The Klimontovich Description of Complex Plasma Systems;

Low Frequency Electrostatic Modes, Spectral Densities of Fluctuations and Collision Integrals

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Doctoral Thesis in Physical Electrotechnology
Stockholm, Sweden 2012
Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie doktorsexamen i fysikalisk electroteknik fredagen den 23 mars 2012 klockan 13.15 i H1, Teknikringen 33, KTH, Stockholm.

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Tryck: Universitetsservice US-AB
Abstract

Plasmas seeded with solid particulates of nanometer to micron sizes (complex plasma systems) are a ubiquitous feature of intergalactic, interstellar and planetary environments but also of plasma processing applications or even fusion devices. Their novel aspects compared with ideal multi-component plasmas stem from (i) the large number of elementary charges residing on the grain surface, (ii) the variability of the charge over mass ratio of the dust component, (iii) the inherent openness and dissipative nature of such systems.

Their statistical description presents a major challenge; On one hand by treating dust grains as point particles new phase space variables must be introduced augmenting the classical Hamiltonian phase space, while the microphysics of interaction between the plasma and the grains will introduce additional coupling between the kinetic equations of each species, apart from the usual fine-grained electromagnetic field coupling. On the other hand complex plasma systems do not always exist in a gaseous state but can also condensate, i.e. form liquid, solid or crystalline states.

In this thesis we study gaseous partially ionized complex plasma systems from the perspective of the Klimontovich technique of second quantization in phase-space, initially, in regimes typical of dust dynamics. Starting from the Klimontovich equations for the exact phase space densities, theory deliverables such as the permittivity, the spectral densities of fluctuations and the collision integrals are implemented either for concrete predictions related to low frequency electrostatic waves or for diagnostic purposes related to the enhancement of the ion density and electrostatic potential fluctuation spectra due to the presence of dust grains. Particular emphasis is put to the comparison of the self-consistent kinetic model with multi-component kinetic models (treating dust as an additional massive charged species) as well as to the importance of the nature of the plasma particle source.

Finally, a new kinetic model of complex plasmas (for both constant and fluctuating sources) is formulated. It is valid in regimes typical of ion dynamics, where plasma discreteness can no longer be neglected, and, in contrast to earlier models, does not require relatively large dust densities to be valid.
Acknowledgements

I would like to express my gratitude to My Tsarina, Svetlana Ratynskaia; her uncompromising attitude, her enthusiasm and dedication to science and her volcanic (bi)polar temperament have made my two year long Ph.D experience a real pleasure. It has been a blessing having a supervisor with whom I share common viewpoints about life and science, that has always been available for scientific discussions regardless of time or workload, that has been supportive in both scientific and personal mishaps, and most of all has managed to tame (partly) my overambitious nature. Always caring not about the amount of papers we can produce but about my overall scientific awareness and ethos, you had been a model supervisor for me, even though you were asphyxiating in those pedagogical courses enforced on you ...

It has been an honor to work with The Professore, Umberto de Angelis, whose work on the kinetic theory of complex plasmas has been a rare source of inspiration, but also frustration since such a diamond work going unnoticed makes you wonder...

I am thankful to Lars Blomberg and Nickolay Ivchenko for being co-operative and encouraging and thus making my Department life easier, and Johanna Bergman for helping me to struggle through every day formalities.

Last but certainly not least, I would like to thank my family, friends and all those who stood by me in the tragedy that befell me in the year 2010. Not to mention music, whiskey and silence for keeping me company in those late night working hours that I always cherish.
Στην μνήμη του πατέρα μου
Σπυρίδωνα Γεωργίου Τόλια

Τα πάντα ρεί και ουδέν μένει
List of Papers

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Chapter 1

Introduction

Complex (dusty) plasmas are non-ideal plasma systems seeded with solid particulates. Such systems are ubiquitous in astrophysical environments [Goertz, 1989; Whittet, 2002; Draine, 2003], laboratory applications [Bouchoule, 1999; Boufendi and Bouchoule, 2002] or even fusion devices [Krasheninnikov et al., 2010]. Their novel aspects compared with ideal multi-component plasmas stem from the fact that the charge over mass ratio of the dust species is not constant. In particular, the dust charge 'breathes' with the local plasma parameters and its quasi-stationary equilibrium value is set up by the balance of the plasma fluxes absorbed and emitted by the grain. Moreover, complex plasmas are open systems, they exchange particles and energy with the ambient environment in order for the grains to maintain their charges. Through the continuous charging process grains serve as a sink of plasma particles and/or radiation which should be replenished by an external source, whose nature plays an important role in the kinetics and thermodynamics of the system [Tsytovich et al., 2008].

Complex plasma systems are most commonly met in nature in a gaseous state, where the species kinetic energy greatly exceeds the interaction energy. The Klimontovich technique of second quantization in phase-space [Klimontovich, 1958] offers a self-consistent approach for the description of such systems [Tsytovich and de Angelis, 1999] and also a unifying framework for the study and interpretation of diverse topics such as the spectral densities of fluctuations and the scattering of radiation, wave dispersion and stochastic acceleration, hydrodynamic equations and transport coefficients, collective dust interaction inducing transitions from disordered weakly coupled to ordered strongly coupled states.

The main motivation behind this thesis is twofold. Initially, the thesis is focused on the comprehensive study of the previously "unexplored" kinetic models of partially ionized complex plasmas [Tsytovich et al., 2005]. More specifically, different kinetic models are formulated in presence of neutrals and compared to identify regimes where the self-consistent description is necessary, low frequency electrostatic waves are investigated self-consistently and a new long-wavelength mode is
discovered in presence of strong electron impact ionization of neutrals, the spectral densities of fluctuations are derived and numerically studied as a new diagnostic tool for sub-micron dust detection and composition. In addition, the thesis is devoted to the formulation and development of a kinetic model of complex plasmas in frequency ranges typical of ion dynamics. Previous kinetic models have laid the theoretical foundations for the extension of the Klimontovich approach for complex plasma systems. However, they endure severe mathematical complexities and have a limited parameter range of validity. They are not only restricted to low frequency regimes typical of dust dynamics, but also they impose a strong restriction on the dust densities, that should be high enough for binary plasma collisions to be neglected in comparison to collisions with dust and simultaneously low enough for the dust species to be in its gaseous phase.

The basic scientific results of this thesis are gathered in six papers.

**Paper I** employs a self-consistent kinetic model of fully ionized complex plasmas for the study of dust acoustic waves and the identification of experimentally accessible parameter regimes where the deviations from the commonly used collisionless hydrodynamic description are most pronounced.

**Paper II** formulates the "standard" and multi-component kinetic models in presence of neutrals and compares with a self-consistent kinetic model of partially ionized complex plasmas in terms of the static and dynamic permittivities. Moreover, the low frequency responses of the dust and plasma species are computed taking into account the effect of neutrals in the ion capture cross-sections in the weakly collisional regime. A criterion for the omission of induced dust density fluctuations in frequencies larger than the dust plasma frequency is numerically investigated.

**Paper III** is focused on the kinetic study of low frequency electrostatic waves. The effect of ionization and neutral pressure in the dust acoustic waves is investigated. A novel long-wavelength mode is discovered and attributed to the interplay between plasma absorption on dust and electron impact ionization of neutrals, a physical mechanism is proposed and its properties are numerically investigated.

**Paper IV** is devoted to the experimental / theoretical study of dust charging in the intermediate collisional regime of ions with neutrals, such a regime lies between the pair-collision and hydrodynamic regimes and analytical expressions do not exist for the ion fluxes to the dust grain. The force balance method is used for the determination of the grain charge in experiments run in the Plasmakristall-4 facility, an interpolation formula is proposed for the ion flux that is in remarkable agreement with the experiments.

**Paper V** deals with the spectral densities of ion density and electrostatic potential fluctuations, that are derived in presence of neutrals. Numerical results are presented investigating the dependence on pressure and electron temperature for the first time, the parameters used refer to realistic quiescent plasma laboratory configurations. The results support the feasibility of the use of the spectral densities as a new diagnostic tool for the determination of the dust composition. A
condition is derived for the omission of plasma discreteness that properly defines the low frequency regime of dust dynamics.

**Paper VI** formulates a kinetic model of complex plasmas in time scales relevant for ion dynamics (for both constant and fluctuating plasma sources). The permittivity, the collision integrals and the spectral densities are derived in their general forms. The structure of the ion kinetic equation is analyzed and applications for both astrophysical and laboratory environments are discussed.
Chapter 2

The Klimontovich Description of Ideal Un-magnetized Plasmas

This chapter can be regarded as a compendium summarizing results from the kinetic description of ideal un-magnetized plasmas. The problem is viewed from the perspective of the Klimontovich approach as implemented by Tsytovich [Tsytovich, 1989; Tsytovich, 1995], this method despite being mathematically elegant is rarely encountered even in specialized textbooks.

We start by reviewing different methods of kinetic description of ideal un-magnetized plasmas and continue by presenting the Klimontovich approach in a generalized fashion that will facilitate the transition to complex plasma systems. The system is assumed to consist of an arbitrary number of negatively and positively charged species, which implies that the results also correspond to simplified multi-component models of complex plasmas.

2.1 Statistical description of plasmas

The first attempt for a statistical description of plasmas originates back to Landau and his derivation of the Landau kinetic equation [Landau, 1937]. The starting point in Landau’s approach is the Boltzmann kinetic equation. The latter, originally derived for rarified gases, incorporates only pair-wise interactions in the collision integral, justified due to the short-range nature of the molecular forces. However, due to the long-range nature of Coulomb forces, such a treatment is not self-consistent for plasmas. This inconsistency manifests itself as a number of divergent integrals arising in the treatment of the collision integral.

1. The collision integral diverges for Coulomb cross-sections and large distances between charged particles. This implies that collisions between charged particles are important at large distances, where the change in momentum and the scattering angle are small. This led Landau to a first order expansion of the collision integral with respect to the momentum transfer
CHAPTER 2. THE KLIMONTOVICH DESCRIPTION OF IDEAL UN-MAGNETIZED PLASMAS

and is similar to the argument underlying the Fokker-Planck description of plasmas; "The cumulative effect of many small angle deflections in the rate of change of a particle’s kinetic energy turns out to be much larger than the effect of the infrequent large angle deflections".

2. **The collision integral diverges logarithmically for both small and large distances.** Heuristically, we can say that the collision integral contains integrals of the form \( \int \frac{1}{k^2} dk \), which result from the Fourier transform of the bare Coulomb potential, \( \phi(k) \propto \frac{1}{k^2} \). In order to encounter this problem Landau added upper and lower integration limits to the wavenumber \( k \). The low integration limit is set on \( k_{\text{min}} = \frac{1}{\lambda_D} \), where \( \lambda_D = \frac{1}{\epsilon} \) is the plasma Debye length, it expresses the fact that the Debye sphere defines the sphere of interaction of many charged particles. It can also stem from substituting the Coulomb potential with the more appropriate Yukawa potential, with a Fourier transform \( \phi(k) \propto \frac{1}{k^2 + \lambda_Y^2} \). The upper integration limit is set on \( k_{\text{max}} = \frac{1}{R_c} \), where \( R_c \) is the Coulomb radius defined by equating the Coulomb potential energy with the particle thermal energy, \( R_c = \frac{e^2}{T} \). It expresses the weak coupling approximation, for the first order expansion to be valid the particle kinetic energy must be weakly perturbed by the fields of the colliding particles, i.e \( r > R_c \). Overall, the wavenumber integral will now give \( \int \frac{1}{k^2} dk \sim \int_{k_{\text{min}}}^{k_{\text{max}}} \frac{1}{k^2} dk \sim \ln \frac{k_{\text{max}}}{k_{\text{min}}} \sim \ln \frac{\lambda_D}{R_c} \), where the argument of the natural logarithm is proportional to the number of particles in the Debye sphere, \( n\lambda_D^3 \gg 1 \). Even though the limits of integration are approximate, the low sensitivity of the logarithm in case of large arguments, ensures the correctness of results.

Landau’s treatment is rather a result of his brilliant physical intuition than of mathematical rigor. However, his treatment essentially includes the physics behind more strict statistical approaches, namely, (i) small momentum transfer in collisions, (ii) static screening of the fields, (iii) omission of non-linear effects. We should also note that in the more precise collision integrals to be derived later, the Landau collision integral can be recovered by using \( \epsilon_{k, k'} \cdot v' \simeq 1 \). This demonstrates that a subtle effect not taken into account is the dynamic plasma polarization during collisions, that can be important especially for systems far from equilibrium.

The strict theoretical foundations of a kinetic theory of gases/plasmas were laid by Bogoliubov [Bogoliubov, 1946]. To illustrate the main ideas, let us assume a system of \( N \) particles, where each can be fully described by a set of \( X = \{ x_1, x_2, ..., x_X \} \) dynamic variables. Every particle is allocated \( X \) coordinates in a \( X \times N \) dimensional phase space, where the system is represented by a single point at any given time instant. Then the exact density of the system will be a product of \( X \times N \) \( \delta \)-functions centered at the solutions of the deterministic equations describing the evolution of the system with given initial conditions.

Statistics can then be made by assuming an ensemble of such systems, each prepared with different initial conditions. Then one can define distribution func-
2.2. MICROSCOPIC PHASE-SPACE DENSITIES AND SMALL PARAMETERS

In the Klimontovich kinetic scheme the starting point is the microscopic single particle distribution function of each species \( f^\alpha(X, t) \), that in the case of point particles is simply a sum of \( \delta \)-functions positioned at exact phase-space trajectories. Here \( X \) denotes the set of phase-space variables, these are independent variables that fully characterize any possible state of the particle and typically refer to the position and momentum. However, for complex systems they can be complemented by the angular momentum and the internal energy - e.g plasma-molecular systems [Klimontovich \textit{et al.}, 1989] or the charge, mass, surface temperature - e.g complex plasma systems [Schram \textit{et al.}, 2003; Tsytovich \textit{et al.}, 2004]. Moreover, in case of open systems, \( f^\alpha(X, t) \) can become more complicated in order to allow for birth/death processes and chemical transformations - e.g partially ionized systems, chemically reactive systems, complex plasmas [Klimontovich \textit{et al.}, 1987].

The phase space variables \( X \) follow sets of deterministic dynamic equations, that, when complemented by the sets of equations describing the fields that are produced by the particles themselves, can fully characterize any future state of the many-body system provided initial conditions are known. For example for ideal magnetized plasmas, the deterministic equations consist of the Hamilton equations...
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of motion and the Maxwell equations for the electromagnetic fields.

Integro-differential equations for \( f^\alpha(X, t) \), the Klimontovich equations, can be constructed (i) either by differentiating their known solution with respect to time and using the dynamic equations (ii) or by repeated integration of the Liouville equation over the phase-space variables of all particles except one, and they represent an exact description of the system [Klimontovich, 1967].

Departure from such a deterministic description is achieved by decomposing the distribution functions and the self-consistent fields in regular and fluctuating parts [Tsytovich, 1995],

\[
    f^\alpha(X, t) = \Phi^\alpha(X, t) + \delta f^\alpha(X, t) \tag{2.1}
\]

where \( \Phi^\alpha(X, t) = \langle f^\alpha(X, t) \rangle \) is the smooth regular part with the brackets denoting the average over the Gibbs ensemble \(^1\) of all discrete species and \( \delta f^\alpha(X, t) \) with \( \langle \delta f^\alpha(X, t) \rangle = 0 \) is the spiky fluctuating part. The latter can be further separated into the natural fluctuations \( \delta f^{\alpha,(0)}(X, t) \), connected directly to the discrete nature of the system, and fluctuations induced by the self-consistent fields \( \delta f^{\alpha,(\text{ind})}(X, t) \). This decomposition has both physical and mathematical grounds, since \( \delta f^{\alpha,(0)}(X, t) \) is the homogeneous solution of the fluctuation equation in absence of fields (free streaming particles) and \( \delta f^{\alpha,(\text{ind})}(X, t) \) is the particular solution.

The scheme proceeds with applying the decomposition in the Klimontovich and the field equations and ensemble averaging. This will yield a partial integro-differential equation for the regular part \( \Phi^\alpha(X, t) \), the kinetic equation, that will also contain ensemble averages of the product of two fluctuating quantities, i.e the collision integral. Subtraction of the averaged part from the decomposed equation, will result to an equation for the fluctuating part which is then Fourier transformed in space and time. The above system of equations is closed and can be easily treated mathematically after invoking the following basic assumptions:

The first basic assumption of the Klimontovich kinetic scheme is that products of fluctuating quantities and their ensemble averages can be omitted in the equations for the fluctuating parts. The omission of high order terms in fluctuations is equivalent to the omission of triple correlation functions. It is essential for the formation of a closed system of equations and its justification is connected with the existence of small dimensionless parameters that characterize the nature of the system [Klimontovich, 1997]. Nevertheless, such an assumption definitely restricts the systems for which the description is applicable, i.e molecular systems should be rarefied and fully ionized systems should be in the gaseous state.

In the case of rarefied gases: the radius \( r_0 \) of action of the molecular forces is small compared to the mean intermolecular distance \( r_{\text{av}} \). Thus, the simultaneous

\(^1\)In statistic descriptions instead of focusing on a unique system, we focus on an ensemble consisting of a very large number of identical realizations of our original system, all prepared subject to whatever conditions are specified. In general, the systems in the ensemble will be in different states and characterized by different macroscopic parameters. By definition, the concept of ensemble is directly linked to the large number of experiments that can be conducted on a macroscopic system and the concept of probability space in mathematics.
2.3. SYSTEMS OF NON-INTERACTING PARTICLES AND THE NATURAL STATISTICAL CORRELATOR

approach of three molecules within the sphere of action \( r_0 \) is a rare event due to the smallness of the parameter \( \epsilon_g = \frac{r_0^3}{r_{av}^3} \). In that sense triple correlation functions can be ignored being second order on \( \epsilon_g \) and a closed system of equations for single particle regular distributions and double correlation functions or equivalently single particle regular distributions and their fluctuations can be constructed.

In the case of plasmas: the charged particles are interacting via long-range Coulomb forces. The effective radius of action of the forces in a plasma is the Debye radius \( \lambda_D \) defined for a multi-component plasma via

\[
\lambda_D^2 = \sum_\alpha \frac{4\pi e^2 n_\alpha}{T_\alpha},
\]

while the small parameter is \( \epsilon_p = \frac{\nu_{coll}}{\omega_p} \), which implies that there are many particles in the Debye sphere (also notice that the ratio of interaction to kinetic energy for a particle is approximately proportional to \( \epsilon_p^{2/3} \) which means that the system is in the gaseous phase). However, the ratio of collisional to collective effects is also approximately proportional to \( \epsilon_p \). Therefore, the pair correlation function will be of the order of \( \epsilon_p \) and the triple correlation function of the order of \( \epsilon_p^2 \) and thus negligible.

The second basic assumption of the Klimontovich kinetic scheme refers to the Bogoliubov hypothesis of a hierarchical structure in the characteristic temporal and spatial scales of variations. We begin with the relaxation time \( \tau_0 \) of a hydrodynamic quantity, it can be approximated by the ratio of the system’s scale \( L \) to the sound velocity \( c_s \), \( \tau_0 = L/c_s \). The characteristic time \( \tau_1 \) for a single particle distribution function to relax to its local equilibrium values is approximated by the ratio of the mean free path in Coulomb collisions to the thermal velocity of the particles, which is the inverse of the collision frequency, \( \tau_1 = l_{mf}/u_{th} = 1/\nu_c \). Finally, the characteristic time \( \tau_2 \) for the relaxation of a pair correlation function is estimated by the average time for a particle to travel over a correlation distance, hence it is approximately the ratio between the Debye length and the thermal velocity, which is the inverse of plasma frequency \( \tau_2 = \frac{\lambda_D}{u_{th}} = \frac{1}{\omega_p} \).

The above lead to the inequalities \( L \gg l_{mf} \gg \lambda_D \gg \tau_0 \gg \tau_1 \gg \tau_2 \), that are always satisfied in a plasma due to the smallness of the parameter \( \epsilon_p = \frac{\nu_{coll}}{\omega_p} \). This space-time scale separation is of paramount importance in the treatment of the equations for the fluctuating parts, enabling us to treat ensemble averaged quantities as constant in the fluctuations space-time scales [Ichimaru, 1992].

2.3 Systems of non-interacting particles and the natural statistical correlator

Before proceeding in the Klimontovich equation for ideal un-magnetized plasmas, we take a small de-tour to analyze systems of non-interacting particles. The application of fluctuation theory for such systems results in a relation for the natural
CHAPTER 2. THE KLIMONTOVICH DESCRIPTION OF IDEAL UN-MAGNETIZED PLASMAS

statistical correlator, a key relation to the Klimontovich description of any system [Tsytovich, 1995].

We assume an unbounded system of non-interacting particles of different species. In lack of particle collisions and in absence of external fields, the system’s Hamiltonian will be $H(p_1, p_2, ..., p_N) = \sum_{1 \leq i \leq N} \frac{p_i^2}{2m_i}$, whereas the momentum of each particle will be independent of time. Therefore, the phase-space variables will be $X = \{r, p\}$ and the microscopic phase-space densities for each species can be written as $f^\alpha_p(r, t) = \sum_{1 \leq i \leq N\alpha} \delta(r - r_i(t))\delta(p - p_i)$ and differentiation with respect to time, together with $\frac{\partial r_i}{\partial t} = v_i(t)$ will yield the Klimontovich equation

\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \right) f^\alpha_p(r, t) = 0 .
\]  \tag{2.2}

In such a system the exact distribution function can be decomposed into regular and fluctuating parts, $f^\alpha_p(r, t) = \Phi^\alpha_p(r, t) + \delta f^\alpha_p(0)(r, t)$, where the latter appear solely as a consequence of the discrete nature of matter. The regular part of the Klimontovich equation is

\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \right) \Phi^\alpha_p(r, t) = 0 ,
\]  \tag{2.3}

while the fluctuating part is

\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \right) \delta f^\alpha_p(0)(r, t) = 0 .
\]  \tag{2.4}

Due to the random character of the natural fluctuations in a gas, one cannot acquire deterministic relations and only a stochastic description is viable. Knowing that their mean value is zero, it suffices to obtain a relation for the second moment, i.e. the natural statistical correlator $\langle \delta f^\alpha_p(0)(r, t)\delta f^\beta_p(0)(r', t') \rangle$ or more conveniently in Fourier space $\langle \delta f^\alpha_p(k, \omega)\delta f^\beta_p(k', \omega') \rangle$.

In absence of an interaction potential, the fluctuating part of the distribution function is stationary in space and time, hence all correlations between fluctuations are functions of the time difference and the difference between the spatial coordinates only, $\langle \delta f^\alpha_p(0)(r, t)\delta f^\beta_p(0)(r', t') \rangle = \langle \delta f^\alpha_p(0)\delta f^\beta_p(0) \rangle(r - r', t - t')$. Fourier
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transforming in space and time we get

$$\langle \delta f^\alpha_{p,k,\omega} \delta f^{\beta}_{p',k',\omega'} \rangle = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta f^\alpha_{p}(r,t) e^{i(\omega \cdot r - \phi \cdot t)} \; \delta f^{\beta}_{p'}(r',t') e^{i(\omega' \cdot r' - \phi' \cdot t')} \; dt \; dr \; dt' \; dr'$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta f^\alpha_{p}(r,t) \delta f^{\beta}_{p'}(r',t') e^{i(\omega + \omega' \cdot r + \omega' \cdot r')} \; dt \; dr \; dt' \; dr'$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle f^\alpha_{p}(R,\tau) e^{i(\omega + \omega' \cdot R)} \; f^{\beta}_{p'}(R',\tau' e^{i(\omega + \omega' \cdot R)} \; dt \; dR \; d\tau \; dt' \; dR' \; d\tau'$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle f^\alpha_{p}(R,\tau) e^{i(\omega + \omega' \cdot R)} \; f^{\beta}_{p'}(R',\tau' e^{i(\omega + \omega' \cdot R)} \; dt \; dR \; d\tau \; dt' \; dR' \; d\tau'$$

$$= \langle f^\alpha_{p}(R,\tau) e^{i(\omega + \omega' \cdot R)} \; f^{\beta}_{p'}(R',\tau' e^{i(\omega + \omega' \cdot R)} \; dt \; dR \; d\tau \; dt' \; dR' \; d\tau'$$

$$= \langle f^\alpha_{p}(R,\tau) e^{i(\omega + \omega' \cdot R)} \; f^{\beta}_{p'}(R',\tau' e^{i(\omega + \omega' \cdot R)} \; dt \; dR \; d\tau \; dt' \; dR' \; d\tau'$$

$$= \langle f^\alpha_{p}(R,\tau) e^{i(\omega + \omega' \cdot R)} \; f^{\beta}_{p'}(R',\tau' e^{i(\omega + \omega' \cdot R)} \; dt \; dR \; d\tau \; dt' \; dR' \; d\tau'$$

By Fourier transforming the fluctuating part of the Klimontovich equation in space and time we get

$$-i\omega \delta f^\alpha_{p,k,\omega} + i(v \cdot k) \delta f^\alpha_{p,k,\omega} = 0$$

$$\langle \omega - k \cdot v \rangle \delta f^\alpha_{p,k,\omega} = 0$$

$$\delta f^\alpha_{p,k,\omega} \propto \delta(\omega - k \cdot v),$$

which is the only non-trivial solution of the algebraic equation. The $\delta-$function simply states that non-interacting particles stream freely with a constant phase velocity $v$.

Combining we get

$$\langle \delta f^\alpha_{p,k,\omega} f^\beta_{p',k',\omega'} \rangle \propto \langle f^\alpha_{p}(R,\tau) f^{\beta}_{p'}(R',\tau' e^{i(\omega + \omega' \cdot R)} \; dt \; dR \; d\tau \; dt' \; dR' \; d\tau'$$

$$\propto \delta(\omega + \omega' \cdot R) \; \delta(\omega + \omega' \cdot R') \; \delta(\omega - k \cdot v) \; \delta(\omega - k' \cdot v).$$

Another relation can be found by employing the relation for independent events in any statistics. Let $A$ be a large number of statistically independent elements, then the mean square of fluctuations is equal to the mean value, i.e $\langle (\delta A)^2 \rangle = \langle A \rangle$ with $\delta A = A - \langle A \rangle$. By choosing $A$ to correspond to the number of particles of one species that have their momenta in the range $\{p,p + dp\}$, $A$ will simply be the exact distribution function $f^\alpha_p$ and $\langle (\delta f^\alpha_p)^2 \rangle = \Phi^\alpha_p$, which results to

$$\langle \delta f^\alpha_{p,k,\omega} f^{\beta}_{p',k',\omega'} \rangle \propto \Phi^\alpha_p \delta(\omega + \omega' \cdot R) \; \delta(\omega - k \cdot v).$$

The only undetermined part is a numerical factor, that stems from the solution of the fluctuation equation. It is found by using the normalization condition
\[(\langle \delta N \rangle^2) = \langle N \rangle, \text{ where } N \text{ is the number of particles in a volume } V. \] Since the natural correlator uses Fourier components, which refer to an infinite gas volume, whereas the normalization condition is valid for a finite volume, we should use a Fourier series expansion in a finite cubic volume of edge \(L\) and then consider the limit \(L \rightarrow \infty\). The procedure is trivial and yields unity for the numerical factor.

Thus, the natural statistical correlator is given by

\[
\langle \delta f^\alpha_{p,k,\omega} \delta f^\beta_{p',k',\omega'} \rangle = \delta_{\alpha \beta} \Phi^{\alpha}_{p} \delta(p - p') \delta(\omega + \omega') \delta(k + k') \delta(w - k \cdot v). \tag{2.5}
\]

It is important to notice that \(\Phi^{\alpha}_{p}\) is an arbitrary solution of the regular Klimontovich equation and does not have to be thermal; the system of non-interacting particles does not have to be in equilibrium for the above relation to be valid.

### 2.4 The Klimontovich equation and its decomposition

In ideal un-magnetized plasmas the particles interact through Coulomb potentials. In absence of external fields the Hamiltonian is

\[
H(r_1, r_2, ..., r_N, p_1, p_2, ..., p_N) = \sum_{1 \leq i \leq N} \frac{p_i^2}{2m_i} + \frac{1}{2} \sum_{1 \leq i \neq j \leq N} U(|r_i - r_j|), \text{ where } U(|r_i - r_j|) = \frac{e_i e_j}{|r_i - r_j|} \text{ is the unscreened Coulomb potential energy.}
\]

The phase-space variables will be \(X = \{r, p\}\) and the exact distribution function for a species \(\alpha\) will \(f^\alpha_p(r, t) = \sum_{1 \leq i \leq N^\alpha} \delta(r - r_i(t)) \delta(p - p_i(t))\), where the dynamics of the particles are governed by the classical equations of motion for the electric force,

\[
\frac{\partial r_i(t)}{\partial t} = v_i(t) \quad \text{and} \quad \frac{\partial p_i(t)}{\partial t} = e_i E(r_i(t), t), \quad \text{with} \quad E \text{ denoting the fine-grained electrostatic field produced self-consistently by the point-particles themselves and described by the Poisson equation } \nabla \cdot E(r, t) = 4\pi \sum_{\alpha} e_\alpha \int f^\alpha_p(r, t) \frac{d^3p}{(2\pi)^3}.
\]

We note that in the equation of motion of each particle the portion of the fields produced by the particle itself should not be taken into account. We also note that the Poisson equation can be used in order to determine the exact fields in terms of the particle orbits, whereas the set of dynamic equations of motion can be solved in order to determine the exact particle phase-space orbits in terms of the fields. Hence, in this exact self-consistent description, if the fields and the particle phase-space coordinates are known at one time, then we can compute them for all following times.

Differentiation with respect to time, together with the dynamic equations of motion and the properties \(\frac{\partial}{\partial a} f(a - b) = -\frac{\partial}{\partial a} f(a - b), \frac{\partial}{\partial t} f[a(t)] = \frac{\partial f}{\partial a} \frac{\partial a}{\partial t}, a \delta(a - b) = \)
b\delta(a-b) gives
\[
\frac{\partial f_\alpha^a}{\partial t} = \sum_{1 \leq i \leq N_a} \frac{\partial}{\partial t} \left[ \delta(r - r_i(t)) \delta(p - p_i(t)) \right] \\
= \sum_{1 \leq i \leq N_a} \delta(r - r_i(t)) \frac{\partial p_i(t)}{\partial t} \delta(p - p_i(t)) + \sum_{1 \leq i \leq N_a} \delta(p - p_i(t)) \frac{\partial}{\partial t} \delta(r - r_i(t)) \\
= \sum_{1 \leq i \leq N_a} \delta(r - r_i(t)) \frac{\partial p_i(t)}{\partial t} \cdot \frac{\partial}{\partial p} \delta(p - p_i(t)) + \\
\sum_{1 \leq i \leq N_a} \delta(p - p_i(t)) \frac{\partial r_i(t)}{\partial t} \cdot \frac{\partial}{\partial r} \delta(r - r_i(t)) \\
= \sum_{1 \leq i \leq N_a} \delta(r - r_i(t)) \frac{\partial p_i(t)}{\partial t} \cdot \frac{\partial}{\partial p} \delta(p - p_i(t)) - \\
\sum_{1 \leq i \leq N_a} \delta(p - p_i(t)) \frac{\partial r_i(t)}{\partial t} \cdot \frac{\partial}{\partial r} \delta(r - r_i(t)) \\
= -\sum_{1 \leq i \leq N_a} \delta(r - r_i(t)) \frac{\partial p_i(t)}{\partial t} \cdot \frac{\partial}{\partial p} \delta(p - p_i(t)) - \sum_{1 \leq i \leq N_a} \delta(p - p_i(t)) \frac{\partial r_i(t)}{\partial t} \cdot \frac{\partial}{\partial r} \delta(r - r_i(t)) \\
= -\sum_{1 \leq i \leq N_a} \delta(r - r_i(t)) \frac{\partial p_i(t)}{\partial t} \cdot \frac{\partial}{\partial p} f_\alpha^a(r, t) - \sum_{1 \leq i \leq N_a} \delta(p - p_i(t)) \frac{\partial r_i(t)}{\partial t} \cdot \frac{\partial}{\partial r} f_\alpha^a(r, t) \\
= -v \cdot \frac{\partial}{\partial r} f_\alpha^a(r, t) - e_\alpha E(r, t) \cdot \frac{\partial}{\partial p} f_\alpha^a(r, t),
\]
which yields the Klimontovich equation
\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + e_\alpha E(r, t) \cdot \frac{\partial}{\partial p} \right) f_\alpha^a(r, t) = 0. \tag{2.6}
\]
In absence of external fields, for \( f_\alpha^a(r, t) = \Phi_\alpha^a(r, t) + \delta f_\alpha^a(r, t) \) and \( E(r, t) = \delta E(r, t) \) the decomposition scheme will yield the plasma kinetic equation
\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \right) \Phi_\alpha^a(r, t) = -e_\alpha \left\langle \delta E(r, t) \cdot \frac{\partial \delta f_\alpha^a(r, t)}{\partial p} \right\rangle, \tag{2.7}
\]
the quasi-neutrality condition
\[
\sum_\alpha e_\alpha \int \Phi_\alpha^a(r, t) \frac{d^3p}{(2\pi)^3} = \sum_\alpha e_\alpha n_\alpha(r, t) = 0, \tag{2.8}
\]
and the equations for the fluctuating quantities
\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \right) \delta f_\alpha^a(r, t) = -e_\alpha \delta E(r, t) \cdot \frac{\partial \Phi_\alpha^a(r, t)}{\partial p}, \tag{2.9}
\]
\[ \nabla \cdot \delta \mathbf{E}(r, t) = 4\pi \sum_{\alpha} e_{\alpha} \int \delta f_{\alpha}^p(r, t) \frac{d^3p}{(2\pi)^3}. \quad (2.10) \]

In the following, for simplicity we will omit the \((r, t)\) dependence from the ensemble average quantities.

### 2.5 The fluctuation equation and the permittivity

We start by Fourier transforming the fluctuating part of the Klimontovich equation in space and time,

\[ (\omega - \mathbf{k} \cdot \mathbf{v}) \delta f_{\alpha}^p_{\mathbf{k}, \omega} = -ie_{\alpha} \delta \mathbf{E}_{\mathbf{k}, \omega} \cdot \frac{\partial \Phi_{\alpha}^p}{\partial p}. \quad (2.11) \]

The solution of the above algebraic equation is the sum of the homogeneous and the particular solution. This separation is carried out not only due to mathematical reasoning but also bears physical grounds; the solution of the homogeneous equation is acquired in the limit of uncharged non-interacting particles, \(e_{\alpha} \rightarrow 0\) in Eqs.\((2.9,2.10)\), and represents natural fluctuations due to particle discreetness, whereas the solution of the inhomogeneous equation is directly proportional to the field fluctuations and represents induced fluctuations. Due to the self-consistent closure by the Poisson equation, eventually the field fluctuations are also produced by the natural fluctuations. Therefore, the ultimate source of fluctuations is particle discreteness and by combining Eqs.\((2.9,2.10)\) one can express all fluctuating quantities as linear functions of the natural fluctuations in Fourier space.

To proceed we regard longitudinal fields only, i.e \(\delta \mathbf{E}_{\mathbf{k}, \omega}^l = \frac{k}{\mathbf{k}} \delta \mathbf{E}_{\mathbf{k}, \omega}\) and also add an infinitesimally small positive imaginary part \(+i0\) in \(\omega - \mathbf{k} \cdot \mathbf{v}\) [Klimontovich, 1998]. The addition of the infinitesimally positive imaginary part in the frequency is done in order to regularize the divergence of the integral at the resonance \(\omega = \mathbf{k} \cdot \mathbf{v}\); by going into the complex domain Cauchy’s formula can be utilized by an integration contour encircling the pole with the direction of de-touring chosen in accordance with the causality principle. This technique is also known as adiabatic switching of the interaction, since \(\omega = \omega + i0\) means that the field fluctuations are switched on slowly from \(t = -\infty\) instead of switched on abruptly on \(t = 0\), e.g by assuming a monochromatic fluctuation we see that the magnitude \(\lim \limits_{(\nu, t) \rightarrow (0^+, -\infty)} |A \exp (-i\omega t + \nu t)| = \lim \limits_{(\nu, t) \rightarrow (0^+, -\infty)} A \exp (\nu t) = 0\), and increases slowly until \(t = 0\), where it reaches its nominal value. The reason for the choice of the sign in \(+i0\) can also be seen by artificially introducing an infinitesimal dissipation/growth in the system by means of the relaxation time approximation \(\mp \nu (f_{\alpha}^p - N_{\alpha} \Phi_{\alpha}^p, \text{eq})\), respectively. In that case, the integral responses would be of
the form
\[
\lim_{\nu \to 0^+} \int \frac{f(x)}{x - x_0 + \nu} \, dx = \lim_{\nu \to 0^+} \int \frac{f(x)(x - x_0 + \nu)}{(x - x_0)^2 + \nu^2} \, dx \\
= \lim_{\nu \to 0^+} \int \frac{f(x)(x - x_0)}{(x - x_0)^2 + \nu^2} \, dx + \nu \lim_{\nu \to 0^+} \int \frac{f(x)\nu}{(x - x_0)^2 + \nu^2} \, dx \\
= \int \lim_{\nu \to 0^+} \frac{f(x)(x - x_0)}{(x - x_0)^2 + \nu^2} \, dx \\
\pm i \pi \int f(x) \lim_{\nu \to 0^+} \left( \frac{1}{\pi (x - x_0)^2 + \nu^2} \right) \, dx \\
= P.V. \int \frac{f(x)}{x - x_0} \, dx \mp i \pi \int f(x) \delta(x - x_0) \, dx \\
= P.V. \int \frac{f(x)}{x - x_0} \, dx \mp i \pi f(x_0),
\]
with the minus sign in the final formula corresponding to the physically expected dissipation and +i0.

\[
\delta f_{p,k,\omega}^\alpha = \delta f_{p,k,\omega}^{\alpha,(0)} + \delta f_{p,k,\omega}^{\alpha,(ind)}
= \delta f_{p,k,\omega}^{\alpha,(0)} - \frac{ie_\alpha}{k} \left( \frac{1}{\omega - k \cdot v + i0} k \cdot \frac{\partial \Phi_{p,\omega}^\alpha}{\partial p} \right) \delta E_{k,\omega}.
\]

We now integrate over the momentum space, define the fluctuating plasma densities \( \delta n_{k,\omega}^{\alpha,(0)} = \int \delta f_{p,k,\omega}^{\alpha,(0)} \, d^p p \) and \( \delta n_{k,\omega}^{\alpha,(ind)} = \int \delta f_{p,k,\omega}^{\alpha,(ind)} \, d^p p \), and also define the plasma species susceptibilities \( \chi_{k,\omega}^\alpha = \frac{4\pi e_\alpha^2}{k} \int \frac{1}{\omega - k \cdot v + i0} \left( k \cdot \frac{\partial \Phi_{p,\omega}^\alpha}{\partial p} \right) \, d^p p \).

\[
\delta n_{k,\omega}^{\alpha,(0)} + \delta n_{k,\omega}^{\alpha,(ind)} = \delta n_{k,\omega}^{\alpha,(0)} - \frac{i k}{4\pi e_\alpha} \chi_{k,\omega}^\alpha \delta E_{k,\omega}.
\]

The Fourier transform of the fluctuating part of the Poisson equation yields
\[
\begin{align*}
\text{i} k \delta E_{k,\omega} &= 4\pi \sum_\alpha e_\alpha \int \delta f_{p,k,\omega}^{\alpha,(0)} \, d^3 p \\
\text{i} k \delta E_{k,\omega} &= 4\pi \sum_\alpha e_\alpha \delta n_{k,\omega}^{\alpha,(0)} - i k \sum_\alpha \chi_{k,\omega}^\alpha \delta E_{k,\omega} \\
\text{i} k \left( 1 + \sum_\alpha \chi_{k,\omega}^\alpha \right) \delta E_{k,\omega} &= 4\pi \sum_\alpha e_\alpha \delta n_{k,\omega}^{\alpha,(0)} \\
\delta E_{k,\omega} &= \frac{4\pi}{i k \epsilon_{k,\omega}} \sum_\alpha e_\alpha \int \delta f_{p,k,\omega}^{\alpha,(0)} \, d^3 p,
\end{align*}
\]
with
\[
\epsilon_{k,\omega} = 1 + \sum_\alpha \chi_{k,\omega}^\alpha = 1 + \frac{4\pi}{k^2} \sum_\alpha e_\alpha^2 \int \frac{1}{\omega - k \cdot v + i0} \left( k \cdot \frac{\partial \Phi_{p,\omega}^\alpha}{\partial p} \right) \, d^3 p.
\]
denoting the permittivity of the system (the longitudinal part of the dielectric tensor).

Finally, by substituting Eq. (2.14) into Eq. (2.13) we can also express the total plasma density fluctuations as a linear function of the natural fluctuations,

$$\delta n_\omega^\alpha = \left(1 - \frac{X_{k,\omega}}{\epsilon_{k,\omega}}\right) \int \delta f_{p,k,\omega}^{\alpha,0} \frac{d^3 p}{(2\pi)^3} - \frac{X_{k,\omega}}{\epsilon_{k,\omega}} \sum_{\beta \neq \alpha} e_\beta \int \delta f_{p,k,\omega}^{\beta,0} \frac{d^3 p}{(2\pi)^3}. \quad (2.15)$$

2.6 The plasma kinetic equation and the collision integral

The plasma kinetic equation has the form

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r}\right) \Phi_p^\alpha = J_p^\alpha, \quad (2.16)$$

where $J_p^\alpha = -e_\alpha \frac{\partial}{\partial p} \cdot \langle \delta E(r,t) \delta f_p^\alpha(r,t) \rangle$ is the collision integral. Since for each species we have both induced and natural fluctuating parts, the collision integral will consist of two terms, $J_p^{\alpha,1} = -e_\alpha \frac{\partial}{\partial p} \cdot \langle \delta E(r,t) \delta f_p^{\alpha,0}(r,t) \rangle$ and $J_p^{\alpha,2} = -e_\alpha \frac{\partial}{\partial p} \cdot \langle \delta E(r,t) \delta f_p^{\alpha,(ind)}(r,t) \rangle$.

For the first term: (i) we inverse Fourier transform in space and time, (ii) we substitute for the electric field fluctuations as functions of natural fluctuations only, applying Eq. (2.14), (iii) we use the properties of the natural statistical correlator to discard the exponentials and the trivial integrations, (iv) we use the reality of the collision integral in real space, (v) we use the Sokhotskyi-Plemelj formula, to separate the permittivity into real and imaginary parts, for the imaginary part we get \( \Im\{\epsilon_{k,\omega}\} = -\frac{4\pi^2}{k^2} \sum_{\beta} e_{\beta}^2 \int \delta(\omega - k \cdot). \)
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$$v') \left( k \cdot \frac{\partial \Phi_p^\beta}{\partial p'} \right) \frac{d^3p'}{(2\pi)^3}$$

Hence, overall

$$J_{p}^{\alpha_1} = -e_\alpha \frac{\partial}{\partial p} \cdot \int \frac{k}{k} \delta E_{k_\omega} \delta f_{p,k_\omega}^{\alpha_1(0)} e^{-i(\omega + \omega')t} e^{i(k + k') \cdot r} d\omega d\omega' d^3k d^3k'$$

$$= -\sum_\beta e_\alpha e_\beta \frac{\partial}{\partial p} \cdot \int \frac{k}{k} \frac{4\pi}{k + k e_{k_\omega}} \delta f_{p',k_\omega}^{\beta(0)} \delta f_{p,k_\omega}^{\alpha_1(0)} e^{-i(\omega + \omega')t}$$

$$e^{i(k + k') \cdot r} d\omega d\omega' d^3k d^3k' \frac{d^3p'}{(2\pi)^3}$$

$$= + \sum_\beta e_\alpha e_\beta \frac{\partial}{\partial p} \cdot \int \frac{k}{k} \frac{4\pi}{k + k e_{k_\omega}} \delta f_{p',k_\omega}^{\beta(0)} \delta f_{p,k_\omega}^{\alpha_1(0)} e^{-i(\omega + \omega')t}$$

$$e^{i(k + k') \cdot r} d\omega d\omega' d^3k d^3k' \frac{d^3p'}{(2\pi)^3}$$

For the second term: (i) we inverse Fourier transform both fluctuating quantities in space and time, (ii) we substitute for the induced fluctuations as functions of the collision integral in real space, (iii) we substitute for the induced fluctuations in real space as functions of natural fluctuations only, applying Eq.(2.14), (iv) we use the properties of the natural statistical correlator to discard the exponentials and the trivial integrations, (v) we use the reality of the collision integral in real space, (vi) in the presence of $$\delta(k + k')\delta(\omega + \omega')$$ we have $$\frac{1}{\omega - k \cdot v + i\epsilon_0} = -\frac{1}{\omega - k \cdot v}$$, (vii) the total permittivity is a real quantity is the $$(r,t)$$ space, thus for the Fourier transform we have $$\epsilon_{-k,-\omega} = \epsilon_0$$ and hence $$\epsilon_{-k,-\omega} e_{k_\omega} = |\epsilon_{k_\omega}|^2$$, (viii) we decompose $$\frac{1}{\omega - k \cdot v - i\epsilon_0}$$ into imaginary and real parts with the Sokhotskyi-Plemelj formula, the real part...
will be \(-\pi \delta(\omega - k \cdot v)\).

\[
J_{\mu}^{\alpha,2} = -e_{\alpha} \frac{\partial}{\partial p} \int \frac{k}{k} \left( \delta E_{\omega,k} \delta f_{p,k^\prime,-\omega} \right) e^{-i(\omega + \omega')} e^{i(k + k') \cdot r_{dw,\omega d\omega'} d^3 k \cdot k'}
\]

\[
= +e_{\alpha} \frac{\partial}{\partial p} \int \frac{k}{k} \left( \frac{1}{\omega - k' \cdot v + 0} k' \cdot \frac{\partial \Phi_{\mu}^\alpha}{\partial p} \right) e^{-i(\omega + \omega')} e^{i(k + k') \cdot r_{dw,\omega d\omega'} d^3 k \cdot k'}
\]

\[
= -\sum_\beta \sum_\gamma e_{\beta} e_{\gamma}^2 \frac{\partial}{\partial p} \int \frac{k}{k} \frac{1}{k^2} \left( \frac{1}{\omega - k' \cdot v + 0} k' \cdot \frac{\partial \Phi_{\mu}^\beta}{\partial p} \right) \delta E_{\omega,k} \delta f_{p,k^\prime,-\omega} d^3 k \cdot k' \cdot d^3 p \cdot d^3 p' \left( 2\pi \right)^3 \left( 2\pi \right)^3
\]

\[
= -\sum_\beta \sum_\gamma e_{\beta} e_{\gamma}^2 \frac{\partial}{\partial p} \int \frac{k}{k} \frac{1}{k^2} \left( \frac{1}{\omega - k' \cdot v + 0} k' \cdot \frac{\partial \Phi_{\mu}^\beta}{\partial p} \right) \delta E_{\omega,k} \delta f_{p,k^\prime,-\omega} d^3 k \cdot k' \cdot d^3 p \cdot d^3 p' \left( 2\pi \right)^3 \left( 2\pi \right)^3
\]

Combining the above we get

\[
J_{\mu}^{\alpha,2} = 2 \sum_\beta e_{\beta} e_{\alpha} \int \left( k \cdot \frac{\partial}{\partial p} \right) \frac{1}{k^4} \frac{\delta(k \cdot v - k' \cdot v')}{k^4} \frac{\partial \Phi_{\mu}^\beta}{\partial p} \frac{\partial \Phi_{\mu}^\alpha}{\partial p'} \frac{d^3 k}{d^3 k'} \frac{d^3 p}{d^3 p'} \left( 2\pi \right)^3 \left( 2\pi \right)^3,
\]

Equivalently using Einstein’s convention of summation over repeated indices we
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have

\[ J^\alpha_p = \frac{1}{2} \frac{\partial}{\partial p_i} \sum_k k_i k_j \left( \frac{4 e_\alpha e_\beta (2\pi)^3}{k^4 |\epsilon_{k,k',v'}|^2} \delta(k \cdot v - k \cdot v') \right) \]

\times \left[ \Phi^\beta_{p'} \frac{\partial \Phi^\alpha_p}{\partial p_j} - \Phi^\alpha_p \frac{\partial \Phi^\beta_{p'}}{\partial p'_j} \right] \frac{d^3 p'}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3}. \quad (2.18) \]

Hence, we end up with the Lenard-Balescu form of the plasma kinetic equation [Lenard, 1960; Balescu, 1960]

\[ \left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \right) \Phi^\alpha_p = 2 \sum_\beta e_\alpha^2 e_\beta^2 \int \left( k \cdot \frac{\partial}{\partial p} \right) \delta(k \cdot v - k \cdot v') \]

\times \left[ \Phi^\beta_{p'} \left( k \cdot \frac{\partial \Phi^\alpha_p}{\partial p} \right) - \Phi^\alpha_p \left( k \cdot \frac{\partial \Phi^\beta_{p'}}{\partial p'} \right) \right] \frac{d^3 p'}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3}. \quad (2.19) \]

Below we discuss the physics lying behind the structure of the collision integral and its main properties;

1. The collision integral can be conveniently rewritten in the Fokker-Planck form and therefore it can be interpreted as the superposition of a drift process in momentum space (due to a friction force \( F^\alpha_p \)) and of a diffusive process in momentum space (with a diffusion coefficient tensor \( D^\alpha_p \)). The friction force stems from the correlator \(-e_\alpha \frac{\partial}{\partial p} \langle \delta E(r, t) \delta f^\alpha_p(r, t) \rangle\) and the diffusion term from the correlator \(-e_\alpha \frac{\partial}{\partial p} \langle \delta E(r, t) \delta f^\alpha_{p,ind}(r, t) \rangle\) that involves the spectral density of the electric field fluctuations \( \langle \delta E_{k,\omega} \delta E_{k',\omega'} \rangle \). The Fokker-Planck form is [Ichimaru, 1973; Aleksandrov et al., 1983]

\[ J^\alpha_p = \frac{\partial}{\partial p} D^\alpha_p \cdot \frac{\partial \Phi^\alpha_p}{\partial p} + \frac{\partial}{\partial p} \left( F^\alpha_p \Phi^\alpha_p \right), \quad (2.20) \]

with the components of the diffusion coefficient given by

\[ D^\alpha_{p,i,j} = \frac{1}{2} \sum_\beta 4 e_\alpha^2 e_\beta^2 (2\pi)^3 \int \frac{k_i k_j}{k^4 |\epsilon_{k,k',v'}|^2} \delta(k \cdot v - k \cdot v') \Phi^\beta_{p'} \frac{d^3 p'}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3}. \quad (2.21) \]

and the components of the friction force given by

\[ F^\alpha_{p,i} = -\frac{1}{2} \sum_\beta 4 e_\alpha^2 e_\beta^2 (2\pi)^3 \int \frac{k_i}{k^4 |\epsilon_{k,k',v'}|^2} \delta(k \cdot v - k \cdot v') \]

\times \left( k \cdot \frac{\partial \Phi^\beta_{p'}}{\partial p'} \right) \frac{d^3 p'}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3}. \quad (2.22) \]
2. The $\delta$–function, $\delta(k \cdot v - k \cdot v')$, in the collision integral is a consequence of the conservation of energy in the collision process. Let us assume two particles colliding with initial velocities $v, v'$ and that the electrostatic interaction takes place through the intermediary of a field, i.e a virtual wave that carries momentum $\hbar k$ and energy $\hbar \omega$. We also assume small momentum transfer, $|\hbar k| \ll |p|$, and non-relativistic velocities. For the first particle conservation of energy/momentum gives [Tsytovich, 1995]

$$\epsilon_p = \epsilon_{p-k} + \hbar \omega$$

$$\frac{p^2}{2m} = \frac{(p - \hbar k)^2}{2m} + \hbar \omega$$

$$\frac{\hbar (k \cdot p)}{m} = \hbar \omega \Rightarrow \omega = k \cdot v.$$  

(2.23)

Similarly for the second particle we have $\epsilon_{p'} = \epsilon_{p'+k} - \hbar \omega$, which results in $\omega = k \cdot v'$. Combining we get $k \cdot v = k \cdot v'$. Hence, the presence of $\delta(k \cdot v - k \cdot v')$ expresses energy conservation in Coulomb collisions in case of small momentum transfer.

