On the Validity of the Fixed Point Equation and Decoupling Assumption for Analyzing the 802.11 MAC Protocol

[Extended Abstract]

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ABSTRACT

Performance evaluation of the 802.11 MAC protocol is classically based on the decoupling assumption, which hypothesizes that the backoff processes at different nodes are independent. A necessary condition for the validity of this approach is the existence and uniqueness of a solution to a fixed point equation. However, it was also recently pointed out that this condition is not sufficient; in contrast, a necessary and sufficient condition is a global stability property of the associated ordinary differential equation. Such a property was established only for a specific case, namely for a homogeneous system (all nodes have the same parameters) and when the number of backoff stages is either 1 or infinite and with other restrictive conditions. In this paper, we give a simple condition that establishes the validity of the decoupling assumption for the homogeneous case. We also discuss the heterogeneous and the differentiated service cases and show that the uniqueness condition is not sufficient; we exhibit one case where the fixed point equation has a unique solution but the decoupling assumption is not valid.

1. PROBLEM STATEMENT

Most existing work on performance evaluation of the 802.11 MAC protocol [2,4,5,7] relies on the “decoupling approximation” which was first adopted in the seminal work of Bianchi [2]. Essentially, it assumes that all nodes in the same network experience the same time-invariant collision probability, which in turn amounts to the assumption that the backoff processes are independent. This assumption is unavoidable primarily because the stationary distribution of the original Markov chain cannot be explicitly written [5,6].

Once we assume that the decoupling assumption holds, the analysis of the 802.11 MAC protocol leads to a fixed point equation [5], also called Bianchi’s formula. Kumar et al. discussed conditions under which the fixed point equation has a unique solution in [5, Theorem 5.1]; if the attempt probability in backoff stage $k$ is nonincreasing in $k$, the fixed point equation has a unique solution.

However, it is pointed out in [1], using a mean field convergence method, that the uniqueness of a solution to the fixed point equation does not necessarily lead to the validity of the decoupling assumption in stationary regime. Instead, one should verify that the associated Ordinary Differential Equation (ODE), obtained when applying mean field convergence theory, is globally stable, which we define in the following sense: there is a unique limit point to which all trajectories converge. This unique limit point, if it exists, is the solution to the fixed point equation, but the converse is not true; we give in Section 3 an example where there is a unique fixed point but the ODE is not globally stable.

Therefore, to establish the validity of Bianchi’s formula, one needs to prove that the associated ODE is globally stable. We study in this paper the case of a single cell network. In [3, Theorem 5.4], Bordenave et al. studied the homogeneous case (all nodes have same per-stage backoff probabilities) for the case where the number of backoff stages is infinite. They found the following sufficient condition for global stability of the ODE, hence for validity of the decoupling assumption:

$$q_0 < \ln 2 \quad \text{and} \quad q_{k+1} = q_k/2, \quad \forall k \geq 0$$

where $q_k$ is the re-scaled attempt probability (defined in Section 2) for a node in backoff stage $k$. In this paper, we focus on the case where the total number of backoff stages $K + 1$ is finite, as this is true in practice and in Bianchi’s formula. Sharma et al. [8] obtained a result for $K = 1$ and mentioned the difficulty to go beyond. In this paper, we solve the issue to a large extent: In the homogeneous case (Section 2), we prove that a surprisingly simple condition on the re-scaled attempt probabilities, namely $q_{k} \leq 1$ for all $k$, is sufficient for global stability. For the heterogeneous (Section 3) and AIFS differentiated services cases, we formulate the ODE by appealing to a recent result [1, Theorems 1 & 2] but are not able to provide such a simple result. In contrast, we give a counterexample; it also serves as an illustration of the fact that there may be a unique solution to the fixed point equation whereas the decoupling assumption does not hold.

2. THE HOMOGENEOUS CASE

2.1 Basic Operation of DCF Mode
Time is slotted. Each node following the randomized access procedure of 802.11 distributed coordination function (DCF) generates a backoff value after receiving the Short Inter-Frame Space (SIFS) if it has a packet to send. This backoff value is uniformly distributed over \( \{0, 1, \cdots, 2b_0-1\} \) or \( \{1, 2, \cdots, 2b_0\} \) where \( 2b_0 \) is the initial contention window.

Whenever the medium is idle for the duration of a Distributed Inter-Frame Space (DIFS), a node unfreezes (starts) its countdown procedure of the backoff and decrements the backoff by one per every time-slot. It freezes the countdown procedure as soon as the medium becomes busy. There exist \( K+1 \) backoff stages whose indices belong to the set \( \{0, 1, \cdots, K\} \) and we assume \( K > 0 \). If two or more wireless nodes finish their countdowns at the same time-slot, there occurs a collision between RTS (ready to send) packets. If there is a collision, each node who sees the collision avoidance (PCF) generates a Distributed Inter-Frame Space (DIFS), a node unfreezes (starts) its countdown procedure of the backoff and decrements the countdown during channel activity, the total time over which the other nodes transmit. Since all nodes freeze their countdown, the average attempt rate and backoff value is cut in half. Therefore, it is sufficient to analyze the backoff process in order to investigate the performance of single-cell networks. This technique has been adopted in many works including [1][3][5][7].

