Optimal Heating
in the Smart Electrical Grid

ELIN NORDAHL

Master’s Degree Project
Stockholm, Sweden December 2012

XR-EE-RT 2012:034
Abstract

This thesis focuses on time flexible demand of electrically heated buildings, suggesting that load can be manipulated to maintain balance between supply and demand of electricity, and reduce cost with no or small impact on temperature comfort. Taking off from a consumer perspective, associated values are discussed and energy management strategies of electrically heated buildings are evaluated.
# Contents

## 1 Introduction

1.1 Residential Heating and Residential Electric Heating .......................... 3
1.1.1 Demand Response, Electrically Heated Households - Field Study ............ 4
1.1.2 A Changing Electricity Tariff .................................................. 5
1.2 Problem Formulation and Contributions ............................................. 6
1.2.1 Contributions ................................................................. 6

## 2 Dynamic Model and Test Scenarios

2.1 Thermal Model of a Household - Consumer Demand .......................... 8
2.1.1 System Description .......................................................... 10
2.2 The Electricity Market and Electricity Tariffs ................................... 11
2.2.1 Sets of Electricity Tariffs .................................................... 13
2.3 Sets of Outdoor Temperatures ....................................................... 13

## 3 Consumer Demand - Electrically Heated Buildings

3.1 Time Based Control Law, Inspired by Elforsk Field Study .................. 14
3.1.1 Defining Input of Time Based Control Law .................................. 15
3.1.2 Simulating Time Based Control Law ........................................ 18
3.1.3 Results of Time Based Control Law ................................-------- 19
3.1.4 Conclusion of Time Based Control Law ..................................... 19
3.2 Receding Horizon Control - Cost Minimization of Direct Electrical Heating .................. 21
3.2.1 Objective Function and Constraints ....................................... 21
3.2.2 Remedy for Infeasible Solutions .......................................... 23
3.2.3 Simulating Receding Horizon Control - Cost Minimization of Direct Electrical Heating .................................................. 25
3.2.4 Results of Receding Horizon Control - Cost Minimization of Direct Electrical Heating .................................................. 27
3.2.5 Conclusion of Receding Horizon Control - Cost Minimization of Direct Electrical Heating .................................................. 29
3.3 Receding Horizon Control - Cost Minimization of the use of a Heat Pump .................................................. 34
### Contents

#### 3.3.1 System Description
- Simulation of Receding Horizon Control - Cost Minimization of a Heat Pump

#### 3.3.3 Results of Receding Horizon Control - Cost Minimization of a Heat Pump

#### 3.3.4 Conclusion of Receding Horizon Control - Cost Minimization of a Heat Pump

#### 4 Cost minimization - Distributed Setting
- Decomposition - Three Separable Subsystems
- System Description
- Receding Horizon Control via Dual Decomposition
- Solving with The Price Adjustment Algorithm
- Simulating Distributed Setting - System of Three Subsystems
- Conclusion of Distributed Setting

#### 5 Conclusion and Future Work
- Conclusion
- Future Work

#### Appendices
- Optimization Problems and Receding Horizon Control
- Convexity
- Nonlinear Programming
- First Order Necessary Conditions
- Lagrange Function and Convexity
- Dynamic Model and Constraints
- Objective function
- Problem Formulation, Standard Form
- Feasibility
- Disturbances
- Stability
- Distributed setting, general framework
- The Price Adjustment Algorithm
Chapter 1

Introduction

An increasing share of renewable energy sources on the energy market are assumed to cause imbalance between energy supply and demand, resulting in highly variable energy prices. The introduction of these intermittent energy sources should rely on the flexibility of consumers, shifting consumption when production is high. Real time pricing and demand response are often promoted as keys for a successful large scale integration of renewables, preferably implemented in some kind of net, where flexibility of distribution is high. Consistency between supply and demand would be maintained with real time information, engaging energy producers and consumers to distribute and use distributed energy resources in an optimal way, in measures of economy and comfort.

This thesis takes off from a consumer perspective and under the assumption that demand response and consumer energy efficiency rely on introducing mechanisms and incentives for consumers to change energy consumption behavior. One of these incentives may be using distributed energy resources in such a way that cost is minimized. There are several types of cost minimizing applications that could be considered in this smart electrical grid, either by directly changing consumer habits or by the exploitation of indirect load shifts, shifting consumption where supply exceeds demand. Examples of such smart services could be using the natural thermal storage of refrigerators, air conditioning or a battery, loading in price valleys and reloading in price peaks allowing full consumer flexibility utilizing energy whenever and to whatever purpose.

There is an extensive literature treating different aspects of smart electrical grids. A general introduction to cost minimizing algorithms such as receding horizon control or model predictive control, suitable for such implementations is presented in [JYS10]. [MRRPW+07] presents an energy minimizing system for a household, discussing minimizing actions implementing model predictive control of a micro combined heat and power plant in combination with heat and electricity storages. In [MAV09] a controller is proposed, predicting wind power in the combination of optimizing state of charge of a battery, smoothing variations in wind power generation.
This thesis will pursue the idea of a thermal model of an electrically heated building, exploiting thermal capacity in order to minimize cost. Assuming an available prediction horizon, this energy planning problem may be subject to more advanced optimization algorithms such as receding horizon control.

### 1.1 Residential Heating and Residential Electric Heating

The large consumption of energy used on residential heating has long been recognized. According to the Swedish Energy Agency, an average Swedish household of 142 m² used 13.5 MWh on residential heating in 2009, [Ene11b]. This is 56 percent of its total energy consumption. Although the share of total consumption should be at its largest for single houses, this indicates a general consumption pattern and is assumed to hold for most traditional heating systems. This large amount of energy may hold an energy saving potential. As a response to this, a reduction of energy consumption within residential heating with 20 percent by 2020 and 50 percent by 2050, compared to 1995 was suggested, in Mars 2006 [GÅ08].

The future demand of central heating in general, was analyzed and summarized in [GÅ08]. Two important conclusions, relevant for this project were found. One states the importance of using heat recovery from ventilation exhaust air, an exhaust air heat pump is suggested. The report [GÅ08] also states that residential heating holds a large energy saving potential and that it could be accomplished quite easily, by using appropriate heating systems. A thorough analysis considering recent consumption trends within heating is given in [ene11a]. The report states that 27 percent of all one or two dwelling buildings in Sweden were heated using electricity. Electricity was converted into heat by warming water or directly using a resistor to convert electrical energy into heat. In 2009, approximately 14.4 TWh of electricity was used on electrical heating in Sweden.

#### 1.1.1 Demand Response, Electrically Heated Households - Field Study

Field work on the willingness of the consumer to adapt consumption behavior to a time-varying price was conducted and presented in [PES09], 2009. The intention was to investigate tariff models and control structures, enabling an increased price sensitivity of demand. The study was carried through by Elforsk, [PES09], and the report presents the result of two field studies. The first study covers results using a test group of 21 customers, where 10 customers were followed during two years. The second study covers results of 15 customers where 11 customers were followed during two years.

