Stabilized Max-Min Flow Control Using PID and PII² Controllers* 

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SUMMARY This paper describes an analytical framework for the weighted max-min flow control of elastic flows in packet networks using PID and PII² controller when flows experience heterogeneous round-trip delays. Our algorithms are scalable in that routers do not need to store any per-flow information of each flow and they use simple first come first serve (FCFS) discipline, stable in that the stability is proven rigorously when there are flows with heterogeneous round-trip delays. We first suggest two closed-loop system models that approximate our flow control algorithms in continuous-time domain where the purpose of the first algorithm is to achieve the target queue length and that of the second is to achieve the target utilization. The slow convergence [1] of many rate-based flow control algorithms, which use queue lengths as input signals, can be resolved by the second algorithm. Based on these models, we find the conditions for controller gains that stabilize closed-loop systems when round-trip delays are equal and extend this result to the case of heterogeneous round-trip delays with the help of Zero exclusion theorem. We simulate our algorithms with optimal gain sets for various configurations including a multiple bottleneck network to verify the usefulness and extensibility of our algorithms.

key words: delayed systems, control theory, flow control

1. Introduction

Recently many efforts have been devoted to provide a framework for designing best-effort service networks that can offer low-loss, low-delay data services where flow control plays a major role in controlling congestion as well as allocating bandwidth among users by enforcing users to adjust their transmission rate in a certain way in response to congestion in their path. The potential advantages of such networks would be the ability to offer even real-time services without the need for complicated admission control, resource reservation or packet scheduling mechanisms.

Flow control is a distributed algorithm to fairly share network bandwidth among competing data sources while maximizing the overall throughput without incurring congestion. The most common understanding of fairness for a best-effort service network is max-min fairness as defined in [2]. The intuition behind the max-min bandwidth sharing is that any flow is entitled to as much as bandwidth use as is for any other flow with the assumption that all flows have equal priority. This intuition naturally leads to the idea of maximizing the bandwidth use of flows with minimum bandwidth allocation, thus giving rise to the term max-min flow control.

This paper concerns the design of minimum plus weighted max-min flow control [3], a generalization of max-min flow control, where each flow is associated with two parameters, its weight \( w_i \) and minimum rate requirement \( m_i \), such that the minimum rate of each flow is guaranteed as requested during the entire holding time of the flow and the bandwidth unused after allocating the minimum rates is shared by all flows in the weighted max-min sense. An increase in the weight of a flow leads to an increase in the bandwidth share of the flow with the assumption that users pay more for a higher weight. Let us define flow \( i \)'s source rate, say \( a_i \), to be \( a_i = w_i f_i + m_i \) where \( f_i \) is the max-min fair share of the bandwidth unused after allocating the minimum rates to all flows. Let us denote the set of all links, the set of all flows and the set of flows traversing through link \( l \) by \( L \), \( N \) and \( N(l) \), respectively. Then, the weighted max-min fairness can be defined as follows.

**Definition 1:** A rate vector \( < a_1, ..., a_{|N|} > \) is said to be feasible if it satisfies \( a_i \geq 0 \), \( \forall i \in N \) and \( \sum_{i \in N(l)} a_i \leq \alpha_f^{l} \mu^{l} \), \( \forall l \in L \).

**Definition 2:** A rate vector \( < a_1, ..., a_{|N|} > \) where \( a_i = w_i f_i + m_i \) is said to be weighted max-min fair if it is feasible, and for each \( i \in N \) and feasible fair rate vector \( < f_1, ..., f_{|N|} > \) for which \( f_i < f_i \), there exists some \( i' \) with \( f_i = f_i' > f_i \).

Here \( \mu^l \) denotes the capacity of link \( l \) and \( \alpha_f^{l} \) is a constant defining target link utilization \( (0 < \alpha_f^{l} \leq 1) \). Note that admission control is necessary to ensure \( \sum_{i \in N(l)} m_i < \alpha_f^{l} \mu^{l} \) for all \( l \in L \) so that the minimum rate of each flow is guaranteed as requested during the entire holding time of the flow. Definition 2 can be restated more informally as follows: a rate vector \( < a_1, ..., a_{|N|} > \) is said to be weighted max-min fair if it is feasible and for each user \( i \in N \), its fair rate \( f_i \) cannot be increased while maintaining feasibility without decreasing the rate fair \( f_i' \) for some user \( i' \) for which \( f_i < f_i' \).

In this paper, our goal is to provide a control-theoretic framework based on deterministic fluid models that reveals not only the existence of such a distributed iterative algorithm but also an explicit stability condition of the algorithm in presence of flows with heterogeneous round-trip delays.

1.1 Our Contributions

We propose two control-theoretic max-min flow control
models and algorithms. The first algorithm satisfies Definition 2 for $a_T^2 = 1$ such that in the steady-state, bandwidth at every bottleneck link is used to the full while the minimum plus weighted max-min fairness is maintained. Moreover, the queue length at every bottleneck link converges to the target value, say $q_f^l$, thereby achieving constant queueing delay expressed by $\frac{q_f^l}{\mu_l}$. In contrast, the second algorithm satisfies Definition 2 for $0 < a_T^f < 1$ such that in the steady-state, every bottleneck link achieves its target utilization $(\alpha_T^f + \mu_l)$ and hence virtually zero queueing delay while the minimum plus weighted max-min fairness is maintained. The motivation behind the second algorithm is making the queueing delay at each link to be virtually zero and improving transient performance by absorbing transient overshoots occurring before convergence at the expense of reduced link utilization. But the major advantage of the second algorithm is that the slow adaptation of source rates traversing routers with empty buffers is overcome with this algorithm. The sluggishness of PI controllers based on queue length is also pointed out in [1]. Therefore, the former can offer zero-loss, constant-delay data services at full utilization of bottleneck links whereas the latter can offer zero-loss, zero-delay data services and faster rate adaptation at the expense of reduced bottleneck link utilization.

In the former, the difference between queue length and target queue length, i.e., $q_f^l(t) - q_f^l$, is used as a congestion measure at each link $l$ and the max-min fair rate $f_i$ is computed by a PID (proportional integral double integral) controller of this queue-length based congestion measure. In the latter, $\sum_{i \in N(l)} a_i(= w_if_i + m_i) - a_T^f\mu_l$ is used as a congestion measure at each link $l$ and the max-min fair rate $f_i$ is computed by a PI$^2$ (proportional integral double integral) controller of this aggregate-flow based congestion measure. We show that the closed-loop characteristics of the network under these two different algorithms are actually identical, yielding the identical stability condition. By appealing to the Nyquist stability criterion [4] and the Zero exclusion theorem in robust control theory [5], we derive the sufficient and necessary condition for the asymptotic stability of the network as an explicit and usable function of the upper bound $\bar{\tau}$ of all round-trip delays ($\bar{\tau} \geq \tau_i$ for all $i \in N$ where $\tau_i$ is the round-trip delay of flow $i$). Moreover, we find optimal controller gains for both PID and PI$^2$ controllers to maximize the asymptotic decay rate of the closed-loop dynamics, thereby achieving faster convergence. Finally, both PID and PI$^2$ controllers are highly scalable in that the computational complexity of the link algorithm is $O(1)$ with respect to number of flows passing through a link and no per-flow queueing implementation is necessary at any link.

