Remarks on the quality of GPS precise point positioning using phase observations

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Masoud Shirazian
Abstract

GPS processing, like every processing method for geodetic applications, relies upon least-squares estimation. Quality measures must be defined to assure that the estimates are close to reality. These quality measures are reliable provided that, first, the covariance matrix of the observations (the stochastic model) is well defined and second, the systematic effects are completely removed (i.e., the functional model is good).

In the GPS precise point positioning (PPP) the stochastic and functional models are not as complicated as in the differential GPS processing. We will assess the quality of the GPS Precise Point Positioning in this thesis.

To refine the functional model from systematic errors, we have 1) used the phase observations to prevent introducing any hardware bias to the observation equations, 2) corrected observations for all systematic effects with amplitudes of more than 1cm, 3) used undifferenced observations to prevent having complications (e.g. linearly related parameters) in the system of observation equations.

To have a realistic covariance matrix for the observations we have incorporated the ephemeris’ uncertainties into the system of observation equations.

The above-mentioned technique is numerically tested on the real data of some of the International GNSS Service stations. The results confirm that undifferenced stochastic-related properties (e.g. degrees of freedom) can be reliable means to recognize the parameterization problem in differenced observation equations. These results also imply that incorporation of the satellite ephemeris uncertainties might improve the estimates of the station positions.
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My deepest thanks go to my parents for their support and patience in whole my life. They have always been my shelter and my reliable guides to find the right path of life.

Last, I would like to express my sincerest appreciation to my wife and daughters, Farzaneh, Marzie and Arnavaz for their understanding and patience during the years we were away from our lovely country and families.
# List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>AC</td>
<td>Analysis Center</td>
</tr>
<tr>
<td>ANTEX</td>
<td>ANTenna EXchange format</td>
</tr>
<tr>
<td>DCB</td>
<td>Differential Code Bias</td>
</tr>
<tr>
<td>ERP</td>
<td>Earth Rotation Parameter</td>
</tr>
<tr>
<td>GNSS</td>
<td>Global Navigation Satellite System</td>
</tr>
<tr>
<td>GPS</td>
<td>Global Positioning System</td>
</tr>
<tr>
<td>IAG</td>
<td>International Association of Geodesy</td>
</tr>
<tr>
<td>IERS</td>
<td>International Earth Rotation Service</td>
</tr>
<tr>
<td>IGS</td>
<td>International GNSS Service</td>
</tr>
<tr>
<td>ITRF</td>
<td>International Terrestrial Reference Frame</td>
</tr>
<tr>
<td>LC</td>
<td>Linear Combination</td>
</tr>
<tr>
<td>LS</td>
<td>Least Squares</td>
</tr>
<tr>
<td>PPP</td>
<td>Precise Point Positioning</td>
</tr>
<tr>
<td>RINEX</td>
<td>Receiver INdependent EXchange format</td>
</tr>
<tr>
<td>SD</td>
<td>Single Difference</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
</tr>
<tr>
<td>SP3</td>
<td>Standard Product 3</td>
</tr>
<tr>
<td>WGS84</td>
<td>World Geodetic System 84</td>
</tr>
<tr>
<td>ZD</td>
<td>Zero Difference</td>
</tr>
<tr>
<td>ZHD</td>
<td>Zenith Hydrostatic Delay</td>
</tr>
<tr>
<td>ZTD</td>
<td>Zenith Total Delay</td>
</tr>
<tr>
<td>ZWD</td>
<td>Zenith Wet Delay</td>
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1 Introduction

1.1 Background

In data processing for geodetic applications, we usually rely upon least-squares estimation. To assess how close the estimators are to reality, we describe the quality for these estimators. These quality measures are defined by proper propagation of the observation uncertainties to the estimates of the unknowns. The defined quality measures are reliable provided that:

- The covariance matrix of the observations (the stochastic model) is well defined.
- The systematic effects are completely removed (i.e., the functional model is good).

Global positioning system (GPS) data processing is based on the least-squares estimation too. Unfortunately, many of current GPS processing software, which use simple diagonal elevation dependent covariance matrix for the raw observations, lead to the quality of the estimators that is too optimistic (see e.g. Tiberius et al. 1999; Krynski and Zanimonskyi 2003; Becerra 2008 and Kutterer 2002).

GPS precise point positioning (PPP) has attracted great attention during the last decade. In this technique, the stochastic and functional models are not as complicated as in the differential GPS processing. Therefore, in this thesis we will assess the quality of the GPS Precise Point Positioning.

1.2 Thesis objectives

PPP is the processing of the ionosphere-free code and phase data of a single station while the satellite coordinates and clocks are fixed. Centimeter level of accuracy is claimed for the static PPP (see e.g. Kouba and Héroux 2001 and Martín et al. 2011). To validate this accuracy, we study the discrepancies between the estimators and their values from the International Terrestrial Reference Frame (ITRF) solutions. Generally, the obtained uncertainties for the station positions from the PPP have always been less than their corresponding discrepancies. Therefore, in this thesis, to assess the quality of the PPP we will:
- improve the existing covariance matrix definition for the observations by introducing satellite coordinates and clocks uncertainties to the observation equations;
- find and remove the systematic effects in the functional model as much as we can;
- test and verify the correctness of the above strategies by conducting numerical studies.

### 1.3 Contributions of this thesis

The contributions of the research, conducted for this thesis are as follows:

- Solving the PPP problem using only phase observations
- Studying the possible problems in differenced phase ambiguity parameters and particularly, the problem of introducing linearly dependent ambiguity parameters to the system of observation equations
- Improving the covariance matrix of the observations by incorporation of the uncertainties of the satellite orbits and clocks into the system of observation equations
- Numerical implementation of the above theories

### 1.4 Outline of thesis

This dissertation will consist of 5 chapters.

In chapter 2, background of GPS processing is discussed. Some issues like processing strategies are focused in this chapter. The systematic effects in GPS processing are of special concern in this chapter too.

Chapter 3 consists of a comparison between differenced and undifferenced processing strategies. The problems with the differenced phase observations and their effect on the estimates are discussed in this chapter.

Chapter 4 is about the covariance matrix of the observations. We improve this matrix by incorporating satellite coordinates and clocks uncertainties into the system of observation equations.

In chapter 5 we will discuss the results of the preceding chapters in the form of conclusions and remarks. A To Do part (future work) will be part of this chapter too.

### 1.5 Software implementation and limitations

In this thesis, MATLAB is the main programming language used for the numerical studies.
The difficulties and restrictions in this thesis are as follows:

- Due to the use of satellite ephemeris and clock files published by the IGS (International GNSS service) and the fact that they are tabulated for either 15 or 5 minutes time intervals, the limitation to follow this time intervals exists in all numerical studies in this thesis. This means that we have just picked the observations of the epochs at the above-mentioned time intervals and not used the rest of the observations.

- Because of the latency in publishing the above-mentioned files, the processing is restricted to the post-process mode.

- The other limitation in the numerical studies is that we have used only phase observations for precisely estimating the unknowns. The reason of this limitation is explained in chapter 2.

- Ionospheric delay dithers are the other limiting phenomena. Even in good ionosphere conditions, these dithers happen in polar zones and tropical areas. These dithers are similar to the cycle slips and could be mistaken by the processing software when cycles slip detection. Therefore, we process the observations of the stations located in mid-latitude areas to prevent having problem of these dithers in the observations.

- Ocean tide loading is another limiting factor. The closer to the coastlines you are the more ocean loading effect you will have. Hence, we have selected points far enough from the coastlines to reduce this effect on the station position estimates.

- In this research, only static solution is used for the numerical studies.
2 Positioning by GPS

This chapter is a concise collection of the concepts and principles of GPS, which can be found in many textbooks, e.g. Hofmann-Wellenhof et al. (2007), Leick (2003), Misra and Enge (2006), Parkinson and Spilker (1996) and Xu (2007).

2.1 GPS observables

Point positioning by GPS is the solution of a spatial resection of which the distances from the GPS satellites to the receiver antenna are measured. The GPS satellites transmit electromagnetic signals to enable distance measurement at the receiver. For this measurement to be done, some pseudo random noises (PRNs) and navigation signals, created at the GPS satellites, modulate some carrier waves constructing the GPS signals together. Table 2.1 shows the particulars of the GPS signals (Verhagen 2005).

<table>
<thead>
<tr>
<th>Frequency band</th>
<th>Frequency [MHz]</th>
<th>Civil PRN</th>
<th>Precise PRN</th>
<th>Military PRN (planned for modernized GPS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>1575.42</td>
<td>C/A</td>
<td>P1</td>
<td>M</td>
</tr>
<tr>
<td>L2</td>
<td>1227.60</td>
<td>C/A</td>
<td>P2</td>
<td>M</td>
</tr>
<tr>
<td>L5</td>
<td>1176.45</td>
<td>I+Q</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The code or pseudorange observable is derived from demodulating the signals and comparing the transmitted PRNs with the analogue PRNs, generated in the receiver. This, called code correlation, gives us the transition time of the signal from satellite to receiver. Therefore, the code observable is (Misra and Enge 2006):

\[ p_{r,j}(t) = c \left[ t_r(t) - t^s(t - \tau_{r}^s) \right] + \varepsilon_{p,j} \]  \hspace{1cm} (2.1)

where:

\[ p_{r,j}^s \] : code observable at receiver \( r \) from satellite \( s \) on frequency \( j \) [m]
The code observable is a coarse distance measurement between satellite and receiver. On contrary, the phase observable is a very precise measurement of this distance. The drawback of this observable is that it is an ambiguous measure of the geometric satellite-receiver distance. The phase observable is equal to the difference between carrier signal phase generated at the satellite at the transmission time and the one generated at the receiver at the reception time. The phase observable is then:

\[ \varphi_{r,j}^s(t) = \phi_{r,j}(t) - \phi_{j}^s(t - \tau_r^s) + N_{r,j}^s + \epsilon_{\phi} \]  

(2.2)

where:

- \( \varphi \) : carrier phase observable [cycles]
- \( N \) : integer carrier phase ambiguity [cycles]
- \( \epsilon \) : (random) phase measurement error (noise) [cycles]

We note that:

\[ \phi_{r,j}(t) = f_j t_r(t) + \phi_{r,j}(t_0) \]  

(2.3)

\[ \phi_{j}^s(t - \tau_r^s) = f_j t^s(t - \tau_r^s) + \phi_{j}^s(t_0) \]  

(2.4)

with:

- \( f \) : nominal frequency of carrier wave [s\(^{-1}\)]
- \( \phi_{r,j}(t_0) \): initial receiver phase at zero time (the beginning of signal tracking) [cycles]
- \( \phi_{j}^s(t_0) \): initial satellite phase at zero time [cycles]

Therefore, the carrier phase observable will be:

\[ \varphi_{r,j}^s(t) = f_j t_r(t) - f_j t^s(t - \tau_r^s) + [\phi_{r,j}(t_0) - \phi_{j}^s(t_0)] + N_{r,j}^s + \epsilon_{\phi} \]  

(2.5)
To transform the phase observable unit to metre, it must be multiplied by the wavelength \( \lambda_j = \frac{c}{f_j} \). Thus, Eq. (2.5) becomes:

\[
\phi^s_{r,j}(t) = \lambda_j \varphi^s_{r,j}(t) = c[t_r(t) - t^s(t - \tau^s_r)] + \lambda_j[\varphi_{r,j}(t_0) - \varphi^s_{j}(t_0)] + \lambda_j N^s_{r,j} + \epsilon_{\phi_j} \tag{2.6}
\]

where \( \epsilon_{\phi} \) is the phase observation noise in meters. Both receiver and satellite clocks are not exactly synchronized with the GPS time. Thus, the respective clock errors \( dt_r \) and \( dt^s \) must be accounted for:

\[
t_r(t) = t + dt_r(t)
\]

\[
t^s(t - \tau^s_r) = t - \tau^s_r + dt^s(t - \tau^s_r)
\]

Substitution of (2.7) and (2.8) into (2.1) and (2.6) yields:

\[
p^s_{r,j}(t) = c t^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \epsilon_{p_j}
\]

\[
\phi^s_{r,j}(t) = c t^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \lambda_j[\varphi_{r,j}(t_0) - \varphi^s_{j}(t_0)] + \lambda_j N^s_{r,j} + \epsilon_{\phi_j}
\]

Setting \( \rho^s_r = c t^s_r \), which is the geometric distance between satellite and receiver, the code and phase observables will be:

\[
p^s_{r,j}(t) = \rho^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \epsilon_{p_j} \tag{2.9}
\]

\[
\phi^s_{r,j}(t) = \rho^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \lambda_j[\varphi_{r,j}(t_0) - \varphi^s_{j}(t_0)] + \lambda_j N^s_{r,j} + \epsilon_{\phi_j} \tag{2.10}
\]

Considering the fact that the GPS signals are subject to atmospheric and some other systematic effects (see next sections), the complete observation equations for the code and phase observations will be:

\[
p^s_{r,j}(t) = \rho^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + da^s_{r,j}(t) + d\eta^s_{r,j}(t) + \epsilon_{p_j} \tag{2.11}
\]

\[
\phi^s_{r,j}(t) = \rho^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \lambda_j[\varphi_{r,j}(t_0) - \varphi^s_{j}(t_0)] + \lambda_j N^s_{r,j} + da^s_{r,j}(t) + d\eta^s_{r,j}(t) + \epsilon_{\phi_j} \tag{2.12}
\]

with:
\( da \): atmospheric error
\( d\eta \): other systematic effects (multipath, phase wind-up, relativity, solid Earth tide and ocean tide loading, antenna phase centre and hardware delays which are discussed in the sequel).