In the first field study, customers used waterborne electric heating. Consumption was managed remotely with direct load control. All changes in consumption were made directly and remotely coordinated, adapting to low and high prices by a net company, distributing energy. In the second study customers used direct electric heating, adjusting their consumption manually. These consumers were given
an estimate of time intervals where electricity tariffs were assumed to be low or high but also updates of current electricity tariffs, with the possibility to adjust consumption to changes in electricity tariffs outside predetermined time intervals. Near future electricity prices as well as past consumption and costs were given on-line.

The results presented in [PES09], concludes that the group of consumers in the field study did not suffer from any loss of comfort considering adjustments in effect due to electricity tariff variations. No conclusions were made considering leaving the actual decision power of consumption to some kind of coordinator, in this case the net company. For consumers who manually adjusted their consumption, the awareness of price changes was high as well as the willingness to, in some extent, adapt their behavior.

The conclusions stated in [PES09], suggests that these two groups of customers, participating in the studies mentioned above, were willing to adapt their consumption, making use of a volatile electricity tariff, without any loss of comfort. This thesis will extend these conclusions to comprise an arbitrarily large group of customers, voluntarily subject to energy management of electrically heated residential buildings.

1.1.2 A Changing Electricity Tariff

Ambitious goals regarding an efficient end-use of energy was proposed by a national program in Mars 2006, Sweden. An energy and climate program was also presented, aiming on an increase of renewable energy sources consumed with 49 percent by 2020, [GÅ08].

Renewable sources such as wind and solar power can be considered intermittent and difficult sources of energy to forecast, nor can they be controlled upon demand. Assuming that the price mechanisms of supply and demand hold, one would suggest that the introduction of such renewable sources on the energy market should change the dynamics of electricity tariffs. A more volatile electricity tariff is therefore to be expected, [PES09]. The effect of a changing electricity tariff with respect to volatility, should have impact on all consumption, in some way using electricity as input, including electrically supplied residential heating systems. With reliable energy planning the cost aware consumer could make considerable gain, making use of electricity tariff valleys and avoid consumption during electricity tariff peaks.
1.2 Problem Formulation and Contributions

The expected increase of renewable energy sources on the energy market is assumed to change the dynamics of electricity tariffs. A less volatile electricity tariff, revealing a well known consumption pattern with easily predictable peaks and valleys, is assumed to be replaced by a more volatile electricity tariff with new dynamics. One challenge, ensuring a successful introduction of renewable energy sources, should be how to engage the consumer to make maximum use of a changing electricity tariff. Smart meters, executing cost minimizing decisions may be one solution for such a challenge.

This thesis provides with a thermal model of a household, with changeable parameters. The aim is using this model and by simulations address potential benefits and potential problems introducing cost minimizing agents. Minimum achievable cost is evaluated in a set of test scenarios, defined by electricity tariffs of different volatility and outdoor temperature.

Applying real time optimization of residential heating in the electrical grid, three heating strategies are tried. By first only considering consumer demand possible savings when implementing a more advanced automatic control, such as receding horizon control, is compared against a time based control law defined by predetermined time intervals. Both control strategies are tried in a set of test scenarios. Input is defined by electricity directly converted into heating. Secondly, a heat pump is introduced. As before, receding horizon control is applied in different test scenarios. The cost function is described by electricity tariffs of different volatility and the time varying coefficient of performance. The relation between peaks and valleys of electricity tariff and outdoor temperature is acknowledged and the responsiveness to changes in volatility is compared. A third heating strategy implements dual decomposition in a distributed setting. A producer is included in a defined energy system, locally interacting with energy consumers.

In the first two heating strategies consumers are assumed to find current forecast values of electricity tariffs and outdoor temperatures centrally available, from which consumer cost functions are constructed. In the third heating strategy consumers are assumed to share a common resource. Forecast values of outdoor temperature may still be centrally available, but accessibility of electricity tariffs are decentralized, and energy distributed by exploiting the principles of shadow pricing.

1.2.1 Contributions

Taking off in the field work of [PES09] with centrally available information, simulations shows that it is economically justified, using an automatic cost minimizing algorithm such as receding horizon control on direct electrically heated residential buildings. This thesis uses simulation to show that a volatile electricity tariff is economically preferable for the consumer, compared to a less volatile electricity tariff.
tariff. Implementing receding horizon control on direct electric heating results in a reduced cost compared to the time based control law defined in this thesis. Simulations also show that possible relative gain, utilizing receding horizon control, is larger as the impact of outdoor temperature increases, making the control assign input to price valleys more accurately.

When implementing receding horizon control on the use of a heat pump, cost is reduced further. Outdoor temperature is included in the cost function, with the time varying coefficient of performance, resulting in an input assigned more continuously over the prediction horizon compared to direct electric heating. As outdoor temperature increases, the effect of a highly volatile electricity tariff is reduced. Input assigned, follows the variations of outdoor temperature for large enough outdoor temperatures. Added work defined as electricity is minimized when outdoor temperature approaches indoor temperature, i.e., for a large coefficient of performance of the heat pump, $COP_{hp}$.

This thesis show how consumers sharing a common resource can utilize the principles of shadow pricing, in a decentralized manner, via dual decomposition.

The concept of two types of structural models is presented. These models distinguish by the way they make information accessible to the consumers. A structural model with centrally available forecast values on a prediction horizon $N$, is compared against a decomposed structural model where information is distributed via public or interfacing variables.
Chapter 2
Dynamic Model and Test Scenarios

In this section, an energy consumer described by a simplified thermal model of a household is derived. Consumer demand is modeled as required heating to ensure indoor temperature comfort and addressed with the objective to reduce consumption cost, defined by the product of electricity consumed and cost of electricity for that period of time. Throughout the thesis, this model will be used and tested in different test scenarios defined by electricity tariff and outdoor temperature.

Thermal properties of the building are described in subsection 2.1. Electricity tariffs are described in subsection 2.2 and the outdoor temperatures used are described in subsection 2.3.

2.1 Thermal Model of a Household - Consumer Demand

The model used in this thesis considers an enclosed volume, neighboring an outer thermal zone of time varying temperature. The volume is equivalent to an insulated apartment of 45 m² or a small building separated from neighboring thermal zones, other than those defined by outdoor temperature. Outdoor temperature is considered as a measurable well behaved disturbance. No other disturbances, such as solar radiation or wind, affecting the system are considered.

Given a space with a thermal resistive enclosure, thermal characteristics can be described using SI-units.

- $T_{ia}(t)$ [K] or $y(t)$ is the temperature inside the thermal zone, to withhold within a desired interval of comfort.

- $T_{out}(t)$ [K] or $v(t)$ is the time varying outdoor temperature. In this case it is considered as a disturbance, assumed to be well behaved and continuous.

The three sets of outdoor temperature considered are described in subsection 2.3 depicted in Figure 2.3.
• $u(t)$ [kW] is the power input of the heating system. Input is assumed to react immediately.