3. The factor $\frac{1}{k^4 |\epsilon_{k,k \cdot v'}|}$ in the collision integral expresses the dynamic screening of the fields of the colliding particles, that is the screening that occurs due to all the other particles through the dependence of the permittivity on the regular distribution functions of all species. A simple demonstration of this argument can be done by considering the electrostatic self-field of a unit charge moving with constant velocity $v$ in a dispersive dielectric medium with complex permittivity $\epsilon_{k,\omega}$. Gauss electric law will read as $\nabla D = 4\pi \rho_q$, where the charge density is $\rho_q(r,t) = \delta(r-vt)$, the displacement field is $D(r,t) = -\epsilon(r,t) \nabla \left( G(r,t) \right)$ and $G(r,t)$ is the Green function of the potential of the self-field. A Fourier transform in space and time will yield the solution $G_{k,\omega} = \frac{\delta(\omega-k \cdot v)}{2 \pi k^2 \epsilon_{k,\omega}}$ and the integral over the frequencies will give $G_k = \int G_{k,\omega} d\omega = \frac{1}{2\pi k^2 |\epsilon_{k,k \cdot v'}|}$. The presence of $|G_k|^2 \propto \frac{1}{k^4 |\epsilon_{k,k \cdot v'}|}$, is due to the screening of the fields of both particles. The appearance of the permittivity in the collision integral clearly expresses collective effects in collisions.

4. The probability for a collision to occur can be determined by methods of detailed balancing in phase-space and comparison with the collision integral. It is of great interest whether the collision integral can be found by a procedure similar to the one adopted by Boltzmann in his derivation of the kinetic equation of dilute gases, especially since the omission of high order fluctuation terms is close to Boltzmann’s assumption of molecular chaos [Landau and Lifshitz, 1980]. The answer is no; the collective effects in collisions can only be taken into account properly through fluctuation theory and two particle-scattering processes will inevitably ignore such effects. However, such methods are important for the estimation of the collision probability. Let us denote the probability for a binary collision per unit time
2.6. THE PLASMA KINETIC EQUATION AND THE COLLISION INTEGRAL

to occur between a particle of a species $\alpha$ with initial momentum $p$ and a particle of species $\beta$ with initial momentum $p'$ with a momentum transfer $k$ within the range $\{k, k + dk\}$ by $w_{p,p'}^{\alpha \beta}(k) \frac{d^3k}{(2\pi)^3}$. Then it is straightforward that the decrease per unit time of particles of a species $\alpha$ with momentum $p$ will be equal to the probability $w_{p,p'}^{\alpha \beta}(k)$ multiplied by the number of particles $\alpha$ with momentum $p$ and by the number of particles $\beta$ with momentum $p'$:

$$\left[ \frac{\delta \Phi}{\delta t} \right]_{c^-} = \sum_{\beta} \int w_{p,p'}^{\alpha \beta}(k) \Phi_{p}^{\alpha} \Phi_{p'}^{\beta} \frac{d^3p'}{(2\pi)^3} \frac{d^3k}{(2\pi)^3}.$$  

While the inverse process will lead to increase per unit time of particles $\alpha$ with momentum $p$:

$$\left[ \frac{\delta \Phi}{\delta t} \right]_{c^+} = \sum_{\beta} \int w_{p,k-p',-k}^{\alpha \beta}(k) \Phi_{p-k}^{\alpha} \Phi_{p'+k}^{\beta} \frac{d^3p'}{(2\pi)^3} \frac{d^3k}{(2\pi)^3}.$$  

Symmetry considerations in the collision probability and first order expansions in the small momentum transfer $k$ both in the probability and the distribution functions will result in a collision integral of the form

$$J_{\alpha p} = \left[ \frac{\delta \Phi_{p}^{\alpha}}{\delta t} \right]_{c^+} - \left[ \frac{\delta \Phi_{p}^{\alpha}}{\delta t} \right]_{c^-} = \frac{1}{2} \frac{\partial}{\partial p_i} \sum_{\beta} \int \Phi_{p-k}^{\alpha} \Phi_{p'+k}^{\beta} \frac{d^3p'}{(2\pi)^3} \frac{d^3k}{(2\pi)^3}.$$  

Finally, by comparing with Eq.(2.18) we get the following expression for the zero order probability for a collision

$$w_{p,p'}^{\alpha \beta(0)} = \frac{4 e_\alpha e_\beta (2\pi)^3}{k^4 |\mathbf{k} \cdot \mathbf{v} - \mathbf{k} \cdot \mathbf{v}'|^2} \delta(k \cdot v - k \cdot v').$$  

5. The particle number, the momentum and the energy are conserved locally in collisions, integration of the related moments of the collision integral over the momentum space gives zero. The quantities $\{1, p, \frac{p^2}{2m}\}$ are known as collisional invariants and can be used also for the construction of solutions of the kinetic equation. It is quite straightforward to prove that the number of particles of each kind is conserved, while the total (for all species $\alpha$) local momentum and kinetic energy are conserved [Liboff, 1990],

$$\int J_{\alpha p} \frac{d^3p}{(2\pi)^3} = 0,$$  

$$\sum_{\alpha} \int p J_{\alpha p} \frac{d^3p}{(2\pi)^3} = 0,$$  

$$\sum_{\alpha} \int \frac{p^2}{2m_{\alpha}} J_{\alpha p} \frac{d^3p}{(2\pi)^3} = 0.$$  

(2.26)  

(2.27)  

(2.28)
6. As far as the solutions of the plasma kinetic equation are concerned, they bear a number of remarkable properties: [Nicholson, 1983; Ichimaru, 1992]

(I) For $\Phi_\alpha^p \geq 0$ at $t = 0$, then $\Phi_p \geq 0$ for all successive times. (II) Any Maxwellian distribution is a stationary solution, for thermal distributions the drift and diffusion processes in momentum space balance each other. (III) As the time approaches infinity, any solution approaches a Maxwellian distribution. Hence, these are the only stationary solutions. This property is strongly connected with the second law of thermodynamics and Boltzmann’s H-theorem, the entropy will increase in time until the closed system reaches thermodynamic equilibrium.

2.7 The spectral densities of fluctuations

Combining the expressions of the fluctuating quantities as functions of the natural fluctuations with the natural statistical correlator, we can compute the spectral densities of the fluctuating quantities. These have the general form $\langle \delta k, \omega \delta k', \omega' \rangle$ and are measurable quantities that can provide vast information on fundamental plasma parameters [Ichimaru, 1964].

In this section we will compute the spectral density of electrostatic field fluctuations $S^E_{k, \omega} = \langle \delta E_{k, \omega}, \delta E_{k', \omega'} \rangle$ (measurable as the voltage power spectrum at the terminals of dipole or monopole antennas), the spectral density of electrostatic potential fluctuations $S^\phi_{k, \omega} = \langle \delta \phi_{k, \omega}, \delta \phi_{k', \omega'} \rangle$ (measurable as the floating potential of conventional electrostatic probes) and the spectral densities of plasma density fluctuations $S^\alpha_{k, \omega} = \langle \delta n^{\alpha, (0)}_{k, \omega}, \delta n^{\alpha, (0)}_{k', \omega'} \rangle$ (measurable as the ion/electron saturation currents of Langmuir probes).

We point out that for the experimental measurement of the spectral densities of fluctuations (that are related to the discrete nature of the plasma components), quiescent plasma configurations are needed so that the fluctuation level is not masked by turbulence, waves or instabilities. This is not typical in laboratory discharges, since plasmas that require an electric field for their generation are usually unstable and exhibit pronounced spatial fine structures. Rare exceptions are the magnetic cusp device [Spinicchia et al., 2006], the brush cathode discharge [Persson, 1965; Bingham, 1967] and its variants, such as the reflex brush cathode [Allison and Chambers, 1966], the inverse brush cathode [Musel, 1966], the large V-groove cathode [Caron, 1971]. On the other hand space plasmas can be quiescent, there the spectral densities of electrostatic field fluctuations have been used for the determination of plasma parameters (electron density, electron temperature, ion bulk speed) for decades through the established diagnostic of Quasi-thermal noise spectroscopy [Meyer-Vernet, 1979; Meyer-Vernet and Perche, 1989].

Before proceeding to the calculations, it is necessary to define the term natural spectral functions, these are integrals of the natural correlator of any species over the momentum space i.e. $\mu(v) \int f^{\alpha, (0)}_{p, k, \omega} \delta f^{\alpha, (0)}_{p', k', \omega'} \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3}$ with $\mu(v)$ a dimensionless quantity that is function of the phase-space speed. For multi-component full-
2.7. THE SPECTRAL DENSITIES OF FLUCTUATIONS

ionized plasmas, \( \mu(n) = 1 \), and the only spectral functions are of the form \( S_{k,\omega}^{\alpha,(0)} = \int \langle \delta f_{p,k,\omega}^{\alpha,(0)} \delta f_{p',k',\omega'}^{\alpha,(0)} \rangle \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \). In this case they can also be interpreted as the spectral density of the natural plasma density fluctuations \( S_{k,\omega}^{\alpha,(0)} = \langle \delta n_{k,\omega}^{\alpha,(0)} \delta n_{k',\omega'}^{\alpha,(0)} \rangle \) since \( \delta n_{k,\omega}^{\alpha,(0)} = \int \delta f_{p,k,\omega}^{\alpha,(0)} \frac{d^3 p}{(2\pi)^3} \).

We start by calculating the spectral densities of the electrostatic field fluctuations. Using Eq.(2.14), the natural correlator \( \delta \)-function properties and the reality condition for the permittivity we have

\[
S_{k,\omega}^{E} = \langle \delta E_{k,\omega} \delta E_{k',\omega'} \rangle \\
= -\left( \frac{4\pi}{k\epsilon_{k,\omega} k'\epsilon_{k',\omega'}} \right) \sum_{\alpha} \left( e_{\alpha} \int \delta f_{p,k,\omega}^{\alpha,(0)} \frac{d^3 p}{(2\pi)^3} \right) \sum_{\beta} \left( e_{\beta} \int \delta f_{p',k',\omega'}^{\beta,(0)} \frac{d^3 p'}{(2\pi)^3} \right) \\
= -\frac{16\pi^2}{k^2 \epsilon_{k,\omega} \epsilon_{k',\omega'}} \sum_{\alpha} \sum_{\beta} e_{\alpha} e_{\beta} \int \langle \delta f_{p,k,\omega}^{\alpha,(0)} \delta f_{p',k',\omega'}^{\beta,(0)} \rangle \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \\
= -\frac{16\pi^2}{k^2 \epsilon_{k,\omega} \epsilon_{k',\omega'}} \sum_{\alpha} e_{\alpha}^2 \int \langle \delta f_{p,k,\omega}^{\alpha,(0)} \delta f_{p',k',\omega'}^{\alpha,(0)} \rangle \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \\
= \frac{16\pi^2}{k^2 \epsilon_{k,\omega} \epsilon_{k',\omega'}} \sum_{\alpha} e_{\alpha}^2 \int \langle \delta f_{p,k,\omega}^{\alpha,(0)} \delta f_{p',k',\omega'}^{\alpha,(0)} \rangle \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \\
= \frac{16\pi^2}{k^2 \epsilon_{k,\omega} \epsilon_{k',\omega'}} \sum_{\alpha} e_{\alpha}^2 S_{k,\omega}^{E} \\
= \frac{16\pi^2}{k^2 \epsilon_{k,\omega} \epsilon_{k',\omega'}} \sum_{\alpha} e_{\alpha}^2 S_{k,\omega}^{E}.
\]

(2.29)

For the spectral density of the electrostatic potential fluctuations, we simply use the definition \( \delta E = -\nabla \phi \) and Eq.(2.29) to obtain

\[
S_{k,\omega}^{\phi} = \langle \delta \phi_{k,\omega} \delta \phi_{k',\omega'} \rangle = -\left( \frac{1}{k^2} \delta E_{k,\omega} \delta E_{k',\omega'} \right) \\
= \frac{1}{k^2} \langle \delta E_{k,\omega} \delta E_{k',\omega'} \rangle = \frac{1}{k^2} S_{k,\omega}^{E} \\
= \frac{16\pi^2}{k^4 \epsilon_{k,\omega} \epsilon_{k',\omega'}} \sum_{\alpha} e_{\alpha}^2 S_{k,\omega}^{E}.
\]

(2.30)

The spectral densities of plasma density fluctuations will contain both natural and induced parts. We use Eq.(2.15), the natural correlator \( \delta \)-function properties
and the reality condition for the permittivity and the susceptibilities to obtain

\[ S_{k,\omega}^\alpha = \langle \delta n_{k,\omega}^\alpha \delta n_{k',\omega'}^\alpha \rangle \]

\[ = \left\{ \left( 1 - \frac{\chi_k^\alpha}{\epsilon_k^*} \right) \delta n_{k,\omega}^\alpha(0) - \frac{\chi_{k',\omega'}^\alpha}{\epsilon_{k',\omega'}^*} \sum_{\beta \neq \alpha} e_\beta \delta n_{k',\omega'}^\beta(0) \right\} \times \]

\[ \left\{ \left( 1 - \frac{\chi_{k',\omega'}^\alpha}{\epsilon_{k',\omega'}^*} \right) \delta n_{k',\omega'}^\alpha(0) - \frac{\chi_{k,\omega}^\alpha}{\epsilon_{k,\omega}^*} \sum_{\beta \neq \alpha} e_\beta \delta n_{k,\omega}^\beta(0) \right\} \]

\[ = \left( 1 - \frac{\chi_k^\alpha}{\epsilon_k^*} \right) \left( 1 - \frac{\chi_{k,\omega}^\alpha}{\epsilon_{k,\omega}^*} \right) S_{k,\omega}^\alpha(0) + \frac{\chi_{k,\omega}^\alpha}{\epsilon_{k,\omega}^*} \frac{\chi_{k,\omega}^\alpha}{\epsilon_{k,\omega}^*} \sum_{\beta \neq \alpha} e_\beta \delta n_{k',\omega'}^\beta(0) \]

\[ = \left( 1 - \frac{\chi_k^\alpha}{\epsilon_k^*} \right)^2 S_{k,\omega}^\alpha(0) + \frac{\chi_k^\alpha}{\epsilon_k^*} \sum_{\beta \neq \alpha} e_\beta \delta n_{k',\omega'}^\beta(0) \]

\[ = \left( 1 - 2 \Re \left( \frac{\chi_k^\alpha}{\epsilon_k^*} \right) \right) S_{k,\omega}^\alpha(0) + \frac{\chi_k^\alpha}{\epsilon_k^*} \sum_{\beta \neq \alpha} e_\beta \delta n_{k',\omega'}^\beta(0) \]

Finally, we compute the spectral density of the natural plasma density fluctuations

\[ S^{\alpha, (0)}_{k,\omega} = \int \int \int f_{p,\omega}^{\alpha, (0)} f_{p',\omega'}^{\alpha, (0)} \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \]

\[ = \int \int \Phi_{p,\omega}^{\alpha} \delta(\omega - \omega') \delta(k + k') \delta(p - p') \delta(\omega - k \cdot v) \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \]

\[ = \left( \frac{1}{(2\pi)^3} \right) \int \Phi_{p,\omega}^{\alpha} \delta(\omega - k \cdot v) \frac{d^3 p}{(2\pi)^3} \]

\[ = \left( \frac{1}{(2\pi)^3} \right) \int \Phi_{p,\omega}^{\alpha} \delta(\omega - k \cdot v) d^3 v \]
For the speed integral we use the transformation $y = \Phi$.

In case of Maxwellian distributions we have

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For the computation we choose spherical coordinates and $\mathbf{k}/\mathbf{\hat{z}}$, the azimuthal integration is trivial, while for the integration over the elevation angle we use the transformation $x = \cos \theta$. Furthermore, we use the property $\delta(ax) = \frac{1}{|a|}\delta(x)$ of Dirac’s function, the property $H(a) = H(x)$ of Heaviside’s step function and

$$\int_{-b}^{+b} \delta(x - |x_0|) dx = \left\{ \begin{array}{ll}
1, & |x_0| \leq b \\
0, & |x_0| > b
\end{array} \right. = H(b - |x_0|).$$

For the speed integral we use the transformation $y = \frac{m_\alpha v^2}{2T_\alpha}$ and $y_{\text{min}} = \frac{m_\alpha v^2}{k^2}$,

$$S_{k,\omega}^{\alpha,0} = \frac{1}{(2\pi)^3} \int \Phi^\alpha(v) \delta(\omega - k \cdot v) d^3 v$$

$$= \frac{2\pi n_\alpha}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty \int_0^\pi v^2 \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) \delta(\omega - kv \cos \theta) \sin \theta dv d\theta$$

$$= \frac{2\pi n_\alpha}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty \int_{-1}^1 v^2 \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) \delta(\omega - kv) dv dx$$

$$= \frac{2\pi n_\alpha}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty v^2 \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) \int_{-1}^1 \delta[-kv(x - \frac{\omega}{k}v)] dx dv$$

$$= \frac{1}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty v \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) \int_{-1}^1 \delta(x - \omega \frac{v}{k}) dx dv$$

$$= \frac{1}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty v \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) H(1 - \frac{\omega}{k}v) dv$$

$$= \frac{1}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty v \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) H(v - \frac{\omega}{k}) dv$$

$$= \frac{1}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_0^\infty v \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right) dv$$

$$= \frac{1}{(2\pi)^3} \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \int_{y_{\text{min}}}^\infty e^{-y} dy$$

$$= \frac{1}{(2\pi)^{7/2}} \frac{n_\alpha}{kv T_\alpha} \int_{y_{\text{min}}}^\infty e^{-y} dy$$

$$= \frac{1}{(2\pi)^{7/2}} \frac{n_\alpha}{kv T_\alpha} \exp \left( -\frac{\omega^2}{2k^2 v_T^2} \right).$$

(2.33)
Chapter 3

The Klimontovich Description of Complex Plasmas

Ideal multi-component plasmas consist of species with constant charge over mass ratio, \( \epsilon_\alpha/m_\alpha \). On the other hand, in complex plasmas the dust species bears the characteristic of charge variability. The dust charge depends on the local plasma parameters and is determined by the balance between plasma fluxes flowing towards the grain and charged particles emitted by the grain. The dust charge and its variability essentially determine all the novel features of complex plasma systems [Tsytovich, 1999; Tsytovich et al., 2008; Fortov et al., 2009],

- **Open systems**: Thermodynamically, a system is defined as 'open' when it exchanges energy with the ambient in order to maintain its equilibrium. In fact, dust grains constantly collect plasma particles and radiation in order to maintain their equilibrium charges, these have to be sustained by external sources. A demonstration of the openness of the complex plasma systems is the fact that the Maxwellian distribution is not a stationary solution of the plasma kinetic equations. This can be intuitively understood for electrons; in typical configurations with negatively charged dust grains, energetic electrons overcome the repulsive potential barrier and get absorbed by the grains, which will lead to the depletion of the high energy tails of thermal distributions.

- **Non-hamiltonian systems**: Due to charge variability the electrostatic forces acting on the grains are of non-potential nature, i.e \( \nabla \times (Z_d\vec{E}) \neq 0 \). Strictly speaking, such a system cannot be described by a Hamiltonian, due to the non-conservative interaction between dust particles that is ultimately caused by the dynamic dust-plasma interactions.

- **Dissipative systems**: There is strong and continuous absorption of plasma fluxes on the grains. The dissipative nature of the system is clearly exhibited by the presence of \( \omega - \vec{k} \cdot \vec{v} + \nu_{d,\alpha}(\vec{v}) \) instead of \( \omega - \vec{k} \cdot \vec{v} + i0 \) in the plasma
CHAPTER 3. THE KLIMONTOVICH DESCRIPTION OF COMPLEX PLASMAS

responses. High dissipation together with the openness of the system lead to an enhanced probability for the formation of self-organized structures.

- **Systems with highly charged constituents:** In contrast to ordinary plasmas, where the components are rarely multiply charged, grain charges typically reach hundreds or even thousands of elementary charges, which causes strong grain-plasma and grain-grain interactions which lead to phase transitions and strongly coupled states.

However, the presence of dust does not always imply that it should be treated as a distinct plasma species [de Angelis, 1992]. This can be determined by the comparison of two fundamental lengths, the mean inter-grain separation \( r_{av} = \left( \frac{3}{4 \pi n_d} \right)^{1/3} \) and the plasma Debye radius \( \lambda_D \). In the case \( r_{av} \gg \lambda_D \), the grain self-field is completely screened out by the plasma before its closest neighboring grain can "feel" its presence and then the grains are essentially isolated (e.g dust in fusion, dust in magnetospheres/comets). In the case \( r_{av} < \lambda_D \) the effects of the neighboring grain interaction as well as other collective effects can be important and dust should be treated as an additional plasma component. Finally, in cases of strongly coupled dusty systems and \( r_{av} \ll \lambda_D \) one can even treat only dust as a species while treating the plasma as the ambient medium providing grain charging and potential screening [Fortov et al., 2005].

In addition when treating the effect of dust particles in plasma kinetics, in frequency regimes typical of ion or electron dynamics, one can also avoid treating dust dynamically (through kinetic theory or hydrodynamic approximations) due to the fast temporal scales and short spatial scales involved.

### 3.1 The effect of charge variability in the structure of the Klimontovich kinetic scheme

The variability of dust charge has deep imprints on the structure of the Klimontovich kinetic scheme [Tsytovich and de Angelis, 1999];

(i) Foremost, the charge can be regarded as a new phase space variable for dust. The charge extended phase space will be 7-dimensional \( \{r, p, q \} \) which will lead to a charge derivative term in the Liouville and the Klimontovich equations, yet they will both still be in a continuity form.

(ii) Since any new phase space variable should be accompanied by a dynamic equation for the deterministic description of the system to be viable, the charging equation will now complement the equations of motion.

(iii) Moreover, it is necessary to obtain a new relation for the natural statistical correlator for dust, \( \langle \delta f_{p,k,\omega}(q) \delta f_{p',k',\omega'}(q') \rangle \) to account for the charge. This will be found from first principles and from the homogeneous solution of the fluctuating part of the dust Klimontovich equation in absence of fields.

(iv) Furthermore, the dust species serves as a sink for plasma particles which also...
3.2. BASIC COMPLEX PLASMA PARAMETERS

brings out the necessity for a source to sustain the plasma. A rigorous kinetic description should provide results for the absorption cross-sections by itself through dedicated collision integrals [Schram et al., 2000]. However, one can also adopt a sink term description with cross-sections given by pair-particles collision models.

(v) The microscopic phase space densities for the dust particles will still be in the form of a sum of products of $\delta$-functions, including a term for the charge. On the other hand in the microscopic phase space densities for the plasma particles the product of $\delta$-functions should contain extra Heaviside (step) functions that account for the particles vanishing in phase space after absorption or appearing after generation.

It is worth noticing that one can define more microscopic variables for dust: the angular momentum (spinning dust in presence of strong magnetic fields), the dust surface temperature (in case of inhomogeneous plasma fluxes), the mass of the grains (in systems with strong neutral absorption), the internal energy of the grains. However, unlike the dust charge, they can only be important in very specific scenarios.

One question that arises, though, is what is the reason that makes all these kinetic variables appear? What makes dust so special? The answer is quite intuitive, when approximating a constituent that has a classical inner structure with a point particle, then self-consistency demands that all the inner degrees of freedom should complement \{\textbf{r}, \textbf{p}\} as phase-space variables.

An established example can be found in the application of the Klimontovich approach for plasma-molecular systems [Klimontovich et al., 1989]. There molecules are treated classically as strongly coupled subsystems consisting of pairs of electrons/ions bound by a harmonic potential, the point particle description of molecules then leads to a 12-dimensional phase space instead of a 6-dimensional, with the microscopic variables corresponding to position/momentum of the motion of the pair as a whole with the total momentum and to position/momentum of a fictitious particle with reduced mass in the center of mass frame.

3.2 Basic complex plasma parameters

In this section, we introduce the dusty plasma parameter notations and some conventions used in the forthcoming presentation of the kinetic model.

The notations we will follow for the equilibrium dust charge is $q_{eq} = -eZ_d$, where $Z_d$ is the characteristic charge number (with values near 1000 for sub-micron dust grains). The dust radius will be denoted by $a$. We also introduce average kinetic particle energies and denote them by $T_e$, $T_i$, $T_d$ for electrons, ions and dust particles respectively, these quantities are useful in order of magnitude estimations regarding the range of validity of our main assumptions. Common laboratory values are $T_e \simeq (1 - 3)\text{eV}$, $T_i \simeq 0.03\text{eV}$, $T_d \simeq T_i$.

The main parameters describing the system are the following dimensionless
CHAPTER 3. THE KLIMONTOVICH DESCRIPTION OF COMPLEX PLASMAS

quantities:

\[ z = \frac{Z_d e^2}{a T_e}, \quad P = \frac{n_d Z_d}{n_e}, \quad \tau = \frac{T_i}{T_e}, \quad \tau_d = \frac{T_d (1 + P)}{T_d Z_d P} \]  

(3.1)

The dimensionless charge parameter, \( z \), has values of the order of unity, this implies that the equilibrium dust charge is proportional to the dust size. The ion to electron temperature ratio, \( \tau \), has the values \( \tau \simeq 1 \) in Q-machines and tokamaks and \( \tau \simeq 0.01 \) in most discharge plasmas. The dimensionless density parameter, \( P \), not to be confused with the Havnes parameter defined by a normalization on ion density \( P_{\text{hav}} = \frac{n_d Z_d}{n_i} \), gives the quasineutrality condition \( n_i = n_e (1 + P) \).

We also introduce the thermal velocities for each species \( v_{T\alpha} = \sqrt{T_{\alpha} m_{\alpha}} \), the Debye lengths \( \lambda_{D\alpha}^2 = \frac{T_{\alpha}}{4 \pi e^2 n_{\alpha}} \) and the plasma frequencies \( \omega_{p\alpha}^2 = \frac{4 \pi n_{\alpha} e^2}{m_{\alpha}} \). The total plasma Debye length will be given by \( \lambda_D^{-2} = \sum \lambda_{D\gamma}^{-2} \). We should also note that the subscript \( \alpha \) implies only the plasma species \( \alpha = \{i, e\} \). All fluctuations will be denoted by \( \delta \) in front of the physical quantity, the normal components of the distributions will be denoted by \( \Phi \).

3.3 Basic kinetic assumptions and their critical assessment

1. The system is considered infinite and consists of electrons, ions and dust particles.

We refer to systems of large size compared to the length scale of the physical phenomena of interest. Thus, boundary effects can be ignored and the Fourier transform can be used in the treatment of the fluctuating quantities.

2. All components are in the gaseous state, where the kinetic energy substantially exceeds the interaction energy.

This assumption is strongly related with the existence of small parameters and the omission of high order terms in fluctuations. For the plasma components it is always satisfied, since it coincides with the definition of plasma and namely the existence of many particles inside the Debye screening sphere. However, for the dust component, due to large value of the dust charge, this is not always self-evident, especially for micron-dust.

We assume that the mean inter-grain distance is \( r_{\text{av}} \simeq n_d^{-1/3} \) and that the interaction potential is a Yukawa potential with the effective screening length in small distances equal to the ion Debye length \( (\lambda_{\text{scr}} \simeq \lambda_{Di}) \). Then, the ratio of interaction to kinetic energy for the dust grains, known as the coupling parameter, can be written as \( \Gamma \simeq \frac{Z_d^2 e^2}{n_d^{1/3} T_d} r_d^{-4/3} \lambda_{Di}^{-1} \) and the condition for the dust component to be in the gaseous state will simply be \( \Gamma < 1 \).

3. The external electric and magnetic fields are considered to be zero.

In absence of external electric and magnetic fields and for the mean thermal
3.3. BASIC KINETIC ASSUMPTIONS AND THEIR CRITICAL ASSESSMENT

velocity of the particles much less than the speed of light, a number of simplifications is possible. We use the Ampere-Maxwell equation and the Gauss electric law for the fine-grained electromagnetic fields $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ with the plasma/dust particles acting as sources, $\gamma = \{i, c, d\}$,

\[
\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} + \frac{4\pi}{c} \sum_{\gamma} e_{\gamma} \int v f_{\gamma}^{p}(\mathbf{r}, t) \frac{d^3p}{(2\pi)^3},
\]

\[
\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 4\pi \sum_{\gamma} e_{\gamma} \int f_{\gamma}^{p}(\mathbf{r}, t) \frac{d^3p}{(2\pi)^3}.
\]

Roughly, we have $|\mathbf{B}| |\mathbf{E}| \sim |\delta \mathbf{B}| |\delta \mathbf{E}| \sim |\mathbf{v}| c \mathbf{E} \ll 1$, which implies that the magnetic field fluctuations can be ignored compared to the electric field fluctuations. Moreover, from Faraday’s law

\[
\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c} \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t},
\]

we conclude that the rotational part of the electric field follows the scaling

\[
\frac{|\mathbf{E}_{\text{rot}}|}{|\mathbf{E}_{\text{long}}|} \sim \left( \frac{|\mathbf{v}| c}{\mathbf{E}} \right)^2 \ll 1
\]

and therefore the electric field can be considered longitudinal and purely potential, i.e

\[
\nabla \times \mathbf{E}(\mathbf{r}, t) = 0, \quad \mathbf{E}(\mathbf{r}, t) = -\nabla \varphi(\mathbf{r}, t).
\]

4. **The dust grains are fully characterized by position, momentum and charge.**

The dust charge is an independent variable and is generally not a function of the position of the grain.

The dust particles are assumed spherical, mono-disperse, non-spinning with constant mass, internal energy and surface temperature. The variables that fully characterize any possible state of the dust grains are therefore position ($\mathbf{r}$), moment ($\mathbf{p}$) and charge ($q$).

5. **The size of the dust grains is small compared to the plasma Debye radius.**

The dust grains can then be treated as point particles and their exact distribution function will be a sum of δ-functions. This assumption makes it necessary to use model cross-sections for plasma absorption on dust, that are derived by pair-particle collision approaches. Note that such cross-sections will be dependent on the radius and hence some deliverables of the model like the dust fluctuations, the interaction potential, the forces between dust particles will be explicitly dependent on the radius too. The finite size of the radius is very important and there exist forces essential for the grain dynamics that vanish for zero grain radius. Only in their electromagnetic fluctuation treatment are the grains behaving as point particles, cross-sections or collision frequencies used outside the kinetic model can and should be dependent on the radius.
CHAPTER 3. THE KLIMONTOVICH DESCRIPTION OF COMPLEX PLASMAS

The condition \( a \ll \lambda_D \) is nearly always satisfied in engineered dusty plasma ground or micro-gravity experiments and in naturally produced laboratory dust (typically sub-micron). It is also also typically satisfied in astrophysical environments, since there the Debye lengths are large, due to the depleted plasma densities.

6. The dust velocity is much less than the electron/ion velocities.
   
   In any exact computation of the charging cross-sections we should have a result of the form \( \sigma_{\alpha}(q, v_r) \) with \( v_r = |v_\alpha - v_d| \) the magnitude of the relative velocity. Due to the massive dust grains the characteristic dust velocities are much smaller than the plasma velocities, therefore \( v_d \ll v_\alpha \) and for the cross-sections \( \sigma_{\alpha}(q, v_\alpha) \), which simplifies the derivations significantly. Moreover, the dynamic screening factor \( |\epsilon_{k,k'}^{eff}w|^{-2} \) in the dust-dust collision integrals can be evaluated at zero frequencies.

7. The sources of plasma particles and the currents emitted by the grain are considered non-fluctuating.
   
   The plasma sources, \( s(r, t) \), can be varying in space and time but only slowly when compared to the temporal and spatial scales of the fluctuations. The existence of clearly distinguishable hierarchical scales is what justifies the decomposition into regular and fluctuating parts. Such sources can be e.g constant radiation sources leading to photo-ionization of neutrals. The currents emitted by the grain (photo-emission, thermionic emission, field emission or secondary emission) are also considered non-fluctuating and are denoted by \( I_{ext}(r, t) \).

8. Only the discreteness of the dust component is taken into account while the electrons and ions are treated as continuous Vlasov fluids in phase space.
   
   This means that in the treatment of fluctuations, for the fluctuating part of the dust distribution function we use \( \delta f^d_P(q,r,t) = \delta f^{d,(ind)}_P(q,r,t) + \delta f^{d,(0)}_P(q,r,t) \), while for the electrons and ions we only take into account the fluctuating parts induced by electric field fluctuations, i.e \( \delta f^\alpha_P(r,t) = \delta f^{\alpha,(ind)}_P(r,t) \) [Tsytovich and de Angelis, 1999]. In fact, this procedure has been described as a two-step averaging [Tsytovich et al., 2004]: a complete self consistent averaging procedure would involve simultaneous ensemble averaging over the total discreteness of the system in which the collision integrals describing the charging process would emerge through the fluctuation theory alone, on the contrary here in the first step we average over the plasma discreteness and use charging model cross-sections for plasma absorption on dust (like the Orbit Motion Limited approach), while in the second step we average over the dust discreteness. Physically, the omission of plasma discreteness is connected to the fact that the frequencies related to dust discreteness are much smaller than those related to plasma discreteness due to the much larger grain mass, smaller char-
3.3. BASIC KINETIC ASSUMPTIONS AND THEIR CRITICAL ASSESSMENT

Academic dust velocities and number densities. The assumption limits the validity of the kinetic model in the low frequency regime of dust dynamics, that can roughly be defined by \( \omega \ll k v_{T_i} \) or more accurately by \( \omega / k < \Lambda_{\alpha} v_{T_d} \) with \( \Lambda_{\alpha} \) typically of the order of a few and slightly dependent on plasma/dust parameters. The assumption also implies further restrictions in the model; (i) omission of plasma discreteness leads to omission of plasma-plasma collision integrals in the kinetic equations which means that plasma binary collisions should be neglected compared to Coulomb/inelastic collisions with dust, (ii) omission of plasma discreteness leads to the omission of non-collective dust charge fluctuations, which should be smaller than collective dust charge fluctuations. These consequences will be addressed in the last two assumptions. This assumption is also crucial mathematically, leading to great simplifications, since now it is possible to express all fluctuating quantities through the natural dust fluctuations and compute collision integrals and spectral densities with the aid of the natural statistical correlator for dust.

9. The dust density parameter must be large enough for the electron/ion binary Coulomb collisions to be neglected when compared to dust/plasma elastic or inelastic collisions.

Coulomb collisions between plasma species can be of three types: ion-ion collisions, ion-electron collisions and electron-electron collisions, each with its own mean collision frequency. The largest collision frequency is the ion-ion collision frequency given by

\[
\bar{\nu}_{ii} = \frac{4}{3} \frac{\sqrt{\pi} n_i e^4 \Lambda}{\sqrt{m_i T_i^{3/2}}}
\]

with \( \Lambda = \int_{\lambda_{min}}^{\Lambda_{\alpha}} \frac{dt}{\Lambda_{\alpha}} = \ln \left( \frac{\Lambda_{\alpha}}{\lambda_{min}} \right) \) the Coulomb logarithm representing the cumulative effect of all Coulomb collisions within a Debye sphere for impact parameters ranging from the distance of closest approach to the Debye length. For Coulomb collisions of ions with dust one can similarly acquire

\[
\bar{\nu}_{id} = \frac{4}{3} \frac{\sqrt{\pi} n_d Z_i^2 e^4 \Lambda'}{\sqrt{m_i T_i^{3/2}}}
\]

where the Coulomb logarithm will be different in general, due to moderate or even strong ion-dust coupling. However, the differences will be small and it can also be demonstrated that neither large-angle scattering nor non-linear scattering can change the above result significantly. Demanding \( \bar{\nu}_{ii} < \bar{\nu}_{id} \) for binary plasma collisions to be negligible we end up with the condition \( PZ_d > 1 \) [Tsytovich, 1998].

Such a condition clearly sets severe limitations; On one hand the dust densities must be high enough so that collisions with dust dominate over plasma binary collisions, whereas on the other hand the dust densities must be low enough
so that the dust component is in the gaseous state. This restricts the validity of the kinetic model in engineered weakly coupled dusty plasma experiments.

10. The charge on the grain is sufficiently large \( Z_d \gg 1 \) and the dust charge fluctuations are small.

Dust grains embedded in a plasma are generally multiply charged, \( Z_d \gg 1 \). However, for grain sizes below tens of nanometers \( Z_d \) could be relatively low, such cases cannot be treated classically, since then quantum mechanical effects for electrons in charging usually become important: thermionic emission, field-assisted tunneling of grain surface electrons to the surrounding plasma, quantum tunneling onto the grain of plasma electrons overcoming the repulsive potential barrier.

Due to fluctuations associated with plasma or dust discreteness the surface potential and the charge of the dust grain will also fluctuate. Dust charge fluctuations can be either non-collective (referring to individual dust grains) or collective (referring to ensembles of dust grains) [Tsytovich et al., 2002].

The source of non-collective dust charge fluctuations is the discreteness of the charging process itself associated with discrete impacts of individual electrons and ions on the grain. It can be theoretically treated as a one-step Markov process with the accompanying master equation describing the generation and depletion of dust particles with characteristic charge number \( q/e \) that can only vary in a \( \pm 1 \) stepwise fashion [Matsoukas and Russell, 1995]. Alternatively, it can be treated by considering the Klimontovich decomposition of the charging equation with the natural fluctuating parts of the plasma distribution functions taken into account only, together with the natural statistical correlator and the use of form factors in the capture cross-sections [Tsytovich and de Angelis, 2002]. Both approaches are equivalent and in case of O.M.L cross-sections the result is \( \langle (\delta Z_d)^2 \rangle = \frac{\tau + z}{Z_d (1+z)} \) which for \( \tau \ll 1 \) and \( z \approx 1 \) results in \( \frac{\langle (\delta Z_d)^2 \rangle}{Z_d^2} \approx \frac{1}{Z_d (1+z)} \approx \frac{1}{Z_d} \). We therefore conclude that provided that \( Z_d \gg 1 \) the dust charge fluctuations are small compared to the quasi-equilibrium dust charge, i.e \( \langle (\delta q)^2 \rangle \ll q_{eq}^2 \).

The source of collective dust charge fluctuations is the discreteness of the dust component. Results from the Klimontovich approach have revealed that induced plasma fluctuations have a negligible effect, hence one can take into account only induced and natural dust fluctuations [Tsytovich and de Angelis, 2002]. An approximate result in terms of dusty plasma parameters is \( \frac{\langle (\delta Z_d)^2 \rangle}{Z_d^2} \approx n_d a^2 \lambda D_1 \frac{z^2}{(1+z)^2} \), which for typical values is always much less than unity (but simultaneously larger than the non-collective part).

We have so far demonstrated that for grains with sizes above tens of nanometers, the kinetic assumption \( \langle (\delta q)^2 \rangle \ll q_{eq}^2 \) strictly holds. We know investigate the necessity for such an assumption: Owing to the charge expansion of the phase-space, the Liouville and consequently the Klimontovich equation for dust contains an extra charge derivative term. Thus, the Fourier transformed
3.4. THE KLIMONTOVICH EQUATIONS FOR THE DUST/PLASMA COMPONENTS

fluctuation equation for dust will be a first order inhomogeneous differential equation with respect to the charge instead of an algebraic equation. This will alter both the natural statistical correlator and the treatment of the induced dust fluctuations. The approximation of small deviations from the equilibrium charge leads to physical results and less cumbersome Green’s functions.

3.4 The Klimontovich equations for the dust/plasma components

The Liouville phase-space for a system consisting of $N_d$ dust particles will be $7N_d$-dimensional and the exact dust particle density in such a phase-space will have the form $N(r_1, p_1, q_1, ..., r_{N_d}, p_{N_d}, q_{N_d}, t) = \prod_{i=1}^{N_d} \delta(r_i - X_i(t))\delta(p_i - p_i(t))\delta(q_i - Q_i(t))$.

Liouville’s equation, the expression of conservation of probability in the phase space, will now obtain the form

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_d} \nabla_r(r_i, N) + \sum_{i=1}^{N_d} \nabla_p(p_i, N) + \sum_{i=1}^{N_d} \frac{\partial}{\partial q_i} (q_i, N) = 0 \quad (3.2)$$

The density $N$ represents the joint probability that the dust particle 1 has coordinates between $(r_1, p_1, q_1)$ and $(r_1 + dr_1, p_1 + dp_1, q_1 + dq_1)$, the dust particle 2 has coordinates between $(r_2, p_2, q_2)$ and $(r_2 + dr_2, p_2 + dp_2, q_2 + dq_2)$... and the dust particle $N_d$ has coordinates between $(r_{N_d}, p_{N_d}, q_{N_d})$ and $(r_{N_d} + dr_{N_d}, p_{N_d} + dp_{N_d}, q_{N_d} + dq_{N_d})$. Hence, if we integrate over the phase-space coordinates of all particles except one, we will end up with the reduced probability distribution of one particle $f_d(r, p, q, t)$ that is the same with the Klimontovich microscopic phase-space density and will satisfy the equation [Tsytovich and de Angelis, 1999]

$$\frac{\partial f_d(r, p, q, t)}{\partial t} + \nabla_r(\dot{r} f_d^d(r, p, q, t)) + \nabla_p(\dot{p} f_d^d(r, p, q, t)) + \frac{\partial}{\partial q} (\dot{q} f_d^d(r, p, q, t)) = 0 \quad (3.3)$$

So far, we only did an abstract extension to the phase-space. The physical behavior of the dust particles is governed by a set of three dynamic equations. These are the definition of momentum, Newton’s second law of motion for the Lorentz force and the charging equation.

$$\dot{r} = p/m_d; \quad \dot{p} = qE + (q/m_d)p \times B; \quad I = \dot{q} = I_{ext} + \sum_{\alpha} I_\alpha \quad (3.4)$$

If we use the fact that momentum/space are independent variables, neglect the magnetic field and substitute for all terms, we end up with the generalized form of the Klimontovich equation for dust particles,

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + qE \cdot \frac{\partial}{\partial p}\right) f^d_d(r, p, q, t) + \frac{\partial}{\partial q} \left[ I_{ext}(r, t) + \sum_{\alpha} I_\alpha(q, r, t) \right] f^d_d(r, q, t) = 0 \quad (3.5)$$
The current $I_\alpha(q, r, t)$ is the current of plasma particles collected by the grain, it will be the sum of current fluxes on the grain, thus we just integrate the current flux $e_\alpha \sigma_\alpha(q,v) v$ all over the distribution function of the plasma species,

$$I_\alpha(q, r, t) = \int e_\alpha \sigma_\alpha(q,v) v f^\alpha_p(r,t) \frac{d^3 p}{(2\pi)^3}.$$  \hspace{1cm} (3.6)

In case of the plasma species, the phase space will remain $6N_\alpha$-dimensional, the difference with the multi-component kinetic theory is that probability is not conserved in the usual way, ions/electrons are lost due to the charging process with the dust particles acting as sinks for the plasma species distribution and hence, an external source of plasma particles will be added, which we assume that only has a regular component. The generalized Klimontovich equation for the plasma species will be given by [Tsytovich and de Angelis, 1999]

$$\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + e_\alpha E \cdot \frac{\partial}{\partial p} \right) f^\alpha_p(r,t) = s_\alpha(r,t) - \left( \int \sigma_\alpha(q,v) v f^\alpha_p(r,q,t) dq \frac{d^3 p}{(2\pi)^3} \right) f^\alpha_p(r,t).$$  \hspace{1cm} (3.7)

Finally, the system of equations is self-consistently closed by Maxwell’s equations with the particles acting as the sources of the fine-grained electromagnetic fields. In case of un-magnetized plasmas and in absence of external electric fields these degenerate in the Poisson equation, that will have the form

$$\nabla \cdot E(r,t) = 4\pi \left( \sum_\alpha e_\alpha \int f^\alpha_p(r,t) \frac{d^3 p}{(2\pi)^3} + \int q \delta f^d_p(r,q,t) dq \frac{d^3 p}{(2\pi)^3} \right).$$  \hspace{1cm} (3.8)

While in multi-component models the Klimontovich equations are coupled only through the microscopic electrostatic fields, here there is additional coupling due to the charging equation and the sink term. This introduces the expressions for the fluctuating quantities and the permittivity, introduces new collision integrals and in general results in cumbersome algebra.

In the latter we will omit the $r,t$ dependencies for simplicity. Moreover, the dust momentum will be denoted by $p'$ when in the same expression with the plasma momentum $p$.

### 3.5 The decomposition of the Klimontovich equations

The quantities are decomposed into regular (ensemble averaged) and fluctuating (with zero average) parts. Initially, the Klimontovich equations are ensemble averaged, which results in the kinetic equation for the regular part. The regular part is subtracted from the Klimontovich equation and second order fluctuation terms are neglected as well as their averages, which results in the equation for the fluctuating part. This equation is then Fourier transformed in space and time, where due to the different space-time scales the regular components can be considered constant when treating the fluctuating terms.
We apply the methodology for the dust particles, with \( E = \delta E, f^d_p(q) = \Phi^d_p(q) + \delta f^d_p(q), I_\alpha(q) = \langle I_\alpha(q) \rangle + \delta I_\alpha(q) \), where the fluctuations in the particle current are related with the fluctuations in the particle density distribution via

\[
\delta I_\alpha(q) = e_\alpha \int v \sigma_\alpha(q, v) \delta f^d_p(q) \, d^3p / (2\pi)^3. \tag{3.9}
\]

Moreover, the external currents are considered to have a regular part only. For the dust regular component we obtain

\[
\left\{ \frac{\partial}{\partial t} + v \cdot \nabla + \frac{\partial}{\partial q} \left[ \left( I_{\text{ext}} + \sum_\alpha \langle I_\alpha(q) \rangle \right) \right] \right\} \Phi^d_p(q) = -q \frac{\partial}{\partial p} \langle \delta E \delta f^d_p(q) \rangle - \frac{\partial}{\partial q} \sum_\alpha \langle \delta I_\alpha(q) \delta f^d_p(q) \rangle.
\]

The terms on the left side of the equation: vary smoothly in the \( \{ r, p, q \} \) phase-space and represent collective effects, the third term describes collective effects stemming from the charging process and is absent in the multi-component kinetic model. The terms on the right side of the equation: are spiky quantities and represent collisional effects, the first term describes dust-dust collisions and dust-plasma particle collisions in the presence of dust charge fluctuations, the second term describes the collisional effects of the charging process and has no analogy in previous kinetic models.

For the random dust component, we Fourier transform in space and time (\( \partial/\partial t \rightarrow -i\omega, \nabla \rightarrow ik \)) and we use the separate time/space-scale assumption to avoid convolutions in the Fourier transforms of products of functions. The result will be:

\[
v(\omega - k \cdot v)\delta f^d_{p, k, \omega}(q) - \frac{\partial}{\partial q} \left[ \left( I_{\text{ext}} + \sum_\alpha \langle I_\alpha(q) \rangle \right) \delta f^d_{p, k, \omega}(q) \right] = q \delta E_{k, \omega} \cdot \frac{\partial}{\partial p} \Phi^d_p(q)
+ \frac{\partial}{\partial q} \left( \sum_\alpha \delta I^0_{k, \omega}(q) \Phi^d_p(q) \right). \tag{3.11}
\]

Homogeneous equation: when the right hand side is zero, we have a differential equation that depends on the spiky quantity \( \delta f^d_{p, k, \omega}(q) \) only, the solution will give the free dust particle fluctuations due to discreetness. Inhomogeneous equation: when the right hand side is non-zero, we have to find a particular solution, this will be dependent on the spiky quantities \( \delta I^0_{k, \omega}(q), \delta E_{k, \omega} \) and will describe fluctuations induced by the fluctuating electric field and by fluctuations of the currents in the charging of dust, the term \( \delta I^0_{k, \omega}(q) \) will couple this equation to the random plasma component equation. The total solution will be the sum of the homogeneous and the particular solution, \( \delta f^d_{p, k, \omega}(q) = \delta f^d_{p, k, \omega}^{(0)}(q) + \delta f^d_{p, k, \omega}^{\text{ind}}(q) \), in analogy with the multi-component kinetic models.
QUANTITATIVE PHYSICS AND INSTRUMENTATION

3.6 The assumption of small deviations from the dust equilibrium charge

In this part, we will use the approximation that the deviations from the equilibrium charge in both the regular and the fluctuating parts of the dust particle distribution function are small [Tsytovich and de Angelis, 1999]. The quasi-equilibrium charge is given by the condition that the average net particle flux on the dust particles vanishes. Hence, we regard the charging equation for the ensemble averaged parts...
and set the total current $dq/dt$ equal to zero, this yields

$$I_{ext} + \sum_{\alpha} \langle I_{\alpha}(q_{eq}) \rangle = 0 \Rightarrow I_{ext} + \sum_{\alpha} \int e_{\alpha} \sigma_{\alpha}(q_{eq}, v) v \Phi_{\alpha}^{(0)} \frac{d^3 p}{(2\pi)^3} = 0$$  \hspace{1cm} (3.15)

We assume small deviations of the dust charges around the equilibrium value, $q = q_{eq} + \Delta q$. We use the Taylor expansion for the particle current on the dust particle and we keep the first order terms only.

$$\sum_{\alpha} \langle I_{\alpha}(q) \rangle = \sum_{\alpha} \langle I_{\alpha}(q_{eq}) \rangle + \Delta q \frac{\partial}{\partial q} \left( \sum_{\alpha} \langle I_{\alpha}(q_{eq}) \rangle \right) \approx \sum_{\alpha} \langle I_{\alpha}(q_{eq}) \rangle + \nu_{ch} \Delta q \Rightarrow \nu_{ch} \Delta q = -\sum_{\alpha} \left( \frac{\partial}{\partial q} \langle I_{\alpha}(q_{eq}) \rangle \right) \Delta q, \hspace{1cm} (3.16)$$

where we defined the charging frequency $\nu_{ch} = -\frac{\partial}{\partial q} \left( \sum_{\alpha} \langle I_{\alpha}(q_{eq}) \rangle \right) \bigg|_{q=q_{eq}}$. We substitute $q = q_{eq} + \Delta q$ in the charging equation and use the Taylor expansion and the quasi-equilibrium condition, together with Eq.(3.15),

$$\frac{\partial}{\partial t} (q_{eq} + \Delta q) = I_{ext} + \sum_{\alpha} \langle I_{\alpha}(q_{eq}) \rangle - \nu_{ch} \Delta q \Rightarrow$$

$$\frac{\partial}{\partial t} \Delta q = -\nu_{ch} \Delta q \Rightarrow \frac{\partial (\Delta q)}{\Delta q} = -\nu_{ch} \frac{\partial t}{\partial t} \Rightarrow$$

$$\int_{0}^{t} \frac{\partial (\Delta q)}{\Delta q} = -\nu_{ch} t \Rightarrow \ln \frac{\Delta q(t)}{\Delta q_{0}} = -\nu_{ch} t \Rightarrow \Delta q(t) = \Delta q_{0} \exp (-\nu_{ch} t), \hspace{1cm} (3.17)$$

where $\Delta q_{0}$ is the initial value of the deviation at $t = 0$. It is obvious that the charges are relaxing to their equilibrium values with a time constant $\tau_{rel} = 1/\nu_{ch}$. Furthermore, the initial sign of the deviations does not change until equilibrium is reached. Therefore, by normalizing the deviations with their initial value we will always obtain a positive quantity, $\bar{q} = \Delta q/\Delta q_{0} > 0$.

