A PADÉ APPROXIMANT RECONSTRUCTION SCHEME FOR FAST RESOLUTION OF STRUCTURAL-ACOUSTIC FINITE ELEMENT MODELS INCLUDING POROUS MATERIALS

R. Rumpler$^{1,2*}$, J.-F. Deü$^1$ and P. Göransson$^2$

$^1$ LMSSC, Conservatoire National des Arts et Métiers
292 rue Saint-Martin, case 2D6R10, 75141 Paris Cedex 03, France
Email: deu@cnam.fr

$^2$ MWL, Department of Aeronautical and Vehicle Engineering
Royal Institute of Technology (KTH), SE-100 44 Stockholm, Sweden
Email: pege@kth.se, rumpler@kth.se

ABSTRACT

In this work, an efficient solution strategy is investigated for the resolution of multi-frequency structural-acoustic problems including 3D modelling of poroelastic materials. The finite element method is used, together with a combination of a modal-based reduction of the poroelastic domain and a Padé-based reconstruction approach. It thus takes advantage of the reduced-size of the problem while further improving the computational efficiency by limiting the number of frequency resolutions of the original problem. An adaptive procedure is proposed for the discretization of the frequency range into frequency intervals of reconstructed solution. The validation is presented on a 3D poro-acoustic example.

1 INTRODUCTION

During the last decade, porous materials have been widely used in order to reduce noise for interior domain in the transports industry. In the design process, several detailed structural-acoustic analyses need to be performed. However, modelling vibroacoustic problems with dissipative interfaces, such as porous materials described with the Biot-Allard theory [1], can lead to prohibitive sizes of Finite Element (FE) models. It is therefore important to consider efficient solution strategies.

Among the enhancements proposed, use of equivalent acoustic impedances [2] proved to be very efficient, but limited by strong assumptions. In the scope of 3D FE modelling, use of a mixed displacement-pressure porous formulation for the solid and fluid phases respectively [3] downsized the number of degrees of freedom (dofs) per node from 6, when using a standard solid and fluid phases displacement formulation, to 4 dofs. Hierarchical elements also proved to reduce the number of dofs needed to model the porous media [4]. Alternatively, modal reduction techniques have been proposed and applied to standard linear poroelastic finite elements, in an attempt to keep a fine and complex 3D modelling of the problem in the scope of low frequency applications [5, 6].

In the present approach, a reconstruction based on Padé approximants [7] is combined with a modal-based reduction of the costly poroelastic domain [6, 8]. This combination takes advantage...
of the best out of each method: the Padé approach limits the number of frequencies at which the full computation has to be performed while the reduced-sized model enhances the reconstruction of the solution around these main frequencies. Consequently, substantial efficiency improvements are expected. Furthermore, the trade-off between the computational time and precision is discussed on a 3D poro-acoustic validation case.

In the first three sections, the FE formulation, the modal-based reduction of the poroelastic domain as well as the Padé approximation method are recalled. An adaptive procedure is then proposed for the automatic decomposition of the frequency range into contiguous frequency intervals. The last section is dedicated to the validation and discussion of the approach on a 3D example.

2 FE FORMULATION FOR THE PORO-ACOUSTIC PROBLEM

A poro-acoustic problem is considered, whose description and notations are presented on Figure 1. The acoustic fluid and the porous media occupy the domains \( \Omega_F \) and \( \Omega_P \) respectively. The compressible fluid is described using pressure fluctuation \( p \) as primary variable (Subsection 2.1.1), while fluid and solid phases homogenized displacements \( (u_s, u_f) \) are retained for the porous media (Subsection 2.1.2). The domains boundaries are separated into contours of (i) imposed Dirichlet boundary conditions \( \partial_1 \Omega_F \) and \( \partial_1 \Omega_P \), (ii) prescribed Neumann boundary conditions \( \partial_2 \Omega_F \) and \( \partial_2 \Omega_P \), and (iii) coupling interface between acoustic fluid and porous media \( (\Gamma_{FP}) \). The FE formulation is presented for a stationary harmonic response at angular frequency \( \omega \).

![Figure 1: Description ans notations of the poro-acoustic interaction problem](image)

Table 1: List of material parameters

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_s )</td>
<td>Density of the material constituting the frame</td>
</tr>
<tr>
<td>( (\lambda; \mu) )</td>
<td>Lamé parameters for the solid frame</td>
</tr>
<tr>
<td>( \rho_f )</td>
<td>Ambient fluid density</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Ambient fluid viscosity</td>
</tr>
<tr>
<td>( P_0 )</td>
<td>Ambient fluid standard pressure</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Heat capacity ratio for the ambient fluid</td>
</tr>
<tr>
<td>( Pr )</td>
<td>Prandtl number for the ambient fluid</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Porosity</td>
</tr>
<tr>
<td>( \alpha_\infty )</td>
<td>Tortuosity</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Static flow resistivity</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>Viscous characteristic length</td>
</tr>
<tr>
<td>( \Lambda' )</td>
<td>Thermal characteristic length</td>
</tr>
</tbody>
</table>

2.1 Dynamic equations and constitutive laws

2.1.1 Compressible fluid \( (p) \)

The internal fluid within cavities is assumed compressible and inviscid, satisfying the Helmholtz equation derived from the motion, continuity, and constitutive equations

\[
\Delta p + \frac{\omega^2}{c_0^2} p = 0 \quad \text{in} \quad \Omega_F
\]

where \( c_0 \) is the constant speed of sound in the fluid, and \( p \) the pressure fluctuation field.