• $D$ [kWK$^{-1}$] describes the thermal conductance of the enclosed thermal zone, defined as $UA$. The heat transfer coefficient is denoted $U$ and defined as $U = k/L$, where $k$ is the thermal conductivity and $L$ is the thickness of the enclosing material. The heat transfer coefficient is the quantity of heat that passes in unit time through unit area, measured in [Wm$^{-2}$K$^{-1}$]. A low $D$ implies a large thermal resistivity which describes a well insulated building. The total thermal conductance is the summation over all surfaces, enclosing the space, with corresponding material.

$$ D_{tot} = \sum_m U_mA_m $$

The parameter is thus highly dependent upon the area, the material of that area and what thermal zone neighboring that area. In this case, $U_m$ is set to an approximate mean and $A = \sum_mA_m$ is the area neighboring the outer thermal zone.

Considering the heat transfer coefficient of a house, $U_m$ is set to an approximate mean value for a well insulated buildings; 0.45 [Wm$^{-2}$K$^{-1}$], [DP09] and [MKDB12]. The enclosing area is set to $A = 145$ m$^2$.

• $C$ [kJK$^{-1}$] is the thermal capacitance or thermal capacity of the volume. A large $C_{th}$ implies that the space is capable of storing a larger amount of heat. In this case $C = C_{th}$ is a lumped parameter that characterizes the amount of heat required to change the temperature by one unit of all the components defining the space. The heat capacity is the summed heat capacity. This parameter includes air, interior and walls not interacting with outdoor walls. $V_k$ is the volume, $\rho_k$ is the density and $c_{p,k}$ is the specific heat of the material.

$$ C_{th} = \sum_k V_k\rho_k c_{p,k} $$

Considering thermal capacity of the volume assumed, numerical values from [DP09] and [MKDB12] are scaled and used in all simulation throughout the thesis. For an indoor space of 90 [m$^3$], $C_{th} = 3 \cdot 10^4$ [kJK$^{-1}$].

The thermal model derived, describes a dynamically slow system, meaning that a change of input does not have immediate effect on output. Increasing heating will slowly and gradually increase indoor temperature, and reducing heating will slowly decrease indoor temperature. The rate of cooling is defined by (2.1.1).

$$ \eta = \frac{\sum_m U_mA_m}{\sum_k V_k\rho_k c_{p,k}} $$ (2.1.1)
(2.1.1) yields an estimated value of $\eta = 0.008 \, [h^{-1}]$.

2.1.1 System Description

A heat balance equation will describe the changes in indoor temperature. When heat supplied equals heat loss, indoor temperature will be constant. In this case, output is converted to Celsius degrees. The system is described in continuous time as,

$$\frac{d}{dt} C_{th} T_{ia}(t) = Q_{supplied} - Q_{loss}$$ \hspace{1cm} (2.1.2)

$$Q_{loss} = D_{tot} (T_{ia}(t) - T_{oa}(t))$$

$$Q_{supplied} = u(t)$$

$$\frac{d}{dt} C_{th} T_{ia}(t) = u(t) - D_{tot} (T_{ia}(t) - T_{oa}(t)) \, [\degree C]$$ \hspace{1cm} (2.1.3)

By using forward Euler discretization, the heat balance equation (2.1.3) is represented in discrete time setting $\Delta t$ to one hour.

$$A = 1 - \frac{D_{tot}\Delta t}{C_{th}} \quad B = \frac{\Delta t}{C_{th}} \quad V = \frac{D\Delta t}{C_{th}}$$

$$T(t+1) = AT_{ia}(t) + Bu(t) + V T_{oa}(t) \, [\degree C]$$ \hspace{1cm} (2.1.4)

Indoor temperature, $T_{io}(t)$ or $y(t)$ will also be referred to as a measurable output of (2.1.4). Outdoor temperature, $T_{oa}(t)$ or $v(t)$ will be referred to as a time varying measurable disturbance, [TL07]. The model is linear, where $u(t)$ is the control signal to be executed.

$$y(t+1) = Ay(t) + Bu(t) + Vv(t) \, [\degree C]$$ \hspace{1cm} (2.1.4)

Numerical Values:

$$D_{tot} = 0.065 \, [kWK^{-1}], \, C_{th} = 3 \cdot 10^4 \, [kJK^{-1}], \, \Delta t = 1 \, [h].$$

2.2 The Electricity Market and Electricity Tariffs

The electricity price on the Nordic Market is set on Nord Pool Spot, and the price is determined based upon a trading. The cheapest types of productions are used first and thereafter, the more expensive production is included to meet demand. The electricity price is governed by the basic laws of economics, set by the interchange of supply and demand. Every hour, demand from energy traders are matched with energy producers and price determined.
The Nord Pool Spot market (Elspot) is a day-ahead market, where electricity tariff is determined by bids, predicting supply and demand 12 to 36 hours in advance of delivery, [Nor11]. Market clearing price is thereafter determined for each hour by the Nord Pool power exchange. Variable costs such as fuel costs and production taxes differ between different kinds of energy production. As opposed to the more traditional energy sources such as water or nuclear power, renewable energy sources are nearly cost free, yet not so reliable. A sudden increase of demand, or failure of a production plant, may cause heavy variations in tariffs, forcing production to a more costly power plant, [RIPO3]. Volatility is in this case, a measure of these variations, defined by the rate of which prices fluctuate. High volatility implies that prices fluctuate widely, while a low volatility does not. Assuming that time flexible consumers could balance inconsistency between supply and demand, load curve could be flatten with corresponding response on electricity tariffs, preventing unnecessary use of environmentally costly energy sources. A linear model including stability criteria of electricity tariffs, based on the one consumer one supplier case, can be found in [Fer97].

2.2.1 Sets of Electricity Tariffs

To simulate how a heating system would respond on electricity tariffs of different volatility caused by the factors mentioned above, two sets of electricity tariffs are used. One is based upon the tariffs on the Nordic Electricity Market, i.e. the spot price from Nord Pool, [Nor11], where volatility still is quite low, Figure 2.2. In an other case, the highly volatile electricity tariff of New York City is applied, [NYI11]. The two sets of electricity tariffs used throughout this thesis is plotted together in Figure 2.1, illustrating difference of volatility.

The electricity tariffs used, show a general consumption pattern. Although volatility is different, both electricity tariffs peak around hour 8 – 14 and 17 – 20, which are common time intervals of high consumption. Both tariffs are considerably low during night or early morning, at about hour 1 – 8. To simplify comparison, all tariffs are converted to Euros. 1 USD = 0.728 EUR.

Forecast values of electricity tariffs on a prediction horizon $N$, are assumed estimates based on historical data, shifting one time step ahead as one hour passes. In the reminder of this thesis, current electricity tariffs will be denoted as $p(t)$ and the set of available forecast values used, depicted in Figure 2.2, will be collected in a vector $f$.

$$f = \{p_{t1}, \ldots, p_{t+N-1}\}$$

By considering the electricity tariff of Nord Pool and New York City, over a horizon $N$, approximately the same mean value over the time period, is obtained by finding the weight $\rho$, defined in (2.2.1). In this way cost may be compared when volatility is the only parameter differing the two tariffs. $p_2$ denotes the set of values
Figure 2.1: New York City electricity tariff [EUR] and Nord Pool electricity tariff [EUR], plotted together, February 11th. The plot illustrates peaks of the New York City electricity tariff and Nord Pool electricity tariff, implying the heavy variations of the New York City electricity tariff compared to the Nord Pool electricity tariff.