### 2.2 Errors and systematic effects

As the expected accuracy level for the PPP is cm level, all systematic effects with the magnitude of one cm or more must be removed from the observations. In this section such systematic effects are discussed and explained briefly.

#### 2.2.1 Atmospheric effects

GPS signals on their way from satellite to receiver must pass through the Earth’s atmosphere. Due to the presence of neutral and ionized particles in the atmosphere, the travel path of the signals is not a straight line and therefore the signals are delayed. The author refers to Odijk (2002), Kleijer (2004) and Memarzadeh (2009) for more details about the atmospheric effects on the GPS signals.

One can classify the layers of the atmosphere depending on their effect on the GPS signals. The two important layers in GPS processing are \textit{ionosphere} and \textit{troposphere}. Therefore, \( da \) in the equations (2.11) and (2.12) will be divided to the ionospheric and tropospheric parts. The new observation equations for the code and phase observations read then:

\[
P_{\tau,j}^s(t) = \rho_r^s + c[dt_r(t) - dt^s(t - \tau_r^s)] + \gamma_{pj}I_{\tau,j}^s(t) + T_{\tau,j}^s(t) + d\eta_{\tau,j}^s(t) + \epsilon_{\phi_j} \tag{2.13}
\]

\[
\phi_{\tau,j}^s(t) = \rho_r^s + c[dt_r(t) - dt^s(t - \tau_r^s)] + \lambda_j[\phi_{\tau,j}(t_0) - \phi_j^s(t_0)] + \lambda_jN_{\tau,j}^s + \gamma_{\phi_j}I_{\tau,j}^s(t) + T_{\tau,j}^s(t) + d\eta_{\tau,j}^s(t) + \epsilon_{\phi_j} \tag{2.14}
\]

where:

- \( l \): ionospheric delay of L1 code [m]
- \( T \): tropospheric delay [m]
- \( \gamma \): ionospheric dispersion factor

The ionosphere is a dispersive medium. This means that the ionospheric dispersion factor is a function of frequency and consequently, differs for each GPS frequency.
\[ \gamma_{pj} = -\gamma_p = \frac{r_j^2}{r_p^2}, \quad j = 1,2,5 \] (2.15)

Moreover, the ionosphere affects code and phase observables in different ways. This effect occurs as a delay on code observable and as an advance in phase observable. This is why the sign of the effect is different for code and phase observables. The values of \( \gamma \) are listed in Table 2.2 for each of the GPS signals.

**Table 2.2:** Dispersion factor for each of the GPS signals

<table>
<thead>
<tr>
<th></th>
<th>Code</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>L2</td>
<td>1.647</td>
<td>-1.647</td>
</tr>
<tr>
<td>L5</td>
<td>1.793</td>
<td>-1.793</td>
</tr>
</tbody>
</table>

A way to remove the ionospheric delay (almost completely) is to use a dual (triple) frequency receiver and form a special linear combination of the signals, called *ionosphere-free* linear combination. Following are the ionosphere-free linear combinations of the code and phase observations respectively:

\[
\begin{align*}
p_{r,1f}(t) &= \frac{r_j^2}{r_j^2 - r_j^2} p_{r,1}(t) - \frac{r_j^2}{r_j^2 - r_j^2} p_{r,j}(t) \\
\phi_{r,1f}(t) &= \frac{r_j^2}{r_j^2 - r_j^2} \phi_{r,1}(t) - \frac{r_j^2}{r_j^2 - r_j^2} \phi_{r,j}(t)
\end{align*}
\] (2.15a)

To mitigate the effect of ionosphere on the GPS data one can use some models e.g. Klobuchar (Klobuchar 1996), NeQuick (Radicella et.al. 2003) and GIM (Schaer 1999).

Contrary to ionosphere, the troposphere is a non-dispersive medium. Therefore, the tropospheric delay is the same on all GPS observables and frequencies. Usually, this delay is computed in the zenith direction at the receiver and then mapped on the satellite-receiver vector. The most famous models to find zenith delays are Saastamoinen (Saastamoinen 1972, 1973) and Hopfield (Hopfield 1969, 1970, 1972) and as mapping function, Ifadis (Ifadis 1992) and Niell (Niell 1996) are the ones which are commonly in use.
Depending on the behavior of troposphere, we divide the tropospheric delay into two parts; wet and dry tropospheric delays. Dry (or hydrostatic) part, ranging about 2.20-2.40 m, is possible to compute accurately enough for geodetic purposes using existing models and ground meteorological measurements. Wet part, ranging from 0 to 0.40 m, can only be determined with the accuracy of 2-5 cm (Kleijer 2004).

The height component of the receiver position is highly correlated with the zenith tropospheric delay (Kleijer 2004). Thus, the residual error of the model, will strongly affect the height component. Consequently, for precise geodetic purposes, the zenith wet delay must be considered as an additional parameter to the system of observation equations. Therefore, the new observation equations for code and phase observations read:

\[
p^{s}_{r,j}(t) = \rho^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \gamma p_j I^s_{r,j}(t) + M_w(z^s_r(t))D_w(t) + d\eta^s_{r,j}(t)
\]

\[+ \epsilon^p_{p_j} \tag{2.16}\]

\[
\phi^{s}_{r,j}(t) = \rho^s_r + c[dt_r(t) - dt^s(t - \tau^s_r)] + \lambda_j [\varphi_{r,j}(t_0) - \varphi^s_{j}(t_0)] + \lambda_j N^s_{r,j} + \gamma \phi^s_{r,j}(t)
\]

\[+ M_w(z^s_r(t))D_w(t) + d\eta^s_{r,j}(t) + \epsilon^p_{\phi_j} \tag{2.17}\]

where:

- \(M_w\) : mapping function
- \(z^s_r\) : zenith angle of satellite \(s\) from receiver \(r\)
- \(D_w\) : zenith wet delay [m]

Note that in the equations (2.16) and (2.17) the tropospheric delay of the dry part is lumped to \(\eta^s_r\). From these new observation equations, we can estimate the wet delay and it will not have any influence on the receiver position.

### 2.2.2 Phase wind-up effect

GPS signals are (right-handed) circularly polarized waves. This means that the electric field makes a spiral movement from transmitter to receiver (see fig. 2.1).
In Fig. 2.1, $E$ is the vector of electric field, $H$ is the vector of magnetic field and $j$ is the vector of propagation direction. For such a wave, when transmitter or receiver rotates, the phase of the received wave will change. This change, a systematic effect in the GPS phase observation, is called phase wind-up effect. Theoretically, it ranges from zero to a number of cycles of the carrier phase. GPS satellites bodies need to orient towards the Sun to use their photo-cells for supplying energy. Therefore they must rotate around their body frame axes and this rotation results in the satellite phase wind-up. For more details about this effect and how to remove it we refer to Wu et al. (1993). The phase wind-up effect due to receiver rotation is absorbed by the receiver clock error if only phase observation is used (Wu et al. 1993).

### 2.2.3 Multipath

Multipath effect is an error source for both code and phase observations. This error happens when the receiver antenna receives direct and indirect (reflected from objects nearby the receiver) signals from the GPS satellites (see Fig. 2.2).
As shown in Fig. 2.2, multipath depends only on the local environment around the antenna. The indirect signal depends on the reflecting surface and the position of satellite. The reflecting surface is usually static whereas the satellite moves with time. Thus the multipath effect varies with time.

Theoretically, the multipath effect may reach up to 15 meters for P-code and 150 meters for C/A-code and a few centimeters for phase measurements (Xu 2007).

For more details about multipath and how to mitigate its effect on the GPS signals, we refer to Xu (2007), Braasch (1996) and Langley (1998).

The reader should note that the multipath effect does not influence code and phase observations equally. As this effect could be lumped to the satellite clock error, then we will have different satellite clock errors for code and phase observations. This must be accounted for when using code and phase observations together for the GPS processing. To avoid this inconsistency between the code and phase observations, it is highly recommended to use just phase observations for precise applications of GPS.

In this research, we have used only phase observations of the IGS permanent stations. As these stations are located in good places so that multipath is ignorable (no reflecting surfaces are around the points), this effect is treated as stochastic error (as part of the observation noise $\epsilon$ in equation 2.17) through this research.
2.2.4 The relativistic effect

If two observers determine the distance between two points or the time difference between two events, they will measure different lengths and times if (Fliegel and DiEsposti 1996):
- they are moving with respect to each other
- in the same gravitational field one is higher or lower than the other one
- one is accelerating with respect to the other

GPS observations must be corrected accordingly for the above-mentioned effects. Many contributions so far are done to study the relativistic effect on the GPS observations. We refer to Ashby (2003) and Kouba (2002, 2004) for more details. In GPS processing, only periodic relativity correction must be applied (IS-GPS-200). This correction is:

\[ \Delta t_{rel} = -\frac{2r_\text{s} \cdot \overline{V}_\text{s}}{c^2} \]  

(2.18)

where:

- \( \Delta t_{rel} \): periodic relativity correction [s]
- \( r_\text{s} \): satellite position vector
- \( \overline{V}_\text{s} \): satellite velocity vector

The same convention has also been adopted by IGS, i.e., all the IGS satellite clock solutions are consistent with this convention (Kouba and Héroux 2000).

2.2.5 Antenna phase center variations

The antenna phase center variation is a small effect, which must be accounted for when precise positioning is in demand. This error occurs because the geometrical antenna center (the point to which the antenna coordinates are attributed) is not coincident with the point at which the signals are received/transmitted. This effect depends on the frequency of the signal. Therefore it is different for L1 and L2. It also depends on the azimuth and elevation of the satellite with respect to the receiver. For more details we refer to Xu (2007).

To help the GPS users remove the effect, the International GNSS service (IGS) publishes text files (ATX files) in which one can find all needed parameters for correcting the observations. These files are the source of applying the correction for the antenna phase center variations in this research.
2.2.6 Hardware delays (biases)

When generating/receiving GPS signals, they must pass through some electronic paths which cause delays (biases) in transmission/reception of the signals. These delays are usually lumped to the satellite/receiver clock errors. These delays (biases) vary from one signal to another. Consequently, different satellite/receiver clock errors, derived from processing different signals, have different values and are not equal to each other. These biases on code observations vary in time up to ±4ns for satellite hardware (Schaer 1999) whereas on phase observations they are small and negligible.

As mentioned before, different code observations have different code biases. This means that if we compute satellite/receiver clock errors using different code observations, we get different estimates for them. Receiver clock error is always estimated (not fixed) by solving the system of observation equations, but satellite clock errors (or simply satellite clocks) are fixed in many applications of GPS signals. Therefore, the question; which satellite clock (obtained from which code observation) should be used, arises. As an accepted rule, the satellite clocks, obtained from processing ionosphere-free linear combination of code observations, are the clock representative and their biases are assumed to be zero. Thus, the biases for the rest of code observations are represented as differential biases with respect to ionosphere-free one. These biases, called differential code biases (DCBs), are used for correcting the fixed satellite clocks in the observation equations when a code observation other than ionosphere-free linear combination is used (see chapter 3 for more details).

Analogous to the multipath effect, hardware delays have different effects on code and phase observations. Therefore, this must be accounted for when high accurate estimation of the unknowns is in demand. Through this research, only phase observations have been used to avoid this inconsistency between the hardware delays of code and phase observations. The hardware delay on phase observation is treated as stochastic error (as part of the observation noise $\epsilon$ in equation 2.17) in this thesis.

