\partial^2 \psi}{\partial \varphi^2}.$$ \hfill (4.14)

We will assume that the relevant equation for the KK spectrum and states is the eigenvalue equation for the LB operator on the hyperbolic disc, $\Delta_{LB}\psi = m^2 \psi$. To prove this rigorously involves expanding the equations of motion to first order in $M^{-2}$ and making proper gauge choices to bring out the physical states of the metric perturbation. Because of the complicated structure of the internal space, as compared to the ADD model, these computations are quite complicated and will not be performed in this thesis. This is also discussed in Ref. [39].

### 4.4 Kaluza-Klein spectrum

In order to obtain the KK spectrum, we need to solve the eigenvalue equation $\Delta_{LB}\psi = m^2 \psi$ for the LB operator on the hyperbolic disc. In contrast to the ADD model, it is not possible to obtain an exact analytic expression for the spectrum of the hyperbolic disc model. Therefore, it will be valuable to first consider some general features of the spectrum.

#### 4.4.1 General properties of the Kaluza-Klein spectrum

Consider a smooth manifold $M$ with a boundary $\partial M \neq \emptyset$. Let $L^2(M)$ be the Hilbert space of functions on $M$ of finite norm with inner product

$$\langle f, g \rangle = \int d^4x \sqrt{|g|} f(x)g(x).$$ \hfill (4.15)

Consider the eigenvalue equation for the LB operator on $M$. Its solutions, $\psi$, are required to be second order differentiable on $M$ and continuous on the closure of $M$, which we denote by $\overline{M}$. In order to completely specify the eigenvalue equation, a choice of boundary conditions has to be made. We consider the following three classes of boundary conditions:

1. Dirichlet conditions, requiring the function to vanish on $\partial M$. 

operator is the correct operator acting on $\tilde{h}_{\mu\nu}$. The LB operator will be denoted by $\Delta_{LB}$. In a coordinate system $(x^n)$, its action on a scalar field is
2. Neumann conditions, requiring the normal derivative of the function to vanish on $\partial M$.

3. Mixed conditions, i.e., Neumann conditions on some open subset $N$ of $\partial M$ and Dirichlet conditions on $\partial M \setminus N$.

We will consider two theorems concerning the spectrum of the LB operator. The first of these is concerned with the general features of the spectrum and eigenstates.

**Theorem 4.4.1.** Let the closure $\overline{M}$ of the manifold $M$ be connected and compact. The following facts for the eigenvalues and eigenmodes of the LB operator on $M$, subject to one of the classes of boundary conditions (1)–(3), hold:

1. The set of eigenvalues consists of a sequence $0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \infty$, and each associated eigenspace is finite-dimensional.

2. Eigenspaces, belonging to distinct eigenvalues, are orthogonal in $L^2(M)$, which is the direct sum of all the eigenspaces.

3. Each eigenfunction is $C^\infty$ on $M$.

*Proof.* See Ref. [42].

The first statement says that, like the spectrum for a flat internal space, the spectrum is discrete. Also, if zero is an eigenvalue, then there is a gap between the zero eigenvalue and the first non-zero one. This gap is of great importance in connection to the astrophysical constraints associated with the model. The second statement says that it is indeed possible to make an expansion of the perturbation along the extra dimensions in terms of the eigenfunctions, which is of fundamental importance.

The second result, which is known as Weyl’s asymptotic formula, describes the behavior of the eigenvalues in the asymptotic limit of large eigenvalues.

**Theorem 4.4.2.** For each of the classes of boundary conditions (1)–(3), let $N(\lambda)$ denote the number of eigenvalues of the LB operator on $M$ that are less than or equal to $\lambda$, with each eigenvalue being counted with multiplicity. Then,

$$
\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\omega_d V_d}{(2\pi)^d}.
$$

(4.16)

Here, $d$ is the dimensionality of $M$, $\omega_d$ is the volume of the unit disc in $\mathbb{R}^d$ and $V_d$ is the volume of $M$.

*Proof.* See Ref. [43]. For a proof in English which is not complete, but enough for the purposes of this thesis, see also Ref. [44].
Note that this theorem implies that, for fixed dimensionality, the spectrum behaves asymptotically in the same way for any geometry. The only difference is in the constant of proportionality, $\omega_d V_d/(2\pi)^d$. The theorem is actually easy to motivate heuristically. In the present context, the eigenvalue is a mass squared. A large eigenvalue corresponds to a small distance, and for distances much smaller than the curvature radius, the manifold will look flat. Hence, the spectrum should reduce to the spectrum of a flat manifold in the limit of large eigenvalues, which is exactly the statement of the theorem. Because of this theorem, we expect that any “strange” behavior in the spectrum of the hyperbolic disc, as compared to that of a flat space, occurs at the lower end of the spectrum.

By Theorem 4.4.1, we can order the eigenvalues in a countable increasing series. By Theorem 4.4.2, we obtain asymptotically for the $n$th eigenvalue

$$n = N(\lambda_n) = \frac{\omega_d V_d \lambda_n^{d/2}}{(2\pi)^d} \quad \Leftrightarrow \quad \lambda_n = \left[ \frac{(2\pi)^d n}{\omega_d V_d} \right]^{2/d}.$$  \hspace{1cm} (4.17)

In the hyperbolic disc model, we have $\lambda_n = m_n^2$, $d = 2$, $\omega_d = \pi$ and, by Eq. (4.7), $V_d = 4\pi v^{-2} \sinh^2(vL/2)$, which means that we obtain

$$m_n = \frac{v}{\sinh(vL/2)} \sqrt{n}, \quad n \to \infty.$$  \hspace{1cm} (4.18)

### 4.4.2 Solution of the eigenvalue equation

We now turn to the solution of the eigenvalue equation (4.19). It turns out that, for our purposes, the only possible class of boundary conditions is Neumann conditions. It can be proved [42] that an eigenfunction has the eigenvalue zero if and only if the function is constant. But a constant function satisfying Dirichlet or mixed boundary conditions is identically equal to zero. This means that in order to include a massless graviton in the KK spectrum, the only possible choice is Neumann conditions. This choice will not be motivated from a physical point of view, but can be seen as a limitation imposed by the fact that the theory has to reproduce the conventional results in the low-energy limit.

The Neumann condition only gives one condition in the radial direction, whereas two are needed for a second order differential equation. The second condition that we need to impose is that the solution is regular everywhere. This condition is easy to motivate. A perturbation of the metric having a singularity at some point is in no way small in the vicinity of that point. Hence, such a solution violates the assumptions that motivated the linearization of the equations of motion in the first place.

In the angular direction, the requirement that the solutions are single-valued means that they have to be periodic with period $2\pi$. 
In polar coordinates \((r, \phi)\), the eigenvalue equation for the LB operator on the hyperbolic disc is given by
\[
-\frac{1}{\sinh(vr)} \frac{\partial}{\partial r} \left[ \sinh(vr) \frac{\partial \psi}{\partial r} \right] - \frac{v^2}{\sinh^2(vr)} \frac{\partial^2 \psi}{\partial \phi^2} = m^2 \psi.
\] (4.19)

We introduce the dimensionless variable \(\tau \equiv vr\) to obtain the equation
\[
-\frac{1}{\sinh(\tau)} \frac{\partial}{\partial \tau} \left[ \sinh(\tau) \frac{\partial \psi}{\partial \tau} \right] - \frac{1}{\sinh^2(\tau)} \frac{\partial^2 \psi}{\partial \phi^2} = k^2 \psi,
\] (4.20)

where we have also introduced the dimensionless eigenvalue \(k^2 \equiv m^2/v^2\).

Next, we make a separation of variables, setting \(\psi(\tau, \phi) \equiv T(\tau)\Phi(\phi)\) and dividing the equation by \(\psi\), to obtain
\[
\sinh(\tau) \frac{\partial}{\partial \tau} \left[ \sinh(\tau) \frac{\partial T}{\partial \tau} \right] + k^2 \sinh^2(\tau) = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} \equiv \ell^2,
\] (4.21)

where \(\ell\) is a separation constant, which has to be determined from the boundary conditions. The equation for \(\Phi\) is
\[
\frac{d^2 \Phi}{d\phi^2} + \ell^2 \Phi = 0,
\] (4.22)

which has the solutions
\[
\Phi_\ell(\phi) = e^{i\ell \phi}.
\] (4.23)

Because of the periodic boundary conditions \(\Phi_\ell(\phi + 2\pi) = \Phi_\ell(\phi)\), \(\ell\) must be an integer.

The equation for \(T\) is
\[
\frac{1}{\sinh(\tau)} \frac{d}{d\tau} \left[ \sinh(\tau) \frac{dT}{d\tau} \right] + \left[ k^2 - \frac{\ell^2}{\sinh^2(\tau)} \right] T = 0.
\] (4.24)

In order to solve this equation, we make the change of variables \(x \equiv \cosh(\tau)\). Then, \(\sinh^2(\tau) = x^2 - 1\), and we obtain the equation
\[
\frac{d}{dx} \left[ (1 - x^2) \frac{dT}{dx} \right] + \left[ \nu(\nu + 1) - \frac{\ell^2}{1 - x^2} \right] T = 0,
\] (4.25)

where we have introduced \(\nu(\nu + 1) \equiv -k^2\), in order to obtain the equation on its standard form. This is known as Legendre’s associated equation. It turns up e.g. as the equation for the polar angle in the theory of angular momentum in ordinary quantum mechanics. In that context, however, the variable \(x \equiv \cos(\theta) \in [-1, 1]\), whereas in this case, \(x \equiv \cosh(\tau) \in [1, \infty)\). This has the consequence that, in contrast to the context of angular momentum, the parameter \(\nu\) is not restricted to
positive integer values, but can take any complex values, which are determined by the boundary conditions.

