Methods for reliability based design optimization of structural components

Tomas Dersjö

Doctoral thesis no. 78, 2012
KTH School of Engineering Sciences
Department of Solid Mechanics
Royal Institute of Technology
SE - 100 44 Stockholm Sweden
Akademisk avhandling som med tillstånd av Kungliga Tekniska Högskolan i Stockholm framlägges till offentlig granskning för avläggande av teknisk doktorsexamen måndagen den 12 mars kl. 10.15 i sal F3, Kungliga Tekniska Högskolan, Lindstedtsvägen 26, Stockholm.
Abstract

Cost and quality are key properties of a product, possibly even the two most important. One definition of quality is *fitness for purpose*. Load-bearing products, *i.e.* structural components, lose their fitness for purpose if they fail. Thus, the ability to withstand failure is a fundamental measure of quality for structural components. Reliability based design optimization (RBDO) is an approach for development of structural components which aims to minimize the cost while constraining the probability of failure. However, the computational effort of an RBDO applied to large-scale engineering problems has prohibited it from employment in industrial applications. This thesis presents methods for computationally efficient RBDO.

A review of the work presented on RBDO algorithms reveals that three constituents of an RBDO algorithm has rendered significant attention; *i*) the solution strategy for and numerical treatment of the probabilistic constraints, *ii*) the surrogate model, and *iii*) the experiment design. A surrogate model is ”a model of a model”, *i.e.* a computationally cheap approximation of a physics-based but computationally expensive computer model. It is fitted to responses from the physics-motivated model obtained via a thought-through combination of experiments called an experiment design.

In Paper A, the general algorithm for RBDO employed in this work, including the sequential approximation procedure used to treat the probabilistic constraints, is laid out. A single constraint approximation point (CAP) is used to save computational effort with acceptable losses in accuracy. The approach is used to optimize a truck component and incorporates the effect that production related design variables like machining and shot peening have on fatigue life.

The focus in Paper B is on experiment design. An algorithm employed to construct a novel experiment design for problems with multiple constraints is presented. It is based on an initial screening and uses the specific problem structure to combine one-factor-at-a-time experiments to a several-factors-at-a-time experiment design which reduces computational effort.

In Paper C, a surrogate model tailored for RBDO is introduced. It is motivated by applied solid mechanics considerations and the use of the first order reliability method to evaluate the probabilistic constraint. An optimal CAP is furthermore deduced from the surrogate model.

In Paper D, the paradigm to use sets of experiments rather than one experiment at a time is challenged. A new procedure called experiments on demand (EoD) is presented. The EoD procedure utilizes the core of RBDO to quantify the demand for new experiments and augments it by a *D*-optimality criterion for added robustness and numerical stability.
Preface

The research presented in this thesis has been conducted at the Department of Solid Mechanics, Royal Institute of Technology (KTH Hållfasthetslära) and at the Department of Dynamics and Strength Analysis, Truck Chassis Development, Scania CV AB. The research has been financed by Scania CV AB and the Swedish Governmental Agency for Innovation Systems (VINNOVA). The opportunity the funding provided has been great and I am grateful to have been given it.

The work has been highly rewarding from a scientific as well as a personal point of view. This is due to a number of people. It would lead to far to thank all in person. Thus; to those who have contributed to my personal or scientific development I send my gratitude. However, a special thanks to those who have given significant contributions to this thesis is in order.

First and foremost, to my academic advisor Prof. Mårten Olsson whose encouragement, advice and patience I value greatly. I am much obliged. Also, the interest and encouragement shown by the steering committee of this project is highly appreciated. A special mention of my former manager Mr. Martin Edberg who employed and fully supported me during his time as head of our department is in order. My current manager Mr. Ola Rugeland has continued on the path of complete trust and support and for that I am thankful.

Going to work has been a pleasure during this time. For that, I wish to express my gratitude to my colleagues at both my work places. Leaving work has been a pleasure as well. I blame my friends and family for that.

Finally, to my mother and father: Although I do not say it often, I could not have wished for better parents.