We can now return to the equation of the fluctuating part of the dust distribution function and apply the above approximation. Initially, we regard the homogeneous differential equation in order to acquire a relation for the dust natural statistical correlator,

$$\ast \omega - k \cdot v \delta f_{p,k,\omega}^{d,(0)}(q) - \frac{\partial}{\partial q} \left[ I_{ext} + \sum_{\alpha} \langle I_{\alpha}(q) \rangle \right] \delta f_{p,k,\omega}^{d,(0)}(q) = 0$$

$$(\omega - k \cdot v) \delta f_{p,k,\omega}^{d,(0)}(\Delta q) + \frac{\partial}{\partial \Delta q} \left( \nu_{ch} \Delta q \delta f_{p,k,\omega}^{d,(0)}(\Delta q) \right) = 0. \hspace{1cm} (3.18)$$
The above equation is a first order homogeneous differential equation of the form,

\[ A(x) f(x) + \frac{d}{dx} (B(x)f(x)) = 0 \]

\[ A(x) f(x) + B(x) \frac{df(x)}{dx} + f(x) \frac{dB(x)}{dx} = 0 \]

\[ B(x) \frac{df(x)}{dx} = - \left[ A(x) + \frac{dB(x)}{dx} \right] f(x) \]

\[ \frac{df(x)}{f(x)} = - \frac{A(x)}{B(x)} \frac{dB(x)}{B(x)} dx \]

\[ \ln f(x) = - \ln B(x) - \int \frac{A(x)}{B(x)} dx \]

\[ f(x) = C' B(x) \exp \left( - \int \frac{A(x)}{B(x)} dx \right), \]

where \( C' \) is the integration constant. For \( x = \Delta q \), \( A(x) = \nu (\omega - k \cdot v) \), \( B(x) = \nu_{ch} \Delta q \), \( C = \frac{C'}{\nu_{ch}} \), this gives

\[ \delta f^{d,(0)}_{p,k,\omega} (\Delta q) = \frac{C}{\Delta q} \exp \left[ \ln (\Delta q) - \frac{\nu (\omega - k \cdot v)}{\nu_{ch}} \right] \]

\[ \delta f^{d,(0)}_{p,k,\omega} (\Delta q) = \frac{C}{\Delta q} \exp \left[ - \frac{\nu (\omega - k \cdot v)}{\nu_{ch}} \ln (\Delta q) \right] \] (3.19)

In the case of \( \Delta q < 0 \) we should find an alternative expression for the solution due to the undefinable negative logarithm. Since \( \text{sgn}(\Delta q) = \text{sgn}(\Delta q_0) \), we can normalize with respect to the initial deviation, \( \bar{\eta} = \Delta q / \Delta q_0 > 0 \) and resolve the ill-defined logarithm issue,

\[ \nu (\omega - k \cdot v) \delta f^{d,(0)}_{p,k,\omega} (\Delta q) + \frac{\partial}{\partial \Delta q} \left( \nu_{ch} \Delta q_0 \frac{\Delta q_0}{\Delta q} \delta f^{d,(0)}_{p,k,\omega} (\Delta q) \right) = 0 \]

\[ \nu (\omega - k \cdot v) \delta f^{d,(0)}_{p,k,\omega} (\Delta q) + \frac{\partial}{\partial (\Delta q_0)} \left[ \nu_{ch} \Delta q_0 \delta f^{d,(0)}_{p,k,\omega} (\Delta q) \right] = 0 \]

\[ \nu (\omega - k \cdot v) \delta f^{d,(0)}_{p,k,\omega} (\bar{\eta}) + \frac{\partial}{\partial \bar{\eta}} \left[ \nu_{ch} \bar{\eta} \delta f^{d,(0)}_{p,k,\omega} (\bar{\eta}) \right] = 0 \]

\[ \delta f^{d,(0)}_{p,k,\omega} (\bar{\eta}) = \frac{C}{\bar{\eta}} \exp \left[ - \frac{\nu (\omega - k \cdot v)}{\nu_{ch}} \ln (\bar{\eta}) \right] \] (3.20)

Moreover, we can choose the constant \( C \) to correspond to the position of the charge at \( t = 0 \) being \( r = r_0 \), this delta function condition is equivalent to \( e^{-ik \cdot r_0} \) in the Fourier space. Overall, we have

\[ \delta f^{d,(0)}_{p,k,\omega} (\bar{\eta}) = e^{ik \cdot r_0} \frac{1}{\bar{\eta}} \exp \left[ - \frac{\nu (\omega - k \cdot v)}{\nu_{ch}} \ln (\bar{\eta}) \right]. \] (3.21)
3.6. THE ASSUMPTION OF SMALL DEVIATIONS FROM THE DUST EQUILIBRIUM CHARGE

This expression for the natural dust fluctuations can be expressed in an approximate but more convenient form if we switch back to the time-space domain. We use the inverse Fourier transform, split the resulting quadruple integral into a product of integrals and use the properties of the delta function

\[ \delta(t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{+ik \cdot r} d^3k \] and

\[ \delta(r) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \] and

\[ \delta f_d(0) \propto \delta(\Delta q - \Delta q_0 \exp(-\nu_{ch} t)) \delta (r - r_0 - vt) \] (3.24)

The conclusions from this relation are: any charge moves in space with constant velocity \( v \) and reaches its equilibrium value \( \Delta q = 0 \) rapidly with the time determined by the ratio 1/\( \nu_{ch} \), which is small. If we have low-frequency fluctuations in frequencies less than the charging frequency, we can neglect the exponential factor in Eq.(3.23), hence for these fluctuations the equilibrium charge is reached almost instantaneously,
We return to Fourier space,
\[\delta f^{(0)}_{p,k,\omega}(\mathbf{q}) \propto \delta(\Delta q) e^{i\mathbf{q} \cdot \mathbf{r}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta (\mathbf{r} - \mathbf{r}_0 - vt) e^{-i\mathbf{k} \cdot \mathbf{r}} dt d\mathbf{r} d\mathbf{r}_0\]
\[\propto \delta(\Delta q) e^{i\mathbf{q} \cdot \mathbf{r}_0} \int_{-\infty}^{\infty} e^{-i\mathbf{k} \cdot \mathbf{r}_0} dt \]
\[\propto \delta(\Delta q) e^{-i\mathbf{k} \cdot \mathbf{r}_0} \int_{-\infty}^{\infty} e^{i(\mathbf{k} \cdot \mathbf{v} - \omega)t} dt \]
\[\propto e^{-i\mathbf{k} \cdot \mathbf{r}_0} \delta(\Delta q) \delta(\omega - \mathbf{k} \cdot \mathbf{v}). \tag{3.25}\]

It is obvious that this solution still satisfies the homogeneous equation of fluctuations, the first term will vanish due to \(\delta(\omega - \mathbf{k} \cdot \mathbf{v})\) and the second due to \(\delta(\Delta q)\). One could also reach this solution by using a limiting procedure for \(\nu_{ch} \rightarrow 0\) in Eq.(3.21) and some properties of generalized functions.

We can now average the product of two free particle fluctuations with respect to the charge position \(\mathbf{r}_0\), we use that \(\delta(\Delta q)\delta(\Delta q') = \delta(q - q')\delta(q - q)\) and \((e^{i\mathbf{k} \cdot \mathbf{r}_0} e^{i\mathbf{k} \cdot \mathbf{r}_0}) = \int \int \exp(i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}_0) d\mathbf{r}_0 = \delta(\mathbf{k} + \mathbf{k}')\),
\[\langle \delta f^{(0)}_{p',k',\omega}(\mathbf{q}) \delta f^{(0)}_{p,k,\omega}(\mathbf{q}) \rangle \propto \delta(\Delta q)\delta(\Delta q')\delta(\omega - \mathbf{k} \cdot \mathbf{v})\delta(\omega' - \mathbf{k}' \cdot \mathbf{v}')\delta(p - p')\]
\[\propto \delta(q - q')\delta(q - q)\delta(q - q')\delta(q - q')\delta(\omega - \mathbf{k} \cdot \mathbf{v})\delta(\omega' - \mathbf{k}' \cdot \mathbf{v}')\delta(p - p')\delta(\mathbf{k} + \mathbf{k}')\]
\[\propto \delta(q - q')\delta(q - q)\delta(q - q')\delta(q - q')\delta(\omega - \mathbf{k} \cdot \mathbf{v})\delta(\omega' + \mathbf{k} \cdot \mathbf{v})\delta(p - p')\delta(\mathbf{k} + \mathbf{k}'). \tag{3.26}\]

Finally, the proper normalization of the average should be the average part of the dust distribution function, which yields
\[\langle \delta f^{(0)}_{p',k',\omega}(\mathbf{q}) \delta f^{(0)}_{p,k,\omega}(\mathbf{q}) \rangle = \Phi^d_{\nu}(\mathbf{q}) \delta(q - q')\delta(q - q)\delta(\omega - \mathbf{k} \cdot \mathbf{v})\delta(\omega' + \mathbf{k} \cdot \mathbf{v})\delta(p - p')\delta(\mathbf{k} + \mathbf{k}'). \tag{3.27}\]

This is a generalization of the natural statistical correlator of the multi-component kinetic theory,
\[\langle \delta f^{(0)}_{p,k',\omega}(\mathbf{q}) \delta f^{(0)}_{p,k,\omega}(\mathbf{q}) \rangle = \Phi^d_{\nu} \delta(\mathbf{p} - \mathbf{p}')\delta(\omega - \mathbf{k} \cdot \mathbf{v})\delta(\omega' + \mathbf{k} \cdot \mathbf{v})\delta(p - p')\delta(\mathbf{k} + \mathbf{k}'), \tag{3.28}\]

where the differences are: the omitted Kronecker delta due to the fact that there is only one kind of dust particles and the addition of charge related delta functions.

We now find Green’s function of the inhomogeneous equation for the fluctuating part of the dust distribution function. It can be conveniently rewritten as
\[\left[\frac{\omega - \mathbf{k} \cdot \mathbf{v}}{\nu_{ch} + \frac{\partial}{\partial \mathbf{q}}}\right] \delta f^{(0)}_{p,k,\omega}(\mathbf{q}) = \frac{1}{\nu_{ch}} R_{p,k,\omega}(\mathbf{q}), \tag{3.29}\]
where \(R_{p,k,\omega}(\mathbf{q})\) refers to the inhomogeneous part, that is dependent on the electric field and particle current fluctuations. For a general solution, we should first find
the Green’s function that satisfies the differential equation

\[
\left[ \frac{i(\omega - k \cdot v)}{v_{ch}} + \frac{\partial}{\partial q} \right] G(q, q', \omega - k \cdot v) = \frac{1}{\nu_{ch}} \delta(q - q').
\]  

(3.30)

The solution will be the same as in the homogeneous equation, but the constant will be dependent on \(q'\), thus

\[
G(q, q') = \frac{C(q')}{q} \exp \left( -i \frac{\omega - k \cdot v}{v_{ch}} \ln q \right)
\]

(3.31)

In the theory of Green’s functions in the second order differential equations, the Green’s function,\( G(q, q') \), will be dependent on \(q'\).

\[
G(q, q') = \begin{cases} 
C_1(q') \exp \left( -i \frac{\omega - k \cdot v}{v_{ch}} \ln q \right), & q' > q \\
C_2(q') \exp \left( -i \frac{\omega - k \cdot v}{v_{ch}} \ln q \right), & q' < q 
\end{cases}
\]

(3.32)

Now let \(\epsilon > 0\), with \(\epsilon\) being arbitrarily small, then \(q'_{+} = q' + \epsilon, q'_{-} = q' - \epsilon\) and we integrate the differential equation in the interval \((q', q'_{+}, q'_{-})\) using the discontinuity of the Green’s function,

\[
\frac{i(\omega - k \cdot v)}{v_{ch}} \int_{q'_{-}}^{q'_{+}} G(q, q') dq' + \int_{q'_{-}}^{q'_{+}} \frac{\partial}{\partial q} G(q, q') dq' = \frac{1}{\nu_{ch}} \int_{q'_{-}}^{q'_{+}} \delta(q - q') dq' \Rightarrow
\]

\[
\frac{i(\omega - k \cdot v)}{v_{ch}} \int_{q_{+}}^{q_{-}} G(q, q') dq' + \int_{q_{-}}^{q_{+}} \frac{\partial}{\partial q} G(q, q') dq' = \frac{1}{\nu_{ch}} \Rightarrow \left[ \frac{\partial G(q, q')}{\partial q} \right]_{q_{-}}^{q_{+}} = \frac{1}{\nu_{ch}} \Rightarrow
\]

\[
\frac{\partial}{\partial q} \left[ G_{+}(q', q) - G_{-}(q', q) \right] = \frac{1}{\nu_{ch}} \Rightarrow C_1(q') - C_2(q') = \frac{1}{\nu_{ch}} \exp \left( +i \frac{\omega - k \cdot v}{v_{ch}} \ln q' \right)
\]

(3.33)

Causality arguments lead to the conclusion that for \(q' < q\) the response must be zero. Hence, \(G_{-}(q', q) = 0 \Rightarrow C_{2}(q') = 0\), and with the use of the above relation \(C_1(q') = -\frac{1}{\nu_{ch}} \exp \left( +i \frac{\omega - k \cdot v}{v_{ch}} \ln q' \right)\). Overall, we have

\[
G(q, q') = \begin{cases} 
-\frac{1}{\nu_{ch}} \exp \left( -i \frac{\omega - k \cdot v}{v_{ch}} \ln (q) - \ln (q') \right), & q' > q \\
0, & q' < q
\end{cases}
\]

(3.34)

Knowledge of the Green’s function of the problem means that the solution of the inhomogeneous equation with an inhomogeneous term \(R_{p,k,\omega}(q)\) will be

\[
\delta f_{p,k,\omega}^{d, (ind)}(q') = \int G(q', q'') R_{p,k,\omega}(q'') dq''.
\]

(3.35)
CHAPTER 3. THE KLIMONTOVICH DESCRIPTION OF COMPLEX PLASMAS

We note that the inhomogeneous term consists of a "non-charge derivative" source term \( R_{\text{I}}^{I}(p,k,\omega)(q) = q \delta E_{k,\omega} \cdot \frac{\partial}{\partial p} \Phi_{p}(q) \) and a "full charge derivative" source term \( R_{\text{II}}^{II}(p,k,\omega)(q) = \frac{\partial}{\partial q} \left( \sum_{\alpha} \delta I_{\text{II}}^{\alpha}(q) \Phi_{p}(q) \right) \).

We can now investigate the basic properties of the Green’s function. In the treatment of fluctuations and the collision integrals various integrals of the induced fluctuating part of the dust distribution function will appear. In general they can be categorized into two groups. In the first case, the integral will have the form

\[
I_{\text{col1}} = \int A(q,q') \delta f_{p,k,\omega}^{d,\text{ind}}(q')dq',
\]

with \( A(q,q') \) some weighting function and the source term not a full derivative with respect to the charge, then we can use the approximation \( q' \simeq q_{eq} \) in the weighting function, \( A(q,q') \simeq A(q,q_{eq}) \). In the second case, the integral will have the form

\[
I_{\text{col2}} = \int D(q,q') \delta f_{p,k,\omega}^{d,\text{ind}}(q')dq',
\]

with \( D(q,q') \) some weighting function and the source term a full derivative with respect to the charge, \( R_{\text{II}}^{II}(p,k,\omega)(q) = \frac{\partial B(q)}{\partial q} \). Use of the same approximation, would lead to a zero result, to avoid the trivial result, we use the Taylor expansion of the weighting function around the equilibrium charge and keep the two first linear terms.

For the first case, we substitute Eq. (3.35) in the integral, separate the multiple integrals, use the equilibrium approximation and substitute for the Green’s function

\[
I_{\text{col1}} = \int A(q,q') \delta f_{p,k,\omega}^{d,\text{ind}}(q')dq' = \int R_{p,k,\omega}^{I}(q'')dq'' \int A(q,q')G(\vec{q}',\vec{q}'')d\vec{q}'
= \int R_{p,k,\omega}^{I}(q'')A(q,q_{eq})dq'' \int G(\vec{q}',\vec{q}'')d\vec{q}'
= -\int R_{p,k,\omega}^{I}(q'')A(q,q_{eq})dq'' \int \frac{1}{\nu_{ch} q} \exp \left( -\frac{\omega - k \cdot \nu_{ch}}{\nu_{ch}} \left[ \ln (\vec{q}') - \ln (\vec{q}'') \right] \right) d\vec{q}'
= -\int R_{p,k,\omega}^{I}(q'')A(q,q_{eq})dq'' \int \frac{1}{\nu_{ch} q} \exp \left( -\frac{\Omega}{\nu_{ch}} \left[ \ln (\vec{q}') - \ln (\vec{q}'') \right] \right) d\vec{q}'
\]

(3.36)

where we have set \( \Omega = \omega - k \cdot \nu \) for convenience. The Green’s function is zero for \( \vec{q}' < \vec{q}'' \) and non-zero for \( \vec{q}' > \vec{q}'' \), hence the upper limit in the \( d\vec{q}' \) integration must be \( \vec{q}'' \). Since the normalized charge is always positive the lower boundary of integration will be zero. Moreover, use of the transformation \( \ln \vec{q} = u \) in the same
3.6. THE ASSUMPTION OF SMALL DEVIATIONS FROM THE DUST EQUILIBRIUM CHARGE

integral will result in \( \frac{q_f'}{q_f} \) and the integration limits will become \((-\infty, \ln \eta')\).

\[
I_{\text{col}1} = -\int R_{p,k,w}^1(q'') A(q, q_{eq}) dq'' \int_{0}^{\eta''} \frac{1}{\nu_{ch} \eta} \exp \left( -\frac{\Omega}{\nu_{ch}} \ln (\eta) - \ln (\eta'') \right) d\eta' \\
= \frac{1}{\nu_{ch}} \int R_{p,k,w}^1(q'') A(q, q_{eq}) \exp \left( \frac{\Omega}{\nu_{ch}} \ln \eta'' \right) \exp \left( -\frac{\Omega}{\nu_{ch}} u \right) du \\
= \frac{1}{\Omega} \int R_{p,k,w}^1(q'') A(q, q_{eq}) \exp \left( \frac{\Omega}{\nu_{ch}} \ln \eta'' \right) \exp \left( -\frac{\Omega}{\nu_{ch}} \ln \eta'' \right) dq'' \\
= \frac{1}{\Omega} \int R_{p,k,w}^1(q'') A(q, q_{eq}) dq'' \\
= \frac{A(q, q_{eq})}{\iota(\omega - k \cdot v + \iota 0)} \int R_{p,k,w}^1(q'') dq'' , \tag{3.37}
\]

where \(+\iota 0\) was added in the denominator due to causality. For the second case, let us use the same approximation for the weighting function \( D(q, q') \approx D(q, q_{eq}) \) with \( R_{p,k,w}^1(q') = \frac{\partial B(q)}{\partial q} \). We use the property of Eq.(3.37) and obtain

\[
I_{\text{col}2} = A(q, q_{eq}) \frac{1}{\iota(\omega - k \cdot v)} \int \frac{\partial B(q'')}{\partial q''} dq'' \\
I_{\text{col}2} = A(q, q_{eq}) \frac{1}{\iota(\omega - k \cdot v)} \left[ B(q'') \right]_{-\infty}^{+\infty} = 0 , \tag{3.38}
\]

since the physical quantity \( B(q) \) must vanish at infinity. This implies that in the zeroth order approximation the integral is zero, thus, we must use the first order approximation for the weighting function \( D(q, q') \) to avoid the trivial result. We expand in Taylor series and set \( \frac{\partial D(q, q')}{\partial q'} |_{q'=q_{eq}} = C(q) \),

\[
D(q, q') = D(q, q_{eq} + \Delta q_0 \eta') \Rightarrow D(q, q') \simeq D(q, q_{eq}) + \Delta q_0 \eta' \frac{\partial D(q, q')}{\partial q'} |_{q'=q_{eq}} \Rightarrow \\
D(q, q') \simeq D(q, q_{eq}) + \Delta q_0 \eta' C(q) . \tag{3.39}
\]

When we substitute in the integral, following the above discussion the zeroth order term will give a trivial result, following the same procedure as with the first case,
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and using integration by parts at the end we have:

\[ I_{col2} = \int R_{p,k,\omega}^{(i)}(q'') dq'' \int \Delta q_0 C(q) \frac{q}{\nu_{ch}} G(\bar{q}, \bar{q}') dq' \]

\[ = -\Delta q_0 C(q) \int R_{p,k,\omega}^{(i)}(q'') dq'' \int_{-\infty}^{0} \frac{q}{\nu_{ch}} \exp \left[-i \frac{\Omega}{\nu_{ch}} (\ln \bar{q} - \ln \bar{q}')\right] dq' \]

\[ = -\Delta q_0 C(q) \int R_{p,k,\omega}^{(ii)}(q'') dq'' \int_{-\infty}^{0} \frac{1}{\nu_{ch}} \exp \left[-i \frac{\Omega}{\nu_{ch}} (\ln \bar{q} - \ln \bar{q}') + \ln \bar{q}''\right] dq' \]

\[ = -\Delta q_0 C(q) \nu_{ch} \int R_{p,k,\omega}^{(i)}(q'') \exp \left(i \frac{\Omega}{\nu_{ch}} \ln \bar{q}''\right) dq'' \int_{-\infty}^{0} \exp \left[-i \frac{\Omega}{\nu_{ch}} + 1\right] du \]

\[ = -\Delta q_0 C(q) \nu_{ch} \int R_{p,k,\omega}^{(i)}(q'') \exp \left(i \frac{\Omega}{\nu_{ch}} \ln \bar{q}''\right) \left\{ \exp \left[-i \frac{\Omega}{\nu_{ch}} + 1\right] \right\}_{-\infty}^{0} \]

\[ = \frac{C(q)}{-\Omega + \nu_{ch}} \int R_{p,k,\omega}^{(i)}(q'') q'' dq'' \]

\[ = \frac{C(q)}{-\Omega + \nu_{ch}} \int B(q'') dq'' \]

\[ = \frac{C(q)}{-\Omega + \nu_{ch}} \int B(q'') dq'' \]

\[ = \frac{C(q)}{i(\omega - k \cdot \nu + \nu_{ch}) \int B(q'') dq''} \]

\[ = \frac{1}{i(\omega - k \cdot \nu + \nu_{ch}) \int B(q'') dq''} \]

An additional property can be derived by exploiting the narrowness of the regular part of the dust distribution function around the equilibrium dust charge, using the
3.7. THE PERMITTIVITY

reduced dust distribution function $\Phi_{p}^{d}$ defined through $\Phi_{p}^{d}(q) = \delta(q - q_{eq}) \Phi_{p}^{d}$ and integrating Eq.(3.41) over the dust momentum space

$$
\int F(q, q') \delta f_{p, k, \omega}^{d,(ind)}(q') dq' = \frac{F(q, q_{eq})}{i(\omega - k \cdot v + iv_{ch})} \int q'' \delta E_{k, \omega} \frac{\partial \Phi_{p}^{d}(q'' - q_{eq})}{\partial p} dq'' - \frac{1}{i(\omega - k \cdot v + iv_{ch})} \left( \frac{\partial F(q, q')}{\partial q'} \right)_{q' = q_{eq}} \int \sum_{\alpha} \delta f_{k, \omega}^{d}(q'') \Phi_{p}^{d}(q'' - q_{eq}) dq''
$$

$$
\int F(q, q') \delta f_{p, k, \omega}^{d,(ind)}(q') dq' = \frac{F(q, q_{eq})}{i(\omega - k \cdot v + iv_{ch})} \frac{k}{i} \frac{\partial \Phi_{p}^{d}}{\partial p} \delta E_{k, \omega} - \frac{1}{i(\omega - k \cdot v + iv_{ch})} \left( \frac{\partial F(q, q')}{\partial q'} \right)_{q' = q_{eq}} \sum_{\alpha} \delta f_{k, \omega}^{d}(q_{eq}) \Phi_{p}^{d}
$$

$$
\int F(q, q') \delta f_{p, k, \omega}^{d,(ind)}(q') dq' = -\frac{i}{4\pi q_{eq}} \frac{k}{k} \int \frac{1}{\omega - k \cdot v + iv_{ch}} \left( \frac{\partial F(q, q')}{\partial q'} \right)_{q' = q_{eq}} \sum_{\alpha} \delta f_{k, \omega}^{d}(q_{eq}) \left( \frac{1}{\omega - k \cdot v + iv_{ch}} \Phi_{p}^{d} \right)
$$

where we also used the definition of the dust susceptibility ($\chi_{d, eq}$) and the dust charging process response ($\chi_{d, ch}$)

$$
\chi_{d, eq}^{\alpha} = \frac{4\pi q_{eq}^{2}}{k^{2}} \int \frac{1}{\omega - k \cdot v + iv_{ch}} \left( k \frac{\partial \Phi_{p}^{d}}{\partial p} \right) d^{3}p \left( \frac{2\pi}{\hbar} \right)^{3},
$$

$$
\chi_{d, ch}^{\alpha} = \int \frac{1}{\omega - k \cdot v + iv_{ch}} \Phi_{p}^{d} d^{3}p \left( \frac{2\pi}{\hbar} \right)^{3}.
$$

3.7 The permittivity

In order to acquire the permittivity we must express all fluctuating quantities as functions of the natural dust fluctuations. We start from the expression for the fluctuating part of the particle currents to the grains $\delta f_{k, \omega}^{d}(q)$ and substitute for the induced plasma fluctuations $\delta f_{p, k, \omega}^{d,(ind)}$, the fluctuating part of the capture frequency
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\[ \delta v_{d,\omega}(k, \omega, v) \] and the longitudinal field \( \delta E_{k,\omega} = \frac{k}{\pi} \delta E_{k,\omega} \).

\[
\delta I_{k,\omega}(q) = \sum_{\alpha} \int e_{\alpha} v_{\sigma_{\alpha}}(q, v) \delta f_{p, k, \omega}^{(ind)} \frac{d^3 p}{(2\pi)^3} \\
= \sum_{\alpha} \int \frac{e_{\alpha} v_{\sigma_{\alpha}}(q, v)}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \left( e_{\alpha} \delta E_{k,\omega} \cdot \frac{\partial \Phi_{p}^{\alpha}}{\partial p} + \delta v_{d,\alpha}(k, \omega, v) \Phi_{p}^{\alpha} \right) \frac{d^3 p}{(2\pi)^3} \\
= \left( \sum_{\alpha} \int \frac{e_{\alpha} v_{\sigma_{\alpha}}(q, v)}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \delta E_{k,\omega} \left( k \cdot \frac{\partial \Phi_{p}^{\alpha}}{\partial p} \right) \frac{d^3 p}{(2\pi)^3} \right) \\
+ \sum_{\alpha} \int \frac{e_{\alpha} v_{\sigma_{\alpha}}(q, v)}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \delta v_{d,\alpha}(k, \omega, v) \Phi_{p}^{\alpha} \frac{d^3 p}{(2\pi)^3} \\
= \hat{S}_{k,\omega}(q) \delta E_{k,\omega} \\
+ \int \left( \sum_{\alpha} \int \frac{e_{\alpha} v_{\sigma_{\alpha}}(q, v) \sigma_{\alpha}(q', v)}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \Phi_{p}^{\alpha} \frac{d^3 p}{(2\pi)^3} \right) \delta f_{p', k, \omega}^{(ind)}(q') \frac{d^3 q'}{(2\pi)^3} \\
= \hat{S}_{k,\omega}(q) \delta E_{k,\omega} + \int \tilde{\hat{S}}_{k,\omega}(q, q') \delta f_{p', k, \omega}^{(ind)}(q') \frac{d^3 q'}{(2\pi)^3} \\
+ \int \tilde{\hat{S}}_{k,\omega}(q, q') \delta f_{p', k, \omega}^{(ind)}(q') \frac{d^3 q'}{(2\pi)^3), \tag{3.45}
\]

where we used the definition of the mixed responses

\[
\hat{S}_{k,\omega}(q) = \sum_{\alpha} \int \frac{e_{\alpha} v_{\sigma_{\alpha}}(q, v)}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \left( k \cdot \frac{\partial \Phi_{p}^{\alpha}}{\partial p} \right) \frac{d^3 p}{(2\pi)^3}; \tag{3.46}
\]

\[
\tilde{\hat{S}}_{k,\omega}(q, q') = \sum_{\alpha} \int \frac{e_{\alpha} v_{\sigma_{\alpha}}(q, v) \sigma_{\alpha}(q', v)}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \Phi_{p}^{\alpha} \frac{d^3 p}{(2\pi)^3}. \tag{3.47}
\]

For the evaluation of the second adder of the above expression we will apply the property of Eq.(3.42) for \( F(q, q') = \tilde{S}_{k,\omega}(q, q') \).

\[
\int \tilde{\hat{S}}_{k,\omega}(q, q') \delta f_{p', k, \omega}^{(ind)}(q') \frac{d^3 q'}{(2\pi)^3} = - \frac{k \chi_{k,eq}^{d,eq}}{4\pi q_{eq}} \tilde{S}_{k,\omega}(q, q_{eq}) \delta E_{k,\omega} \\
+ \frac{\partial \tilde{\hat{S}}_{k,\omega}(q, q')}{\partial p} \chi_{k,\omega}^{d,eh} \sum_{\alpha} \delta I_{k,\omega}^{\alpha}(q_{eq}) \\
= - \frac{k \chi_{k,eq}^{d,eq}}{4\pi q_{eq}} \tilde{S}_{k,\omega}(q, q_{eq}) \delta E_{k,\omega} \\
+ \tilde{\hat{S}}_{k,\omega}(q, q_{eq}) \chi_{k,\omega}^{d,eh} \sum_{\alpha} \delta I_{k,\omega}^{\alpha}(q_{eq}) \tag{3.48}
\]
with the mixed response $\tilde{S}_{k,\omega}(q, q')$ given by

$$
\tilde{S}_{k,\omega}(q, q') = \sum_\alpha \int \frac{\nu_\alpha v^2 \sigma_\alpha(q, v) \sigma_\alpha'(q', v)}{i(\omega - k \cdot v + i\nu_\alpha(v))} \Phi_{\nu}^0 \frac{d^3p}{(2\pi)^3} \quad (3.49)
$$

Overall, we get an algebraic equation for $\delta I_{k,\omega}(q_e, q)$ by setting $q = q_e$

$$
\delta I_{k,\omega}(q) = S_{k,\omega}(q)\delta E_{k,\omega} - \frac{i k \chi_{k,\omega}}{4\pi q_e} \tilde{S}_{k,\omega}(q, q_e) \delta E_{k,\omega} + S_{k,\omega}(q, q_e) \chi_{k,\omega}^{d, ch} \sum_\alpha \delta I_{\alpha}(q_e) 
$$

$$
+ \int \tilde{S}_{k,\omega}(q, q') \delta f^{(0)}_{p', k, \omega}(q') \frac{d^3p'dq'}{(2\pi)^3}
$$

$$
\delta I_{k,\omega}(q_e) = S_{k,\omega}(q_e)\delta E_{k,\omega} - \frac{i k \chi_{k,\omega}}{4\pi q_e} \tilde{S}_{k,\omega}(q_e, q_e) \delta E_{k,\omega} + \tilde{S}_{k,\omega}(q_e, q_e) \chi_{k,\omega}^{d, ch} \delta I_{k,\omega}(q_e) 
$$

$$
+ \int \left(1 - \tilde{S}_{k,\omega}(q_e, q_e) \chi_{k,\omega}^{d, ch}\right)^{-1} \tilde{S}_{k,\omega}(q_e, q') \delta f^{(0)}_{p', k, \omega}(q') \frac{d^3p'dq'}{(2\pi)^3}
$$

$$
\delta I_{k,\omega}(q_e) = \beta_{k,\omega} \delta E_{k,\omega} + \int \gamma_{k,\omega}(q') \delta f^{(0)}_{p', k, \omega}(q') \frac{d^3p'dq'}{(2\pi)^3} \quad (3.50)
$$

with the auxiliary responses $\beta_{k,\omega}$ and $\gamma_{k,\omega}$ defined by

$$
\beta_{k,\omega} = \frac{S_{k,\omega}(q_e) - \frac{i k \chi_{k,\omega}}{4\pi q_e} \tilde{S}_{k,\omega}(q_e, q_e)}{1 - \tilde{S}_{k,\omega}(q_e, q_e) \chi_{k,\omega}^{d, ch}} \quad (3.51)
$$

$$
\gamma_{k,\omega}(q) = \frac{\tilde{S}_{k,\omega}(q_e, q)}{1 - \tilde{S}_{k,\omega}(q_e, q_e) \chi_{k,\omega}^{d, ch}} \quad (3.52)
$$

The latter result simplifies the property of Eq.(3.42) expressing the integral as a function of the electric field fluctuations and the natural dust fluctuations only,

$$
\int F(q, q') \delta f^{(0)}_{p', k, \omega}(q') \frac{d^3p'dq'}{(2\pi)^3} = \left[ \frac{\partial F(q, q')}{\partial q'} \right]_{q'=q_e} \chi_{k,\omega}^{d, ch} \beta_{k,\omega} - \frac{i k \chi_{k,\omega}^{d, eq}}{4\pi q_e} F(q, q_e) \delta E_{k,\omega} + \int \gamma_{k,\omega}(q') \delta f^{(0)}_{p', k, \omega}(q') \frac{d^3p'dq'}{(2\pi)^3} \quad (3.53)
$$

We now return to the fluctuating parts of the plasma distribution functions, we integrate them over the plasma momentum space, $\delta n_{k,\omega} = \int \delta f_{p, k, \omega} \frac{d^3p}{(2\pi)^3}$, use the
property of Eq. (3.53) and acquire

\[
\begin{align*}
\delta n_{k,\omega}^p &= - \frac{i k}{4\pi} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{\omega - k \cdot v + \nu_{d,\alpha}} \left( k \cdot \frac{\delta \Phi_{p}^{\alpha}}{\partial p} \right) \right] \delta E_{k,\omega} \\
&+ \int \frac{d^3p}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \Phi_{p}^{\alpha} \frac{d^3p}{(2\pi)^3} \int \sigma_{\alpha}(q',v) \delta f_{p',k,\omega}(q') \frac{d^3p' d^3q'}{(2\pi)^3} \\
&= - \frac{i k \chi_{k,\omega}^{d,eq}}{4\pi} \delta E_{k,\omega} + \int \frac{d^3p}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \Phi_{p}^{\alpha} \frac{d^3p}{(2\pi)^3} \int \sigma_{\alpha}(q',v) \delta f_{p',k,\omega}(q') \frac{d^3p' d^3q'}{(2\pi)^3} \\
&= - \frac{i k \chi_{k,\omega}^{d,eq}}{4\pi} \delta E_{k,\omega} + \int \frac{d^3p}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \Phi_{p}^{\alpha} \frac{d^3p}{(2\pi)^3} \int \sigma_{\alpha}(q',v) \delta f_{p',k,\omega}(q') \frac{d^3p' d^3q'}{(2\pi)^3} +
\end{align*}
\]

where we used the definitions of the responses \( \chi_{k,\omega}^{\alpha} \) denoting the plasma susceptibilities altered from the multi-component expressions due to the presence of the dissipative frequency \( \nu_{d,\alpha}(v) \) in the denominator, and of the charging responses \( \tilde{\gamma}_{k,\omega}^{\alpha}(q) \) and \( \tilde{\beta}_{k,\omega}^{\alpha}(q) \)

\[
\begin{align*}
\chi_{k,\omega}^{\alpha} &= \frac{4\pi e_{\alpha}^2}{k^2} \int \frac{1}{\omega - k \cdot v + \nu_{d,\alpha}} \left( k \cdot \frac{\delta \Phi_{p}^{\alpha}}{\partial p} \right) d^3p \frac{1}{(2\pi)^3} \\
\tilde{\gamma}_{k,\omega}^{\alpha} &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{i(\omega - k \cdot v + \nu_{d,\alpha}(v))} \Phi_{p}^{\alpha} \frac{d^3p}{(2\pi)^3}.
\end{align*}
\]
3.7. THE PERMITTIVITY

\[
\bar{\beta}_{k,\omega}^\alpha = \int \frac{e_q v_\alpha'(q, v)}{i(\omega - k \cdot v + iw_{d,\alpha}(v))} \Phi_\alpha^\alpha \left( \frac{d^3 p}{(2\pi)^3} \right),
\]  

(3.57)

Another important quantity to be evaluated is \( \int q' \delta f_{p',k,\omega}^{d,(ind)} \frac{d^3 p' dq'}{(2\pi)^3} \). With the use of Eq.(3.53) we obtain

\[
\int q' \delta f_{p',k,\omega}^{d,(ind)} \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) = \left[ \frac{\chi_{d,ch}^{\omega}}{\alpha_k \omega} \bar{\beta}_{k,\omega}^\alpha - \frac{i k \chi_{d,eq}^{\omega}}{4\pi} \right] \delta E_{k,\omega}
\]

\[+ \gamma_{k,\omega}(q') \delta f_{p',k,\omega}^{d,(0)}(q') \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) \].  

(3.58)

The fluctuating part of the Poisson equation will now lead to an expression of the electrostatic field fluctuations as a function of the natural dust fluctuations only. Consequently all other fluctuating quantities can now be expressed as functions of the dust discreteness. Using Eqs.(3.54,3.59) and \( \bar{\beta}_{k,\omega}^\alpha = \sum_{\alpha} \bar{\beta}_{k,\omega}^\alpha \), \( \bar{q}_{k,\omega} = \sum_{\alpha} \overline{q}_{k,\omega} \)

we get

\[i k \delta E_{k,\omega} = \frac{4\pi}{\alpha_k} \left( \sum_{\alpha} e_q v_\alpha'(q, v) + \int q' \delta f_{p',k,\omega}^{d,(0)}(q') \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) \right) \delta E_{k,\omega}
\]

\[= 4\pi \left( \sum_{\alpha} e_q v_\alpha'(q, v) + \int q' \delta f_{p',k,\omega}^{d,(0)}(q') \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) \right) \delta E_{k,\omega}
\]

\[+ \frac{i k \chi_{d,eq}^{\omega}}{4\pi} \sum_{\alpha} \overline{q}_{k,\omega}(q) \delta E_{k,\omega} + \frac{4\pi \chi_{d,ch}^{\omega} \bar{\beta}_{k,\omega}^\alpha}{\alpha_k} \delta E_{k,\omega} + \frac{4\pi \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q) \delta E_{k,\omega}}{\alpha_k} \delta E_{k,\omega}
\]

\[+ 4\pi \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q) \delta E_{k,\omega} + \frac{4\pi \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q) \delta E_{k,\omega}}{\alpha_k} \delta E_{k,\omega}
\]

\[+ \frac{4\pi \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q) \delta E_{k,\omega}}{\alpha_k} \delta E_{k,\omega} + \frac{4\pi \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q) \delta E_{k,\omega}}{\alpha_k} \delta E_{k,\omega}
\]

\[= -i k \sum_{\alpha} \overline{q}_{k,\omega}(q) + \chi_{d,eq}^{\omega} + \frac{4\pi \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q) \delta E_{k,\omega}}{\alpha_k} \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) \]

\[+ 4\pi \left\{ \frac{q' + \bar{q}_{k,\omega}(q') + \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q') + \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q') \delta f_{p',k,\omega}^{d,(0)}(q') \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) \right\}
\]

\[+ 4\pi \left\{ q' + \bar{q}_{k,\omega}(q') + \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q') + \chi_{d,ch}^{\omega} \bar{q}_{k,\omega}(q') \delta f_{p',k,\omega}^{d,(0)}(q') \left( \frac{d^3 p' dq'}{(2\pi)^3} \right) \right\}
\]
CHAPTER 3. THE KLIMONTOVICH DESCRIPTION OF COMPLEX PLASMAS

For the second term in brackets we have

\[ q_{k_{eff}}^f(q) = q + \tilde{q}_{k_{eq}}(q) + \chi_{d,ch}^{d,ch} \gamma_{k_{eq}}(q) + \chi_{d,ch}^{d,ch} (1 + \tilde{\beta}_{k_{eq}}(q_{eq})) \]

\[ = q + \tilde{q}_{k_{eq}}(q) + \chi_{d,ch}^{d,ch} \gamma_{k_{eq}}(q) \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right) \]

\[ = q + \tilde{q}_{k_{eq}}(q) + \chi_{d,ch}^{d,ch} \gamma_{k_{eq}}(q) \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right) \cdot \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right), \]

while for the first term in brackets

\[ \alpha_{k_{eq}} = 1 + \sum_o x_{k_{eq}}^o + \sum_o x_{d,ch}^{d,ch} \gamma_{k_{eq}}(q_{eq}) + \sum_o \frac{4\pi}{k} \chi_{d,ch}^{d,ch} \tilde{\beta}_{k_{eq}}(q_{eq}) - \frac{4\pi}{k} \chi_{d,ch}^{d,ch} \beta_{k_{eq}} \]

\[ = 1 + \sum_o x_{k_{eq}}^o + \gamma_{k_{eq}}(q_{eq}) + \gamma_{k_{eq}}(q_{eq}) + \frac{4\pi}{k} \chi_{d,ch}^{d,ch} \tilde{\beta}_{k_{eq}}(q_{eq}) \]

\[ = 1 + \sum_o x_{k_{eq}}^o + \gamma_{k_{eq}}(q_{eq}) + \gamma_{k_{eq}}(q_{eq}) + \frac{4\pi}{k} \chi_{d,ch}^{d,ch} \tilde{\beta}_{k_{eq}}(q_{eq}) \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right) \]

\[ = 1 + \sum_o x_{k_{eq}}^o + \gamma_{k_{eq}}(q_{eq}) + \gamma_{k_{eq}}(q_{eq}) + \frac{4\pi}{k} \chi_{d,ch}^{d,ch} \tilde{\beta}_{k_{eq}}(q_{eq}) \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right) \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right) \]

Therefore, overall we have

\[ \delta E_{k_{eq}} = \frac{4\pi}{k} \int q_{k_{eff}}^f(q) \delta f_{p'}^{(0)}(q) \frac{d^3p'dq}{(2\pi)^3} \]  

(3.59)

where the effective charge is defined by

\[ q_{k_{eff}}^f(q) = q + \tilde{q}_{k_{eq}}(q) + \frac{\tilde{S}_{k_{eq}}(q_{eq}, q) \chi_{d,ch}^{d,ch}}{1 - S'_{k_{eq}}(q_{eq}, q_{eq}) \chi_{d,ch}^{d,ch}} \left( 1 + \tilde{\beta}_{k_{eq}}(q_{eq}) \right) \]  

(3.60)
3.8. THE COLLISION INTEGRALS

and the permittivity is defined by

\[ \epsilon_{k,\omega} = 1 + \sum_{\alpha} \chi_{k,\omega}^\alpha \frac{q_{\text{eff}}}{q_{\text{eq}}} + \frac{4\pi}{k} \frac{S_{k,\omega}(q_{\text{eq}}) \chi_{k,\omega}^d}{1 - S_{k,\omega}(q_{\text{eq}}) \chi_{k,\omega}^d} \left( 1 + \tilde{\beta}_{k,\omega}(q_{\text{eq}}) \right). \]  

(3.61)

The latter quantities determine the interactions in dusty plasmas.

In order to simplify the cumbersome expressions for the collision integrals, we define a number of new responses. The effective permittivity, that determines the dynamic screening of the fields of colliding particles, is defined by

\[ \epsilon_{k,\omega}^{\text{eff}}(q) = \frac{q_{\text{eq}} \epsilon_{k,\omega}}{q_{\text{eff}}(q)}. \]  

(3.62)

The effective particle permittivity is defined through

\[ \epsilon_{k,\omega}^{\text{eff}(P)} = \epsilon_{k,\omega}^{\text{eff}} + \chi_{k,\omega}^d. \]

In terms of integral responses it will be given by

\[ \epsilon_{k,\omega}^{\text{eff}(P)} = \frac{q_{\text{eq}}}{q_{k,\omega}^{\text{eff}}(q_{\text{eq}})} \left( 1 + \sum_{\alpha} \chi_{k,\omega}^\alpha + \frac{4\pi}{k} \frac{S_{k,\omega}(q_{\text{eq}}) \chi_{k,\omega}^d}{1 - S_{k,\omega}(q_{\text{eq}}) \chi_{k,\omega}^d} \left( 1 + \tilde{\beta}_{k,\omega}(q_{\text{eq}}) \right) \right). \]  

(3.63)

The charging response \( \Lambda_{k,\omega} \), that is related to the charging process only, is defined by

\[ \Lambda_{k,\omega} = r\gamma_{k,\omega} \epsilon_{k,\omega}^{\text{eff}}(q_{\text{eq}}) + \frac{4\pi}{k} \beta_{k,\omega}. \]  

(3.64)

3.8 The collision integrals

We start with the presentation of the dust collision integral [Tsytovich and de Angelis, 1999]

\[ J_{p}^d(q) = -q \frac{\partial}{\partial p} \left( \frac{\partial}{\partial q} \langle \delta f_{p}^d(q) \rangle \right) - \frac{\partial}{\partial q} \left( \sum_{\alpha} \langle \delta I_{\alpha}(q) \delta f_{p}^d(q) \rangle \right), \]  

(3.65)

The first term describes the interaction of the electrostatic micro-field fluctuations with the fluctuating part of the dust distribution function and hence refers to the dust-dust and dust-plasma collisions in presence of dust-charge fluctuations, such a term is also present in multicomponent plasmas. The second term describes the effect of the charging process in dust kinetics and has no analogy in multicomponent kinetic models.

Expressions of all fluctuating quantities as functions of the natural correlator, use of the natural statistical correlator and Green’s function (but without substituting for it) will yield the explicit form

\[ J_{p}^d(q) = \frac{\partial}{\partial p} \left( \int D_{p}^d(q, q') \cdot \frac{\partial \Phi_{p}^d(q')}{\partial q'} dq' \right) + \frac{\partial}{\partial q} \left( F_{p}^d(q, q') \cdot \frac{\partial \Phi_{p}^d(q'}{\partial q'} dq' \right) \]

\[ + \frac{\partial}{\partial q} \left( \int q' F_{p}^d(q, q') \cdot \frac{\partial \Phi_{p}^d(q')}{\partial q'} dq' \right) - \frac{\partial}{\partial q} \left( \delta(1) \Phi_{p}^d(q) \right) + \frac{\partial}{\partial q} \left( q \int I_{\omega}(q, q') \cdot \frac{\partial \Phi_{p}^d(q')}{\partial q'} dq' \right). \]  

(3.66)
The first adder describes diffusion in momentum space with the diffusion tensor being an integral over the charge and the integrand given by the tensor

\[
D_{p,l,m}^d(q, q') = -\frac{2}{\pi} \Im \left\{ \int \frac{k_l k_m q' q''_e}{k^4 |\epsilon_{k,k'}(q_e)|^2} G(q, q', k \cdot v' - k \cdot v) \Phi_F^{d} \frac{d^3 p' d^3 k}{(2\pi)^3} \right\}.
\]

The structure of the diffusion tensor resembles strongly the diffusion tensor for multi-component plasmas with some major differences: (i) the diffusion tensor is integrated over the charge space in order to account for dust charge variability during the interaction, (ii) dust variability also leads to \(q''_e p q''_d\) for the charge instead of \(q''_e q''_d\) that would be present in absence of dust charge fluctuations, (iii) the usual \(\delta(k \cdot v - k \cdot v')\) function is substituted by \(G(q, q', k \cdot v - k \cdot v')\), this is to be expected since \(\delta(k \cdot v - k \cdot v')\) is Green’s function for free streaming particles in a \(\{r, p\}\) phase-space and \(G(q, q', k \cdot v - k \cdot v')\) is Green’s function for free streaming particles in an extended \(\{r, p, q\}\) phase-space, (iv) the dynamic screening of the fields of colliding particles is now represented by \(1/k^4 |\epsilon_{k,k'}(q_e)|^2\) instead of \(1/k^4 |\epsilon_{k,k'}|\).

The second adder stems from the ensemble average \(-q_{\pi q} \cdot \langle \delta E \delta f_p^d(0, q) \rangle\) and represents a drift process in momentum space. The relevant friction force is given by

\[
F_{p,l}^d(q) = \frac{q_{eq} q''_e}{2\pi^2} \Re \left\{ \int \frac{k_l}{k^4 |\epsilon_{k,k'}(q_e)|^2} d^3 k \right\}.
\]

It is the same friction force that is present in multi-component kinetic models. There, however the imaginary (dissipative) part of the permittivity enables the use of the Plemelj-Sokhotskiy formula for further simplifications and ultimately the more compact form of the Lennard-Balescu collision integral.