We now summarize those properties of the solutions of Eq. (4.25) that will be important for the subsequent work. All of these results can be found in e.g. Ref. [45].

Since Eq. (4.24) is a second order ordinary differential equation, it has two linearly independent solutions. They are known as the associated Legendre functions of the first and second kind, and are usually denoted by $P_{\nu}^{\ell}(x)$ and $Q_{\nu}^{\ell}(x)$, respectively. However, the associated Legendre functions of the second kind are singular at the point $x = 1$, corresponding to $r = 0$. Hence, these solutions do not obey the boundary conditions at the origin and must be rejected. The associated Legendre functions of the first kind, on the other hand, are regular everywhere. These are the physically acceptable solutions. From now on, we will simply refer to these as Legendre functions.

It is convenient to make yet another reparametrization by introducing the parameter $\rho$, which is related to the parameter $\nu$ by the equation

$$-\nu(\nu + 1) = \frac{1}{4} + \rho^2 \quad \Rightarrow \quad \nu = -\frac{1}{2} \pm i\rho. \quad (4.26)$$

The solution of Eq. (4.24) is given by

$$T_{\nu}^{\ell}(r) = P_{-\frac{1}{2} + i\rho}^{\ell} \cosh(\nu r), \quad (4.27)$$

and the eigenvalue is

$$m = v \sqrt{\frac{1}{4} + \rho^2}. \quad (4.28)$$

For $\nu$ real or $\nu = -1/2 + i\rho$ with $\rho$ real, $P^{\ell}_{\nu}$ is real.

One thing about the spectrum is important to note. For real $\rho$, Eq. (4.28) implies that the eigenvalues lie in the interval $[v/2, \infty)$. However, the parametrization of $m$ in terms of $\rho$ is arbitrary, and there is nothing that says that $\rho$ must be real. Since the spectrum of the LB operator is non-negative and the eigenvalue zero corresponds to a constant function, what remains to be examined is the interval $(0, v/2)$, i.e., $\nu \in (-1, 0)$. For the $\ell = 0$ modes, it can be proved that the derivatives of the Legendre functions have no zeroes in the interval $x \in (1, \infty)$ for real $\nu$. The proof is based on the identity

$$P_{\nu}^{m}(x) = (x^2 - 1)^{m/2} \frac{d^{m} P_{\nu}^{0}(x)}{dx^{m}}, \quad m = 1, 2, 3, \ldots \quad (4.29)$$

For $m = 1$, we obtain

$$\frac{dP_{\nu}^{0}(x)}{dx} = (x^2 - 1)^{-1/2} P_{\nu}^{1}(x). \quad (4.30)$$

It has been proved in Ref. [46] that the functions $P_{\nu}^{m}(x)$ have no zeroes for real $\nu$ and integer $m$. Hence, since $(x^2 - 1) \neq 0$ for $x > 1$, the derivative of $P_{\nu}^{0}$ has
no zeros in the interval \( x \in (1, \infty) \) for real \( \nu \). We have not been able to prove the corresponding result for general \( \ell \). However, numerical studies have revealed no such zeroes.

We can still argue that there is a large gap between the zero and the first non-zero KK masses. The inverse of the eigenvalues are interpreted as the wavelengths of the corresponding eigenmodes. This means that the size of the inverted eigenvalues cannot exceed the longest linear distance on the internal space, \( i.e., \) its radius \( L \). This is also true for the ADD model, where it is explicit from the formula \( m_n = |n|/R \geq 1/R \). However, in the ADD model, this radius is typically very large, which means that the eigenvalues can be very small. As has been discussed, this is not the case in the hyperbolic model, where the radius is typically much smaller. This means that we will still obtain a significant energy gap, bounded from below by \( L^{-1} \).

In any calculations that will be performed in this thesis, \( v/2 \) will be larger than \( L^{-1} \). Typically, we will have \( v = \mathcal{O}(100)L^{-1} \). This means that the condition \( m > v/2 \) is stronger than the condition \( m > L^{-1} \). For computational purposes, we will assume that the spectrum (excluding the zero mode) is completely contained in the interval \([v/2, \infty)\). As \( v \) is typically much smaller than the energy scales that will be considered in this thesis, any deviations from this assumption should not affect the results that will be derived in Ch. 5 very much.

Returning to the properties of the Legendre functions, they can be expressed as

\[
P_{\nu}^\ell(x) \propto \left(\frac{x + 1}{x - 1}\right)^{|\ell|/2} F\left(-\nu, \nu + 1, 1 + |\ell|; \frac{1}{2} - \frac{x}{2}\right),
\]

(4.31)

where \( F \) denotes the hypergeometric function. This function is usually denoted \( _2F_1 \), but we skip the subscripts in order to make the notation less messy. Since the eigenfunctions are only defined up to normalization, we have not included any constant overall factors. The normalization will be discussed in Sec. 4.5.

With \( x = \cosh(vr) \) and \( \nu = -1/2 + i\rho \), Eq. (4.31) becomes

\[
P_{\frac{1}{2} + i\rho}^\ell[\cosh(vr)] \propto \tanh|\ell| \left[\frac{\sinh^2(vr)}{2}\right].
\]

(4.32)

The hypergeometric function is defined as the analytic continuation of the power series

\[
F(a, b, c, z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},
\]

(4.33)

where \((a)_n \equiv \Gamma(a + n)/\Gamma(a)\) is known as the rising factorial or Pochhammer symbol. For the function to be single-valued, a cut in the complex plane has to be made. For the solutions that are valid in the real interval \([1, \infty)\) this cut is made along the interval \((-\infty, 1]\). This will be of no concern in this thesis.

According to Eq. (4.33), \( F(a, b, c, 0) = 1 \). Since \( \tanh(0) = 0 \), we have \( T_0^\ell(0) = 0 \) whenever \( \ell \neq 0 \), \( i.e., \) the eigenfunctions are equal to zero at the origin. For \( \ell = 0 \),
we have $T^n_0(0) = 1$. This will have consequences for the phenomenology if the brane is placed at $r = 0$, which will be discussed in Ch. 5.

To obtain the mass spectrum, we impose the Neumann boundary conditions

$$\left. \frac{dT^\ell}{d\tau} \right|_{\tau = \tau_{\text{max}}} = 0,$$

where $\tau_{\text{max}} \equiv vL$. The derivative of the hypergeometric function can be expressed as

$$\frac{d}{dx} F(a, b, c, x) = \frac{ab}{c} F(a + 1, b + 1, c + 1, x).$$

With our parameters, we obtain

$$\frac{d}{d\tau} F\left[ \frac{1}{2} - i\rho, \frac{1}{2} + i\rho, 1 + |\ell|, -\sinh^2 \left( \frac{\tau}{2} \right) \right]$$

$$= -\frac{1}{2} \sinh(\tau) \frac{1}{1 + |\ell|} F\left[ \frac{3}{2} - i\rho, \frac{3}{2} + i\rho, 2 + |\ell|, -\sinh^2 \left( \frac{\tau}{2} \right) \right].$$

Hence, the derivative of Eq. (4.32) can be written in terms of the hypergeometric function

$$\frac{dP^\ell}{d\tau} \left[ \frac{1}{2} + i\rho \cosh(v\tau) \right]$$

$$= \tanh^{i\ell} \left( \frac{\tau}{2} \right) \left[ \frac{1}{\sinh(\tau)} F\left( \frac{1}{2} - i\rho, \frac{1}{2} + i\rho, 1 + |\ell|, -\sinh^2 \left( \frac{\tau}{2} \right) \right) \right]$$

$$- \frac{1}{2} \sinh(\tau) \frac{1}{1 + |\ell|} F\left[ \frac{3}{2} - i\rho, \frac{3}{2} + i\rho, 2 + |\ell|, -\sinh^2 \left( \frac{\tau}{2} \right) \right].$$

This is as far as analytical calculations can take us. The solutions of Eq. (4.34) are not known analytically. This presents some serious problems. To solve the equation numerically for a fixed set of parameters is straightforward. However, if the KK states have $O(\bar{M}_{Pl}^{-1})$ couplings to SM matter, as in the ADD model, then in order to obtain an observable signal we would at least need an order of $10^{30}$ eigenvalues. It is impossible to numerically solve for such a huge number of eigenvalues. This means that we have to find some different approach. We can solve for a reasonable number of the lowest eigenvalues and use Weyl’s formula for the higher ones, for which it should be a good approximation.

Another problem is related to the numerical calculations. We have to calculate values of the Legendre functions for very large values of the parameter $\rho$ and $\ell$. The problem is that for too large values, the numerical computation on the functions become very demanding and time consuming. This problem could perhaps be resolved by developing more efficient algorithms, but this will not be considered in this thesis. Instead, we will use an approximate expression for the Legendre functions.
4.4.3 Constraints on the parameter space

In order to be able to make any numerical calculations, we have to decide on some set of values for the parameters of the model. We first consider some constraints that have to be imposed on the parameter space.

In the hyperbolic disc model, there are three parameters $M_*$, $v$, and $L$. However, they are related by Eq. (4.10), which means that only two of them are independent. We will choose these to be $M_*$ and $v$, since the constraints that we will consider are most easily expressed in terms of these variables.