Stockholm, February 2012
List of appended papers

**Paper A:** Reliability based design optimization using a single constraint approximation point
T Dersjö and M Olsson
*Journal of Mechanical Design 133 (3), 2011, 031006*

**Paper B:** Efficient design of experiments for structural optimization using significance screening
T Dersjö and M Olsson
*Structural and Multidisciplinary Optimization 45(2), 2012, 185–196*

**Paper C:** A directional surrogate model tailored for efficient reliability based design optimization
T Dersjö and M Olsson
*Report 518, Department of Solid Mechanics, KTH Engineering Sciences, Royal Institute of Technology, Stockholm, Sweden*
To be submitted

**Paper D:** Reliability based design optimization with experiments on demand
T Dersjö and M Olsson
*Report 519, Department of Solid Mechanics, KTH Engineering Sciences, Royal Institute of Technology, Stockholm, Sweden*
To be submitted
# Contents

## Introduction
- Background and rationale ................................................. 13
- Simulation-driven development ........................................... 15
- Surrogate models ............................................................. 17
- Design of experiments ....................................................... 18
- Problem formulation ......................................................... 20
- Treatment of probabilistic constraints .................................. 22

## Summary of appended papers

## Bibliography
Methods for reliability based design optimization of structural components
Introduction

This thesis presents methods for reliability based design optimization intended for structural applications. The methods are presented in detail in the appended papers. However, the introductions given in the papers are brief and they thus require some background for a thorough understanding. This introductory part is meant to provide the reader with that background by placing it in a context. Furthermore, all studies need limitations. Some aspects that are relevant to structural optimization with computer models but are not the focus of the appended papers will be shortly mentioned here. Finally, some parts which are treated in the appended papers but in a condensed manner will be elaborated upon here.

Background and rationale

Cost and quality are key properties of a product, possibly even the two most important. One definition of quality is fitness for purpose. Load-bearing products, i.e. structural components, loose their fitness for purpose if they fail. An illustrative example of the possible consequences of a mechanical failure is given in Fig. 1.

Figure 1: Example of mechanical failure leading to significant loss of fitness for purpose.
Methods for reliability based design optimization of structural components

Figure 2: Schematic illustration of the relation between a design variable $x$ and the production cost, failure cost, and total cost of a product, respectively.

Thus, the ability to withstand failure is a fundamental requirement for attaining quality for structural components. A schematic illustration of the conflict between quality and cost which is often encountered in product development is given in Fig. 2. However, the relation between failure cost and failure probability is a complex one. Hence, an approach frequently taken in optimization is to set a required reliability constraint which the design must at least meet. Reliability based design optimization (RBDO) is an approach for development of structural components which aims to minimize the cost while observing constraints for the probability of failure. However, the computational effort of an RBDO applied to complex engineering problems has prohibited it from becoming an every day tool in industry. This thesis presents methods which aim to reduce the computational effort associated with an RBDO.

A review of the work presented on RBDO algorithms reveals that three components of an RBDO algorithm has rendered significant attention; i) the solution strategy for and numerical treatment of the probabilistic constraints, ii) the surrogate model, and iii) the experiment design. A surrogate model is "a model of a model", i.e. a computationally cheap approximation of a physics-based but computationally expensive computer model. It is fitted to responses from the physics-motivated model obtained via a thought-through combination of experiments called an experiment design. Contributions to these three areas are made in the papers appended in this thesis. Thus, some background and further reading on these subjects are presented in this introductory part of this thesis. However, in order to present the reader with a bigger picture, the field of RBDO will first be placed in a broader context; that of simulation-driven development of structural components.
Simulation-driven development

Development of load-bearing or otherwise stressed component is an iterative procedure. In each loop, more or less refined computations are verified with tests on physical prototypes. Since physical prototypes as well as test rigs and man hours are costly, companies strive to keep the physical testing to a minimum. With the advent of powerful computers, the events that can be simulated are steadily increasing. Simulation is now the paradigm for development of structural components. Research has traditionally been aimed at improvement of models and formulations of governing equations. Over the last decades, there has been considerable emphasis on how to use these high-fidelity models in a more automated way than the trial-and-error type iterations where the success depends highly on the skill of the individual engineer. Optimization is a type of automation of the development work which, for obvious reasons, has gained considerable attention. Optimization is a vast field even when limited to mechanical applications. A distinction between different types of optimization will be made here in order to put the appended work in a context.