1.2 Related Works

Although continual growth of data applications has triggered off the theoretical development of flow control algorithms, there are several major problems that are only partially solved. One of them is round-trip delay caused when feedback congestion signal traverses along its route to deliver itself to the corresponding source. This delay is unpredictable and worse still, can be variational. If all delays of flows are known, we may use optimal control theory [6] in minimizing some performance measure and in globally stabilizing the network. But, this requires that every router knows the variational round-trip delays and that means per-flow information of variational round-trip delays should be stored in every router.

A number of fair rate allocation algorithms [7], [8] have been proposed for ABR service in ATM networks. Since they are performance-oriented heuristic algorithms, they cannot guarantee the asymptotic stability of networks in the presence of round-trip delays. Benmohamed and Meerkov [9] formulated the rate-based flow control problem as a discrete-time feedback control problem with delays. It is notable that they have shown that their proposed algorithm can place the poles of the closed-loop system at arbitrary position in complex plane, yet it still requires that routers know the number of bottlenecked flows for each round-trip delay. In the sequel, we need a scalable and stable flow control algorithm that does not require routers know either per-flow information nor global topology information. Moreover, gain values used for flow control should not be set to conservative values to avoid degradation of overall performance.

In [10], authors proposed a simple proportional integral (PI) flow control algorithm where users’ sending rates and the network queues are asymptotically stabilized at a unique equilibrium point at which max-min fairness and target queue lengths of links are achieved. Although the stability of the closed-loop system was analyzed that was restricted to the case where all round-trip delays are equal. In [11], authors proposed a PID flow control algorithm and found equivalent stability conditions in discrete time domain but did not find an explicit stability region and optimal controller gains. Our paper extends these two approaches to PID and PI$^2$ flow control algorithms in addition to unifying many works achieving max-min fairness into one analytical framework.

2. Network Model and Controllers

In this section, we propose network models and controllers which achieve weighted max-min fairness. The network architecture with multiple sources and links is depicted in Fig. 1. Let us consider a bottleneck link $l \in L$. Then, the dynamics of the buffer of the link can be written by

$$q_i^l(t) = \left\{ \begin{array}{ll} \frac{\sum_{i \in N(l)} a_i(t - \tau_i^l) - \mu_l}{\sum_{i \in N(l)} a_i(t - \tau_i^l) - \mu_l^i}, & q_i^l(t) > 0 \\ \frac{\mu_l^i}{\sum_{i \in N(l)} a_i(t - \tau_i^l) - \mu_l}, & q_i^l(t) = 0 \end{array} \right. \tag{1}$$

where $a_i(t)$ is the sending rate of source $i$, $\tau_i^l$ is the forward-path delay from source $i$ to link $l$, $\mu_l$ is the link capacity of the link and the saturation function $[.]^+ = \max\{, 0\}$ represents that the $q_i^l(t)$ cannot be negative.

A source $i$ sends packets according to fair rate value as-
signed by the network. To achieve weighted max-min fairness, let us assume that the source sends packets according to the minimum value among the fair rate values assigned by the links along the path of its flow. Thus we assume the following source algorithm.

\[ a_i(t) = m_i + w_i \min_{l \in L(i)} \left[ f^l(t - \tau_{i}^{bf}) \right], \]

where \( L(i) \) is the set of links through which flow \( i \) traverses, \( f^l(t) \) is the rate value assigned by the link \( l \) on the path of flow \( i \) and \( \tau_{i}^{bf} \) is the backward-path delay from link \( l \) to source \( i \). Because \( \min[\cdot] \) operation is taken over a finite number of links, there should exist at least one link \( l \) such that \( f^l = \min[\cdot] \). Therefore, each flow \( i \) has at least one bottleneck \( l \in L(i) \). There are several assumptions employed for the analysis of the network model.

A.1. We assume that the sources are persistent until the closed-loop system reaches steady state. By persistent, we mean that the source always has enough data to transmit at the allocated rate.

A.2. We assume that the available link capacity \( \mu^l \) is constant until the system reaches steady state. Also, the buffer size at this link is assumed infinite.

A.3. There are two delays, say, the forward-path delay \( \tau_{i}^f \) and the backward-path delay \( \tau_{i}^{bf} \), which include propagation, queuing, and transmission and processing delays. We denote the sum of two delays by \( \tau_i \) and assume that this is constant.

2.1 The PID Control Model

To control flows and to achieve weighted max-min fairness, we use a PID link controller at each link. In PID link controller model, there is a specified target queue length \( q^l_T \) to avoid underutilization of the link capacity. Because we have a nonzero target queue length \( q^l_T \), PID model implies that \( \alpha^l_T = 1 \) in Definition 1. Each link calculates the common feedback rate value \( f^l(t) \) for all flows traversing through the link according to PID control mechanism.

In general, a proportional term increases the convergence speed of transient responses and reduces errors caused by disturbances. An integral term is necessary to eliminate steady state error and it decreases the size of stability region. A derivative term adds some damping and extends the area of stability region and it also improves the performance of transient periods.

Let us denote the set of flows bottlenecked at link \( l \) and its cardinality by \( Q^l \) and \( |Q^l| \). The link algorithm with the PID controller that uses the difference between \( q^l(t) \) and \( q^l_T \) as its input is given by

\[ f^l(t) = \left[ 1 - \frac{1}{|Q^l|} \left( g_p e^l(t) + g_i \int_0^t e^l(t)dt + g_d \dot{e}^l(t) \right) \right]^+ \]

where \( e^l(t) = q^l(t) - q^l_T \) is the error signal between control target and current output signal and, \( g_p > 0 \) and \( g_i, g_d \geq 0 \). Here, \( |Q^l| \) denotes the sum of locally bottlenecked flows’ weights, i.e., \( |Q^l| \equiv \sum_{i \in Q^l} w_i \). For convenience in deriving our results, we use the definition \( \rho_i \equiv w_i/|Q^l| \). Then it is satisfied that \( \sum_{i \in Q^l} \rho_i = 1 \) and \( \rho_i > 0 \) because we require \( w_i > 0 \).