2.1.2 Porous media Biot theory \( (u_s, u_f) \)

At angular frequency \( \omega \), the poroelastic media satisfies the following elastodynamic linearized equations, derived in the Biot-Allard theory [1], taking into account inertia and viscous coupling
effects between solid and fluid phases,

\[
\text{div } \sigma_s - i \omega \bar{b}(\omega) (u_s - u_t) + \omega^2 \left[ (1 - \phi) \rho_s + \rho_a \right] u_s - \rho_a u_t = 0 \quad \text{in } \Omega_p, \tag{2a}
\]

\[
\text{div } \sigma_t - i \omega \bar{b}(\omega) (u_t - u_s) + \omega^2 \left[ -\rho_s u_t + (\phi \rho_f + \rho_a) u_t \right] = 0 \quad \text{in } \Omega_p, \tag{2b}
\]

where \( u_s \) and \( u_t \) are respectively the solid phase and fluid phase averaged displacements in the sense of Biot theory. \( \bar{b}(\omega) \) (henceforth denoted \( \tilde{b} \), where \( \tilde{\cdot} \) refers to a complex-valued quantity) and \( \rho_a \) are respectively the complex frequency-dependent viscous drag and the inertia coupling parameter, based on the standard notations of material parameters (see Table 1), and given by

\[
\tilde{b} = \sigma \phi^2 \left[ 1 + \frac{4i \omega \alpha_0 \eta \rho_f}{\sigma^2 \Lambda^2 \phi^2} \right]^{\frac{1}{2}}, \quad \text{and } \rho_a = \phi \rho_f (\alpha_\infty - 1). \tag{3}
\]

\( \sigma_s \) and \( \sigma_t \) are the averaged stress tensors for the solid and fluid phases respectively. In [6], it was shown that they satisfy the Lagrangian stress-strain relations developed by Biot, rewritten in the following form using Voigt notation,

\[
\sigma_s = D_s^{(1)} \varepsilon(u_s) + (\tilde{K}_f - P_0) D_s^{(2)} \varepsilon(u_s) + D_s^{(1)} \varepsilon(u_t) + (\tilde{K}_f - P_0) D_s^{(2)} \varepsilon(u_t), \tag{4a}
\]

\[
\sigma_t = D_{sf}^{(1)} \varepsilon(u_s) + (\tilde{K}_f - P_0) D_{sf}^{(2)} \varepsilon(u_s) + D_t^{(1)} \varepsilon(u_t) + (\tilde{K}_f - P_0) D_t^{(2)} \varepsilon(u_t), \tag{4b}
\]

where \( \varepsilon(u_s) \) and \( \varepsilon(u_t) \) are the strain tensors associated to the averaged displacements fields \( u_s \) and \( u_t \) respectively. \( \tilde{K}_f(\omega) \) is the effective bulk modulus of the fluid phase (henceforth denoted \( \tilde{K}_f \)),

\[
\tilde{K}_f = \frac{\gamma P_0}{\gamma - (\gamma - 1) \left[ 1 + \frac{8 \eta}{\omega \rho_f \Lambda^2 \rho_f} \left( 1 + \frac{\omega \rho_f \Lambda^2 \rho_f}{16 \eta} \right)^2 \right]^{\frac{1}{2}}}.
\tag{5}
\]

\( D_s^{(1),(2)}, D_t^{(1),(2)} \) and \( D_{sf}^{(1),(2)} \) are constant real-valued constitutive matrices given in [6].

### 2.2 Finite element discretized problem

The test-function method is used to derive the variational formulation of the coupled problem. Details can be found in [6, 9]. Thus, using the Helmholtz equation (1), the elastodynamic equations (2a) and (2b), the constitutive expressions (4a) and (4b), as well as the acoustic velocity source excitation and coupling conditions, the following discretized system of equations arises,

\[
\begin{align*}
&\begin{bmatrix}
    K_F & 0 & 0 \\
    -(1 - \phi)A_{Fs}^T & K_{ss}^{(1)} & K_{sf}^{(1)} \\
    -\phi A_{ff}^T & K_{sf}^{(1)} & K_{ff}^{(1)}
\end{bmatrix} + \begin{bmatrix}
    0 & 0 & 0 \\
    0 & K_{ss}^{(2)} & K_{sf}^{(2)} \\
    0 & K_{sf}^{(2)} & K_{ff}^{(2)}
\end{bmatrix} \begin{bmatrix}
    0 \rho_a & 0 \\
    0 & \phi A_{Fs}^T \\
    0 & \phi A_{ff}^T
\end{bmatrix} \begin{bmatrix}
    P \\
    M_F \\
    M_{ss}
\end{bmatrix} = \begin{bmatrix}
    \omega^2 U_{Fi} \\
    M_s \\
    M_{sf}
\end{bmatrix}
\end{align*}

\tag{6}
\]

which can be symmetrized by dividing the acoustic equation by \( \omega^2 (\omega \neq 0) \).