The third and fourth adders involve mixed second order partial derivatives over the momentum and the charge and they stem from the \(-q_{\pi} \cdot \langle \delta E \delta f_p^d(q) \rangle\) and \(-q_{\pi} \cdot \langle \delta I_{\alpha}(q) \rangle\) ensemble averages respectively. They appear due to the fact that in dust dust collisions both the momenta and the charges of the dust particles change. Notice that in a stochastic description this implies a correlation between the stochastic momentum term of the Langevin equation and the noise term of the charging equation. The friction forces are given by

\[
F_{p,l,m}^{d,q}(q, q') = \frac{1}{(2\pi)^2} \Re \left\{ \int \frac{k_l q_{eq}}{k^2 |\epsilon_{k,k'}(q_e)|^2} \left( \frac{4\pi q_{eq}}{k_l |\epsilon_{k,k'}(q_e)|} \beta_{k,k'}(q_e) + \gamma_{k,k'}(q_e) \right) \right\} G(q, q', k \cdot v' - k \cdot v) \Phi_F^{d} \frac{d^3 p' d^3 k}{(2\pi)^3},
\]

(3.69)
The fifth adder involves a first order derivative with respect to the charge and stems from the ensemble average $-\frac{\partial}{\partial q} \left( \sum_{\alpha} \langle \delta I_{\alpha}(q) \delta \Phi_{p}^{d}(q) \rangle \right)$. The physics behind this term will become more transparent if we regard the charge derivative term in the left hand side of the kinetic equation and apply the approximation of small deviations from the equilibrium charge. We will then obtain $\frac{\partial}{\partial q} \left[ \left( \sum_{\alpha} I_{\alpha}(q) + I_{ext} \right) \Phi_{p}^{d}(q) \right] = -\frac{\partial}{\partial q} \left[ \nu_{ch}(q - q_{eq}) \Phi_{p}^{d}(q) \right]$, which when combined with the present adder results in $+\frac{\partial}{\partial q} \left[ \nu_{ch}(q - q_{eq}) - \delta(I) \right] \Phi_{p}^{d}(q)$. It is now obvious that it describes collective corrections to the charging currents due to shadowing of plasma fluxes due to the presence of neighboring grains. It is given by

$$\delta(I) = \Re \left\{ \int \left( \gamma_{k,k'} v' + \frac{4\pi q_{eq}}{\nu_{ch}(q - q_{eq})} \beta_{k,k'} v' \right) G(q,q',k\cdot v' - k\cdot v) \Phi_{p}^{d}(q') \frac{d^{3}k'}{(2\pi)^{3}} \right\} .$$

(3.71)

The sixth adder involves a second order derivative in the charge and essentially describes a diffusion process in the charge space. The diffusion coefficient will be a scalar given by

$$I_{ch}(q,q') = \frac{1}{(2\pi)^{3}} \Re \left\{ \int \frac{4\pi q_{eq}}{\nu_{ch}(q - q_{eq})} \beta_{k,k'} v' + \gamma_{k,k'} v' \right\}^{2} G(q,q',k\cdot v' - k\cdot v) \Phi_{p}^{d}(q') \frac{d^{3}k'}{(2\pi)^{3}} \right\} .$$

(3.72)

Combining all the above, the kinetic equation of the regular part of the dust distribution function will be

$$\left\{ \frac{\partial}{\partial t} + v \frac{\partial}{\partial \nu} \right\} \Phi_{p}^{d}(q) = \frac{\partial}{\partial \nu} \left( \int D_{p}(q,q') \cdot \frac{\partial \Phi_{p}^{d}(q')}{\partial \nu} \, dq' \right) + \frac{\partial}{\partial \nu} \left( F_{p}^{d}(q) \Phi_{p}^{d}(q) \right) + \frac{\partial}{\partial \nu} \left( \int q' F_{p}^{d}(q,q') \cdot \frac{\partial \Phi_{p}^{d}(q')}{\partial \nu} \, dq' \right) + \frac{\partial}{\partial \nu} \left( \int \delta I_{ch}(q,q') \cdot \frac{\partial \Phi_{p}^{d}(q)}{\partial \nu} \, dq \right) .$$

(3.73)

We now proceed to the presentation of the plasma collision integral [Tsytovich and de Angelis, 1999]

$$J_{p}^{\alpha} = -\varepsilon_{\alpha} \frac{\partial}{\partial \nu} \cdot \langle \delta E \delta f_{p}^{\alpha} \rangle - \langle \delta v_{d,\alpha}(v) \delta f_{p}^{\alpha} \rangle$$

$$= -\varepsilon_{\alpha} \frac{\partial}{\partial \nu} \cdot \langle \delta E \delta f_{p}^{\alpha} \rangle - \int \nu_{\alpha}(q,v) \langle \delta f_{p}^{\alpha} \delta f_{p}^{d}(q) \rangle \frac{d^{3}p'}{d\nu} \frac{d^{3}q}{(2\pi)^{3}} .$$

(3.74)
The second ensemble average is new compared to multi-component theories and is obviously a consequence of dust acting as a sink of plasma particles. The explicit form of the collision integral is

$$J^\alpha_p = \frac{\partial}{\partial p} \cdot D^\alpha_p \cdot \frac{\partial \Phi^\alpha}{\partial p} + \frac{\partial}{\partial p} \cdot \left( F^\alpha_p \Phi^\alpha_p + \nu^\alpha_{d,\alpha}(v) \Phi^\alpha_p \right).$$  \hfill (3.75)

The first adder describes a diffusion process in momentum space with the diffusion tensor given by

$$D^{\alpha,d}_{p,i,m} = 2e_o q_e q(q_e) (2\pi)^3 \int \frac{k_i k_m}{k^4 \nu^\alpha_{d,\alpha}(q_e)} \left\{ \frac{1}{\pi} \left( \frac{\nu_{d,\alpha}(v)}{(k \cdot v - k \cdot v')^2 + \nu_{d,\alpha}^2(v)} \right) \right\} \Phi^\alpha \frac{d^3p' d^3k}{(2\pi)^6}.$$  \hfill (3.76)

Compared to the diffusion tensor of multi-component plasmas we notice: (i) due to the omission of plasma discreteness the tensor consists of one instead of three adders, since only collisions of plasma particles with dust are taken into account, (ii) the \( \delta \)-function \( \delta(k \cdot v - k \cdot v') \) which represents energy/momentum conservation in Coulomb collisions is substituted by the function \( \frac{1}{\pi} \left( \frac{\nu_{d,\alpha}(v)}{(k \cdot v - k \cdot v')^2 + \nu_{d,\alpha}^2(v)} \right) \) which expresses the inelasticity of collisions due to plasma absorption. It is a Lorentz line nascent \( \delta \)-function with the property

$$\lim_{\nu_{d,\alpha}(v) \rightarrow 0} \left( \frac{1}{\pi} \left( \frac{\nu_{d,\alpha}(v)}{(k \cdot v - k \cdot v')^2 + \nu_{d,\alpha}^2(v)} \right) \right) = \delta(k \cdot v - k \cdot v').$$

The width of the Lorentzian broadening is, as expected, determined by the dissipation, i.e. the plasma capture frequency \( \nu_{d,\alpha}(v) \), (iii) the dynamic screening of the fields of the colliding particles is now represented by the effective permittivity in the expression \( \frac{\nu_{d,\alpha}(v)}{k^4 \nu^\alpha_{d,\alpha}(q_e)} \), (iv) due to the characteristic dust velocities being much smaller than the plasma phase space velocities one can neglect the dust velocity \( v' \) in the expression \( k \cdot v - k \cdot v' \) and also evaluate the responses in their static limit.

The second adder describes a drift in momentum space with the friction force given by

$$F^{\alpha,d}_{p,i} = -e_o q_e v \sigma(q_e, v) \frac{2\pi^2}{k} \Re \left\{ \int \frac{k_i}{k^4 \nu^\alpha_{d,\alpha}(q_e)} \left( \frac{k \cdot v - k \cdot v' + \nu_{d,\alpha}(v)}{(k \cdot v - k \cdot v')^2 + \nu_{d,\alpha}^2(v)} \right) \left( \frac{\nu_{d,\alpha}(v)}{k^4 \nu^\alpha_{d,\alpha}(q_e)} \right) \right\} \frac{d^3p' d^3k}{(2\pi)^6}.$$  \hfill (3.77)

In contrast to multi-component plasmas the friction force is determined completely by the charging process. This is partly due to the omission of the plasma discreteness, since in its presence there would at least be one additional non-vanishing contribution from the ensemble average \(-e_o \frac{\partial}{\partial p} \delta E \delta F^{\nu}(0)\) which would be of the form

$$F^{\alpha}_{p,i} = \frac{e_o}{2\pi} \Re \left\{ \int \frac{k_i}{k^2 \epsilon_{k,k',v'}} \cdot d^3k \right\}.$$
3.9. The spectral densities of fluctuations

The third adder is the collective absorption frequency $\nu_{d,\alpha}^f(v)$. This term competes with the $-\nu_{d,\alpha}(v)\Phi_{p}^\alpha$ term of the kinetic equation, which describes absorption processes where individual particles participate and are significantly altered in the presence of many particles, e.g. shadowing of fluxes to the dust grains due to absorption/scattering from other neighboring grains. Notice that now the total capture frequency of the plasma kinetic equation will be $\nu_{d,\alpha}(v) - \nu_{d,\alpha}^f(v)$, with relevant implications for hydrodynamic descriptions.

Combining all the above, the kinetic equation of the regular part of the plasma distribution function will be

$$
\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} \Phi_{p}^\alpha = s_{\alpha} - \left( \nu_{d,\alpha}(v) - \nu_{d,\alpha}^f(v) \right) \Phi_{p}^\alpha + \frac{\partial}{\partial p} \cdot \hat{D}_{p}^{\alpha,d} \cdot \frac{\partial \Phi_{p}^\alpha}{\partial p} + \frac{\partial}{\partial p} \cdot \left( F_{p}^{\alpha,d} \Phi_{p}^\alpha \right). \tag{3.78}
$$

3.9 The spectral densities of fluctuations

For the spectral density of the electrostatic field fluctuations we use Eq. (3.59) and the natural statistical dust correlator to acquire

$$
S_{k,\omega}^{E} = \frac{16\pi^2 q_{eq}^2}{k^2 |e_{eff,k,\omega}(q_{eq})|^2} S_{k,\omega}^{d,(0)} \tag{3.79}
$$

Similarly for the spectral density of the electrostatic potential fluctuations we use $\delta\phi_{k,\omega} = \frac{16\pi^2 q_{eq}^2}{k^2 |e_{eff,k,\omega}(q_{eq})|^2}$ and get

$$
S_{k,\omega}^{\phi} = \frac{16\pi^2 q_{eq}^2}{k^4 |e_{eff,k,\omega}(q_{eq})|^2} S_{k,\omega}^{d,(0)} \tag{3.80}
$$

where we notice the spectral enhancement in low frequencies due to the presence of $Z^2_d$ in the nominator of both expressions.

For the spectral density of the plasma density fluctuations we use Eqs. (3.54,3.59) to express the fluctuating part of the plasma densities as a function of dust natural fluctuations only,

$$
\delta n_{k,\omega}^\alpha = \int N_{k,\omega}^\alpha(q) \delta f_{p,k,\omega}(q) \frac{d^3 p dq}{(2\pi)^3}, \tag{3.81}
$$

with $N_{k,\omega}^\alpha(q) = - \frac{q_{eq}}{e_{\alpha} e_{eff,k,\omega}(q)} M_{k,\omega}^\alpha(q) + \Lambda_{k,\omega}^\alpha(q)$ where

$$
M_{k,\omega}^\alpha(q) = \chi_{k,\omega}^\alpha + \frac{\tilde{\gamma}_{k,\omega}(q_{eq})}{q_{eq}} \chi_{k,\omega}^{d,eq} + \frac{4\pi^4}{k} \chi_{k,\omega}^{d,ch} \beta_{k,\omega} \tilde{\gamma}_{k,\omega}^\alpha(q_{eq}) \chi_{k,\omega}^{d,ch},
$$

$$
\Lambda_{k,\omega}^\alpha(q) = \frac{\tilde{\gamma}_{k,\omega}(q_{eq})}{e_{\alpha}} \chi_{k,\omega}^{d,ch}(q_{eq}) \gamma_{k,\omega}^{d,ch}. \tag{3.81}
$$
Hence, with the use of the natural statistical dust correlator we get

\[ S^d_{k,\omega} = |N^d_{k,\omega}(q_{eq})|^2 S^{d,(0)}_{k,\omega}. \]

We note the dependence of \( S^\alpha_{k,\omega} \) on \( Z_d^2 \), since \( N^\alpha_{k,\omega}(q) \propto Z_d \) and \( \Lambda^\alpha_{k,\omega}(q) \propto Z_d \), the latter through \( \tilde{\rho}_{k,\omega}(q_{eq}) \propto Z_\alpha \) and \( \gamma_{k,\omega}(q_{eq}) \propto Z_d \). This also demonstrates the significant enhancement of the plasma fluctuation spectra for low frequencies typical of dust dynamics.

Finally, for the spectral densities of dust density fluctuations we can integrate the equation for \( \delta f^{d,(ind)}_{p', k, \omega}(q) \) over the dust momentum / charge space or alternatively we can set \( F(q, q') = 1 \) in the general property of Eq.(3.42),

\[
\int \delta f^{d,(ind)}_{p', k, \omega}(q) \frac{d^3 p'}{dq} = -\frac{1}{4\pi q_{eq}} \frac{k}{\epsilon_{k,\omega}} \delta E_{k,\omega}
\]

\[
\delta n^{d,(ind)}_{k,\omega} = -\int \frac{\chi_{k,\omega}}{\epsilon_{k,\omega}} \delta f^{d,(0)}_{p', k, \omega}(q) \frac{d^3 p'}{dq}
\]

\[
\delta n^{d,(ind)}_{k,\omega} + \delta n^{d,(0)}_{k,\omega} = -\int \left( \frac{\chi_{k,\omega}}{\epsilon_{k,\omega}} - 1 \right) \delta f^{d,(0)}_{p', k, \omega}(q) \frac{d^3 p'}{dq}
\]

\[
\delta n^{d}_{k,\omega} = -\int N^d_{k,\omega}(q) \delta f^{d,(0)}_{p', k, \omega}(q) \frac{d^3 p'}{dq}
\]

with \( N^d_{k,\omega}(q) = -\left( \frac{\chi_{k,\omega}}{\epsilon_{k,\omega}} - 1 \right) \). The natural statistical correlator will now yield

\[ S^d_{k,\omega} = |N^d_{k,\omega}(q_{eq})|^2 S^{d,(0)}_{k,\omega}. \]

### 3.10 Kinetic phenomena unique in complex plasmas

Below we will focus on a number of novel results that stem from the kinetic model of complex plasmas and manifest the effects of dust charge and dust charge fluctuations.

*Stochastic heating as a consequence of dust charge variability:* One of the basic consequences of dust charge fluctuations is that the energy is not conserved in dust-dust interactions. This can lead to a growth of the mean local energy (*temperature*)
of the dust particles in time, coined as stochastic heating of dust particles. The energy source for this instability is ultimately the external source of plasma particles, necessary for dust grains to maintain their equilibrium charges [de Angelis et al., 2005].

The generic nature of Eq.(3.73) can be demonstrated by integrating over the dust charge space and defining the reduced distribution function \( \Phi^d_p = \int \Phi^d_p(q) \) [Tsytovich and de Angelis, 2001]. This will yield a kinetic equation for \( \Phi^d_p \) that is relatively simplified, since the last terms of Eq.(3.73) vanish being full derivatives with respect to the charge. In a zero order approximation with respect to \( \frac{\omega}{w_{\text{re}}} \ll 1 \) and a first order approximation with respect to \( \frac{\omega}{w_{\text{ch}}} \ll 1 \), together with the application of the Plemelj-Sokhotskyi formula and the properties of Green’s function \( G(q, q', \omega - k \cdot v) \) it can be demonstrated that the first two terms will express energy conservation and provide relaxation to equilibrium containing \( \delta(k \cdot v' - k \cdot v) \).

On the other hand the last remaining term will be non-conservative and will express the effect of dust charge fluctuations.

In that case the equation for the mean energy \( \epsilon = \int \frac{p^2}{2m} \Phi^d_p \frac{d^3p}{(2\pi)^3} \) will be

\[
\frac{d\epsilon}{dt} = \int \frac{p^2}{2m} I^{NC}_{dd}(p) \frac{d^3p}{(2\pi)^3}
\]

with

\[
I^{NC}_{dd}(p) = \frac{\partial}{\partial p} \cdot \int q F^{d,q}(q, q') \frac{\partial \Phi^d_p(q')}{\partial q'} dq dq',
\]

that can ultimately be approximated by \( \frac{d\epsilon}{dt} = \nu_\epsilon \epsilon \) with \( \nu_\epsilon > 0 \).

It is worth reporting that similar results have been reported from a Fokker-Planck approach to systems with variable charges [Ivlev et al., 2004] and from the stochastic theory of a harmonic oscillator with random frequency [Marmolino, 2011]. Stochastic heating has been recently implemented to address the acceleration of small grains in the interstellar medium, where it is particularly important for the processes of shattering and coagulation of dust [Ivlev et al., 2010]. In a totally different parameter regime, the presence of hyper-velocity particles in tokamak edge plasmas [Ratynskaia et al., 2008] can possibly be explained by the same mechanism [Marmolino et al., 2008].

Dust charge distributions: Integration of Eq.(3.73) over the dust momentum space together with the introduction of the reduced dust distribution function (normalized to unity) \( \Phi^d(r, q, t) = \int \Phi^d_p(r, q, t) \frac{d^3p}{(2\pi)^3} = n_d f_q(q) \) will yield an equation for the evolution of dust charge distribution. The first three terms of Eq.(3.73) vanish being full derivatives with respect to the momentum and in the stationary and
homogeneous case we have [Tsytovich and de Angelis, 2002]
\[
\frac{\partial}{\partial q} \left( \nu_{ch}(q - q_{eq}) - \delta(I) \right) f_d(q) + \int q' F^d_p(q, q') \frac{\partial \Phi_p^d(q')}{\partial p} \frac{d^3 p}{(2\pi)^3} dq' + \int q' I_{ch}(q, q') \frac{\partial \Phi_p^d(q')}{\partial q} dq' \frac{d^3 p}{(2\pi)^3} = 0.
\]

Approximate solutions, in the parameter regime of laboratory applications show that \(f_d(q)\) has a Lorentz shape, with the width coinciding with the width of dust charge fluctuations, i.e
\[
f_d(q) = \frac{1}{\pi} \frac{\sqrt{\langle (\Delta q)^2 \rangle}}{q_{eq}^2 + \langle (\Delta q)^2 \rangle}.
\]

Therefore, within the approximation of small deviations from the equilibrium dust charge, the above function will reduce to a \(\delta\)-function \(\delta(q - q_{eq})\), which brings out the approximation \(\Phi_p^d(r, q, t) = \Phi_p^d(r, t) \delta(q - q_{eq})\).

**Collective dust charge fluctuations** are related with the dust discreteness and can be found by the present theory by the ensemble average of the square of the first \(\Delta q\) moment of the fluctuating dust charge distribution function (normalized by the dust density) [Tsytovich and de Angelis, 2002],
\[
\langle (\Delta q)^2 \rangle = \frac{1}{n_d^4} \left\langle \int \Delta q \delta f^d(r, q, t) dq \int \Delta q' \delta f^d(r, q', t) dq' \right\rangle.
\]

With use of \(\delta f^d(r, q', t) = \int \delta f^d_p(r, q', t) \frac{d^3 p}{(2\pi)^3}\), Fourier transforms in space / time and decomposition in natural and induced parts one can find approximate relations for the collective dust charge fluctuations. It also be shown that they usually exceed the non-collective dust charge fluctuations related to plasma discreteness, a result that might be important for the system’s condensation to a strongly coupled state.

**Spectral densities as a diagnostic tool for sub-micron dust:** Due to the large number of elementary charges residing on dust \(Z_d \gg 1\), there is an important enhancement of the spectral densities of plasma and electrostatic field fluctuations in the low frequency regime \(Z_d^2 S^d_k(\omega)\). The spectral enhancement is roughly proportional to \(n_d a^{7/2}\) which implies a proportionality \(n_d a^{7/2}\) and therefore strong dependence on the dust density and radius. Similarly, the region of spectral enhancement is strongly depending on the radius of the dust grains. Hence, the measurement of the spectral densities and comparison with the theoretical results can provide rich information about the dust density and composition [Ratynskaia et al., 2007; Ratynskaia et al., 2010]. Such a diagnostic is particularly important for in situ detection of sub-micron dust particles that cannot be monitored by fast cameras.
Chapter 4

The Klimontovich Description of Partially Ionized Complex Plasmas

Complex plasmas are a thermodynamically open system: When embedded into plasma, the dust grains are fast charged to an equilibrium value with a time constant equal to the inverse of the charging frequency. In order to maintain that charge, they continuously absorb plasma particles/radiation, depleting the system. Therefore, plasma or radiation sources are needed to replenish the plasma density and the system exchanges particles/energy with the background medium [Tsytovich et al., 2008]. These sources can be either constant in the temporal/spatial scales of the fluctuations (e.g. radiation sources) or fluctuating (e.g. electron impact ionization of neutrals).

Laboratory complex plasmas are engineered in low temperature discharges, where in most cases the background gas is not fully ionized. In a full kinetic model of partially ionized complex plasmas, the system consists of four distinct species interacting with each other: ions, electrons, dust particles and neutral gas. For a self-consistent description, the Klimontovich equations of all species should be analyzed and closure should be provided by the Maxwell equations [Tsytovich et al., 2005].

4.1 The Bhatnagar-Gross-Krook collision integral

Before proceeding to the Klimontovich description of the system we present the Bhatnagar-Gross-Krook (BGK) collision integral [Bhatnagar et al., 1954]. This is a model collision integral that cannot be derived from first principles. However, it can be viewed as an approximation of the Boltzmann collision term for systems close to thermodynamic equilibrium [Liboff, 1990].

The Boltzmann equation describes the evolution of the smooth average part of the single particle distribution function of a rarefied gas within the assumptions, (i) three body and higher correlations are neglected, (ii) the collision time is much
CHAPTER 4. THE KLIJMONTOVITCH DESCRIPTION OF PARTIALLY IONIZED COMPLEX PLASMAS

larger than the duration of collisions, (iii) collisions are elastic and their dynamics are well-described by classical mechanics,

\[
\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + F \cdot \frac{\partial}{\partial p} \right\} f_1 = \int d\Omega \int d^3p_2 \sigma(\Omega, |v_1 - v_2|)|v_1 - v_2|(f_1^I f_2^I - f_1 f_2),
\]

(4.1)

with \( F \) denoting the external force acting on the particles, where \( f_1 = f(r, p_1, t) \) refers to the first colliding particle before the collision event, \( f_2 = f(r, p_2, t) \) refers to the second colliding particle before the collision event and the primes refer to the particles after the collision event with \( p_1 + p_2 = p_1' + p_2' \) and \( |p_1|^2 + |p_2|^2 = |p_1'|^2 + |p_2'|^2 \), whereas \( \sigma(\Omega, |v_1 - v_2|) \) is the differential cross-section for scattering in a solid angle \( \Omega \) in the center of mass frame. In a compact form the Boltzmann collision term can be rewritten as \( \dot{J}(f_1) = \int f' f_1^I d\mu_1 - \int f f_1' d\mu_1 \) with \( \mu_1 = \sigma(\Omega, |v - v_1|)|v - v_1| d\Omega d^3p_1 \).

Near equilibrium the system is close to a local Maxwellian state. As inferred by the H-theorem, the main effect of collisions is to cause a quick relaxation to equilibrium and therefore the primed components (referring to the after-collision interval) can be assumed as local Maxweilians, \( \int f' f_1^I d\mu_1 \simeq \int f^0 f_1^0 d\mu_1 \). Since local Maxweilians nulify the collision integral we also get \( \dot{J}(f^0) = 0 \Rightarrow \int f^0 f_1^0 d\mu_1 = \int f^0 f_1^0 d\mu_1 \). Moreover, due to the conservation of particles, momentum and kinetic energy in the collision \( f_1^0 \) and \( f_1 \) will have the first three moments equal, which leads us to the reasonable assumption \( \int f_1' d\mu_1 = \int f_1^0 d\mu_1 \). Overall, we have

\[
\dot{J}(f_1) = \int f' f_1^0 d\mu_1 - \int f f_1' d\mu_1
\]

\[
\simeq \int f^0 f_1^0 d\mu_1 - \int f f_1 d\mu_1
\]

\[
\simeq f^0 \int f_1^0 d\mu_1 - f \int f_1 d\mu_1
\]

\[
\simeq f^0 \int f_1^0 d\mu_1 - \int f^0 f_1 d\mu_1
\]

\[
\simeq f^0 \int f_1^0 d\mu_1 - f \int f_1 d\mu_1
\]

\[
\simeq f^0 \int f_1 d\mu_1 (f^0 - f)
\]

\[
\simeq \nu(v) (f^0 - f),
\]

where the collision frequency is defined by \( \nu(v) = \int f_1^0 d\mu_1 = \int f^0 \sigma(\Omega, |v - v_1|)|v - v_1| d\Omega d^3p_1 \). This is the form of the BGK collision integral, which we rewrite as

\[
\frac{\delta f}{\delta t} = -\nu(v) (f - f^0(v)).
\]

(4.2)

Its physical meaning becomes transparent by assuming a spatially homogeneous gas in absence of external forces and with an initial velocity distribution \( f(0) \). In
4.2. THE EFFECT OF NEUTRALS IN THE STRUCTURE OF THE KLIMONTOVICH EQUATIONS

In a classical statistical description, neutral atoms can be treated as strongly coupled subsystems consisting of electrons and a multiply charged ion bound by some model potential. In that sense, collisions with ions / electrons such as short range polarization scattering can be treated collectively, but not collisions of quantum-mechanical nature like ion-atom resonant charge transfer collisions and electron
impact ionization, which also happen to be the dominant collisional processes in typical discharges [Liebermann and Lichtenberg, 1994].

However, neutrals can also be treated as point particles without internal structure provided that the collisional frequencies are given by external models (e.g. pair particle approaches). Moreover, for relatively large pressures, one can safely assume that neutral-neutral, ion-neutral and dust-neutral collisions are frequent enough to keep the neutrals in thermodynamic equilibrium. To simplify things more and enable focusing on plasma / dust kinetics, neutrals can be treated as continuous fluids (omission of their discrete nature) following a Maxwellian distribution $\Phi_p$ (normalized to unity) with an average kinetic energy $T_n$. They are essentially the ambient medium not only providing dissipation and relaxation to equilibrium through collisions but also providing the source of plasma particles through electron impact ionization [Tsytovich et al., 2005].

Collisions of neutrals with electrons can be neglected, with their frequencies being typically two orders of magnitude less than those of ion-neutral collisions. Collisions with ions and dust grains, will be treated by implementing the BGK relaxation time approximation in the Klimontovich equations. In case of ions, the addition will be $-\nu_{n,i}(f_p^N - N_i \Phi_p)$, where $N_i = n_i + \delta n_i^{(\text{ind})}$ and it is assumed that the thermalization is towards the neutral distribution function due to $T_i \simeq T_n$ and $m_i \simeq m_n$, while $\nu_{n,i}$ is velocity independent and usually $\nu_{n,i} = n_i v_{\text{r},i} \sigma_{n,i}$ with $\sigma_{n,i}$ the velocity independent charge exchange collision cross-sections. In case of dust, the addition will be $-\nu_{n,d}(f_p^N - N_d \Phi_p^{d,eq})$, where $N_d = n_d + \delta n_d^{(0)} + \delta n_d^{(\text{ind})}$ and $\nu_{n,d}$ a velocity independent collision frequency usually described by the Epstein kinetic model due to the large radius of the grains compared to the size of the neutrals, with $\Phi_p^{d,eq}$ the normalized to unity distribution function making the dust collision integral zero. Notice that there is a degree of arbitrariness in the choice of $\Phi_p^{d,eq}$; there is a variety of collisional processes each leading the dust distribution to its equilibrium form with a different rate, for example, in case dust-neutral collisions are dominating, $\Phi_p^{d,eq}$ can be reasonably assumed to be the Maxwellian

$$
\Phi_p^{d,eq} = \left(\frac{m_p}{2\pi kT_{d,\text{eq}}}\right)^{3/2} \exp\left(-\frac{m_p v^2}{2kT_{d,\text{eq}}}\right) \text{ with } T_{d,\text{eq}} = \frac{n_d T_n + m_d T_d}{m_n + m_d}.
$$

Electron impact ionization of neutrals will be treated as an instantaneous process generating new ions assumed to have the same velocity distribution with their ‘parent’ neutrals $\Phi_p^M$ (normalized to unity). The related source term in the ions will have the form $s_{iz} = \nu^I n_i(r, t) \Phi_p^M$, where $\nu^I = \frac{1}{m_e (2\pi \sigma v)} \int \nu^I(v) f_p^M \frac{d^3 v}{(2\pi)^3}$. Since equal numbers of electrons and ions are generated by ionization we end up with $s_{iz} = \tilde{\nu}^I \Phi_p^M$, where $\tilde{\nu}^I = \int \nu^I(v) f_p^M \frac{d^3 v}{(2\pi)^3}$. It is obvious that the source of plasma particles is fluctuating following the fluctuations of the electron distribution function $f_p^e$.

Finally, neutrals alter the plasma capture cross-sections on dust $\sigma_n(q, v)$. This implies that the equilibrium dust charge, the charging frequency and the responses related to charging and absorption will be altered. This is not apparent from the structure of the Klimontovich equations and their decomposition though, since they...
4.3 THE KLIMONTOVICH EQUATIONS FOR THE DUST/PLASMA COMPONENTS

can be formulated with arbitrary cross-sections. Notice that the above assumptions regarding the description of neutrals complement the basic kinetic assumptions analyzed in the previous chapter.

4.3 The Klimontovich equations for the dust/plasma components

Due to (i) continuous absorption of ions on dust grains, (ii) generation of ions in electron impact ionization of neutrals and (iii) inelastic collisions with neutrals, the distribution function of the ion species is no longer constant in time as measured along the orbit of a hypothetical particle in phase space, \( \frac{D}{Dt} f_i(r, p, t) \neq 0 \). The Klimontovich equation should also contain sink and source terms due to these processes.

\[
\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + e_i E \cdot \frac{\partial}{\partial p} \right\} f_i^p = \left( \int v^t(v) f_e^p (\frac{d^3 p}{(2\pi)^3}) \phi^M_p - \nu_{n,i}(f_i^p - N_i \Phi^M_p) \right) - \left( \int \sigma_i(q, v) v f_i^d(q) \frac{dq^d(p')}{(2\pi)^3} \right) f_i^p. \quad (4.4)
\]

Since there are no sinks/sources for the dust particles, the Klimontovich equation for dust should have a continuity form in the phase-space. There are two main differences from the standard form, (i) The charge variability leading to a \( \{p, r, q\} \) augmentation of the Hamiltonian phase space, which adds a charge derivative term to the Liouville and subsequently the Klimontovich equation (ii) The presence of dust-neutral collisions: treated approximately via the BGK collision integral. In the un-magnetized case the Klimontovich equation for the dust species has the form, omitting the \( (r, t) \) dependence,

\[
\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + q E \cdot \frac{\partial}{\partial p} \right\} f_d^p(q) + \frac{\partial}{\partial q} \left[ (I_{ext} + \sum_\alpha I_\alpha(q)) f_d^p(q) \right] = -\nu_{n,d} \left( f_d^p(q) - N_d \Phi_d^{eq}(q) \right). \quad (4.5)
\]

The Klimontovich equation for the electron species, will essentially have the same form with the Klimontovich equation for the ion species, with the exception that electron-neutral collisions will be due to elastic polarization scattering. Due to the very different temporal and spatial scales of the electron dynamics compared to the dynamics of the massive ions/dust particles, in low frequencies, typical of dust dynamics, a full kinetic description can be avoided. Instead of the Klimontovich equation, it can be assumed that the electron distribution function has a dependence on the local potential of the form

\[
f_e^p(r, t) = \Phi_e^p(r, t) \left[ 1 + \frac{e \phi(r, t)}{T_e(r, t)} \right], \quad (4.6)
\]
where $\Phi_{\text{e}}^e(r,t)$ is the equilibrium distribution function at the local mean energy $T_e(r,t)$ (adiabatic electrons). Since there is no equation for $\Phi_{\text{e}}^e(r,t)$, its form has to be assumed a priori. In case it is assumed Maxwellian, it is obvious that $f_{\text{e}}^p(r,t)$ is just the linearized Maxwell-Boltzmann distribution.

Since we are describing the motion of charged particles in their own fields, self-consistent closure of the system will stem from the Poisson equation with plasma/dust charges treated as sources,

\[ \nabla \cdot E(r,t) = 4\pi \left( \sum_\alpha e_\alpha \int f_{\text{p}}^\alpha \frac{d^3p}{(2\pi)^3} + \int q' f_{\text{p}}^e(q') \frac{dq'd^3p'}{(2\pi)^3} \right). \] (4.7)

### 4.4 Decomposition in regular and fluctuating parts

The strategy is to decompose the Klimontovich equations for all species together with the Poisson equation, then express all fluctuating quantities as a function of the natural dust fluctuations (which can be considered as the only discreet component of the system in sufficiently low frequencies). This way one can derive expressions for the total permittivity and the effective charge, quantities that define interactions in complex plasmas. Afterwards, using the natural statistical correlator, the collision integrals can be computed in the Boltzmann kinetic equation of each component and also the spectral densities of fluctuations.

The equation for the fluctuating part of the dust distribution function will be

\[
\begin{align*}
&\left(\frac{\partial}{\partial t} + v' \cdot \frac{\partial}{\partial r}\right) \delta f_{\text{p}}^d(q) + \frac{\partial}{\partial q} \left[ \left( I_{\text{ext}} + \sum_\alpha \langle I_\alpha(q) \rangle \right) \delta f_{\text{p}}^d(q) \right] + \\
&\quad q \delta E(t,r) \cdot \frac{\partial}{\partial p'} \Phi_{\text{p}}^d(q) + \frac{\partial}{\partial q} \left[ \sum_\alpha \delta I_\alpha \Phi_{\text{p}}^d(q) \right]
\end{align*}
\]

Since dust discreteness is taken into account, the dust particle density fluctuations will have both natural and induced parts, i.e $\delta n_d = \delta n_d^{(0)} + \delta n_d^{(\text{ind})}$. We Fourier transform in space and time, with $-i\omega + i\vec{k} \cdot \vec{v}' + \nu_{n,d} = -i(\omega - \vec{k} \cdot \vec{v}' + \nu_{n,d})$ we have

\[
\begin{align*}
-I(\omega - \vec{k} \cdot \vec{v}' + \nu_{n,d}) \delta f_{\text{p},k,\omega}^d(q) + &\frac{\partial}{\partial q} \left[ \left( I_{\text{ext}} + \sum_\alpha \langle I_\alpha \rangle \right) \delta f_{\text{p},k,\omega}^d(q) \right] = \\
-q \delta E_{k,\omega} \cdot \frac{\partial}{\partial p'} \Phi_{\text{p}}^d(q) - &\frac{\partial}{\partial q} \left[ \sum_\alpha \delta I_{\text{k},\omega} \Phi_{\text{p}}^d(q) \right] + \nu_{n,d} \left( \delta n_{\text{k},\omega}^{d,(0)} + \delta n_{\text{k},\omega}^{d,(\text{ind})} \right) \Phi_{\text{p}}^d(q).
\end{align*}
\]

We know that $\delta n_{\text{k},\omega}^{d,(\text{ind})} = \int \delta f_{\text{p},k,\omega}^d(q) \frac{dq'd^3p'}{(2\pi)^3}$, thus the BGK description of collisions clearly introduces a feedback in the equation. In order to express $\delta n_{\text{k},\omega}^{d,(\text{ind})}$, as a function of the processes that induce it (electric field, charging plasma current,
natural fluctuations) we need to integrate the equation over the charge and momentum space. Furthermore, since we take into account the inhomogeneous part of the equation, we have \( \int \int \delta f_{p',k,\omega}(q) \frac{d^3 p'}{(2\pi)^3} = \delta n_{k,\omega}^{d,(0)} \) only. When integrating over the charge variable, terms that are full derivatives of the charge will vanish, because the distribution function of dust in any physical complex plasma system must tend to zero at \( q \to \pm \infty \) faster than any powers of the charge,

\[
\int \delta f_{p',k,\omega}(q) dq = \int \frac{q}{i(\omega - k \cdot \mathbf{v}' + \nu_{n,d})} \delta E_{k,\omega} \cdot \frac{\partial}{\partial p'} \Phi_p^d(q) dq
\]

\[
- \nu_{n,d} \left( \delta n_{k,\omega}^{d,(0)} + \delta n_{k,\omega}^{d,(ind)} \right) \int \frac{\Phi_{p',eq}^d(q)}{i(\omega - k \cdot \mathbf{v}' + \nu_{n,d})} dq.
\]

Since we are examining longitudinal fields \( \delta E_{k,\omega} = \hat{k} \delta E_{k,\omega} = \frac{1}{k} \delta E_{k,\omega} \). We also use the narrowness of the dust distribution function around the equilibrium charge \( \Phi_p^d(q) = \Phi_p^d(0) \delta(q - q_{eq}) \) in the evaluation of the charge integrals,

\[
\int \int \delta f_{p',k,\omega}(q) \frac{d^3 p'}{(2\pi)^3} = - \nu_{eq} \delta E_{k,\omega} \int \frac{\hat{k} \cdot \frac{\partial}{\partial p'} \Phi_p^d(q_{eq})}{2(2\pi)^3} \frac{d^3 p'}{\omega - k \cdot \mathbf{v}' + \nu_{n,d}}
\]

\[
+ \nu_{n,d} \left( \delta n_{k,\omega}^{d,(0)} + \delta n_{k,\omega}^{d,(ind)} \right) \int \frac{\Phi_{p',eq}^d(q_{eq})}{\omega - k \cdot \mathbf{v}' + \nu_{n,d}} \frac{d^3 p'}{(2\pi)^3}.
\]

We define the responses \( \chi_{k,\omega}^{d,eq}, d_{k,\omega}^{eq} \) and \( a_{k,\omega}^{nd} \),

\[
\chi_{k,\omega}^{d,eq} = \frac{4\pi q_{eq}^2}{k^2} \int \frac{1}{\omega - k \cdot \mathbf{v}' + \nu_{n,d}} \left( \hat{k} \cdot \frac{\partial \Phi_p^d(q)}{\partial p'} \right) \frac{d^3 p'}{(2\pi)^3},
\]

(4.8)

\[
d_{k,\omega}^{eq} = \int \frac{\Phi_{p',eq}^d(q_{eq})}{\omega - k \cdot \mathbf{v}' + \nu_{n,d}} \frac{d^3 p'}{(2\pi)^3},
\]

(4.9)

\[
a_{k,\omega}^{nd} = \frac{1}{1 - \nu_{n,d} d_{k,\omega}^{eq}},
\]

(4.10)

and end up with

\[
\delta n_{k,\omega}^{d,(ind)} = - \frac{1}{4\pi q_{eq}} \chi_{k,\omega}^{d,eq} \delta E_{k,\omega} + \nu_{n,d} d_{k,\omega}^{eq} \left( \delta n_{k,\omega}^{d,(0)} + \delta n_{k,\omega}^{d,(ind)} \right)
\]

\[
\left( 1 - \nu_{n,d} d_{k,\omega}^{eq} \right) \delta n_{k,\omega}^{d,(ind)} = - \frac{1}{4\pi q_{eq}} \chi_{k,\omega}^{d,eq} \delta E_{k,\omega} + \nu_{n,d} d_{k,\omega}^{eq} \delta n_{k,\omega}^{d,(0)}
\]

\[
\delta n_{k,\omega}^{d,(ind)} = - \frac{1}{4\pi q_{eq}} \delta_{k,\omega}^{nd} \chi_{k,\omega}^{d,eq} \delta E_{k,\omega} + \nu_{n,d} d_{k,\omega}^{eq} \delta_{k,\omega}^{nd} \delta n_{k,\omega}^{d,(0)}.
\]

(4.11)
Returning to the equation for the dust component and substituting for the induced dust density fluctuations, we now have

$$-i \left( \omega - k \cdot v' + \nu_{n,d} \right) \delta f_{p',k,\omega}^d(q) + \frac{\partial}{\partial q} \left( I_{ext} + \sum_{\alpha} I_\alpha(q) \right) \delta f_{p',k,\omega}^d(q) = -R_{p',k,\omega}(q),$$

where the inhomogeneous term is given by

$$R_{p',k,\omega}(q) = q \delta E_{k,\omega} \cdot \frac{\partial}{\partial p} \Phi_{p'}^d(q) + \frac{\partial}{\partial q} \left[ \sum_{\alpha} \nu_{n,d} a_{k,\omega}^{nd} \delta f_{p',k,\omega}^d(q) \right]$$

$$+ \nu_{n,d} \left( \frac{k}{4 \pi q_{eq}} \delta E_{k,\omega} - \nu_{n,d} a_{k,\omega}^{nd} \delta f_{p',k,\omega}^d(q) \right)$$

$$+ \nu_{n,d} a_{k,\omega}^{nd} \left( \frac{k}{4 \pi q_{eq}}, \omega \delta E_{k,\omega} + i \delta n_{k,\omega}^{d(0)} \frac{4 \pi q_{eq}}{k} \right) \Phi_{p'}^d(q),$$

due to $$-\nu_{n,d} a_{k,\omega}^{nd} \delta f_{p',k,\omega}^d + i = -\nu_{n,d} a_{k,\omega}^{nd} \delta f_{p',k,\omega}^d + i = \frac{-\nu_{n,d} a_{k,\omega}^{nd} + i + \nu_{n,d} a_{k,\omega}^{nd}}{1 - \nu_{n,d} a_{k,\omega}^{nd}} = ia_{k,\omega}^{nd}$$. The homogeneous equation has the form

$$-i \left( \omega - k \cdot v' + \nu_{n,d} \right) \delta f_{p',k,\omega}^d(q) + \frac{\partial}{\partial q} \left( I_{ext} + \sum_{\alpha} I_\alpha(q) \right) \delta f_{p',k,\omega}^d(q) = 0.$$
same Taylor expansion in the homogeneous equation, will result to

\[ \int (\omega - k \cdot v' + i\nu_{n,d}) \delta f_{p',k,\omega}^d(\Delta q) + \frac{\partial}{\partial \Delta q} (\nu_{ch} \Delta q \delta f_{p',k,\omega}^d(\Delta q)) = 0. \]

The only difference with the case of fully ionized complex plasmas is the presence of neutral-dust collision frequency, since it is part of a constant factor in the charge differential equation, all previous results can be extended to our case with the substitution \( \omega - k \cdot v' \rightarrow \omega - k \cdot v' + i\nu_{n,d} \). The solution of the homogeneous equation will be given by Eq.(3.21), the Green’s function of the problem will be given by Eq.(3.27) and the properties of integrals of the induced dust distribution over the charge described by Eqs.(3.37,3.40). Knowledge of the Green’s function (delta response) of the problem, implies that the solution of the inhomogeneous equation with a source term \( R_{p',k,\omega}(q) \) will be \( \delta f_{p',k,\omega}^{d,(ind)}(q) = \int G(q,q')R_{p',k,\omega}(q')dq' \). Let \( F(q) \) be an arbitrary differentiable weighting function of the charge, let us wish to compute the integral of the form \( \int F(q)\delta f_{p',k,\omega}^{d,(ind)}(q) dq \). It is obvious that it should be evaluated for charges close to the equilibrium charge. In case the source term is not a full derivative of the charge, evaluation at \( q = q_{eq} \) would suffice, the Green’s function would then reduce to \( \frac{1}{i(\omega - k \cdot v' + i\nu_{n,d})} \) (property of Eq.(3.37)).

In case the source term is a full derivative of the charge, evaluation at \( q = q_{eq} \) gives zero contribution and first order deviations from equilibrium should also be taken into account, the Green’s function would then reduce to \( \frac{1}{i(\omega - k\cdot v' + i\nu_{n,d} + i\nu_{d})} \) (property of Eq.(3.40)). From Eq.(4.13) we notice that the actual source term can be decomposed in both terms,

\[
\int F(q)\delta f_{p',k,\omega}^{d,(ind)}(q) dq = \int F(q)G(q,q')R_{p',k,\omega}(q') dq dq' \\
= \int F(q)G(q,q')R_{1,p',k,\omega}(q') dq dq' \\
+ \int F(q)G(q,q')R_{2,p',k,\omega}(q') dq dq' \\
= \frac{\int F(q_{eq})}{i(\omega - k \cdot v' + i\nu_{n,d})} \int R_{1,p',k,\omega}(q') dq' \\
- \frac{\partial F(q_{eq})}{\partial q} \Bigg|_{q=q_{eq}} \int R_{2,p',k,\omega}(q') dq' \\
= \frac{\int F(q_{eq})}{i(\omega - k \cdot v' + i\nu_{n,d})} \left( \delta E_{k,\omega} \int \frac{q'}{k} k \cdot \frac{\partial}{\partial p} \Phi_{p'}^{eq}(q') dq' \right) \\
+ \nu_{n,d} a_{n,\omega}^{ind} \left( \frac{k}{i\nu_{eq}} \delta E_{k,\omega} + \nu_{d} n_{\omega}^{eq} \right) \int \Phi_{p'}^{eq}(q') dq' \\
- \frac{\partial F(q_{eq})}{\partial q} \Bigg|_{q=q_{eq}} \int \sum_{n} \delta I_{k,\omega}^{n}(q') \Phi_{p'}^{eq}(q') dq'.
\]

(4.14)
Therefore, they have both fluctuating and average parts. We start from the distribution functions of the electrons and the dust particles respectively. Finally, the kinetic equation for the average part of the dust distribution function and by integrating the above relation over the momentum space, together with the definitions of the response $\chi_{k,\omega}$:

$$
\chi_{k,\omega}^{d,ch} = \int \frac{1}{\omega - k \cdot v' + \nu_{n,d} + \nu_{n,d}} \Phi_p^{d'} \frac{d^3p'}{(2\pi)^3},
$$

we have

$$
\int F(q)\delta f_{p',k,\omega}(q) \frac{d^3p'dq}{(2\pi)^3} = F(q_{eq}) \int \frac{k - \frac{\partial}{\partial p'} \Phi_p^{d'}}{(\omega - k \cdot v' + \nu_{n,d})} \frac{d^3p'}{(2\pi)^3} + F(q_{eq}) \nu_{n,d} \delta f_{p,eq}^{d}.
$$

An additional property can be derived by using the narrowness of the average part of the ionization frequency and the absorption frequency depend

$$
\int \Phi_{p'}(q) \frac{d^3p'}{v'} (2\pi)^3 = I_{ext} + \sum_{\alpha} \nu_{\alpha} \left( \sum \delta I_{\alpha} \frac{d}{dq} \Phi_{p'}^{d,\alpha}(q) \right) - q \left( \frac{\delta E \cdot \delta f_{p'}(q)}{\partial p'} - \frac{\partial}{\partial q} \sum_{\alpha} \delta I_{\alpha} f_{p'}^{d,\alpha}(q) \right).
$$

Finally, the kinetic equation for the average part of the dust distribution function will be

$$
(\frac{\partial}{\partial t} + v' \cdot \frac{\partial}{\partial r}) \Phi_{p'}^{d'}(q) = - \frac{\partial}{\partial q} \left( I_{ext} + \sum_{\alpha} \nu_{\alpha} \left( \sum \delta I_{\alpha} \frac{d}{dq} \Phi_{p'}^{d,\alpha}(q) \right) - q \left( \frac{\delta E \cdot \delta f_{p'}(q)}{\partial p'} - \frac{\partial}{\partial q} \sum_{\alpha} \delta I_{\alpha} f_{p'}^{d,\alpha}(q) \right) \right).
$$

We notice that the ionization frequency and the absorption frequency depend on the distribution functions of the electrons and the dust particles respectively. Therefore, they have both fluctuating and average parts. We start from $\nu_{\alpha}$: From the adiabatic assumption for the electron species, after decomposition we acquire,

$$
\Phi_{p'}^{e} + \delta f_{p'}^{e} = \Phi_{p'}^{e} + \frac{e \delta \phi}{T_e} \Phi_{p'}^{e} \Rightarrow \delta f_{p'}^{e} \Phi_{p'}^{e} = \frac{e \delta \phi}{T_e} \Phi_{p'}^{e} \Rightarrow \delta f_{p',k,\omega}^{e} = \frac{e}{T_e} \Phi_{p'}^{e} \delta E_{k,\omega}.
$$
4.4. DECOMPOSITION IN REGULAR AND FLUCTUATING PARTS

This relation after integration over the momentum space will result in a relation for the electron density fluctuations,

\[ \int \delta f_{e,p,k,\omega} \frac{d^3p}{(2\pi)^3} = \frac{e^2}{T_e k} \int \Phi_{p}^{e} \frac{d^3p}{(2\pi)^3} \delta E_{k,\omega} \Rightarrow \delta n_{k,\omega}^{e} = \frac{e\nu_e}{T_e k} \delta E_{k,\omega}. \]

We set \( n_e \nu_e = \int \nu^{e}(v) \Phi_{p}^{e} \frac{d^3p}{(2\pi)^3} \) for the average part, while we acquire

\[ \delta \nu^{e} = \int \nu^{e}(v) \delta f_{p}^{e} \frac{e^2}{T_e k} \left( \int \nu^{e}(v) \Phi_{p}^{e} \frac{d^3p}{(2\pi)^3} \right) \delta E_{k,\omega} \Rightarrow \delta \nu_{k,\omega}^{e} = \frac{e\nu_e}{T_e k} \delta E_{k,\omega}. \]

For the absorption frequency we directly set

\[ \nu_{d,i}(v) = \frac{\sigma_{\nu}(q,v)\nu_{d,i}}{(2\pi)^3}, \]

\[ \delta \nu_{d,i}^{d,i} = \int \sigma_{\nu}(q,v) \nu \delta f_{p'k,\omega}^{d,i} \frac{d^3p'^{d,i}}{(2\pi)^3}. \]

The equation for the fluctuating part of the ion distribution function will be

\[ \left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial v} \right) \delta f_{p,k,\omega}^{i} + e\delta E \cdot \frac{\partial}{\partial p} \Phi_{p} = \nu_{e,i} \delta f_{p,k,\omega}^{M} - \nu_{e,i} \delta n_{k,\omega}^{M} - \delta \nu_{d,i}^{d,i} \nu_{e,i} \delta f_{p,k,\omega}^{d,i}. \]

We Fourier transform in time and space, we use longitudinal fields and gather all \( \delta f_{p,k,\omega}^{i} \) terms together,

\[ -i (\omega - k \cdot v + i(\nu_{e,i} + \nu_{d,i})) \delta f_{p,k,\omega}^{i} + \frac{e\delta E_{k,\omega}}{k} \frac{\partial}{\partial p} \Phi_{p} = \delta \nu_{d,i}^{d,i} \Phi_{p}^{M} + \nu_{e,i} \delta n_{k,\omega}^{M} \Phi_{p}^{M} - \delta \nu_{d,i}^{d,i} \Phi_{p}^{d,i} \frac{d^3p^{d,i}}{(2\pi)^3}. \]

We rearrange the terms, substitute for \( \delta \nu_{d,i}^{d,i}, \delta \nu_{d,i}^{d,i} \) and solve for the induced fluctuations

\[ \delta f_{p,k,\omega}^{i} = -\frac{1}{i (\omega - k \cdot v + i(\nu_{e,i} + \nu_{d,i}))} \left[ -\frac{e\delta E_{k,\omega}}{k} \frac{\partial}{\partial p} \Phi_{p}^{i} + \frac{i\nu_e}{T_e k} \delta E_{k,\omega} \Phi_{p}^{M} \right. \]

\[ + \left. \nu_{e,i} \delta n_{k,\omega}^{M} \right] \Phi_{p}^{M} - \int \sigma_{\nu}(q,v) \nu_{d,i} \delta f_{p'k,\omega}^{d,i} \frac{d^3p'^{d,i}}{(2\pi)^3}. \]