Figure 2.2: Nord Pool tariff [EUR], February 11th. Shows peaks and valleys of Nord Pool electricity tariff, [Nor11]
on the New York City electricity tariff, and $p_1$ denotes the set of values on the Nord Pool electricity tariff.

$$\bar{p}_i = \frac{1}{N} \sum_{i=1}^{N} p_i$$

(2.2.1)

$$\rho \approx \frac{\bar{p}_1}{\bar{p}_2}$$

$p_2$ is scaled by $\rho$. In this case, $\rho$ is approximated with $\rho \approx 1.21$, yielding a mean value of $\bar{p} \approx 60.92$. Throughout this thesis, tables showing results of the weighted cost is denoted with $N.Y.C_\rho$.

### 2.3 Sets of Outdoor Temperatures

Three sets of outdoor temperatures are considered and presented in Figure 2.3 [SMH11]. The outdoor temperatures describes three test scenarios representing winter, spring and summer. In the reminder of this thesis, current outdoor temperature will be denoted as $v(t)$.

![Sets of outdoor temperatures over N = 24 hours, W](image)

The set of available forecast values used, depicted in Figure 2.3 will be collected in a vector $W$, where $N$ is the prediction horizon. Forecast values of outdoor temperature are assumed predictable and reliable estimates, and if needed updated by the hour.

$$W = \{v_{t+1}, ..., v_{t+N-1}\}$$
Chapter 3

Consumer Demand - Electrically Heated Buildings

Today, residential heating is rarely governed by any conscious decisions of high impact. Load follows the variations of disturbances determined by some rationally governed comfort constraints, defined by indoor temperature rarely deviating much from set-point. Still, the large amount of energy used on heating, makes small adjustments to changes in energy tariffs economically beneficial, justifying some kind of energy management.

Consider a consumer, interacting with an energy supplier. He or she has observed the time varying electricity tariffs offered by the energy supplier and decides to try reduce cost of residential heating. This Chapter will address possible benefits and possible problems that such a cost aware consumer will encounter, charged by an electricity tariff of varying spot price. Different concepts of energy management applied on residential heating in the electrical grid are introduced, using the derived thermal model of a household and the set of test scenarios defined in Chapter 2. The purpose is to illustrate how cost is affected when circumstances of this kind of heating change.

In the first two sections, section 3.1 and section 3.2, two types of control strategies are applied and tried in the same test scenarios. Results are compared and can be found in subsection 3.2.5. In the first case, the customer is assumed adjust energy input by a control law, defined by predetermined time intervals, and in the second case receding horizon control is implemented. Two types of inputs are tried, using receding horizon control. The first case considers cost minimization of direct electric heating, and the second case, section 3.3 considers cost minimization of the use of a heat pump.

In following sections, suggested controls are defined as local controllers, and without interaction with other consumers. Heat balance is assumed to be governed by the properties defined in subsection 2.1. In all cases considered, households are assumed to be equipped with indoor and outdoor sensors measuring temperature, manually or automatically read. Consumers are assumed to find current forecast
$f = \{p_t|t, p_{t+1}|t \ldots p_{t+N-1}|t\}$

$W = \{v_t|t, v_{t+1}|t \ldots v_{t+N-1}|t\}$

Consumer

Figure 3.1: Information governing cost minimizing decision are centrally accessible.

Values of electricity tariffs and outdoor temperatures centrally available and centrally managed, from which consumer cost functions are constructed, Figure 3.1. Cost is calculated based on simulation and compared using the sets of electricity tariffs defined in section 2.2. Outdoor temperature can be found in section 2.3.

3.1 Time Based Control Law, Inspired by Elforsk Field Study

The report [PES09] presents a control law implemented on waterborne electrical heating, adapting to a varying electricity tariff by changing input relative outdoor temperature, during some predetermined time intervals. Assume that a similar control strategy suggested in field study one of [PES09] is implemented; a time based control law with predetermined time intervals adjusting to electricity tariff variations. The control law, presented as the time based control law, can be assumed managed remotely with direct control, or adjusted manually by the consumer. Results are assumed to be the same, and governed by some kind of decision maker implementing software or not. The consumer is informed on how the electricity tariffs will develop the forthcoming day as well as estimated values of outdoor temperature. The consumer will follow a control scheme, such that consumption is reduced in time intervals where the electricity tariffs are expected to be high and preheats when electricity tariffs are expected to be low.

The main concept of this control is that it is fixed within these predetermined time intervals. Cost for the two cases, are achieved by multiplying the input sequence generated with corresponding electricity tariff.

The electricity tariffs used are assumed to follow the general pattern of peaks
and valleys described in subsection 2.2.1 [3.1] shows time intervals where an energy reducing action is recommended, either by reducing heating or preheating, [PES09]. These are the only time intervals considered in this thesis. Using these time intervals, Table 3.1 is designed.

Table 3.1: Table showing time intervals for corresponding control action to be implemented.

<table>
<thead>
<tr>
<th>Hour</th>
<th>Control action</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-7</td>
<td>Preheating due to low spot price.</td>
</tr>
<tr>
<td>7-9</td>
<td>Reduction of effect due to high spot price.</td>
</tr>
<tr>
<td>17-20</td>
<td>Reduction of effect due to high spot price.</td>
</tr>
</tbody>
</table>

\[ u(t) = \begin{cases} 
    u_{\text{max}} & \text{if } 5 \leq t < 7 \\
    u_{\text{min}} & \text{if } 7 \leq t < 9 \\
    u_{\text{min}} & \text{if } 17 \leq t < 20 \\
    u_n & \text{all other times } t 
\end{cases} \]

3.1.1 Defining Input of Time Based Control Law

As mentioned in subsection 2.1.1 indoor temperature will be constant when \( Q_{\text{loss}} \) equals \( Q_{\text{applied}} \). Using the heat balance equation given in (2.1.2), \( u_{\text{max}} \) and \( u_{\text{min}} \) is determined by shifting the expected value of outdoor temperature relative the true value of outdoor temperature. This shift results in a surplus or shortage of heat with corresponding effect on indoor temperature. \( T_{\text{ia}} \) is the indoor temperature and measured before applying the next control action for the next time sequence. \( \bar{T}_{\text{oa}} \) is an estimated mean of outdoor temperature for corresponding time interval, see Figure 2.3 and assumed to be given on-line. Indoor temperature is assumed to be 19 degrees Celsius at midnight, initializing the control sequence.