### 3 MODAL-BASED REDUCTION OF POROELASTIC DOMAIN

As an alternative to the full FE problem (6), the modal-based reduction of the poroelastic domain, as proposed in [6, 8], is used in the present work. The extension of the reduction to the acoustic domain is straightforward and detailed in [9], but not used in this contribution. Thus, non-reduced acoustic dofs are separated into internal ones (subscript \( \tilde{f} \)), and those at interface with the porous
media (subscript $I$). Notations used are presented in Figure 2. The solid and fluid phase dofs (subscripts $s$ and $f$ respectively) are further denoted by a common set of porous dofs (subscript $P$), so that the matrix system of equations (6) may be rewritten as

$$
\begin{bmatrix}
K_{II} - \omega^2 M_{II} & K_{II} - \omega^2 M_{II} & 0 \\
K_{II} - \omega^2 M_{II} & K_{II} - \omega^2 M_{II} & -\omega^2 A_{IP} \\
0 & -A_{IP}^T & K_p^{(1)} + (\tilde{K}_f - P_0) K_p^{(2)} + i \omega \tilde{b} C_p - \omega^2 M_p
\end{bmatrix}
\begin{bmatrix}
P_f \\
F_f \\
U_p
\end{bmatrix}
= \begin{bmatrix}
\omega^2 U_{fb} \\
0 \\
0
\end{bmatrix}
$$

(7)

which can be symmetrized by dividing the acoustic equations by $\omega^2$ ($\omega \neq 0$).

The reduced model is then obtained from a transformation basis including a set of porous dofs (see [6, 8] for further details),

$$
\begin{bmatrix}
P_f \\
F_f \\
U_p
\end{bmatrix}
= \begin{bmatrix}
I_f & 0 & 0 \\
0 & I_f & 0 \\
0 & \Psi_{PI} & \Phi_{pm}
\end{bmatrix}
\begin{bmatrix}
P_f \\
F_f \\
\hat{\alpha}_m
\end{bmatrix},
$$

(8)

where $\hat{}$ denotes an approximation of the original solution. When applied to a symmetrized form of Eq. (7), the transformation leads to following reduced set of equations,

$$
\begin{bmatrix}
\frac{1}{\omega^2} K_{II} - M_{II} \\
\frac{1}{\omega^2} K_{II} - M_{II} \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\omega} K_{II} - M_{II} \\
\frac{1}{\omega} K_{II} - M_{II} \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\Omega_m
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{K}_f - P_0 \\
K_{p}^{(2)} \\
K_{p,m}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
K_p^{(1)} \\
K_p^{(2)} \\
\kappa_m
\end{bmatrix}
$$

(9)

where for porous matrices indexed by subcript $P$, i.e. $B_p \in \{K_p^{(1)}, K_p^{(2)}, C_p, M_p\}$,

$$
B_{p,II} = \Psi_{PI}^T B_p \Psi_{PI},
B_{p,m} = \Psi_{PI}^T B_p \Phi_{pm} = B_{p,m}^T
$$

$I_m$ is a unit matrix of dimension $m$, corresponding to the number of porous modal coordinates. $\Omega_m$ is a diagonal matrix with, on its diagonal, the $m$ lowest eigenfrequencies resulting from the porous eigenvalue problem. $\kappa_m$ and $\zeta_m$ are sparsely populated square matrices of dimension $m$ [6].

4 SOLUTION RECONSTRUCTION USING PADÉ APPROXIMANTS

4.1 Presentation and notations

The previously established sets of equations, both for the non-reduced (symmetrized version of system (6)) and the reduced (Eq. (9)) problems, can be expressed as the following frequency-dependent form,

$$
Z(\omega)u(\omega) = F(\omega),
$$

(11)
with for $\omega \neq 0$,
\[
Z(\omega) = \frac{1}{\omega^2} K_F + K_p^{(1)} + \left( K_f(\omega) - P_0 \right) K_p^{(2)} + i \omega \tilde{b}(\omega) C_p - \omega^2 M_p,
\]  
(12)