2.2.7 Solid Earth tide and ocean tide loading effects

In the field of gravitational attraction of the Sun and the Moon and other celestial bodies, the Earth, as an elastic body is deformed. This deformation results in displacements on the points on the Earth which may reach up to 60cm (Xu 2007). In GPS processing the solid Earth tide is a well-known correction which is formulated in IERS Conventions (2010) as follows:
\[
\Delta \vec{r} = \sum_{j=1}^{2} \frac{\mu_j R_E^4}{\mu j} \left\{ h_2 \vec{r} \left[ \frac{3}{2}(\vec{r}_j \cdot \vec{r})^2 - \frac{1}{2} \right] + 3l_2(\vec{r}_j \cdot \vec{r})[\vec{r}_j - (\vec{r}_j \cdot \vec{r})\vec{r}] \right\}
\]

(2.19)

where:

- \( \Delta \vec{r} \): displacement vector of the station due to degree 2 of tidal potential
- \( \mu \): the gravitational constant of the Earth
- \( R_E \): the equatorial radius of the Earth
- \( \mu_j \): the gravitational constant of the Moon (\( j = 1 \)) and the Sun (\( j = 2 \))
- \( \vec{r}_j \): the geocentric position vector of the Moon (\( j = 1 \)) and the Sun (\( j = 2 \)), \((\vec{r}_j \) is the length of the vector\)
- \( \vec{r} \): the station position vector
- \( h_2 \): Love number of degree 2
- \( l_2 \): Shida number of degree 2

The degree 2 Love and Shida numbers are computed by these formulas:

\[
h_2 = 0.6078 - 0.0006 \frac{3\sin^2 \varphi - 1}{2}; \quad l_2 = 0.0847 + 0.0002 \frac{3\sin^2 \varphi - 1}{2}
\]

(2.20)

There is also a permanent part of tidal displacement included in degree 2 tidal potential. The radial and transverse (in the north direction) components of this part are:

\[
-0.0603(3 \sin^2 \varphi - 1) \quad \text{and} \quad -0.0252 \sin 2 \varphi
\]

(2.21)

The load due to ocean tide influences the Earth surface. The GPS observations must be corrected for the displacement, generated by this load if the station is close to the coast and high accuracy positioning is in demand. The ocean tide loading displacement vector is computed as (IERS Conventions 2010):

\[
\Delta \rho_j = \sum_{i=1}^{11} f_i \cdot \text{amp}_j(i) \cdot \cos[\text{arg}(i, t) - \text{phase}_j(i)]; \quad \text{arg}(i, t) = \omega_i t + \chi_i + u_i
\]

(2.22)

where:

- \( j = 1,2,3 \): displacement in radial, west and south directions respectively
- \( \text{amp}_j(i) \): amplitude of the \( i^{th} \) wave related to the computation point
- \( \text{phase}_j(i) \): phase of the \( i^{th} \) wave related to the computation point
- \( \text{arg}(i, t) \): the argument of the \( i^{th} \) wave at the computing time \( t \)
\( \chi_i \) : the astronomical argument at time of zero hour

\( f_i \) and \( u_i \): lunar node dependent parameters

\( \omega_i \) : the angular velocity of the \( i^{th} \) wave

\( \omega_i, f_i \) and \( u_i \) can be found in Doodson (1928).

Ocean tide loading effect could be neglected for low accuracy kinematic positioning or for the static solution when a 24-hour observation time span is used (Kouba and Héroux 2000). Thus, in this research the ocean tide loading effect is neglected.

### 2.2.8 The error budget for the phase-only PPP

To summarize this section, the error budget for the PPP, when only (ionosphere-free) phase observations are used, will be discussed. Following, we have listed the error sources and their amplitudes affecting each frequency (L1 or L2):

#### Table 2.3: The error budget for the phase observations (\( \sigma \))

<table>
<thead>
<tr>
<th>Category</th>
<th>Error Source</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Satellite</strong></td>
<td>orbit</td>
<td>( \sim 2.5 \text{ cm} )</td>
</tr>
<tr>
<td></td>
<td>clock</td>
<td>( \sim 7.5 \text{ ps (( \sim 2.5 \text{ cm} ))} )</td>
</tr>
<tr>
<td></td>
<td>hardware delays</td>
<td>mm-cm level</td>
</tr>
<tr>
<td><strong>Signal propagation</strong></td>
<td>ionosphere</td>
<td>mm-cm level</td>
</tr>
<tr>
<td></td>
<td>troposphere</td>
<td>mm-cm level</td>
</tr>
<tr>
<td></td>
<td>multipath</td>
<td>mm-cm level</td>
</tr>
<tr>
<td><strong>Antenna</strong></td>
<td>phase center offset</td>
<td>mm-cm level</td>
</tr>
<tr>
<td></td>
<td>phase center variations</td>
<td>mm level</td>
</tr>
<tr>
<td><strong>Receiver</strong></td>
<td>observation noise</td>
<td>mm level (2 mm for L1)</td>
</tr>
<tr>
<td></td>
<td>hardware delay</td>
<td>mm-cm level</td>
</tr>
</tbody>
</table>

Note that the uncertainties of the satellite orbits and clocks are incorporated into the system of observation equations (see section 4.2.3) and the rest of the errors in the table are treated as stochastic errors in the observations. The ionospheric error in this table consists of higher order delays and the bending effect (since we eliminate the first order
delay by forming ionosphere-free linear combination of the phase observations). The reader should note that the above-mentioned errors (except hardware delays and orbit and clock errors) increase when the satellite elevation angle decreases. This means that the least errors happen at the zenith direction at the station. Therefore, we find $\sigma_0$, in Eq. (4.3), which is the ionosphere-free phase uncertainty at the zenith direction. If we assume that all the above errors (excluding orbit and clock uncertainties), be 1 mm at zenith and $\epsilon_{\phi1} = 2$ mm (the observation noise of L1) and $\sigma_{\phi2} = \frac{f_1}{f_2} \sigma_{\phi1}$ (Leick 2003) then:

$$\begin{align*}
\sigma_{\phi1} &= \sqrt{\sum_{i=1}^{8} \sigma_i^2} \approx 3.3 \text{ mm} \\
\sigma_{\phi2} &= \frac{f_1}{f_2} \sigma_{\phi1} \approx 4.2 \text{ mm}
\end{align*}$$

Substituting the values of $f_1$ and $f_2$ in Eq. (2.15a) yields:

$$\phi_3 = 2.547 \phi_1 - 1.547 \phi_2$$

Thus:

$$\sigma_0 = \sigma_{\phi3} = \sqrt{2.547^2 \sigma_{\phi1}^2 + 1.547^2 \sigma_{\phi2}^2} \approx 10.6 \text{ mm}$$

### 2.3 Processing strategies

Depending on our demand we process GPS observations in different ways. One can classify these ways (strategies) as follows:

#### 2.3.1 Code positioning and phase positioning

The accuracy of the observations is one of the criteria for classification of the strategies. The precision of code observations is at decimeter level whereas this accuracy for phase observation is at millimeter level. Therefore, depending on the desired accuracy one can choose one of these kinds of positioning.

#### 2.3.2 Absolute and relative positioning

The terms “Absolute positioning”, “Single point positioning” or “Stand alone positioning” convey the same meaning, which is to find a single point position using just one receiver, observing some satellites.
On the contrary, “Relative positioning” or “differential positioning” means finding points positions using at least two receivers, observing some satellites simultaneously. In relative positioning we find vector of the point coordinate differences instead of point positions (see chapter 3 for more information).

2.3.3 Static and kinematic

If the receiver (in absolute positioning) is stationary, the positioning is static. If the receiver moves with time the positioning is kinematic.
In relative positioning if one of the receivers is moving, it is called kinematic positioning. Otherwise, it is static.

2.3.4 Real-time processing and post processing

The procedure in which observing the satellites and computing the position are done at the same time, is called real-time processing. If observation collection stage at the field is finished before computation of position, it is called post processing. Generally, the code observations enable us to do real-time positioning. This is because the code observations are not ambiguous. The ambiguity in the observations takes time to be solved.

2.4 Satellite orbits and clocks

In GPS positioning, the satellites often play the role of known points. Therefore, for most of the GPS applications, satellite orbits and clocks are necessary as known parameters. As the main purpose of GPS was the standard positioning in real-time with the aid of code observations, the satellite orbits and clocks are broadcasted along with the signals. These satellite orbits and clocks are called “broadcast ephemeris”. However, this ephemeris is predicted at control segment of GPS and is not accurate enough for most of precise GPS applications. To provide the GPS users with the precise ephemeris (as well as other data and products), the International GNSS Service (IGS) was established in 1994. In the IGS analysis centers, high accuracy GNSS products are computed using the data of more than 300 permanent GPS stations world widely and publish them through the internet.

For precise GPS applications we use the IGS precise ephemeris. This ephemeris is published in the form of “standard product #3” (sp3 or recently sp3c files) consisting of satellite coordinates and clock corrections (and their uncertainties) at 15 minutes time interval.
2.4 Concluding remarks

So far, we introduced the basic knowledge of GPS. To summarize this chapter and emphasize the main points, we pay more attention to the following remarks:

- Special care must be taken when using both code and phase observations together, because the systematic effects influencing each of them are different and of different magnitudes. Particularly, we can name multipath effect and the DCBs.
- To incorporate satellite orbits and clocks uncertainties into the system of observation equations, the uncertainties must be available at each epoch. The lack of this information (uncertainties) is a problem when interpolated orbits are used.
3 Differenced and undifferenced observations

3.1 Introduction

GPS measurements are relative observations from the GPS satellites to receivers. The accuracy of the position estimates depend on the random and systematic errors of these observations. If two (or more) receivers are observing the satellites simultaneously, the influence of the common errors decreases significantly by differencing the observations. Thus, practically the accuracy of the solution for the base-line vector formed between the two terminal observing points is larger than the accuracy of the position estimators from the absolute positioning.

Although in practice, differencing of observations yields better estimators, the mathematical model of the system of differenced observation equations is complicated. The implementation aspects of the differencing techniques require special care when high accuracies are in demand from the GPS data processing. In this chapter, a comparative discussion is presented to find ways to decrease the risk of having poor solutions due to the implementation complexity of the differencing techniques.

3.2 Undifferenced observation equations

In the preceding chapter, Equations (2.16) and (2.17) represent the undifferenced code and phase observation equations respectively. To account for the hardware delays, however, the above-mentioned equations must be modified (see Parkinson and Spilker 1996). Both GPS satellite and receiver are dealing with the hardware delays and they depend on the signal types and frequencies (Schaer 1999). Thus, the new observation equations for code and phase observations read:

\[
\begin{align*}
    \rho_{r,j}^s(t) &= \rho_{r,j}^s + c[dt_r(t) + \mathbb{B}_{r,j}^s - dt^s(t - \tau_r^s) - \mathbb{B}_j^s] + \gamma_{pj} l_{r,j}^s(t) + M_w(z_r^s(t))D_w(t) + d\eta_{r,j}^s(t) + \epsilon_{p,j} \\
    \phi_{r,j}^s(t) &= \phi_{r,j}^s + c[dt_r(t) + \mathbb{C}_{r,j} - dt^s(t - \tau_r^s) - \mathbb{C}_j^s] + \lambda_j [\phi_{r,j}(t_0) - \phi_j^s(t_0)] + \lambda_j N_{r,j}^s + \\
    &\quad \gamma_{\phi j} l_{r,j}^s(t) + M_w(z_r^s(t))D_w(t) + d\eta_{r,j}^s(t) + \epsilon_{\phi j}
\end{align*}
\]

(3.1)

(3.2)

where:

\( \mathbb{B}_{r,j} : \) receiver code bias for the \( j^{th} \) frequency  
\( \mathbb{B}_j^s : \) satellite code bias for the \( j^{th} \) frequency
Generally, hardware delays are considered as nuisance parameters in GPS processing. These delays in the phase observations are small and could be treated as observation noise (Liu et al. 2004). On the contrary, these delays in the code observations (code biases) are very important in absolute positioning. They vary from decimeter level to meter level (Schaer 1999). Different code observables have different code biases. Therefore, the derived clocks (satellite or receiver) from each code observation are contaminated with its bias. In practice, the clock derived from the ionosphere-free linear combination of P1 and P2 codes is considered as the clock representative. This means that users of single frequency receivers must apply a special correction (Differential Code Bias or DCB correction) for consistency. For more information about DCBs and the procedure of applying them, we refer to Schaer (1999).

In the published files (e.g. by the IGS), the values for orbits and clocks refer to the ionosphere-free phase observations. This implies that although we do not consider any bias for ionosphere-free code, there is still a discrepancy between the clocks derived from ionosphere-free code and phase observations. Therefore, this discrepancy must be accounted for when using both code and phase data together.