### 2.2 Mean Field Analysis

Let \( N \) be the number of nodes and assume that a node in stage \( k \) attempts transmission with probability \( q_k/N \). The parameter \( q_k \) is the re-scaled attempt probability; this form of scaling is required in [3] to avoid saturation when the \( k \)-th stage is open. Denote by \( X_n(t) \) the backoff stage of node \( n \) at time-slot \( t \); the occupancy measure (or empirical measure) of backoff stage \( k \) at time-slot \( t \) is defined as \( \Phi_k(t) := \frac{1}{N} \sum_{n=1}^{N} 1\{X_n(t) = k\} \), where \( 1\{\cdot\} \) is the indicator function. It is shown in [3] that \( \Phi(Nt) \) converges in probability to \( \phi(t) \) which is the solution of the ODE:

\[
\begin{align*}
\frac{d\phi_k(t)}{dt} &= \bar{q}(t)(1 - \gamma(t)) - q_k\phi_k(t) + q_k\phi(t)(\gamma(t)), \\
\frac{d\phi_0(t)}{dt} &= q_{k-1}\phi_{k-1}(t)(\gamma(t)) - q_k\phi_k(t), \quad k \in \{1, \cdots, K\}.
\end{align*}
\]

Here \( \bar{q}(t) := \sum_{k=0}^{K} q_k \phi_k(t) \) is the mean field limit of the average attempt rate and \( \gamma(t) = 1 - e^{-\bar{q}(t)} \) is that of the collision probability. It is important to note that the above system is degenerate \( \text{deg} \) because we also have a manifold relation \( \phi_0(t) = 1 - \sum_{k=1}^{K} \phi_k(t) \), which can be plugged into (2) to eliminate the first equation with respect to \( \phi_0(t) \). Therefore, we can use the reduced version which corresponds to the second equation of (2), along with the manifold relation. We call this system homogeneous because all nodes have the same parameter set \( q_k \) and \( K \).

A necessary condition for \( \phi \) to be an equilibrium point is obtained by equating the right-hand sides to 0, which gives

\[
\phi_k = \frac{d_k}{\bar{q}_k} \phi_0 \quad \text{and} \quad \phi_0 = \frac{q_0}{\bar{q}_0} \sum_{k=0}^{K} \frac{\gamma_k}{\bar{q}_k} \gamma
\]

which is Bianchi’s fixed point equation. It is shown in [1] that if (2) is globally stable, then in the limit of large \( N \), the decoupling assumption holds in stationary regime and the probability that a tagged node is in stage \( k \) converges to \( \phi_k \), which is then necessarily the unique solution of (3). It is also shown that the converse is not necessarily true. We show in the full version the following results, using monotonicity and spectral analysis:

**Theorem 1.** If \( q_k \leq 1 \) for all \( k \in \{0, \cdots, K\} \), the ODE in (2) is globally stable.

**Proposition 1.** If the sequence \( q_k \) is monotonic nonincreasing with \( k \), the fixed point equation in (3) has a unique solution.

### 2.3 Discussion

Theorem 1 shows that if \( q_k \leq 1 \) for all \( k \), then the decoupling assumption and therefore Bianchi’s method are asymptotically valid. Note that \( q_k \) is the re-scaled attempt probability, the attempt probability for a given \( N \) being \( p_k = q_k/N \). If we take the backoff value in 802.11 to be exponentially distributed, we can see from Section 2.1 that \( p_k = 1/(2b_k - 1/2) \). Note that if the hypothesis of Theorem 1 holds, necessarily the fixed point equation in (3) has a unique solution.

Proposition 1 gives a complementary result, as existence and uniqueness of a solution to the fixed point equation in (3) may not be sufficient to warrant that the ODE is globally stable. Note that the case in (1) satisfies the hypotheses of both Theorem 1 and Proposition 1 so, in some sense, our result is an extension of Theorem 5.4.

It is still open whether the hypothesis in Proposition 1 implies global stability when the hypothesis of Theorem 1 does not hold.