An optimization problem could have multiple objectives or a single objective. In multi-objective optimization, the goal is to establish the best compromise between conflicting objectives. The best trade-offs are a set of designs that define the Pareto frontier. The work in this thesis uses a single objective and a fixed target failure probability but it would most certainly be of interest for decision makers in industry to know how much the extra reliability costs.

Another classification of optimization types is in problems where analytical information, in particular analytical gradients, can be attained and cases where they need to be obtained through numerical estimation. The gradients under consideration are the derivatives with respect to design variables. To be able to attain analytical gradients, one must be familiar with the governing equations of the system under study and the problem must be smooth, or at least $C^1$. This is a typical situation for topology and topography optimization. In what might be called parameterized optimization, e.g. optimization of thicknesses, radii, modulus of elasticity etc, the physics-based model is instead often viewed as a black-box; see Shan and Wang (2010). Both approaches have advantages and disadvantages. Access to analytical gradients makes the methods more efficient and should be preferred when applicable but they are less general. It makes it harder to use stand alone software for optimization as well as limits the application range to smooth problems. The black-box approach on the other hand is more expensive even for problems with few design variables and in addition significantly suffers from the problem of size, see Koch et al. (1999). Since analytical gradients are not assumed to be readily available, surrogate model coefficients have to be estimates from multiple experiments. The number of experiments needed generally increase with the number of variables, at worst in an exponential manner. This is the reason why so much effort has been spent on finding optimal experiment designs. In this work, it is assumed that no detailed
analytical information about the system is available and that designed experiments must be used to conduct the optimization.

In many design situations, a single evaluation of the physics-motivated model is computationally costly. In Gu (2001), it is stated that one crash simulation on a full passenger car takes from 36 to 160 hours. Although these numbers are now a decade old, there is a tendency to use more sophisticated models the more computer power there is available. In fact, high computational cost of a single evaluation is still the main reason for much of the research conducted in the mechanical optimization area. To alleviate the burden, approximations of the physics-based model are often used. The approximations have been called various things, where response surfaces, meta models and surrogate models are the most common. They all denote "a model of a model" and the term surrogate model will be used in the following. With the introduction of a surrogate model, an error is introduced. The trade-off between computational efficiency and accuracy is a delicate matter and should be decided for each problem. However, the primary objective of the research presented in this thesis is to lower the computational cost of an RBDO. The use of surrogate models is one mean to achieve that and it has been used here.

Although optimization is key to stay competitive in simulation-driven development of structural components, the definition of what is ‘optimal’ is not self-evident. Conventional deterministic optimization renders designs that are optimal for a specific variable setting, usually this setting is called the nominal conditions. In reality, manufacturing- and usage-induced variability in design variables and parameters cause variation in the structural integrity of components. In the case of trucks, the structural integrity varies from one component to the other due to variation in such design variables as material properties and geometrical dimensions, and such design parameters as driver behavior and road profile. In a situation like this, a deterministically optimal design is hazardous, since the variation in performance may render catastrophic consequences. Therefore, optimization formulations which acknowledge the stochastic nature of design variables and parameters have gained increasing attention. Within this field, further distinctions will be made in the Section on the problem formulation. The classification of optimization types in structural applications made here is summarized in Table 1.

**Table 1:** Classification of properties which vary in structural optimization problems. The problems considered in the work presented in this thesis all share the properties presented in bold text.

<table>
<thead>
<tr>
<th>Objective(s)</th>
<th>Function type</th>
<th>Gradients</th>
<th>Surrogate Models</th>
<th>Design variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple</td>
<td><strong>Smooth</strong></td>
<td>Analytical</td>
<td><strong>Warranted</strong></td>
<td>Deterministic</td>
</tr>
<tr>
<td>Single</td>
<td>Discontinuous</td>
<td><strong>Numerical estimates</strong></td>
<td>Unwarranted</td>
<td><strong>Stochastic</strong></td>
</tr>
</tbody>
</table>