- \( u_n \) is the input when no control action relative electricity tariff is applied and defined as (3.1.1).

\[ u_n = D_{\text{tot}}(T_{\text{ia}} - \bar{T}_{\text{oa}}) \quad (3.1.1) \]

\( u_n \) is applied between hour 0 \( \leq u(t) < 5 \), 9 \( \leq u(t) < 17 \) and 20 \( \leq u(t) < 24 \).
• $u_{\text{max}}$ is assigned in a predetermined time interval when electricity tariffs are assumed to be low. Increased effect, $u_{\text{max}}$, is calculated out of a shift of $-5$ degrees Celsius, relative outdoor temperature for corresponding time interval and day of simulation. As when calculating $u_n$, $\bar{T}_{\text{oa}}$, is an estimated mean of outdoor temperature during this time interval.

\[ u_{\text{max}} = D_{\text{tot}}(T_{ia} - (\bar{T}_{\text{oa}} - 5)) \] (3.1.2)

• $u_{\text{min}}$ is assigned in a predetermined time interval when electricity tariffs are assumed to be high. Reduced effect, $u_{\text{min}}$ is calculated out of a shift of $+15$ degrees Celsius relative outdoor temperature for corresponding time interval and day of simulation. As when calculating $u_n$, $\bar{T}_{\text{oa}}$, is an estimated mean of outdoor temperature during this time interval.

\[ u_{\text{min}} = D_{\text{tot}}(T_{ia} - (\bar{T}_{\text{oa}} + 15)) \] (3.1.3)

To ensure comfort, output should be within given bounds. By knowing expected values of outdoor temperature, comfort can be ensured by using the equations above. If comfort constraints are violated, input is adjusted and gradually increased or decreased, such that indoor temperature hits the nearest bound of the comfort constraints, (3.1.4).

\[ y \leq y(t) \leq \bar{y} \] (3.1.4)

The procedure can be summarized as follows: The consumer observe electricity tariff dynamics and checks that they relate to recommended predefined time intervals with corresponding temperature shift, described in (3.1.1), (3.1.2) and (3.1.3). After each time interval, indoor temperature is measured and adjusted to the nearest bound if comfort constraint is violated.

**Numerical Values:**

$u_{\text{min}}$, $u_n$, and $u_{\text{max}}$, see (3.1.1), (3.1.2) and (3.1.3). $y = 19$ °C, $\bar{y} = 22$°C, $y_0 = 19$°C.
3.1.2 Simulating Time Based Control Law

As seen, simulating the time based control, input assigned manages to avoid electricity tariff peaks, Figure 3.2. Reduction of power during electricity tariff peaks are more evident than preheating during electricity tariff valleys, resulting in an indoor temperature close to the lower bound.

Figure 3.2: The time based control law designed in this case, keeps output within given bounds. As seen, the effect of recommended control action does not have a considerable effect on indoor temperature. The left column shows indoor temperature generated from corresponding input, shown to the right.
3.1.3 Results of Time Based Control Law

Table 3.2 shows possible costs when implementing the time based control law. In the third column, the New York City electricity tariff is scaled with $\rho$.

<table>
<thead>
<tr>
<th>Month</th>
<th>Cost Nord Pool</th>
<th>Cost N.Y.C</th>
<th>Cost $N.Y.C_{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>February</td>
<td>2.51</td>
<td>2.01</td>
<td>2.42</td>
</tr>
<tr>
<td>April</td>
<td>0.98</td>
<td>0.75</td>
<td>0.91</td>
</tr>
<tr>
<td>June</td>
<td>0.34</td>
<td>0.23</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Applying the time based control law designed in this case, shows that a more volatile electricity tariff decreases cost for the time period considered.

3.1.4 Conclusion of Time Based Control Law

The control law consisting of a scheme of recommended control actions, presented in [16509], regards the relation between low electricity tariff and low outdoor temperature, assigning a higher load during these hours. Furthermore, it reduces load when tariffs are high and outdoor temperature is high. Looking at the simulations, load curves generated are quite similar considering the three different sets of outdoor temperature. Output i.e., indoor temperature is quite flat and close to the lower bound of the comfort constraint. The effect of preheating is not very visible, Figure 3.2.

As seen in Table 3.2 the time based control law, shows that there is a possibility to save money on a more volatile electricity tariff. The control sequence defined in Table 3.1 is quite easy to implement and software is not necessary, decisions should be manageable with or without any kind of remote or automatic control, given that data over the prediction horizon is known and does not change during execution.

In the simulations shown before, the peaks and valleys of the electricity tariffs of New York City and Nord Pool are assumed to be within the same time intervals. Refining input to fit the New York City electricity tariff more accurately, keeping the same values of $u(t)$, makes output violate comfort constraints. These simulations are not included in the report, but this should indicate that changing heating strategy by adjustments to fit new observed dynamics of time varying parameters may be difficult. The control scheme can of course be refined and reduced cost improved considerably, depending on how cost aware the consumer is assumed to be,
but such adjustments are time consuming and not very practical. This argument includes updating information of indoor and outdoor temperature measurements and as well as re-estimations of intervals if the predetermined valleys and peaks of electricity tariffs do not fit.

If one assumes that the control law is governed by manual adjustments by the consumer, one mentionable advantage of this heating strategy is that full control is preserved and managed by the consumer. The result is only dependent upon how active he or she and for how long this attentiveness lasts. One major drawback of manual adjustments is loss of reliable load predictions. This may not be a problem for a single consumer, but it should complicate his or hers integration in an energy system, dependent on reliable predictions of consumer demand.
3.2 Receding Horizon Control - Cost Minimization of Direct Electrical Heating

Assume now that the same consumer as in section 3.1 decides to use a more advanced automatic control, cost minimizing heating input and aiming on at least as good results as when using the time based control law. The same test scenarios described in subsection 2.2 and subsection 2.3 are used together with the thermal model described in (2.1.4). All is automated and processed in software. As in the control strategy presented before, all desirable information on a time horizon $N$ is provided.

In this case a cost minimizing agent is introduced, predicting system dynamics considering time varying parameters and accounting for given constraints. The algorithm implemented is receding horizon control, also known as model predictive control, which is well suited for earlier mentioned comfort constraints and assumed objective, i.e. to minimize heating cost. An overview of relevant theory is given in Appendix A. The dynamics are given from the derived model in (2.1.2) and presented as (2.1.4).

$$y(t+1) = Ay(t) + Bu(t) + Vv(t)$$

As in (2.1.2), temperature is constant when supplied heat equals heat loss. This relationship holds for all times on the prediction horizon. Input is defined as electricity required to supply the system with heat.

3.2.1 Objective Function and Constraints

The objective is to determine actions such that cost is minimized and simultaneously fulfilling given constraints i.e., to minimize energy input $u_{t+k}$ at spot price $p_{t+k}$ given time $t$, such that the minimizing decision satisfies given constraints of the system. The subscripts on $p$ and $u$ corresponds corresponding variable at time $t+k$, which is an estimate for a range defining the prediction horizon given time $t$. $t = 0$ for the first iteration, and after corresponding optimization problem is solved updated with $t+1$ for a time period $M$ of interest. $\sum_{k=0}^{N-1} p_{t+k} u_{t+k}$ defines the running cost over the prediction horizon. The control will thus penalize a large product of electricity tariff and required energy input.