The most important constraint is the one coming from the astrophysics, which was considered in Sec. 3.8. The constraints coming from the supernova SN1987A are only relevant if sufficiently many KK modes are available at the maximum temperature of the supernova, $T \approx 30$ MeV. If the parameter $v$ is chosen so that the first non-zero mass of the KK spectrum is larger than this temperature, then the constraints from SN1987A are completely avoided. Assuming that there are no eigenvalues in the interval $(0, v/2)$, this is achieved by choosing $v > 100$ MeV. Note that this bound has not been optimized. In order to do so, a more careful analysis would be needed, including the coupling constants of the KK mode in the hyperbolic disc model. The lower bound of 100 MeV merely gives an idea of the order of magnitude of this constraint. We will only consider values of $v$ larger than this bound. An upper bound on $v$ is given by the fundamental mass scale $M_*$, since it is taken to be the cutoff energy of the effective model. Hence, we cannot consider values of $v$ that are larger than $M_*$.

Like in the ADD model, we must have $M_* > 1$ TeV in order not to contradict experimental results [33], which have found no signs of quantum gravitational effects. In order for the model to provide a solution to the hierarchy problem, we also do not consider values of $M_*$ that are much larger than the TeV scale.

4.4.4 Approximation of the Kaluza-Klein modes

Before turning to the numerical solution of the KK spectrum, it is useful to consider the eigenfunctions in some more detail. Weyl’s formula only gives information on the spectrum, not the eigenfunctions. In order to further analyze the eigenfunctions we can use a certain trick, which has been adapted from Ref. [47]. This is done by rewriting Eq. (4.24) as follows. We introduce the auxiliary function $u^\ell_\rho(\tau)$ as

$$T^\ell_\rho(\tau) \equiv \frac{u^\ell_\rho(\tau)}{\sqrt{\sinh(\tau)}}.$$  \hfill (4.38)

In terms of $u^\ell_\rho$, Eq. (4.24) becomes

$$\frac{d^2u^\ell_\rho}{d\tau^2} + \frac{\ell^2 - 1/4}{\sinh^2(\tau)} u^\ell_\rho = \rho^2 u^\ell_\rho.$$  \hfill (4.39)

This equation has exactly the form of a one-dimensional Schrödinger equation with a potential $V(\tau) = (\ell^2 - 1/4)/\sinh^2(\tau)$ and energy $E = \rho^2$. This form of the
The solutions of the Schrödinger equation have different qualitative behavior depending on the relation between the potential $V(\tau)$ and the energy $E$. For $E > V(\tau)$, the solution is oscillatory, while for $E < V(\tau)$, there is one solution $u_\ell^\prime \propto \sinh^{-1/2+\ell}(\tau)$ and another $u_\ell^\prime \propto \sinh^{-1/2-\ell}(\tau)$ [47]. For each $\ell$, there is one increasing and one decreasing solution. Since we have shown that the eigenfunctions are zero at the origin, it is the increasing solution that we are interested in, while the decreasing solution corresponds to the singular solution. The turning point $\tau_0$ between the increasing and the oscillatory region, defined by $E = V(\tau_0)$, is

$$\tau_0 = \arcsinh \left( \sqrt{\frac{\ell^2 - 1/4}{\rho^2}} \right). \quad (4.40)$$

For $0 \leq \tau \leq \tau_0$, the solution is increasing, and for $\tau_0 \leq \tau < \infty$, it is oscillatory.

A second important advantage of the comparison with the Schrödinger equation is that there is a well-known scheme for approximating its solutions. This is known as the Wente-Kramers-Brillouin (WKB) approximation. For a derivation of this approximation, see e.g. [48]. The main result is the following approximate expression for the solutions

$$u_\ell^\prime(\tau) \propto \frac{\sin(\Theta(\tau))}{\left[ \rho^2 - \frac{\ell^2 - 1/4}{\sinh^2(\tau)} \right]^{1/4}}, \quad \tau > \tau_0, \quad (4.41)$$

where

$$\Theta(\tau) = \int_{\tau}^{\infty} d\tau' \sqrt{\rho^2 - \frac{\ell^2 - 1/4}{\sinh^2(\tau')}} \approx \rho \tau + \varphi_0. \quad (4.42)$$

The corresponding approximate expression for $T_\ell^\prime$ is

$$T_\ell^\prime(\tau) \propto \frac{\sin(\rho \tau + \phi_0)}{\left[ \rho^2 \sinh^2(\tau) - (\ell^2 - 1/4) \right]^{1/4}}, \quad \tau > \tau_0. \quad (4.43)$$

The condition for the WKB approximation to be valid is

$$\frac{1}{2\pi} \left| \frac{d\lambda}{d\tau} \right| \ll 1, \quad (4.44)$$

where

$$\lambda(\tau) = \frac{2\pi}{\left[ E - V(\tau) \right]^{1/2}}. \quad (4.45)$$

Inserting the expressions for the potential and the energy, we obtain

$$\lambda(\tau) = \frac{2\pi \sinh(\tau)}{\left[ \rho^2 \sinh^2(\tau) - (\ell^2 - 1/4) \right]^{1/2}}. \quad (4.46)$$

This function has a singularity at $\tau = \tau_0$, and around this point, the approximation cannot in general be trusted. However, with the further assumptions concerning
the eigenfunctions that we will make, the agreement is still quite good even in this region. This will be demonstrated when we have specified the expressions for the functions completely and can compare them to the exact solutions.

As the eigenfunctions are very small for \( \tau < \tau_0 \), we will approximate them to be zero in this region. However, they are not exactly zero, and it is assumed that there are no zeroes of their derivatives in this region. For \( \tau \geq \tau_0 \), we use the WKB approximation. Even though this is not a very good approximation close to \( \tau_0 \), it is the best approximation that we have. In order for the functions to be continuous at \( \tau = \tau_0 \), we have to choose \( \phi_0 \) so that \( \sin(\rho(\tau - \tau_0)) = 0 \), i.e., \( \phi_0 = -\rho \tau_0 \). The final expression that we will use for the eigenfunctions is

\[
T^\ell_\rho(\tau) = \begin{cases} 
\frac{\sin[\rho(\tau - \tau_0)]}{[\rho^2 \sinh^2(\tau) - (\ell^2 - 1/4)]^{1/4}}, & \tau \geq \tau_0, \\
0, & \tau < \tau_0
\end{cases}
\] (4.47)

In Fig. 4.2, this approximation is compared to the corresponding exact solutions for some values of the parameters \( \rho \) and \( \ell \). In Fig. 4.3, the function

\[
|T^\ell_\rho(\tau)|^2 \frac{\rho^2 \sin^2(\tau) - (\ell^2 - 1/4)}{[\rho^2 \sinh^2(\tau) - (\ell^2 - 1/4)]^{1/2}}
\] (4.48)

has been plotted in order make the comparison more clear. Because the functions are rapidly oscillating for large values of \( \rho \), the functions are plotted only for small values of this parameter. The agreement is quite good, especially after one period of oscillation of the sine. Observe that the agreement becomes better as the parameters \( \rho \) and \( \ell \) increase.

We can draw some conclusions regarding the spectrum by using the properties of these approximate eigenfunctions. In the region \( \tau < \tau_0 \), there are no zeroes of the derivative of \( T^\ell_\rho \). This means that any solution of Eq. (4.34) has to fulfill the condition

\[\rho^2 \sin^2(\tau_{\text{max}}) \geq \ell^2 - 1/4,\] (4.49)

giving some constraints on the spectrum.

### 4.4.5 Numerical analysis of the Kaluza-Klein spectrum

The lower part of the KK spectrum can be obtained numerically for some different sets of values for the parameters \( M_\star \) and \( v \). A numerical solution is easy to implement, using the expression (4.37). For this lower part of the spectrum, the eigenvalues are obtained as follows. For fixed \( \ell \), the difference in \( \rho \)-space between the first and second eigenvalue is approximately \( \Delta \rho = \pi/vL \). On the other hand, if the parameter \( \ell \) is increased, the difference between two eigenvalues is very small, increasing only approximately logarithmically. This means that in solving for the low eigenvalues, we only have to consider the lowest zeroes for each \( \ell \), since the higher ones will be much larger. This simplifies the numerical calculations, since all that needs to be done is to find the first zeroes of a set of functions depending
on a single parameter $\ell$. Also, for $\ell \neq 0$, each eigenvalue is double degenerate, since the eigenfunctions only depend on $|\ell|$. This means that we only need to consider $\ell \geq 0$. The results of the numerical solutions are presented in Fig. 4.4. We have also fitted a logarithmic function to these points. At this low end of the spectrum, the increase is very slow. For large $m$, we expect Weyl’s formula to hold. In this region, the spectrum is

$$m_n = \frac{v}{\sinh(vL/2)} \sqrt{n},$$

(4.50)

where $m_n$ is the $n$th eigenvalue, and the spectrum is sorted in increasing order. The numerical results that have been obtained show a much slower rate of increase than that described by this expression. However, this is only natural. Since there is a mass gap in the hyperbolic disc model, which is not reflected in Weyl’s formula, the actual increase of the spectrum has to be slow at the lower end of the spectrum, in order for the asymptotic formula to “catch up”. In fact, if the numerical solution is extrapolated to higher $n$, then it is almost constant in comparison to Weyl’s formula. This is demonstrated in Fig. 4.5. We will approximate the spectrum as

Figure 4.2. The approximate eigenfunctions (4.47) compared to the exact eigenfunctions (4.32). Upper left panel: $\rho = 2$, $\ell = 2$. Upper right panel: $\rho = 5$, $\ell = 2$. Lower left panel: $\rho = 2$, $\ell = 50$. Lower right panel: $\rho = 5$, $\ell = 50$. 