16
Surrogate models

Simulation-driven development of structural components involves evaluations of physics-motivated, or mechanistic, computer models. Today, the computations are usually implemented as Finite Element (FE) models. An evaluation of a simulation model is, in line with physical testing, called an experiment, or, when further clarification is needed, a computer experiment. A thought-through combination of experiments is called an experiment design and the study of the subject is called Design of Experiments (DoE). The computational effort associated with a single experiment may be large. Optimization as well as evaluation of stochastic constraints requires a significant amount of experiments. Thus, if the simulation model is to be used for each experiment, stochastic optimization becomes unfeasible from a practical point of view. Instead, a surrogate model, which is an approximating functional expression, is used to approximate the simulation model. It is not necessarily formulated with respect to the physics under study. The surrogate models are fitted to responses from experiments. A classification in interpolating and smoothing surrogate models can be made, where the former coincide with the simulation model at the experiment points whereas the latter are not required to. Smoothing surrogate models are extensively used in the robust design methodology and the closely related response surface methodology (RSM). RSM applies to physical tests, where the experiments are non-deterministic. The application to deterministic experiment data has however been questioned, see Sacks et al. (1989) and Simpson et al. (2001). In Simpson et al. (2001), it is stated that "since there is no random error it is not justifiable to smooth across data points; instead the model should hit each point exactly and interpolate between them". This opinion is reasonable. Indeed, there is no random error if the computer code is deterministic. However, the use of smoothing surrogate models is by an efficiency standard justified if the optimization converges fast. The surrogate model proposed in Paper C is in that sense justified for the problem type it was tailored for. For one example, convergence is achieved in one step even if the surrogate model does not interpolate all experiments.

A number of studies on surrogate models and their application in mechanical engineering have been performed. Some of the most common surrogate models are polynomial (often referred to as response surfaces), moving least squares, kriging, neural networks, and radial basis functions. For extensive studies of response surfaces, moving least squares models, and kriging models; see Myers and Montgomery (2002), Salkauskas and Lancaster (1986), and Martin and Craig (2005), respectively.

A surrogate model is, however, only valid in a sub-domain of the whole design domain. In this work, the surrogate model validity domain is called the Region of Confidence (RoC). First a global, permissible design space domain must be set. The permissible design domain is often set by constraints originating from requirements other than those under study. Examples of such aspects are geometrical restrictions, production-induced restrictions and performance requirements other than structural integrity. The design domain is often taken
Methods for reliability based design optimization of structural components

Figure 3: Design space and recursively updated Region of Confidence (RoC). The solid box is the permissible domain in design space and the dashed boxes are the RoCs. Red dots indicate experiment designs used to obtain responses to which the surrogate model coefficients are fitted.

as a hyperrectangle. In an optimization, the surrogate model is recursively updated as the design evolves in design space. An example of recursively updated RoCs and thereto pertaining experiment designs is shown in Fig. 3. The first RoC is often set to a (subjective) fraction of the design domain. In the simplest form, the RoC is only translated, or panned, throughout the design space as the design evolves. Another option is to use the entire design domain as the first RoC. The RoC is then updated through zooming. A simple yet effective "pan-and-zoom" algorithm has been proposed by Stander and Craig (2002).

Design of experiments

Design of experiments (DoE) is a field of applied mathematics (particularly statistics) concerned with combining experiments so that they are in various senses optimal for experiments where scatter is present. For a thorough treatment of the subject, see Myers and Montgomery (2002). Design of experiments is considered an important part of the Japanese post-war industrial development. The design for six-sigma and robust design philosophies, see Taguchi (1993), are to a high extent based on design of experiments and the more wide-spanning RSM. In the industrial development problems where design of experiments have been used, the field of application has often been to enhance efficiency of production plants and similar. There are, however, some important differences between optimization of physical production processes and simulation-driven development of structural components. In most traditional applications, a vaguely understood process is studied. Furthermore, it is practically impossible to avoid scatter in physical experiments. If an experiment is repeated using, to the best of our knowledge, identical values for those inputs that can be controlled, the response will not be the same as for the prior experiments. Experiments are simply not deterministic and it is only possible to regard the expected response. Thus, the traditional "black-box" view, where a polynomial and a stochastic error term describe the response of the studied process in a given domain of the design space, is a reasonable approach, and one advocated by heuristics. The
A polynomial term is then assumed to describe the expected response and the stochastic error term accounts for the response noise caused by the scatter in uncontrollable parameters. One of the aims of design of experiments is to minimize the uncertainty in polynomial coefficient estimates. In simulation-driven development, to the best of our knowledge, the simulation model, e.g. the FE model, is an accurate description of the physics of the phenomenon under study. Also, all parameters affecting the response can be controlled. Hence, there are reasons to use interpolating surrogate models. In a simulation context, the discrepancy between a smoothing surrogate model and the corresponding simulation model cannot be attributed to noise but rather shortcomings in the surrogate model. Consequently, the statistically founded experiment design optimality does not apply for simulation purposes. This does not mean that the scatter should be neglected but rather that there are more efficient ways to treat it. Other aspects than statistics can be regarded to form optimal computer experiment designs. In the work presented in this thesis, an experiment design that requires a minimum of simulations for fitting an interpolating surrogate model is used. Moreover, a novel approach to DoE which takes advantage of the specific problem’s structure, called constraint-orthogonal experiment design, is presented in Paper B. Examples of experiment designs are given in Fig. 4, including an example of the reduced design presented in Paper B.