### 3. THE HETEROGENEOUS CASE

#### 3.1 Mean Field Analysis

The above analysis can be extended to model mechanisms provided by the enhanced distributed channel access (EDCA) of the 802.11e standard. The first mechanism, collision window (CW) differentiation, amounts to a per-class setting of \( q_0 \) and \( K \), on the assumption that \( q_k = q_0/2^k \) for \( k \in \{0, \cdots, K\} \). We extend this feature by allowing per-class setting of \( K \) and \( q_k \) for any \( k \in \{0, \cdots, K\} \) for the sake of generality and notational aesthetics. The second mechanism, called AIFS differentiation, offers a soft preemptive prioritization to a certain class by holding back other classes from attempting transmissions for \( \Delta \) time-slots. For simplicity, the analysis here is presented for two classes, i.e.,

\[1\]A degenerate system has a singular Jacobian matrix which means that its linearization cannot determine the local stability of the system.
Class H (high) and Class L (low). Let us call the time-slots reserved for Class H reserved slots, which will correspond to the superscript R. We call the remaining slots following reserved slots common slots, corresponding to the superscript C. Note that both Class H and Class L users can access the channel during common slots, whereas the backoff procedures of Class L users are suspended during reserved slots.

The per-class parameters and occupancy measures are denoted by \( q^R_k, q^L_k, K^H, K^L, \Phi^H_k(t) \) and \( \Phi^L_k(t) \). Using the approach in [1, Theorems 1 & 2], one finds that the system can be approximated by the solution of the ODE

\[
\begin{align*}
\frac{d\Phi^H_k(t)}{dt} &= q^H_k - q^H_{k-1}\Phi^H_{k-1}(t) - q^H_k\Phi^H_k(t), \\
\frac{d\Phi^L_k(t)}{dt} &= \pi^C(t) \left( q^L_{k-1}\Phi^L_{k-1}(t) - q^L_k\Phi^L_k(t) \right)
\end{align*}
\]

whose corresponding fixed point equation is

\[
\begin{align*}
\Phi^H &= \sigma^H \sum_{k=0}^{N^H} (\gamma^H)^k, \quad \Phi^L = \sigma^L \sum_{k=0}^{N^L} (\gamma^L)^k, \\
\gamma^H &= \pi^R \left( 1 - e^{-\Phi^H} \right) + \pi^C \left( 1 - e^{-\Phi^L} \right), \\
\pi^R &= \frac{\sum_{k=0}^{N^R} (1 - \gamma^R)^k}{\sum_{k=0}^{N^R} (1 - \gamma^R)^k}, \\
\pi^C &= \frac{\sum_{k=0}^{N^C} (1 - \gamma^C)^k}{\sum_{k=0}^{N^C} (1 - \gamma^C)^k}.
\end{align*}
\]

Here \( \sigma^H \) and \( \sigma^L \) are the constants which respectively denote the proportions of Class H and L users to population \( N \).

### 3.2 A Counterexample

We present an example where the fixed point equation [5] has a unique solution but the decoupling assumption does not hold. It is a heterogeneous case without AIFS differentiation, i.e., \( \Delta = 0 \), which in turn leads to \( \pi^R = 0 \) and \( \pi^C = 1 \). There are two classes H and L such that population of each class is \( N^H = N^L = 640 \). The numbers of backoff stages are equal, i.e., \( K^H + 1 = K^L + 1 = 21 \). The attempt probability at each backoff stage is:

\[
\begin{align*}
(p_0^H, p_1^H, \ldots, p_{20}^H) &= \left( \frac{1}{2400}, \frac{1}{480}, \frac{1}{40}, \ldots, \frac{m^{19}}{40} \right), \\
(p_0^L, p_1^L, \ldots, p_{20}^L) &= \left( \frac{1}{3840}, \frac{1}{64}, \frac{1}{64}, \ldots, \frac{1}{64} \right)
\end{align*}
\]

where \( m = 4/5 \). The fixed point equation has a unique solution with \( \gamma^H = \gamma^L = \gamma^C = \gamma_1 = 0.912 \). Since there is only one solution, one might be much inclined to hazard the conjecture by Bianchi et al. [2,5] that the collision probability is approximately \( \gamma_1 \). However, there is a stable limit cycle around this equilibrium with a stable oscillation.

The average collision probability obtained through simulations is 0.869 which is less than \( \gamma^H \) or \( \gamma^C \). The decoupling assumption does not hold; in contrast, nodes are coupled by the oscillations of the occupancy measure, an emerging property of the system dynamics.

### 3.3 Uniqueness Results

We are not able to prove the equivalent of Theorem 4 which would justify the decoupling assumption in the heterogeneous case. However, we can prove the following weaker results:

\[ \text{Proposition 2. If } q^H_k \leq 1 \text{ and } q^L_k \leq 1 \text{ for all } k, \text{ the fixed point equation } \Phi \text{ has a unique solution.} \]

\[ \text{Proposition 3. If } q^H_k \text{ and } q^L_k \text{ are nonincreasing in } k, \text{ the fixed point equation } \Phi \text{ has a unique solution.} \]

### 4. REFERENCES