Chapter 4. Hyperbolic disc model
4.4. Kaluza-Klein spectrum

\[ |S_{rh}(r)|^2 \equiv |T_{rh}(r)|^2 \frac{v^2 \sinh^2(r) - (\ell^2 - 1/4)^{1/2}}{v^2} \]

for the approximate eigenfunctions (4.47) and the exact eigenfunctions (4.32). Upper left panel: \( \rho = 2, \ell = 2 \). Upper right panel: \( \rho = 5, \ell = 2 \). Lower left panel: \( \rho = 2, \ell = 50 \). Lower right panel: \( \rho = 5, \ell = 50 \).

The density of states is
\[ N(m) = \frac{\sinh^2(vL/2)}{v^2} m^2 \Theta(m - m_1), \]
where \( \Theta(x) \) is the Heaviside step function and \( m_1 = v/2 \). In reality, the transition between the slow increase in the lower limit and the increase according to Weyl’s formula is probably more smooth. However, in our approximation, the transition typically takes place for \( n \) at least of the order of \( 10^{20} \). Since it is not possible to
solve numerically for that many eigenvalues, there is not much that can be said about the transition between the two regions, so we have to resort to this simple approximation.

4.5 Interaction Lagrangian

In Sec. 3.7, we worked out the part of the action describing the coupling of the KK modes of the graviton to the SM fields. We now repeat this work for the hyperbolic disc model. The only difference from the ADD model is the geometry of the internal space, or in the four-dimensional picture, the KK modes and masses. Thus, the
interaction part of the action is

\[ S^{(\rho, \ell)}_{\text{int}} = \frac{1}{M^2} \int d^4x T^{\mu\nu}(x) \frac{1}{\|\psi_\rho\|} h^{(\rho, \ell)}_{\mu\nu}(x) \psi_\rho(y_0). \]  \hspace{1cm} (4.53)

In the ADD model, the normalization factor \(1/(2\pi R)^{d/2}\) could be combined with the factor \(M^{-d/2-1}\) to give an overall coupling factor \(\bar{M}_P^{-1}\). In the hyperbolic disc model, the normalization factor \(\|\psi_\rho\|\) is not that simple. In particular, it is not independent of the KK indices, and it is generally not possible to calculate it analytically. In Ch. 5, it will turn out to be useful to have the factor \(\bar{M}_P^{-1}\) explicitly displayed also for the hyperbolic disc model. To achieve this, we write

\[
\frac{1}{M^2\|\psi_\rho\|} = \frac{1}{M^2} \frac{1}{\|\psi_\rho\|} = \frac{1}{\bar{M}_P \|\psi_\rho\|} \frac{V^{1/2}}{V^{1/2}} \hspace{1cm} (4.54)
\]

and obtain

\[ S^{(\rho, \ell)}_{\text{int}} = \frac{c_\rho^{(\rho, \ell)}}{M_P} \int d^4x T^{\mu\nu}(x) h^{(\rho, \ell)}_{\mu\nu}(x), \hspace{1cm} (4.55) \]

where

\[ c_\rho^{(\rho, \ell)} \equiv \frac{\psi_\rho^{(\rho, \ell)}(y_0)V^{1/2}}{\|\psi_\rho\|}, \hspace{1cm} (4.56) \]

which is a dimensionless number characterizing the coupling strength.
Chapter 5

Collider phenomenology

In order to test the models that have been described, it is important to work out some of their experimental consequences. In principle, large extra dimensions could be tested by studies of the gravitational force at very small distances, searching for deviations from Newton’s inverse-square law. However, in practice it is hard to perform such experiments. The most precise experiments to date have only been able to probe gravity down to distances of 55 µm [32]. An alternative is to investigate experimental signatures of extra dimensions in particle colliders. In this setting, it turns out to be possible to obtain signals that are large enough to be detected in the next generation of colliders.

The two most important classes of particle colliders are lepton colliders (e.g., electron-positron colliders such as the LEP at CERN) and hadron colliders (such as the Tevatron at Fermilab). At present, a new proton-proton collider known as the Large Hadron Collider (LHC) is being constructed at CERN. It is scheduled to become operative in 2008. When completed, it will reach higher energies than any other collider, having a center-of-mass (CM) energy of 14 TeV. Also, there are plans on building a new lepton collider, known as the International Linear Collider (ILC). However, the ILC is not yet confirmed, and its completion lies far away in the future. Because of this, we will only consider proton-proton collisions, which are relevant for the LHC, in this thesis.

With the results on the KK spectra of the ADD and hyperbolic disc models at hand we may now turn to investigate the experimental signatures of these models. We begin with a brief review of collider phenomenology, with an aim towards LHC processes. This necessarily includes a review of the methods used to make calculations on hadron collisions, in particular the parton model. Next, we sketch the calculations of cross sections for the ADD model, starting from the interaction Lagrangian that was derived in Ch. 3. Then, we modify these results to the hyperbolic disc model. The cross sections that are calculated are the most important results of this thesis. Finally, we discuss background contributions of SM processes.
and the possibility to distinguish the experimental signatures from those of other beyond-the-SM processes, such as supersymmetry.

5.1 LHC physics

The LHC is a circular hadron collider. When completed, it will accelerate two proton beams to a CM energy of 14 TeV. The LHC includes a number of collision points and detectors. Most suited for searches of extra dimensions are the two general purpose detectors CMS and ATLAS. We will only need a few parameters of the LHC in this thesis, and hence, we will not discuss any further technical details.

5.1.1 Collider phenomenology

From a theoretical point of view, particle collisions are conveniently described in terms of cross sections. The cross section for a particular process is defined as the ratio of the rate of reactions and the rate of incoming particles per unit transverse area. Cross sections have the dimension length squared. They are usually measured in units of barns, where 1 b = $10^{-32}$ m$^2$, or in particle physics in terms of 1 fb = $10^{-15}$ b. A useful conversion factor is 1 GeV$^{-2}$ $\approx$ 3.9 mb. It is also possible to specify one or more of the properties for the final state particles, e.g. their direction. The resulting cross section is then an infinitesimal quantity, and one usually introduces the differential cross section, which is defined as the cross section per unit solid angle. In any real experiment, the differential cross section has to be integrated over some small angular region, corresponding to the resolution of the detector. By integrating the differential cross section over all angles, one obtains the total cross section.

An advantage of cross sections is that they do not depend on the particular setup of an experiment. This means that they can be used to compare results of different experiments.

For an introduction on the calculation of cross sections from fundamental principles, see e.g. Ref. [13]. The cross sections that are relevant for this thesis have been calculated in the literature for the ADD model and are easily modified to the hyperbolic disc model.

From an experimental point of view, what is actually measured by a detector is the number of observed events, e.g. detected photons. It is important to be able to translate cross sections to something which can be compared to these data. For this purpose, the most important parameter of an accelerator is its luminosity. The luminosity is defined as the rate of incoming particles per unit transverse area. The integrated luminosity is the total number of incoming particles per transverse area during some time interval, e.g. one year. Multiplying the cross section by the luminosity gives the reaction rate and multiplying by an integrated luminosity gives the total number of reactions during the corresponding time interval. This makes it easy to compare theoretical cross sections to actual experimental data.
5.1.2 Hadron collisions and the parton model

There are several complications related to hadron collisions, as compared to e.g. lepton collisions. In hadron collisions, the fundamental interactions involve quarks and gluons, and are described by QCD. Because of color confinement in QCD, quarks cannot exist as free particle states. As two or more quarks are separated, the attractive force between them increases, and finally becomes large enough that quark-antiquark pairs can be produced. The process continues in this way, and the quarks are combined into hadrons. In this way, the quarks are fragmented into hadrons, which appear as narrow jets in detectors.

The situation is further complicated by the fact that at low energies the strong coupling constant is large, so that perturbation theory is not valid. However, QCD is an asymptotically free theory, which means that the coupling constant decreases with increasing energy. Because of this, it is possible to use perturbation theory for high-energy processes. The energy limit separating the perturbative and non-perturbative regimes is the QCD scale, $\Lambda_{\text{QCD}} \approx 217$ MeV.

The strength of the strong coupling constant also means that for strong interactions vacuum fluctuations become important. In particular, when dealing with proton collisions, the proton cannot be modeled as a static object consisting of two up quarks and one down quark. Because of strong vacuum fluctuations, one has to take into account the effects of virtual states of quarks, and antiquarks. This is usually treated using the parton model.

In the parton model, a hadron is treated as a “gas” of approximately free so-called partons, which are to be identified with the quarks, antiquarks and gluons. This approximation is motivated for high-energy collisions which occur on time scales that are much smaller than the typical time scale for parton interactions. A review of the parton model and its relation to QCD is included in Ref. [49].

The distribution of partons in a given species of hadrons is described by the parton distribution functions (PDFs) of the hadron, which give the probability density for a parton to have a fraction $x$ of the total momentum $P$ of its parent hadron. The PDFs represent the non-perturbative aspects of QCD and can only be obtain numerically from experiments such as deep inelastic scattering processes [49]. Sticking strictly to the parton model, the PDFs are independent of the energy scale of the process, i.e., the transferred energy in the collision, which will be denoted by $Q$. However, taking higher-order QCD effects into account this is no longer true. The PDFs are then functions of this energy scale.