A schematic description of experiment design through the use of an initial one-factor-at-a-time design (Koshal) followed by a screening of the significance of effects from input variables on the response leading up to an optimal several-factors-at-a-time experiment design is shown in Fig. 5

In Paper D, an experiments-on-demand procedure is presented.
Figure 5: Schematic description of experiment design through the use of an initial one-factor-at-a-time design followed by a screening of the significance of effects from input variables on the response leading up to an optimal several-factors-at-a-time experiment design.

**Problem formulation**

The overall aim in optimization under uncertainty is to find a design which, accounting for uncertainties, is optimal. Common to all branches of optimization under uncertainty is to find a solution which avoids undesired performance for a large portion of a population of components. Following Taguchi (1993), an optimal design should minimize the societal cost. The societal cost includes all costs; the direct customer cost, e.g. the purchasing and repair cost, the manufacturer cost, e.g. warranty cost, and the third party cost, e.g. costs related to failures, such as medical costs. Formulating a cost function which correctly incorporates all these costs is a difficult task. Instead, requirements on stochastic properties of the performance, for instance the variance or failure probability, which are easier to quantify, are introduced. The assumption behind this appears to be that, within certain bounds, the societal, or at least customer and manufacturer costs, increases with these properties, even if the exact relation is unknown.

Two main branches of optimization under can be distinguished; robust design and reliability based design optimization. Robust design aims at finding a design which, while meeting optimality conditions, is insensitive to noise. Noise in this sense is undesired variations in design variables and parameters. This approach is beneficial for irregular functions, where, for a number of nominal designs, the expected responses are in the same order but the same noise cause largely different variations in performance. In contrast, reliability based design optimization (RBDO), has mostly been applied to problems with strictly decreasing or increasing performance. For this class of problems, a sufficient distance or safety factor, formulated mathematically as a probabilistic constraint on the design, is sought. The RBDO optimization problem is stated as
Figure 6: Probability density function (blue) and cumulative distribution function (black) for a normally distributed variable $X$ with mean $\mu = 0$ and standard deviation $\sigma = 0.5$.

\[ \begin{align*}
\min_{\mathbf{x}} & \quad C(\mathbf{x}) \\
\text{subject to} & \quad \left\{ \begin{array}{l}
p_{f,j}(\mathbf{x}) \leq \alpha_{req,j}, \ j = 1, 2, \ldots, N_C \\
x^L_i \leq x_i \leq x^U_i, \ i = 1, 2, \ldots, N_X
\end{array} \right.
\end{align*} \] (1)

where $\mathbf{x}$ is a vector of design variables $x_i$, $i = 1, 2, \ldots, N_X$ and $C$ is the cost function. The denotation $p_{f,j}$, $j = 1, 2, \ldots, N_C$ is the probability of failure with respect to constraint $j$ and $\alpha_{req,j}$ is the required (maximum allowed) probability of failure for constraint $j$. Finally, $x^L_i$ and $x^U_i$ are stochastic design variable $x_i$’s lower and upper bound, respectively. In RBDO, a distinction is often made between deterministic design variables, stochastic design variables and stochastic design parameters. In this work, the term design variable will generally be used except when further clarification is needed. Design variable $x_i$ is a continuous variables and it has an associated probability density function $f_{X_i}(x_i|\theta)$ and cumulative distribution function $P(X_i \leq x_i) = F_{X_i}(x_i|\theta)$, where $\theta$ are distribution coefficients. Examples of commonly used distribution types are the normal distribution, the log-normal distribution, and the Weibull distribution. For a normally distributed variable, the coefficients $\theta_i = [\mu_i, \sigma_i]$ needed to describe the distribution is the mean, $\mu_i$, and the standard deviation, $\sigma_i$. An example of a normally distributed variable, $X \sim \mathcal{N}(0, 0.5)$ is shown in Fig. 6.