To ensure a satisfying indoor temperature, constraints are imposed on the system, such that (3.2.1) defines the interval of temperature comfort. Bounds given in (3.2.2) defines a limit on achievable energy input. All simulations are assumed to start with an indoor temperature of 19 degrees Celsius.

$$\underline{y} \leq y_{t+k} \leq \bar{y} \quad (3.2.1)$$

$$0 \leq u_{t+k} \leq \bar{u} \quad (3.2.2)$$
Assuming full information at state $y_t$, the problem can be represented as a constrained optimization problem.

$$\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} p_{t+k|t} u_{t+k|t} \\
\text{subject to} & \quad y_{t+k|t} = A y_{t+k|t} + B u_{t+k|t} + V v_{t+k|t} \\
& \quad y \leq y_{t+k|t} \leq \bar{y} \\
& \quad 0 \leq u_{t+k|t} \leq \bar{u} \quad (3.2.3) \\
& \quad y_{t0} = y(t)
\end{align*}$$

By using forward recursion over the prediction horizon, system dynamics can be forecasted. This includes how the system will respond on disturbances for the forthcoming time.

$$\begin{align*}
y_{t+1|t} &= A y_{t|t} + B u_{t|t} + V v_{t|t} \\
y_{t+2|t} &= A y_{t+1|t} + B u_{t+1|t} + V v_{t+1|t} \\
y_{t+3|t} &= A y_{t+2|t} + B u_{t+2|t} + V v_{t+2|t} \\
y_{t+4|t} &= A y_{t+3|t} + B u_{t+3|t} + V v_{t+3|t} \\
& \vdots \\
y_{t+N|t} &= A y_{t+N-1|t} + B u_{t+N-1|t} + V v_{t+N-1|t} \quad (3.2.4)
\end{align*}$$

Expanding the equations given in (3.2.4) for the prediction horizon of choice, yields explicit constraints on system dynamics, given in (3.2.5).

$$\begin{align*}
y_{t+k|t} &= A^k y_{t|t} + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j|t} + \sum_{j=0}^{k-1} A^j V v_{t+k-1-j|t} & k = 1, \ldots, N. \quad (3.2.5)
\end{align*}$$

The decision variables can be defined as,

$$Y = \{y_{t+1|t}, \ldots, y_{t+N|t}\}$$

$$U = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$$

The estimated measurable disturbance, i.e., outdoor temperature is defined in the vector $W$.

$$W = \{v_{t|t}, \ldots, v_{t+N-1|t}\}$$
The vector \( z \) contains all decision variables and the vector \( f \), contains the forecast values of the electricity tariff used. Using the vector of decision variables and associated constraints, the problem may be formulated as in (3.2.6).

\[
z = \{y_{t+1|t}, \ldots, y_{t+N|t}, u_{t|t} \ldots u_{t+N-1|t}\}
\]

\[
f = \{p_{t|t}, \ldots, p_{t+N-1|t}\}
\]

The matrix \( G \) is a \( 6N \times 2N \) matrix containing predictions of system dynamics and boundary constraints for corresponding decision variable. The matrix \( S \) is a \( 6N \times 1 \) matrix and contains all constants for corresponding decision variable. The matrix \( E \) is a \( 6N \times 1 \) matrix associated with state measurement.

\[
\begin{align*}
\text{minimize} & \quad f^T z \\
\text{subject to} & \quad Gz \leq S + Ey_{t|t} \\
& \quad G \in \mathbb{R}^{6N \times 2N}, \quad S \in \mathbb{R}^{6N \times 1}, \quad E \in \mathbb{R}^{6N \times 1}
\end{align*}
\]

The basic receding horizon control law is described by the following algorithm.

**Algorithm 1** Algorithm, receding horizon control

```plaintext
for \( t = 1 \rightarrow M \) do
    Measure the state \( y_{t|t} \).
    Find current forecast values of tariff and outside temperatures for the control period \( N \).
    Solve the optimization problem (3.2.6) and find the minimizing set, \( U^* = \{u_{t|t}^*, \ldots, u_{t+N-1|t}^*\} \).
    if \( U^* = \emptyset \) then
        Stop. Problem infeasible, find remedy.
    end if
    Apply only \( u_{t|t}^* \).
    Wait for the new sampling time \( t + 1 \).
end for
```

### 3.2.2 Remedy for Infeasible Solutions

Adding slack variables to constraints involving states preserves feasibility. In this case, energy input is restricted by the capacity of the heaters installed and the solution to (3.2.6) is infeasible for low enough outdoor temperatures. Introducing
slack variables ε and ε allow violation of output constraints resulting in a small deviation from \( y \) and \( y \), if needed. The slack variables are positive only when the output constraints are violated. In order to keep the slack variables close to zero they are included in the cost function and penalized hard, here denoted with \( \gamma \). The problem is now transformed such that (3.2.7) is solved instead of solving (3.2.3).

\[
\begin{align*}
\min_{z} & \quad \sum_{k=0}^{N-1} p_{t+k|\ell} u_{t+k|\ell} + \gamma \sum_{k=1}^{N} (\varepsilon_{t+k|\ell} + \bar{\varepsilon}_{t+k|\ell}) \\
\text{subject to} & \quad y_{t+k+1|\ell} = A y_{t+k|\ell} + B u_{t+k|\ell} + V v_{t+k|\ell} \\
& \quad y - \varepsilon_{t+k|\ell} \leq y_{t+k|\ell} \leq y + \bar{\varepsilon}_{t+k|\ell} \\
& \quad 0 \leq u_{t+k|\ell} \leq \bar{u} \\
& \quad \varepsilon_{t+k|\ell} \geq 0 \\
& \quad \bar{\varepsilon}_{t+k|\ell} \geq 0 \\
& \quad y_{t|\ell} = y(t) \\
\end{align*}
\]

(3.2.7)

Rearranging the constraints, the matrix \( G \) is now \( 6N \times 3N \). The matrix \( S \), containing all constants is a \( 6N \times 1 \) and the matrix \( E \), is a \( 6N \times 1 \) matrix. The new vector of variables \( z \) can be found below.

\[
\begin{align*}
- \sum_{j=0}^{k-1} A^j B u_{t+k-1-j|\ell} - \varepsilon_{t+k|\ell} + \bar{\varepsilon}_{t+k|\ell} & \leq \sum_{j=0}^{k-1} A^j V v_{t+k-1-j|\ell} - y + A^k y_{t|\ell}, \quad k = 1, \ldots, N. \\
\sum_{j=0}^{k-1} A^j B u_{t+k-1-j|\ell} + \varepsilon_{t+k|\ell} - \bar{\varepsilon}_{t+k|\ell} & \leq - \sum_{j=0}^{k-1} A^j V v_{t+k-1-j|\ell} + \bar{\varepsilon}_{t+k|\ell} - A^k y_{t|\ell}, \quad k = 1, \ldots, N. \\
- \varepsilon_{t+k|\ell} \leq 0, \quad -\bar{\varepsilon}_{t+k|\ell} \leq 0, \quad -u_{t+k|\ell} \leq 0, \quad u_{t+k|\ell} \leq \bar{u} \\
z = \{ u_{t|\ell}, \ldots, u_{t+N-1|\ell}, \varepsilon_{t|\ell}, \ldots, \varepsilon_{t+N|\ell}, \bar{\varepsilon}_{t|\ell}, \ldots, \bar{\varepsilon}_{t+N|\ell} \} \\
f_w = \{ p_{t|\ell}, \ldots, p_{t+N-1|\ell}, \gamma, \ldots, \gamma \}
\end{align*}
\]