In the hadron collision processes that we will consider, two hadrons with opposite momenta $P$ and $-P$ (in the CM frame) collide. The interacting partons have fractions of the momenta of their parent hadrons equal to $x_1$ and $x_2$, respectively. Since the energy of a process in which the parton model is applicable is much larger than the masses of the partons, these are approximated as being massless. This means that the CM energy squared of the proton-proton system is $s = (P_1 + P_2)^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2 \approx 2P_1 \cdot P_2$. The corresponding quantity in the parton-parton system is $\hat{s} = (x_1 P_1 + x_2 P_2)^2 \approx x_1 x_2 s$. 

For our purposes, the most important result of the parton model is that a cross section for a high-energy hadron collision can be written as a convolution of a parton-level cross section with the PDFs of the hadron. The parton-level cross sections can be calculated in perturbative QCD, using Feynman diagrams. The non-perturbative effects are contained in the PDFs. In order to obtain the total cross section, the partons are summed over. The result for the cross section for a process \( A + B \to X \) is given by

\[
\sigma_{A+B\to X}(s) = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2 f_{a/A}(x_1, \hat{s}) f_{b/B}(x_2, \hat{s}) \hat{\sigma}_{a+b\to X}(\hat{s}),
\]

where \( \hat{s} = x_1x_2s \), \( \hat{\sigma}_{a+b\to X} \) is the partonic cross section for the subprocess \( a+b \to X \), and \( f_{a/A} \) is the PDF for the parton \( a \) in the hadron \( A \). The sums over \( a \) and \( b \) are taken over all quarks, antiquarks, and gluons.

From now on, the only hadron that will be considered is the proton. The intrinsic quark content of the proton are up and down quarks. Vacuum fluctuations also give rise to gluons, strange quarks, and up, down, and strange antiquarks. Since the charm, bottom, and top quarks are too heavy to be important in vacuum fluctuations, they are usually excluded in the parton model. To simplify the notation for the PDFs, we will suppress the hadron species, and name the PDFs after the parton species, i.e., \( u(x, Q) \equiv f_{u/p}(x, Q) \). One usually introduces the valence quarks \( u_v = u - \bar{u} \) and \( d_v = d - \bar{d} \), which represent the intrinsic quark content of the proton, and the sea quarks \( u_{sea} = \bar{u} \), \( d_{sea} = \bar{d} \), and \( s_{sea} = s = \bar{s} \), which together with the gluon represent the vacuum fluctuations.

At the time of writing of this thesis, the most recently published PDFs are the CTEQ6 sets [50]. However, in this thesis, we will use the leading order (LO) PDFs of Ref. [51], as these are parametrized in \( (x, Q) \)-space and are easy to work with. In Fig. 5.1, these PDFs have been plotted for a fixed value of \( Q = 100 \text{ GeV} \).

### 5.1.3 Collisions involving gravitons

In particle collision processes, the KK modes of the graviton could appear either as real particle states in the initial or final state, or as intermediate virtual states. In the context of the ADD model, the experimental implications have been studied extensively for both of these cases [31, 52–58]. The main result of this thesis is to produce corresponding results for the hyperbolic disc model which was considered in Ch. 4. We will only consider real KK state production.

Since the KK modes have couplings to the SM fields that are suppressed by \( M_P^{-1} \), processes involving production of individual KK modes have cross sections which are completely negligible compared to the background [31]. This means that there is no possibility to study interactions with individual KK modes experimentally. However, it is possible to study the production of any KK mode in some process, e.g., \( e^- + e^+ \to \gamma + G \). In the higher-dimensional picture, this corresponds to the production of a graviton with any momentum. At high energies, a large number
of KK modes are kinematically available. Because of this, it is possible to obtain signals that are large enough to be observable.

Due to the small coupling of the produced graviton, it will escape detection. Since it carries away energy and momentum, it will appear that the process violates conservation of these quantities. This is denoted by the symbol $E$, i.e., the reaction is written as $A + B \rightarrow X + E$. Missing energy in itself is not a measurable quantity, and therefore, it is useful to consider reactions in which the graviton is produced together with some other particles which can be measured, e.g., a photon or a hadronic jet. These particles can be detected and their momenta and directions can be measured. At the LHC, there are two relevant such reactions, $p + p \rightarrow \text{jet} + G$ and $p + p \rightarrow \gamma + G$. The reaction $p + p \rightarrow Z^0 + G$ is also possible, but does not give a sufficiently strong signal, since the $Z$ boson can only be observed through its leptonic decay at the LHC.

An important question to consider is whether the production of a single KK mode is even allowed. Naively, such a reaction seems to violate conservation of momentum along the extra dimensions. However, this conservation law follows from the translational invariance of space, and along the extra dimensions, translational invariance is broken by the presence of the brane. In a more refined model, where the dynamics of the brane are taken into account, the brane would recoil as a KK mode is emitted, and in this way, conservation of momentum is insured. Hence, emission of a single KK mode is indeed allowed.
Chapter 5. Collider phenomenology

5.2 The ADD model

We now turn to the calculation of cross sections for models of large extra dimensions. We begin by studying the ADD model. The results that we obtain are easily modified to the hyperbolic disc model. We give an outline of the procedure for the case of photon plus jet production, following the work of Ref. [31], where more detailed derivations can be found. For the case of graviton plus jet production, only the results are given.

5.2.1 Graviton plus photon production

In Ch. 3, the interaction part, \( S_{\text{int}} \), of the action for the ADD model was derived. From this, the relevant Feynman rules involving gravitons can be obtained. However, we first need the energy-momentum tensor for QED. It is given by the definition (3.4), applied to the Lagrangian (3.17). The result is

\[
T_{\mu\nu} = \frac{i}{4} \bar{\psi} (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) \psi - \frac{i}{4} (\partial_\mu \bar{\psi} \gamma_\nu + \partial_\nu \bar{\psi} \gamma_\mu) \psi + \frac{1}{2} e Q \bar{\psi} (\gamma_\mu A_\nu + \gamma_\nu A_\mu) \psi + F_{\mu\lambda} F^{\lambda\nu} + \frac{1}{4} \eta_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}.
\]

(5.2)

The four-dimensional propagator for a graviton of mass \( m \) is

\[
\frac{iP_{\mu\nu\alpha\beta}}{k^2 - m^2},
\]

(5.4)

where

\[
P_{\mu\nu\alpha\beta} = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta})
- \frac{1}{2m^2} (\eta_{\mu\alpha} k_\nu k_\beta + \eta_{\nu\beta} k_\mu k_\alpha + \eta_{\mu\beta} k_\nu k_\alpha + \eta_{\nu\alpha} k_\mu k_\beta)
+ \frac{1}{6} \left( \eta_{\mu\nu} + \frac{2}{m^2} k_\mu k_\nu \right) \left( \eta_{\alpha\beta} + \frac{2}{m^2} k_\alpha k_\beta \right).
\]

(5.5)

The spin-sum rule for the polarization tensors of the graviton is

\[
\sum_s \epsilon_{\mu\nu}(k, s) \epsilon_{\alpha\beta}(k, s) = P_{\mu\nu\alpha\beta}(k).
\]

(5.6)

The vertices that are relevant for graviton plus photon production are of the forms

\[\text{Diagram}
\]

\[\text{Diagram}
\]

\[\text{Diagram}
\]
5.2. The ADD model

The graviton has been represented by a zigzagged line. Notice that these are a fermion propagator, a photon propagator, and the fundamental QED vertex, modified by the addition of a graviton. The vertex factor for the first diagram is

\[
- \frac{1}{4M_{Pl}} [(k_1 + k_2)_\mu \gamma_\nu + (k_1 + k_2)_\nu \gamma_\mu,]
\]

where the \( k_i \) are the momenta of the fermions. The second factor is

\[
- \frac{i}{M_{Pl}} (W_{\mu\alpha\beta} + W_{\nu\mu\alpha\beta}),
\]

where

\[
W_{\mu\nu\alpha\beta} = \frac{1}{2} \eta_{\mu\nu}(k_1^\beta k_2^\alpha - k_1 \cdot k_2 \eta_{\alpha\beta}) + \eta_{\nu\alpha}(k_1^\beta k_2^\nu - k_1 \cdot k_2 \eta_{\nu\alpha}) - \eta_{\alpha\beta} k_1^\mu k_2^\nu + \eta_{\mu\alpha}(k_1 \cdot k_2 \eta_{\nu\beta} - k_1^\beta k_2^\nu) - \eta_{\mu\beta} k_1^\nu k_2^\alpha.
\]

The third one is

\[
- \frac{i}{2M_{Pl}} eQ(\gamma_\mu \eta_{\nu\alpha} + \gamma_\nu \eta_{\mu\alpha}).
\]

The subprocess giving rise to photon plus graviton production is \( q + \bar{q} \rightarrow \gamma + G \). To order \( \alpha_s/M_{Pl} \), there are four diagrams contributing to this process. They are of the forms

\[
\begin{align*}
\text{(a)} & \quad \text{(b)} & \quad \text{(c)} & \quad \text{(d)}
\end{align*}
\]

From these diagrams, the individual differential cross sections can be calculated. This has been performed in Ref. [31]. The result, which is averaged over initial states and summed over final states with respect to color and spin, is

\[
\frac{d\sigma_m}{dt}(q + \bar{q} \rightarrow \gamma + G) = \frac{\alpha Q^2}{48} \frac{1}{sM_{Pl}^2} F_1(t/s, m^2/s),
\]

where the function \( F_1 \) is defined as

\[
F_1(x, y) = \frac{1}{x(y - 1 - x)} \left[ -4x(1 + x)(1 + 2x + 2x^2) ight. \\
+ y(1 + 6x + 18x^2 + 16x^3) - 6y^2x(1 + 2x) \\
+ y^3(1 + 4x) \left. \right].
\]
5.2.2 Graviton plus jet production

The treatment of graviton plus jet production is closely analogous to graviton plus photon production. For this case, there are additional vertices. These are essentially the conventional QCD propagators and vertices, modified by the addition of gravitons. The Feynman rules for these vertices are given in Ref. [31].