The constraints on $\mathbf{x}$ in Eq. (1) should be interpreted as constraints on the distribution coefficients $\theta$ rather than $\mathbf{x}$, since a design variable $X_i$ may in general take any value on the real axis, i.e. $x_i \in ]-\infty, \infty[$. In statistics, a distinction between aleatory and epistemic uncertainty, where the former means scatter or intrinsic variation and the latter refers to lack of information, is sometimes made. Attributing a normal distribution to a design variable is in this context a recognition of aleatory uncertainty. However, estimating the coefficients needed to describe a normal distribution is associated with epistemic uncertainty. This uncertainty may in general be significant. Only aleatory uncertainty is regarded in the work presented here but the difficulties adherent to epistemic uncertainty is acknowledged. It is one of the main
Methods for reliability based design optimization of structural components

reasons why computational efficiency is the focus of the methods in this thesis, even at the expense of a slight accuracy loss. The assumption behind the work here is that of all the errors in the prediction of failure probability, the errors introduced by the computational methods are smaller than those related to epistemic uncertainty. In this work, the cost function does not include the cost of failures. Instead, there is a constraint on \( p_f \). It can be said with some certainty that the total cost of the component would increase if the failure probability was higher than the required failure probability used in this work. However, in Paper A, a suggestion on how to include a more detailed estimate of a component’s production costs than just the cost of its mass in the optimization is made. Moreover, it has been assumed that the manufacturing precision cannot be altered. Thus, those coefficients in \( \theta \) that are related to the spread of a design variable, e.g. the standard deviation for a normally distributed variable, are fixed. The mean, the median, and the mode of a stochastic variable are all measures of location. In this work, only the means are considered to be design variables. Also, the same required probability of failure has been used for all constraints. The RBDO formulation thus reduces to

\[
\min_{\mu} \quad C(\mu) \\
\text{subject to} \quad \left\{ \begin{array}{l} p_{f,j}(x) \leq \alpha_{\text{req}}, \ j = 1, 2, \ldots, N_C \\
\mu_i^L \leq \mu_i \leq \mu_i^U, \ i = 1, 2, \ldots, N_X \end{array} \right.
\]

(2)

Treatment of probabilistic constraints

The probability of failure with respect to constraint \( j \), \( p_{f,j} \) can be stated as

\[
p_{f,j} = P(G_j(X) \leq 0) = \int_{G_j(X) \leq 0} f_X(x)dx,
\]

(3)

where \( G_j, \ j = 1, 2, \ldots, N_C \) is a performance or failure function and \( G_j \leq 0 \) means failure. A graphic representation of the failure probability constraint is shown in Fig. 7.

In computational solid mechanics problems, \( G_j \) is almost without exception constituted by an FE model, which is computationally demanding to evaluate. Therefore, a surrogate model \( \hat{G}_j \) is used to approximate it. Since the optimization is performed with respect to the mean values \( \mu \), a relation between the probability of failure and the means is needed. Two main approaches can be identified for evaluation of Eq. (3); sampling based algorithms, such as Monte Carlo simulation, see Rubinstein (1981), and developments of it, and semi-analytical evaluations, such as the First and Second Order Reliability Method (FORM and SORM, respectively), see Madsen et al. (1986). The computational effort associated with the sampling-based estimates of failure probability increases as the target failure probability decreases. For low probabilities, the computational effort can be comparable to that of the FE model evaluation. Also, the error associated with the sampling based methods is stochastic.
Stochastic results are less advantageous than bias errors from a convergence point of view. For these reasons and ease of implementation, sampling based procedures have not been used to predict the failure probability in this work. Early works on analytical evaluations of failure probability introduced the first order second moment reliability index, see Cornell (1969) and Hasofer and Lind (1974). First order refers to the order of the Taylor approximation of the failure function whereas second moment refers to the statistical measure used to describe the stochastic variables. For linear failure functions and normally distributed failure functions, the first order second moment approach is exact. A logical next step would be to use a complete description, i.e. to attribute a distribution function, for the stochastic variables. The method of approximating the failure probability using first order failure functions and complete distribution functions is called the First Order Reliability Method (FORM), see Madsen et al. (1986), and it is commonly used in RBDO algorithms presented. In FORM, an isoprobabilistic transformation of the stochastic variables $X_i$ to standard normally distributed variables $U_i$ is carried out as