It is worth mentioning when estimating the clock errors (either for satellite or receiver) in the system of observation equations one can lump DCBs to the clock parameters for the system to be consistent. Thus, we do not see any parameter for DCBs in the system of observation equations.

### 3.2.1 Gauss-Markov form of the Undifferenced observation equations

Equations (3.1) and (3.2) are nonlinear because of the existence of term (the satellite-receiver geometric range) in both of them. After linearization of the equations, fixing all satellite dependent parameters and applying all corrections for the systematic errors, the Gauss-Markov form of the Undifferenced observation equations will be:

\[ E[\Delta y] = Ax \quad D[\Delta y] = Q_y \]  

(3.3)

where:

- \( E[] \) : mathematical expectation operator
- \( \Delta y \) : linearized undifferenced observation vector
\( A \): design (Jacobian) matrix
\( x \): unknown vector
\( D(\cdot) \): dispersion operator
\( Q_y \): covariance matrix of the observations

This form for the code observations at a particular epoch \( k \) is then:

\[
E\left\{ \begin{pmatrix}
\Delta p^1_r \\
\Delta p^2_r \\
\vdots \\
\Delta p^m_r
\end{pmatrix}
\right\} = \begin{pmatrix}
-e^1_r & c & \gamma_{pj} & 0 & \cdots & 0 \\
-e^2_r & c & 0 & \gamma_{pj} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-e^m_r & c & 0 & 0 & \cdots & \gamma_{pj}
\end{pmatrix}
\begin{pmatrix}
M_w(z^1_r) \\
M_w(z^2_r) \\
\vdots \\
M_w(z^m_r)
\end{pmatrix} \begin{pmatrix}
\Delta x_r \\
dt_r \\
I^1_{r,j,k} \\
I^2_{r,j,k} \\
\vdots \\
p^m_{r,j,k} \\
D_{wk}
\end{pmatrix};
\quad D\{\Delta y_k\} = \sigma^2_p q_{y_k}
\]

(3.4)

In this equation, \( \Delta p^s_r \) is the linearized code observation from satellite \( s \) to receiver \( r \) and 
\( e^s_r = (e^s_{r1}, e^s_{r2}, e^s_{r3})^T \) is the vector of \( \frac{\overrightarrow{p^s_r}}{||\overrightarrow{p^s_r}||} \) (unit satellite-receiver vector) at epoch \( k \). \( \sigma_{pj} \) is the noise of the code observable of the \( j \)th frequency at the zenith direction. The covariance matrix \( q_{y_k} \) in this equation is:

\[
q_{y_k} = diag(\sec z^i_r); \quad i = 1, 2, \ldots, m
\]

(3.4a)

where \( z^i_r \) is the zenith angle of satellite \( s \) from the receiver \( r \). The diagonal arrangement of \( q_{y_k} \) is based on the assumption that the noise of the GPS observables are normally distributed (Tiberius and Borre 1999). Moreover, Euler and Goad (1991); Gerdan (1995); Jin (1996); Tiberius (1999) and Barnes (2000) represented that the standard deviations of the GPS observations are elevation dependent.

Usually, in GPS processing, we model or eliminate the atmospheric effects. Then the corresponding unknowns and design matrix entries to the tropospheric effects components disappear. To process the ionosphere-free code observations collected during \( N \) epochs in one batch, the matrix form for static mode is then:
\begin{equation}
E\left\{ \begin{pmatrix}
\Delta p_{r1}^{s_i} \\
\Delta p_{r2}^{s_i} \\
\vdots \\
\Delta p_{rN}^{s_i}
\end{pmatrix}
\right\} =
\begin{pmatrix}
-E_{r1}^{s_i} & cU_1 & 0 & \cdots & 0 \\
-E_{r2}^{s_i} & 0 & cU_2 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-E_{rN}^{s_i} & 0 & 0 & \cdots & cU_N
\end{pmatrix}
\begin{pmatrix}
\Delta x_r \\
dt_{r1} \\
dt_2 \\
\vdots \\
dt_{rN}
\end{pmatrix};
\end{equation}

\begin{equation}
D\{\Delta y\} = Q_y = \sigma_{p3}^2 \text{diag} \begin{pmatrix}
q_{y1} \\
q_{y2} \\
q_{yN}
\end{pmatrix}
\end{equation}

where $\Delta p_{ri}^{si}$ is the set of the linearized ionosphere-free code observations, $s_i$ is the number of satellites and $dt_{ri}$ indicates the receiver clock error at the $i^{th}$ epoch respectively. The vector $U_i$ is of order $(s_i \times 1)$ and its all entries are unity. $E_{ri}^{si}$ is a $(s_i \times 3)$ matrix as follows:

\begin{equation}
E_{ri}^{si} = \begin{pmatrix}
e_{ri1}^{iT} \\
e_{ri2}^{iT} \\
\vdots \\
e_{riN}^{iT}
\end{pmatrix}
\end{equation}

The covariance matrix $Q_y$ in Eq. (3.5) is diagonal because time-correlation between the observations of two consecutive epochs is negligible if the time interval is more than 20s (Bona 2000). $\sigma_{p3}$ is the noise of the ionosphere-free code observation at the zenith direction.

Analogously, this matrix form for the ionosphere-free code observation equations in kinematic mode will be:

\begin{equation}
E\left\{ \begin{pmatrix}
\Delta p_{r1}^{s_i} \\
\Delta p_{r2}^{s_i} \\
\vdots \\
\Delta p_{rN}^{s_i}
\end{pmatrix}
\right\} =
\begin{pmatrix}
-\Delta x_r \\
\Delta x_r \\
\vdots \\
\Delta x_r
\end{pmatrix};
\end{equation}
\[
D[\Delta y] = Q_y = \sigma_{y}^2 diag \begin{pmatrix} q_{y1} \\ q_{y2} \\ \vdots \\ q_{yN} \end{pmatrix}
\]  \hspace{1cm} (3.6)

One can easily construct the matrix form for the phase observation equations just by adding the phase ambiguity unknowns and their corresponding part to the design matrix. Therefore, the system of observation equations at epoch \( k \) is:

\[
E\left\{ \begin{pmatrix} \Delta \phi^1_r \\ \Delta \phi^2_r \\ \vdots \\ \Delta \phi^m_r \end{pmatrix} \right\} = E_{\Delta y_k} = \begin{pmatrix} -e^1_r & c & Y & 0 & \cdots & 0 & M_w(z^1_r) & \lambda_j & 0 & \cdots & 0 \\ -e^2_r & c & 0 & Y & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -e^m_r & c & 0 & 0 & \cdots & Y & M_w(z^m_r) & 0 & 0 & \cdots & \lambda_j \end{pmatrix}_{A_k} \begin{pmatrix} \Delta x_r \\ dt_{rk} \\ I^1_{r,j,k} \\ I^2_{r,j,k} \\ \vdots \\ I^m_{r,j,k} \\ D_{wk} \\ a^1_{r,j} \\ a^2_{r,j} \\ \vdots \\ a^m_{r,j} \end{pmatrix}_{\Delta \tau} 
\]

\[
D[\Delta y_k] = \sigma_{y}^2 q_{y_k} 
\]  \hspace{1cm} (3.7)

Where \( \Delta \phi^r \) is the linearized phase observation, \( \sigma_{y} \) is the noise of the phase observable of the \( j \)th frequency at the zenith direction and:

\[
a^r_{r,j} = N^r_{r,j} + \varphi_{r,j}(t_0) - \varphi^r_{j}(t_0) 
\]  \hspace{1cm} (3.8)

For processing the ionosphere-free phase observations collected during \( N \) epochs in one batch, the matrix form for static mode is then (troposphere is modeled):
\[
E\left(\begin{pmatrix}
\Delta \Phi_{s_1}^{r_1} \\
\Delta \Phi_{s_2}^{r_2} \\
\vdots \\
\Delta \Phi_{s_N}^{r_N}
\end{pmatrix} \right) = \begin{pmatrix}
-E_{r_1}^{s_1} cU_1 & 0 & \cdots & 0 & \lambda_3 \Xi_1 \\
-E_{r_2}^{s_2} & cU_2 & \cdots & \lambda_3 \Xi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-E_{r_N}^{s_N} & 0 & \cdots & cU_N & \lambda_3 \Xi_N
\end{pmatrix} \begin{pmatrix}
\Delta x_r \\
dt_{r_1} \\
dt_{r_2} \\
\vdots \\
dt_{r_N} \\
a_{r_1}^1 \\
a_{r_2}^2 \\
\vdots \\
a_{r_j}^\zeta
\end{pmatrix}
\]

\[
D\{\Delta y\} = Q_y = \sigma_{\phi_3}^2 \text{diag} \begin{pmatrix}
q_{y_1} \\
q_{y_2} \\
\vdots \\
q_{y_N}
\end{pmatrix}
\tag{3.9}
\]

where \(\zeta\) is the number of ambiguity unknowns and \(\Xi_i\) is a matrix of size \((s_i \times \zeta)\) whose entries are 0 and 1 and the setting of the entries is done according to the order of the ambiguity unknowns and \(\lambda_3\) is the so-called narrow-lane wavelength which is around 107mm (Schaer 1999). \(\Delta \Phi_{r_i}^{s_i}\) is the set of the linearized phase observations at the \(i^{th}\) epoch. \(\sigma_{\phi_3}\) is the noise of the ionosphere-free phase observation at the zenith direction.

The matrix form for the phase observation equations in kinematic mode is:

\[
E\left(\begin{pmatrix}
\Delta \Phi_{s_1}^{r_1} \\
\Delta \Phi_{s_2}^{r_2} \\
\vdots \\
\Delta \Phi_{s_N}^{r_N}
\end{pmatrix} \right) = \begin{pmatrix}
-E_{r_1}^{s_1} & 0 & \cdots & 0 & cU_1 & 0 & \cdots & 0 & \lambda_3 \Xi_1 \\
0 & -E_{r_2}^{s_2} & \cdots & \vdots & 0 & cU_2 & \cdots & \lambda_3 \Xi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -E_{r_N}^{s_N} & 0 & 0 & \cdots & cU_N & \lambda_3 \Xi_N
\end{pmatrix} \begin{pmatrix}
\Delta x_r \\
dt_{r_1} \\
dt_{r_2} \\
\vdots \\
dt_{r_N} \\
a_{r_1}^1 \\
a_{r_2}^2 \\
\vdots \\
a_{r_j}^\zeta
\end{pmatrix}
\]

\[
D\{\Delta y\} = Q_y = \sigma_{\phi_3}^2 \text{diag} \begin{pmatrix}
q_{y_1} \\
q_{y_2} \\
\vdots \\
q_{y_N}
\end{pmatrix}
\tag{3.10}
\]
3.3 Singularities

A system of observation equations (as in 3.3) is called regular when \( Ax = 0 \) if and only if \( x = 0 \). This means that if \( Ax = 0 \) and \( x \neq 0 \) then the system is singular. Here, we briefly discuss the singularities in undifferenced observation equations and refer to Tiberius (1998) for more details.

3.3.1 Singularity in the undifferenced code observation equations

Let in Eq. (3.4) \( x = \left( 0,0,-1,\frac{c}{\gamma_j},\frac{c}{\gamma_j},...\right)^T \). Then \( Ax = 0 \) and the system of equations is singular. To remove the singularity, one way is to fix the ionospheric delays in the equations to their values from a model (e.g. Klobuchar model). The other way is to use vertical delay instead of the slant delay.

3.3.2 Singularity in the undifferenced phase observation equations

Suppose we have eliminated the ionospheric delay in the Eq. (3.7). Therefore, it reduces to the following equation:

\[
E \left[ \begin{array}{c}
\Delta \phi_r^1 \\
\Delta \phi_r^2 \\
\vdots \\
\Delta \phi_r^m \\
\end{array} \right] = \begin{pmatrix}
-e_r^T & M_w(z_r^1) & \lambda_3 & 0 & \cdots & 0 \\
-e_r^T & 0 & \lambda_3 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-e_r^T & M_w(z_r^m) & 0 & 0 & \cdots & \lambda_3 \\
\end{pmatrix} \begin{pmatrix}
\Delta x_r \\
dt_{rk} \\
D_{wk} \\
a_{rj}^1 \\
a_{rj}^2 \\
\vdots \\
a_{rj}^m \\
\end{pmatrix} + D\{ \Delta y_k \} = \sigma_{\phi_r}^2 q_{y_k}
\]

(3.11)

Now, let \( x = \left( 0,0,-1,\frac{c}{\lambda_3},\frac{c}{\lambda_3},...\right)^T \). Then \( Ax = 0 \) and the system of equations is singular. To remove the singularity, one way is to lump the receiver clock error to the first phase ambiguity unknown. The observation equations read then:
If one uses Eq. (3.12) to form the batch process of Eq. (3.9) or Eq.(3.10), the resulting system will not be singular anymore.