where $K_F$, $K_p^{(1)}$, $K_p^{(2)}$, $C_p$ and $M_p$ are symmetric, frequency-independent and real-valued matrices for the coupled poro-acoustic problem. Under this form, $K_F$ includes the acoustic “stiffness”, $K_p^{(1)}$ the acoustic “mass”, the poro-acoustic coupling terms and the zero-frequency porous stiffness limit, while $K_p^{(2)}$, $C_p$ and $M_p$ correspond to the higher-frequency porous stiffness, viscous and mass matrices of the porous media respectively. In the following, these global assembled matrices refer to either their non-reduced or reduced form. One benefit of the choice made for the poroelastic formulation is manifest in the Padé-based reconstruction scheme described in the following section. In fact, having the frequency-dependence borne by scalar functions simplifies and enhances the application of the Padé approximation compared to a formulation for which matrix derivatives would have to be considered. For instance, use of a displacement-pressure poroelastic formulation, known to be computationally efficient, would imply solid and fluid phase-dependent derivatives, thus involving a more complex implementation of the reconstruction scheme.

The procedure, detailed in the following sections, can be decomposed into two main steps: (i) the decomposition of the frequency range of interest into frequency intervals, which will be addressed in Section 5 and (ii) for each interval, the resolution of the full problem at a main frequency, followed by the reconstruction of the solution around this point. The latter is presented in the next three sections.

### 4.2 Determination of the Padé approximants

Given a solution vector $U(\omega_0)$, the solution around this point can be estimated, for each dof $j$, by a rational function of Taylor series expansion. Thus, each dof solution function (further denoted $u \rightarrow u_j \rightarrow u$ in simplified notations), can be written as
\[
u(\omega_0 + \Delta \omega) \approx \frac{P_L(\Delta \omega)}{Q_M(\Delta \omega)},
\]  
(13)

with $P_L(\Delta \omega)$ and $Q_M(\Delta \omega)$ being two truncated power series in the variable $\Delta \omega$, to the order $L$ and $M$ respectively, and defined as
\[
P_L(\Delta \omega) = \sum_{k=0}^{L} p_k(\Delta \omega)^k, \quad \text{and} \quad Q_M(\Delta \omega) = \sum_{k=0}^{M} q_k(\Delta \omega)^k.
\]  
(14)

The approximation of $\nu(\omega_0 + \Delta \omega)$ is given by the unique determination of the coefficients $p_k$ and $q_k$. These coefficients can be determined in several ways [10], but a straightforward approach is to consider the Padé approximation as a re-arrangement into a rational function of a Taylor series expansion to the order $L + M$,
\[
\frac{P_L(\Delta \omega)}{Q_M(\Delta \omega)} = A_{L+M}(\Delta \omega) \quad \text{with} \quad A_{L+M}(\Delta \omega) = \sum_{k=0}^{L+M} a_k(\Delta \omega)^k \quad \text{and} \quad a_k = \frac{u^{(k)}(\omega_0)}{k!},
\]  
(15)

allowing for poles in the rational function ($Q_M(\Delta \omega) = 0$). Follows that the $p_k$ and $q_k$ coefficients are solution of a system of linear equations established from the constraint
\[
P_L(\Delta \omega) - A_{L+M}(\Delta \omega) Q_M(\Delta \omega) = 0,
\]  
(16)

where the coefficients of equal order in $\Delta \omega$ are extracted to form a set of $(L + M + 1)$ equations. However, this underdetermined system of Eqs. (16) gives only a solution of the $(L + M + 2)$
coefficients to within a multiplicative constant. Therefore, these are usually normalized so that the zero-order coefficient of the denominator, \( q_0 \), is set to 1 [10]. Subsequently, the system of equations arising from the constraint (16) can be written explicitly in a compact form, involving the \((L+M)\) first derivatives of \( u(\omega) \) at \( \omega_0 \),

\[
k! p_k - \left( \sum_{l=1}^{k} \frac{k!}{l!} (k-l)! q_l \right) u^{(k-l)}(\omega_0)q_l = u^{(k)}(\omega_0), \text{ for } k = 0, \ldots, L+M, \text{ with } \begin{cases} p_k = 0 & \text{if } k > L \\ q_l = 0 & \text{if } l > M \end{cases},
\]

(17)

where \( \binom{k}{l} \) represents the binomial coefficients. This set of equations can be solved numerically, for each dof \( j \), thus involving the resolution of \( N \) small problems of dimension \((L+M+1)\),

\[
[A]^{(j)} = \begin{bmatrix} p_0 \\ \vdots \\ p_L \\ q_1 \\ \vdots \\ q_M \end{bmatrix}^{(j)} = \begin{bmatrix} u(\omega_0) \\ \vdots \\ u^{(L)}(\omega_0) \\ u^{(L+1)}(\omega_0) \\ \vdots \\ u^{(L+M)}(\omega_0) \end{bmatrix}^{(j)}, \text{ for } j = 1, \ldots, N.
\]

(18)

Similarly to an approximation of the solution using Taylor series expansion, the resolution of systems of Eqs. (18) requires the determination of the \((L+M)\) derivatives of the solution vector \( u(\omega) \) at \( \omega_0 \). This aspect is developed in the following section.