There are three partonic subprocesses giving rise to graviton plus jet production. These are

\[ q + g \rightarrow q + G, \quad q + \bar{q} \rightarrow g + G, \quad \text{and} \quad g + g \rightarrow g + G. \]

The cross sections for these subprocesses are

\[
\begin{align*}
\frac{d\sigma_m}{dt}(q + \bar{q} \rightarrow q + G) &= \alpha_s \frac{1}{36} s M_{Pl}^2 F_1(t/s, m^2/s), \\
\frac{d\sigma_m}{dt}(q + g \rightarrow q + G) &= \alpha_s \frac{1}{96} s M_{Pl}^2 F_2(t/s, m^2/s), \\
\frac{d\sigma_m}{dt}(g + g \rightarrow g + G) &= \frac{3\alpha_s}{16} s M_{Pl}^2 F_3(t/s, m^2/s).
\end{align*}
\]

The function \( F_1 \) is given by Eq. (5.12), while \( F_2 \) and \( F_3 \) are defined as

\[
\begin{align*}
F_2(x, y) &= \frac{1}{x(y - 1 - x)} \left[ -4x(1 + x^2) + y(1 + x)(1 + 8x + x^2) \\
&\quad - 3y^2(1 + 4x + x^2) + 4y^3(1 + x) - 2y^4 \right], \\
F_3(x, y) &= \frac{1}{x(y - 1 - x)} \left[ 1 + 2x + 3x^2 + 2x^3 + x^4 \\
&\quad - 2y(1 + x^3) + 3y^2(1 + x^2) - 2y^3(1 + x) + y^4 \right].
\end{align*}
\]

5.2.3 Summation over the Kaluza-Klein spectrum

The cross section for production of any available KK mode of the graviton is obtained by summing over the individual cross sections, up to some maximum mass \( m_{\text{max}} \), which is decided by the available collision energy. At the LHC, \( m_{\text{max}} \) is the CM energy of 14 TeV. Since production of different KK modes result in physically distinct final states, the summation is to be performed on the cross section level, rather than on the amplitude level. Because the mass splitting \( \Delta m = |\bar{n}|/R \) between the KK modes is much smaller than any other physical dimension in the context, this sum can be approximated by an integral

\[
\sum_m \frac{d\sigma_m}{dt} \approx \int dm n(m) \frac{d\sigma_m}{dt},
\]

where \( n(m) \) is the density of states. It is given by

\[
n(m) = S_{d-1} \frac{M_{Pl}^2}{M_2^{d+2} m^{d-1}},
\]

\( M_2 \) is the mass of the graviton.
5.2. The ADD model

where \( S_{d-1} \) is the area of the surface of the unit sphere in \( d \) dimensions. We obtain the result

\[
\frac{d^2 \sigma}{dt \, dm} = n(m) \frac{d\sigma_m}{dt} = S_{d-1} \frac{M^2_{Pl}}{M^{2d}_{*}} m^{d-1} \frac{d\sigma_m}{dt}.
\] (5.21)

5.2.4 Total cross section

We now have all the pieces needed to calculate the cross sections for production of a graviton together with a photon or a hadronic jet. However, it is advantageous to first rewrite the differential cross section (5.21) in terms of physically observable variables. We will only consider the case of photon production in some detail. We choose the set \((x_\gamma, \cos(\theta))\), where \(x_\gamma \equiv 2E_\gamma/\sqrt{s}\) and \(\theta\) is the angle between the outgoing photon and the beam direction in the CM frame. These are related to the variables \((t, m)\) by the equations

\[
t = \frac{s x_\gamma^2}{2} [\cos(\theta) - 1],
\]

\[
m = \sqrt{s(1 - x_\gamma)}.
\] (5.22)

The Jacobian for the change of variables is \(s^{3/2}x_\gamma(1 - x_\gamma)^{-1/2}/4\). Expressing the differential cross section for photon plus graviton production in terms of these variables, we obtain

\[
\frac{d^2 \sigma}{dx_\gamma \, d\cos \theta} = \frac{\alpha}{64} S_{d-1} \left( \frac{\sqrt{s}}{M_*} \right)^{d+2} \frac{1}{8} f(x_\gamma, \cos(\theta)),
\] (5.24)

where

\[
f(x, y) = \frac{2(1 - x)^{d/2 - 1}}{x(1 - y^2)^2} \left[ (2 - x)^2(1 - x + x^2) - 3y^2x^2(1 - x) - y^4x^4 \right].
\] (5.25)

An immediate consequence of this result is that the cross section diverges as \(x_\gamma \to 0\) or \(\theta \to 0, \pi\). This is known as a collinear divergence and arises because of the massless photon in the final state. The same divergence appears in the case of jet production, in the approximation that the jet is massless. To avoid the divergence, we need to impose a minimum transverse momentum of the photon or jet, i.e., we consider only photons and jets that satisfy the condition \(p_T \equiv p \sin \theta \geq p_{T\text{min}}\).

Because of limitations of the detector, we also have to impose a maximal allowed value for the longitudinal rapidity of the outgoing photons and jets. Following Ref. [31], we choose \(|\eta_\gamma| < 2.5\) and \(|\eta_{\text{jet}}| < 3.0\).

Since the colliding hadrons are equally massive, the CM frame for the hadrons coincides with the lab frame. However, when considering the partonic subprocesses, this is not true, since the two partons in general carry different fractions \(x_i\) of the momenta of their parent hadrons. In order to perform calculations in the CM system, we have to perform a boost along the beam direction. In the lab system,
the total energy of the parton-parton system is \( P(x_1 + x_2) \) and the total momentum is \( P(x_1 - x_2) \), where \( 2P \) is the total energy of the proton-proton system. From these expressions, the rapidity, \( \Delta \eta \), of this boost is obtained by using the equation

\[
- \sinh(\Delta \eta) E_{\text{lab}} + \cosh(\Delta \eta) p_{\text{lab}} = p_{\text{CM}} = 0,
\]

which implies that

\[
\Delta \eta = \text{artanh} \left( \frac{p_{\text{lab}}}{E_{\text{lab}}} \right) = \text{artanh} \left( \frac{x_1 - x_2}{x_1 + x_2} \right).
\]

Hence, it is important to consider the transformation properties of cross sections under Lorentz transformations. Total cross sections are invariant under Lorentz transformations along the beam direction [13]. However, since we imposed cuts on the integrations, what we consider are not true total cross sections. The point is that the limits of the integration may not be Lorentz invariant.

The momentum transverse to the beam direction is invariant under boosts along the beam direction, so this causes no problems. However, the longitudinal rapidity transforms as \( \eta \to \eta + \Delta \eta \). Hence, the limit \( |\eta| < \eta_{\text{max}} \), which is to be imposed in the lab frame, is expressed in CM frame as \( -\eta_{\text{max}} + \Delta \eta < \eta < \eta_{\text{max}} + \Delta \eta \).

Finally, we need to remember that we are dealing with an effective theory, which is only valid for energies smaller than the fundamental mass scale, \( M_* \). This means that we should not trust any cross section results, where the effective CM energy is larger than \( M_* \). In a lepton collider, where the CM energy is fixed, any problems can be avoided by only considering reactions with \( \sqrt{s} < M_*^2 \). However, in a hadron collider, the CM energy for the fundamental interactions is not constant, which makes it harder to analyze the validity of the results. In Ref. [31], this has been solved by plotting two cross section—the ordinary one and one which has been put to zero whenever \( \hat{s} > M_*^2 \). In the regions where these two cross sections give essentially the same result, most of the contribution to the cross section has to come from collisions with \( \hat{s} < M_*^2 \), and in these regions the results can be trusted. With “essentially the same”, it is meant that the full cross sections are no more than 50% larger than the cross section which is calculated with the cut.

5.3 The hyperbolic disc model

The cross sections for the ADD model are easily modified to the hyperbolic disc model. The possible differences between the two models are most easily analyzed in the four-dimensional picture. From that point of view, we are dealing with a tower of KK states of different masses that couple to the SM fields. The couplings of individual KK modes are completely determined by the interaction parts of the action, Eq. (4.55). The structure of these are mathematically the same as for the ADD model, Eq. (3.42). This means that we obtain the same kinds of couplings between the KK modes and the SM fields. The only possible difference lies in the coupling constants.
The information on the coupling strength is contained in the constants $c_\rho$, Eq. (4.56). Since these are constant, they only contribute with an overall factor to any vertex involving gravitons. The cross sections involve the magnitude of the amplitude squared, which means that these constants will affect the individual cross sections by a constant overall factor $|c_\rho|^2$, assuming that we only consider diagrams with exactly one graviton vertex, which is the case.