$$u_i = \Phi^{-1}(F_{X_i}(x_i | \mu_i)). \tag{4}$$

The relation in Eq. (4) holds for independent variables. Examples of the transformation can be found in Table 2.

If the variables are not independent, the Rosenblatt transformation, see Rosenblatt (1952), can be used to obtain independent standard normally distributed variables. If the individual (marginal) distributions and the covariances of the design variables are known, the Nataf
Table 2: Isoprobabilistic transformations of stochastic variables.

<table>
<thead>
<tr>
<th>Type</th>
<th>PDF</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>$X \sim \mathcal{N}(\mu, \sigma)$</td>
<td>$x = \mu + \sigma u$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$X \sim \mathcal{U}(a, b)$</td>
<td>$x = \mu - \frac{b-a}{2}(1 + 2\Phi(u))$</td>
</tr>
<tr>
<td>3-par. Weibull</td>
<td>$X \sim \mathcal{W}(\theta, \lambda, \beta)$</td>
<td>$x = \theta + \frac{-\ln(1-\Phi(u))}{\lambda^{1/\beta}}$</td>
</tr>
</tbody>
</table>

transformation, see Liu and Kiureghian (1986), can be used for the same purpose. The constraint in Eq. (3) can, after the transformation, equivalently be expressed as

$$p_{f,j} = P(\hat{G}_j(U) \leq 0) = \int_{\hat{G}_j(U,\mu) \leq 0} f_U(u) du,$$

where also the use of a surrogate model $\hat{G}_j$ has been introduced. The transformation from design space to standard normal space is interpreted graphically in Fig. 8. Due to the hyperrotatability of the multivariate standard normal distribution, the shortest distance from the origin to the failure limit state $\hat{G} = 0$ determines the probability of failure for linear integration domains. The point $u^*$ on the limit state which is closest to the origin is also the point on the limit state function where the integrand $f_U(u)$ is largest. Thus, it is often referred to as the Most Probable Point of Failure (MPFP). Finding the MPFP is in itself an optimization problem. The MPFP can be found by minimizing $u^T u$ subject to $\hat{G} = 0$, where $\hat{G}$ does not need to be linear. Obviously, for the optimality conditions to be satisfied, the MPFP $u^*$ need to be proportional to the partial derivatives of the failure function at the MPFP $\left.\frac{\partial \hat{G}}{\partial u}\right|_{u=u^*}$.

In FORM, the failure probability is approximated by a linearization of the failure function at the MPFP.

![Figure 8: Transformation of the failure probability constraint from design space to standard normal space.](image-url)
A number of solution strategies for RBDO have been presented. A thorough review is given in Schueller and Jensen (2008). In the work presented in this thesis, a sequential approximation procedure involving two nested optimization loops is used to solve the optimization problem. A flowchart of the RBDO algorithm is presented in Fig. 9.

The outer loop \((k)\), is related to updates of the surrogate model. The inner loop \((l)\) is related to linearizations of the failure function. In each iteration \((k,l)\), the MPP estimate is updated and, as the optimization converges, so does the MPP and the estimated failure probability. Also, a single point of multiple constraint approximation is proposed. The point of constraint approximation is formulated with respect to the probabilistic constraints and reduces the computational effort for a multiply constrained RBDO to that of a single constraint RBDO problem. For the examples used, only minor errors in constraint satisfaction are introduced compared to FE model errors and epistemic uncertainties typical for industrial applications in the experience of the author.
Methods for reliability based design optimization of structural components
Summary of appended papers

Paper A: *Reliability based design optimization using a single constraint approximation point.*