Since ions are treated as continuous Vlasov fluids in the phase space, there will be no natural ion density fluctuations, \( \delta n_{k,\omega}^{i} = \delta \nu_{d,i}^{i,ind} \). We notice again that treatment of collisions via the BGK collision integral creates a feedback in the equation. We integrate all over the momentum space, to find a solution for the ion density fluctuations \( \delta n_{k,\omega}^{i} = \int \delta f_{p,k,\omega}^{i} \frac{d^3p}{(2\pi)^3} \). We also use the definitions of the responses \( \chi_{k,\omega}^{i}, G_{k,\omega} \) and \( \delta \overline{n}_{k,\omega}(q) \),

\[ \chi_{k,\omega}^{i} = \frac{4\pi e^2}{k^2} \int \frac{1}{\omega - k \cdot v + \nu_{d,i}(v) + \nu_{e,i}} \left( k \cdot \frac{\partial \Phi_{p}}{\partial p} \right) \frac{d^3p}{(2\pi)^3}, \quad (4.19) \]
\[ G_{k,\omega} = \int \frac{\Phi^M_p}{i(\omega - k \cdot v + w_{d,i}(v) + w_{n,i})} \frac{d^3p}{(2\pi)^3}, \quad (4.20) \]

\[ \tilde{q}_{k,\omega}(q) = \int \frac{e\nu_i(q,v)\Phi^d_p}{i(\omega - k \cdot v + w_{d,i}(v) + w_{n,i})} \frac{d^3p}{(2\pi)^3}, \quad (4.21) \]

and acquire

\[ \delta n^i_{k,\omega} = \frac{e\delta E_{k,\omega}}{ik} \int \frac{1}{\omega - k \cdot v + i(\nu_{n,i} + \nu_{d,i})} \frac{k \cdot \partial}{\partial p} \Phi^M_p \frac{d^3p}{(2\pi)^3} \]
\[ - \frac{\nu_{n,i} \nu_e}{T_{ke}} \delta E_{k,\omega} \int \frac{1}{i(\omega - k \cdot v + i(\nu_{n,i} + \nu_{d,i}))} \frac{d^3p}{(2\pi)^3} \]
\[ - \frac{\nu_{n,i} \delta n^i_{k,\omega}}{i} \int \frac{1}{\omega - k \cdot v + i(\nu_{n,i} + \nu_{d,i})} \frac{d^3p}{(2\pi)^3} \]
\[ + \int \left( \int \frac{\sigma_i(q,v)\Phi^d_p}{i(\omega - k \cdot v + i(\nu_{n,i} + \nu_{d,i}))} \frac{d^3p}{(2\pi)^3} \right) \delta f^d_{p',k,\omega}(q) \frac{dq d^3p'}{(2\pi)^3} \]
\[ = - \frac{i k \chi^i_{k,\omega}}{4 \pi e} \delta E_{k,\omega} - \frac{\nu_{n,i} \nu_e}{T_{ke}} \delta E_{k,\omega} \frac{d^3p}{(2\pi)^3} \]
\[ - \frac{1}{1 + \nu_{n,i} G_{k,\omega}} \frac{1}{1 + \nu_{n,i} G_{k,\omega}} \frac{d^3p}{(2\pi)^3} \]
\[ + \int \left( \int \frac{\delta n^i_{k,\omega}(q)}{e} \delta f^d_{p',k,\omega}(q) \frac{dq d^3p'}{(2\pi)^3} \right) \quad (4.22) \]

Overall, we substitute back in \( \delta f^i_{p,k,\omega} \), we also set for our convenience \( D_{k,\omega}(v) = \omega - k \cdot v + w_{n,i} + w_{d,i}(v) \),

\[ \delta f^i_{p,k,\omega} = \frac{1}{i D_{k,\omega}(v)} \frac{e \delta E_{k,\omega}}{k} \frac{1}{k} \frac{\partial \Phi^d_p}{\partial p} - \frac{\nu_{n,i} \nu_e}{k T_e} \frac{\Phi^M_p}{i D_{k,\omega}(v)} \]
\[ + \frac{\nu_{n,i}}{1 + \nu_{n,i} G_{k,\omega}} \frac{d^3p}{(2\pi)^3} \frac{\Phi^M_p}{i D_{k,\omega}(v)} \]
\[ + \frac{\nu_{n,i}}{1 + \nu_{n,i} G_{k,\omega}} \frac{\nu_e}{T_{ke}} \frac{\Phi^M_p}{i D_{k,\omega}(v)} \]
\[ - \frac{\nu_{n,i}}{1 + \nu_{n,i} G_{k,\omega}} \int \frac{\delta n^i_{k,\omega}(q)}{e} \delta f^d_{p',k,\omega}(q) \frac{dq d^3p'}{(2\pi)^3} \frac{\Phi^M_p}{i D_{k,\omega}(v)} \]
\[ + \int \frac{\sigma_i(q,v)\Phi^d_p}{i D_{k,\omega}(v)} \delta f^d_{p',k,\omega}(q) \frac{dq d^3p'}{(2\pi)^3} \quad (4.23) \]

Finally, the kinetic equation for the average part of the ion distribution function
4.5. THE PERMITTIVITY

The relation for the plasma current flowing to the dust particles is

\[ \sum_\alpha I_\alpha(q) = \int e_\alpha v_F(q,v) \frac{d^3p}{(2\pi)^3}. \]

Therefore, the fluctuating plasma current will be given by

\[ \sum_\alpha \delta I_\alpha(q) = \sum_\alpha \int e_\alpha v_\sigma(q,v) \delta f_p^\alpha \frac{d^3p}{(2\pi)^3}. \]

We will express \( \sum_\alpha \delta I_\alpha(q) \) as a function of the electric field fluctuations and the natural fluctuations using Eq.\((4.18,4.23)\) and the property described by Eq.\((4.16)\),

\[ \sum_\alpha \delta I_\alpha(q) = - \int e v_\sigma(q,v) \frac{d^3p}{(2\pi)^3} + \int e v_\sigma(q,v) \frac{d^3p}{(2\pi)^3}. \]
CHAPTER 4. THE KLIMONTOVICH DESCRIPTION OF PARTIALLY IONIZED COMPLEX PLASMAS

It is obvious that the final expression consists of seven terms, we shall evaluate each term separately,

\[
A_1 = - \int e v \sigma e(q, v) \delta f_e d^3 p = - \int e v \sigma e(q, v) \frac{e^2 \Phi e}{k T_e} d^3 p \frac{d^3 p}{(2\pi)^3} = - \frac{e^2 e}{k T_e} \int e v \sigma e(q, v) \Phi e \frac{d^3 p}{(2\pi)^3} \delta E_{k,\omega} = S_{II}^{I}(q) \delta E_{k,\omega},
\]

where we used the definition of the response \( S_{II}^{I}(q) \),

\[
A_2 = \int e v \sigma i(q, v) e \delta E_{k,\omega} \frac{k \cdot \partial \Phi i}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} = \left[ e^2 \int e v \sigma i(q, v) \frac{k \partial \Phi i}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} \right] \delta E_{k,\omega} = S_{I}^{I}(q) \delta E_{k,\omega},
\]

where we used the definition of the response \( S_{I}^{I}(q) \),

\[
A_3 = - \int e v \sigma i(q, v) \frac{e \nu n i}{k T_e} \delta E_{k,\omega} \frac{\Phi p}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} = - \frac{e \nu n i}{k T_e} \left[ \int e v \sigma i(q, v) \frac{\Phi p}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} \right] \delta E_{k,\omega} = \left[ - \frac{e \nu n i}{k T_e} \lambda_{k,\omega}(q) \right] \delta E_{k,\omega} = S_{III}^{I}(q) \delta E_{k,\omega},
\]

where we used

\[
\lambda_{k,\omega}(q) = \int \frac{e v \sigma i(q, v)}{s (\omega - k \cdot v + iD_{k,\omega}(v))} \frac{\Phi p}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} \tag{4.27}
\]

and the definition of the response \( S_{III}^{I}(q) \),

\[
A_4 = \int e v \sigma i(q, v) \frac{\nu n i}{1 + \nu n i G_{k,\omega}} \frac{i k \chi_{k,\omega}^{I}}{4 \pi e} \frac{\Phi p}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} = \frac{i k \chi_{k,\omega}^{I}}{4 \pi e} \left[ \frac{\nu n i}{1 + \nu n i G_{k,\omega}} \int \frac{e v \sigma i(q, v) \Phi p}{k D_{k,\omega}(v)} \frac{d^3 p}{(2\pi)^3} \right] \delta E_{k,\omega} = \left[ \frac{\nu n i}{1 + \nu n i G_{k,\omega}} \frac{i k \chi_{k,\omega}^{I}}{4 \pi e} \right] \lambda_{k,\omega}^{I}(q) \delta E_{k,\omega} = \Delta S_{k,\omega}^{I} \lambda_{k,\omega}^{I}(q) \delta E_{k,\omega},
\]
where we used the definition of the response $\Delta S_{k,\omega}^I$.

\[
A_5 = \int ev\sigma_i(q, v) \frac{\nu_{n,i}}{1 + \nu_{n,i}G_{k,\omega}} \frac{\nu_{n,i}eG_{k,\omega}}{T_c k} \Phi_p^M \delta E_{k,\omega} \frac{d^3p}{(2\pi)^3} \\
= \frac{\nu_{n,i}eG_{k,\omega}}{T_c k} \left[ \int ev\sigma_i(q, v) \Phi_p^M \frac{d^3p}{(2\pi)^3} \right] \frac{d^3p}{(2\pi)^3} \delta E_{k,\omega} \\
= \frac{\nu_{n,i}eG_{k,\omega}}{1 + \nu_{n,i}G_{k,\omega}} \lambda^I_{k,\omega}(q) \delta E_{k,\omega} = \Delta S_{k,\omega}^I \lambda^I_{k,\omega}(q) \delta E_{k,\omega},
\]

where we used the definition of the response $\Delta S_{k,\omega}^I$.

\[
A_6 = -\int ev\sigma_i(q, v) \frac{\nu_{n,i}}{1 + \nu_{n,i}G_{k,\omega}} \int \tilde{q}_{k,\omega}(q') \frac{d^3p}{(2\pi)^3} \Phi_p^M \frac{d^3p'}{(2\pi)^3} \\
= -\frac{\nu_{n,i}eG_{k,\omega}}{1 + \nu_{n,i}G_{k,\omega}} \int \lambda^I_{k,\omega}(q) \frac{\tilde{q}_{k,\omega}(q')}{e} \frac{d^3p'}{(2\pi)^3} \\
= \int \frac{\nu_{n,i}eG_{k,\omega}}{1 + \nu_{n,i}G_{k,\omega}} \frac{\lambda^I_{k,\omega}(q)}{e} \frac{\tilde{q}_{k,\omega}(q')}{\delta f_{p',k,\omega}(q')} \frac{d^3p'}{(2\pi)^3} \\
= \int \tilde{S}_{k,\omega}^I(q, q') \frac{d^3p'}{(2\pi)^3},
\]

where we used the definition of the response $\tilde{S}_{k,\omega}^I(q, q')$.

\[
A_7 = \int ev\sigma_i(q, v) \int \frac{\nu_{n,i}(q', v)\Phi_p^I}{iD_{k,\omega}(v)} \delta f_{p',k,\omega}(q') \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \\
= e \left[ \int \frac{v^2\sigma_i(q, v)\sigma_i(q', v)\Phi_p^I}{iD_{k,\omega}(v)} \delta f_{p',k,\omega}(q') \frac{d^3p}{(2\pi)^3} \right] \frac{d^3p'}{(2\pi)^3} \\
= \int \tilde{S}_{k,\omega}^I(q, q') \frac{d^3p'}{(2\pi)^3},
\]

where we used the definition of the response $\tilde{S}_{k,\omega}^I(q, q')$.

Overall, we have $\sum_{\alpha} \delta I_\alpha(q) = \sum_{i=1}^7 A_i$, using the definition of the responses $S_{k,\omega}(q)$, $\Delta S_{k,\omega}$ and $\tilde{S}_{k,\omega}(q, q')$ (note that q refers to the charge dependence of the plasma
current, \( q' \) refers to already integrated charge)

\[
S_{k,\omega}(q) = e^2 \int \frac{v \sigma_i(q,v)}{i(\omega - k \cdot v + \nu_{d,i}(v) + \nu_{n,i})} \left( k \cdot \frac{\partial \Phi_p}{\partial p} \right) \frac{d^3p}{(2\pi)^3}
- \frac{i e^2}{kT_e} \int v \sigma_i(q,v) \Phi_p \frac{d^3p}{(2\pi)^3} - \frac{e \nu_{n,i} \nu_e}{kT_e} \lambda_{k,\omega}(q),
\]

(4.28)

\[
\Delta S_{k,\omega} = \frac{\nu_{n,i}}{1 + \nu_{n,i} G_{k,\omega}} \left( \frac{k \chi_{k,\omega}}{4\pi e} + \frac{e \nu_e}{kT_e} G_{k,\omega} \right),\]

(4.29)

\[
\bar{S}_{k,\omega}(q,q') = e \int \frac{v^2 \sigma_i(q,v) \sigma_i(q',v)}{i(\omega - k \cdot v + \nu_{d,i}(v) + \nu_{n,i})} \Phi_p \frac{d^3p}{(2\pi)^3}
- \frac{\nu_{n,i}}{1 + \nu_{n,i} G_{k,\omega}} \lambda_{k,\omega}(q) \bar{q}_{k,\omega}(q'),\]

(4.30)

we obtain

\[
\sum_{\alpha} \delta I_{\alpha}(q) = (S^I_{k,\omega}(q) + S^{II}_{k,\omega}(q) + S^{III}_{k,\omega}(q)) \delta E_{k,\omega} + (\Delta S^I_{k,\omega} + \Delta S^{II}_{k,\omega}) \lambda_{k,\omega}(q) \delta E_{k,\omega}
+ \int \left( \bar{S}^I_{k,\omega}(q,q') + \bar{S}^{II}_{k,\omega}(q,q') \right) \delta f_{p',k,\omega}(q') \frac{dq'd^3p'}{(2\pi)^3}
= S_{k,\omega}(q) \delta E_{k,\omega} + \Delta S_{k,\omega} \lambda_{k,\omega}(q) \delta E_{k,\omega} + \int \bar{S}_{k,\omega}(q,q') \delta f_{p',k,\omega}(q') \frac{dq'd^3p'}{(2\pi)^3}
= (S_{k,\omega}(q) + \Delta S_{k,\omega} \lambda_{k,\omega}(q)) \delta E_{k,\omega} + \int \bar{S}_{k,\omega}(q,q') \delta f_{p',k,\omega}(q') \frac{dq'd^3p'}{(2\pi)^3}.
\]

But since the dust distribution function has both induced and natural fluctuations, we have

\[
\sum_{\alpha} \delta I_{\alpha}(q) = (S_{k,\omega}(q) + \Delta S_{k,\omega} \lambda_{k,\omega}(q)) \delta E_{k,\omega} + \int \bar{S}_{k,\omega}(q,q') \delta f_{p',k,\omega}^{(0)}(q') \frac{dq'd^3p'}{(2\pi)^3}
+ \int \bar{S}_{k,\omega}(q,q') \delta f_{p',k,\omega}^{(ind)}(q') \frac{dq'd^3p'}{(2\pi)^3}.
\]

(3.31)

In order to evaluate the last adder of Eq.(4.31), we apply the property of Eq.(4.16) by setting \( q \to q' \) and \( F(q') \to \bar{S}_{k,\omega}(q,q') \). We also define the responses \( \bar{\beta}_{k,\omega}(q) = \frac{\partial \bar{S}_{k,\omega}(q)}{\partial q} \) and \( \bar{\beta}_{k,\omega}(q,q') = \frac{\partial \bar{S}_{k,\omega}(q,q')}{\partial q'} \),

\[
\bar{\beta}_{k,\omega}(q) = \int \frac{ev \sigma_i(q,v)}{i(\omega - k \cdot v + \nu_{d,i}(v) + \nu_{n,i})} \Phi_p \frac{d^3p}{(2\pi)^3},
\]

(4.32)
4.5. The Permittivity

\[ \tilde{S}_{k,\omega}(q,q') = e \int \frac{v^2 \sigma_i(q,v) \sigma_i'(q',v)}{i(\omega - k \cdot v + \nu_n d_i(v) + \nu_n,\omega)} \Phi_i' \frac{d^3 p}{(2\pi)^3} \]
\[ - \frac{\nu_{n,i}}{1 + \nu_{n,i} G_{k,\omega}} \frac{\lambda_{k,\omega}(q)}{e} \tilde{\beta}_{k,\omega}'(q'), \]

(4.33)

and obtain

\[ \int \tilde{S}_{k,\omega}(q,q') \delta f_{p',k,\omega}(q') \frac{dq' dp'}{(2\pi)^3} = -i \tilde{S}_{k,\omega}(q,q_{eq}) \frac{k \chi_{k,\omega}^{d,eq}}{4\pi q_{eq}} \delta E_{k,\omega} n_{k,\omega}^{ind} + \]
\[ \sum_{\alpha} \delta I_{k,\omega}^{\alpha}(q_{eq}) \times \delta E_{k,\omega} + \int \left\{ \frac{\alpha_{k,\omega}^{d,eq} \tilde{S}_{k,\omega}(q_{eq},q_{eq})}{1 - \delta f_{p',k,\omega}^{d,eq}} \right\} \frac{dp'}{(2\pi)^3} \]

We substitute in Eq.(4.31), set \( q = q_{eq} \) and we solve for the fluctuating plasma current,

\[ \sum_{\alpha} \delta I_{k,\omega}^{\alpha}(q_{eq}) = \left[ \frac{S_{k,\omega}(q_{eq}) + \Delta S_{k,\omega} \lambda_{k,\omega}^{d,eq}(q_{eq})}{1 - S_{k,\omega}'(q_{eq},q_{eq}) \chi_{k,\omega}^{d,eq}} \right] \frac{ik a_{k,\omega}^{d,eq}}{4\pi q_{eq}} \tilde{S}_{k,\omega}(q_{eq},q_{eq}) \]
\[ \times \delta E_{k,\omega} + \int \left\{ \frac{\alpha_{k,\omega}^{d,eq} \tilde{S}_{k,\omega}(q_{eq},q_{eq})}{1 - \delta f_{p',k,\omega}^{d,eq}} \right\} \frac{dp'}{(2\pi)^3} \]

Finally, we use the definitions of \( \gamma_{k,\omega}(q,q') \), \( \beta_0^{k,\omega}(q) \) and \( \beta_{k,\omega}(q) \)

\[ \gamma_{k,\omega}(q,q') = \frac{\tilde{S}_{k,\omega}(q,q')}{1 - S_{k,\omega}'(q_{eq},q_{eq}) \chi_{k,\omega}^{d,eq}} a_{k,\omega}^{n,d} \]
(4.34)
\[ \beta_0^{k,\omega}(q) = \frac{S_{k,\omega}(q) + \lambda_{k,\omega}(q) \Delta S_{k,\omega}(q_{eq})}{1 - S_{k,\omega}'(q_{eq},q_{eq}) \chi_{k,\omega}^{d,eq}} \]
(4.35)
\[ \beta_{k,\omega}(q) = \frac{\beta_0^{k,\omega}(q)}{4\pi q_{eq}} \chi_{k,\omega}^{d,eq} \gamma_{k,\omega}(q,q_{eq}) \]
(4.36)

and acquire

\[ \sum_{\alpha} \delta I_{\alpha}(q_{eq}) = \beta_{k,\omega}(q_{eq}) \delta E_{k,\omega} + \int \gamma_{k,\omega}(q_{eq},q_{eq}) \delta f_{p',k,\omega}^{d,(0)} \frac{dp'}{(2\pi)^3} \]
(4.37)

We shall use the decomposed Poisson equation, Eq.(4.25), together with the expressions for the induced plasma density fluctuations, the property of Eq.(4.16) and the relation for the plasma current, Eq.(4.37), to express the electric field via the natural dust fluctuations only.

\[ i k \delta E_{k,\omega} = 4\pi e \delta n_{k,\omega}^d - 4\pi e \delta n_{k,\omega}^s + 4\pi \int q \delta f_{p',k,\omega}^{d,(0)}(q) \frac{dp'^3 dq}{(2\pi)^3} + 4\pi \int q \delta f_{p',k,\omega}^{d,(ind)}(q) \frac{dp'^3 dq}{(2\pi)^3} \]
Before proceeding in a term by term evaluation, we apply Eq.(4.16) for \( F(q) = q \) and \( F(q) = \tilde{q}_{k,\omega}(q) \), using the definition of the response \( \beta_{k,\omega}^d(q) = \frac{\partial \tilde{q}_{k,\omega}(q)}{\partial q} \).

\[
\int q^2 f_{p',k,\omega}(q) \frac{d^3 p'}{2\pi^3} = -\frac{i}{4\pi} k_{k,\omega} \delta E_{k,\omega} a_{k,\omega}^{eq} + q_{eq} \nu_n a_{k,\omega}^{n,d} q_{eq}^{d,(0)} d_{k,\omega}^{d,(0)} + \sum \delta \Gamma_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} \\
= -\frac{i}{4\pi} k_{k,\omega} \delta E_{k,\omega} a_{k,\omega}^{eq} + q_{eq} \nu_n a_{k,\omega}^{n,d} q_{eq}^{d,(0)} d_{k,\omega}^{d,(0)} + \beta_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} \delta E_{k,\omega} + \gamma_{k,\omega}(q_{eq}, q_{eq}) \delta n_{k,\omega}^{d,(0)} \chi_{k,\omega}^{d,ch} \\
= \left\{ \beta_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} - \frac{i}{4\pi} k_{k,\omega} \delta E_{k,\omega} \right\} \delta n_{k,\omega}^{d,(0)}. \tag{4.38}
\]

\[
\int \tilde{q}_{k,\omega}(q) f_{p',k,\omega}(q) \frac{d^3 p'}{2\pi^3} = -\frac{i}{4\pi} \tilde{q}_{k,\omega}(q_{eq}) k_{k,\omega}^{d,eq} \delta E_{k,\omega} a_{k,\omega}^{eq} \\
+ \tilde{q}_{k,\omega}(q_{eq}) \nu_n a_{k,\omega}^{n,d} q_{eq}^{d,(0)} d_{k,\omega}^{d,(0)} + \tilde{q}_{k,\omega}^{d}(q_{eq}) \sum \delta \Gamma_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} \\
= -\frac{i}{4\pi} \tilde{q}_{k,\omega}(q_{eq}) k_{k,\omega}^{d,eq} \delta E_{k,\omega} a_{k,\omega}^{eq} \\
+ \tilde{q}_{k,\omega}(q_{eq}) \nu_n a_{k,\omega}^{n,d} q_{eq}^{d,(0)} d_{k,\omega}^{d,(0)} + \tilde{q}_{k,\omega}^{d}(q_{eq}) \beta_{k,\omega}(q_{eq}) \delta E_{k,\omega} \chi_{k,\omega}^{d,ch} \\
+ \tilde{q}_{k,\omega}(q_{eq}) \gamma_{k,\omega}(q_{eq}, q_{eq}) \delta n_{k,\omega}^{d,(0)} \chi_{k,\omega}^{d,ch} \\
= \left\{ \tilde{q}_{k,\omega}(q_{eq}) \beta_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} - \frac{i}{4\pi} \tilde{q}_{k,\omega}(q_{eq}) k_{k,\omega}^{d,eq} a_{k,\omega}^{eq} \right\} \\
\times \delta E_{k,\omega} + \left\{ \tilde{q}_{k,\omega}^{d}(q_{eq}) \gamma_{k,\omega}(q_{eq}, q_{eq}) \chi_{k,\omega}^{d,ch} + \tilde{q}_{k,\omega}(q_{eq}) \nu_n a_{k,\omega}^{n,d} q_{eq}^{d,(0)} d_{k,\omega}^{d,(0)} \right\} \delta n_{k,\omega}^{d,(0)}. \tag{4.39}
\]
We start by evaluating the first adder,

\[
4\pi e\delta n_{k,\omega} = -\frac{1}{1 + \nu_n, G_{k,\omega}} \frac{\chi_{k,\omega}}{T_k} \delta E_{k,\omega} - \frac{4\pi e \nu_n \nu_e}{T_e k} G_{k,\omega} \delta E_{k,\omega} + \frac{4\pi}{1 + \nu_n, G_{k,\omega}} \int \tilde{q}_{k,\omega}(q) \delta f^{(0)}_{p',k,\omega}(q) \frac{dqd^3p'}{(2\pi)^3} + \frac{4\pi}{1 + \nu_n, G_{k,\omega}} \int \tilde{q}_{k,\omega}(q) \delta f^{(ind)}_{p',k,\omega}(q) \frac{dqd^3p'}{(2\pi)^3} = -\frac{1}{1 + \nu_n, G_{k,\omega}} \frac{\chi_{k,\omega}}{T_k} \delta E_{k,\omega} - \frac{4\pi e \nu_n \nu_e}{T_e k} G_{k,\omega} \delta E_{k,\omega} + \frac{4\pi}{1 + \nu_n, G_{k,\omega}} \left( \int \tilde{q}_{k,\omega}(q) \delta f^{(0)}_{p',k,\omega}(q) \frac{dqd^3p'}{(2\pi)^3} + \left\{ \tilde{\beta}_{k,\omega}(q_{eq}) \tilde{\gamma}_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} - \tilde{q}_{k,\omega}(q_{eq}) \frac{k_{k,\omega}^{d,eq}}{4\pi q_{eq}} \tilde{\gamma}_{k,\omega}(q_{eq}) \delta E_{k,\omega} + \left\{ \tilde{\beta}_{k,\omega}(q_{eq}) \tilde{\gamma}_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,ch} + \tilde{q}_{k,\omega}(q_{eq}) \nu_n, \nu_e, \tilde{a}_{k,\omega}^{d,eq} \delta E_{k,\omega} \right\} \right\} \delta E_{k,\omega} \right)
\]

\[
\begin{align*}
&= -\frac{k}{1 + \nu_n, G_{k,\omega}} + \frac{G_{k,\omega}}{1 + \nu_n, G_{k,\omega}} \frac{\nu_e}{k^2 \lambda_{De}^2} + \frac{\tilde{q}_{k,\omega}(q_{eq}) \tilde{\gamma}_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,eq}}{1 + \nu_n, G_{k,\omega}} \delta E_{k,\omega} + 4\pi \int \left\{ \tilde{q}_{k,\omega}(q_{eq}) \frac{dqd^3p'}{(2\pi)^3} \right\} \delta E_{k,\omega} + 4\pi \int \left\{ \tilde{q}_{k,\omega}(q_{eq}) \frac{dqd^3p'}{(2\pi)^3} \right\} \delta E_{k,\omega} + 4\pi \int \left\{ \tilde{q}_{k,\omega}(q_{eq}) \frac{dqd^3p'}{(2\pi)^3} \right\} \\
&= -\frac{k}{1 + \nu_n, G_{k,\omega}} + \frac{G_{k,\omega}}{1 + \nu_n, G_{k,\omega}} \frac{\nu_e}{k^2 \lambda_{De}^2} + \frac{\tilde{q}_{k,\omega}(q_{eq}) \tilde{\gamma}_{k,\omega}(q_{eq}) \chi_{k,\omega}^{d,eq}}{1 + \nu_n, G_{k,\omega}} \delta E_{k,\omega} + 4\pi \int \left\{ \tilde{q}_{k,\omega}(q_{eq}) \frac{dqd^3p'}{(2\pi)^3} \right\} \delta E_{k,\omega} + 4\pi \int \left\{ \tilde{q}_{k,\omega}(q_{eq}) \frac{dqd^3p'}{(2\pi)^3} \right\} \delta E_{k,\omega} + 4\pi \int \left\{ \tilde{q}_{k,\omega}(q_{eq}) \frac{dqd^3p'}{(2\pi)^3} \right\} \delta E_{k,\omega}
\end{align*}
\]

For the second adder we have

\[
-4\pi e\delta n_{k,\omega} = -4\pi e \frac{\nu_n \nu_e}{T_e k} \delta E_{k,\omega} = -k \frac{4\pi e^2 \nu_e}{T_e k^3} \delta E_{k,\omega} = -k \left[ \frac{1}{k^2 \lambda_{De}^2} \right] \delta E_{k,\omega},
\]
where the third adder is already in the desired form. The last adder becomes

$$4\pi \int q \pi f^{(ind)}_{q_p, k} (q) \frac{d^3 p'}{2\pi^3} = -ik \left\{ \chi^{d, eq}_{k, \omega} a^{n, d}_{q_k, q_{eq}} + \frac{4\pi t}{k} \beta_{k, \omega} (q_{eq}) \chi^{d, ch}_{k, \omega} \right\} \delta E_{k, \omega} +$$

$$4\pi \int \left\{ \nu_{eq} \nu'_{n, d} a^{n, d}_{k, \omega} \chi^{eq}_{k, \omega} + \gamma_{k, \omega} (q_{eq}, q_{eq}) \chi^{d, ch}_{k, \omega} \right\} \delta f^{(0)}_{q_p', k} (q) \frac{d^3 p'}{2\pi^3} \cdot (4.42)$$

Finally, combining we end up with

$$ik \left[ 1 + \frac{1}{k^2 \chi^2_{De}} + \frac{1}{k^2 \chi^2_{De}} \frac{1}{1 + \nu_{n, i} G_{k, \omega}} + \frac{\chi_{k, \omega}}{1 + \nu_{n, i} G_{k, \omega}} + \frac{4\pi t}{k} \beta_{k, \omega} (q_{eq}) \chi^{d, ch}_{k, \omega} + \frac{4\pi t}{k} \beta_{k, \omega} (q_{eq}) \chi^{d, ch}_{k, \omega} \right] \frac{\nu_{eq} a^{n, d}_{k, \omega} \chi^{d, ch}_{k, \omega} + \gamma_{k, \omega} (q_{eq}, q_{eq}) \chi^{d, ch}_{k, \omega}}{1 + \nu_{n, i} G_{k, \omega}} \delta E_{k, \omega} =$$

$$4\pi \int \left\{ q + \frac{2}{1 + \nu_{n, i} G_{k, \omega}} \frac{\gamma_{k, \omega} (q_{eq}, q_{eq}) \chi^{d, ch}_{k, \omega}}{1 + \nu_{n, i} G_{k, \omega}} + \frac{\nu_{eq} \nu'_{n, d} a^{n, d}_{k, \omega} \chi^{eq}_{k, \omega} + \gamma_{k, \omega} (q_{eq}, q_{eq}) \chi^{d, ch}_{k, \omega} \delta f^{(0)}_{q_p', k} (q) \frac{d^3 p'}{2\pi^3} \cdot (4.42)$$

Since, the total permittivity and the effective charge are defined through the relation

$$ik \epsilon_{k, \omega} \delta E_{k, \omega} = 4\pi \int q_{eff} (q) \delta f^{(0)}_{q_p', k} (q) \frac{d^3 p'}{2\pi^3} \cdot (4.43)$$

a direct comparison will give us their final expressions,

$$q_{eff} (q) = q - q_{eq} \chi^{eq}_{k, \omega} + \frac{1}{1 + \nu_{n, i} G_{k, \omega}} \left[ \frac{\gamma_{k, \omega} (q_{eq}, q_{eq})}{1 + \nu_{n, i} G_{k, \omega}} \right], \quad (4.44)$$

$$\epsilon_{k, \omega} = \epsilon^p_{k, \omega} + \nu_{eq} a^{n, d}_{k, \omega} \left[ 1 + \frac{\chi_{k, \omega} \gamma_{k, \omega} (q_{eq}, q_{eq})}{1 + \nu_{n, i} G_{k, \omega}} \right], \quad (4.45)$$

where

$$\chi^{eq}_{k, \omega} = -\nu_{n, d} a^{n, d}_{k, \omega} \left[ 1 + \frac{\gamma_{k, \omega} (q_{eq}, q_{eq})}{1 + \nu_{n, i} G_{k, \omega}} \right], \quad (4.46)$$

$$\epsilon^p_{k, \omega} = 1 + \frac{1}{k^2 \chi^2_{De}} \left[ 1 + \frac{\nu_{eq} a^{n, d}_{k, \omega}}{1 + \nu_{n, i} G_{k, \omega}} \right], \quad (4.47)$$
Chapter 5

Charging of Non-emitting Grains in Presence of Neutrals

The problem of dust charging is central to the field of complex plasmas, since the high dust charge values and their fluctuations are responsible for the most fundamental processes. Here, we are interested in the charging of non-emitting dust grains embedded in isotropic partially ionized plasmas.

The dust charge will be studied in its normalized form \( z = \frac{Z_d e^2}{aT_e} \) which is also the normalized dust surface potential \( \phi_s = -\frac{Z_d e}{a} \). For the comprehensive study of the effect of pressure it is important to define the plasma collisionality \( \alpha_c = \frac{\lambda_D}{l_{n,i}} \), that is the ratio of the dust screening length approximated by the ion Debye length to the mean free path in ion collisions with neutrals [Khrapak and Morfill, 2009]. It determines whether the ions are collisionless (pair-particle approach) or collisional (hydrodynamic approach) in the perturbed sphere around the grain. It is important to point out, that the collision cross-sections of electrons with neutrals are at least two orders of magnitude smaller than the collision cross-sections of ions with neutrals (typical values for Neon are \( \sigma_{n,e} = 10^{-16} \text{ cm}^2 \) and \( \sigma_{n,i} = 10^{-14} \text{ cm}^2 \)), this means that there exists a wide range of pressures where ions are hydrodynamically treated while electrons are collisionless.

Before proceeding to quantitative results we briefly present a qualitative view of the problem: A grain immersed in a plasma will initially become negatively charged due to the higher mobility of the light electrons. The quasi-stationary charge will then be set up by the balance of electron and ion currents flowing to the grain and it will remain negative. For very low pressures, the Orbit Motion Limited (O.M.L.) approach can be implemented for both the electron and the ion species [Allen, 1992; Allen et al., 2000]. As the pressure gradually increases, collisions start to affect the ion flux. Effective charge exchange collisions lead to the trapping of low energy ions in the strong field region around the grain and to eventual absorption on the grain surface [Lampe et al., 2003]. This leads to an increase in the ion flux and hence a decrease in dust charge. For larger pressures, ion transport will become collision
dominated, ion mobility will become suppressed by collisions with neutrals and the ion flux will reduce, leading to an increase in the dust charge, which also implies the existence of a minimum for $l_{n,i} \sim \lambda_D$. For even larger pressures, electron transport will also become collisional, both electron and ion fluxes will simultaneously reduce and the dust charge saturates with respect to the gas pressure.

For a better understanding of the problem we can identify the following regimes according to the value of the plasma collisionality,

1. Collisionless regime, $l_{n,e}, l_{n,i} \gg \lambda_D$.
2. Weakly collisional regime, $l_{n,i} > \lambda_D$.
3. Intermediate collisional regime, $l_{n,i} \sim \lambda_D$.
4. Strongly collisional regime, $l_{n,i} < \lambda_D$.
5. Fully collisional regime, $l_{n,i}, l_{n,e} \ll \lambda_D$.

We point out that gas discharges typically operate in the weakly and intermediate collisional regimes.

### 5.1 Charging in the collisionless regime

The collisionless regime is defined by $l_{n,e}, l_{n,i} \gg \lambda_D$, the Orbit Motion Limited (O.M.L.) cross-sections can then be used for the computation of the electron and ion currents. The basic assumptions of the O.M.L. approach are: (i) the dust grain is isolated, i.e. other dust grains do not affect the motion of electrons and ions in its vicinity; (ii) electrons and ions do not experience collisions during the approach to the grain, (iii) there are no barriers in the effective potential [Kennedy and Allen, 2003].

The heavy grain is considered to be an infinitely massive scattering center for the electrons and ions colliding with it. Due to the conservation of angular momentum of the plasma particle we have $m_a u b = m_a r^2 \dot{\phi}$, where $u$ is the velocity of the particle when it enters the mean free path sphere and $b$ is the impact parameter. Conservation of the total energy of the particle also imposes the requirement that

$$
\frac{1}{2} m_a v^2 + e_a \phi_\infty = \frac{1}{2} m_a r^2 + \frac{1}{2} m_a r^2 \dot{\phi}^2 + e_a \phi_s, \tag{5.1}
$$

where $\phi_\infty \simeq 0$ is the potential on the mean free path surface, $\phi_s$ is the potential at the surface of the grain. At the closest approach $r = r_{min} = a$ and $\dot{r} = 0$, eliminating the angular velocity and solving for the impact parameter will result in $b^2 = b_{max}^2 = (1 - \frac{2e_a \phi_s}{m_a v^2})a^2$. The absorption cross-sections will be $\pi b_{max}^2$.

In the case that the currents emitted from the grain are negligible, the grain will be negatively charged and we will have $\phi_s < 0$. Hence, there will be an attractive potential for the ions, which implies that an ion with any velocity can be absorbed.
5.2. CHARGING IN THE WEAKLY COLLISIONAL REGIME

In the weakly collisional regime, $l_{n,i} > \lambda_D$, electrons are still collisionless, whereas collisions of ions with neutrals have an effect on charging. In order to demonstrate this; Let us assume an ion in the perturbed plasma region in the vicinity of the grain and that it undergoes a resonant charge exchange collision with a neutral. The high-energy orbiting ion will be substituted by a low-energy ion that will be -with high efficiency- trapped in the region of high attractive potential and eventually absorbed by the grain (unless it undergoes subsequent collisions). This will lead to an increase in the ion flux to the grain and hence a decrease in the dust charge [Zakrzewski and Kopieczynski, 1974].

We consider the sphere of strong ion-dust interaction around the grain, with its radius defined by equating the ion kinetic and interaction energies, i.e $|U(R_0)| = T_i$. The expected number of collisions with neutrals can be approximated by $\frac{R_0}{n_i} < 1$ and the probability of $n$ occurrences will be given by the Poisson distribution.
CHAPTER 5. CHARGING OF NON-EMITTING GRAINS IN PRESENCE OF NEUTRALS

\( P(n) = \frac{1}{n!} \left( \frac{R_0}{l_{n,i}} \right)^n \exp \left( -\frac{R_0}{l_{n,i}} \right) \). The probability of no collisions will be \( P(0) = \exp \left( -\frac{R_0}{l_{n,i}} \right) \simeq 1 - \frac{R_0}{l_{n,i}}, \) the probability of a single collision \( P(1) = \frac{R_0}{l_{n,i}} \exp \left( -\frac{R_0}{l_{n,i}} \right) \simeq \frac{R_0}{l_{n,i}} \), while the probability of multiple collisions \( P(n > 1) \simeq \left( \frac{R_0}{l_{n,i}} \right)^2 \simeq 0. \) We essentially have two independent ion populations

- Collisionless ions appearing with \( P_0 = 1 - \frac{R_0}{l_{n,i}} \) and having an O.M.L charging current \( I_0 = \sqrt{8\pi e a^2 n_i v_{Ti}} (1 + \frac{z}{\tau}). \)

- Ions colliding once with \( P_1 = \frac{R_0}{l_{n,i}} \) and then being absorbed with 100% efficiency, which means that the charging current will be the random current in the \( R_0 \) sphere \( I_1 = \sqrt{8\pi e R_0^2 n_i v_{Ti}}. \)

The total ion current will then be

\[
I_i = \sum_{i=0,1} P_i I_i = \sqrt{8\pi e a^2 n_i v_{Ti}} (1 + \frac{z}{\tau}) \left( 1 - \frac{R_0}{l_{n,i}} \right) + \sqrt{8\pi e R_0^2 n_i v_{Ti}} \frac{R_0}{l_{n,i}} \\
\simeq \sqrt{8\pi e a^2 n_i v_{Ti}} (1 + \frac{z}{\tau}) + \sqrt{8\pi e R_0^2 n_i v_{Ti}} \\
\simeq \sqrt{8\pi e a^2 n_i v_{Ti}} \left( 1 + \frac{z}{\tau} + \frac{R_0}{a^2 l_{n,i}} \right).
\]

In the above equation \( R_0 \) is still undetermined, it is depending on the potential around the grain. The simplest solution can be found by assuming a Yukawa potential with the screening length given by the ion Debye length. In that case, with \( \beta_T = \frac{z}{\tau} \frac{a}{\lambda_{Di}} \) the thermal scattering parameter,

\[
|U(R_0)| = T_i \Rightarrow Z de^2 \frac{R_0}{\lambda_{Di}} \exp \left( -\frac{R_0}{\lambda_{Di}} \right) = T_i \\
\frac{z a}{\tau} \frac{R_0}{\lambda_{Di}} \exp \left( -\frac{R_0}{\lambda_{Di}} \right) = \frac{R_0}{\lambda_{Di}} \\
\beta_T \exp \left( -\frac{R_0}{\lambda_{Di}} \right) = \frac{R_0}{\lambda_{Di}},
\]

and \( R_0 = \lambda_{Di} x_0 \) with \( x_0 \) the root of the transcendental equation \( \beta_T e^{-x} = x. \)

A more self-consistent calculation involves the solution of the Poisson equation with electrons following the Boltzmann distribution and ions following the Gurevich distribution [Alpert et al., 1965], with the latter taking into account absorption on the grain (but with the effects of barriers in the effective potential and trapped ions still neglected). Numerical solution [Ratynskaia et al., 2006] yielded a fit with good accuracy in the Yukawa form with an effective screening length \( \lambda = (1+0.2\sqrt{\beta_T})\lambda_{Di}, \) and \( R_0(\beta_T) = (-0.1 + 0.8\sqrt{\beta_T})\lambda_{Di}, \) for typical \( \beta_T \approx 1 - 13. \)
Finally, since the electrons are still collisionless, \( I_e = -\sqrt{8\pi e a^2} n_e v_{Te} e^{-z} \) and the dust charge will be given by the solution of

\[
n_{i}v_{Ti} \left( 1 + \frac{z}{\tau} + \frac{R_i^2 R_0}{a^2 l_{n,i}} \right) = n_e v_{Te} e^{-z}.
\]

\[ (5.6) \]

5.3 Charging in the intermediate collisional regime

In the intermediate collisional regime, \( l_{n,i} \sim \lambda_D \) while the electrons are still collisionless. In such a regime neither a pair-particle nor a hydrodynamic approach can be implemented and therefore the ion current can only be approximated via interpolation formulas [Khrapak and Morfill, 2008].

The requirements an interpolation formula are (i) in the limit \( l_{n,i} > \lambda_D \) it should coincide with the ion current for the weakly collisional regime \( I_{iWC} \), (ii) in the limit \( l_{n,i} \ll \lambda_D \) it should asymptotically reach the hydrodynamic limit \( I_{iSC} \), (iii) it should reproduce the theoretically predicted and experimentally verified minimum in the dust charge, therefore \( I_i \) should have a pronounced maximum. Such an interpolation formula is

\[
I_{iC}^i = \left( \left( \frac{1}{I_{iSC}} \right)^{\gamma} + \left( \frac{1}{I_{iWC}} \right)^{\gamma} \right)^{-1/\gamma},
\]

where \( \gamma > 0 \) is the charging index. It can either be a function of the plasma parameters, chosen to best fit numerical results or molecular dynamics data as in Hutchinson’s [Hutchinson and Patacchini, 2007] and Zobnin’s fits [Zobnin et al., 2000], or a constant.

5.4 Charging in the strongly collisional regime

In the limit of strong collisionality, \( l_{n,i} < \lambda_D \ll l_{n,e} \), electron transport to the grain is still collisionless, while ion transport to the grain is collision dominated [Su and Lam, 1963; Khrapak et al., 2006]. In this case a fluid description can be adopted for the ion component. In a steady state the momentum equation and the continuity equation will read us

\[
\Gamma_i = n_i \mu_i E - D_i \nabla n_i,
\]

\[
\nabla \cdot \Gamma_i = Q_{si} - Q_{Li},
\]

where \( Q_{si} \) and \( Q_{se} \) represent the source and loss rates of the ions (ionization, volume recombination, absorption on dust etc), \( \mu_i \) and \( D_i \) are the ion mobility and diffusivity respectively (with \( \mu_i \) considered independent of the electric field and the Einstein relation assumed to hold) and \( \Gamma_i = n_i v_i \) the ion flux.

By assuming that the characteristic ionization and recombination lengths are much larger than the plasma Debye length, the continuity equation will yield \( \nabla \cdot \).
\( \Gamma_i = 0 \) and the ion flux will be conserved. In spherical coordinates it will only have a constant radial component satisfying
\[
\Gamma_i = 4\pi r^2 \left( n_i(r) \mu_i \frac{d\phi(r)}{dr} + D_i \frac{dn_i(r)}{dr} \right).
\]

On the other hand, for the electron density distribution around the negatively charged grain, one can assume that it follows a Boltzmann relation
\[
n_e(r) = n_0 \exp \left( \frac{e\phi(r)}{T_e} \right),
\]
where \( n_{e0} = n_{i0} = n_0 \) are the electron/ion densities in the quasi-neutral unperturbed plasma. The system of equations is closed by the Poisson equation,
\[
d^2 \phi/dr^2 + \frac{2}{r} \frac{d\phi}{dr} = -4\pi e(n_i(r) - n_e(r)),
\]
with the boundary conditions \( \phi(\infty) = 0 \), \( \phi(a) = -Zde/a = \phi_s \), \( n_i(a) = n_e(a) = 0 \) and \( n_i(\infty) = n_e(\infty) = n_0 \). We should notice that electron Boltzmann relation produces a slight inconsistency not satisfying the boundary conditions for \( \phi(a) \) and \( n_e(a) \) simultaneously, it yields \( n_e(a) = n_0 e^{-z} \).

The above system can only be solved numerically. However, in the limit \( a \ll \lambda_D \) and close to the surface of the grain, a major simplification can be made; Since the densities of the electrons and ions asymptotically tend to zero at the grain surface, it can be assumed that the geometrical terms of the Poisson equation dominate in the vicinity of the grain. In that case the Poisson equation will take its spherical vacuum form, which has the bare Coulomb potential as a solution \( \phi(r) = -Zde/r \).

In that case the equations are decoupled and the ion flux conservation relation can be solved to yield
\[
I_i = e\Gamma_i = 4\pi e n_0 D_i \frac{(z/\tau)}{1 - \exp(-z/\tau)}.
\]

For the collisionless electrons the O.M.L current will be
\[
I_e = -\sqrt{8\pi e} a^2 n_0 v_{Te} e^{-z}. \tag{5.8}
\]

The dimensionless charge \( z \) as usual will be found from the current balance (we assume \( n_e \approx n_i \approx n_0 \))
\[
4\pi e n_0 D_i \frac{(z/\tau)}{1 - \exp(-z/\tau)} = \sqrt{8\pi e} a^2 n_0 v_{Te} e^{-z},
\]
\[
\sqrt{2\pi} D_i \frac{(z/\tau)}{1 - \exp(-z/\tau)} = av_{Te} e^{-z},
\]
\[
z e^2 \frac{(z/\tau)}{1 - \exp(-z/\tau)} = a\nu_{Te} m_i v_{ni,1} \sqrt{2\pi I_i},
\]
\[
z e^2 \frac{(z/\tau)}{1 - \exp(-z/\tau)} = \frac{\tau}{\sqrt{2\pi}} \frac{v_{Te}}{I_{ni,1} v_{Ti}}.
\]

In gas discharges typically \( \tau \approx 0.01 \) and therefore \( \exp(-z/\tau) \rightarrow 0 \), which yields
\[
z e^2 = \frac{\tau}{\sqrt{2\pi}} \frac{v_{Te}}{I_{ni,1} v_{Ti}}. \tag{5.8}
\]
5.5 Charging in the fully collisional regime

In the fully collisional regime, \( l_{n,i}, l_{n,e} \ll \lambda_D \), thus, both electron and ion transport to the grain will be collision dominated [Chang and Laframboise, 1976; Khrapak et al., 2006]. In this regime the electron fluxes will also be determined by the hydrodynamic equations. Flux continuity will result in

\[
\Gamma_e = 4\pi r^2 \left( -n_e(r) \frac{d\phi(r)}{dr} + D_e \frac{dn_e(r)}{dr} \right),
\]

while the ion flux relation and the Poisson equation will remain the same. Employing the same approximations in the limit of infinitesimally small dust grains, \( a \ll \lambda_D \), we end up with the currents

\[
I_i = e\Gamma_i = 4\pi e a n_i D_i \frac{(z/\tau)}{1 - \exp(-z/\tau)},
\]

\[
I_e = -e\Gamma_e = -4\pi e a n_e D_e \frac{ze^{-z}}{1 - e^{-z}}.
\]

Current balance, with the use of \( \exp(-z/\tau) \to 0 \), results in (we assume \( n_e \approx n_i \approx n_0 \))

\[
4\pi e a n_0 D_i \frac{(z/\tau)}{1 - \exp(-z/\tau)} = 4\pi e a n_0 D_e \frac{ze^{-z}}{1 - e^{-z}}
\]

\[
\frac{z}{1 - \exp(-z/\tau)} = \frac{D_e}{D_i} \frac{ze^{-z}}{1 - e^{-z}}
\]

\[
1 = \tau \frac{D_e}{D_i} \frac{e^{-z}}{1 - e^{-z}}
\]

\[
z \simeq \ln \left( 1 + \tau \frac{D_e}{D_i} \right)
\]

\[
z \simeq \ln \left( 1 + \tau \frac{v_T e \sigma_{n,i}}{v_T e \sigma_{n,e}} \right).
\]

We conclude that in the fully collisional regime the dust charge is independent of the pressure.

5.6 Maximum of the charge as a function of pressure

From the behavior of the charge as a function of the plasma collisionality and its saturation for large pressures, it is obvious that it will be maximum either in
CHAPTER 5. CHARGING OF NON-EMITTING GRAINS IN PRESENCE OF NEUTRAL

Figure 5.1: The dimensionless dust charge as a function of the plasma collisionality. Adopted from Khrapak and Morfill (2009).

The determination of the charge maximum is not a trivial issue, since the flux balance equation for O.M.L. currents is transcendental.

The approximate solution for the charge in the fully collisional regime, in the case \( n_e \neq n_i \), can be further simplified by using \( \tau \simeq 0.01, \frac{\sigma_{ni}}{\sigma_{ne}} \simeq 100 \) and \( n_e v_{Te} \gg n_i v_{Ti} \), with the latter valid for moderate dust densities, so that the electron density is not severely depleted.

\[
z_{SC} \simeq \ln \left( 1 + \frac{n_e v_{Te} \sigma_{ni}}{n_i v_{Ti} \sigma_{ne}} \right) \simeq \ln \left( 1 + \frac{n_e v_{Te}}{n_i v_{Ti}} \right) \approx -z_{SC} \simeq \frac{n_i v_{Ti}}{n_e v_{Te}}.
\]

Let us assume a function of the normalized charge \( f(z) = n_e v_{Te} e^{-z} - n_i v_{Ti} \left( 1 + \frac{z}{\tau} \right) \).

Table 5.1: The plasma collisionality and the expressions for the ion and electron fluxes.