The second possible modification lies in the distribution of masses appearing in the models, i.e., the KK spectrum. This was worked out in Sec. 4.4.5.

In the ADD model, we used the fact that the mass splittings between the KK modes are small, so that the sum over the spectrum could be approximated by an integral. In the hyperbolic disc model, the mass splittings $\Delta m \lesssim v/\sinh(vL/2)$ are typically even smaller, and the same approach can be used. The number of KK modes of mass less than or equal to $m$ is given by

$$N(m) = \frac{\sinh^2(vL/2)}{v^2} m^2 \Theta(m - m_1) = \pi \frac{M_{Pl}^2}{M^4} m^2 \Theta(m - m_1),$$

where we have used Eq. (4.10). This gives the density of states

$$n(m) = \frac{dN}{dm} = \pi \frac{M_{Pl}^2}{M^4} [2m \Theta(m - m_1) + m^2 \delta(m - m_1)]$$

and the differential cross section

$$\frac{d^2\sigma}{dt\,dm} = \pi \frac{M_{Pl}^2}{M^4} [2m \Theta(m - m_1) + m^2 \delta(m - m_1)] \frac{d\sigma_m}{dt}.$$

The final expression for the partonic differential cross section for photon plus graviton production in the hyperbolic disc model is

$$\frac{d^2\sigma}{dt\,dm} = \pi \frac{|c|^2}{M^4} [2m \Theta(m - m_1) + m^2 \delta(m - m_1)] \frac{\alpha Q^2}{48 s} F_1(t/s, m^2/s).$$

In the ADD model, the coupling strength is independent of KK indices. This is not the case in the hyperbolic disc model. In fact, the coupling strength is rapidly varying, because of the factor $\sin[\rho(\tau - \tau_0)]$ in Eq. (4.43). The coupling constants, considered as functions of $\rho$, oscillate with a period $\Delta \rho \approx \pi/\tau_b$, where $\tau_b$ is the position of the brane in the radial direction.

This leads to some technical problems. When integrating over the KK spectrum, we only need the density of states as a function of $m$. However, the coefficients $|c_\rho|^2$ are now included in the integrand, and these are only available as functions of $\rho$ and $\ell$, not of $m$.

Another problem arises since the differential cross section (5.31) has to be integrated numerically in order to obtain the total cross section. Because of the rapid variation of the coupling constants, this numerical integration has to be performed with a very high accuracy in order to give reliable results. This can be handled...
by replacing the rapidly varying function by a smooth function that is obtained by averaging over the rapid variations, reflecting only variations on larger scales. This approximation is valid if the size of the variations that are averaged over is small compared to the general variation of the integrand.

This means that we need to average over $\ell$ in the coefficients $|c_{\ell}\rho|^2$. We can then use the relation (4.28) to translate this result into an averaged value over $m$. We will denote the averaged squared coupling constants as functions of $m$ by $|c_m|^2$. We will simply refer to these as the coupling constants. The averaging has been performed numerically, and the results are presented in Fig. 5.2. The high peaks are due to numerical uncertainties and can be reduced by increasing the numerical precision. However, the calculations then become more time consuming, and here, we have only plotted the coupling constants with a lower precision in order to demonstrate their general behavior.

![Figure 5.2. The mean value of the coupling constants as a function of $m$. The radial position of the brane is chosen to be $\tau_b = \tau_{\text{max}}/2$. The high peaks are due to numerical uncertainties.](image)

It turns out that the average coupling as a function of $m$ is essentially constant. This can be explained as follows: The way in which the coupling varies as a function of $\ell$ does not matter, since different values of $\ell$ are essentially uncorrelated to different values of $m$. The oscillatory behavior with $\rho$ is averaged over and does not matter either. What is left is the $\rho$-dependence of $\tau_0$, the point separating the zero region of the eigenfunctions from the oscillatory region. However, what is important is the fraction of eigenmodes at a given value of $\rho$ that are non-zero at $\tau = \tau_b$. This fraction is essentially given by the ratio of the number of allowed values for the parameter $\ell$ that give non-zero coupling constants and the total number of allowed values for $\ell$. It is given by

$$\frac{\rho^2 \sinh^2(\tau_{\text{max}}) + 1/4}{\rho^2 \sinh^2(\tau_b) + 1/4} \approx \frac{\sinh^2(\tau_{\text{max}})}{\sinh^2(\tau_b)},$$

which is independent of $\rho$. 

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The mean value of the couplings also depend on the position of the brane on the disc, \( \tau_b \), as a parameter. In general, one might expect the mean value to vary significantly with \( \tau_b \). However, this variation is quite smooth. Consider the coupling for some fixed \( \rho \). A measure of the number of non-zero couplings is given by \( \sqrt{\rho^2 \sinh^2(\tau_b) + 1/4} \approx \rho \sinh(\tau_b) \). This number goes as \( \sinh(\tau_b) \). However, the individual coupling constants decrease with \( \tau_b \) as \( \left[ \rho^2 \sinh^2(\tau_b) - (\ell^2 - 1/4) \right]^{-1/2} \approx \rho \sinh^{-1}(\tau_b) \propto \sinh^{-1}(\tau_b) \), which means that these the effects cancel each other.

The results are presented in Fig. 5.3. In order to simplify comparison between the different sets of parameters, the coupling constants have been plotted against the normalized radial coordinate \( \tau/\tau_{\text{max}} \). The variation of the coupling constant with \( \tau_b \) is small, except close to the boundary at \( \tau = \tau_{\text{max}} \), where there is a rapid increase. This can be explained heuristically as follows. For small \( \tau_b \), the boundary of the disc lies far away from the brane and does not cause very much variation of the coupling constants. However, as the radial coordinate approaches \( \tau_{\text{max}} \), the brane comes closer to the boundary, which then becomes more important and causes a larger variation. Hence, the variation displayed in Fig. 5.3 is what could be expected. Notice also that the coupling constants behave in a very similar way for each of the choices of the parameters \( v \) and \( M_* \).

![Figure 5.3](image_url)

**Figure 5.3.** The mean value of the coupling constant as a function of the radial position \( \tau_b \) of the brane. Note that the coupling constants are plotted against the normalized radial coordinate \( \tau/\tau_{\text{max}} \) in order to simplify the comparison among the different sets of parameters. The high peaks are due to numerical uncertainties.

### 5.4 Analysis of background contributions

The experimental signature that is observed in a detector is a photon or a hadronic jet with anomalous energy. However, such signatures could also arise from SM
processes, which then give rise to background contributions disturbing the signal. According to Ref. [31], the largest such background contribution comes from processes with a photon or jet together with a $Z$ boson in the final state, with the $Z$ boson decaying into neutrinos, which are not observed by the detector. In order to analyze the possibility of making any relevant observations at the LHC, it is important to consider these background contributions and how to minimize their effects. However, this is highly problematic in the hyperbolic disc model. The reason for this is that the overall factor coming from the coupling constants is not known, and varies with all parameters. This means that the possibility to analyze the signal as compared to the background is highly model dependent. Because of this, we will not perform any detailed analysis in this thesis.

In principle, other beyond-the-SM physics could give rise to experimental signatures similar to those that we have worked out. An important task of the LHC is to restrict the different possibilities of such models. Hence, it is important to be able to distinguish between signals from different models, e.g. the present model and other models of extra dimensions or models of supersymmetry. This has also been mentioned in Ref. [31], and the analysis is perfectly applicable to the hyperbolic disc model, since the same kinds of reactions are considered. As an example, we mention that in supersymmetry, the important reactions involve missing energy and two hadronic jets. In this way, such signatures can be distinguished from signatures coming from extra dimensions.

### 5.5 Results for the hyperbolic disc model

We now give the main results of this thesis, in the form of cross sections. We have chosen to plot the cross sections against different variables that could be varied in an experimental setting, at least in principle.

The coupling constants contribute to each cross section with an overall factor, depending on the parameters. As can be observed from Fig. 5.3, this constant could vary with several orders of magnitude for different choices of the position of the brane, $\tau_b$. Hence, the question of what this position could be is an important one. The most symmetric position of the brane would be at $\tau = 0$. In this case, all the coupling constants vanish except those for which $\ell = 0$. Since the largest allowed value for $\ell$ is at least of the order of $10^{24}$ for the choices of parameters that we consider, the $\ell = 0$ modes constitute only a negligibly small fraction of the KK modes. Hence, we conclude that if the position of the brane is at $\tau = 0$, then there is no possibility to obtain a measurable signal. For definiteness, we have chosen $\tau_b = \tau_{\text{max}}/2$ for each choice of the parameters $v$ and $M_\ast$.

Naively, it would seem to be illustrative to demonstrate one plot of the cross sections for each allowed order of magnitude of the parameter $v$. However, for $v$ smaller than about an order of magnitude less than the fundamental mass scale $M_\ast$, there is practically no difference in the cross sections between different $v$. The reason for this is that if $v$ is much smaller than the relevant energy scales, which
are here of the same order of magnitude as $M_*$, then the mass gap $v/2$ becomes negligible in the cross sections, which are summed over all the KK states. Also, the coupling constants do not vary very much with $v$. Because of this, we have chosen to plot the cross sections for the minimum value $v = 100$ MeV, the maximum value $v = M_*$, and also for the value $v = M_*/2$. All cross sections that we have considered decrease with increasing $v$.