Suppose that the closed-loop system has an equilibrium point at which the derivatives of the system variables are zero, i.e., \( \lim_{t \to \infty} q^l(t) = 0, \lim_{t \to \infty} q^l_T = q^l_T, \lim_{t \to \infty} \alpha^l_T = \alpha^l_T \). For convenience in deriving our results, we use the definition \( \rho_i \equiv w_i/|Q^l| \). Then it is satisfied that \( \sum_{i \in Q^l} \rho_i = 1 \) and \( \rho_i > 0 \) because we require \( w_i > 0 \).

Using (6), and the definitions (4) and (5), we obtain

\[ \sum_{i \in Q^l} (w_i f_{i}^{l} + m_{i}) + \sum_{i \notin Q^l} \alpha^l_T m_{i} = \mu^l \]

which establishes that the PID control model achieves the following weighted max-min fairness property

\[ r_{is} = w_i \left( \frac{w_i}{|Q^l|} \left( \mu^l - \sum_{i \notin Q^l} w_i \alpha^l_T - \sum_{i \notin Q^l} m_i \right) \right). \]
1. In this case, one can use PI1 control model as follows because we now use rate error signal instead of queue error signal.

\[
f^f(t) = \left[ -\frac{1}{Q_w} \left( h_P e_2^f(t) + h_I \int_0^\infty e_2^f(t) dt \right) + h_P \int_0^\infty e_2^f(t) dt \right]^+ (8)
\]

where \( h_P, h_I \geq 0 \) and \( h_I > 0 \). It should be remarked that PID and PI1 models are not identical because \( e_1^f(t) = \dot{q}(t) = \sum_{i \in N(0)} a_i (t - t_i^{1,f}) - \mu t \neq e_2^f(t) \) for \( q(t) > 0 \). In this model, the purpose of control is to achieve the target utilization, \( \alpha_T \). In PI1 model, note that \( \dot{q}(t) = -(1 - \alpha_T^T) \mu t < 0 \) when \( q(t) > 0 \) and \( e_2^f(t) = 0 \). Therefore, this model controls flows so that the queue length at steady state becomes zero at the cost of some degree of underutilization. In PID model, note that \( e_1^f(t) \) cannot be smaller than \( -d_1 \) because \( q(t) \) cannot be negative. Thus, one axiomatic advantage of PI1 model is that the control dynamics are not saturated at \( q(t) = 0 \) because the controller uses the rate error signal as its input instead of the queue error signal. Thus the main physical saturation nonlinearity of the PID model can be overcome by this model.

For steady state analysis, following a similar way given in Sect. 2.1 except that the saturation nonlinearity of the PID model can be overcome instead of the queue error signal. Thus the main physical saturation nonlinearity of the PID model can be overcome by this model.

\[
r_{st} \equiv w_1 f^f_s = \frac{w_1}{Q_w} \left( \alpha_T^T \mu - \sum_{i \in N(0)} w_i f^p_i(t) \right) - \sum_{i \in N(0)} m_i \right). (9)
\]

This shows that the PI1 control model also achieves weighted max-min fairness property.

3. Stability Analysis

Although we presented a multiple bottleneck network architecture in Sect. 2, rigorous stability analysis of these kinds of models has been shown to be very difficult in [12] due to the dynamics coupling among links employing FCFS (first come first serve) discipline. In [12], though such dynamics coupling exists in theory, the effect of coupling was shown to be negligible through simulations. Recently, Wydowski et al. [13] also showed that the dynamics coupling is of a very weak form. Thus, in this section, we drop the superscript l and the analysis is focused on a single bottleneck model. We conjecture that our analytical results can be extended to multiple bottleneck models without significant modification.

We describe the stability conditions for controller gains for two network models when the saturation functions employed in (1), (3) and (8) are relaxed. The main contribution of our analysis is that we find the equivalent stability condition in continuous-time domain for the case flows experience heterogeneous round-trip delays, and the stability condition depends only on a given upper bound of round-trip delays. Due to space limitation, we concentrate on the PID model and similar arguments for the PI12 model are given in Sect. 3.4.

3.1 Homogeneous-Delay Case

To analyze the homogeneous-delay case of the PID model, we simply set \( |Q| \) to be 1, then there is only one flow which is bottlenecked at the link. We also regard this case as the situation where all round-trip delays of flows are equal and \( \rho_i \) are chosen to be \( \rho_i = \frac{w_1}{|Q|}, \forall i \in Q \). This case allows us to drop the subscript of \( \tau_i \), so that the round-trip delay of flow 1 be \( \tau \). By the homogeneous-delay assumption, \( \rho_i \) can be set as follows.

\[
\rho_1 = \frac{w_1}{|Q|} = 1 \text{ and } \rho_i = 0, \forall i > 1. (10)
\]

By Eq. (3) and plugging Eq. (2) into Eq. (1), we can get the following equations.

\[
\dot{e}_1(t) = w_1 \dot{f}(t - \tau),
\]

\[
\dot{f}(t) = -\frac{1}{|Q|} \left[ g_D \dot{e}_1(t) + g_I e_1(t) + g_D \dot{e}_1(t) \right].
\]

Then the Laplace transform of the open-loop system is given by

\[
G(s) \equiv \left( \frac{g_D + \frac{g_I}{s} + \frac{g_I}{s^2}}{G_0(s)} \right) \exp(-\tau s)
\]

which corresponds to the open-loop transfer function of the PID model. By \( s = j\omega \), the following equations, to which we now apply Nyquist stability criterion [4], are obtained.

\[
G(j\omega) = G_0(j\omega) \exp(-j\tau \omega),
\]

\[
G_0(j\omega) = g_D - j \frac{g_I}{\omega} - \frac{g_I}{\omega^2}. (12)
\]

Note that the Nyquist plot of \( G_0(j\omega) \), which is depicted in Fig. 2 starts in the third quadrant and ends at \( g_D \) where \( \omega = +\infty \). Inferring from Fig. 2, we can see that the condition
$|g_D| < 1$ is necessary because the Nyquist plot of $G(j\omega)$ will encircle or touch $-1 + j0$ unless the condition is satisfied. 

Let us denote by $P$ and $\omega$ the point at which the Nyquist plot of $G_0(j\omega)$ intersects with the unit circle and the value of $\omega$ at $P$, respectively. As shown in Fig. 2, $\phi$ is the angle between $P$ and $-1 + j0$. More precisely,

$$\phi = \arccos(-\text{Re}[G_0(j\omega)]).$$

(13)

Since the Nyquist plot of $G(j\omega)$ is the Nyquist plot of $G_0(j\omega)$ rotated by $\tau\omega$ in the clockwise direction, it is required by Nyquist stability criterion that $\tau\omega < \phi$. Before proving the theorem for homogeneous-delay case, we need the following proposition. (Its proof is in Appendix A.1.)