<table>
<thead>
<tr>
<th>regime</th>
<th>( \frac{\Delta d}{d_{oom}} )</th>
<th>( I_i )</th>
<th>( I_e )</th>
<th>( z(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL ( \ll 1 )</td>
<td>( \sqrt{8\pi a^2 e n_i v_{Ti}(1 + \frac{z}{\tau})} )</td>
<td>( -\sqrt{8\pi a^2 e n_e v_{Te} e^{-z}} )</td>
<td>const.</td>
<td></td>
</tr>
<tr>
<td>WC ( &lt; 1 )</td>
<td>( \sqrt{8\pi e a^2 n_i v_{Ti}} \left( 1 + \frac{z}{\tau} + \frac{R_0^2}{R_{conf}} \right) )</td>
<td>( -\sqrt{8\pi e a^2 n_e v_{Te} e^{-z}} )</td>
<td>( \sim )</td>
<td></td>
</tr>
<tr>
<td>IC ( \sim 1 )</td>
<td>( \left( \frac{1}{\tau} \right)^\gamma + \left( \frac{1}{\tau} \right)^{\gamma-1} )</td>
<td>( -\frac{\sqrt{8\pi e a^2 n_e v_{Te} e^{-z}}}{\exp(-\frac{z}{\tau})} )</td>
<td>min.</td>
<td></td>
</tr>
<tr>
<td>SC ( &gt; 1 )</td>
<td>( 4\pi e a n_i D_i \frac{1}{1 - \exp(-z/\tau)} )</td>
<td>( -\frac{\sqrt{8\pi e a^2 n_e v_{Te} e^{-z}}}{1 - z/\tau} )</td>
<td>( \sim )</td>
<td></td>
</tr>
<tr>
<td>FC ( \gg 1 )</td>
<td>( 4\pi e a n_i D_i \frac{1 - \exp(-z/\tau)}{z/\tau} )</td>
<td>( -4\pi e a n_e D_e \frac{z}{1 - z/\tau} )</td>
<td>const.</td>
<td></td>
</tr>
</tbody>
</table>
the first derivative with respect to $z$ is $\frac{df}{dz} = -n_e v_T e^{-z} - \frac{n_i v_T}{\tau} < 0$, the function is monotonically decreasing and hence it can only have one zero. That zero will by definition be the O.M.L. charge, $z_{OML}$. We evaluate the function for $z = 0$ and $z = z_{SC}$,

\[
f(z_{SC}) = n_e v_T e^{-z_{SC}} - n_i v_T i \left(1 + \frac{z_{SC}}{\tau}\right) = n_e v_T e \frac{n_i v_T i}{n_e v_T} - n_i v_T i - n_i v_T i \frac{z_{SC}}{\tau}
\]

\[
= n_i v_T i - n_i v_T i - n_i v_T i \frac{z_{SC}}{\tau} = -n_i v_T i \frac{z_{SC}}{\tau} = -\frac{n_i v_T i}{n_i v_T i} \ln \left(\frac{n_e v_T e}{n_i v_T i}\right) < 0,
\]

\[
f(0) = n_e v_T e^0 - n_i v_T i = n_e v_T e - n_i v_T i > 0
\]

Due to the continuity of $f(z)$, by Bolzano’s theorem, for $f(0)f(z_{SC}) < 0$, the only root of $f(z)$ must lie between $(0,z_{SC})$, and hence $z_{OML} < z_{SC}$, and $z_{SC}$ is the maximum of the charge.
Chapter 6

Dust Charging Experiments in PK-4

6.1 The discharge tube and the microgravity experiments

The project Plasma-Kristall 4 (PK-4) is a continuation of the PK series of complex plasma experiments performed onboard the Russian Mir Space Station and the International Space Station (ISS). Unlike its predecessors PKE-Nefedov and PK-3 plus [Nefedov et al., 2003] that were using a planar rf capacitive discharge, the PK-4 experiment is run in a long cylindrical chamber with a combined dc/rf discharge [Fortov et al., 2005].

PK-4 is expected to fly on the ISS after 2013. Earth’s gravity exerts an external force on the massive dust component in ground based laboratory experiments. This force prohibits the levitation of the dust grains in the bulk quasi-neutral plasma and can only be avoided under microgravity conditions. Such conditions can be achieved by parabolic flights, sounding rockets or orbital flights.

The plasma is produced by applying a voltage of about 1000 V to the dc electrodes and the operating gas is either neon or argon at pressures between 10 and 500 Pa, while the DC current can vary until 5 mA. The dc discharge is produced in a glass tube with a length of 35 cm and a diameter of 3 cm, while an rf discharge can also be applied by external rf coils. Basic discharge modes are: (i) pure dc discharge, (ii) two rf inductive discharges, (iii) combinations of the dc discharge with one or two rf inductive discharges, (iv) combination of the dc discharge with one rf capacitive discharge. Inlet of dust grains in the discharge chamber is possible through a set of four dust dispensers. The dust grains are melamine formaldehyde monodisperse spherical particles of diameters between 0.5 and 11 µm, while nanoparticles can also be synthesized directly in the discharge. For their observation the dust grains are illuminated by a laser sheet and observed by two fast charge coupled device (CCD) video cameras.

A number of experiments are scheduled to be performed, they include: charging
CHAPTER 6. DUST CHARGING EXPERIMENTS IN PK-4

of dust particles and dust-acoustic waves, determination of the ion drag force, observation of the critical point in liquid-gas phase transitions, cloud collisions and lane formation, observations of the transition from laminar flow to turbulence at a microscopic level, investigation of Laval nozzle flows, experiments on solitons, agglomeration and particle growth [Thoma et al., 2007].

6.2 Dust charging experiments

When inserted into the discharge plasma, the dust particles get negatively charged and start drifting against the dc electric field. As long as the flow is stable and inter-grain forces can be neglected (which is valid for sufficiently low numbers of dust particles), the drift velocity of the dust grains is simply set up by the balance of the forces acting on them: (i) the electrostatic force, (ii) the neutral drag force, (iii) the ion drag force, (iv) the electron drag force. The electron drag force is always negligible due to the small ion to electron temperature ratio ($\tau \leq 0.01$), while the ion drag force is usually much smaller than the other two forces. It is evident that measurement of the dust drift velocity by analysis of recorded camera frames, together with measurement of the plasma parameters by Langmuir probes and the force balance equation can yield experimental values of the charge. This simple approach is known as the force balance method [Ratynskaia et al., 2004; Khrapak et al., 2005].

Alternatively, the charge can be experimentally measured by the onset of the dust acoustic wave instability [Ratynskaia et al., 2004]. For large numbers of dust particles there is an easily identifiable transition from stable flow to unstable flow (with a clear wave behavior). Responsible for this sharp transition (with an experimental error of 1 Pa only) is the relative drift between the grains and the ions,

Figure 6.1: Sketch of the PK-4 experimental facility.
6.3. COMPARISON WITH CHARGING MODELS

which triggers the ion-dust streaming instability and excites dust acoustic waves at a certain threshold pressure around 50 Pa. The dust-acoustic waves tend to have very amplitudes and are clearly non-linear. However, the onset of the low-frequency self-excited waves can be well described by linear hydrodynamic dispersion relations with collisionless Boltzmann electrons, static drifting collisional ions and counter-drifting collisional dust particles. The threshold pressure can be defined as the pressure for which the mode is driven from stable to unstable and will be dependent on the dust charge.

6.3 Comparison with charging models

The operating pressure range of the PK-4 discharge (10 – 500 Pa), spans the weakly collisional and the intermediate collisional regimes for ions. Hence, the experimental determination of the charge presented above can be used for the verification of the charging models of these regimes.

In the weakly collisional regime both independent methods can be used for a comparison. The experimental data have shown a remarkable agreement with both the charging model and molecular dynamics simulations for a variety of grain radii.

On the other hand, in the intermediate collisional regime only the force balance method can be used. However, the existence of a pronounced minimum in the charge well within the PK-4 operating pressures can facilitate a comparison with the interpolation formula. The experimental data have shown a very good agreement with the exception of small systematic errors that can be contributed to an underestimation of the ion drag force.
Chapter 7

Summary

7.1 Paper I: Regimes for experimental tests of kinetic effects in dust acoustic waves

Dust acoustic waves (DAW) are waves similar to conventional ion acoustic waves, with the dust grains participating in the wave dynamics [Rao et al., 1990]. They are ubiquitous features of complex plasmas produced in laboratory discharges, being excited by the ion-dust streaming instability [Merlino, 2009]. Involving the motion of massive dust grains, the DAW propagate in the low frequency regime defined by $k v_T d \ll \omega \ll k v_T i$, typical frequencies are in the range of tens of Hz. They have received particular attention since they can be observed using laser light scattering and charge coupled device (CCD) cameras, which provides the unique opportunity of investigations of wave-particle interactions at a particle level [e.g., Barkan et al., 1995; Thompson et al., 1997]. The physics of the mode are quite intuitive; the heavy dust species provides the inertia and the plasma pressure coupled to the electric field provides the restoring force.

Linear weakly coupled dust acoustic waves have mostly been studied through fluid models - from collisionless approaches to the inclusion of collisions with neutrals [Pieper and Gorce, 1996], particle drifts [Ivlev et al., 1999], plasma sources [D'Angelo, 1997], plasma absorption on dust, dust charge fluctuations [e.g., Melandso et al., 1993; Jana et al., 1993; Varma et al., 1993]- or through the multi-component [Rosenberg, 1993] and standard kinetic models [Shukla and Mamun, 2002]. However, for the self-consistent inclusion of the effects of dust charge fluctuations and plasma absorption the use of the "full" kinetic model is necessary [Tsytovich et al., 2002].

In this paper the DAW waves are studied through the "full" kinetic model in the fully ionized case. Aim of the work is to identify experimentally accessible parameter regimes where charging effects / plasma absorption have a strong effect on the DAW dispersion, which will result in strong deviations from the traditionally used collisionless hydrodynamic approach.
CHAPTER 7. SUMMARY

The theoretical advances of this paper include

1. The derivation of analytical asymptotic relations for the DAW phase velocities in wavelength regimes where (i) plasma absorption can be neglected, (ii) ion absorption is important, (iii) both ion and electron absorption are strong. The results indicate a steep increase of the DAW phase velocity for large wavelengths and are also confirmed numerically.

2. The derivation of the first order in $\frac{k}{\nu Ti}$ imaginary parts of the low frequency plasma responses and their implementation for the determination of the damping rate of the waves under the small growth rate approximation.

3. The derivation of semi-analytical closed solutions of the dispersion equation that are first order in $\frac{k v_T}{\omega}$.

Results for concrete parameters (PK-4 facility, magnetic cusp) are presented and criteria are formulated for the deviations of the DAW behavior from the collisionless fluid approach to be maximum;

- possibility for the observation of large wavelengths (feasible in elongated discharge tubes)
- non-thermal plasmas with $T_e \gg T_i$ (typical for low temperature discharges)
- dense dust clouds
- high plasma densities (feasible in magnetic confined low-temperature discharges)
- grains with large grain radius levitating in the bulk quasineutral discharge plasma (feasible in microgravity).

7.2 Paper II: Kinetic models of partially ionized complex plasmas in the low frequency regime

Continuous absorption of plasma fluxes on the dust grains brings out the necessity of an external source to replenish the plasma, so that the grains can sustain their quasi-equilibrium charges. The source can be either non-fluctuating or fluctuating. An example of fluctuating source, encountered in most laboratory discharges, is electron impact ionization of neutrals, where the source fluctuations follow the fluctuations of the exact electron distribution function. In partially ionized complex plasmas the system’s components are electrons, ions, dust grains and neutrals. The latter can be treated as a medium in thermodynamic equilibrium providing dissipation through collisions, the source of plasma particles and effects in grain charging [Tsytovich et al., 2005].
In this paper kinetic models of partially ionized complex plasmas are formulated in the low frequency regime, analyzed and compared in terms of their permittivity and static permittivity. Theoretical advances include

1. The low frequency integral plasma/dust responses, the static permittivity and the permittivity are derived for the "full" kinetic model of partially ionized complex plasmas.

2. The standard and "multi-component" kinetic models are formulated for partially ionized complex plasmas by employing the Bhatnagar-Gross-Krook relaxation time approximation for the description of collisions with neutrals.

3. Absorption cross-sections are derived for ions taking into account the effect of neutrals in the weakly collisional regime, An analytical expression for the new charging frequency is also derived.

4. The low frequency integral plasma/dust responses of all models are calculated for thermal distributions.

5. A simple condition is derived for the convergence of the multi-component with the "full" kinetic model.

6. The ratio of induced to natural dust density fluctuations is derived for the "full" kinetic model, depending on its value great simplifications in the structure of the kinetic equations are feasible.

On the numerical side, the permittivity and static permittivity of all models is calculated for typical laboratory parameters. Novel results reveal
• The real part of the permittivity of the "full" kinetic model has multiple roots, which opens up the possibility for another low frequency electrostatic mode in addition to the dust acoustic. This mode does not exist for the standard and multi-component kinetic models.

• The ratio of induced to natural dust density fluctuations is a decaying function of the frequency. It is well below unity, already before the dust plasma frequency $\omega_{pd}$, for all parameter ranges investigated. This implies that for kinetic models focusing on regimes typical of ion dynamic, induced dust fluctuations can be ignored. This will lead to major simplifications in the theory, since the induced part of the dust distribution function is the most cumbersome term in the mathematical treatment.

7.3 Paper III: Low frequency electrostatic modes in partially ionized complex plasmas; a kinetic approach

In this paper the low frequency electrostatic modes in partially ionized complex plasmas are investigated employing the full kinetic model. This is the most self-consistent attempt to date. The results indicate that, apart from the dust acoustic mode (DA), a novel long-wavelength (LW) mode exists. The characteristics of the LW mode are determined by the interplay between electron impact ionization of neutrals and plasma absorption on dust. It propagates for long wavelengths and "induces" cut-offs in the DA mode.

The damping rates and dispersion relations of both modes are found from the solution of the coupled system of equations $\Im\{\epsilon_{k,\omega_{r},\omega_{i}}\} = 0$, $\Re\{\epsilon_{k,\omega_{r},\omega_{i}}\} = 0$ for realistic laboratory parameters;

• A numerical scheme is devised, based on the strength of dissipation by neutrals, that significantly simplifies the formidable numerical task with a negligible error.

• From physical and mathematical grounds, the multiple roots of the permittivity are classified to each mode.

• The sensitivity of the dispersion relations of both modes on basic plasma and discharge parameters is investigated (gas pressure, dust density, dust size, electron temperature, operating gas).

• The DA mode kinetic dispersion is compared to hydrodynamic dispersion relations.

Extensive numerical investigation of the long-wavelength mode has revealed its main characteristics:
7.3. **PAPER III: LOW FREQUENCY ELECTROSTATIC MODES IN PARTIALLY IONIZED COMPLEX PLASMAS; A KINETIC APPROACH**

Figure 7.2: The dispersion relation for the long-wavelength mode as a function of the pressure for PK-4 density and temperature profiles ($n(p), T_e(p)$).

1. It is determined by the interplay between electron impact ionization and plasma absorption.

2. It does not exist for: low electron temperatures compared to the gas first ionization energy, relatively low pressures and relatively low dust densities.

3. Its dispersion relation appears to be adequately fitted by $\omega_r(k) = A_1 + \frac{A_2}{k^2 + A_3}$, where the coefficients $A_1$, $A_2$ and $A_3$ depend on the discharge, dust grain and plasma parameters, while $A_1$ determines the cut-off frequency of the wave.

4. It is present for large wavelengths. However, as the ionization frequency/pressure increases it can propagate at larger wavenumbers.

5. The phase velocities are an order of magnitude larger than the DA phase velocities.

6. The group velocity is negative, a characteristic common in ionization waves [Pekarek, 1968].

7. The presence of the mode in the large-wavelength part of the Fourier spectrum leads to a cut-off of the DA mode in this range. The cut-off frequency/wavenumber of the DA mode is the same with the ones of the LW mode.

8. For very small wavenumbers it reaches time scales characteristic of ion dynamics and the low-frequency kinetic model does not suffice for a self-consistent solution.
Discussion is provided regarding (i) the physical mechanism of the mode, (ii) its relation with the phenomenon of long range collective dust attraction [e.g., Tsytovich and Morfill, 2004; Castaldo et al., 2006; Ratynskaia et al., 2006; Tsytovich et al., 2006; de Angelis et al., 2010], (iii) its emergence in hydrodynamic models [Tsytovich and Watanabe, 2003], (iv) the possibility that it is driven unstable in very small wavenumbers [Else et al., 2009].

Finally, the necessary conditions, that experiments aiming to observe the LW mode should satisfy, are pointed out:

- possibility of observation of long wavelengths,
- relatively high operating pressures,
- elevated electron temperatures,
- operation with gases of low first ionization potential,
- sub-thermal drifts of ion/dust components.

7.4 Paper IV: Grain charging in an intermediately collisional plasma

The dust charge is the most fundamental parameter of complex plasma systems. For engineered complex plasmas, formed by injection of small spherical solid particulates in laboratory discharges, different charging models exist depending on the value of plasma collisionality $\lambda_D$, that is mostly determined by the gas pressure [Khrapak and Morfill, 2009].

Typical plasma discharges, fall either in the weakly collisional regime $l_{n,i} > \lambda_D$ - where charging models have been extensively studied and verified both experimentally and by molecular dynamics simulations [Khrapak et al., 2005] - or in the intermediate collisional regime $l_{n,i} \sim \lambda_D$. In such a regime, neither a pair-particle nor a hydrodynamic approach is applicable to describe the ion transport to the grain and therefore the use of interpolation formulas is inevitable. Interpolation formulas have been proposed based on molecular dynamics simulations, but they have never been tested experimentally.

In this paper we report the first experimental verification of an interpolation formula for the ion current in the intermediate collision regime. Dedicated experiments have been performed in the PK-4 dc discharge prototype in the Max-Planck-Institute for Extraterrestrial Physics in Garching, Germany. The experiments have been performed with Neon gas at elevated pressures $100 - 500$ Pa. The dust grain flow has been illuminated by a laser sheet and the grain motion recorded by two fast video cameras, thus enabling measurements of the grain steady flow and the dust density. Plasma parameter measurements have been carried out by Langmuir probes in absence of dust grains. The aforementioned experimental data can be
used for the determination of the dust charge through the force balance method, with the neutral drag, ion drag and electrostatic forces acting on the grain.

Comparison with the theoretical dust charge found through the flux balance condition, with the ion flux given by \( I_i^{IC} = \left[ \left( \frac{1}{\gamma_{\text{WC}}} \right)^\gamma + \left( \frac{1}{\gamma_{\text{SC}}} \right)^\gamma \right]^{-1/\gamma} \), has shown a remarkable agreement within the experimental errors. The correct asymptotic behavior of the interpolation formula in the weakly collisional regime has also been confirmed by comparison with previous experiments.

7.5 Paper V: Spectra of ion density and potential fluctuations in weakly ionized plasmas in presence of dust grains

Fluctuations are omnipresent at physical systems, even in thermodynamic equilibrium, due to the discrete nature of matter. Initially, they were regarded as noise and were undesirable in experiments. However, the fluctuation-dissipation theorem guarantees that the properties of any system (quantum or classical) can be determined from its intrinsic fluctuations [Kubo, 1966]. This has resulted in the implementation of noise spectroscopic methods as diagnostics in a variety of physical systems [Ichimaru, 1964], e.g. for the determination of the kinetic parameters of a chemical reaction in an electrolytic solution [Feher and Weissman, 1973], for extraction of information about the quantum state of ultra-cold fermionic gases of
alkali atoms [Mihaila, 2006], for the determination of plasma parameters in quiescent space environments [e.g., Meyer-Vernet, 1979; Meyer-Vernet and Perche, 1989; Moncuquet et al., 2006].

The presence of dust grains in plasma systems leads to enhancement of the spectral densities of ion density fluctuations and electrostatic field fluctuations, by orders of magnitude [Ratynskaia et al., 2007]. These important modifications stem from the large number of elementary charges residing on the dust surface and the dissipative nature of charging collisions between dust grains and plasma particles. They are mostly confined in the low frequency regime and are strongly dependent on the dust density and the grain radius. Consequently, it has been recently proposed that the modifications in the plasma spectra can be used as an \textit{in situ} diagnostic tool for sub-micron dust. Proof of principle of the diagnostic has already been confirmed and also verified by post-mortem analysis, yet for the interpretation of the experimental data the multi-component kinetic model has been employed [Ratynskaia et al., 2010].

The spectral densities have been derived for the ‘full’ kinetic model only in the fully ionized case [de Angelis et al., 2006]. However, an important class of quiescent laboratory discharges, the brush cathodes, operate in elevated pressure regimes [Persson, 1965]. In this paper, we derive the spectral densities of ion density and electrostatic potential fluctuations in partially ionized complex plasmas in the low frequency regime. We study the effects of pressure and ionization numerically, for realistic parameters and taking into account the effects of the finite probe size. We also compare with the results of the multi-component model to determine regimes where deviations are significant.

Theoretical advances include:

1. The derivation of the low frequency spectral densities of the electrostatic field, electrostatic potential and ion density fluctuations for the ‘full’ and multi-component models of partially ionized complex plasmas. This enables a comparison that shows the necessity of the ‘full’ kinetic model.

2. The calculation of the low frequency spectral densities of the electrostatic field, electrostatic potential and ion density fluctuations for weakly ionized plasmas in absence of dust grains. This facilitates the estimation of the order of magnitude enhancement of the spectral densities due to the presence of dust.

3. The derivation of a criterion for the omission of plasma discreteness based on the natural spectral densities for thermal distributions. Such a criterion properly defines the low frequency regime and hence the range of validity of the kinetic model as $\omega/k < \Lambda_\alpha v_T d$, where $\Lambda_\alpha$ is a dimensionless quantity of the order of a few (that is a weak function of the dust and plasma parameters).

The numerical investigations have lead to the conclusions below:
The results of both kinetic models reveal orders of magnitude enhancement (five to ten orders of magnitude) of the fluctuation level due to the presence of dust, similar to previously reported spectral changes for fully ionized plasmas.

Neutral gas density (pressure) can be responsible for significant modifications of spectral density magnitudes, despite the opposite behavior of the equilibrium dust charge number $Z_d$. Hence the inclusion of the effect of neutrals is essential for a more realistic comparison with experiments.

The "full" self-consistent model deviates from the multi-component model significantly (above typical experimental errors) already for typical dust densities $n_d \approx 10^5 \text{ cm}^{-3}$ and dust radii $a > 100 \text{ nm}$, values that are common for in situ produced dust.

The effect of electron temperature variation in the spectral densities is non-monotonic and can be attributed to the strength electron impact ionization.

7.6 Paper VI: Effects of dust particles in plasma kinetics; ion dynamics time scales

Application of the Klimontovich approach for complex plasma systems has so far led to self-consistent kinetic models in the low frequency regime of dust dynamics, due to treatment of electrons/ions as continuous Vlasov fluids [Tsytovich and de Angelis, 1999]. This also implies some further restrictions. Namely, the dust densities should be (i) large enough so that binary plasma collisions can be neglected when compared to collisions with dust [Tsytovich, 1998], (ii) low enough so that the dust component is its weakly coupled gaseous state. This limits the parameter regime of the above models mostly in weakly coupled engineered complex plasma experiments [Fortov et al., 2005].

Aim of this paper is the development of a self-consistent kinetic model valid in frequency regimes typical of ion dynamics. In such frequency ranges the discreteness of the ion and the electron component can no longer be neglected. A central assumption leading to great simplifications in the cumbersome algebra involved is that the induced fluctuating part of the dust distribution function can be neglected when compared to the natural fluctuating part. In that case a full dynamic description of dust can be avoided and dust charge fluctuations can be found independently by first order expansion of the charging equation.

Theoretical advances include:

1. The formulation of the basic assumptions of a new kinetic model of complex plasmas valid in frequency ranges characteristic of ion dynamics.

2. The derivation of the general forms of the integral responses, the effective charges and the permittivity of the system in such frequency ranges.
3. The derivation of the ion collision integrals, the ion kinetic equations and the detailed analysis of their structure.

4. The derivation of the spectral densities of the ion density fluctuations and their physical interpretation.

The new kinetic model has been formulated in a generic fashion in order to account for both fluctuating and non-fluctuating plasma sources, hence it can be applied for both space and engineered complex plasma systems. It is expected to give new interpretations to a variety of phenomena of both laboratory and astrophysical context

- ion stochastic acceleration [de Angelis et al., 2006],
- dust-ion acoustic waves [Shukla and Silin, 2002],
- spectra modification due to the presence of dust [Ratynskaia et al., 2010],
- scattering of electromagnetic radiation in dust clouds [de Angelis et al., 2002],
- self-consistent non-thermal distribution functions [Ricci et al., 2001],
- contribution of complex plasma populations to the distortion of the cosmic microwave background, the Sunyaev-Zeldovich effect [Sunyaev and Zeldovich, 1970].
8.1 Self-consistent treatment of electrostatic waves in complex plasmas

Addition of drifting components in the treatment of dust acoustic waves
One of the basic assumptions of the kinetic models is the absence of external electric fields, $\langle E \rangle = 0$. However, most laboratory discharge plasmas need an electric field for their creation, which implies drifting dust and plasma components. For DA waves the electrons can always be considered in Boltzmann equilibrium, while the ion drift can be sub-thermal, $v_i \ll v_T$, on the other hand the massive highly charged grains are typically supra-thermal, $v_d > v_T$. Therefore, the influence of the external electric field can be important.

In fact, the relative drift between the ion and dust components can drive the DA waves unstable [Rosenberg, 1993]. This is the typical excitation mechanism of the acoustic perturbations observed in the experiments [Merlino, 2009]. Even for relatively high pressures, ionization and ion drag force can be destabilizing factors that are sufficient to overcome the strong collisional damping.

Thus, the addition of a dc electric field in the self-consistent kinetic description is important, it will shed light in the competition between the stabilizing (collisionless damping, collisions with neutrals, dust charge fluctuations, plasma absorption) and the destabilizing (ionization, relative drift) factors. We notice that this will not only change the plasma / dust kinetic equations but also the expression for the permittivity since the equations for the fluctuating parts will now contain the extra term $e\langle E \rangle \cdot \frac{\partial \delta f_i}{\partial p}$.

Treatment of the $k \to 0$ limit of the long-wavelength mode
The long-wavelength mode, theoretically predicted in Paper III, is attributed to the competition between electron impact ionization of neutrals and absorption of ions on the grains and is always damped in the low frequency regime. Plots of the dispersion relation $\omega_r(k)$ of the LW mode, show that in the $k \to 0$ limits the mode
eigenfrequencies are larger than the dust plasma frequency, which makes a low frequency kinetic model inadequate for its description in very small wavenumbers. It is obvious that the mode is dependent on both ion and dust dynamics. However, the permittivity derived from the low frequency kinetic model is valid for all frequency regimes; The permittivity depends only on fluctuations that are directly proportional to the electric field fluctuations. None of these terms have been omitted, since all the components’ induced fluctuations are regarded, whereas the omission of plasma discreetness will affect the effective charge but not the permittivity. Thus, we conclude that the same permittivity expression can be applied for the analysis of the LW mode in the whole frequency range, provided that the frequency is no longer omitted in the ion integral responses.

Such a treatment of the long-wavelength mode is very important. In case of an ionization source and a ground state with no directed motion, analysis of the self-consistent hydrodynamic equations for partially ionized complex plasmas has revealed that the system is unstable for ion scale perturbations [Else et al., 2009; Khrapak and Morfill, 2010]. The instability length is very long corresponding to $k \to 0$, which can have consequences on feasibility of the long-range attraction mechanism, a paradigm for the transition of the system to strongly coupled states [Tsytovich, 2006].

**Self-consistent treatment of the dust-ion acoustic waves**

Dust-ion acoustic waves (DIA) are electrostatic waves similar to ion sound waves propagating in the regime $k v_T_i \ll \omega \ll k v_T_e$; the ions provide the inertia, the electron pressure provides the restoring force, while dust is essentially at rest [Shukla and Silin, 1992]. In the fully ionized case, the collisionless hydrodynamic equations for $k^2 \lambda_{De}^2 \ll 1$ yield the well-known dispersion relation $\omega = \omega_{pi} \lambda_{De} k$ or alternatively $v_{ph} = \sqrt{1+P} v_{Ti}$. Therefore, even though dust does not participate in the wave dynamics, due to its effect in the ground state of the system through the quasi-neutrality condition, it leads to an increase of the phase velocity of the ion sound waves.

DIA waves can be self-consistently treated either through the general permittivity expression derived in Paper II, or through the permittivity of the kinetic model of Paper VI. In that sense, DIA waves can lead to a first indirect verification of the validity of the new kinetic model.

#### 8.2 Development of an *in situ* dust diagnostic based on the fluctuation spectra

**Effect of chemical reactions in the fluctuation spectra**

*In situ* dust formation takes place in a complicated environment involving chemical transformations [e.g., Bouchoule et al., 1991; Berndt et al., 2003; Benedikt, 2010], (i) breaking of molecular bonds of the precursor gases, (ii) formation of reactive
8.3. APPLICATIONS OF THE NEW KINETIC MODEL

radicals leading to larger molecules, (iii) avalanche condensation of molecules for
the formation of a solid core, (iv) surface attachment of radicals. It is expected
that such processes will have an effect on the fluctuation spectra [Uddholm, 1983].
But will the effect be strong enough to mask the order of magnitude enhancement
of the ion density fluctuation spectra due to the effect of dust?
The answer to this question is crucial to the development of the method as a di-
agnostic. Apparently, a rigorous classical kinetic treatment is formidable, since
the system involves strongly coupled subsystems and chemical bond formation of
quantum nature. However, estimates of the effect can be made by including ap-
proximate collision operators in the Klimontovich equations, that in a relaxation
time approximation will just be sink terms with a frequency equal to the chem-
ical reaction rate. In that sense they will contribute a chemical damping factor
\( (\omega - k \cdot v + i(\nu_{d,i} + \nu_{n,i} + \nu_{chem})) \) in the fluctuations, that bears a resemblance with
the effect of neutrals in the B.G.K. approximation.

Continuous and discrete size distributions for dust
After their formation, dust grains continue to grow in size through coagulation
(attachment collision of small dust particles forming a larger grain) and through
surface growth (deposition of outer layers with dust acting as a spherical substrate).
The radius of the dust grains is hence a dynamic variable, that can be approximated
either as a discrete variable (when coagulation dominates) or as a continuous vari-
able (when surface growth dominates).
Each treatment will have an effect both in the results and the tolerance errors of the
outputs (dust density and grain size) of the diagnostic. So far, experimental results
have been compared with theoretical models assuming discrete dust components,
that were treated in a multi-component fashion.

Multi-component or "full" kinetic model?
So far only multi-component kinetic models have been used for the theoretical inter-
pretation of the results. Treatment of multiple dust species within the framework
of the "full" kinetic model will definitely result in cumbersome results, due to the
strong coupling of the kinetic equations. For dust radii lower than 100 nm, the
effects of absorption and dust charge fluctuations in the spectra will be negligible,
however, dust grains will diameters approaching the micron range are also possible
to exist.

8.3 Applications of the new kinetic model

Self consistent solutions of the ion kinetic equation
The numerical solution of the ion kinetic equation will provide self-consistent reg-
ular ion distribution functions and describe their temporal evolution. These are
expected to be non-thermal and with a strong dependence on the source of plasma
particles [Ricci et al., 2001].
The solution of the integro-differential equation for $\Phi^i_p$ is a formidable numerical task, not only due to the products of the distribution functions of the colliding particles present in the collision integral, but also due to the dependence of the permittivity on $\Phi^i_p$ through the integral responses $\chi_{k,\omega}$, $\tilde{\eta}_{k,\omega}$, $\tilde{S}_{k,\omega}$, $\tilde{\tilde{S}}_{k,\omega}$.

**Stochastic ion acceleration**

The collision integrals can be used for the study of stochastic acceleration of ions due to their interaction with dust grains [de Angelis et al., 2005]. For dust particles it has been shown that charge fluctuations in dust-dust interactions produce an instability which can lead to stochastic heating of the dust grains, under astrophysical [Ivlev et al., 2010], laboratory [Marmolino et al., 2009] and fusion conditions [Marmolino et al., 2008]. The linear stage of this instability can be well described by the dust kinetic equation resulting from fluctuation theory. It is of great interest whether a similar effect could be predicted for ions, in that case the second moment of the ion kinetic equation would be cast into a form $\frac{d\epsilon}{dt} = \nu \epsilon$ where $\epsilon$ is the mean local energy of ions and $\nu$ the energy growth rate.

The new kinetic model, that is applicable for both astrophysical and laboratory conditions, can be employed to investigate the conditions for $\nu > 0$, whether the dust charge should be positive or negative for the instability threshold to be reached, or even how dissipation by neutrals can quench this instability. Already, by simple inspection of the ion kinetic equation, one can see that the term describing dissipation due to absorption on dust can be reduced due to collective effects, and that the fluctuating ionization source also contributes to the positivity of $\nu$.

**Extension to frequencies characteristic of electron dynamics**

In the new kinetic model, the adiabatic assumption has been employed for the electrons. This confines the frequency range of validity in $\omega_{pd} < \omega \ll k v_T e$. An extension to frequencies typical of electron dynamics can easily be implemented by using a full dynamic description of electrons similar to the ion description. This will enable the self-consistent description of the scattering of electromagnetic radiation through complex plasma configurations and will provide the modifications of the relevant cross-sections as functions of the dust parameters. The latter could have major astrophysical implications, i.e. the effects of space dust populations in the distortion of the cosmic microwave background (Sunyaev - Zeldovich effect) [Sunyaev and Zeldovich, 1970].
Appendices
Appendix A

Generalized Approach in the Computation of the Integral Responses

The approach in the computation of the integral responses is based on the fact that for isotropic distribution functions, spherical coordinates and with the appropriate choice of the rotation of the coordinate system; the integrations over the azimuthal and elevation angles can be computed analytically, leaving only the speed integral to be evaluated numerically (preferably in a dimensionless form). The methodology is valid for all frequencies and is performed for Maxwellian distributions. Nevertheless, it can be followed for any isotropic distribution.

A.1 Calculation of the ion responses

The calculation of the ion responses for thermal distributions is performed following the methodology below;

- Since we are interested in the temporal attenuation of the waves, we use $\omega = \omega_r + i\omega_i$ (initial value problem). In case we were interested in the spatial attenuation, we would have $\mathbf{k} = k_r + ik_i$ (boundary value problem).

- We decompose in real and imaginary parts. Note that only in the low frequency regime $\omega_r \ll kv_T$, the ion responses will be either purely real or purely imaginary, due to the omission of $\omega_r$.

- We choose a coordinate system such that $\hat{k} \parallel \hat{z}$, hence $\mathbf{k} \cdot \mathbf{v} = kv_z$.

- For the computation of the integrals we choose spherical coordinates, $d^3v = \sin \theta v^2 d\phi d\theta dv$ and $kv_z = kv \cos \theta$.

- The integration over the azimuthal angle $\phi$ is trivial, $d^3v = 2\pi \sin \theta v^2 d\theta dv$. 
APPENDIX A. GENERALIZED APPROACH IN THE COMPUTATION OF THE INTEGRAL RESPONSES

- The integration over the elevation angle $\theta$ is always analytical and can be performed using a number of auxiliary integrals, defined below. The results will be complicated functions of the speed either of logarithmic or inverse tangent nature.

- The integration over the speed $v$ is performed via the transformation $y = \frac{m_i v^2}{2T_i}$. It is convenient to define a number of auxiliary functions in the dimensionless $y$-space in order to simplify the cumbersome results. These are the logarithmic and inverse tangent kernels, the total imaginary part of denominator of the responses, the ion capture frequencies, the ion capture cross-sections and their charge derivative.

**Auxiliary Integrals**

The angular integrals can be classified in one of the following forms, with $A = \omega_r$, $B = kv$ and $C = \nu_{tot}$,

\[
I_1 = \int_0^\pi \frac{\sin \theta}{(A - B \cos \theta)^2 + C^2} d\theta = \frac{1}{BC} \left[ \arctan \left( \frac{A + B}{C} \right) - \arctan \left( \frac{A - B}{C} \right) \right]
\]

\[
I_2 = \int_0^\pi \frac{\cos \theta \sin \theta}{(A - B \cos \theta)^2 + C^2} d\theta = -\frac{1}{2B^2} \ln \left[ \frac{(A + B)^2 + C^2}{(A - B)^2 + C^2} \right]
\]

\[
+ \frac{A}{B^2C} \left[ \arctan \left( \frac{A + B}{C} \right) - \arctan \left( \frac{A - B}{C} \right) \right]
\]

\[
I_3 = \int_0^\pi \frac{\cos^2 \theta \sin \theta}{(A - B \cos \theta)^2 + C^2} d\theta = \frac{2}{B^2} - \frac{A}{B^3} \ln \left[ \frac{(A + B)^2 + C^2}{(A - B)^2 + C^2} \right]
\]

\[
+ \frac{A^2 - C^2}{B^3C} \left[ \arctan \left( \frac{A + B}{C} \right) - \arctan \left( \frac{A - B}{C} \right) \right]
\]

\[
I_4 = \int_0^\pi \frac{B \sin \theta \cos \theta (A - B \cos \theta)}{(A - B \cos \theta)^2 + C^2} d\theta = -2 + \frac{C}{B} \left[ \arctan \left( \frac{A + B}{C} \right) - \arctan \left( \frac{A - B}{C} \right) \right]
\]

\[
+ \frac{A}{2B} \ln \left[ \frac{(A + B)^2 + C^2}{(A - B)^2 + C^2} \right]
\]

\[
I_5 = \int_0^\pi \frac{(A - B \cos \theta) \sin \theta}{(A - B \cos \theta)^2 + C^2} d\theta = \frac{1}{2B} \ln \left[ \frac{(A + B)^2 + C^2}{(A - B)^2 + C^2} \right]
\]

The integrals can be easily computed with the transformation $\cos \theta = x$. Other useful integrals are the following

\[
I_6 = \int_0^\infty (A + By)e^{-y}dy = A + B,
\]

\[
I_7 = \int_0^\infty y^2e^{-y^2}dy = \frac{\sqrt{\pi}}{4},
\]

both easily computed by employing integration by parts.
A.1. CALCULATION OF THE ION RESPONSES

Ion capture cross-sections in the weakly collisional regime

For the collision enhanced collection model a generalization of our discussion about the charging currents in terms of cross-sections yields; that there is a collisional ion population occurring with probability \( \frac{R_0}{l_{n,i}} \) that has the geometrical cross-section \( \pi R_0^2 \), there is an independent collisionless ion population with probability \( 1 - \frac{R_0}{l_{n,i}} \) that has the O.M.L cross-section \( \pi a^2 \left( 1 - \frac{2qe}{am_i v^2} \right) \).

\[
\sigma_i(q, v) = \pi R_0^2 \left( \frac{R_0}{l_{n,i}} \right) + \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 - \frac{2qe}{am_i v^2} \right).
\]  

(A.1)

In order to compute the radius of the perturbed plasma region \( R_0 \) self-consistently, the electrostatic potential around a test grain should be known. It can be found through the Poisson equation, which in the general case (taking into account absorption of particles on the grain, trapped ions, potential barriers, collisions with neutrals and other non-linearities) will be an integro-differential equation to be solved numerically. Such complexity can be avoided by assuming that the interaction potential is of the Yukawa type, i.e \( \phi(r) = \frac{q}{r} \exp \left( -\frac{r}{\lambda} \right) \), with the effective screening length \( \lambda \) regarded as an unknown parameter. Curve fitting of numerical solutions to the Yukawa form, in various scenarios, has shown that \( \lambda \) is slightly larger than the ion Debye length \( \lambda_{Di} \), hence \( \lambda \approx \lambda_{Di} \) is still a viable approximation.

From the definition of \( R_0 \) we have

\[
|U(R_0)| = T_i \Rightarrow -\frac{qe}{R_0} \exp \left( -\frac{R_0}{\lambda} \right) = T_i,
\]  

(A.2)

since in absence of emission from the grain, the dust charge is always negative. The above equation is a transcendental equation for \( R_0 \) that should be solved numerically. It is obvious that \( R_0 \) is charge dependent \( (R_0(q)) \). In equilibrium, for \( q = -Z_d e \),

\[
\frac{Z_d e^2}{\lambda} \exp \left( -\frac{R_0}{\lambda} \right) = T_i \Rightarrow \frac{Z_d e^2}{T_i} \exp \left( -\frac{R_0}{\lambda} \right) = R_0
\]

\[
\frac{Z_d e^2 a}{\lambda} \exp \left( -\frac{R_0}{\lambda} \right) = \frac{R_0}{\lambda} \Rightarrow \frac{Z_d e^2 T_e a}{T_i} \exp \left( -\frac{R_0}{\lambda} \right) = R_0
\]

\[
\frac{z a}{\tau} \exp \left( -\frac{R_0}{\lambda} \right) = \frac{R_0}{\lambda} \Rightarrow \beta_T \exp \left( -\frac{R_0}{\lambda} \right) = \frac{R_0}{\lambda},
\]  

(A.3)

where \( \beta_T = \frac{z}{\tau} \frac{z}{\lambda} \) is the scattering parameter at the ion thermal velocity. Using normalization over the effective screening length \( x = \frac{R_0}{\lambda} \), the transcendental equation for the radius \( R_0 \) becomes \( \beta_T \exp(x) = x \). In the case that \( \beta_T \ll 1 \), we have that

\[
x \ll 1 \Rightarrow \exp(x) \simeq 1 \Rightarrow x \simeq \beta_T \Rightarrow \frac{R_0}{\lambda} = \frac{z a}{\tau} \Rightarrow R_0 = \frac{za}{\tau}.
\]
APPENDIX A. GENERALIZED APPROACH IN THE COMPUTATION OF
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and \( R_0 \ll \lambda \).

The dependence of \( R_0 \) on the charge will introduce an extra factor in the charging frequency. We compute the first derivative of \( R_0(q) \) by direct differentiation of the transcendental equation,

\[
\frac{\partial}{\partial q} \left[ -\frac{q e}{R_0(q)} \exp \left( -\frac{R_0(q)}{\lambda} \right) \right] = \frac{\partial T_i}{\partial q} \Rightarrow \frac{\partial}{\partial q} \left[ -\frac{q}{R_0(q)} \exp \left( -\frac{R_0(q)}{\lambda} \right) \right] = 0
\]

\[
\frac{1}{R_0} \exp \left( -\frac{R_0(q)}{\lambda} \right) - \frac{q}{R_0^2} \exp \left( -\frac{R_0(q)}{\lambda} \right) \left. \frac{\partial R_0}{\partial q} \right|_{q=q_i} = 0
\]

Finally, the derivative of the equilibrium charge will be

\[
\left. \frac{\partial R_0}{\partial q} \right|_{q=q_i} = -\frac{R_0}{Z_{qe}} \left( \frac{1}{1+R_0/\lambda} \right).
\]

(A.4)

The new flux balance condition will be given by the equality of electron and ion currents, we also use the approximation \( (1 - \frac{R_0}{\lambda}) \approx 1 \) valid in the weakly collisional regime,

\[
\sqrt{8\pi e n_{eUT_e} a^2} \exp \left( \frac{q e}{a T_e} \right) = \sqrt{8\pi e n_{iUT_i} a^2} \left( 1 - \frac{q e}{a T_i} \right) + \sqrt{8\pi e n_{iUT_i} a^2} \left( 1 - \frac{q e}{a T_i} + \frac{R_0^2}{2 a T_i} \left( \frac{3}{2 a T_i} - \frac{3 R_0^2}{2 a T_i} \right) \right)
\]

(A.5)

For the computation of the charging frequency one should not forget to take into account the dependence of \( R_0 \) on the equilibrium charge,

\[
\nu_{ch} = -\frac{\partial}{\partial q} \sum_{\alpha} (I_{\alpha}(q)) = -\frac{\partial I_e(q)}{\partial q} - \frac{\partial I_i(q)}{\partial q}
\]

\[
= \sqrt{8\pi e n_{eUT_e} a^2} \exp \left( \frac{q e}{a T_e} \right) + \sqrt{8\pi e n_{iUT_i} a^2} \left( 1 - \frac{q e}{a T_i} + \frac{R_0^2}{2 a T_i} \left( \frac{3}{2 a T_i} - \frac{3 R_0^2}{2 a T_i} \right) \right)
\]

\[
= \sqrt{8\pi e n_{iUT_i} a^2} \left( \frac{e}{a T_i} \left[ 1 - \frac{q e}{a T_i} + \frac{R_0^2}{2 a T_i} \right] + \left( \frac{3}{2 a T_i} - \frac{3 R_0^2}{2 a T_i} \right) \right)
\]

\[
= \frac{\sqrt{8\pi e^2 n_{iUT_i} a^2}}{T_i} \left\{ \frac{T_i}{T_e} \left[ 1 + \frac{z}{\tau} + \frac{R_0^2}{a^2 l_{n,i}} \right] + \left( 1 - \frac{a T_i}{e a^2 l_{n,i}} \right) \right\}
\]

\[
= \frac{\sqrt{8\pi e^2 n_{iUT_i} a^2}}{T_i} \left\{ 1 + \frac{z}{\tau} + \frac{R_0^2}{a^2 l_{n,i}} - \frac{a T_i}{e a^2 l_{n,i}} \right\}
\].
A.1. Calculation of the Ion Responses

where all the quantities are evaluated in equilibrium charge and we substituted for the exponential from the equilibrium condition. Moreover, we use

\[
\sqrt{8\pi e^2 n_{i} u_{T_{i}} a} = \frac{4\pi n_{i} e^2 u_{T_{i}} a}{\sqrt{2\pi}} = \frac{u_{T_{i}} a}{\sqrt{2\pi} \lambda_{D_{i}}} = \frac{a}{\sqrt{2\pi} \lambda_{D_{i}}},
\]

\[
- \frac{a T_{i}}{e} \frac{3 R_{0}^{2} \partial R_{0}}{a^{2} l_{n,i}} \frac{\partial q}{\partial q} = \frac{a T_{i}}{e} \frac{3 R_{0}^{2} R_{0}}{a^{2} l_{n,i}} \frac{Z_{i} e^{2}}{a^{2} l_{n,i}} \frac{1}{1 + R_{0}/\lambda} = \frac{a T_{i}}{Z_{D} e^{2}} \frac{3 R_{0}^{3}}{a^{2} l_{n,i}} \frac{1}{1 + R_{0}/\lambda}
\]

we substitute and acquire

\[
\nu_{\text{ch}} = \frac{a}{\sqrt{2\pi} \lambda_{D_{i}}} \left\{ 1 + \tau + z + \frac{R_{0}^{2} R_{0}}{a^{2} l_{n,i}} \left[ \tau + \frac{3\tau^{3}}{z} \left( \frac{1}{1 + R_{0}/\lambda} \right) \right] \right\}.
\]

(A.6)

Comparing with the charging frequency in absence of ion-neutral collisions \((l_{n,i} \rightarrow \infty)\), \(\nu_{\text{ch}} = \frac{a}{\sqrt{2\pi} \lambda_{D_{i}}} \left\{ 1 + \tau + z \right\}\), we notice that the charging frequency increases. There are two additional terms: one due to the altered equilibrium condition and the other due to the dependence of the radius \(R_{0}\) on the charge.

Finally, an exact computation can be made in the case \(\beta_{T} \ll 1 \Rightarrow \frac{z_{a}}{2} \ll \lambda\), then \(R_{0} = \frac{z_{a}}{2}\) and \(R_{0} \ll \lambda \Rightarrow 1 + R_{0}/\lambda \simeq 1\),

\[
\nu_{\text{ch}} = \frac{a}{\sqrt{2\pi} \lambda_{D_{i}}} \left\{ 1 + \tau + z + \frac{z^{3} a^{3}}{\tau^{3} l_{n,i} a^{2}} \left[ \tau + \frac{3\tau}{z} \left( \frac{1}{1 + R_{0}/\lambda} \right) \right] \right\}.
\]

(A.7)

**Auxiliary functions for the ion capture cross-sections**

Using the narrowness of the dust distribution around the equilibrium charge we acquire \(\nu_{d,i} = \int \nu_{\sigma_{i}}(q, v) \Phi_{i}^{d}(q) dq d^{3}p' \simeq \nu_{\sigma_{i}}(q_{eq}, v) \int \Phi_{i}^{d}(q) dq d^{3}p' \simeq n d \nu_{\sigma_{i}}(q_{eq}, v)\). For the capture cross-sections of ions on dust, we can either use the O.M.L. model or the collision enhanced collection model (C.E.C).
APPENDIX A. GENERALIZED APPROACH IN THE COMPUTATION OF
THE INTEGRAL RESPONSES

We now express both cross-sections in the $y$-space,

$$
\sigma_i(q_{eq}, y) = \pi a^2 \frac{R_0}{l_{n,i}} + \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 + \frac{2Za^2}{am_i v^2} \right)
$$

$$
= \pi a^2 \frac{R_0}{a^2} \frac{R_0}{l_{n,i}} + \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 + \frac{Za^2 T_e}{T_y \pi} \right)
$$

$$
= \pi a^2 \frac{R_0}{a^2} \frac{R_0}{l_{n,i}} + \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \frac{z}{\tau y} \left( 1 + \frac{\tau}{z} y \right)
$$

$$
= \pi a^2 \frac{z}{\tau y} \left( \frac{\pi a^2}{z} \frac{R_0}{l_{n,i}} + \left( 1 - \frac{R_0}{l_{n,i}} \right) \left( 1 + \frac{\tau}{z} y \right) \right)
$$

while for the O.M.L. cross-sections we use the limit $l_{n,i} \to \infty$ and acquire $\sigma_i(q_{eq}, y) = \pi a^2 \frac{z}{\tau y} \left( 1 + \frac{\tau}{z} y \right)$. We can combine both expressions in a form convenient for the computation of the integral responses, $\sigma_i(q_{eq}, y) = \pi a^2 \frac{z}{\tau y} \sigma_0(y)$, where $\sigma_0(y)$ is the dimensionless $y$-transformed cross-section function defined by

$$
\sigma_0(y) = \begin{cases} 
1 + \frac{z}{\tau} y, & \text{O.M.L} \\
\pi a^2 \frac{z}{\tau y} \left( \frac{\pi a^2}{z} \frac{R_0}{l_{n,i}} + \left( 1 - \frac{R_0}{l_{n,i}} \right) \left( 1 + \frac{\tau}{z} y \right) \right), & \text{C.E.C.}
\end{cases}
$$

(A.8)

In some responses the charge derivative of the cross-section, evaluated at the equilibrium charge, is present. We compute for both charging models and transform in $y$-space,

$$
\frac{\partial \sigma_i(q, v)}{\partial q} = \frac{\partial}{\partial q} \left( \pi R_0^2 \frac{R_0}{l_{n,i}} \right) + \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 - \frac{2Ze a^2}{am_i v^2} \right)
$$

$$
= -3\pi a^2 R_0^2 \frac{\partial R_0}{\partial q} \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 - \frac{2Ze a^2}{am_i v^2} \right)
$$

$$
= -3\pi a^2 R_0^2 \frac{Za^2}{l_{n,i}} \left( \frac{1}{1 + R_0/\lambda} \right) \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 + \frac{2Ze a^2}{am_i v^2} \right) + \frac{R_0}{Za^2} \frac{1}{l_{n,i}} \pi a^2 \left( 1 + \frac{2Ze a^2}{am_i v^2} \right)
$$

$$
= -\pi a^2 R_0^2 \frac{Za^2}{l_{n,i}} \left( \frac{1}{1 + R_0/\lambda} \right) \pi a^2 \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 + \frac{2Ze a^2}{am_i v^2} \right)
$$

$$
= -\pi a^2 \frac{Za^2}{l_{n,i}} \left( \frac{1}{1 + R_0/\lambda} \right) \pi a^2 \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 + \frac{2Ze a^2}{am_i v^2} \right)
$$

while for O.M.L. cross-sections we use the limit $l_{n,i} \to \infty$ and acquire $\sigma_i'(q_{eq}, y) = -\pi a^2 \frac{z}{\tau y} \sigma_0'(y)$, where $\sigma_0'(y)$ is the dimensionless $y$-transformed cross-section derivative function defined by

$$
\sigma_0'(y) = \begin{cases} 
\frac{1}{Za^2} \frac{R_0}{l_{n,i}} \left( \frac{1}{1 + R_0/\lambda} \right) \pi a^2 \left( 1 - \frac{R_0}{l_{n,i}} \right) \pi a^2 \left( 1 + \frac{2Ze a^2}{am_i v^2} \right), & \text{O.M.L} \\
\pi a^2 \frac{z}{\tau y} \left( \frac{\pi a^2}{z} \frac{R_0}{l_{n,i}} + \left( 1 - \frac{R_0}{l_{n,i}} \right) \left( 1 + \frac{\tau}{z} y \right) \right), & \text{C.E.C.}
\end{cases}
$$

(A.9)
A.1. **CALCULATION OF THE ION RESPONSES**

**Auxiliary Functions**

We now define a number of auxiliary functions that appear in the ion integral responses and give their expressions in the \( -y \)-space.