For each of the cross sections, we have also plotted the corresponding cross section for the two-dimensional ADD model, with the same fundamental mass scale. Except for the factor coming from the coupling constants, the cross sections of the hyperbolic disc model essentially reduce to the ADD cross sections for relatively small $v$. For the position of the brane at $\tau = \tau_{\text{max}}/2$, the coupling constants are typically of the order of 0.05, meaning that the hyperbolic disc model cross sections are smaller than the corresponding ones of the ADD model, or even smaller for larger $v$. However, as we have mentioned, the coupling constants vary significantly with the position of the brane, and it is possible to obtain cross sections larger than in the ADD model for other choices of $\tau_b$. For this reason, it is hard to compare the absolute magnitudes of the cross sections between the models. However, the shapes of the cross sections are independent of the coupling constants and can be compared.

As mentioned earlier, for small $v$, the cross sections have the same shape as in the ADD model. Since the cross sections of the ADD model have the fundamental mass scale dependence $\sigma \propto M_*^{-4}$, the overall factor of these cross sections can be varied by varying $M_*$. Hence, for small $v$, given any set of parameters in the hyperbolic disc model, the same cross sections could be obtained for the ADD model for some choice of $M_*$. This is a problem, since it makes it hard to distinguish the signals of these two models.

The general difference between jet and photon production is that jet production gives a stronger signal. This result has been obtained for the ADD model in Ref. [31]. However, this is not true for the differential cross section in Fig. 5.6. This could depend on the different choices of parameters. Note that although the cross sections are presented in pairs, the scales differ between the two figures. Note also that the two figures of each pair are presented for different sets of parameters.

Figure 5.4 shows the total cross section for both graviton plus jet production and graviton plus photon production in the hyperbolic disc model as a function of the transverse momentum cutoff $E^{\text{min}}_T$. The relevant parameters have been given the same values as in Ref. [31]. Since a higher momentum cutoff means that less data is taken into consideration, the cross sections decrease with increasing $E^{\text{min}}_T$.

Figure 5.5 shows the total cross sections as functions of $\sqrt{s}$. Since more KK modes are kinematically available at higher $\sqrt{s}$, the cross section is increasing. Note that because of the variable mass of the graviton, there is no peak at the threshold energy $v/2$. Since the total CM energy of the LHC is completely fixed at 14 TeV, it is not possible to compare this result to experimental data from the LHC alone. However, it could still give some information if data from different sources at different CM energies are compared.
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Figure 5.4. The total cross sections as functions of the transverse momentum cutoff. Left panel: graviton plus jet production, $M_* = 5$ TeV. Right panel: graviton plus photon production, $M_* = 2$ TeV.

Figure 5.5. The total cross sections as functions of the fundamental mass scale. Left panel: graviton plus jet production, $M_* = 5$ TeV, $E_{\text{T},\gamma}^{\text{min}} = 1$ TeV. Right panel: graviton plus photon production, $M_* = 2$ TeV, $E_{\text{T},\gamma}^{\text{min}} = 400$ GeV

In Fig. 5.6, we have plotted the differential cross sections with respect to $\cos(\theta)$. For this plot, we have chosen not to include the result for the ADD model, since this is much larger than the hyperbolic disc model cross sections and would make the plot hard to read. The reason for choosing $\cos(\theta)$ instead of e.g. the transverse momentum is that the angular distribution of outgoing particles is easy to obtain with good accuracy in the detector, as it only depends on geometry. Measurements on momentum, on the other hand, require calibration of the calorimeters and are harder to perform. Since the experimental setup is completely symmetric with respect to the two directions of the incoming beams, the differential cross section is a function only of $|\cos(\theta)|$.

The differential cross section for jet production starts decreasing for large values of $|\cos(\theta)|$. The reason for this is the cutoff on the transverse momentum the we
impose on the outgoing beam. The same kind of behavior can be observed for photon production for other choices of parameters. Apart from this, the differential cross sections increases with $|\cos(\theta)|$. This is what should be expected, since the differential cross section diverges for $|\cos(\theta)| \to 1$ if no cutoff on the transverse momentum is imposed.

Figure 5.6. The differential cross sections with respect to $\cos(\theta)$. Left panel: graviton plus jet production, $M_* = 5$ TeV, $E_T^{min} = 1$ TeV. Right panel: graviton plus photon production, $M_* = 2$ TeV, $p_T^{min} = 400$ GeV.
Chapter 6

Summary and conclusions

In this thesis, we have studied the subject of extra dimensions in physics. In particular, we have paid attention to the class of models known as large extra dimensions, with the ADD model as the canonical example. The novel results of the thesis are phenomenological results for a model with a two-dimensional hyperbolic disc as internal space.

In Ch. 1, we gave a brief introduction to theoretical physics, particle physics, and extra dimensions. In Ch. 2, we described the SM of particle physics and introduced extra dimensions in the setting of UEDs. We gave an introduction to KK decomposition and how to relate the fundamental higher-dimensional theory to the low-energy four-dimensional picture. In Ch. 3, we introduced large extra dimensions through the ADD model. We studied general relativity in four dimensions and in the setting of extra dimensions, and also an effective quantum theory of gravity. Then, we considered the KK decomposition of the higher-dimensional graviton, and the couplings between the KK modes and the SM fields. We also discussed some problems related to the ADD model. In Ch. 4, we introduced the hyperbolic disc model. We considered a number of exact and approximate properties of the KK modes and spectrum of this model, as well as the couplings to the SM fields. Finally, in Ch. 5, we considered the collider phenomenology of extra dimensions, starting with a general introduction to particle collisions relevant for the LHC. Then, we briefly outlined the derivation of the cross sections for the ADD model. We considered production of a real graviton together with a photon or a hadronic jet. These results were modified to the hyperbolic disc model. Finally, the background contributions for the processes were considered, and the results for the cross sections in the hyperbolic disc model were given.

The main result of the thesis is to provide concrete cross sections for the hyperbolic disc model. Since it is not possible to obtain the KK spectrum analytically, some approximations have been made, both for the spectrum and for the KK modes. To the knowledge of the author, no previous attempts have been made to calculate cross sections for models with hyperbolic extra dimensions.
Chapter 6. Summary and conclusions

There are two main motivations for studying hyperbolic geometry in the context of extra dimensions. First, since the volume of a hyperbolic manifold grows exponentially with its radius, it is possible to obtain a large volume, and hence, a small fundamental mass scale, from a small radius. This means that the problem of the hierarchy between the fundamental mass scale and the radius of the internal space that the ADD model suffers from can be avoided. Second, for a hyperbolic geometry, there is a mass gap between the zero mode and the first massive mode in the KK spectrum. This means that the constraints coming from e.g. astroparticle physics that are important for the ADD model can be completely avoided.

The problems of the hyperbolic disc model are mainly phenomenological ones. In the ADD model, there is only one free parameter, the fundamental mass scale. In the hyperbolic disc model, on the other hand, we also need to consider the curvature of the internal space and the position of the brane in the radial direction.

Since all cross sections are multiplied by an overall factor depending on the parameters, it is hard to make any definite predictions in this model. To separate the signatures of the model from e.g. the ADD model, one could study the principal shapes of the different cross sections. However, if the curvature is smaller than the fundamental mass scale by more than about an order of magnitude, these two models give predictions which are so similar that they are inseparable in any experimental setting.

These two facts mean that the phenomenology of the hyperbolic disc model is highly model dependent. It is thus an important task to try to restrict the allowed region of the parameter space further.

On the theoretical side, a shortcoming of the hyperbolic disc model is that we have not been able to find a physical motivation for the energy-momentum tensor (4.9). However, this does not mean that there is no such motivation. This has been discussed in the similar setting of a closed hyperbolic manifold in Ref. [39].

For a continuation of this work, a number of improvements could be made. The approximations regarding the spectrum and modes could be refined. With more computational power and more refined algorithms, it might be possible to work with the exact eigenmodes, instead of the approximate ones that have been employed in this thesis. Another improvement would be to incorporate more recent parton distribution functions, e.g. the CTEQ6 sets [50].

An interesting area of study would be to search for other reactions that could be studied at the LHC. Another possibility is to consider indirect signals with the graviton appearing as a virtual intermediate state. However, in both of these cases, the phenomenological problems of the reactions considered in this thesis would probably be present.

It would also be interesting to study closed hyperbolic internal spaces in more detail. For such spaces, the metric perturbation satisfies periodic boundary conditions, which in some sense are more natural than those of the disc model, since they immediately follow from the geometry and do not have to be put in by hand. However, such spaces are far more complicated to handle, both mathematically and computationally. In Ref. [39], closed hyperbolic extra dimensions are discussed.
from a more theoretical point of view. In Ref. [59], numerical studies of the spectra of closed hyperbolic manifolds are discussed.

Perhaps the most interesting continuation of the work that has been performed in this thesis would be to look for properties of some experimental signatures that could distinguish the hyperbolic disc model from the ADD model. One difference between the models is the mass gap in the hyperbolic disc model, which is not present in the ADD model. However, there are two problems which restrict the usefulness of this mass gap in distinguishing between the models. First, since the mass of the produced graviton is continuous, there is no peak in the cross section at the threshold energy. The cross section starts out very small, and hence, it is hard to make any measurements in this energy region. Second, to consider this mass gap, one would need to study the cross sections as functions of the CM energy. This is not possible at the LHC, where the CM energy is fixed at 14 TeV.
Bibliography