The second way to cope with the singularity is to use (ionosphere-free) phase and code data together. This is the way, which is usually used in the traditional GPS Precise Point Positioning (PPP). The system of the equations in this case is:

\[
E\left\{ \begin{pmatrix}
\Delta \phi^1_r \\
\Delta \phi^2_r \\
\vdots \\
\Delta \phi^m_r \\
\end{pmatrix}
\right\} = \begin{pmatrix}
-e^{1T}_r & 0 & M_w(z^1_r) & \lambda_3 & 0 & \cdots & 0 \\
-e^{2T}_r & c & M_w(z^2_r) & 0 & \lambda_3 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-e^{mT}_r & c & M_w(z^m_r) & 0 & 0 & \cdots & \lambda_3 \\
\end{pmatrix}
\begin{pmatrix}
\Delta \phi^1_r \\
\Delta \phi^2_r \\
\vdots \\
\Delta \phi^m_r \\
\end{pmatrix}
\] 

\[
D\{\Delta y_k\} = \text{diag}\left(\begin{pmatrix}
\sigma^2_{p3}q_{y_k} \\
\sigma^2_{p3}q_{y_k} \\
\end{pmatrix}\right)
\] 

(3.13)

Remark: As mentioned before, the satellite clock errors derived from code or phase observations are different. When they are supposed to be fixed in the observation equations, some bias terms must be added to the equations for consistence. To represent this in the matrix form of the observation equations one can write:
where $B_k^s$ is the considered satellite clock bias at epoch $k$. One cannot estimate the unknowns of these equations unless by lumping these bias terms into the ambiguity parameters. As there is no guarantee for constancy of the biases, the phase ambiguity may change with time. Thus, phase ambiguities constancy during continuously tracking of the satellites becomes lost. Therefore, phase-only data is recommended for precise GPS applications.

The other way to remove the singularity is the use of differenced observations. This issue is explained in the sequel.

### 3.4 Differenced observation equations

Differenced observations are formed by subtracting some of the entries of the observation vector from each other. If we classify the parameters to two classes of desired and nuisance, some of the nuisance parameters, common or highly correlated in the two observations to be subtracted from each other, are eliminated or significantly reduced. The other advantage of differencing is that it removes singularities in the system of the observation equations. For more details on differencing, we refer to Tiberius (1998) and Wells et al. (1986).

**Note:** Because of the aforementioned reasons in the previous section, from now on, the focus will be on the phase observations and code observations will be discussed only when
necessary. Atmospheric effects are also ignored in the next equations. They are assumed to be fixed to their values obtained from correction models.

### 3.4.1 Single differences

#### 3.4.1.1 Between-satellite single differences

Let the observation vector be the phase observations from \( m \) satellites to the receiver \( r \). The between-satellite single differenced observations form if one subtracts an observation from the others in the undifferenced observation vector. This happens when pre-multiplying the undifferenced observation vector by the following \((m - 1) \times m\) matrix, called between-satellite differencing operator (de Jonge 1998):

\[
R_{bs}^T = (-U_{m-1}, I_{m-1})
\]  

(3.15)

Thus, the Eq. (3.11) converts to:

\[
E[\Delta Y_k] = R_{bs}^T E\{\Delta y_k\} = R_{bs}^T A_k \begin{pmatrix} \Delta x_r \\ dt_{rk} \\ a^1_{rj} \\ a^2_{rj} \\ \vdots \\ a^m_{rj} \end{pmatrix}; \quad D[\Delta Y_k] = \sigma^2_{\phi j} R_{bs}^T a y_k R_{bs}
\]

(3.16)

Pre-multiplying \( A_k \) by \( R_{bs}^T \), results in an \((m - 1) \times (m + 4)\) matrix in which is as follows:

\[
R_{bs}^T A_k = \begin{pmatrix} (e_r^1 - e_r^2)^T & 0 & -\lambda_j & \lambda_j & 0 & \cdots & 0 \\ (e_r^1 - e_r^3)^T & 0 & -\lambda_j & 0 & \lambda_j & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ (e_r^1 - e_r^m)^T & 0 & -\lambda_j & 0 & 0 & \cdots & \lambda_j \end{pmatrix}
\]

(3.17)

As one can see all the entries corresponded to \( dt_{rk} \) are equal to zero. This means that we can ignore the second column of \( R_{bs}^T A_k \) and eliminate \( dt_{rk} \) from the vector \( \Delta x \). Matrix \( A'_k \) is an \((m - 1) \times (m + 2)\) matrix which is obtained from \( R_{bs}^T A_k \) after eliminating the zero column and reparametrizing the phase ambiguity unknowns, which reads:
observation equations for the merged undifferenced observation vector are:

\[
A'_k = \begin{pmatrix}
(e_r^1 - e_r^2)^T & \lambda_j & 0 & \cdots & 0 \\
(e_r^1 - e_r^3)^T & 0 & \lambda_j & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
(e_r^1 - e_r^m)^T & 0 & 0 & \cdots & \lambda_j
\end{pmatrix}
\]  

(3.18)

and:

\[
E\{\Delta y_k\} = R_{bs}^T E\{\Delta y_k\} = A'_k \begin{pmatrix}
\Delta x_r \\
\alpha_{r1}^{12} \\
\vdots \\
\alpha_{r1}^{1m} \\
\Delta x'
\end{pmatrix} ; \quad D\{\Delta y_k\} = \sigma_{y_k}^2 R_{bs}^T q_{y_k} R_{bs}
\]  

(3.19)

where \(\Delta x'\) is the reparametrized version of \(\Delta x\) (i.e. \(a_{rj}^{kl} = a_{rj}^k - a_{rj}^l\)). Note that there is no receiver clock unknown in \(\Delta x'\).

3.4.1.2 Between-receiver single differences

Assume that the two receivers 1 and 2 observed \(m\) satellites simultaneously. Then the observation equations for the merged undifferenced observation vector are:

\[
E\{\Delta y_k\} = A_k \begin{pmatrix}
\Delta x_1 \\
\Delta x_2 \\
dt_{1k} \\
dt_{2k} \\
a_{1j}^1 \\
a_{1j}^2 \\
\vdots \\
a_{1j}^m \\
da_{2j}^1 \\
da_{2j}^2 \\
\vdots \\
da_{2j}^m \\
\Delta x'
\end{pmatrix} ; \quad D\{\Delta y_k\} = \sigma_{y_k}^2 q_{y_k}
\]  

(3.20)
The between-receiver single differenced observations are formed if one subtracts the observations of the receiver 1 from the observations of the receiver 2 in the undifferenced observation vector. This happens when pre-multiplying the undifferenced observation vector by the following $m \times 2m$ matrix, called between-receiver differencing operator (de Jonge 1998):

$$R_{sd}^T = (-I_m, I_m)$$  \hspace{1cm} (3.21)

Thus, the Eq. (3.20) converts to:

$$E\{\Delta Y_k\} = R_{sd}^T E\{\Delta y_k\} = R_{sd}^T A_k \Delta x; \quad D\{\Delta Y_k\} = \sigma_{\phi_j}^2 R_{sd}^T q_{y_k} R_{sd}$$  \hspace{1cm} (3.22)

Pre-multiplying $A_k$ by $R_{sd}^T$, results in an $m \times (2m + 8)$ matrix in which is as follows:

$$\begin{pmatrix}
    e_1^T & -c & -\lambda_j & 0 & \ldots & 0 & \lambda_j & 0 & \ldots & 0 \\
    e_1^T & -c & 0 & -\lambda_j & \ldots & \vdots & 0 & \lambda_j & \ldots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    e_n^T & -c & 0 & 0 & \ldots & -\lambda_j & 0 & 0 & \ldots & \lambda_j \\
\end{pmatrix}$$  \hspace{1cm} (3.23)

Reparametrizing the unknowns so that $\Delta x_{12} = \Delta x_2 - \Delta x_1, \Delta d_{t12k} = dt_{2k} - dt_{1k}$ and $\sigma_{a_{12j}}^2 = a_{1j}^s - a_{2j}^s$ and letting leads to the new design matrix $A_{sd}$ which is of order $m \times (m + 4)$ reads then:

$$A_{sdk} = \begin{pmatrix}
    (e_1^1 - e_1^2)^T & c & \lambda_j & 0 & \ldots & 0 \\
    (e_1^2 - e_2^2)^T & c & 0 & \lambda_j & \ldots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    (e_n^m - e_2^m)^T & c & 0 & 0 & \ldots & \lambda_j \\
\end{pmatrix}$$  \hspace{1cm} (3.24)

and the matrix form of the observation equations will be:
\[
E\{\Delta Y_k\} = R_{sd}^TR_{sd}E\{\Delta y_k\} = A_{sdk} \begin{pmatrix}
\Delta x_{12} \\
\Delta t_{12k} \\
a_{12j}^1 \\
a_{12j}^2 \\
\vdots \\
a_{12j}^m \\
\Delta x
\end{pmatrix} ;
D\{\Delta Y_k\} = \sigma^2_{\phi j}R_{bs}^TR_{sd}q_{yk}R_{sd}R_{bs}
\] (3.25)

### 3.4.2 Double differences

Double differenced observations are formed when taking between-satellite difference of between-receiver single differenced observations. This means that if one pre-multiply \( \Delta y_k \) in the Eq. (3.20) by \( R_{bs}^TR_{sd} \), the resulting \((m-1)\)-vector \( \Delta Y_k \) is the vector of the double differenced phase observations. Analogous to the single differences, after reparametrizing the phase ambiguity unknowns \((a_{12j}^1 = a_{12j}^2 - a_{12j}^1)\), eliminating \( \Delta t_{12k} \) from the unknown vector and its correspondent column in the design matrix and letting \( e_{kl}^{ij} = e_k^j - e_l^j - e_k^j + e_l^j \), one gets to the following observation equations:

\[
E\{\Delta Y_k\} = R_{bs}^TR_{sd}E\{\Delta y_k\} = A_{ddk} \begin{pmatrix}
\Delta x_{12} \\
a_{12j}^1 \\
a_{12j}^2 \\
\vdots \\
a_{12j}^m \\
\Delta x
\end{pmatrix} ;
D\{\Delta Y_k\} = \sigma^2_{\phi j}R_{bs}^TR_{sd}q_{yk}R_{sd}R_{bs}
\] (3.26)

where:

\[
A_{ddk} = \begin{pmatrix}
e_{12}^{1T} & \lambda_j & 0 & \ldots & 0 \\
e_{12}^{13T} & 0 & \lambda_j & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
e_{12}^{1mT} & 0 & 0 & \ldots & \lambda_j
\end{pmatrix}
\] (3.27)

The most important property of the double differences is that their phase ambiguity parameters are intrinsically integer numbers. Looking into the double differenced phase ambiguities and considering Eq. (3.8), imply that the double differenced ambiguity:

\[
a_{12j}^2 = a_{12j}^2 - a_{12j}^1 = a_{2j}^2 - a_{1j}^1 + a_{1j}^1 = N_{2j}^2 - N_{1j}^2 - N_{1j}^1 + N_{1j}^1
\] (3.28)
is obtained from summation and subtracting of four integer numbers which becomes an integer. This is extra information for data processing and enables ambiguity resolution (see Teunissen et al. 1997). Ambiguity resolution helps reducing the observation time span significantly. Double differencing is widely used now in GPS network processing software. The GPS software solves the baselines accurately using double differenced phase observations. Then the baselines are used for other geodetic purposes, e.g. network processing or direct and inverse geodetic problems. For more details, we refer to de Jonge (1998) and Sjöberg and Shirazian (2012).

3.4.3 Triple differences

Taking differences between two consecutive epochs of the double difference phase observations (from the same satellites in both epochs) gives triple differences. Due to the constancy and independence of the phase ambiguity in time, triple differencing cancels out the phase ambiguities. Thus, in the unknown vector, there are just baseline components. For more details see Hofmann-Wellenhof et al. (2007).

3.5 The equivalence theorem for the linear models with and without nuisance parameters (or equivalence theorem in brief)

When the system of observation equations is manipulated (e.g. by differencing), we must assure that the new system results in the same estimates as of the former system. This issue is discussed and proved in Schaffrin and Grafarend (1986). We prove this equivalence in a more straightforward way by using the theory of the adjustment of partitioned models (Teunissen 1999-a).