For the inverse tangent kernel we have

\[
\Psi_{\pm}(y) = \arctan \left[ \frac{\omega_r \pm kv}{\omega_i + n_n \sigma_{n,i} v_{Ti} + n_d \sigma_i (q_{eq}, y)} \right]
\]

\[
= \arctan \left[ \frac{\omega_r \pm kv}{\omega_i + n_n \sigma_{n,i} v_{Ti} + n_d \sqrt{2/T_i} \sqrt{\sigma_i (q_{eq}, y)}} \right]
\]

\[
= \arctan \left[ \frac{\omega_r \pm k \sqrt{2v_{Ti} \sqrt{y}}}{\omega_i + n_n \sigma_{n,i} v_{Ti} + n_d \sqrt{2v_{Ti} \sqrt{y} \sigma_i (q_{eq}, y)}} \right]
\]

\[
= \arctan \left[ \frac{\omega_r \pm k \sqrt{2v_{Ti} \sqrt{y}}}{\omega_i + n_n \sigma_{n,i} v_{Ti} + n_d \pi a^2 \tau \sigma_0 (y)} \right]
\]

The above expression appears in the responses as \( \Psi_{\text{tot}} = \Psi_+ - \Psi_- \). In the low frequency limit, \( \omega_r \ll kv_{Ti} \), which yields

\[
\Psi_{\text{tot}}^{LF} = 2 \arctan \left[ \frac{ky}{\sqrt{2v_{Ti}} \sqrt{y} + n_n \pi a^2 \frac{\tau}{y} \sigma_0 (y)} \right]
\]

In the fully ionized case, in absence of collisions with neutrals we additionally have \( \sigma_{n,i} \to 0 \), \( \omega_i \to 0 \) and \( \sigma_0 (y) = 1 + \frac{\tau}{y} \), which yields

\[
\Psi_{\text{tot}}^{LF,f} = 2 \arctan \left[ \frac{ky}{n_d \pi a^2 \frac{\tau}{y} \sigma_0 (y)} \right]
\]

\[
= 2 \arctan \left[ \frac{ky}{n_d \pi a^2 \frac{\tau}{y} (1 + \frac{\tau}{y})} \right]
\]

\[
= 2 \arctan \left[ \frac{\kappa_i y}{1 + \frac{\tau}{y}} \right]
\]

In the latter, we have defined the dimensionless wavenumber \( \kappa_i = \frac{k}{n_d \pi a^2 \frac{\tau}{y}} \), where \( l_{id} = (n_d \pi a^2 \frac{\tau}{y})^{-1} \) is approximately equal to the characteristic ion absorption length.
ions on dust is not important.

In the low frequency limit the expression will remain the same being independent
to define $C$ which clearly vanishes in the low frequency limit.

For the logarithmic kernel we have

$$L(y) = \ln \frac{(\omega_r + k v)^2 + (\nu_i + \nu_{n,i} + \nu_{d,i}(v))^2}{(\omega_r - k v)^2 + (\omega_i + \nu_{n,i} + \nu_{d,i}(v))^2}$$

$$= \ln \frac{(\omega_r + k v)^2 + (\omega_i + \nu_{n,i} + \nu_{d,i}(v))^2}{(\omega_r - k v)^2 + (\omega_i + \nu_{n,i} + \nu_{d,i}(v))^2}$$

$$= \ln \frac{(\omega_r + k \sqrt{2} v T_i \sqrt{\overline{g}})^2 + (\omega_i + n n_{n,i} v T_i + n d \sqrt{2} v T_i \sqrt{\overline{g}} \sigma_i(q_{eq}, y))^2}{(\omega_r - k \sqrt{2} v T_i \sqrt{\overline{g}})^2 + (\omega_i + n n_{n,i} v T_i + n d \sqrt{2} v T_i \sqrt{\overline{g}} \sigma_i(q_{eq}, y))^2}$$

which clearly vanishes in the low frequency limit.

For the total imaginary part of the expression $\omega - k \cdot v + \nu_i(v)$ we have

$$C(y) = \omega_i + \nu_{n,i} + \nu_{d,i}(v)$$

$$= \omega_i + n n_{n,i} v T_i + n d \sigma_i(q_{eq}, y)$$

$$= \sqrt{2} v T_i \left( \frac{\omega_i}{\sqrt{2}} + \frac{n n_{n,i} \sqrt{\overline{g}}}{\sqrt{2}} + n d \pi a^2 \frac{\sigma_i}{\tau} y \right) .$$

For convergence to the expressions of the low frequency responses it is convenient
to define $C(y) = \sqrt{2} v T_i C_k(y)$ or equivalently

$$C_k(y) = \frac{\left( \frac{\omega_i}{\sqrt{2}} + \frac{n n_{n,i} \sqrt{\overline{g}}}{\sqrt{2}} + n d \pi a^2 \frac{\sigma_i}{\tau} y \right)}{k \sqrt{\overline{g}}} .$$

In the low frequency limit the expression will remain the same being independent
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of the frequency, while in the fully ionized case

\[ C_k(y) = \frac{n_d \pi a^2 \hat{z} \sigma_0(y)}{k \sqrt{y}} \]

\[ = \frac{n_d \pi a^2 \hat{z} (1 + \frac{\tau}{y})}{k \sqrt{y}} \]

\[ = 1 + \frac{\tau}{\kappa_i \sqrt{y}}. \]

The integral response \( \chi_{k,\omega} \)

The response is the generalization of the ion susceptibility in collisional plasmas, it is defined by

\[ \chi_{k,\omega} = \frac{4 \pi e^2}{k^2} \int \frac{1}{\omega - k \cdot v + \omega_i(v)} \left( k \cdot \frac{\partial \Phi_p^i}{\partial p} \right) \frac{d^3p}{(2\pi)^3}. \quad (A.10) \]

We use the Maxwellian derivative property, \( \omega = \omega_r + \omega_i \), set \( C(v) = \omega_i + \nu_{n,i} + \nu_{d,i}(v) \) and decompose in real and imaginary parts.

\[ \chi_{k,\omega} = \frac{4 \pi e^2}{k^2} \int \frac{1}{\omega - k \cdot v + \omega_i(v) + \nu_{n,i}} \left( k \cdot \frac{\partial \Phi_p^i}{\partial p} \right) \frac{d^3p}{(2\pi)^3} \]

\[ = \frac{4 \pi e^2}{k^2} \int \frac{1}{\omega_r - k \cdot v + i(\omega_i + \nu_{d,i}(v) + \nu_{n,i})} \left( k \cdot \frac{\partial \Phi_p^i}{\partial p} \right) \frac{d^3p}{(2\pi)^3} \]

\[ = \frac{4 \pi e^2}{T_i k^2} \int \frac{k \cdot v}{\omega_r - k \cdot v + i C(v)} \Phi^i(v) d^3v \]

\[ = \frac{4 \pi e^2}{T_i k^2} \int \frac{(k \cdot v)(\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi^i(v) d^3v \]

\[ + \frac{4 \pi e^2}{T_i k^2} \int \frac{(k \cdot v) C(v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi^i(v) d^3v. \]
For the real part we use spherical coordinates and evaluate the integral through $I_4$ and $I_7$

\[
\Re(\chi_{k_{\omega},\omega}) = -\frac{4\pi e^2}{T_i k^2} \int_{\mathbb{R}^3} \frac{(k \cdot v)(\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi(v) d^3 v
\]

\[
= -\frac{4\pi e^2 }{T_i k^2} \left[ \int_0^\infty \left( \frac{\int_0^\infty k v \sin \theta \cos \theta (\omega_r - kv \cos \theta)}{(\omega_r - kv \cos \theta)^2 + C^2(v)} dv \right) \frac{2\pi v^2 \Phi(v) dv}{\omega_r} \right]
\]

\[
= (4\pi)^2 e^2 \frac{\omega_r}{T_i k^2} \int_0^\infty \frac{v^2 \Phi(v) dv}{\omega_r} - \frac{8\pi^2 e^2}{T_i k^3} \int_0^\infty \frac{v C(v) (\Psi_+(v) - \Psi_-(v)) \Phi(v) dv}{\omega_r} - \frac{8\pi^2 e^2}{T_i k^3} \int_0^\infty \frac{v C(v) |\Psi_+(v)| \Phi(v) dv}{\omega_r} - \frac{8\pi^2 e^2}{T_i k^3} \int_0^\infty \frac{v C(v) (\Psi_+(v) - \Psi_-(v)) \Phi(v) dv}{\omega_r}
\]

\[
= \frac{1}{k^2 \lambda^2_{D_i}} - \frac{1}{\lambda^2_{D_i} k^2} \frac{1}{2\pi k v} \int_0^\infty C(y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy
\]

\[
= \frac{1}{k^2 \lambda^2_{D_i}} - \frac{1}{\lambda^2_{D_i} k^2} \frac{1}{2\pi k v} \int_0^\infty e^{-y} C_k(y) (\Psi_+(y) - \Psi_-(y)) dy
\]

\[
= \frac{1}{k^2 \lambda^2_{D_i}} \left\{ 1 - \frac{1}{\sqrt{v}} \int_0^\infty e^{-y} C_k(y) (\Psi_+(y) - \Psi_-(y)) dy - \frac{\omega_r}{2\sqrt{v} k v T_i} \int_0^\infty e^{-y} L(y) dy \right\}.
\]
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For the imaginary part we evaluate the angular integral through $I_2$

$$
\Im\{\chi_{k,\omega}\} = \frac{4\pi e^2}{T_i k^2} \int_0^\infty \frac{(k \cdot v) C(v)}{(\omega_i - k \cdot v)^2 + C^2(v)} \Phi'(v) dv
$$

$$
= \frac{4\pi e^2}{T_i k^2} \int_0^\infty \left( \frac{\cos \theta \sin \theta}{(\omega_i - kv \cos \theta)^2 + C^2(v)} \right) 2\pi kv C(v) \Phi'(v) dv
$$

$$
= \frac{4\pi e^2}{T_i k^2} \int_0^\infty -\frac{1}{2k^2 v^2} L(v) + \frac{\omega_i}{k^2 v C(v)} (\Psi_+(v) - \Psi_-(v)) 2\pi kv C(v) \Phi'(v) dv
$$

$$
= \frac{4\pi e^2}{T_i k^2} \int_0^\infty v L(v) C(v) \Phi'(v) dv + \frac{4\pi e^2}{T_i k^2} \int_0^\infty \frac{2\pi \omega_i}{k} \int_0^\infty v (\Psi_+(v) - \Psi_-(v)) \Phi'(v) dv
$$

$$
= -\frac{1}{k^2 \lambda_{Di}^2} \frac{\pi}{2} \frac{m_i}{2\pi T_i} \int_0^\infty v L(v) C(v) \exp\left(-\frac{m_i v^2}{2 T_i}\right) dv
$$

$$
+ \frac{4\pi e^2}{T_i k^2} \frac{2\pi \omega_i}{k} \int_0^\infty v (\Psi_+(v) - \Psi_-(v)) \Phi'(v) dv
$$

$$
= -\frac{1}{k^2 \lambda_{Di}^2} \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-y} C(y) L(y) dy
$$

$$
+ \frac{1}{k^2 \lambda_{Di}^2} \frac{2\pi \omega_i}{k} \sqrt{2\pi T_i} \frac{m_i}{2\pi T_i} \int_0^\infty v (\Psi_+(v) - \Psi_-(v)) \exp\left(-\frac{m_i v^2}{2 T_i}\right) dv
$$

$$
= -\frac{1}{k^2 \lambda_{Di}^2} \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-y} C(y) L(y) dy
$$

$$
+ \frac{1}{k^2 \lambda_{Di}^2} \frac{1}{2\sqrt{\pi}} \frac{\omega_i}{\sqrt{2\pi k T_i}} \int_0^\infty e^{-y (\Psi_+(y) - \Psi_-(y))} dy
$$

$$
= \frac{1}{k^2 \lambda_{Di}^2} \left\{ \frac{\omega_i}{\sqrt{2\pi k T_i}} \int_0^\infty e^{-y (\Psi_+(y) - \Psi_-(y))} dy - \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-y} C(y) L(y) dy \right\}.
$$

Overall, we have

$$
\chi_{k,\omega} = \frac{1}{k^2 \lambda_{Di}^2} \left\{ 1 - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} C(y) (\Psi_+(y) - \Psi_-(y)) dy - \frac{\omega_i}{2\sqrt{2\pi k T_i}} \int_0^\infty e^{-y} L(y) dy \right\}
$$

$$
+ \frac{i}{k^2 \lambda_{Di}^2} \left\{ \frac{i \omega_i}{\sqrt{2\pi k T_i}} \int_0^\infty e^{-y (\Psi_+(y) - \Psi_-(y))} dy - \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-y} C(y) L(y) dy \right\}.
$$

(A.11)
In the low frequency limit the imaginary part of the response will vanish, while for the real part we get

\[ \chi_{i,k,\omega} = \frac{1}{k^2 \lambda_D^2} \left\{ 1 - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} C(y) \Psi_{LF}^{\text{tot}} dy \right\} \]

\[ = \frac{1}{k^2 \lambda_D^2} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-y} \left( \frac{\omega_i \sqrt{\pi}}{\sqrt{2} \tau_i} + \frac{n_a \pi a^2 \sqrt{\pi}}{\sqrt{2} \tau_a} \right) \right\} \times \arctan \left[ \frac{ky}{\sqrt{\pi}} + \frac{n_a \pi a^2 \sqrt{\pi}}{\sqrt{2} \tau_a} \right] \]

where the second part of the response expresses deviations from Debye screening due to absorption on dust and collisions with neutrals.

In the fully ionized case, the relation will be further simplified

\[ \chi_{i,k,\omega} = \frac{1}{k^2 \lambda_D^2} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-y} \left( \frac{\omega_i \sqrt{\pi}}{\sqrt{2} \tau_i} + \frac{n_a \pi a^2 \sqrt{\pi}}{\sqrt{2} \tau_a} \right) \right\} \times \arctan \left[ \frac{ky}{\sqrt{\pi}} + \frac{n_a \pi a^2 \sqrt{\pi}}{\sqrt{2} \tau_a} \right] \]

The integral response \( G_{k,\omega} \)

The response is defined by

\[ G_{k,\omega} = \int \frac{1}{i(\omega - k \cdot v + \nu_i(v))} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} \]  

(A.14)

We use \( \omega = \omega_r + i\omega_i \), set \( C(v) = \omega_i + \nu_{n,i} + \nu_{d,i}(v) \) and decompose in real and imaginary parts,

\[ G_{k,\omega} = \int \frac{1}{i(\omega - k \cdot v + \nu_i(v))} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} \]

\[ = - \int \frac{1}{(\omega_i + \nu_{d,i}(v) + \nu_{n,i}) - i(\omega_r - k \cdot v)} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} \]

\[ = - \int \frac{1}{(\omega_i + \nu_{d,i}(v) + \nu_{n,i}) + i(\omega_r - k \cdot v)^2} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} \]

\[ = - \int \frac{C(v) + i(\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} \]

\[ = - \int \frac{C(v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} - i \int \frac{\omega_r - k \cdot v}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_{M}^{P} \frac{d^3p}{(2\pi)^3} \]
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For the real part we use spherical coordinates and evaluate the angular integral through \( I_1 \)

\[
\Re\{G_{k,\omega}\} = -\int \frac{C(v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_p \frac{d^3p}{(2\pi)^3} \\
= -\int_0^\infty \left( \int_0^\pi \frac{\sin \theta}{(\omega_r - k v \cos \theta)^2 + C^2(v)} d\theta \right) 2\pi v^2 C(v) \Phi^M(v) dv \\
= -\frac{2\pi}{k} \int_0^\infty v (\Psi_+(v) - \Psi_-(v)) \Phi^M(v) dv \\
= -\frac{2\pi}{k} \frac{m_i}{2\pi T_i} \frac{3/2}{2} \int_0^\infty v (\Psi_+(v) - \Psi_-(v)) \frac{d\theta}{2\pi T_i} e^{-y} dy \\
= -\frac{1}{\sqrt{2\pi kvT_i}} \int_0^\infty (\Psi_+(v) - \Psi_-(v)) e^{-y} dy
\]

For the imaginary part we evaluate the angular integral through \( I_5 \)

\[
\Im\{G_{k,\omega}\} = -\int \frac{\omega_r - k \cdot v}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_p \frac{d^3p}{(2\pi)^3} \\
= -\int_0^\infty \left( \int_0^\pi \frac{\cos \theta}{(\omega_r - k v \cos \theta)^2 + C^2(v)} d\theta \right) 2\pi v^2 C(v) \Phi^M(v) dv \\
= -\frac{\pi}{k} \int_0^\infty v L(v) \Phi^M(v) dv \\
= -\frac{\pi}{k} \frac{m_i}{2\pi T_i} \frac{3/2}{2} \int_0^\infty v L(v) \frac{d\theta}{2\pi T_i} e^{-y} dy \\
= -\frac{1}{2k} \sqrt{\frac{m_i}{2\pi T_i}} \int_0^\infty L(y) e^{-y} dy \\
= -\frac{1}{2\sqrt{2\pi kvT_i}} \int_0^\infty L(y) e^{-y} dy.
\]
Overall, we get

$$G_{k,\omega} = -\left\{ \frac{1}{\sqrt{2\pi kv_{T_i}}} \int_0^\infty e^{-y} (\Psi_+(y) - \Psi_-(y)) \, dy \right\} - i \left\{ \frac{1}{2\sqrt{2\pi kv_{T_i}}} \int_0^\infty e^{-y} L(y) \, dy \right\} .$$

(A.15)

In the low frequency regime the imaginary part will vanish, while for the real part we have

$$G_{k,\omega} = -\frac{1}{\sqrt{2\pi kv_{T_i}}} \int_0^\infty e^{-y} \Phi_{\text{tot}}(y) \, dy$$

$$= -\frac{2}{\sqrt{2\pi kv_{T_i}}} \int_0^\infty e^{-y} \arctan \left[ \frac{ky}{\omega_i \sqrt{\nu}} + \frac{n_a \sigma_n \sqrt{\nu}}{\sqrt{2}} + n_d \pi a^2 \frac{1}{\tau_0(y)} \right] \, dy$$

$$= -\frac{1}{kv_{T_i}} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y} \arctan \left[ \frac{ky}{\omega_i \sqrt{\nu}} + \frac{n_a \sigma_n \sqrt{\nu}}{\sqrt{2}} + n_d \pi a^2 \frac{1}{\tau_0(y)} \right] \, dy .$$

(A.16)

For the fully ionized case, the response does not exist, being dependent on the distribution function of the neutrals $\Phi_{\text{tot}}^M$. The response is directly related to collisions with neutrals via the B.G.K approximation and electron impact ionization, being always multiplied by $\nu_{n,i}$ or $\nu_e$.

**The integral response $\overline{q}_{k,\omega}(q)$**

The response is defined by

$$\overline{q}_{k,\omega}(q) = \int \frac{e^{i\sigma_1(q,v)}}{i(\omega - k \cdot v + i\nu_i(v))} \Phi_p^i \frac{d^3p}{(2\pi)^3} .$$

(A.17)

It is evaluated for $q = q_{eq}$. Initially, we use $\omega = \omega_r + i\omega_i$, set $C(v) = \omega_i + \nu_{n,i} + \nu_{d,i}(v)$ and decompose in real and imaginary parts,

$$\overline{q}_{k,\omega}(q) = -\int \frac{e^{i\sigma_1(q,v)}}{(\omega_r - k \cdot v + i(\omega_i + \nu_{n,i} + \nu_{d,i}(v)))} \Phi_p^i \frac{d^3p}{(2\pi)^3}$$

$$= -\int \frac{e^{i\sigma_1(q,v)}}{(\omega_r - k \cdot v + iC(v))} \Phi_p^i \frac{d^3p}{(2\pi)^3}$$

$$= -\int \frac{e^{i\sigma_1(q,v)} (C(v) - (\omega_r - k \cdot v))}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_p^i \frac{d^3p}{(2\pi)^3}$$

$$= -\int \frac{e^{i\sigma_1(q,v)} C(v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_p^i \frac{d^3p}{(2\pi)^3} - i \int \frac{e^{i\sigma_1(q,v)} (\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_p^i \frac{d^3p}{(2\pi)^3} .$$

For the real part we use spherical coordinates and evaluate the angular integral through $I_1$. We also use the dimensionless wavenumber $\kappa_i = \frac{k}{n_a \pi a^2 \tau_i}$ and the
relation \(-\frac{en_i}{n_d} = q_{eq} \frac{1 + P}{P}\)

\[\Re\{q_{k,\omega}^2\} = - \int \frac{e v \sigma_i(q, v) C(v)}{(\omega_e - k \cdot v)^2 + C^2(v)} \Phi_i \frac{d^3p}{(2\pi)^3}\]

\[= - \int_0^\infty \left( \int_0^\pi \frac{\sin \theta}{(\omega_e - k v \cos \theta)^2 + C^2(v)} d\theta \right) 2\pi e v \sigma_i(q, v) C(v) \Phi_i(v) dv\]

\[= - \int_0^\infty \frac{2\pi e v \sigma_i(q, v) C(v)}{k v C(v)} (\Psi_+(v) - \Psi_-(v)) \Phi_i(v) dv\]

\[= - \frac{2\pi e}{k} \int_0^\infty v^2 \sigma_i(q, v) (\Psi_+(v) - \Psi_-(v)) \Phi_i(v) dv\]

\[= - \frac{2\pi e n_i}{k} \frac{3}{2} \int_0^\infty v \sigma_i(q, v) (\Psi_+(v) - \Psi_-(v)) \exp \left(-\frac{m_i v^2}{2T_i}\right) dv\]

\[= - \frac{2\pi e n_i}{k} \frac{\sqrt{2}}{2\pi \sqrt{2}} \int_0^\infty \sqrt{\gamma} \sigma_i(q, y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy\]

\[= - \frac{en_i}{n_d} \frac{1}{\sqrt{\pi}} \frac{\pi a^2 z}{\tau} \frac{1}{\kappa \sqrt{\gamma}} \int_0^\infty \frac{\sigma_0(y)}{\sqrt{y}} (\Psi_+(y) - \Psi_-(y)) e^{-y} dy\]

\[= q_{eq} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y) e^{-y}}{\kappa \sqrt{\gamma}} [\Psi_+(y) - \Psi_-(y)] dy.\]

For the imaginary part we evaluate the angular integral through \(I_5\)

\[\Im\{q_{k,\omega}^2\} = - \int \frac{e v \sigma_i(q, v)(\omega_e - k \cdot v)}{(\omega_e - k \cdot v)^2 + C^2(v)} \Phi_i \frac{d^3p}{(2\pi)^3}\]

\[= - \int_0^\infty \left( \int_0^\pi \frac{\omega_e - k v \cos \theta}{(\omega_e - k v \cos \theta)^2 + C^2(v)} d\theta \right) 2\pi e v \sigma_i(q, v) C(v) \Phi_i(v) dv\]

\[= - \int_0^\infty \frac{2\pi e v \sigma_i(q, v) C(v)}{k v} L(v) \Phi_i(v) dv\]

\[= - \frac{\pi e n_i}{k} \frac{3}{2} \int_0^\infty v \sigma_i(q, v) L(v) \exp \left(-\frac{m_i v^2}{2T_i}\right) dv\]

\[= - \frac{en_i}{n_d} \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{\gamma} \sigma_i(y) L(y) e^{-y} dy\]

\[= - \frac{en_i}{n_d} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{n_d a^2 z}{\kappa \sqrt{\gamma}} \frac{\sigma_0(y)}{y} L(y) dy\]

\[= q_{eq} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y) e^{-y}}{\kappa \sqrt{\gamma}} L(y) dy.\]
Overall, we get
\[ \tilde{q}_{k,\omega} = \left\{ \frac{q_{eq}}{\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} \left[ \Psi_+(y) - \Psi_-(y) \right] dy \right\} + i \left\{ \frac{q_{eq}}{2\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} L(y) dy \right\}. \] (A.18)

In the low frequency regime the imaginary part will vanish, while for the real part we have
\[ \tilde{q}_{k,\omega} = \frac{q_{eq}}{\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} \Psi_{LF} dy \]
\[ = \frac{2q_{eq}}{\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} \arctan \left[ \frac{ky}{\omega_i\sqrt{\pi} + \frac{n_0\sigma_y\sqrt{\pi}}{\sqrt{2}} + n_d\frac{a^2}{\pi} \sigma(y)} \right] dy. \] (A.19)

In the fully ionized case the response will become
\[ \tilde{q}_{k,\omega} = \frac{2q_{eq}}{\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty e^{-y} \frac{1 + \frac{\kappa_i}{\kappa_i\sqrt{y}}}{\sqrt{\pi}} \arctan \left[ \frac{\kappa_i\sqrt{y}}{1 + \frac{\kappa_i}{\sqrt{y}}} \right] dy. \] (A.20)

### The integral response \( \lambda_{k,\omega}(q) \)

The response is defined by
\[ \lambda_{k,\omega}(q) = \int \frac{e^{i\kappa_i(q,v)}}{i(\omega - k \cdot v + i\nu_1(v))} \Phi_M \frac{d^3p}{(2\pi)^3}. \] (A.21)

It is evaluated for \( q = q_{eq} \). From a simple inspection we notice that it is exactly the same with \( \tilde{q}_{k,\omega}(q) \), but instead of the ion distribution we have the normalized distribution of the neutrals. Since, \( \Phi_M = \frac{\Phi_k}{n_i} \), we will also have \( \lambda_{k,\omega} = \frac{\tilde{q}_{k,\omega}}{n_i} \), i.e
\[ \lambda_{k,\omega} = \left\{ \frac{q_{eq}}{n_i\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} \left[ \Psi_+(y) - \Psi_-(y) \right] dy \right\} + i \left\{ \frac{q_{eq}}{2n_i\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} L(y) dy \right\}. \] (A.22)

In the low frequency regime the imaginary part will vanish, while for the real part we have
\[ \lambda_{k,\omega} = \frac{q_{eq}}{n_i\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} \Psi_{LF} dy \]
\[ = \frac{2q_{eq}}{n_i\sqrt{\pi}} \frac{1 + P}{P} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\kappa_i\sqrt{y}} \arctan \left[ \frac{ky}{\omega_i\sqrt{\pi} + \frac{n_0\sigma_y\sqrt{\pi}}{\sqrt{2}} + n_d\frac{a^2}{\pi} \sigma(y)} \right] dy. \] (A.23)
A.1. CALCULATION OF THE ION RESPONSES

For the fully ionized case, the responses do not exist, being dependent on the distribution function of the neutrals \( \Phi^i_n \). The response is directly related to collisions with neutrals via the B.G.K approximation and electron impact ionization, being always multiplied by \( \nu_n \) or \( \nu_e \).

**The integral response \( \tilde{\beta}_{k,\omega}(q) \)**

The response is defined by

\[
\tilde{\beta}_{k,\omega}(q) = \int \frac{e \nu v \sigma^i(q,v)}{i (\omega - k \cdot v + i \nu_i(v))} \Phi^i_p \frac{d^3p}{(2\pi)^3}.
\]  

(A.24)

It is evaluated for \( q = q_{eq} \). We decompose in real and imaginary parts.

\[
\Re\{\tilde{\beta}_{k,\omega}(q)\} = - \int \frac{e \nu v \sigma^i(q_{eq},v)}{(\omega - k \cdot v)^2 + \nu^2} \Phi^i_p \frac{d^3p}{(2\pi)^3} \]

\[
= -e \int_0^\infty \left\{ \int_0^{\pi} \frac{\sin \theta}{(\omega_
u - kv \cos \theta)^2 + \nu^2} d\theta \right\} 2\pi v^3 \sigma^i(q_{eq},v) C(v) \Phi^i(v) dv
\]

\[
= -e \int_0^\infty \frac{1}{k C(v)} (\Psi_+(v) - \Psi_-(v)) 2\pi v^3 \sigma^i(q_{eq},v) C(v) \Phi^i(v) dv
\]

\[
= -\frac{2\pi e}{k} \int_0^\infty (\Psi_+(v) - \Psi_-(v)) v^2 \sigma^i(q_{eq},v) \Phi^i(v) dv
\]

\[
= -\frac{2\pi n_e}{k} \int_0^\infty \frac{1}{2\pi T_i^{3/2}} \left( \int_0^\infty (\Psi_+(v) - \Psi_-(v)) \nu \sigma^i_n(q_{eq},v) v \exp \left( -\frac{m_i v^2}{2T_i} \right) dv \right) dv
\]

\[
= -\frac{2\pi a_i e}{k} \frac{1}{2\pi \sqrt{\pi}} \int_0^\infty (\Psi_+(y) - \Psi_-(y)) \sqrt{\nu} \sigma^i_n(q_{eq},y) e^{-\nu} dy
\]

\[
= \frac{n_i}{Z_{d,n,k}} \frac{n_d n_e a^2}{\sqrt{\pi}} \int_0^\infty (\Psi_+(y) - \Psi_-(y)) \nu \sigma^i_n(y) e^{-\nu} dy
\]

\[
= \frac{n_i n_e}{Z_{d,n,k}} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\sigma^i_n(y) e^{-\nu}}{\nu \sqrt{\pi}} (\Psi_+(y) - \Psi_-(y)) dy
\]

\[
= \frac{1 + P}{P} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\sigma^i_n(y) e^{-\nu}}{\nu \sqrt{\pi}} (\Psi_+(y) - \Psi_-(y)) dy.
\]
APPENDIX A. GENERALIZED APPROACH IN THE COMPUTATION OF THE INTEGRAL RESPONSES

For the imaginary part we evaluate the angular integral through $I_5$

$$\Im\{\tilde{\beta}_{k,\omega}(q)\} = -\int \frac{\sin(\theta)}{\omega - kv \cos \theta} \left\{ \frac{e^{i q (\omega - k \cdot v)}}{(\omega - k \cdot v)^2 + C^2(v)} \Phi^i_p \right\}^2 (2\pi)^3$$

$$= -2\pi e^\frac{1}{2} \int_0^\infty \frac{1}{2\pi} L(v) v^3 \sigma_i'(q_{eq}, v) \Phi^i(v) dv$$

$$= -\frac{\pi n_i}{k} \left( m_i + \frac{1}{2\pi k} \right)^{3/2} \int_0^\infty v L(v) \sigma_i'(q_{eq}, v) \exp\left( -\frac{m_i v^2}{2T_i} \right) dv$$

$$= -\frac{\pi n_i}{k} \frac{1}{2\pi \sqrt{\pi}} \int_0^\infty \sqrt{\pi} \sigma_i'(q_{eq}, y) \exp\left( -y \right) L(y) dy$$

$$= \frac{n_i}{k} \frac{1}{2\pi \sqrt{\pi}} \int_0^\infty \frac{\sigma_i'(y) e^{-y}}{\sqrt{\pi}} L(y) dy$$

Overall, we have

$$\tilde{\beta}_{k,\omega}(q) = 1 + \frac{P}{\sqrt{\pi}} \left\{ \int_0^\infty \frac{\sigma_0'(y) e^{-y}}{\sqrt{\pi}} \Psi_{tot} dy \right\} + i \left\{ \int_0^\infty \frac{\sigma_0'(y) e^{-y}}{\sqrt{\pi}} L(y) dy \right\}$$

In the low frequency regime the imaginary part will vanish, while for the real part we have

$$\tilde{\beta}_{k,\omega}(q) = 1 + \frac{P}{\sqrt{\pi}} \int_0^\infty \frac{\sigma_0'(y) e^{-y}}{\sqrt{\pi}} \Psi_{tot} dy$$

$$= 1 + \frac{P}{\sqrt{\pi}} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sigma_0'(y) e^{-y}}{\sqrt{\pi}} \arctan \left[ \frac{ky}{\sqrt{2\pi} \tau} + \frac{n_i \sigma_i \sqrt{\pi}^2}{\sqrt{2}} + n_d \pi \alpha^2 \pi \sigma_0(y) \right] dy.$$  

(A.25)

In the fully ionized case the response can be further simplified

$$\tilde{\beta}_{k,\omega}(q) = 1 + \frac{P}{\sqrt{\pi}} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y}}{\sqrt{\pi}} \arctan \left[ \frac{ky}{1 + \frac{3}{y}} \right] dy.$$  

(A.26)
A.1. CALCULATION OF THE ION RESPONSES

The integral response $S_{k, \omega}^I(q)$

The response is defined by

$$S_{k, \omega}^I(q) = e^2 \int \frac{v \sigma_i(q, v)}{i(\omega - k \cdot v + \nu_i(v))} \frac{k}{k} \frac{\partial \Phi_i}{\partial p} \frac{d^3 p}{(2\pi)^3} \tag{A.28}$$

It is evaluated for $q = q_{eq}$. We use the Maxwellian derivative property and decompose in real and imaginary parts.

$$S_{k, \omega}^I(q) = \frac{e^2}{kT_i} \int \frac{v \sigma_i(q_{eq}, v)(k \cdot v)}{i(\omega - k \cdot v + i(\nu_{d,v}(v) + \nu_{n,v}))} \Phi_i(v) d^3 v$$

$$= \frac{e^2}{kT_i} \int \frac{v \sigma_i(q_{eq}, v)(k \cdot v)}{i(\omega_r - k \cdot v - C(v))} \Phi_i(v) d^3 v$$

$$= \frac{e^2}{kT_i} \int \frac{v \sigma_i(q_{eq}, v)(k \cdot v)(C(v) + i(\omega_r - k \cdot v))}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_i(v) d^3 v.$$ 

For the imaginary part we evaluate the angular integral through $I_4$ and $I_6$

$$\Im(S_{k, \omega}^I(q)) = \frac{e^2}{kT_i} \int \frac{v \sigma_i(q_{eq}, v)(k \cdot v)(\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi_i(v) d^3 v$$

$$= \frac{e^2}{kT_i} \int_0^\infty \left\{ \int_0^\pi \frac{k v \cos \theta \sin \theta (\omega_r - k v \cos \theta)}{(\omega_r - k v \cos \theta)^2 + C^2(v)} d\theta \right\} 2\pi v^3 \sigma_i(q_{eq}, v) \Phi_i(v) d v$$

$$= \frac{e^2}{kT_i} \int_0^\infty \left\{ -2 + \frac{C(v)}{k v} (\Psi_+(v) - \Psi_-(v)) + \frac{\omega_r}{2 k v} L(v) \right\} 2\pi v^3 \sigma_i(q_{eq}, v) \Phi_i(v) d v$$

$$= \frac{e^2}{kT_i} \int_0^\infty 4\pi v^3 \sigma_i(q_{eq}, v) \Phi_i(v) d v$$

$$+ \frac{e^2}{kT_i} \int_0^\infty \left\{ \frac{C(v)}{k v} (\Psi_+(v) - \Psi_-(v)) + \frac{\omega_r}{2 k v} L(v) \right\} 2\pi v^3 \sigma_i(q_{eq}, v) \Phi_i(v) d v$$

$$= \frac{4\pi^2 n_i}{kT_i} (\frac{m_i}{2\pi T_i})^{3/2} \int_0^\infty v^2 \sigma_i(q_{eq}, v) \exp \left(-\frac{m_i v^2}{2T_i}\right) dv$$

$$+ \frac{e^2}{kT_i} \int_0^\infty \left\{ \frac{C(v)}{k v} (\Psi_+(v) - \Psi_-(v)) + \frac{\omega_r}{2 k v} L(v) \right\} 2\pi v^3 \sigma_i(q_{eq}, v) \Phi_i(v) d v$$

$$= -\frac{1}{k\lambda_D^2} \sqrt{\frac{T_i}{m_i}} \frac{2}{2\sqrt{\pi}} \int_0^\infty y \sigma_i(q_{eq}, y) e^{-y} dy$$

$$+ \frac{e^2}{kT_i} \int_0^\infty \left\{ \frac{C(v)}{k v} (\Psi_+(v) - \Psi_-(v)) + \frac{\omega_r}{2 k v} L(v) \right\} 2\pi v^3 \sigma_i(q_{eq}, v) \Phi_i(v) d v.$$
APPENDIX A. GENERALIZED APPROACH IN THE COMPUTATION OF THE INTEGRAL RESPONSES

\[
\Im(S_k^I(q)) = -\frac{1}{\kappa \lambda^2_{D_1}} \sqrt{\frac{\tau_1}{m_1}} \frac{2}{2\pi \sqrt{2\pi}} \frac{\pi a^2 z}{\tau} \int_0^\infty \sigma_0(y)e^{-y}dy \\
+ \frac{\omega^2}{k^2 T_i} \int_0^\infty \left\{ C(v) \left( \Psi_+ (v) - \Psi_- (v) \right) + \frac{\omega^2}{2k^2} L(v) \right\} 2\pi r^3 \sigma_{r}(q_{eq}, v) \Phi(v)dv \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) - \frac{2\pi r^2}{k^2 T_i} \int_0^\infty C(v) \left( \Psi_+ (v) - \Psi_- (v) \right) v^2 \sigma_{r}(q_{eq}, v) \Phi(v)dv \\
+ \frac{\pi \omega^2}{k^2 T_i} \int_0^\infty L(v)v^2 \sigma_{r}(q_{eq}, v) \Phi(v)dv \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) - \frac{2\pi m_1 \omega^2}{k^2 T_i} \left( \frac{m_1}{2\pi T_i} \right)^{3/2} \int_0^\infty C(v) \Phi_+(v) \\
- \Psi_-(v) \sigma_{r}(q_{eq}, v) \exp \left( -\frac{m_1 \omega^2}{2k^2} \right) dv + \frac{\pi \omega^2}{k^2 T_i} \int_0^\infty L(v)v^2 \sigma_{r}(q_{eq}, v) \Phi_+(v)dv \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) - \frac{1}{2k^2 \lambda_{D_1}^2} \frac{1}{2\pi^{3/2}} \int_0^\infty C(y) \Phi_+(y) \\
- \Psi_- (y) \sigma_0(y) e^{-y} dy + \frac{\pi \omega^2}{k^2 T_i} \int_0^\infty L(y)v^2 \sigma_{r}(q_{eq}, v) \Phi_+(y)dv \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) + \frac{1}{2k^2 \lambda_{D_1}^2} \frac{1}{2\pi^{3/2}} \int_0^\infty C(y) \Phi_+(y) \\
- \Psi_- (y) \sigma_0(y) e^{-y} dy + \frac{\omega^2}{4k^4 \lambda_{D_1}^4} \left( \frac{m_1}{2\pi T_i} \right)^{3/2} \int_0^\infty L(y) \sigma_{r}(q_{eq}, v) \exp \left( -\frac{m_1 \omega^2}{2k^2} \right) dv \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) + \frac{1}{2k^2 \lambda_{D_1}^2} \frac{1}{2\pi^{3/2}} \int_0^\infty L(y) \sigma_{r}(q_{eq}, v) \exp \left( -\frac{m_1 \omega^2}{2k^2} \right) dv \\
- \Psi_- (y) \sigma_0(y) e^{-y} dy + \frac{\omega^2}{4k^4 \lambda_{D_1}^4} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} C_k(y) \Phi_+(y) \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) - \frac{1}{2k^2 \lambda_{D_1}^2} \frac{1}{2\pi^{3/2}} \int_0^\infty L(y) \sigma_{r}(q_{eq}, v) \exp \left( -\frac{m_1 \omega^2}{2k^2} \right) dv \\
- \Psi_- (y) \sigma_0(y) e^{-y} dy + \frac{\omega^2}{4k^4 \lambda_{D_1}^4} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} L(y)dy \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) + \frac{1}{2k^2 \lambda_{D_1}^2} \frac{1}{2\pi^{3/2}} \int_0^\infty L(y) \sigma_{r}(q_{eq}, v) \exp \left( -\frac{m_1 \omega^2}{2k^2} \right) dv \\
- \Psi_- (y) \sigma_0(y) e^{-y} dy + \frac{\omega^2}{4k^4 \lambda_{D_1}^4} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} L(y)dy \\
= -\frac{1}{\kappa \lambda^2_{D_1}} \frac{a^2}{\sqrt{2\pi}} \int_0^\infty \sigma_0(1) - \frac{1}{2} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} C_k(y) \Phi_+(y) - \Psi_- (y) \sigma_0(y) e^{-y} dy \\
- \frac{\omega^2}{4} \frac{1}{\sqrt{2k^2 T_i}} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} L(y)dy \right\} .
A.1. Calculation of the Ion Responses

For the real part we evaluate the angular integral through $I_2$

$$
\Re\{ S_{k,q}(q) \} = \frac{e^2}{kT_i} \int_0^\infty \frac{\sigma_i(q,v)(k \cdot v)C(v)}{\omega_r - k \cdot v + C^2(v)} \Phi'(v) dv
$$

$$
= \frac{e^2}{kT_i} \int_0^\infty \left\{ \int_0^\pi \frac{\cos \theta \sin \theta}{(\omega_r - k \cdot v \cos \theta)^2 + C^2(v)} d\theta \right\} 2\pi k v^2 \sigma_i(q,v)C(v)\Phi'(v) dv
$$

$$
= \frac{e^2}{k^2T_i} \int_0^\infty \left\{ \int_0^\pi \frac{1}{2} L(v) + \frac{\omega_r}{C(v)} (\Psi_+(v) - \Psi_-(v)) \right\} 2\pi k v^2 \sigma_i(q,v)C(v)\Phi'(v) dv
$$

$$
= -\frac{\pi e^2}{k^2T_i} \int_0^\infty \frac{1}{2} L(v) v^2 \sigma_i(q,v)C(v)\Phi'(v) dv
$$

$$
+ \frac{2\pi \omega_r e^2}{k^2T_i} \int_0^\infty (\Psi_+(v) - \Psi_-(v)) v^2 \sigma_i(q,v)\Phi'(v) dv
$$

$$
= \frac{4\pi m_i e^2}{k^2T_i} \left( m_i \right)^{3/2} \int_0^\infty L(v) v \sigma_i(q,v)C(v) \exp \left( \frac{m_i v^2}{2T_i} \right) dv
$$

$$
+ \frac{2\pi \omega_r e^2}{k^2T_i} \int_0^\infty (\Psi_+(v) - \Psi_-(v)) v^2 \sigma_i(q,v)\Phi'(v) dv
$$

$$
= \frac{1}{4k^2 \lambda_{Dl}^2} \frac{d^2}{d^2} \int_0^\infty L(y) C(y) \frac{\sigma_0(y) e^{-y}}{\sqrt{y}} dy
$$

$$
+ \frac{2\pi \omega_r e^2}{k^2T_i} \int_0^\infty (\Psi_+(v) - \Psi_-(v)) v^2 \sigma_i(q,v)\Phi'(v) dv
$$

$$
= \frac{1}{4k^2 \lambda_{Dl}^2} \frac{d^2}{d^2} \int_0^\infty L(y) C(y) \frac{\sigma_0(y) e^{-y}}{\sqrt{y}} dy
$$

$$
+ \frac{\omega_r}{2k^2 \lambda_{Dl}^2} \frac{1}{2\pi} \int_0^\infty \sqrt{y} \sigma_i(q,v,y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy
$$

$$
= \frac{1}{4k^2 \lambda_{Dl}^2} \frac{d^2}{d^2} \int_0^\infty L(y) C(y) \frac{\sigma_0(y) e^{-y}}{\sqrt{y}} dy
$$

$$
+ \frac{\omega_r}{2k^2 \lambda_{Dl}^2} \frac{1}{2\pi} \int_0^\infty \sqrt{y} \sigma_i(q,v,y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy
$$

$$
= \frac{1}{4k^2 \lambda_{Dl}^2} \frac{d^2}{d^2} \int_0^\infty L(y) C(y) \frac{\sigma_0(y) e^{-y}}{\sqrt{y}} dy
$$

$$
+ \frac{\omega_r}{2k^2 \lambda_{Dl}^2} \frac{1}{2\pi} \int_0^\infty \sqrt{y} \sigma_i(q,v,y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy
$$

$$
= \frac{1}{2k \lambda_{Dl}^2} \frac{d^2}{d^2} \int_0^\infty \frac{\omega_r}{\sqrt{2kT_i}} \int_0^\infty \sqrt{y} \sigma_i(q,v,y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy
$$

$$
+ \frac{\omega_r}{2kT_i} \int_0^\infty \sqrt{y} \sigma_i(q,v,y) (\Psi_+(y) - \Psi_-(y)) e^{-y} dy
$$

$$
= \frac{1}{2k \lambda_{Dl}^2} \frac{d^2}{d^2} \int_0^\infty \frac{\sigma_0(y) e^{-y}}{\sqrt{y}} (\Psi_+(y) - \Psi_-(y)) dy
$$

$$
+ \frac{\omega_r}{2kT_i} \int_0^\infty \frac{\sigma_0(y) e^{-y}}{\sqrt{y}} (\Psi_+(y) - \Psi_-(y)) dy
$$
Overall, we get

\[ S_{k,\omega}(q) = \frac{1}{2k\lambda_1} \frac{a^2}{\sqrt{2\pi}} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} C_k(y)L(y)dy 
+ \frac{\omega}{\sqrt{2\pi}} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} (\Psi_+(y) - \Psi_-(y))dy 
- \frac{1}{2} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} (\Psi_+(y) - \Psi_-(y))dy 
- \frac{1}{2} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} L(y)dy. \]

(A.29)

In the low frequency regime the real part will vanish, while for the imaginary part we have

\[ S_{k,\omega}(q) = -i \left\{ \frac{1}{2k\lambda_1} \frac{a^2}{\sqrt{2\pi}} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} C_k(y)\Psi^{PF}_{kd,i}dy \right\} 
= -i \left\{ \frac{1}{2k\lambda_1} \frac{a^2}{\sqrt{2\pi}} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} \left( \frac{\omega\sqrt{y}}{\sqrt{2\pi}} + \frac{n_{n,i}\sqrt{y}}{\sqrt{2}} + n_d\sigma^2 \frac{\gamma}{\pi} \right)dy \right\} \times \arctan \left[ \frac{ky}{\sqrt{(\omega\sqrt{y})^2 + n_{n,i}\sqrt{y} + n_d\sigma^2 \frac{\gamma}{\pi}}} \right]. \]

(A.30)

In the fully ionized case the response will be further simplified

\[ S_{k,\omega}(q) = -i \left\{ \frac{1}{2k\lambda_1} \frac{a^2}{\sqrt{2\pi}} \int_0^{\pi} \int_0^\infty \frac{\sigma_0(y)e^{-y}}{\sqrt{y}} (1 + \frac{\gamma}{\kappa y}) \arctan \left[ \frac{\kappa y}{1 + \frac{\gamma}{\kappa y}} \right] dy \right\}. \]

(A.31)

The integral response \( \tilde{S}_{k,\omega}(q, q') \)

The response is defined by

\[ \tilde{S}_{k,\omega}(q, q') = \int \frac{\sigma^2 \sigma_s(q, v) \sigma_s(q', v)}{i(\omega - k \cdot v + iv_i(v))} \Phi^i_p \frac{d^3p}{(2\pi)^3}. \]

(A.32)

It is evaluated for \( q = q' = q_{eq} \). We decompose in real and imaginary parts.

\[ \tilde{S}_{k,\omega}(q, q') = -e \int \frac{\sigma^2 \sigma_s^2(q_{eq}, v)}{(\omega_r - k \cdot v + i(\omega_i + \nu_{in,i} + \nu_{i}(v)))} \Phi^i_p \frac{d^3p}{(2\pi)^3} \]

\[ = -e \int \frac{\sigma^2 \sigma_s^2(q_{eq}, v)}{(\omega_r - k \cdot v - iC(v))} \Phi^i_p \frac{d^3p}{(2\pi)^3} \]

\[ = -e \int \frac{\sigma^2 \sigma_s^2(q_{eq}, v)C(v)\Phi^i_p \frac{d^3p}{(2\pi)^3}}{(\omega_r - k \cdot v)^2 + C^2(v) - 1e} \int \frac{\sigma^2 \sigma_s^2(q_{eq}, v)(\omega_r - k \cdot v)\Phi^i_p \frac{d^3p}{(2\pi)^3}}{(\omega_r - k \cdot v)^2 + C^2(v)}. \]
For the real part we evaluate the angular integral through $I_1$

$$\Re\{\tilde{S}_{k,\omega}(q, q')\} = -e \int_0^{\infty} \frac{v^2 \sigma_1^2(q, v) C(v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi'_p d^3p$$

$$= -e \int_0^{\infty} \left( \int_0^{\infty} \left( (\omega_r - k v \cos \theta)^2 + C^2(v) \right. \right. \sin \theta \left. \right) d\theta \right) 2\pi v^4 \sigma_1^2(q, v) C(v) \Phi'(v) dv$$

$$= -e \int_0^{\infty} \left( \int_0^{\infty} \left( (\omega_r - k v \cos \theta)^2 + C^2(v) \right. \right. \sin \theta \left. \right) d\theta \right) 2\pi v^4 \sigma_1^2(q, v) C(v) \Phi'(v) [\Psi_+(v) - \Psi_-(v)] dv$$

$$= -e \int_0^{\infty} \left( \int_0^{\infty} \left( (\omega_r - k v \cos \theta)^2 + C^2(v) \right. \right. \sin \theta \left. \right) d\theta \right) 2\pi v^4 \sigma_1^2(q, v) C(v) \Phi'(v) [\Psi_+(v) - \Psi_-(v)] dv$$

$$= -\frac{2\pi n_i e}{k} \frac{2}{2\pi} \sqrt{T_1 m_i} \int_0^{\infty} y \sigma_1^2(q, y) e^{-y} [\Psi_+(y) - \Psi_-(y)] dy$$