**Proposition 1:** If there exists a unique value $\bar{\omega} \in (0, \pi/\tau)$ such that $|G(j\bar{\omega})| = 1$, $\text{Im}[G(j\bar{\omega})] < 0$, and $|G(j\omega)| > 1$ for all $\omega < \bar{\omega}$, then $\text{Im}[G(j\omega)] < 0$ is satisfied for all $\omega$ in $0 < \omega \leq \bar{\omega}$.

With the help of Proposition 1, the equivalent stability condition for the homogeneous-delay case now can be stated as follows. (Its proof is in Appendix A.2.)

**Theorem 1 (Homogeneous-Delay Case, PID Model):** The closed-loop system of the PID model with a homogeneous delay $\tau \geq 0$ is asymptotically stable if and only if $|g_D| < 1$ and the delay is bounded by

$$0 \leq \tau < \frac{\arccos\left(\frac{g_I}{\bar{\omega}^2} - g_D\right)}{\bar{\omega}}.$$  

(14)

3.2 Explicit Stability Conditions

Although we acquired the equivalent condition for the stability of our closed-loop system, the conditions are implicit and do not allow easy choice of controller gains, $g_P$, $g_I$, and $g_D$. To obtain more explicit stability conditions, we proceed in the following way.

We assume that $\tau$ is fixed to a value and that $g_D \geq 0$, $g_P > 0$ and $g_I \geq 0$. We will find explicit conditions for controller gains. Now, there are three variables, i.e., $g_P$, $g_I$ and $g_D$, concerned with the stability conditions. For mathematical tractability, we will ignore the case $\tau = 0$ and use the following definitions of variables.

$$\omega_1 \equiv \bar{\omega}\tau, \quad G_D \equiv g_D, \quad G_P \equiv g_P, \quad G_I \equiv g_I\tau^2.$$  

If we rewrite Eq. (14) and the condition for $\bar{\omega}$ in terms of new variables assuming $\tau > 0$, it follows that

$$0 < \omega_1 < \arccos\left(\frac{G_I}{\omega_1^2} - G_D\right)$$

(15)

and

$$\left(\frac{G_I}{\omega_1^2} - G_D\right)^2 + \left(\frac{G_P}{\omega_1}\right)^2 = 1.$$  

(16)

**Corollary 1 (Explicit Stability Region):** The stability condition given in Theorem 1 is equivalent to the following equations.

$$0 \leq G_D < 1,$$

$$0 < G_P < \frac{\arccos(-G_D) \sqrt{1 - G_D^2}}{G_D}$$

(17)

$$0 < G_P < \begin{cases} \arccos(-G_D) \sqrt{1 - G_D^2} & \text{if } 0 \leq G_D < -\cos(\omega_1), \\ \omega_1\sin(\omega_1) & \text{if } -\cos(\omega_1) \leq G_D < 1, \end{cases}$$

(18)

$$0 \leq G_I < \omega_1^2(G_D + \cos(\omega_1))$$

$$\omega_1^2(G_D + \cos(\omega_1)) < G_I < \omega_1^2(G_D + \cos(\omega_1))$$

(19)

where $\omega_0 \approx 2.03$ is the value maximizing the function $\omega\sin(\omega)$ over the interval $0 < \omega < \pi$, $\omega_1$ is the unique solution of $G_P = \omega\sin(\omega)$ over the interval $0 < \omega \leq \omega_0$, and $\omega_{12}$ is the unique solution of $G_P = \omega\sin(\omega)$ over the interval $\omega_0 < \omega < \arccos(-G_D)$ which exists only when the condition $\omega_0 < \arccos(-G_D)$ is satisfied.

**Remark 1 (Essential Controller Term):** From Corollary 1, we can see that $G_D$ can be 0. Then the stability condition for controller gains becomes as follows:

$$0 < G_P < \frac{\pi}{2},$$  

$$0 \leq G_I < \omega_1^2\cos(\omega_1).$$

Similarly, the stability condition when $G_D = G_I = 0$ is $0 < G_P < \pi/2$. One can verify that the essential controller term that should be positive is $P$-term and the other two terms are used for performance improvement. In fact, when $G_D = G_I = 0$ and $G_P > 0$, the performance of closed-loop systems is very poor. Since the essential controller term is $P$-term, one can consider any combinations including $P$-term such as $P$, PI, PI$, PIDD$, etc. The main reason for choosing the PID model lies in its simplicity and efficiency. For example, if we consider the PIDD$^2$ model, we have to estimate the second derivative term of the queue length $\dot{e}^2_i(t) = \ddot{q}(t) - q_{1i}$, and the analysis of the PIDD$^2$ model is much harder than that of the PID model. Similarly, the essential controller term for PI$^2$ model is I-term.

The proof of this corollary is in Appendix A.3. This corollary allows us to draw an exact stability region, provided that we are given a value of $G_D$. With the help of Corollary 1, an explicit stability region is depicted in Fig. 3 for various values of $G_D$. Notably, stability region corresponding to $G_D = 0$ is exactly the same to the stability region found in [10] where PI controller was used for flow control.

3.3 Heterogeneous-Delay Case

In this section, we prove a theorem that allows us to control flows with heterogeneous round-trip delays only with the knowledge of a given upper bound of round-trip delays. This point is important because a router may not store round-trip delay values of flows because doing so inevitably compels a router to store per-flow information.
This theorem guarantees that a network is stabilized for all combinations of $0 \leq \tau_i \leq \bar{\tau}$ if routers know only one upper bound of round-trip delays, i.e., $\bar{\tau}$, by choosing a controller gain set $(G_D, G_P, G_I) = (g_D, g_P \bar{\tau}, g_I \bar{\tau}^2)$ contained in stability region depicted in Fig. 3. Observe that the closed-loop dynamics should be better when the $\bar{\tau}$ is more tightly chosen. A method for the estimation of $|Q_w|$ is explored in Sect. 5 because a router without per-flow information cannot know the exact sum of $w_i$. By appealing to Theorem 2, we can see that it is completely safe to overestimate $|Q_w|$, i.e., $|\hat{Q}_w| \geq |Q_w|$ where $|\hat{Q}_w|$ is the estimate of $|Q_w|$, because $\sum_{i \in Q} \rho_i = \sum_{i \in Q} w_i / |Q_w|$ is allowed to be smaller than 1.