4.3 Successive derivatives of the solution vector

The successive \((L+M)\) derivatives of \( u \) at angular frequency \( \omega_0 \) can be derived from differentiating Eq. (11) with respect to \( \omega \). At the order \( k \) of differentiation, the following expression arises,

\[
\sum_{j=0}^{k} \binom{k}{j} Z^{(k-j)}(\omega_0) u^{(j)}(\omega_0) = f^{(k)}(\omega_0), \text{ for } k = 1, \ldots, (L+M),
\]

(19)

where the zero-order derivatives correspond to the non differentiated functions. Extracting the highest-order term from the summation in Eq. (19) leads to the following recursive expression of the \( k \)-order derivative of \( u \) at \( \omega_0 \),

\[
Z(\omega_0) u^{(k)}(\omega_0) = f^{(k)}(\omega_0) - \sum_{j=0}^{(k-1)} \binom{k}{j} Z^{(k-j)}(\omega_0) u^{(j)}(\omega_0), \text{ for } k = 1, \ldots, (L+M).
\]

(20)

This implies that the successive derivatives of \( u \) required for the determination of the Padé approximations can be efficiently computed as the solution of a system of equations of dimension \( N \), with multiple right-hand sides inexpensively built.

4.4 Application to the poro-acoustic problem

4.4.1 Derivatives of \( Z(\omega) \)

The \( k^{th} \) derivative of \( Z(\omega) \) for the poro-acoustic problem is given by,

\[
Z(\omega)^{(k)} = \begin{cases} (-1)^{k} (k+1)! \frac{1}{\omega^{k+2}} K_F + \tilde{K}_f^{(k)} K_p^{(2)} + i \left( \omega \tilde{b}(\omega) \right)^{(k)} C_p - 2\omega M_p & \text{for } k = 1 \\ (-1)^{k} (k+1)! \frac{1}{\omega^{k+2}} K_F + \tilde{K}_f^{(k)} K_p^{(2)} + i \left( \omega \tilde{b}(\omega) \right)^{(k)} C_p - 2M_p & \text{for } k = 2 \\ (-1)^{k} (k+1)! \frac{1}{\omega^{k+2}} K_F + \tilde{K}_f^{(k)} K_p^{(2)} + i \left( \omega \tilde{b}(\omega) \right)^{(k)} C_p & \text{for } k > 2 \end{cases}
\]

(21)
4.4.2 Reconstruction procedure within one frequency interval

Following the derivations presented in the previous sections, the reconstruction of the solution within one frequency interval is done in two main steps:

- First, at a given angular frequency $\omega_0$, the solution as well as its successive derivatives to the order $(L + M)$ are computed according to Eq. (20). This implies, for each frequency around which the solution is to be approximated, the resolution of $(L + M + 1)$ problems of dimension $N$ in a multiple right-hand side scheme.

- Then, the solution around $\omega_0$ is reconstructed, involving the resolution of $N$ problems of dimension $(L + M + 1)$ (Eq. (18)) to determine the Padé approximants corresponding to each degree of freedom. Follows $N$ trivial rational fraction evaluation (Eq. (13)) for each $\Delta \omega$ at which the approximated solution is to be evaluated.

Three points can be further discussed regarding the given description of the procedure. The first one consists in choosing the coarse frequencies at which the solution will be evaluated, i.e. setting an appropriate discretization of the frequency space. Secondly, for each coarse frequency, the order of polynomial expansions and the frequency range of evaluation around $\omega_0$ have to be chosen accordingly, which raises the question of the values of $L$ and $M$. Thirdly, one has to estimate the conditions of efficiency of such an approximation scheme.

Although the second and third points are thoroughly discussed by Avery et al. in [7], for single field structural or acoustic applications, the choice of coarse frequencies has, to the knowledge of the author, not been been given much attention in the literature. A suggestion is made in this sense in Section 5 of the present work. Regarding the order of polynomial expansions to consider for both the numerator $(L)$ and the denominator $(M)$ in Eq. (13), the following constraint, suggested in [7], reduces the set of possibilities, giving satisfying approximations: $M = L + 1$. Furthermore, there is an upper limit to the maximum order that can be set, due to the ill-conditioned matrix that arises for system of Eqs. (18). In the applications considered in this work, the upper limit is set to $L_{max} + M_{max} + 1 = 12$, thus imposing $L_{max} = 5$, $M_{max} = 6$. This has proved extremely efficient when applied to large single field structural or acoustic applications, with an a priori set frequency discretization of the frequency space [7, 11].