The vector \( f_w \) defines the weight on the running cost where \( \gamma \) is the penalty on \( \varepsilon \) and \( \bar{\varepsilon} \). The running cost is updated for every iteration as indicated by the subscripts. The problem is redefined as (3.2.8).
\[
\begin{align*}
\text{minimize} & \quad f_z^T \zeta \\
\text{subject to} & \quad G \zeta \leq S + E y_{t0} \\
G & \in \mathbb{R}^{6N \times 3N}, \quad S \in \mathbb{R}^{6N \times 1}, \quad E \in \mathbb{R}^{6N \times 1}
\end{align*}
\] (3.2.8)

Now, referring back to Algorithm 1, feasibility is recovered. At time \( t \), the optimization is solved with respect to the new decision variables defined in \( z \), in this case containing control inputs and slack variables, penalized for violating given comfort constraints. Feasibility is preserved and the first input \( u^{*}_t \) can be applied. The procedure is repeated for \( M \) time steps. At each time step, the controller predicts future optimal energy input with respect to expected outdoor temperature and expected future cost for a prediction horizon of \( N \). Implementing the first of \( N \) predicted inputs the heat equation (2.1.4) is updated. The optimization problem (3.2.8) has a linear convex objective, subject to affine constraints and solved by calling \texttt{linprog} in Matlab.

**Numerical Values:**
\[ y = 19^\circ C, \quad \bar{y} = 22^\circ C, \quad \bar{u} = 3.8 \text{ kW}, \quad y_0 = 19^\circ C, \quad \gamma = 1000. \]

### 3.2.3 Simulating Receding Horizon Control - Cost Minimization of Direct Electrical Heating

**Heavy effect of disturbance - simulating a day in February**

Figure 3.3 shows simulations of the system, a day in February. If the effect of the disturbance is large, the effect of preheating is relatively large. The effect increases with approximately 0.5 degrees, comparing output when using New York City tariff versus Nord Pool tariff.

**Outdoor temperature rises - simulating a day in April**

Considering the simulation of minimizing cost with the New York tariff in April, input is maximized over an interval between hour 1 and 6. Prices are low compared to the other times on the horizon. The control successfully predicts, that assigning input in this time interval is optimal. Volatility of the New York City electricity tariff is very large compared to the electricity tariff of Nord Pool, the need of preheating is thus not so large, and the control assigns a second input between hour 14 – 17, Figure 3.4.
Figure 3.3: Implementing receding horizon control, February 11th. Although relatively high electricity tariffs, input is assigned in the middle of the day. The left column represents the system when the New York City tariff is used, and the right column represents the system when the less volatile Nord Pool tariff is used. The top row shows indoor temperature for corresponding tariff. The second row shows assigned input where the dashed line shows how outdoor temperature varies during the day. The last row shows the electricity tariffs used.

**High decision freedom - simulating a day in June**

In Figure 3.5, the control acts in a similar manner as in the case presented before, but with a reduced time interval of input. Although outdoor temperature never goes below 11 degrees, input is high when prices are low and maximized for the New York City tariff case, with 3.8 kW for one hour. In the Nord Pool tariff case, input peaks with 2.5 kW between hour 4 to 5 and switched on again around hour 20.
Figure 3.4: Implementing receding horizon control, April 6th. Noticeable effect of outdoor temperature peak. No or less input is needed in the middle of the day, compared to the day in February. The left column represent the system when the New York City tariff is used, and the right column represents the system when the less volatile Nord Pool tariff is used. The top row shows indoor temperature for corresponding tariff. The second row shows assigned input where the dashed line shows how outdoor temperature varies during the day. The last row shows the electricity tariffs used.

3.2.4 Results of Receding Horizon Control - Cost Minimization of Direct Electrical Heating

As a measurement on achievable savings maximum cost is calculated. The results are presented in Table 3.3. Secondly, it should be interesting to compare cost, when the electricity tariffs have the same mean value, but different volatility. This should be an indication on how cost could develop if volatility changes. The results are presented in Table 3.4.
Figure 3.5: Implementing receding horizon control, June 1st. The effect of disturbances is not so heavy. Input in the middle of the day is reduced further and shifted to the valleys around hour 4 to 5 of the electricity tariffs. The left column represents the system when the New York City tariff is used, and the right column represents the system when the less volatile Nord Pool tariff is used. The top row shows indoor temperature for corresponding tariff. The second row shows assigned input where the dashed line shows how outdoor temperature varies during the day. The last row shows the electricity tariffs used.

**Span of achievable savings**

To see under what circumstances the span of possible savings change and how, (3.2.8), is solved as a maximization problem and compared to minimum cost. As opposed to the minimization problem, this cost should illustrate when decisions are managed in worst possible way, in this case illustrating input maximizing cost with respect to electricity tariff and outdoor temperature, keeping output within given range of comfort.
maximize \[ u^T z \] 
subject to \[ Gz \leq S + Ey_{ff} \] (3.2.9)

The span of achievable savings are measured in percent and presented in column four and five for corresponding tariff, Table 3.3. As seen, the span is larger for the volatile New York City tariff, compared to the less volatile Nord Pool tariff.

Table 3.3: The Table shows minimum cost and maximum cost for corresponding optimization problem, illustrating the span of possible savings, for direct electric heating. Cost is measured in Euros, [EUR].

<table>
<thead>
<tr>
<th>Month</th>
<th>Nord Pool</th>
<th>N.Y.C.</th>
<th>Nord Pool</th>
<th>N.Y.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>March 11th</td>
<td>2.48</td>
<td>4.24</td>
<td>1.79</td>
<td>3.58</td>
</tr>
<tr>
<td>April 6th</td>
<td>0.96</td>
<td>2.70</td>
<td>0.51</td>
<td>2.48</td>
</tr>
<tr>
<td>June 1st</td>
<td>0.33</td>
<td>2.05</td>
<td>0.17</td>
<td>2.03</td>
</tr>
</tbody>
</table>

Comparing impact of volatility

Table 3.4 illustrates possible gains, when the New York City tariff is scaled with \( \rho \), defined in subsection 2.2. The results show that minimizing cost, is more beneficial with a volatile electricity tariff. Given two different electricity tariffs with the same mean value over the prediction horizon \( N \), cost is less applying the New York City electricity tariff, compared to applying the Nord Pool electricity tariff. As the effect of the disturbance decreases, the relative gain of using a highly volatile electricity tariff increases. This is illustrated in column four in Table 3.4 as an increased percent of savings.

Table 3.4: Comparing impact of volatility and percent savings using a more volatile electricity tariff minimizing cost with direct electric heating. Cost is measured in Euros, [EUR].