### 3.4 Stability Analysis for the PII$^2$ Model

For the PII$^2$ model, a similar approach we have used for the PID model reveals that the open-loop transfer function of the PII$^2$ model is given by

$$G(s) \equiv \left( h_P + \frac{h_I}{s} + \frac{h_\tau}{s^2} \right) \sum_{i \in Q} \rho_i \exp(-\tau_i s). \quad (21)$$

By comparing Eqs. (20) and (21) carefully, one can observe that the two equations are the same if the following substitutions are used.

$$h_P = g_D, \quad h_I = g_P, \quad h_\tau = g_I. \quad (22)$$

Because the Nyquist stability criterion and Zero exclusion theorem are related only to the open-loop transfer functions, we can now state the following theorem.

**Theorem 3 (PII$^2$ Model):** By using the Eq. (22), the stability conditions of the PII$^2$ model for the homogeneous-delay and heterogeneous-delay case are respectively given by Theorem 1 and 2.

Thus the stability of the PII$^2$ model also can be determined by checking the residence of the gains $G_D = H_P$, $G_P = H_I$ and $G_I = H_\tau$: in the stability region given in Fig. 3 by defining the gains $H_P$, $H_I$ and $H_\tau$: as follows.

$$H_P \equiv h_P, \quad H_I \equiv h_I \bar{\tau}, \quad H_\tau \equiv h_\tau \bar{\tau}^2.$$

### 4. Optimal Controller Gains

Although we found the equivalent conditions for stability, choosing controller gains is still an open problem, because there is no well-established method for choosing gains. In this section, we provide one approach for choosing controller gains where the asymptotic decay rates of closed-loop system are maximized.

At first we focus on the PID model. From Eqs. (1), (2), and (3), we can get the following closed-loop equation.

$$\dot{e}_1(t) + \sum_{i \in Q} \rho_i \left[ g_P \dot{e}_1(t - \tau_i) + g_I e_1(t - \tau_i) + g_\tau e_1(t - \tau_i) \right] = 0,$$  

(23)
For the PII model, we also get the same closed-loop equation with change of variables, i.e., $e_1(t) \rightarrow e_2(t), g_D \rightarrow h_P$, $g_P \rightarrow h_I$ and $g_I \rightarrow h_{II}$. Because all arguments in this section will depend only on the closed-loop equation, we can see they can be applied to the PII model identically.

Generally, Eq. (23) has infinite number of eigenvalues. Because a router without per-flow information cannot know per-flow round-trip delays $\tau_i$, we simply assume that $\tau_i = \bar{\tau}, \forall i \in Q$. Then, with change of variables, $t = \tau \eta$, Eq. (23) becomes

$$\ddot{e}_1(\eta) + G_D \dot{e}_1(\eta - 1) + G_P e_1(\eta - 1) + G_I e_1(\eta - 1) = 0$$  \hspace{1cm} (24)

where $G_D = g_D$, $G_P = g_P \bar{\tau}$ and $G_I = g_I \bar{\tau}^2$. Then its characteristic equation becomes

$$H(z) \equiv z^2 e^z + G_D z + G_P + G_I = 0.$$  \hspace{1cm} (25)

Any solution to the Eq. (24) can be represented by the following series expansion [14], [15],

$$e_1(\eta) = \sum_{n=1}^{\infty} p_n(\eta) \exp(\alpha \eta)$$ \hspace{1cm} (26)

where $p_n(\eta)$ is a suitable polynomial and $z_n$ are the roots of the corresponding characteristic equation (25). Let us consider the principal root, denoted by $z^*$, which is the root having the largest real part. By letting $\dot{\xi} = -\alpha + j\beta$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, it follows from (26) that

$$e_1(\eta) \approx c_1 \exp(z^* \eta),$$

$$\|e_1(\eta)\| \leq c_2 \exp(-\alpha \eta)$$ \hspace{1cm} (27)

for large $\eta$, where $c_1$ and $c_2$ are constants and $\| \|$ denotes the Euclidean norm. In terms of the original variable $t(=\tau \eta)$, it becomes

$$\|e_1(t)\| \leq c_2 \exp(-\frac{\alpha}{\bar{\tau}} t)$$ \hspace{1cm} (28)

Note that $\alpha/\bar{\tau}$ is the asymptotic decay at which the closed-loop system tends to the equilibrium point. We could consider the principal root, denoted by $z^*$, which is the root having the largest real part. By letting $\dot{\xi} = -\alpha + j\beta$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, it follows from (26) that

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$$\|e_1(\eta)\| \leq c_2 \exp(-\alpha \eta)$$ \hspace{1cm} (27)

for large $\eta$, where $c_1$ and $c_2$ are constants and $\| \|$ denotes the Euclidean norm. In terms of the original variable $t(=\tau \eta)$, it becomes

$$\|e_1(t)\| \leq c_2 \exp(-\frac{\alpha}{\bar{\tau}} t)$$ \hspace{1cm} (28)

Note that $\alpha/\bar{\tau}$ is the asymptotic decay at which the closed-loop system tends to the equilibrium point. We could find controller gain sets maximizing the value of $\alpha$ by extensive numerical calculations with the help of graphical analysis. Because the derivative term in PID control model and the proportional term in PII control model are not necessarily required, we can use two-term controllers, i.e., PI controller or PI’ controller, to allow simpler implementation. The PID, PI, PII and PI’ controller gain sets which maximize the values of $\alpha$ are respectively given as

$$G^3_{\text{PID}} \equiv (G_D, G_P, G_I) = (0.242, 0.868, 0.261),$$

$$G^3_{\text{PII}} \equiv (G_D, G_P, G_I) = (0.0482, 0.091),$$

$$G^3_{\text{PI}} \equiv (H_P, H_I, H_F) = G^3_{\text{PID}},$$

$$G^3_{\text{PI’}} \equiv (H_P, H_I, H_F) = G^3_{\text{PID}},$$

where $G^3_{\text{PID}}$ and $G^3_{\text{PII}}$ are for the PID model and $G^3_{\text{PI}}$ and $G^3_{\text{PI’}}$ are for the PII model. For two groups of gain sets, the values of $\alpha$ are 1.16 and 0.56 respectively. Inferring from the values of $\alpha$, we can expect that a system with a three-term controller (PID or PII) converges approximately twice faster than that with a two-term controller (PI or PI’). For two cases, i.e., $G_D = 0.242$ and $G_D = 0$, values of $\alpha$ are depicted in Fig. 4.