**Theorem 1:** Consider the system of linear equations:

$$E\begin{bmatrix} y \\ m \times 1 \end{bmatrix} = A \begin{bmatrix} x \\ m \times 1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ m \times p & m \times (n-p) \end{bmatrix} \begin{bmatrix} X_1 \\ x_2 \end{bmatrix}; \quad D\{y\} = Q_y$$

(3.29)

where \( m > n \).

Let us assume that the two sub-matrices \( A_1 \) and \( A_2 \) are of full ranks, i.e. \( r(A_1) = p \) and \( r(A_2) = n - p \). If we introduce a matrix \( B_2 \) (which is orthogonal to \( A_2 \)) so that:

$$B_2^T A_2 = 0$$

(3.30)
If $\hat{x}_1$ is the Least-squares solution of $x_1$ in equation (3.29) and $\hat{x}'_1$ is the Least-squares solution of $x_1$ from the system of equations:

$$
E(B_2^T y) = B_2^T A_1 x_1; \quad D(B_2^T y) = B_2^T Q_y B_2
$$

then $\hat{x}'_1 = \hat{x}_1$.

**Proof:** From Eq. (3.29) the estimator $\hat{x}_1$ is:

$$
\hat{x}_1 = \left( A_1^T P_{A_2}^\perp Q_y^{-1} P_{A_2}^\perp A_1 \right)^{-1} A_1^T P_{A_2}^\perp Q_y^{-1} y
$$

(3.32)

where the orthogonal projector $P_{A_2}^\perp$ is:

$$
P_{A_2}^\perp = I - A_2 (A_2^T Q_y^{-1} A_2)^{-1} A_2^T Q_y^{-1}
$$

(3.33)

from the properties of the orthogonal projectors, we know that:

$$
Q_y^{-1} P_{A_2}^\perp = P_{A_2}^\perp = P_{A_2}^\perp Q_y^{-1} P_{A_2}^\perp
$$

(3.34)

Now assume that $B_2$ is a basis for the null space of $A_2^T$, meaning that $B_2^T A_2 = 0$. After multiplying of Eq. (3.29) by $B_2^T$ we have:

$$
E(B_2^T y) = B_2^T A_1 x_1; \quad D(B_2^T y) = B_2^T Q_y B_2
$$

(3.35)

Therefore the estimator $\hat{x}'_1$ of $x_1$ reads:

$$
\hat{x}'_1 = \left[ A_1^T B_2 (B_2^T Q_y B_2)^{-1} B_2^T A_1 \right]^{-1} A_1^T B_2 (B_2^T Q_y B_2)^{-1} B_2^T y
$$

(3.36)

Again, from the properties of the orthogonal projectors, we have:

$$
P_{A_2}^\perp = P_{Q_y B_2} = Q_y B_2 (B_2^T Q_y B_2)^{-1} B_2^T \Rightarrow Q_y^{-1} P_{A_2}^\perp = B_2 (B_2^T Q_y B_2)^{-1} B_2^T
$$

(3.37)

From (3.32), (3.34), (3.36) and (3.37) we conclude that $\hat{x}'_1 = \hat{x}_1$. 
Remark: In the GNSS processing, we see that the differencing operator, acts exactly like \( B_2 \) in (3.36). It means that the estimates of the desired parameters, with and without differencing are identical.

### 3.6 Some problems of differencing

Although differencing is a well-known solution for precise positioning, some problems may occur when forming differenced vector of unknowns which must be accounted for. Among these problems, we try to highlight the problem of introducing linearly related parameters to the system of the observation equations. The details of this problem are discussed in the sequel through three examples.

**Example 1: Introducing linearly related parameters**

When we think of “linear relations between some of the parameters”, two questions may arise. The first question is if the system of equations is rank deficient (in presence of this linear relationship). The second question is about how this linear relationship influences the results. By this simple example we will demonstrate those types of problems/questions and their solutions.

To emphasize the influence of linearly related parameters on the quality of the estimators, we try to make a simple example. Suppose we have 10 measurements of phase ambiguity of one satellite at one receiver in 10 epochs; one measurement at each epoch (for instance from subtracting of ionosphere-free code and phase observations). Then the set of observation equations reads:

\[
\mathbf{E}\{y\} = \mathbf{E}\left(\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{array}\right) = \mathbf{U}_{10} \mathbf{a} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \mathbf{a}; \quad \mathbf{D}\{y\} = \mathbf{Q}_y = \sigma^2 \mathbf{I}_{10} \tag{3.38}
\]

Therefore the Least-squares estimate of \( \mathbf{a} \) will be:

\[
\hat{\mathbf{a}} = \left(\mathbf{A}^T \mathbf{Q}_y^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{Q}_y^{-1} \mathbf{y} = \frac{1}{10} \sum_{i=1}^{10} y_i ; \quad \mathbf{Q}_d = \left(\mathbf{A}^T \mathbf{Q}_y^{-1} \mathbf{A}\right)^{-1} = \frac{1}{10} \sigma^2 \tag{3.39}
\]

Now, we assume that after the 5\(^{th}\) epoch a cycle slip has occurred (which actually has not). Then after the estimation we will have: \( \mathbf{E}\{\hat{\mathbf{a}}_1\} = \mathbf{E}\{\hat{\mathbf{a}}_2\} = \mathbf{a} \). The matrix form of the observation equations then reads:
Note that the design matrix $A'$ is of full rank. The Least-squares estimate of the unknowns is:

$$
\hat{a}' = \left( \tilde{a}_1 \right) = \left( A'^T Q_y^{-1} A' \right)^{-1} A'^T Q_y^{-1} y = \left( \frac{1}{5} \sum_{i=1}^{5} y_i \right) ; \quad Q_{\hat{a}'} = \left( A'^T Q_y^{-1} A' \right)^{-1} = \frac{1}{5} \sigma^2 I_2
$$

We see that after introducing the new unknown, the quality of the estimates becomes poorer.

### 3.6.1 The adjustment method

In the ordinary system of observation equations:

$$
E\{y\} = Ax \; ; \quad D\{y\} = Q_y
$$

the best linear unbiased estimation (BLUE) of unknowns is:

$$
\hat{x}_A = \left( A'^T Q_y^{-1} A' \right)^{-1} A'^T Q_y^{-1} y \quad \text{and} \quad Q_{\hat{x}_A} = \left( A'^T Q_y^{-1} A' \right)^{-1}
$$

However, when there is a linear relationship between some of the unknowns, a constraint $B^T x = 0$ must be added to the system of observation equations to preserve the BLUE. The BLUE then reads (see Teunissen 1999-a):

$$
\hat{x} = \left[ I - Q_{\hat{x}_A} B \left( B^T Q_{\hat{x}_A} B \right)^{-1} B^T \right] \hat{x}_A
$$

and:

$$
Q_{\hat{x}} = \left[ I - Q_{\hat{x}_A} B \left( B^T Q_{\hat{x}_A} B \right)^{-1} B^T \right] Q_{\hat{x}_A} \left[ I - Q_{\hat{x}_A} B \left( B^T Q_{\hat{x}_A} B \right)^{-1} B^T \right]^T
$$

where $I$ is the identity matrix.
Let us return to the example 1 and see how we can solve the problem shown in the example. If we discover, somehow that the cycle slip at epoch 5 has not occurred and \( a_1 = a_2 \), then we have the condition \( B^T a' = (1 \ -1)a' = 0 \). Therefore, according to (3.44) and (3.45) the estimate of the unknown reads:

\[
\hat{a} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = [I - Q_{a'}B(B^T Q_{a'}B)^{-1}B^T]\hat{a'} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \hat{a'} = \frac{1}{10} \left( \sum_{i=1}^{10} y_i \right) ; \quad Q_{\hat{a}} = \frac{1}{10} \sigma^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

(3.46)

From \( Q_{\hat{a}} \) and \( Q_{\hat{a}'} \), we see that: \( \sigma_{\hat{a}} < \sigma_{\hat{a}_1} = \sigma_{\hat{a}_2} \).

We infer that, having linearly related parameters in the system of observation equations, the estimates become poorer and less reliable if we use ordinary Least-squares techniques. It means that in such cases, the mathematical model representation must be \( E[y] = Ax, B^T x = 0; D[y] = Q_y \) and the BLUE of \( x \) for this model is \( \hat{x} \) of (3.44).

**Example 2:**

Suppose a receiver observed four satellites as their coverage is shown below:

![Coverage of satellites 1 to 4](image)

**Fig 3.1:** Coverage of satellites 1 to 4

At epoch \( t_1 \), taking satellite 1 as reference and forming between-satellite single differenced ambiguity parameters we have:

\[
\begin{align*}
a_{12} &= a^2 - a^1 \\
a_{13} &= a^3 - a^1 \\
a_{14} &= a^4 - a^1
\end{align*}
\]

(3.47)

Analogously, at epoch \( t_2 \) we take satellite 2 as reference and have:
\[
\begin{align*}
\{a_{23}^{2} &= a_{3}^{3} - a_{2}^{2} \\
a_{24}^{2} &= a_{4}^{4} - a_{2}^{2}
\}
\end{align*}
\] (3.48)

From (3.47) and (3.48) one can easily infer that:

\[
a_{24}^{2} - a_{23}^{2} = a_{1}\_{4}^{14} - a_{1}^{13} \text{ or } a_{23}^{2} = a_{1}\_{3}^{13} - a_{1}^{12} \text{ or } a_{24}^{2} = a_{1}\_{4}^{14} - a_{1}^{12}
\]

(3.49)

This means that just the combinations, which do not result in having linearly related parameters must be used. For this, we have to eliminate a part of some of the observations in the time span. Or we can have full coverage but must be carefully find all relations and add the necessary conditions to the system of the observation equations.

**Example 3:**

To verify the correctness of our arguments on differenced phase ambiguity parameterization, we have tested them on the real global positioning system (GPS) data of an IGS permanent station. The data specifications, results, and discussion are as follows:

**Data specifications**

A set of the ionosphere-free phase observations (L3) from the IGS BRUS station was selected. The set consists of 60 epochs of observations from 3 satellites (PRNs 19, 7 and 22) at one minute time interval (from 21:55 to 22:54). The date of observation is September 4th of 2006. We do not estimate satellite orbits and clocks. Instead, a jet propulsion laboratory (JPL) high-rate SP3 (standard product #3) file, consisting of satellite orbits and clocks is used. A cycle slip occurred for PRN19 after the 2nd epoch and for PRN22 after the 55th epoch.

**Processing method**

Our approach is to first compare degrees of freedom of the two systems of equations (\(df_{ud}\) for undifferenced and \(df_{sd}\) for single-differenced observations). Then we will compare the estimates from the undifferenced system with those from the differenced one before and after applying equations (3.32) and (3.33).

The set of undifferenced phase ambiguity unknowns is: \(a_{ud} = \{a_{r,1}^{1}, a_{r}^{2}, a_{r,1}^{3}, a_{r,2}^{1}, a_{r,2}^{3}\}\). \(a_{r,1}^{1}\) is the phase ambiguity unknown of PRN19 before epoch 3, \(a_{r,2}^{1}\) is its unknown from epoch 3 to epoch 60, \(a_{r}^{2}\) is the ambiguity unknown of PRN7 during the whole observation time span (no cycle slip), \(a_{r,1}^{3}\) is the ambiguity unknown of PRN22 before epoch 56 and \(a_{r,2}^{3}\) its unknown from epoch 56 to epoch 60.
The single-differenced phase ambiguity unknowns set is: \( a_{sd} = \{ a_{r,1}^{12}, a_{r,1}^{13}, a_{r,2}^{12}, a_{r,2}^{13}, a_{r,3}^{13} \} \), as \( a_{r,1}^{12} = a_{r}^2 - a_{r,1}^1, a_{r,2}^{12} = a_{r}^2 - a_{r,2}^1, a_{r,1}^{13} = a_{r}^3 - a_{r,1}^1, a_{r,2}^{13} = a_{r}^3 - a_{r,2}^1, a_{r,3}^{13} = a_{r}^3 - a_{r,3}^1 \). It is worth mentioning that forming such a set of single-differenced phase ambiguity unknowns is necessary when we decide to proceed to double-differences. In the undifferenced case we have 180 observations, 3 unknowns for station coordinates, 5 unknowns for phase ambiguities and 60 unknowns for receiver clock error. We have also a rank defect, equal to 1. Therefore, the number of degrees of freedom are: \( df_{ud} = 180 - 3 - 5 - 60 + 1 = 113 \). In the single-differenced case we have 120 observations, 3 unknowns for station coordinates and 5 unknowns for phase ambiguities. Thus, the number of degrees of freedom are: \( df_{sd} = 120 - 3 - 5 = 112 \). \( df_{ud} \neq df_{sd} \) is contradicting the theorem of equivalence of undifferenced and differenced models (or in brief, the equivalence theorem). We know that the differencing operator \( R_{bs}^T \) in Eq. (3.15), transforms a space \( \mathbb{R}^m \) to a space \( \mathbb{R}^{m-1} \), but in this example \( a_{sd} \) and \( a_{sd} \) have the same order which theoretically is not correct. The matrix form of the relation between \( a_{sd} \) and \( a_{sd} \) reads:

\[
\begin{pmatrix}
a_{r,1}^{12} \\
a_{r,1}^{13} \\
a_{r,2}^{12} \\
a_{r,2}^{13} \\
a_{r,3}^{13}
\end{pmatrix} =
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
a_{r,1}^1 \\
a_{r,1}^2 \\
a_{r,1}^3 \\
a_{r,2}^1 \\
a_{r,3}^1
\end{pmatrix}
\]

(3.50)

From (3.50) we can see that \( L \) (which transforms the vector of undifferenced ambiguity parameters onto the single differenced ambiguity parameters) is rank deficient and we see also that:

\[
a_{r,2}^{13} - a_{r,2}^{12} - a_{r,1}^{13} + a_{r,1}^{12} = 0
\]

(3.51)

If we add condition (3.51) to the set of single-differenced observation equations we will have: \( df_{sd} = df_{sd} = 113 \), showing that the equivalence theorem is preserved.