The computational effort for reliability based design optimization (RBDO) is no longer prohibitive even for detailed studies of mechanical integrity. The sequential approximation RBDO formulation and the use of surrogate models have greatly reduced the amount of computations necessary. In RBDO, the surrogate models need to be most accurate in the proximity of the most probable point. Thus, for multiply constrained problems, such as fatigue design problems, where each finite element (FE)-model node constitutes a constraint, the computational effort may still be considerable if separate experiments are used to fit each constraint surrogate model. This paper presents an RBDO algorithm that uses a single constraint approximation point (CAP) as a starting point for the experiments utilized to establish all surrogate models, thus reducing the computational effort to that of a single constraint problem. Examples of different complexities from solid mechanics applications are used to present the accuracy and versatility of the proposed method. In the studied examples, the ratio of the computational effort (in terms of FE-solver calls) between a conventional method and the single CAP algorithm was approximately equal to the number of constraints and the introduced error was small. Furthermore, the CAP-based RBDO is shown to be capable of handling over 10,000 constraints and even an intermittent remeshing. Also, the benefit of considering other objectives than volume (mass) is shown through a cost optimization of a truck component. In the optimization, fatigue-specific procedures, such as shot peening and machining to reduce surface roughness, are included in the cost as well as in the constraints.

Paper B: *Efficient design of experiments for structural optimization using significance screening.*

When performing structural optimization of large scale engineering problems, the choice of experiment design is important. However, classical experiment designs are developed to deal with undesired but inevitable scatter and are thus not ideal for sampling of deterministic computational results. In this paper, a novel screening and design of computer experiments algorithm is presented. It is based on the concept of orthogonal design variable significances
and is applicable for problems where design variables do not simultaneously have a significant influence on any of the constraints. Examples of fields where such applications arise are fatigue design and crash safety. The algorithm presented uses significance orthogonality to combine several one-factor-at-a-time experiments in one several-factors-at-a-time experiment. The procedure results in a reduced experiment design matrix. In the reduced experiment design, each variable is varied exactly once but several variables may be varied simultaneously, if their significances with respect to the constraints are orthogonal. Moreover, a measure of influence, as well as an influence significance threshold, is defined. In applications, the value of the threshold is left up to the engineer. To assist in this choice, a relation between model simplification, expressed in terms of the significance threshold, and computational cost is established in a screening. The relation between efficiency and loss of accuracy for the proposed approach is discussed and demonstrated. For two solid mechanics type problems studied herein, the necessary number of simulations could be reduced by 25 % and 64 %, respectively, with negligible losses in accuracy.

**Paper C: A directional surrogate model tailored for efficient reliability based design optimization.**

Reliability based design optimization (RBDO) aims at minimizing an objective while constraining the failure probability of structural components. Due to the iterative nature of both the minimization and the failure probability validation, there is considerable computational effort associated with it. In this paper, a computationally inexpensive approach for RBDO is presented. The key contribution is the directional surrogate model and its associated advantages; a balance between accuracy and computational cost, including the possibility to fit model coefficients based on an optimal experiment design, high fidelity modeling of representative structural responses, treatment of multiple constraints without added computational cost, and straightforward sequential linear programming implementation. The directional surrogate model is of power type with non-linearity only in the gradient direction, thus balancing accuracy and computational cost. Moreover, information from prior iterations is used in every iteration in a weighted least squares optimization. When benchmarked against existing approaches from the literature using a well known reference problem, it is shown to be highly efficient. It also shows promising stability and convergence rate for additional challenging problems to which it has been applied.
Paper D: *Reliability based design optimization with experiments on demand.*

In this paper, an algorithm for reliability based design optimization (RBDO) is presented. It incorporates a novel procedure in which experiments are performed one at a time where and when they are needed. The procedure is called experiments on demand. The experiment procedure utilizes properties specific to RBDO and the problem at hand augmented by the concept of $D$-optimality familiar from traditional design of experiments. Furthermore, an adaptive surrogate model fitting scheme is proposed which balances numerical stability and convergence rate as well as accuracy. Benchmarked against algorithms in the literature, the number of experiments needed for convergence was reduced by up to 80% for a frequently used analytical problem and by up to 19% for an application example. The accuracy of the reliability index is in line with the most efficient algorithm against which it was benchmarked but up to 3% lower than the most accurate algorithm.
Methods for reliability based design optimization of structural components
Bibliography


Methods for reliability based design optimization of structural components