In the following, two extensions to this approach are considered: application of the method to i) a coupled problem such as the poro-acoustic example considered in the previous chapter, on which an adaptive frequency discretization scheme is tested, and ii), a modal-based reduced version of this system of equations, thus estimating the trade-off between the information lost via the reduction and the precision needed to establish successive derivatives of the solution. The latter situation potentially presents an interest in the situation where, e.g., a reduced-model would be mostly advantageous for memory allocation purposes, while the Padé-based reconstruction would allow substantial computational time enhancements.

5 ADAPTIVE DECOMPOSITION IN FREQUENCY INTERVALS

In order to avoid setting an a priori choice of main frequencies at which the solution is estimated by a direct computation, a simple adaptive approach is proposed. It enables a discretization of the frequency space according to the estimated capability of the reconstruction scheme adopted. This limits the lack of precision or the loss in computational efficiency that would be induced by a too coarse or too refined choice for the main frequencies. The present approach is based on two aspects: (i) controlling the error of the reconstructed solution using an error estimation, and (ii) using the frequency interval of convergence for each main frequency to anticipate the contiguous frequency interval of convergence.
5.1 Error estimation in the poroelastic material

For a given approximation of the solution, the error committed with respect to the reference FE solution can be estimated from the residue associated to the time-harmonic response. Thus, at a given angular frequency \( \omega \), the approximated solution, e.g. calculated from a Padé reconstruction procedure, is denoted with double hats "\( \hat{\cdot} \)". From this approximated solution, and using the last set of equations in Eq. (7), a residual force vector for the porous domain is computed,

\[
\mathbf{R}_{f_p}(\omega) = \mathbf{A}_{T_p}^{T_p} \hat{\mathbf{T}}_P = \left( \mathbf{K}_{p}^{(1)} + \left( \hat{\mathbf{K}}_I(\omega) - P_0 \right) \mathbf{K}_p^{(2)} + i \omega \hat{\mathbf{b}}(\omega) \mathbf{C}_p - \omega^2 \mathbf{M}_p \right) \hat{\mathbf{U}}_P. \tag{22}
\]

Following, a \( \mathbf{K}_{p}^{(1)} \)-residual displacement vector can be established,

\[
\mathbf{R}_{U_p}(\omega) = \mathbf{K}_{p}^{(1)-1} \mathbf{R}_{f_p}(\omega). \tag{23}
\]

The error estimator used is then chosen by analogy to the strain energy error estimator used in structural dynamics [12]. It is based on the residue, and can be computed at selected frequencies as

\[
\varepsilon(\omega) = \frac{\mathbf{R}_{U_p}(\omega)_{K_p}^{(1)} \mathbf{R}_{U_p}(\omega)}{\hat{\mathbf{U}}_p_{K_p}^{(1)} \hat{\mathbf{U}}_p} = \frac{\mathbf{R}_{U_p}(\omega) \mathbf{R}_{f_p}(\omega)}{\hat{\mathbf{U}}_p_{K_p}^{(1)} \hat{\mathbf{U}}_p}. \tag{24}
\]

5.2 Adaptive discretization in frequency intervals

It is assumed that the frequency interval of convergence for a given central frequency gives a good a priori estimation of the interval of convergence for contiguous intervals. This is of course a strong assumption, which might be very case-dependent. However, it is reinforced by the fact that an approximation by a rational function of power series exhibits a rather smooth divergence. Consequently, an underestimated interval of convergence hinders the computational efficiency by potentially increasing the number of central frequencies needed, whereas an overestimated interval of convergence implies non-contiguous converged intervals which might still render a sensible approximated solution in the gaps. Furthermore, it is also assumed that, given an increasing modal density for increasing frequencies, the interval of convergence is expected to decrease with increasing central frequencies. For these reasons, the adaptive reconstruction scheme is started from the higher frequency range, and propagated to the lower frequencies with a slight over-estimation of the intervals of convergence. The successive steps are illustrated on Figure 3.

(a) From the first central frequency corresponding to \( \omega_0 \), in the higher end of the frequency range, the reconstruction procedure is applied and combined with an error estimation of the approximated solution both toward the lower and the higher frequencies. When the convergence check is not satisfied anymore, the corresponding upper and lower limits \( \omega_{0+} \) and \( \omega_{0-} \) define the interval of convergence \( \Delta \omega_0^{f} \) corresponding to the angular frequency \( \omega_0 \) (the exponent \( f \) indicates the final version of these parameters, in contrast to the initial estimations, denoted by exponent \( i \), introduced a priori for upcoming intervals). In the present approach, it is considered that the upper limit \( \omega_{0+}^{f} \) defines the upper bound of the frequency domain of interest.