For the imaginary part we evaluate the angular integral through $I_5$

$$\Im\{\tilde{S}_{k,\omega}(q, q')\} = -e \int_0^{\infty} \frac{v^2 \sigma_1^2(q, v) (\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + C^2(v)} \Phi'_p d^3p$$

$$= -e \int_0^{\infty} \left( \int_0^{\infty} \left( (\omega_r - k v \cos \theta)^2 + C^2(v) \right. \right. \sin \theta \left. \right) d\theta \right) 2\pi v^4 \sigma_1^2(q, v) \Phi'(v) dv$$

$$= -e \int_0^{\infty} \left( \int_0^{\infty} \left( (\omega_r - k v \cos \theta)^2 + C^2(v) \right. \right. \sin \theta \left. \right) d\theta \right) 2\pi v^4 \sigma_1^2(q, v) \Phi'(v) L(v) \Phi'(v) dv$$

$$= -\frac{\pi n_i e}{k} \frac{2}{2\pi} \sqrt{T_1 m_i} \int_0^{\infty} y \sigma_1^2(q, y) L(y) e^{-y} dy$$

$$= -\frac{n_i e}{n_d} \frac{2}{2\pi} \sqrt{T_1 m_i} \int_0^{\infty} \frac{e^{-y} \sigma_1^2(q, y)}{\kappa_i y} L(y) dy$$

$$= q_{eq} \frac{1 + P}{P} \frac{\pi a^2 v T_1 \frac{\tau_1}{\sqrt{2\pi}}}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-y} \sigma_1^2(q, y)}{\kappa_i y} L(y) dy.$$
Overall, we get

\[ S_{k,\omega}(q,q') = \left\{ q_{eq} \frac{1 + P \frac{2\pi a^2 v_{Ti} \omega}{\sqrt{2\pi}}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-y} \sigma_{0}^{2}(y)}{\kappa_{i} y} \left[ \Psi_{+}(y) - \Psi_{-}(y) \right] dy \right\} \]

\[ + i \left\{ q_{eq} \frac{1 + P \frac{\pi a^2 v_{Ti} \omega}{\sqrt{2\pi}}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-y} \sigma_{0}^{2}(y)}{\kappa_{i} y} L(y) dy \right\} . \]  

(A.33)

In the low frequency regime the imaginary part will vanish, while for the real part we have

\[ S_{k,\omega}(q,q') = q_{eq} \frac{1 + P \frac{2\pi a^2 v_{Ti} \omega}{\sqrt{2\pi}}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-y} \sigma_{0}^{2}(y)}{\kappa_{i} y} \Psi_{L,F} \]  

\[ \times \arctan \left[ \frac{ky}{\omega_{i} \sqrt{a} + n_{e} \sigma_{n} \sqrt{a}} + n_{d} \pi a^2 \sigma_{0}(y) \right] dy . \]  

(A.34)

In the fully ionized case the above expression will be further simplified to

\[ S_{k,\omega}(q,q') = q_{eq} \frac{1 + P \frac{4\pi a^2 v_{Ti} \omega}{\sqrt{2\pi}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} \left( 1 + \frac{2}{y} \right)^{2} \arctan \left[ \frac{\kappa_{i} y}{1 + \frac{2}{y}} \right] dy . \]  

(A.35)

**The integral response** \( S_{k,\omega}(q,q') \)

The response is defined by

\[ S_{k,\omega}(q,q') = e \int \frac{v^2 \sigma_{i}(q,v) \sigma_{i}^{'}(q',v)}{i(\omega_{r} - k \cdot v + iC(v))} \Phi_{i} \frac{d^{3}p}{(2\pi)^{3}} . \]  

(A.36)

It is evaluated for \( q = q' = q_{eq} \). We decompose in real and imaginary parts.

\[ S_{k,\omega}(q,q') = -e \int \frac{v^2 \sigma_{i}(q_{eq},v) \sigma_{i}^{'}(q_{eq},v)}{(\omega_{r} - k \cdot v + iC(v))} \Phi_{i} \frac{d^{3}p}{(2\pi)^{3}} \]

\[ = -e \int \frac{v^2 \sigma_{i}(q_{eq},v) \sigma_{i}^{'}(q_{eq},v) \left( \omega_{r} - k \cdot v - iC(v) \right) \Phi_{i} \frac{d^{3}p}{(2\pi)^{3}}}{(\omega_{r} - k \cdot v)^2 + C^2(v)} \]

\[ = -e \int \frac{v^2 \sigma_{i}(q_{eq},v) \sigma_{i}^{'}(q_{eq},v) C(v) \Phi_{i} \frac{d^{3}p}{(2\pi)^{3}}}{(\omega_{r} - k \cdot v)^2 + C^2(v)} \]

\[ - e \int \frac{v^2 \sigma_{i}(q_{eq},v) \sigma_{i}^{'}(q_{eq},v) \left( \omega_{r} - k \cdot v \right) \Phi_{i} \frac{d^{3}p}{(2\pi)^{3}}}{(\omega_{r} - k \cdot v)^2 + C^2(v)} . \]
For the real part we evaluate the angular integral through $I_1$ and also use \( \frac{\partial}{\partial z_v} \frac{2\pi a^2 v_T}{\pi} \frac{d}{d\pi^2} \)

\[
\Re\{\mathcal{S}_{k,\omega}(q,q')\} = -e \int v^2 \sigma_i(q, q', v) \sigma_i'(q, q', v) C(v) \Phi_i^* \phi^2 \pi^3
\]

\[
= -e \int_0^\infty \left( \int_0^\infty \frac{\sin \theta}{(\omega - k v \cos \theta)^2 + C^2(v)} d\theta \right) \int_0^\infty 2\pi v^4 \sigma_i(q, q, v) \sigma_i'(q, q, v) C(v) \Phi_i^* (v) dv
\]

\[
= -e \int_0^\infty \int_0^\infty 2\pi v^3 \sigma_i(q, q, v) \sigma_i'(q, q, v) \Phi_i^* (v) \Phi_i (v) \left[ \Psi_+(v) - \Psi_-(v) \right] dv
\]

\[
= -2\pi n_i e \rho_m \left( \frac{m_i}{2\pi T_i} \right)^{3/2} \int_0^\infty v^2 \sigma_i(q, q, v) \sigma_i'(q, q, v) \exp \left( -\frac{m_i v^2}{2T_i} \right) \left[ \Psi_+(v) - \Psi_-(v) \right] dv
\]

\[
= -2\pi n_i e \rho_m \frac{2}{2\pi \sqrt{T_i m_i}} \int_0^\infty v^2 \sigma_i(q, q, v) \sigma_i'(q, q, v) \exp \left( -\frac{m_i v^2}{2T_i} \right) \left[ \Psi_+(v) - \Psi_-(v) \right] dv
\]

\[
= \frac{n_i}{Z a n_d} \frac{1}{2\pi \sqrt{\tau T_i}} \frac{a}{2\pi} \int_0^\infty e^{-\gamma v} \sigma_0(v) \sigma_0'(v) \exp \left( -\frac{m_i v^2}{2T_i} \right) \left[ \Psi_+(v) - \Psi_-(v) \right] dv
\]

\[
= \frac{2\pi n_i a}{m_i v T_i} \int_0^\infty e^{-\gamma v} \sigma_0(v) \sigma_0'(v) \exp \left( -\frac{m_i v^2}{2T_i} \right) \left[ \Psi_+(v) - \Psi_-(v) \right] dv
\]

For the imaginary part we evaluate the angular integral through $I_5$

\[
\Im\{\mathcal{S}_{k,\omega}(q,q')\} = -e \int v^2 \sigma_i(q, q', v) \sigma_i'(q, q', v) (\omega - k \cdot v) C(v) \Phi_i^* \phi^2 \pi^3
\]

\[
= -e \int_0^\infty \left( \int_0^\infty \frac{\sin \theta}{(\omega - k v \cos \theta)^2 + C^2(v)} d\theta \right) \int_0^\infty 2\pi v^4 \sigma_i(q, q, v) \sigma_i'(q, q, v) \Phi_i^* (v) dv
\]

\[
= -e \int_0^\infty \int_0^\infty 2\pi v^3 \sigma_i(q, q, v) \sigma_i'(q, q, v) \Phi_i^* (v) L(v) \Phi_i (v) L(v) \exp \left( -\frac{m_i v^2}{2T_i} \right) dv
\]

\[
= -\pi n_i e \frac{m_i}{2\pi T_i} \left( \frac{m_i}{2\pi T_i} \right)^{3/2} \int_0^\infty v^2 \sigma_i(q, q, v) \sigma_i'(q, q, v) L(v) \exp \left( -\frac{m_i v^2}{2T_i} \right) dv
\]

\[
= -\pi n_i e \frac{2}{2\pi \sqrt{2\pi}} \int_0^\infty v^2 \sigma_i(q, q, v) \sigma_i'(q, q, v) L(v) \exp \left( -\frac{m_i v^2}{2T_i} \right) dv
\]

\[
= \frac{n_i e}{\sqrt{2\pi}} \frac{1}{Z a n_d} \frac{1}{2\pi} \int_0^\infty e^{-\gamma v} \Phi_i^* (v) \Phi_i (v) L(v) dv
\]

\[
= \frac{n_i e}{\sqrt{2\pi}} \frac{1}{Z a n_d} \frac{1}{2\pi} \int_0^\infty e^{-\gamma v} \Phi_i^* (v) \Phi_i (v) L(v) dv
\]

\[
= \pi n_i a \frac{1}{m_i v T_i} \frac{a}{2\pi} \int_0^\infty e^{-\gamma v} \Phi_i^* (v) \Phi_i (v) L(v) dv
\]

\[
= \pi n_i a \frac{1}{m_i v T_i} \frac{a}{2\pi} \int_0^\infty e^{-\gamma v} \Phi_i^* (v) \Phi_i (v) L(v) dv
\]
Overall, we get

\[
\tilde{S}_{k,\omega}(q, q') = \left\{ \frac{2\pi e^2 n_i}{m_i v_T i} \right\} \frac{1}{n_d} \int_0^\infty \frac{e^{-y\sigma_0(y)\sigma'_0(y)}}{\kappa_i y} \left[ \Psi_+(y) - \Psi_-(y) \right] dy + \right.
\]
\[
\left. \frac{\pi e^2 n_i}{m_i v_T i} \frac{1}{n_d} \sqrt{2\pi} \int_0^\infty \frac{e^{-y\sigma_0(y)\sigma'_0(y)}}{\kappa_i y} L(y) dy \right\}.
\]

(A.37)

In the low frequency regime the imaginary part will vanish, while for the real part we have

\[
\tilde{S}_{k,\omega}(q, q') = \left\{ \frac{2\pi e^2 n_i}{m_i v_T i} \right\} \frac{1}{n_d} \int_0^\infty \frac{e^{-y\sigma_0(y)\sigma'_0(y)}}{\kappa_i y} dy
\]
\[
= \frac{4\pi e^2 n_i}{m_i v_T i} \frac{1}{n_d} \sqrt{2\pi} \int_0^\infty \frac{e^{-y\sigma_0(y)\sigma'_0(y)}}{\kappa_i y} dy
\]
\[
\times \arctan \left[ \frac{\kappa_i y}{\sqrt{v_T d}} + \frac{n_d \sigma_0(y)}{\sqrt{\sigma_0(y)}} \right] dy.
\]

(A.38)

Whereas in the fully ionized case we get

\[
\tilde{S}_{k,\omega}(q, q') = \frac{4\pi e^2 n_i}{m_i v_T i} \frac{1}{n_d} \int_0^\infty \frac{e^{-y\sigma_0(y)\sigma'_0(y)}}{\kappa_i y} \arctan \left[ \frac{\kappa_i y}{1 + \frac{\tau}{\sqrt{\sigma_0(y)}}} \right] dy.
\]

(A.39)

A.2 Calculation of the dust responses

The methodology employed in the calculation of the dust responses is essentially the same as in the ion responses, apart from the transformation used in the computation of the speed integral. Here we use the transformation \( y = \sqrt{\frac{m_{\text{d}} v_T}{2d}} \) in order to collapse to the plasma dispersion function for the response \( \chi_{k,\omega}^{d,\text{eq}} \) in absence of neutrals. Other useful quantities are the dimensionless phase velocity \( \zeta = \frac{\omega}{\sqrt{v_T d}} \), the dimensionless total effective damping of the dust susceptibility \( \nu_{t1} = \frac{\omega}{\sqrt{2v_T d}} \), and the dimensionless total effective damping of the charging response \( \nu_{t2} = \frac{\omega_{l1} + \nu_{n,d}}{\sqrt{2v_T d}} \).

The integral response \( \chi_{k,\omega}^{d,\text{eq}} \)

The response is the generalization of the dust susceptibility in the presence of neutrals and is defined by

\[
\chi_{k,\omega}^{d,\text{eq}} = \frac{4\pi q_{\text{eq}}^2}{k^2} \int \frac{1}{\omega - k \cdot v + \nu_{n,d}} \frac{\partial \Phi_p^d}{\partial p} \frac{d^3 p}{(2\pi)^3}.
\]

(A.40)
We decompose in real and imaginary parts and we define $\nu_{\text{tot}} = \omega_i + \nu_{n.d.}$, as the total effective damping term present in the response. Using the property $\frac{\partial \Phi}{\partial \nu} \Phi d^3 v = - \nu \Phi d^3 v$ of the Maxwellian distribution we end up with

$$
\chi_{k, \omega}^{d,eq} = \frac{4\pi q_{eq}^2}{T_d k^2} \int \frac{(k \cdot v)}{\omega_r - k \cdot v + i \nu_{\text{tot}}} \Phi d^3 v
$$

$$
= - \frac{4\pi q_{eq}^2}{T_d k^2} \int \frac{(k \cdot v) (\omega_r - k \cdot v - i \nu_{\text{tot}})}{(\omega_r - k \cdot v)^2 + \nu_{\text{tot}}^2} \Phi d^3 v
$$

$$
= - \frac{4\pi q_{eq}^2}{T_d k^2} \int \frac{(k \cdot v) (\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + \nu_{\text{tot}}^2} \Phi d^3 v
$$

$$
+ \frac{4\pi q_{eq}^2}{k^2} \int \frac{(k \cdot v) \nu_{\text{tot}}}{(\omega_r - k \cdot v)^2 + \nu_{\text{tot}}^2} \Phi d^3 v.
$$

For the real part of the integral $I_4$$

$$
\Re \{\chi_{k, \omega}^{d,eq}\} = \frac{4\pi q_{eq}^2}{T_d k^2} \int \frac{(k \cdot v) (\omega_r - k \cdot v)}{(\omega_r - k \cdot v)^2 + \nu_{\text{tot}}^2} \Phi d^3 v
$$

$$
= - \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty \left( \int_0^\pi \frac{k v \sin \theta \cos \theta (\omega_r - k v \cos \theta)}{(\omega_r - k v \cos \theta)^2 + \nu_{\text{tot}}^2} d\theta \right) \frac{2\pi v^2 \Phi d^3 v}{dv}
$$

$$
= - \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty \left( -2 + \frac{\nu_{\text{tot}}}{kv} \left( \arctan \left( \frac{\omega_r + kv}{\nu_{\text{tot}}}ight) - \arctan \left( \frac{\omega_r - kv}{\nu_{\text{tot}}}ight) \right) \right) \frac{2\pi v^2 \Phi d^3 v}{dv}
$$

$$
= \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty 4\pi v^2 \Phi d^3 v
$$

$$
- \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty \frac{\nu_{\text{tot}}}{kv} \left( \arctan \left( \frac{\omega_r + kv}{\nu_{\text{tot}}}ight) - \arctan \left( \frac{\omega_r - kv}{\nu_{\text{tot}}}ight) \right) \frac{2\pi v^2 \Phi d^3 v}{dv}
$$

$$
- \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty \frac{\omega_r}{2kv} \ln \left( \frac{\omega_r + kv + \nu_{\text{tot}}^2}{\omega_r - kv + \nu_{\text{tot}}^2} \right) \frac{2\pi v^2 \Phi d^3 v}{dv}
$$

$$
= \frac{4\pi n q_{eq}^2}{2\pi T_d k^2} \int_0^\infty \frac{4\pi v^2 \exp \left( -\frac{mv^2}{2T_d} \right)}{dv}
$$

$$
- \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty \frac{\nu_{\text{tot}}}{kv} \left( \arctan \left( \frac{\omega_r + kv}{\nu_{\text{tot}}}ight) - \arctan \left( \frac{\omega_r - kv}{\nu_{\text{tot}}}ight) \right) \frac{2\pi v^2 \Phi d^3 v}{dv}
$$

$$
- \frac{4\pi q_{eq}^2}{T_d k^2} \int_0^\infty \frac{\omega_r}{2kv} \ln \left( \frac{\omega_r + kv + \nu_{\text{tot}}^2}{\omega_r - kv + \nu_{\text{tot}}^2} \right) \frac{2\pi v^2 \Phi d^3 v}{dv}.$$
\[ \Re(\chi^d_{k,\omega}) = \frac{1}{k^3\lambda_D^2} \frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{x} e^{-x} \, dx \]

\[ - \left( \frac{4\pi \nu_0^2}{T_d k^2} \int_0^\infty \nu_\text{tot1} \frac{\arctan}{k} \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ - \left( \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]

\[ = \frac{1}{k^3\lambda_D^2} \frac{1}{k^{3/2}} \frac{2}{\sqrt{\pi}} \frac{\nu_\text{tot1}}{k} \int_0^\infty \sqrt{y} e^{-y} \, dy \]

\[ \times \int_0^\infty \left( \arctan \left( \frac{\omega + k\nu}{\nu_\text{tot1}} \right) - \arctan \left( \frac{\omega - k\nu}{\nu_\text{tot1}} \right) \right) ye^{-y^2} \, dy \]

\[ - \frac{4\pi q_0^2}{T_d k^2} \int_0^\infty \omega \ln \left( \frac{(\omega + k\nu)^2 + \nu_\text{tot1}^2}{(\omega - k\nu)^2 + \nu_\text{tot1}^2} \right) \frac{1}{2k} \right) \left( \frac{1}{\nu_\text{tot1}} - \frac{1}{\nu_\text{tot1}} \right) \pi v^2 \Phi^d(v) \, dv \]
A.2. CALCULATION OF THE DUST RESPONSES

For the imaginary part we use the integral $I_2$

\[ \Phi \{ \chi_{k,w} \} = \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ x = \frac{\nu_{tot1} v^2}{2 T_d k} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]

\[ = \frac{1}{k^2 \lambda_{D,eq}^2} \int_0^\infty \zeta \cos y^2 \left( \arctan \left( \frac{\zeta + y}{\nu_{eq1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{eq1}} \right) \right) dy 
\]

\[ - \frac{4\pi q_{eq}^2}{Td k^2} \int_0^\infty \left( \int_0^n \frac{k v \cos \theta \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot1}^2} \right) 2\pi \nu_{tot1} v^2 \Phi^d(v) dv 
\]
Overall, we have

\[ \chi_{d, eq}^{k, \omega} = \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^\infty \nu_{t1} y e^{-y^2} \left( \arctan \left( \frac{\zeta + y}{\nu_{t1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{t1}} \right) \right) dy \right\} 
- \frac{1}{\sqrt{\pi}} \int_0^\infty \zeta y e^{-y^2} \ln \left( \frac{(\zeta + y)^2 + \nu_{t1}^2}{(\zeta - y)^2 + \nu_{t1}^2} \right) dy \]  
(A.41)

\[ + \frac{1}{k^2 \lambda_{Dd}^2} \left\{ \frac{2}{\sqrt{\pi}} \int_0^\infty \zeta ye^{-y^2} \left( \arctan \left( \frac{\zeta + y}{\nu_{t1}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{t1}} \right) \right) dy \right\} 
- \frac{1}{\sqrt{\pi}} \int_0^\infty \nu_{t1} y e^{-y^2} \ln \left( \frac{(\zeta + y)^2 + \nu_{t1}^2}{(\zeta - y)^2 + \nu_{t1}^2} \right) dy \].  
(A.42)

In the fully ionized case, \( \nu_{n,d} = 0 \) and due to the absence of a strong dissipation mechanism \( \omega_i \to 0 \), thus the dimensionless effective damping rate tends to zero, \( \nu_{t1} \to 0 \). In the real part of the response, the second term will vanish, while for the last term (with \( P \) denoting the Cauchy principal value of the integral around its singular point \( \zeta = y \)):

\[ \lim_{\nu_{t1} \to 0} \Re\{\chi_{d, eq}^{k, \omega} \} = \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{1}{\sqrt{\pi}} P. \int_0^\infty \zeta y e^{-y^2} \ln \left[ \frac{\zeta + y}{\zeta - y} \right]^2 dy \right\} \]

We split into two integrals using the properties of the logarithmic function and we employ integration by parts due to \( ye^{-y^2} = -\frac{1}{2} \frac{d}{dy}(e^{-y^2}) \).

\[ \lim_{\nu_{t1} \to 0} \Re\{\chi_{d, eq}^{k, \omega} \} = \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{2\zeta}{\sqrt{\pi}} P. \int_0^\infty ye^{-y^2} \ln \left[ \frac{\zeta + y}{\zeta - y} \right] dy \right\} 
= \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{2\zeta}{\sqrt{\pi}} \int_0^\infty ye^{-y^2} \ln (\zeta + y) dy + \frac{2\zeta}{\sqrt{\pi}} P. \int_0^\infty ye^{-y^2} \ln (\zeta - y) dy \right\} 
\]

\[ = \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 + \frac{\zeta}{\sqrt{\pi}} \int_0^\infty (e^{-y^2})^' \ln (\zeta + y) dy \right\} 
- \frac{\zeta}{\sqrt{\pi}} P. \int_0^\infty (e^{-y^2})^' \ln (\zeta - y) dy \]

\[ = \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{\zeta}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y^2}}{\zeta + y} dy - \frac{\zeta}{\sqrt{\pi}} P. \int_0^\infty \frac{e^{-y^2}}{\zeta - y} dy \right\} 
+ \frac{\zeta}{\sqrt{\pi}} \left[ e^{-y^2} \ln (\zeta + y) \right]_0^\infty - \frac{\zeta}{\sqrt{\pi}} \left[ e^{-y^2} \ln (\zeta - y) \right]_0^\infty \right\}. \]
A.2. CALCULATION OF THE DUST RESPONSES

The limits to infinity are zero with the use of L’Hospital’s rule, while the limits to zero cancel each other, we set $y \to -y$ in the first integral.

$$\lim_{\nu \to 0} \Re \{ \chi_{k,\omega}^{d,eq} \} = \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{\zeta}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y^2}}{\zeta + y} \, dy - \frac{\zeta}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y^2}}{\zeta - y} \, dy \right\}$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{\zeta}{\sqrt{\pi}} \int_{-\infty}^0 \frac{e^{-y^2}}{\zeta - y} \, dy + \frac{\zeta}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y^2}}{\zeta - y} \, dy \right\}$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \left\{ 1 - \frac{\zeta}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{\zeta - y} \, dy \right\} ,$$

which is the real part of the plasma dispersion function as expected.

As far as the imaginary part is concerned, in the limit $\nu \to 0$ the second part vanishes. The first part might seem to vanish since $(\arctan (\infty) - \arctan (\infty)) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, yet for $z = y$, which lies in the parameter space, the argument of the inverse tangent is of indeterminate form. Use of integration by parts, will reveal a nascent delta function sequence of the Lorentz line form, $\delta_d(x) = \frac{1}{\pi a^2}$, with the properties $\delta(x) : \int_{-\infty}^{+\infty} f(x) \delta(x) \, dx = \lim_{a \to 0} \int_{-\infty}^{+\infty} f(x) \delta_d(x) \, dx = f(0)$ or less strictly $\delta(x) = \lim_{a \to 0} \delta_d(x)$. We have

$$\lim_{\nu \to 0} \Im \{ \chi_{k,\omega}^{d,eq} \} = \frac{1}{k^2 \lambda_{Dd}^2} \frac{2 \zeta}{\sqrt{\pi}} \lim_{\nu \to 0} \int_0^\infty ye^{-y^2} \left\{ \arctan \left[ \frac{\zeta + y}{\nu_{\nu_1}} \right] - \arctan \left[ \frac{\zeta - y}{\nu_{\nu_1}} \right] \right\} \, dy$$

$$= -\frac{1}{k^2 \lambda_{Dd}^2} \frac{\zeta}{\sqrt{\pi}} \lim_{\nu \to 0} \int_0^\infty \left( e^{-y^2} \right) \left\{ \arctan \left[ \frac{\zeta + y}{\nu_{\nu_1}} \right] - \arctan \left[ \frac{\zeta - y}{\nu_{\nu_1}} \right] \right\} \, dy$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \frac{\zeta}{\sqrt{\pi}} \lim_{\nu \to 0} \int_0^\infty e^{-y^2} \frac{d}{dy} \left\{ \arctan \left[ \frac{\zeta + y}{\nu_{\nu_1}} \right] - \arctan \left[ \frac{\zeta - y}{\nu_{\nu_1}} \right] \right\} \, dy$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \frac{\zeta}{\sqrt{\pi}} \lim_{\nu \to 0} \int_0^\infty e^{-y^2} \left\{ \frac{1}{1 + \left( \frac{\zeta + y}{\nu_{\nu_1}} \right)^2} + \frac{1}{1 + \left( \frac{\zeta - y}{\nu_{\nu_1}} \right)^2} \right\} \, dy$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \sqrt{\pi} \zeta \int_0^\infty e^{-y^2} \left\{ \frac{1}{\nu_{\nu_1}^2} \frac{\nu_{\nu_1}^2}{\nu_{\nu_1}^2 + (\zeta + y)^2} + \frac{1}{\nu_{\nu_1}^2} \frac{\nu_{\nu_1}^2}{\nu_{\nu_1}^2 + (\zeta - y)^2} \right\} \, dy$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \sqrt{\pi} \zeta \int_0^\infty e^{-y^2} \frac{\delta(\zeta + y) + \delta(\zeta - y)}{dy} \, dy$$

$$= \frac{1}{k^2 \lambda_{Dd}^2} \sqrt{\pi} \zeta \left\{ \int_0^\infty e^{-y^2} \delta(\zeta + y) \, dy + \int_0^\infty e^{-y^2} \delta(\zeta - y) \, dy \right\} .$$

The first integrand is zero, since the negative root does not belong to the integration interval, hence overall

$$\lim_{\nu \to 0} \Im \{ \chi_{k,\omega}^{d,eq} \} = \frac{1}{k^2 \lambda_{Dd}^2} \sqrt{\pi} \zeta e^{-\zeta^2} . \quad (A.43)$$
which is the imaginary part of the plasma dispersion function, after the application of the Plemelj formula.

The integral response $\chi_{k,\omega}^{d,ch}$

The responses is related to the charging process and is defined by

$$\chi_{k,\omega}^{d,ch} = \int \frac{1}{\omega - k \cdot v + i(\nu_{ch} + \nu_{n,d})} \phi^d d^3p (2\pi)^3.$$  \hfill (A.44)

We separate in real and imaginary parts and define $\nu_{tot} = \omega_i + \nu_{ch} + \nu_{n,d}$ as the total effective damping of the response,

$$\chi_{k,\omega}^{d,ch} = \int \frac{1}{\omega_r - k \cdot v + i(\omega_i + \nu_{ch} + \nu_{n,d})} \phi^d d^3p (2\pi)^3$$

$$= \int \frac{1}{(\omega_r - k \cdot v - \nu_{tot})^2 + \nu_{tot}^2} \phi^d d^3p (2\pi)^3$$

$$= \frac{\nu_{tot}}{(\omega_r - k \cdot v)^2 + \nu_{tot}^2} \phi^d d^3p (2\pi)^3 + \int \frac{\omega_r - k \cdot v}{(\omega_r - k \cdot v)^2 + \nu_{tot}^2} \phi^d d^3p (2\pi)^3.$$

For the real part we use the integral $I_1$

$$\Re(\chi_{k,\omega}^{d,ch}) = \int \frac{\nu_{tot}}{(\omega_r - k \cdot v)^2 + \nu_{tot}^2} \phi^d d^3p (2\pi)^3$$

$$= \int_0^\infty \left\{ \int_0^\pi \frac{\sin \theta}{(\omega_r - k \cdot v)^2 + \nu_{tot}^2} d\theta \right\} 2\pi \nu_{tot} v^2 \phi^d(v) dv$$

$$= \int_0^\infty \frac{2\pi \nu_{tot} v^2}{kv_n} \left( \arctan \frac{\omega_r + kv}{v_{tot}} - \arctan \frac{\omega_r - kv}{v_{tot}} \right) \phi^d(v) dv$$

$$= \frac{2\pi \nu_{tot}}{kv_n} \int_0^\infty \left( \arctan \frac{\omega_r + kv}{v_{tot}} - \arctan \frac{\omega_r - kv}{v_{tot}} \right) \Phi^d(v) dv$$

$$= \frac{2\pi \nu_{tot}}{kv_n} \int_0^\infty \left( \arctan \frac{\omega_r + kv}{v_{tot}} - \arctan \frac{\omega_r - kv}{v_{tot}} \right) \exp \left( -\frac{mv^2}{2T_d} \right) dv$$

$$= \frac{\nu_{tot}}{kv_n} \int_0^\infty \left( \frac{y}{\sqrt{2\pi} m_d} \right)^{3/2} \exp \left( -\frac{y^2}{2m_d^2} \right) \Phi^d(y) dy$$

$$= \int_0^\infty y e^{-y^2} \left( \arctan \left( \frac{\omega_r + k \sqrt{\frac{m_d}{\nu_{tot}}} y}{\nu_{tot}} \right) - \arctan \left( \frac{\omega_r - k \sqrt{\frac{m_d}{\nu_{tot}}} y}{\nu_{tot}} \right) \right) dy$$

$$= \int_0^\infty y e^{-y^2} \left( \arctan \left( \frac{\omega_r + k \sqrt{\frac{m_d}{\nu_{tot}}} y}{\nu_{tot}} \right) - \arctan \left( \frac{\omega_r - k \sqrt{\frac{m_d}{\nu_{tot}}} y}{\nu_{tot}} \right) \right) dy.$$
For the imaginary part of the response we use the integral \( I_5 \)

\[
\begin{align*}
\Im\{\chi_{d,ch}\} &= \int \frac{\omega_r - k \cdot v}{(\omega_r - k \cdot v)^2 + \nu_{tot}^2} \Phi_d \frac{d^3 p}{(2\pi)^3} \\
&= \int_0^\infty \left( \int_0^\pi \frac{(\omega_r - k v \cos \theta) \sin \theta}{(\omega_r - k v \cos \theta)^2 + \nu_{tot}^2} \ d\theta \right) 2\pi v^2 \Phi(v) dv \\
&= \int_0^\infty 2\pi v^2 \ln \left( \frac{(\omega_r + k v)^2 + \nu_{tot}^2}{(\omega_r - k v)^2 + \nu_{tot}^2} \right) \Phi(v) dv \\
&= \frac{n_d\pi}{k} \left( \frac{m_d}{2\pi T_d} \right)^{3/2} \int_0^\infty \ln \left( \frac{(\omega_r + k \sqrt{\frac{m_d}{2\pi}} y)^2 + \nu_{tot}^2}{(\omega_r - k \sqrt{\frac{m_d}{2\pi}} y)^2 + \nu_{tot}^2} \right) v \ exp \left( -\frac{m_d v^2}{2T_d} \right) dv \\
&= \frac{n_d}{k} \sqrt{\frac{m_d}{2\pi T_d}} \int_0^\infty ye^{-y^2} \ln \left( \frac{(\omega_r + k \sqrt{\frac{m_d}{2\pi}} y)^2 + \nu_{tot}^2}{(\omega_r - k \sqrt{\frac{m_d}{2\pi}} y)^2 + \nu_{tot}^2} \right) dy \\
&= \frac{n_d}{\sqrt{2\pi kvT_d}} \int_0^\infty ye^{-y^2} \ln \left( \frac{(\zeta + y)^2 + \nu_{tot}^2}{(\zeta - y)^2 + \nu_{tot}^2} \right) dy .
\end{align*}
\]

Overall, we have

\[
\chi_{d,ch}^{\omega} = \sqrt{\frac{2}{\pi k vT_d}} \int_0^\infty ye^{-y^2} \left( \arctan \left( \frac{\zeta + y}{\nu_{tot}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{tot}} \right) \right) dy \\
+ \frac{1}{\sqrt{2\pi kvT_d}} \int_0^\infty ye^{-y^2} \ln \left( \frac{(\zeta + y)^2 + \nu_{tot}^2}{(\zeta - y)^2 + \nu_{tot}^2} \right) dy .
\]

In the low frequency regime we have \( \omega_r < \omega_{pd} \), while for the charging frequency we always have \( \omega_{pd} \ll \nu_{ch} < \omega_i \). Hence, the total effective damping term is dominating the denominator of the response (with the exception of very strong damping, where \( \omega_i \) will acquire large negative values and reduce \( \nu_{tot}^2 \) significantly). Using \( \nu_{tot} \gg y, \zeta \), it is obvious that the imaginary part vanishes, while for the real part we can use the Taylor expansion of the inverse tangent and keep the first order
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term only, $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \simeq x$ for $x \ll 1$.

\[
\chi^{d,ch}_{k,\omega} = \sqrt{\frac{2}{\pi}} \frac{n_d}{kv_{Td}} \int_0^\infty ye^{-y^2} \left\{ \arctan \left( \frac{\zeta + y}{\nu_{t2}} \right) - \arctan \left( \frac{\zeta - y}{\nu_{t2}} \right) \right\} dy 
\]

\[
\simeq \sqrt{\frac{2}{\pi}} \frac{n_d}{kv_{Td}} \int_0^\infty ye^{-y^2} \left\{ \frac{\zeta + y}{\nu_{t2}} - \frac{\zeta - y}{\nu_{t2}} \right\} dy 
\]

\[
\simeq \sqrt{\frac{2}{\pi}} \frac{n_d}{kv_{Td}} \int_0^\infty \frac{2}{\nu_{t2}} e^{-y^2} dy 
\]

\[
\simeq \sqrt{\frac{2}{\pi}} \frac{n_d}{kv_{Td}} \int_0^\infty \frac{2\sqrt{2} kv_{Td}}{\nu_{t2}} e^{-y^2} dy 
\]

\[
\simeq \frac{\sqrt{2} n_d \sqrt{\pi}}{4} 
\]

which in absence of neutrals gives the familiar $\chi^{d,ch}_{k,\omega} \simeq \frac{n_d}{v_{ch}}$. Alternatively, in the initial response, we can ignore the real part of the denominator, which results in

\[
\chi^{d,ch}_{k,\omega} \simeq \int \Phi^d(v) dv \simeq \frac{1}{\nu_{t2}} \int \Phi^d(v) dv \simeq \frac{n_d}{\nu_{ch} + \nu_{n,d} + \omega_i}. \tag{A.46}
\]

The integral response $\delta^{eq}_{k,\omega}$

A response similar to the one we just encountered has the form,

\[
\delta^{eq}_{k,\omega} = \int \frac{\Phi_{\nu_{t2}}(q)}{\omega - \mathbf{k} \cdot \mathbf{v} + \mathbf{v}_{n,d}} dq \frac{d^3p}{(2\pi)^3}. \tag{A.47}
\]

This response is generated by the BGK collision term in the dust Klimontovich equation, due to the narrowness of the dust equilibrium distribution around the equilibrium charge we have $q = q_{eq}$ and the integration is over the momentum space only. The differences from the charging response are that: the only dissipative term is $\nu_{n,d}$ and hence the total effective damping term will be $\nu_{tot1} = \omega_i + \nu_{n,d}$, the distribution is normalized to unity and hence the previous results will be divided by $n_d$, the imaginary factor is missing leading to the previous results multiplication
by $-i$.

\[
\begin{align*}
d^e_{k,\omega} &= \frac{1}{\sqrt{2\pi kT_d \nu}} \int_0^\infty ye^{-y^2} \ln \left[ \frac{(\zeta + y)^2 + \nu_{\nu_1}^2}{(\zeta - y)^2 + \nu_{\nu_1}^2} \right] dy \\
&\quad - i \sqrt{2 \frac{1}{\pi \nu_1}} \int_0^\infty ye^{-y^2} \left\{ \arctan \left[ \frac{\zeta + y}{\nu_1} \right] - \arctan \left[ \frac{\zeta - y}{\nu_1} \right] \right\} dy . \quad (A.48)
\end{align*}
\]

Finally, despite the similarities, the approximate expressions are not valid for this response, since the charging frequency is absent. In the fully ionized case the response does not exist, since it is connected to collisions with neutrals (always multiplied by $\nu_{\nu_1}$).
Appendix B

Fluid Description of Dust Acoustic Waves

As aforementioned, the partially ionized complex plasma system consists of four distinct ‘species’: the ions, the electrons, the dust grains and the neutral gas. In the hydrodynamic description of waves, the perturbed fluid equations are used for the electrons, the ions and the dust species, whereas the neutral gas fluid quantities are usually considered unperturbed from equilibrium. The fluid equations are derived by the moments of the generalized Boltzmann equations for the ensemble averaged part of each distribution function, the equations used are the continuity equation and the momentum equation, thus we have truncation before the second moment and a relation for the pressure tensor has to be assumed. Most commonly, neglecting anisotropy, the pressure is assumed scalar obeying a relation of the form $\frac{p_j}{n_j^\gamma_j} = \text{const}$, with $\gamma_j = 1$ for isothermal species and $\gamma_j = 5/3$ for adiabatic processes.

The hydrodynamic equations for the plasma species have the form, with $\alpha = \{e, i\}$,

\[
\frac{\partial n_\alpha}{\partial t} + \nabla (n_\alpha v_\alpha) = Q_{sa} - Q_{\alpha}\,
\]

\[
\frac{\partial v_\alpha}{\partial t} + (v_\alpha \cdot \nabla) v_\alpha = -\frac{q_\alpha}{m_\alpha} \nabla \Phi - \frac{1}{m_\alpha n_\alpha} \nabla p_\alpha - \frac{v_\alpha}{n_\alpha} Q_{\alpha} - \sum_{j=i,e,d,n} v_{\alpha,j} (v_\alpha - v_j) - \tilde{v}_{\alpha,d} v_\alpha.
\]

where

1. $Q_{sa}$ describes the production of electrons/ions due to ionization processes (electron impact ionization, photo-ionization) or due to emission processes from the dust surface (photoelectric, secondary, thermionic, field emission)
2. $Q_{\text{rel}}$ describes the loss of electrons/ions due to recombination processes in the volume, due to collection on the dust surface after inelastic collisions, due to loss in the boundary surface of the complex plasma configuration.

3. $\frac{\omega}{\omega_{\text{in}}} Q_{\text{rel}}$ describes momentum loss due to particle loss (only for external to the plasma system processes, since inelastic collision losses are accounted in a separate term), while there is no similar term for momentum gain due to particle production because the newly created particles can be assumed at rest (since the dust or neutral thermal velocities are much smaller than the species thermal velocities).

4. The first collisional term describes momentum transfer in elastic Coulomb collisions between the species.

5. The second collisional term describes momentum transfer in inelastic collisions with dust particles.

Similarly, the hydrodynamic equations for the dust particles have the form

$$
\frac{\partial n_d}{\partial t} + \nabla (n_d v_d) = 0 ,
$$

$$
\frac{\partial v_d}{\partial t} + (v_d \cdot \nabla) v_d = - \frac{Z_d e}{m_d} \nabla \Phi - \frac{1}{m_d n_d} \nabla P_d - \sum_{j=i,e,n} \nu_{d,j} (v_d - v_j) + \sum_{\kappa=e,i} \tilde{v}_{d,\kappa} v_{\kappa} ,
$$

where one should note that there is no source or sink of dust particles and that $Z_d$ is the characteristic charge number with its sign. The above system of equations is self-consistently closed by the Poisson equation and the charging equation,

$$
\nabla^2 \Phi = -4 \pi e (-n_e + n_i + Z_d n_d) ,
$$

$$
\frac{\partial Z_d}{\partial t} + (v_d \cdot \nabla) Z_d = \sum_{j} I_{fj} ,
$$

where $I_{fj}$ are the particle fluxes emitted or absorbed by the grain, with the last equation being the equation for the characteristic charge with its sign, that is equivalent to the charging equation. The latter equation is of hydrodynamic nature containing the convective derivative of the charge, it can be derived by multiplying the dust kinetic equation with the charge $q$ and integrating over the momentum space only.

The linearization of the system is achieved by decomposing each quantity to its equilibrium and perturbed value, and by neglecting high order perturbation terms. We assume unmagnetized homogeneous plasma with non-drifting isothermal species and set

$$
p_j = p_{j0} + p_{j1} , \quad \Phi = \Phi_1 , \quad n_j = n_{j0} + n_{j1} ,
$$

$$
v_j = v_{j1} , \quad E = E_1 , \quad Z_d = Z_{d0} + Z_{d1} ,
$$

(B.1)
where the ground state values are determined by the quasi-neutrality condition, the particle flux balance on the grain surface and the absorption / generation balance for the plasma particles. The perturbed Poisson equation will acquire the form

\[ \nabla^2 \Phi_1 = -4\pi e (n_{i1} - n_{e1} + Z_{d1} n_{d0} + Z_{d0} n_{d1}) \]  \hspace{1cm} (B.2)

The presence of sources/sinks as well as of collisional terms are depending on the physical scenario of the complex plasma configuration. Certain types of collisions can be neglected for short wavelengths, where the characteristic mean free paths are longer than the characteristic scale of the problem. For sufficiently short wavelengths, ion-electron Coulomb collisions, species-dust elastic and inelastic collisions as well as electron-neutral collisions can be ignored. On the other hand, ion-neutral and mostly dust-neutral collisions have large corresponding frequencies and short mean free paths and cannot be neglected in most cases.

Since dust-species inelastic collisions are ignored, there is no need for source terms to preserve the plasma. As an approximation the dust charge can be considered non-fluctuating and the charging equation is removed from the system. Moreover, assuming that particle emission from the grain surface can be neglected, the dust charge is negative $Z_d \rightarrow -Z_d$. Therefore, the system of hydrodynamic equation will become, for $s = \{i, e, d\}$,

\begin{align*}
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s v_s) &= 0, \\
\frac{\partial v_e}{\partial t} + (v_e \cdot \nabla) v_e &= \frac{e}{m_e} \nabla \Phi - \frac{1}{m_e n_e} \nabla \rho_e, \\
\frac{\partial v_i}{\partial t} + (v_i \cdot \nabla) v_i &= -\frac{e}{m_i} \nabla \Phi - \frac{1}{m_i n_i} \nabla \rho_i - \nu_{i,n} (v_i - v_n), \\
\frac{\partial v_d}{\partial t} + (v_d \cdot \nabla) v_d &= \frac{Z_de}{m_d} \nabla \Phi - \frac{1}{m_d n_d} \nabla \rho_d - \nu_{d,n} (v_d - v_n). \hspace{1cm} (B.3)
\end{align*}

The strategy is to use the three hydrodynamic equations, linearize and assume wave-like perturbations ($\exp (-i\omega t + ik \cdot r)$), express the perturbed density as a function of the electrostatic potential perturbation by eliminating $v_{s1}, \rho_{s1}$ and substitute in the Poisson equation, searching for a non-trivial solution. In the case of short wavelengths the total permittivity can be expressed as the sum of the contribution of each species, mixed terms that appear in the kinetic theory treatment are absent. Therefore, the dispersion relation will have the form $1 + \chi_e + \chi_i + \chi_d = 0$, comparing with the perturbed Poisson equation that is

\[ \Phi_1 \left\{ 1 + \frac{4\pi e}{k^2 \Phi_1} n_{e1} - \frac{4\pi e}{k^2 \Phi_1} n_{i1} + \frac{4\pi Z_{d} e}{k^2 \Phi_1} n_{d1} \right\} = 0, \]

we end up with

\[ \chi_e = \frac{4\pi e}{k^2 \Phi_1} n_{e1}, \quad \chi_i = -\frac{4\pi e}{k^2 \Phi_1} n_{i1}, \quad \chi_d = \frac{4\pi Z_{d} e}{k^2 \Phi_1} n_{d1}. \hspace{1cm} (B.4) \]
In the case of the dust acoustic wave, the propagation regime is $k v_{Td} \ll \omega \ll k v_{Ti}, k v_{Te}$ and further simplifications can be made. We proceed with a scale analysis between the inertial and the pressure gradient force term,

$$\frac{m_s n_s \omega v_s}{\nabla p_s} \sim \frac{m_s v_s^2}{k n_s T_s} \sim \frac{v_{ph}^2}{v_{Ts}^2},$$

(B.5)

for electrons/ions $v_{ph}^2 \ll v_{Ts}^2$ and hence the two inertial terms in the momentum equation can be neglected, for dust particles $v_{ph}^2 \gg v_{Td}^2$ and hence the pressure gradient force can be neglected.

For the electron species, the momentum equation will be $0 = e n_e \nabla \Phi - \nabla p_e$, for isothermal electrons, a substitution from the ideal gas law will result in the Boltzmann equilibrium relation, $n_e = n_{e0} \exp \left( \frac{e \Phi}{T_e} \right)$. We decompose the density in both sides and use the smallness of the density perturbation to expand the exponential to a Taylor series and keep the first two terms, the result is $n_{e1} = \frac{e n_{e0}}{T_e} \Phi_1$, and the electron permittivity will be

$$\chi_e = \frac{1}{k^2 \lambda_D^2}. \quad \text{(B.6)}$$

For the ion species, the continuity equation will give $n_{i0} k \cdot \mathbf{v}_{i1} = \omega n_{i1}$, while the momentum equation after taking the inner product with the wavenumber will give $n_{i1} = \frac{e n_{i0} k^2 \Phi_1}{m_i (\nu_{i,n} - v_{ph}^2 k^2)}$, and the ion permittivity will be

$$\chi_i = \frac{\omega_{pi}^2}{-\nu_{i,n} \omega + k^2 v_{Ti}^2}. \quad \text{(B.7)}$$

For the dust particles, continuity will give $n_{d0} k \cdot \mathbf{v}_{d1} = \omega n_{d1}$ and the momentum equation $n_{d1} = -\frac{e Z_d}{m_d (\omega + \nu_{d,n})} n_{d0}$ leading to the dust permittivity

$$\chi_d = \frac{\omega_{pd}^2}{\omega (\omega + \nu_{d,n})}. \quad \text{(B.8)}$$

Hence, the form of the dispersion relation will be

$$1 + \frac{1}{k^2 \lambda_D^2} + \frac{\omega_{pi}^2}{k^2 v_{Ti}^2 - \nu_{i,n} \omega} = \frac{\omega_{pd}^2}{\omega (\omega + \nu_{d,n})}. \quad \text{(B.9)}$$

Depending on the strength of the dissipation on neutrals we have the solutions;

1. For $\nu_{i,n} = 0, \nu_{d,n} = 0$, we have the collisionless result with $\omega = \omega_r$. The
The dispersion relation becomes

\[
1 + \frac{1}{k^2 \Delta_2} + \frac{1}{k^2 \lambda_D^2} = \frac{\omega^2_{pd}}{\omega^2_i},
\]

\[
1 + \frac{1}{k^2 \Delta_2} = \frac{\omega^2_{pd}}{\omega^2_i},
\]

\[
\omega_i^2 = \frac{\omega^2_{pd} \Delta_2}{1 + k^2 \lambda_D^2} k^2.
\]

In the short wavelength limit \(k^2 \lambda_D^2 \gg 1\): we get that \(\omega_i \simeq \omega_{pd}\), stating that the dust plasma frequency is the highest frequency the dust acoustic eigenmodes can reach. In the long wavelength limit \(k^2 \lambda_D^2 \ll 1\): we get that \(\omega_i \simeq \omega_{pd} \lambda_D k = \sqrt{\frac{P_Z \Delta_i}{m_d(1 + P + \tau)}} k\), stating that the phase velocity of the dust acoustic waves is approximately constant.

2. For \(\nu_{i,n} = 0, \nu_{d,n} \neq 0\), the dispersion relation in the long wavelength limit becomes

\[
1 + \frac{1}{k^2 \Delta_2} + \frac{1}{k^2 \lambda_D^2} = \frac{\omega^2_{pd}}{(\omega + \nu_{d,n}) \omega} = 0
\]

\[
1 + \frac{1}{k^2 \lambda_D^2} = \frac{\omega^2_{pd}}{(\omega + \nu_{d,n}) \omega} = 0
\]

\[
\omega^2 + \nu_{d,n} \omega - \frac{\omega^2_{pd} \lambda_D^2}{1 + k^2 \lambda_D^2} k^2 = 0
\]

\[
\omega^2 + \nu_{d,n} \omega - \frac{\omega^2_{pd} \lambda_D^2}{1 + k^2 \lambda_D^2} k^2 = 0.
\]

In the case \(\nu^2_{d,n} < 4 \omega^2_{pd} k^2 \lambda_D^2\); the solution to the dispersion relation has both imaginary and real parts with \(\omega_i = \sqrt{\omega^2_{pd} k^2 \lambda_D^2 - \frac{\nu^2_{d,n}}{4}}, \quad \omega_i = -\frac{\nu_{d,n}}{2}\). In the case \(\nu^2_{d,n} > 4 \omega^2_{pd} k^2 \lambda_D^2\); we have high dissipation and the solution to the dispersion relation will be pure imaginary with \(\omega_i = -\frac{\nu_{d,n}}{2} = \sqrt{\frac{\nu^2_{d,n}}{4} - \omega^2_{pd} k^2 \lambda_D^2}\).

There is a critical wavelength, above which the solutions turn from damped oscillations to aperiodic damping, by setting the discriminant equal to zero

\[
\text{we get } k_{crit} = \frac{\nu_{d,n}}{2 \omega_{pd} \lambda_D} = \frac{\nu_{d,n}}{2} \sqrt{\frac{m_d(1 + P + \tau)}{P_Z \Delta_i}}.
\]

3. For \(\nu_{i,n} \neq 0, \nu_{d,n} \neq 0\), by Taylor expanding in the small \(\frac{\omega_{i,n}}{\omega_i} k^2\) quantity and keeping the first order term only, the ion susceptibility will become

\[
\chi_i \sim \frac{\omega_{i,n}}{k^2 \Delta_i^2 (1 - i \frac{\omega_{i,n}}{\omega_i} k^2)} \sim \frac{1}{k^2 \lambda_D^2 (1 - i \frac{\omega_{i,n}}{\omega_i} k^2)} \sim \frac{1}{k^2 \lambda_D^2} (1 + i \frac{\omega_{i,n}}{\omega_i} k^2).
\]

Therefore, the
dispersion relation will now be
\[
1 + \frac{1}{k^2\lambda_D^2} + i\frac{\omega\nu_{i,n}}{k^4\lambda_D^2v_T^2} = \frac{\omega_{pd}^2}{(\omega + i\nu_{d,n})\omega},
\]  
(B.10)
and if we set \(A(k) = 1 + \frac{1}{k^2\lambda_D^2}\) (non-dimensional) and \(B(k) = \frac{1}{k^4\lambda_D^2v_T^2}\) (dimensions of \(s^2\)),

\[
[\nu_{i,n}B(k)]\omega^3 + [A(k) - \nu_{i,n}\nu_{d,n}B(k)]\omega^2 + [\nu_{d,n}A(k)]\omega - \omega_{pd}^2 = 0,
\]
the above equation is a third degree polynomial equation for \(\omega = \omega_r + \omega_i\) which is solvable analytically, through the Cardano formula for monic polynomials. It is also worth noticing that with respect to \(k^2\) the above equation is quadratic.
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