Now, let us see whether the equivalence theorem holds true in numerical GNSS data processing or not.
Results and discussion

After processing the data under different conditions; I) using undifferenced observations, II) using between satellite differences, and III) between satellite differences when constraint (3.51) is added, we got the three sets of estimates for station coordinates and their standard deviations, shown and compared in Table 1.

Table 1: Station coordinates and their variances under conditions I) using undifferenced data, II) using between satellite differences and III) using between satellite differences when constraint (3.51) is added

<table>
<thead>
<tr>
<th>Condition</th>
<th>X (m)</th>
<th>SD X (m)</th>
<th>Y (m)</th>
<th>SD Y (m)</th>
<th>Z (m)</th>
<th>SD Z (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>4027878.233</td>
<td>0.171</td>
<td>307070.699</td>
<td>0.225</td>
<td>4919486.513</td>
<td>0.048</td>
</tr>
<tr>
<td>II</td>
<td>4027879.653</td>
<td>0.186</td>
<td>307068.849</td>
<td>0.245</td>
<td>4919486.989</td>
<td>0.054</td>
</tr>
<tr>
<td>III</td>
<td>4027878.243</td>
<td>0.171</td>
<td>307070.701</td>
<td>0.225</td>
<td>4919486.518</td>
<td>0.048</td>
</tr>
</tbody>
</table>

In Table 1, the first row shows the estimated station coordinates as well as their standard deviations from processing undifferenced data. In the second row, the estimators of the station coordinates with their standard deviations from processing single-differenced data are shown. The third row consists of the station coordinates estimators and their standard deviations from processing single-differenced data after adding constraint (3.51) and solving Equations (3.44) and (3.45) for the estimators. The difference between the results in the first and the second rows, contradicts the equivalence theorem. This contradiction is removed after adding constraint (3.51) according to the results in the third row. They are almost equal to the ones in the first row. This equality complies with the equivalence theorem. Thus, one can infer that the inequality between the first and the second rows is due to the presence of linear relation between some of the unknown parameters.

Remark: Usually many software, dealing with relative positioning, uses double-differenced phase observations. These observations are obtained from taking differences of between-satellite single-differenced observations between receivers. Therefore, the problem, explained in Example 3, is likely to happen when parametrizing double-differenced phase ambiguities. This risk increases when one uses triple-differenced observations to find and allocate cycle slips.
3.7 Conclusions

To sum up this chapter, we mention the following concluding remarks:

- Considering the arguments in comparing undifferenced and differenced observations, it seems reasonable to offer to use only undifferenced phase observations even in network processing. We can benefit from ambiguity resolution by properly combining the ambiguity unknowns and forming their corresponding constraints (see e.g. de Jonge 1998).

- When dealing with the differenced phase observations, an efficient tool to prevent introducing linearly related unknown parameters into the system of the observation equations is to use the equivalence theorem (as explained in Example 3). Verification of fulfillment of the theorem is a clue to discover the existence of linearly related unknown parameters.
4 Effect of satellite ephemeris on the estimated quality of position

4.1 Introduction

In GPS positioning, usually the satellite ephemeris are fixed in the observation equations to their broadcast or published values. Therefore, to have a realistic covariance matrix for the observations one must have a well-defined covariance matrix of the satellite ephemeris too. This chapter is to assess the influence of incorporation of the satellite coordinates and clocks covariance matrix into the system of observation equations on the parameter estimates.

4.2 Incorporation of the satellite ephemeris uncertainties into the system of the observation equations

To find a realistic covariance matrix for the observations, it is necessary to incorporate the satellite orbits and clocks with their covariance matrix into the system of the observation equations. Such information will modify the covariance matrix of the observations. Recently (since 2005) the IGS started publishing orbit and clock files (The Extended Standard Product 3 Orbit Format or sp3c files), which contain uncertainty of satellite coordinates and clocks (see Hilla 2010).

Note: In this chapter, all the theoretical and technical issues will be applied to the precise point positioning (PPP) technique, described in the next parts of the section.

4.2.1 Observations

Undifferenced ionosphere-free linear combinations of L1 and L2 phase data (L3) are used as raw observations. The observation equation for this observation, which is obtained after simplification of Eq. (2.14) and use of Eq. (3.8) reads:

\[
\phi_s^r = \rho_s^r + c(dt_r - dt_s^r) + T_s^r + \lambda a_s^r + \eta + \epsilon_{\phi 3} \quad (4.1)
\]

\(\phi_s^r\) is the L3 carrier phase from satellite s to receiver r, \(\lambda\) is the L3 wave length (10.3 cm), \(a_s^r\) is phase ambiguity, and \(\epsilon_{\phi 3}\) is the stochastic measurement error (as listed in Table 2.3). The term \(\eta\) contains corrections for various systematic effects that are to be computed from known models.
Systematic effects with impact of more than 1 cm must be included in $\eta$ in Eq. (4.1). Sorted with respect to their impact, these are: relativistic effect, tropospheric refraction, satellite and receiver antenna offsets, solid Earth tide and phase wind-up effect. Elaborate explanations about the relativistic and solid Earth tide effects can be found in the IERS Conventions (2010) (Petit and Luzum 2010). The phase wind-up effect is completely explained in Wu et al. (1993). The tropospheric delay is more complicated. There are many ways to treat it. Among them, we have chosen the way described in Krueger et al. (2004). In this approach we have modeled the ZHD (Zenith Hydrostatic Delay), corrected the observations for it and estimated the ZWD (Zenith Wet Delay) on an epoch-by-epoch basis. To preserve the consistence with what IGS does for estimating this effect (ZWD), we used the Niell’s mapping function (see Niell 1996). The IGS ANTEX files (Beutler et al. 1999) are used for receiver antenna phase centre correction. The satellite antenna phase centre correction is applied just on the radial component of the observation vector in the body frame of the satellite to be consistent with the IGS process to estimate orbits and clocks. The size of the correction is 1.023 m.

The satellite coordinates and clocks are computed from the IGS sp3c files (final orbits and clocks). Together with the approximate values for the receiver position and receiver clock, the a-priori troposphere delay, and the corrections mentioned in the previous paragraph, we obtain the computed observation from Eq. (4.1).

It is worth mentioning that for computing the satellite position at time of transmission the travel time and appropriate estimate of receiver clock error (with an accuracy of better than 1 ms) at each epoch taken into account. These estimates for the receiver clock error are obtained from processing the code observations for each epoch.

4.2.2 The system of observation equations

Similar to Eq. (3.11), the linearized observation equations in the form of the Gauss-Markov model, for each epoch are:

$$E\left[ \begin{array}{c} \Delta \phi_r^1 \\ \Delta \phi_r^2 \\ \vdots \\ \Delta \phi_r^m \\ \Delta y_k \end{array} \right] = \begin{pmatrix} -e_r^1 \lambda_j & 0 & \cdots & 0 & c & M_w(z_r^1) \\ -e_r^2 & 0 & \lambda_j & \vdots & \vdots & c & M_w(z_r^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -e_r^m & 0 & 0 & \cdots & \lambda_j & c & M_w(z_r^m) \end{pmatrix} \begin{pmatrix} \Delta x_r \\ a_r^1 \\ a_r^2 \\ \vdots \\ a_r^m \\ d_{rk} \\ D_{wk} \end{pmatrix} ; D\{\Delta y_k\} = q_{y_k}$$ (4.2)
The vector $\Delta y_k$ consists of the original L3 observations minus computed observations from Eq. (4.1), $q_{yk}$ is the diagonal covariance matrix of the observations, of which the diagonal entries are the inverse of cosine of the zenith angle of satellites (see section 3.2.1 for more details on the covariance matrix of the observations).

\[
q_{yk} = \begin{pmatrix}
1/\cos^2 z_t^1 & 0 & \cdots & 0 \\
0 & 1/\cos^2 z_t^2 & \cdots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & 1/\cos^2 z_r^m
\end{pmatrix}
\]

(4.2a)

**4.2.3 Covariance matrix for the observations**

As one can use the satellite orbits and clocks in the form of weighted known parameters for positioning purposes, they must be incorporated into the system of observation equations. Then Eq. (4.2) can be converted to the following equation:

\[
E\{\left(\Delta y_k\right)\} = \begin{pmatrix} A_k & A_k^s \end{pmatrix} \begin{pmatrix} \Delta x \end{pmatrix}; \quad D\{\left(\Delta y_k\right)\} = \begin{pmatrix} q_{yk} & 0 \\
0 & Q_k^s \end{pmatrix}
\]

(4.2b)

where $A_k^s$, the relevant design matrix of orbits and clock at epoch $k$, converting orbit and clock errors into the range domain and $Q_k^s$ is the relevant covariance matrix of the orbits and clocks at epoch $k$. By pre-elimination technique, we can simplify Eq. (4.2b) so that $\Delta x^s$ disappears from the right hand side of Eq. (4.2b) and we avoid estimating it. Multiplying the second row of the equation by $-A_k^s$ and adding the two rows leads to:

\[
E\{\Delta y_k - A_k^s \Delta x^s\} = A_k \Delta x; \quad D\{\Delta y_k - A_k^s \Delta x^s\} = q_{yk} + A_k^s Q_k^s A_k^{sT}
\]

(4.2c)

Then the covariance matrix of observations for each epoch is:

\[
Q_{yk} = \sigma_0^2 q_{yk} + A_k^s Q_k^s A_k^{sT} \tag{4.3}
\]
The covariance matrix consists of two parts. The first part describes the measurement accuracy with $\sigma_0^2 q_y$, the elevation dependent covariance matrix of raw observations at epoch $k$ and the scale $\sigma_0 = 0.01$ (as was explained in section 2.2.8) is chosen so that the standard deviation of the ionosphere-free linear combination in the zenith direction is 1 cm. The second part describes the effect of orbit and clock errors on the observations.

The second point is to decide, which value to use for the variance factor ($\sigma_0^2$). According to Odijk (2002), the ionosphere-free linear combination of dual frequency observations just eliminates the first order ionospheric effects. It means that the higher order and bending effects are present in the L3 linear combination. Total impact of these effects may reach a few centimetres. We chose $\sigma_0^2 = 1 \text{ cm}$, and with this $\sigma_0^2$ there is proper proportion between corresponding observation covariance matrix and the one for the orbits and clocks.

Finally, the covariance matrix for all epochs reads:

$$Q_y = \text{blkdiag}(Q_{y_1}, Q_{y_2}, \ldots, Q_n)$$

(4.3a)

where $n$ is number of epochs. This covariance matrix is diagonal. If the correlations between orbits and clocks were available, we would get to a non-diagonal covariance matrix for the observations and consequently to the different estimates of the unknown parameters.

4.2.4 The data used for numerical study

24-hour sets of observations of five IGS permanent stations are selected to be processed. 4th of January 2006 is the date for Algoquin, Graz1 and Tehran stations, 19th of June 2005 and 15th of September 2006 for Brussels (Brussels1 and 2) and 7th November 2006 for Saskatoon station. To fix the satellite coordinates and clocks, we used the IGS sp3c files (final orbits and clocks) of the aforementioned days. IGS ANTEX files are used for the receiver antenna phase centre correction.