(b) (resp. (e)) From the previously determined interval of convergence \( \Delta \omega_0^{f} \) (resp. \( \Delta \omega_1^{f} \)), the central angular frequency \( \omega_1 \) (resp. \( \omega_2 \)) associated to the lower-frequency contiguous interval is estimated. It is positioned half-a-convergence-interval \( \frac{\Delta \omega_0^{f}}{2} \) (resp. \( \frac{\Delta \omega_1^{f}}{2} \)) lower than the lower limit \( \omega_{0-}^{f} \) (resp. \( \omega_{1-}^{f} \)). Doing so, the anticipation of generally increasing intervals of convergence might lead to an overlap of two contiguous converged intervals. However, this choice is made in order to lower the risk of non-converged gaps between intervals (see Figure 3 (g) and (h)), appearing due to a locally reduced or asymmetric convergence interval.
Figure 3: Adaptive frequency interval decomposition: blue: central frequency; magenta: upper limit; green: lower limit

(c) (resp. (f)) Once the main frequency is established, the upper and lower bounds for the interval, \( \omega_{1+} \) (resp. \( \omega_{2+} \)) and \( \omega_{1-} \) (resp. \( \omega_{2-} \)) respectively, are estimated a priori. They are determined assuming an interval centered on \( \omega_1 \) (resp. \( \omega_2 \)), and of width \((1 + \alpha)\Delta \omega_0^f\) (resp. \((1 + \alpha)\Delta \omega_1^f\)). The parameter \( \alpha \), which accounts for the anticipation of globally increasing intervals of convergence, is arbitrarily chosen in this work, typically smaller than 0.2. However, one could argue that \( \alpha \) may be estimated considering the frequency-dependence of the modal density, as it relates to the expected number of discontinuities per convergence interval.

(d) (resp. (g)) When the upper and lower bounds are estimated, the reconstruction procedure is applied. Simultaneously, the error estimation check is done starting from the lower and higher bounds rather than the central frequency. The actual upper and lower limits of convergence, i.e. \( \omega_{1+}^f \) (resp. \( \omega_{2+}^f \)) and \( \omega_{1-}^f \) (resp. \( \omega_{2-}^f \)) respectively, are determined when the convergence check based on the error estimation is satisfied. While the mostly observed situation is that of overlapping contiguous convergence intervals (Figure 3(d)), a gap of non-converged reconstructed solution can arise (Figure 3(g) and (h)) as mentioned in (b). In case of such gaps, two situations can arise. First, from the error estimations made for frequencies in the gap, and from the observed continuity of the solution at the upper bound of the gap, the approximated solution may be considered acceptable. Otherwise, if a non acceptable discontinuity is manifest, an additional interval needs to be considered.
6 RESULTS AND DISCUSSION

6.1 Presentation of the application

In the scope of this contribution, the combination of a reduced model with the Padé reconstruction strategy is illustrated on a small 3D problem presented on Figure 4.

![Figure 4: Acoustic cavity mesh and dimensions for 3D problem](image)

It consists of an acoustic domain filled with air, bounded by rigid walls, and treated with a porous layer on one wall, which material parameters are given in Table 2. Sliding coupling conditions are set for the porous layer with the side walls and sticking with the back wall.

6.2 Results and discussion

The proposed adaptive reconstruction procedure is presented for two different truncations of the series expansion: \((L = 3; M = 4)\) and \((L = 5; M = 6)\). In both cases, the initial central frequency is chosen at 1975 Hz, while the error estimation limit is kept at 0.1. The results, featuring the reference solution together with the reconstructed solution and its associated intervals, are presented on Figure 5 for the \((L = 3; M = 4)\) truncation, and on Figure 6 for the \((L = 5; M = 6)\) truncation. The Padé-based reconstruction is combined with the non-reduced FE problem, and the porous-reduced model. The reference solution corresponds to the solution without reduction of the poroelastic domain. The Padé reconstruction applied to the non-reduced set of equations is achieved over 21 frequency intervals for the \((L = 3; M = 4)\) truncation (Figure 5a). It exhibits one gap between approximately 615 Hz and 640 Hz, and a solution not converged in the very upper frequency range of interest due to the choice of initial central frequency. However, due to the choice of limit for the error estimation, and the smooth divergence of the method out of its convergence interval, the reconstructed solution accurately matches the reference solution. Thus, using a reduced set of equations for the solution at central frequencies leads to a loss in precision (Figure 5b). Follows an increase in the error estimation, thus achieving a solution reconstructed over 24 intervals to be compared with the 21 intervals previously needed. In addition to the upper bound, 8 gaps have appeared, of which 4 can be neglected. Among the 4 remaining gaps, the one just below 600 Hz is of concern, considering its location around a resonance frequency, which could justify adding an interval to bridge the gap. However, its width is of approximately 10 Hz, it renders a peak in the frequency response, and its upper bound matches the lower bound of the contiguous interval. These, as confirmed by the comparison with the reference solution, very likely indicate a good approximation of the response. Finally, the gap between 160 Hz and 200 Hz is due to the special treatment applied to the lowest-frequency interval, not detailed in this work. Again, the matching solution between the upper bound of the gap and the lower bound of the contiguous interval indicates a good approximation.