5. Estimation of $|Q^i_w|$.

To eliminate the overhead of storing per-flow information in a router, the sum of locally bottlenecked flows’ weights, $|Q^i_w| = \sum_{i \in Q} w_i$, should be estimated. We largely follow the method described in [10] and extend the method for weighted max-min fairness property. When $k$th control packet arrives at link $l$ at time $t'$, it contains values of $w_i$, $m_i$ and $a_i(t' - t'_{i-1})$. Using these values, for $k$th interval, the sum of locally bottlenecked flows’ weights can be approximated by

$$|Q^i_w|^k = \sum_{i \in (k-1)W,W} W \cdot a_i(t' - t'_{i-1})$$

$$\cdot w_i \cdot (a_i(t' - t'_{i-1}) - m_i \geq \delta \cdot w_i f(t'))$$ \hspace{1cm} (29)

where $1 \{ \}$ is the indicator function. $W$ is the time interval used for averaging. CPS is the control packet size. DPS is the average data packet size and it is assumed that for every transmission of DPS · NCP bytes, a source sends a control packet. The portion of control packets is $CPS/(DPS \cdot NCP + CPS)$. The value $\delta = 0.9$ is used to introduce a margin for estimation. As we have shown in Theorem 2, it is safe to overestimate the sum of weights. For suppression of fluctuation in estimation, the value $|Q^i_w|$ is computed as follows.

$$|Q^i_w| \leftarrow \lambda |Q^i_w| + (1 - \lambda) |Q^i_w|^k$$

where $\lambda$ is an averaging factor and it is found that $\lambda$ yields stable and effective estimation of $|Q^i_w|$ when it is set to $\lambda = 0.98$. 

6. Simulation Results

Here we give simulation results in two scenarios to demonstrate the performance of our algorithms and to compare the performance of two models. The simulations are done using the ns-2 simulator [16]. The largest round-trip propagation delay on paths traversing through a link is set to 200 ms. We assume that all packets are enqueued in the same buffer and served simply with FCFS discipline and $w_{\text{max}}$ is set to 3. Other parameters used for simulation are given in Table 1 where $\Delta$ is one data packet transmission time, i.e., $\Delta \equiv \text{DPS}/\mu^l$ and $\beta \equiv \tau^l/\lambda F^l$ determines the queuing delay of each link. Note that the target queue lengths are $q^l_T \approx 50$ kbytes with $\mu^l = 100$ Mbps and the queuing delay of a link with $q^l(t) = q^l_T$ is approximately 4 ms for the PID model. Simulation results for PID, PII$^2$, PI$^2$ and II$^2$ models are respectively denoted by $G^3_{\text{PID}}, G^3_{\text{PII}}, G^2_{\text{PID}}$ and $G^2_{\text{PII}}$.

6.1 Multiple Bottleneck Network

In the first scenario, we investigate various properties of our algorithm. In [12], [13], authors showed through simulations and analysis that the local stability condition derived in the neighborhood works well for the FCFS discipline. By appealing to this result, we here consider a scenario where two bottlenecks exist. The network configuration is shown in Fig. 5 where the bottleneck links 1 and 2 have the link capacity of 100 Mbps. The flow models used in this scenario and theoretical rates of flows satisfying the weighted max-min fairness property for the PID model are summarized in Table 2. For the PII$^2$ model, changes are indicated in parentheses when needed. At $t = 0$, the queue lengths at link 1 and 2 are already stabilized with 4 flows, S1, S2, S4 and S5. For four gain sets, values of $q^l(t)$ and $|Q^l|$ at link 1 and 2, and source transmission rates $a_i(t)$ are shown in Fig. 6. For $G^3_{\text{PID}}$ and $G^2_{\text{PID}}$, the queue lengths are controlled to the target queue length except transient periods. For $G^3_{\text{PII}}$ and $G^2_{\text{PII}}$, the queue length are nearly zero except transient periods, at the cost of 5% underutilization. It can be observed that the queue length overshoot is smaller and the rate adaptation is faster when a three-term controller ($G^3_{\text{PID}}$ or $G^2_{\text{PID}}$) is used instead of a two-term controller ($G^2_{\text{PID}}$ or $G^2_{\text{PII}}$). The overshoots of queue length at $t = 5s$ and $t = 10s$ are mainly due to the smallness of $|Q^l|$ in this scenario. When $|Q^l|$ becomes large, $|Q^l|$ > 1, such overshoots can be reduced. At $t = 50s$, the queue length with the gain set $G^2_{\text{PID}}$ is still being stabilized after the departure of S5 at $t = 20s$ because the error

---

**Table 1** Parameters used for simulation.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a^*_{\beta}$</th>
<th>$\delta$</th>
<th>$\Lambda$</th>
<th>CPS</th>
<th>DPS</th>
<th>NCP</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.95</td>
<td>0.98</td>
<td>40 bytes</td>
<td>500 bytes</td>
<td>30</td>
<td>300 A</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2** Flow models used for Scenario 1 and fair rate. (The units of $m_l$ and fair rate are in Mbps and the units of $D_i$ are in km. The units of arrival and departure time are in seconds.)

<table>
<thead>
<tr>
<th>Src.</th>
<th>$m_l$</th>
<th>$w_l$</th>
<th>$D_i$</th>
<th>Arr.</th>
<th>Dept.</th>
<th>Sink</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>5(4.75)</td>
<td>1</td>
<td>8000</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>Sink1</td>
</tr>
<tr>
<td>S2</td>
<td>15(14.25)</td>
<td>3</td>
<td>16000</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>Sink1</td>
</tr>
<tr>
<td>S3</td>
<td>0</td>
<td>2</td>
<td>4000</td>
<td>5</td>
<td>$\infty$</td>
<td>Sink2</td>
</tr>
<tr>
<td>S4</td>
<td>20(19)</td>
<td>3</td>
<td>8000</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>Sink2</td>
</tr>
<tr>
<td>S5</td>
<td>0</td>
<td>3</td>
<td>12000</td>
<td>$-\infty$</td>
<td>20</td>
<td>Sink2</td>
</tr>
<tr>
<td>S6</td>
<td>0</td>
<td>2</td>
<td>4000</td>
<td>10</td>
<td>15</td>
<td>Sink2</td>
</tr>
</tbody>
</table>

**Fig. 5** Multiple bottleneck network used for Scenario 1.

**Fig. 6** Results of Scenario 1: From top to bottom—Queue length at link 1 ($q^1(t)$), Estimated sum of locally bottlenecked flows’ weights at link 1 ($\hat{Q}^1$), Queue length at link 2 ($q^2(t)$), Estimated sum of locally bottlenecked flows’ weights at link 2 ($\hat{Q}^2$) and source transmission rates ($a_i(t)$).
signal $e^2(t)$ is saturated to $-q^2_f$ due to the saturation nonlinearity at $q^2(t) = 0$. $G_{PIF}^1$ and $G_{PIF}^2$ achieve theoretical fair rates in advance of two gain sets used for the PID model. S3 is bottlenecked at link 2 from $t = 5s$ to $t = 20s$ and bottlenecked at link 1 as S5 stops its transmission at $t = 20s$. The actual source transmission rates approach to the theoretical fair rates given in Table 2 except transient period. We can see that our algorithms work well even if multiple bottlenecks exist.