The receiver at Algoquin station is AOA BENCHMARK ACT and the antenna is AOAD/M_T, the receiver at Brussels is ASHTECH Z-XII3T and the antenna is ASH701945B_M, the receiver at Graz is TRIMBLE NETRS and the antenna is TRM29659.00 and the receiver at Tehran is ASHTECH UZ-12 and the antenna is the same as Brussels’. The receiver at Saskatoon is the same at Tehran and the antenna is ASH701945E_M.

For validation of the estimates of station coordinates, they are compared with the station coordinates obtained from the ITRF solution at the date of observation.

It is necessary to mention that all satellites below 15 degrees of elevation are dismissed (i.e. cut-off angle is 15 degrees).
4.2.5 Processing strategy

As mentioned before, we decided to use the undifferenced ionosphere-free linear combination of L1 and L2 phase data (L3) as our observations. There are the following reasons for this decision. First, due to unavailability of the IGS final orbits and clocks, we are not able to do real-time precise point positioning at cm level of accuracy. Since the code observations are necessary for real-time positioning and PPP is done in post-process, we decided not to use code data except for time synchronization. Second, due to the low accuracy of the code observations, they do not play any significant role in the adjustment stage. This means that they do not improve the estimates significantly. Third, satellite clocks estimated using code observations and the ones estimated using phase observations are different. Fixing satellite clocks to the same values for both code and phase observations, introduces a bias to the process. Since there is no guaranty that the bias remains constant during the whole observation time span, it could not be absorbed by the phase ambiguities. This means, if we lump this bias to the phase ambiguities, we might lose the epoch-independence of the phase ambiguities, and it results in less reliable estimates of the parameters.

To find more realistic covariance matrix of the observations we restricted ourselves to the sampling rate of 15 minutes. It is because the uncertainty of the satellite coordinates and clocks are available only at every 15 minutes in the sp3c files. This time interval allows us to avoid interpolation of the satellite clocks, which degrades their quality (see Montenbruck et al. 2005).

The unknown parameter vector consists of station coordinates (static solution- one set of X, Y, Z coordinates per station, based on 96 epochs), phase ambiguities, receiver clocks and wet part of tropospheric delay. The last two parameters are estimated for each epoch.

Since the time interval is 15 minutes, the number of epochs in a 24-hour time span is 96. Therefore, the number of observations is not so large. These conditions allow us to process all the data in one batch.

To verify the influence of the covariance matrix of satellite orbits and clocks, all the data is processed once with the weight matrix consisting of the inverse of the covariance matrix of the observations only, and next time the covariance matrix of satellite orbits and clocks are incorporated (see Shirazian 2006).

4.2.6 Results and discussion

In this part, the results of the numerical computations are discussed. In Tables 4.1 and 4.2 the discrepancies of the station position components from their ITRF (the International Terrestrial Reference Frame) values in the geocentric and local
geodetic (topocentric) systems are given. The standard deviations for the position components computed by the MATLAB code (written by the author) from the inverse normal matrix (see Appendix for more details on the method used for providing the standard deviations) are also presented in Tables 4.1 and 4.2. For the computations in Tables 4.1 and 4.2 different weight matrices for the observations were used. The weight matrix used in the computations for Table 4.1 is computed only by using the first part of the Eq. (4.3). This means that only the elevation dependent measurement errors are taken into account. Table 4.2 corresponds to the same data, but with a different weight matrix, in which the covariance matrix of the orbits and clocks is incorporated (Shirazian 2006).

**Table 4.1:** Station coordinates discrepancy and their standard deviations in \( cm \) (at 95% level of confidence)

<table>
<thead>
<tr>
<th>Stations</th>
<th>Geocentric coordinate system (ITRF)</th>
<th>Topocentric coordinate system</th>
<th>3Dimensional discrepancy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Delta x )</td>
<td>( \sigma_x )</td>
<td>( \Delta y )</td>
</tr>
<tr>
<td>Algoquin</td>
<td>-1.12</td>
<td>2.88</td>
<td>-4.33</td>
</tr>
<tr>
<td>Brussels1</td>
<td>6.86</td>
<td>3.26</td>
<td>1.94</td>
</tr>
<tr>
<td>Graz</td>
<td>7.73</td>
<td>4.29</td>
<td>1.95</td>
</tr>
<tr>
<td>Saskatoon</td>
<td>-3.01</td>
<td>2.61</td>
<td>-3.47</td>
</tr>
<tr>
<td>Tehran</td>
<td>2.63</td>
<td>3.60</td>
<td>5.97</td>
</tr>
</tbody>
</table>

**Table 4.2:** Station coordinates discrepancy and their standard deviations in \( cm \) (at 95% level of confidence) when covariance matrix of orbits and clocks is incorporated

<table>
<thead>
<tr>
<th>Stations</th>
<th>Geocentric coordinate system (ITRF)</th>
<th>Topocentric coordinate system</th>
<th>3Dimensional discrepancy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \Delta x )</td>
<td>( \sigma_x )</td>
<td>( \Delta y )</td>
</tr>
<tr>
<td>Algoquin</td>
<td>0.93</td>
<td>2.62</td>
<td>-8.29</td>
</tr>
<tr>
<td>Brussels1</td>
<td>5.95</td>
<td>3.13</td>
<td>1.70</td>
</tr>
<tr>
<td>Brussels2</td>
<td>8.91</td>
<td>3.53</td>
<td>-5.03</td>
</tr>
<tr>
<td>Graz</td>
<td>5.17</td>
<td>3.94</td>
<td>2.08</td>
</tr>
<tr>
<td>Saskatoon</td>
<td>-2.97</td>
<td>2.24</td>
<td>-2.40</td>
</tr>
<tr>
<td>Tehran</td>
<td>2.83</td>
<td>3.26</td>
<td>4.28</td>
</tr>
</tbody>
</table>
Comparison between Tables 4.2 and 4.3 shows that in five stations out of six, the estimates of the stations positions are improved (discrepancies are smaller). However, more experiments are needed to verify the improvement in the position estimates due to incorporation of the ephemeris’ uncertainties. Moreover, other quality measures, e.g. reliability or overall model tests could result in better judgment about the influence of these uncertainties.

4.3 Concluding remarks

To sum up this chapter, we mention the following important points:

- Incorporation of the satellite ephemeris uncertainties into the system of observation equations could improve the results

- Other quality measures, e.g. reliability or overall model tests should be done to have better knowledge about the quality of the estimated positions

- Incorporation of covariance of the satellite ephemeris must be tested on data processing. GPS observation noises are assumed normally distributed. However, the covariance between the satellite coordinates and clocks introduces covariance to the observations. This covariance must be accounted for to have more realistic estimators for the station positions.
5 Conclusions and recommendations

This research aims at finding more realistic estimators for GPS precise point positioning. Theoretical and numerical studies conducted in this thesis lead to the conclusions and recommendations which are discussed in the sequel.

5.1 Conclusions and remarks

In chapter 2, the basics of the GPS positioning are explained and discussed. The important outcome of these discussions is that theoretically, the use of phase and code observations together is not recommended for precise positioning applications. Systematic errors e.g. multipath and hardware delays influence phase and code observations in different ways. This difference in the effects must be accounted for to have realistic position estimates. This is the reason we have used only phase observations for the numerical studies thought this thesis.

Although differencing of the observations improves the quality of the estimators, it might cause complications in the system of observation equations. The problem of introducing linearly related unknown parameters is particularly focused in this research. After differencing of the observations, the previous parameters must be replaced by the differenced ones. This may lead to having linearly related parameters in the system of observation equations and thus, poorer estimates for the unknowns. To recognize such problem in the GPS processing, we benefit from the equivalence theorem (see section 3.5). According to this theorem, equivalence of stochastic-related properties (e.g. degrees of freedom of the observation equations) of the undifferenced observations must be properly checked for the differenced observations. For example, the degrees of freedom of the system of observation equations must be the same before and after differencing. This is a clue to find out the presence of linearly related parameters in the observation equations.

In GPS precise point positioning, precise satellite ephemeris are used as known parameters in the observation equations. To have a more realistic covariance matrix for the observations, we must know the uncertainties of the satellite ephemeris and incorporate them into the system of observation equations (see section 4.2.3 for more details). Recently (since 2005), the IGS started publishing the uncertainties of the ephemeris in the precise ephemeris files (sp3c files). Incorporation of these uncertainties into the observation equations might improve the position estimates. However, studies that are more elaborate are needed to assess the influence of the satellite ephemeris on the PPP estimators.
5.2 Recommendations (To Do part)

Based on what is discussed so far in this research, some future work to accomplish the assessment of the quality of the PPP is recommended here.

1- The incorporation of covariances of the satellite ephemeris into the system of observation equations should be studied. Although the GPS observations collected at proper time interval (e.g. intervals more than 20s) are assumed uncorrelated, satellite ephemeris introduce correlations to the system of observation equations where they are used. Such covariances are not available (published) now. A way to find these covariances is to form a data series from the published satellite coordinates and clocks. Then, by means of spectral analysis techniques (e.g. Fourier spectral analysis), one can obtain the required covariances.

2- Other quality measures should be tested on the numerical studies results to assess the quality of the PPP estimators better. For example, overall model test must be done before and after incorporation of the ephemeris uncertainties and covariances to see how they influence the covariance matrix of the observations.

3- The numerical studies in this thesis are limited to the observations with 15min time interval. Shorter time intervals may improve the estimators quality. The existing published precise ephemerides are tabulated as epochs at 15min time interval. Therefore, to use observations with shorter time intervals than 15min, one must interpolate the existing ephemeris. This interpolated ephemeris must be investigated whether it is degraded by interpolating (compared to the original ephemeris) or not. Moreover, proper uncertainties and covariances for this interpolated ephemeris should be defined.

4- In general, dependent on the required accuracy for positioning, one can either model (using e.g. Saastamoinen or Hopfield models) or estimate the tropospheric delay. Estimation of the ZWD could be a way to avoid inaccurate corrections. To estimate the ZWD, one must add a new unknown to the system of observation equations. The question here is that whether or not the ZWD unknown should be added to the observation equations at each epoch. If not, what the optimum time interval between two consecutive ZWD unknown is. To elaborate on this, we can say that if the wet troposphere is not changing fast (the rate of the change is less than the time interval between two epochs), adding a new unknown only causes over-parameterization problem to occur. Therefore, the best parameterization interval for the ZWD must be found.

5- The studies conducted in this thesis and the above recommendations must be tested on several days of observations of different IGS stations. With the results of such a study, one can perform statistical tests and obtain some information e.g. repeatability measure.
References


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Appendix

The method used for determining standard deviations of the station position

Assume the following Gauss-Markov representation of the observation equations:

\[ E\{y\} = Ax \; ; \quad D\{y\} = Q_y \]

Then:

\[ \hat{x} = (A^TQ_y^{-1}A)^{-1}A^TQ_y^{-1}y \quad \text{and} \quad Q_{\hat{x}} = \hat{\sigma}_0^2 (A^TQ_y^{-1}A)^{-1} \]

where:

\[ \hat{\sigma}_0^2 = \frac{\hat{\nu}^TQ_\nu^{-1}\hat{\nu}}{df}, \quad \hat{\nu} = y - A\hat{x}, \] is the vector of the least-squares residuals and \( df \) is degrees of freedom.

Therefore, the standard deviation of the \( i^{th} \) station position component will be:

\[ \sigma_i = \sqrt{Q_{\hat{x}}(i, i)} \]

and the 3-dimensional standard deviation is:

\[ \sigma_{3D} = \sqrt{a^2 + b^2 + c^2} \]

where \( a^2, b^2 \) and \( c^2 \) are the eigenvalues of \( Q_{\hat{x}} \).

To have all the standard deviations at the confidence level of 95% one must multiply them by 1.96.
The overall model test

A test on the above-mentioned $\hat{\sigma}_0^2$, called *overall model test*, is done to determine if the selected weight matrix is acceptable or there are blunders in the observation vector. An overview of the test is given below (Teunissen 2000):

**Hypotheses:**

We require to test the null hypothesis $H_0: E\{y\} = Ax$ versus the alternative hypothesis $H_A: E\{y\} \in \mathbb{R}^m$. This alternative hypothesis means that the redundancy $df$ equals zero and then $\hat{\nu} = 0$.

**Test statistic:**

The test statistic is $\hat{\sigma}_0^2 = \frac{\varphi^T Q \varphi}{df}$.

**Distribution of $\hat{\sigma}_0^2$:**

$\hat{\sigma}_0^2 \sim F(df, \infty, 0)$, where $F$ denotes Fisher distribution.

**Overall model test:**

reject $H_0$ if $\hat{\sigma}_0^2 > k_\alpha$ where $k_\alpha$ is the critical value at the confidence level $1 - \alpha$ and accept $H_0$ otherwise.