A similar analysis can be conducted when analyzing the results for the procedure with a truncation order of \((L = 5; M = 6)\) for the series expansions (Figure 6). Increasing the order of truncation leads to reconstructed solutions over 15 and 18 intervals for the Padé-based procedure,

<table>
<thead>
<tr>
<th>Frame</th>
<th>Fluid</th>
<th>Porous</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_0 = 343 \text{ m/s})</td>
<td>(\phi = 0.96)</td>
<td>(\phi = 0.96)</td>
</tr>
<tr>
<td>(\lambda = 905357 \text{ Pa})</td>
<td>(\gamma = 1.4)</td>
<td>(\sigma = 32 \text{ kNs/m}^4)</td>
</tr>
<tr>
<td>(\mu = 264062 \text{ Pa})</td>
<td>(Pr = 0.71)</td>
<td>(\alpha_{\infty} = 1.7)</td>
</tr>
<tr>
<td>((1 - \phi) \rho_s = 30 \text{ kg/m}^3)</td>
<td>(\rho_f = 1.21 \text{ kg/m}^3)</td>
<td>(\Lambda = 90 \mu\text{m})</td>
</tr>
<tr>
<td>(\eta = 1.84 \cdot 10^{-5} \text{ Ns/m}^2)</td>
<td>(\Lambda' = 165 \mu\text{m})</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5: Adaptive reconstruction procedure applied to 3D problem with $L = 3$, $M = 4$: (a) from non-reduced problem; (b) from reduced problem.

Figure 6: Adaptive reconstruction procedure applied to 3D problem with $L = 5$, $M = 6$: (a) from non-reduced problem; (b) from reduced problem.

when applied to the the non-reduced and reduced sets of equations respectively. This is consistent with the larger interval of convergence expected when increasing the order of truncation. Again, a very good agreement can be observed between the reference solution and the reconstructed solutions. The only mismatch observed, similarly to what can be observed for the lower truncation order (Figure 5b and 6b), is the level of the 3 peaks of resonance above 900 Hz. Regarding those, it can be argued that they do not have prime physical significance considering how little damped
these resonances are, and thus how frequency-shift-dependent their level determination is.

The impact of increasing the order of truncation is further estimated by comparing the CPU computational times. They are presented on Figure 7, both for a truncation order of \((L = 3; M = 4)\) (Figure 7a), and for a truncation order of \((L = 5; M = 6)\) (Figure 7b). Using a Padé-based reconstruction approach leads to a frequency sweep almost 9 times as fast as for a direct approach, considering frequency increments of 2 Hz for this 3D application. Combining it to a modal-based reduced set of equations does not substantially improve the overall computational time, if accounting for the time allocated to establish the reduced problem. However, the averaged time per frequency increment is greatly improved, so that it even compensates for the cost of building the reduced model if the optimized modal basis is used. Regarding the impact of increasing the order of truncation, it appears, when comparing Figure 7a and Figure 7b, that there is a trade-off to be found. In fact, there is a limit above which, increasing the order of truncation does not substantially increase the intervals width of convergence. This imply little reduction in the number of central frequencies at which the complete solution has to be calculated, thus not compensating for the increased size of the Padé coefficient system to solve (Eq. (18)) for each dof. For the applications considered in this work, the order of truncation \((L = 3; M = 4)\), has proved a good compromise for computational efficiency, while ensuring a well-conditioned Padé coefficient system matrix.

7 CONCLUSION

In this contribution, a Padé-based reconstruction method was tested in combination with modal-based reduced problems. Such a combined approach takes advantage of the complementary properties of these methods. First, using a modal-based reduced problem saves memory resources. The reconstruction approach then enhances dramatically the resolution efficiency for the frequency response over the range of interest. While very application-dependent, and shown particularly efficient for very large applications, improvements by an order of magnitude were observed on the 3D validation case considered. The loss in precision due to the reduction implies the need for a finer decomposition in frequency intervals for the reconstruction scheme. However, this increase in the number of main frequencies is shown to be more than compensated by the improved efficiency of the reconstruction, due to the reduced number of dofs. For the considered example, it even compensates for the initial computational cost allocated to establish the reduced model.

Furthermore, an adaptive approach was proposed in order to automatically determine the main frequencies at which the complete direct solutions have to be computed, and around which the solution is approximated. It involves using an error estimation associated to the solution around
each main frequency, thus establishing frequency intervals of converged solution. Such converged intervals are used to estimate a priori the frequency interval of convergence for the neighbor main frequency, before some a posteriori adjustments. This simple approach proved to produce accurate frequency responses in a very computationally efficient way for the configurations tested. Thus, a priori knowledge of the dynamic behavior is not required for the choice of main frequencies. Additionally, it reduces the risk of setting a too coarse or to fine a priori discretization in frequency intervals, that would respectively hamper the solution accuracy or efficiency.

REFERENCES