6.2 Simple Network with Short-Lived Flows

In the second scenario, we investigate the effect of short-lived flows to our algorithm. We use a simple network shown in Fig. 7 where 10 persistent sources with $D_i = i \times 1800$ km, $i \in \{1, 2, ..., 10\}$ and 15 on-off sources with $D_i = (i - 10) \times 1200$ km, $i \in \{11, 12, ..., 25\}$ exist. A on-off source is modelled by a two-state birth-death model where the dwell time periods in on and off state are exponentially distributed with the mean of 5s and 10s respectively. $u_i$ = 1 and $m_i$ = 0 are used. A portion of the simulation results is shown in Fig. 8. Although the results are oscillatory due to short-lived flows, the feedback rate and estimated sum of flows’ weights at link 3 are kept in the neighborhood of 6.67 Mbps(or 6.33 Mbps) and 15 respectively. Note that the PII model is good at tracking ideal fair rates and $G_{PIF}^3$ has the smallest transient queue length. The utilization of link 3 is found to be 0.946, 0.913, 0.947 and 0.945 respectively with the gain set $G_{PIF}^1$, $G_{PIF}^2$, $G_{PIF}^3$, and $G_{PIF}^4$. The low utilization of $G_{PIF}^3$ is caused by its slow rate adaptation when the queue length is zero. The utilization of $G_{PIF}^1$ and $G_{PIF}^2$ is very close to $\alpha_f = 0.95$ due to their fast rate adaptation.

7. Conclusion

In this paper, we provided two network models which satisfy weighted max-min fairness and dispense with any kind of per-flow operation in routers. We found equivalent stability conditions in two network models with heterogeneous round-trip delays. The theorem states that a stabilizing gain found with homogeneous-delay $\tau$ also stabilizes all the networks with heterogeneous delays less than or equal to $\tau$ and overestimation of the sum of flows’ weights is completely safe. We also derived the equivalent condition for the asymptotic stability of the network as an explicit and usable function of the upper bound $\bar{\tau}$ of all round-trip delays.

The PID model achieves not only full utilization but also the target queue length at its equilibrium point. The PII model achieves zero queueing delays and absorbs transient overshoots in links sacrificing some degree of utilization, and $\alpha_f^2$ can be lowered to absorb the transient queues when many short-live flows exist. It also quickly achieves fair rates because the saturation nonlinearity at empty buffers is now eliminated. We believe that our theoretical and experimental results will play an important role in encouraging the usage of more sophisticated flow control algorithms in packet networks.

In our network models, sources induce overshoots of queue lengths because they immediately adapt to the network-assigned data rates. Although our models, especially the three-term controller in the PII model, are good at reducing transient queues when there are many flows, a source behavior should be combined to our models for further research to reduce transient queue overshoots when there are small number of flows or when the portion of short-lived flows is large. We suggest that one may use techniques proposed in XCP [17] to extend our algorithm for TCP-like sources.

Acknowledgment

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References

Since one of two conditions, i.e., \( \sum \Im \omega G \) always exists a value \( \tilde{\omega} \) such that \( \Im \omega G (\tilde{\omega}) = 0 \) or \( \Im \omega G (\tilde{\omega}) = \pi \).

To prove the Proposition, it is sufficient to show that \( \Im G(\omega) < 0 \) for all \( \omega \in (-\infty, \tilde{\omega}) \). By contradiction, assume that there is a value \( \tilde{\omega} \) such that \( \Im G(\tilde{\omega}) = 0 \). Moreover, from \( \Im G(\tilde{\omega}) = 0 \), it follows that \( \Im G(\tilde{\omega}) = 0 \) or \( \Im G(\tilde{\omega}) = \pi \). Since one of two conditions, i.e., \( G(\tilde{\omega}) > 0 \) or \( G(\tilde{\omega}) < -1 \) should be satisfied, let us assume the former case. But, \( \Im G(\tilde{\omega}) \) can not be zero because \( -\pi < \Im G(\tilde{\omega}) < 0 \) and \( -\pi < \omega \tilde{\tau} < -\omega \tilde{\tau} < 0 \). Therefore, this case cannot happen.

Let us assume the latter case, \( G(\tilde{\omega}) < -1 \). We can summarize the angle values of \( G(j\omega) \) as follows.

Since the angle function of \( G(j\omega) \) is continuous, there should be at least three local extrema. The angle function and its derivative which can be obtained from Eq. (12) are given by

\[
\begin{align*}
\psi(\omega) & = \arctan \left( \frac{g_D \omega^2 - g_I}{g_P \omega} \right) - \frac{\pi}{2} - \omega \tau \\
\psi'(\omega) & = -\frac{g_D \omega}{g_P^2 \omega^2 + (g_D^2 \omega^2 - g_I^2)} - \tau.
\end{align*}
\]

Because \( \psi'(\omega) = 0 \) is a biquadratic equation, there are at most two solutions in \( 0 < \omega < \infty \). This is in contradiction with the fact that \( \psi(\omega) \) has at least three local extrema.

A.2 Proof of Theorem 1

Sufficiency. We can see easily from Fig. 2 that the Nyquist plot has a unique intersection with the unit circle in \( 0 < \omega < \infty \). More formally, there is a unique positive solution to the following equation if \( |g_I| < 1, R \geq 1 \) and \( g_I > 0 \) (The case \( g_I = 0 \) also can be treated in a similar way).

\[
|G(j\omega)| = |G_0(j\omega)| = \left( g_D - \frac{g_I}{\omega^2} \right)^2 + \left( \frac{g_P}{\omega} \right)^2 = R^2, \quad (A\cdot1)
\]

because Eq. (A\cdot1) is equivalent to the following biquadratic equation

\[
(R^2 - g_D^2)\omega^4 + (2g_D g_I - g_P^2)\omega^2 - g_I^2 = 0, \quad (A\cdot2)
\]

and the condition for Eq. (A\cdot2) to have a unique positive solution is equivalent to

\[
(2g_D g_I - g_P^2)^2 + 4g_D^2(R^2 - g_D^2) \geq 0
\]

and

\[
g_I^2(R^2 - g_D^2) < 0.
\]
which is already satisfied by the assumption of \( g_D \) and \( R \). This means that the Nyquist plot reaches the unit circle and never depart from it. That is, the part of the Nyquist plot corresponding to \( \omega > \bar{\omega} \) is entirely contained in the unit circle so that it can never contribute to encircling or touching \(-1 + j0\). If the assumption of \( \tau \) is satisfied, we can easily see that the Nyquist plot of \( G(j\omega) \) is always below \(-1 + j0\) for \( 0 < \omega \leq \bar{\omega} \) and all the assumptions of Proposition 1 are satisfied. Therefore, the Nyquist plot neither does encircle nor touch \(-1 + j0\) because \( \text{Im}[G(j\omega)] < 0 \) for all \( \omega \in (0, \bar{\omega}] \). The open-loop pole \( s = 0 \) can be managed with the boundary deviation technique by adding an infinitesimal half-circle. Appealing to the Nyquist stability criterion, we have proven the stability of closed-loop system.

Necessity can be proven trivially. If \(|g_D| \geq 1\) or the condition of Eq. (14) is violated, the Nyquist plot simply encircles or touches \(-1 + j0\).

A.3 Proof of Corollary 1

Here we give a sketch of proof because a detailed proof requires complicated arguments. To find a necessary condition, let us assume that \( 0 < G_D < 1 \) is fixed. From the fact that \( \arccos(\cdot) \) is a monotonically decreasing function, we see that \( G_I < \omega^2 \bar{G}_D + \cos(\omega t) \) and \( \omega < \arccos(-G_D) \). We can consider two cases, i.e., \( 0 < \omega_1 \leq \frac{\pi}{2} \) and \( \frac{\pi}{2} < \omega < \arccos(-G_D) \). Considering carefully both cases and using the fact that the function \( \omega \sin(\omega) \) has a unique maximum over the interval \( 0 < \omega < \pi \), it follows that

\[
G_P = \begin{cases} 
\omega_1 \sin(\omega_1) \text{ over } \frac{\pi}{2} < \omega_1 < \arccos(-G_D) \\
\omega_0 \sin(\omega_0) \text{ if } \arccos(-G_D) < \omega_0 \\
\omega \sin(\omega) \text{ if } \omega_0 < \omega \leq \arccos(-G_D).
\end{cases}
\]

This equation can be used in obtaining a necessary stability condition for \( G_P \) which is given by Eq. (18). The case \( G_D = 0 \) can be easily treated separately and can be found by setting \( G_D = 0 \) in Eq. (18).

Let us assume that \( 0 < G_D < 1 \) is fixed and \( G_P > 0 \) takes a fixed value such that \( G_P = \omega_1 \sin(\omega_1) \) and \( 0 < \omega_1 < \arccos(-G_D) \) which is the same to the range of \( \omega_1 \). When \( \arccos(-G_D) < \omega_0 \), \( G_I \) is simply given by \( G_I < \omega_1^2 (G_D + \cos(\omega_1)) \) from Eq. (15). When \( \omega_0 \leq \arccos(-G_D) \), we have to consider two cases, i.e., \( 0 < \omega_1 < \omega_0 \) and \( \omega_0 < \omega_1 \leq \omega \leq \arccos(-G_D) \), separately. Considering in this way, it can be shown that a necessary stability condition for \( G_I \) is given by Eq. (19).

Although we omit the proof of sufficiency, it can be verified with direct substitution of Eqs. (17), (18) and (19) into (15) and (16).

A.4 Proof of Theorem 2

Proof of necessity is trivial. If the heterogeneous-delay case is asymptotically stable, then we can simply set \( \rho_1 = 1, \rho_i = 0, \forall i \geq 2 \) and \( \tau_1 = \tau \).

For the proof of sufficiency, both conditions of Proposition 2 should be satisfied. We immediately see that the first condition of Proposition 2 is already satisfied because we have a choice of \( 2|Q| \)-tuple vector, i.e., \( \rho_1 = 1, \rho_i = 0, \forall i \geq 2 \) and \( \tau_1 = \tau \) which stabilizes the homogeneous-delay case. For the second condition, let us define a circular sector as follows:

\[
S^\tau_\omega = \{ s \in C | \|s\| \leq 1, \text{ and } -\tau \omega \leq \angle s \leq 0 \}.
\]

Let us define \( z \equiv \sum_{i \in Q} \rho_i \exp(-j\tau_i \omega) \). By rewriting \( z \) in terms of \( \exp(-j\tau \omega) \) and 0 as follows:

\[
z = \sum_{i \in Q} \rho_i \exp(-j\tau_i \omega) + \left( 1 - \sum_{i \in Q} \rho_i \right) \cdot 0,
\]

we see that \( z \in S^0_\omega \) because \( S^0_\omega \) is a convex set and \( z \) is a convex combination of \( |Q| + 1 \) points in \( S^\tau_\omega \). Furthermore, if we define a set \( G_0(j\omega)S^\tau_\omega = \{ G_0(j\omega) \cdot s | s \in S^\tau_\omega \} \) which is depicted in Fig. A-2, we see that \( V(\omega) \subseteq G_0(j\omega)S^\tau_\omega \) for all \( \omega \) because the following equation holds:

\[
G(j\omega, \rho, \tau) = G_0(j\omega)G(j\omega, \rho, \tau) \in G_0(j\omega)S^\tau_\omega.
\]

Since the circular sector \( G_0(j\omega)S^\tau_\omega \) is bounded by two functions, i.e., \( G_0(j\omega)\exp(-j\tau \omega) \) and \( G_0(j\omega) \) and the magnitudes of two functions are equal to \( |G_0(j\omega)| = |G(j\omega)| \), the circular sector \( G_0(j\omega)S^\tau_\omega \) can be expressed as follows:

\[
G_0(j\omega)S^\tau_\omega = \{ z | \eta_1 \leq \angle z \leq \eta_\mu, \|z\| \leq |G_0(j\omega)| \}, (A-4)
\]

\[
\eta_1 \equiv \angle G_0(j\omega) \exp(-j\tau \omega) = \angle G_0(j\omega) - \tau \omega,
\]

\[
\eta_\mu \equiv \angle G_0(j\omega).
\]

For \( \bar{\omega} < \omega \), where \( \|z\| < 1 \) for all \( z \in G_0(j\omega)S^\tau_\omega \), the sector \( G_0(j\omega)S^\tau_\omega \) is completely contained inside the unit circle. Naturally, \( V(\omega) \), which is a subset of \( G_0(j\omega)S^\tau_\omega \) does not touch \(-1 + j0\). For \( 0 < \omega < \bar{\omega} \), \( \text{Im}[G_0(j\omega)] < 0 \) and \( \text{Im}[G_0(j\omega)\exp(-j\tau \omega)] < 0 \) hold by Proposition 1. Thus, \(-\pi < \eta_1 < 0 \) and \(-\pi < \eta_\mu < 0 \) hold. From Eq. (A-4), we see that \( G_0(j\omega)S^\tau_\omega \) is completely contained in the lower half plane. For \( \omega = 0 \), the value set \( V(\omega) \) is infinity. By Proposition 2, we now complete the proof of sufficiency.
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