<table>
<thead>
<tr>
<th>Month</th>
<th>Nord Pool</th>
<th>N.Y.C.</th>
<th>percent savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>March 11th</td>
<td>2.48</td>
<td>2.17</td>
<td>13 %</td>
</tr>
<tr>
<td>April 6th</td>
<td>0.96</td>
<td>0.62</td>
<td>35 %</td>
</tr>
<tr>
<td>June 1st</td>
<td>0.33</td>
<td>0.21</td>
<td>36 %</td>
</tr>
</tbody>
</table>
Table 3.5 shows cost acquired when simulating the system with a constant electricity tariff. By simulating the system and using a constant electricity tariff in the cost function, keeping slack variables as they were, the algorithm accounts for predictions of system dynamics. The result is an input, assigned proportionally and optimally with respect to outdoor temperature. This should be an indication of how cost will develop when disturbances are managed in an optimal way, but all tariff dynamics are neglected. Cost is calculated by multiplying input generated with corresponding electricity tariff. Output is kept constant at the lower bound of the comfort constraint, 19 degrees Celsius.

Table 3.5: Cost acquired when outdoor temperatures are managed in an optimal way but, dynamics of electricity tariffs are neglected. Cost is measured in Euros, [EUR].

<table>
<thead>
<tr>
<th>Month</th>
<th>Nord Pool</th>
<th>N.Y.C p</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 11th</td>
<td>2.51</td>
<td>2.47</td>
</tr>
<tr>
<td>April 6th</td>
<td>0.97</td>
<td>0.94</td>
</tr>
<tr>
<td>June 1st</td>
<td>0.34</td>
<td>0.29</td>
</tr>
</tbody>
</table>

Changing the prediction horizon

So far the simulations have been performed under the assumption of perfect information i.e., no restriction of available information. This should not be the case for real implementation. Table 3.6 and Table 3.7 shows cost acquired when simulating the system with a reduced prediction horizon $N$, applying the Nord Pool electricity tariff and the scaled New York electricity tariff, $N.Y.C_p$.

Table 3.6: Cost acquired when prediction horizon is reduced, simulating with the Nord Pool electricity tariff. Cost is measured in Euros, [EUR].

<table>
<thead>
<tr>
<th>Month</th>
<th>N = 12</th>
<th>N = 6</th>
<th>N = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 11th</td>
<td>2.48</td>
<td>2.48</td>
<td>2.48</td>
</tr>
<tr>
<td>April 6th</td>
<td>0.96</td>
<td>0.96</td>
<td>0.96</td>
</tr>
<tr>
<td>June 1st</td>
<td>0.34</td>
<td>0.34</td>
<td>0.37</td>
</tr>
</tbody>
</table>
Table 3.7: Cost acquired when prediction horizon is reduced, simulating with the Nord Pool electricity tariff. Cost is measured in Euros, [EUR].

<table>
<thead>
<tr>
<th>Month</th>
<th>$N = 12$</th>
<th>$N = 6$</th>
<th>$N = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 11th</td>
<td>1.99</td>
<td>2.05</td>
<td>2.21</td>
</tr>
<tr>
<td>April 6th</td>
<td>0.69</td>
<td>0.77</td>
<td>0.84</td>
</tr>
<tr>
<td>June 1st</td>
<td>0.24</td>
<td>0.24</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Note that cost for a prediction horizon of $N = 12$ and $N = 6$ in Table 3.7, simulating the system in February are less than cost for a prediction horizon of $N = 24$. This does not follow the same trend as the other results of Table 3.6 and Table 3.7, which shows an increasing cost for a reduced prediction horizon. The increasing cost for a predicting horizon of $N = 24$ should be due to large input assigned at the end of the day, planning for an estimated temperature drop at $t = 48$.

**Changing the parameters of the thermal model**

A large time constant allows adjusting consumption, shifting load to more preferable time intervals. By altering the ratio of thermal conductance, $D_{tot}$ and the thermal capacity, $C_{th}$, the time constant changes. Keeping thermal conductance $D_{tot}$ constant, the system is simulated changing the value of thermal capacity for the two electricity tariffs and the different sets of outdoor temperature, starting with a relatively low value of thermal capacity. Simulations show that using the New York City tariff causes rapid changes of output and input. Disturbances are heavy but so is the responding effect of input. This relative dynamically fast system does not store energy which diminishes consumer flexibility and increases cost. Using the same model with a low thermal capacity, but changing tariff to the Nord Pool electricity tariff shows an output, i.e. indoor temperature, following the inverted pattern of the disturbances and as assumed an increased cost. The result is compared using the same parameters as before, but a very large thermal capacity. Simulations show, that input and output follows about the same output pattern and load curve as the original model, but cost is increased, suggesting that a very large thermal capacity looses precision relative time varying parameters.

In a second trial, thermal capacity is kept constant and thermal conductance changed. Keeping thermal capacity constant and increasing thermal conductance, improves insulation of the system, excluding disturbances. Simulations with these parameters show a load curve that accurately finds unique minima with high precision, reducing cost considerably. For brevity, these simulations are not included in the report.
3.2.5 Conclusion of Receding Horizon Control - Cost Minimization of Direct Electrical Heating

Implementing receding horizon control on electrically heated residential buildings, shows a reduced cost for more a volatile electricity tariff, minimizing with respect to price dynamics and constraints. Setting the prediction horizon to 24 hours, the controller successfully avoids price peaks. The control considers changes of volatility and effect of disturbances simultaneously. If volatility is high, the effect of preheating is larger.

Table 3.3 illustrates relative gains comparing minimum cost and maximum cost for the two tariffs considered. The result shows that the span of possible savings is larger with a highly volatile electricity tariff. As seen in Table 3.4 possible gains implementing receding horizon control on electrically heated residential buildings with the New York City electricity tariff, grows with outdoor temperature. As outdoor temperature increases, decision freedom of when to assign heating input grows. The disturbance does no longer affect the system so heavily, which allow the control to assign input to price valleys more accurately. Simulating the system and comparing the plots for respectively time period, applying different electricity tariffs, also show that when volatility is large the control uses the advantage of preheating to a greater extent.

Table 3.6 and Table 3.7 addresses the importance of reliable predictions, showing an increased cost for a reduced prediction horizon.

Comparing result of time based control law and receding horizon control

In section 3.1 a heating strategy similar to the one presented in [PES09] was implemented. Results using the time based control law presented in Table 3.2 is compared with the results using receding horizon control on the same system, Table 3.4. Table 3.8 shows possible gains in percent, using receding horizon control on temperature control of electrically heated residential buildings, compared to using the time based control law.

As a second instrument of comparison, illustrating the reduced cost of implementing receding horizon control with full system information, cost acquired in Table 3.4 is used and compared with cost acquired when neglecting the weight of electricity tariffs, Table 3.5. This should be a fair comparison, showing what actually is to gain when making adjustments to electricity tariffs as opposed to only considering disturbances.

As outdoor temperature increases, the economical advantage of using receding horizon control grows.

One mentionable advantage of introducing receding horizon control is the actual possibility for the consumer to define comfort constraints. Using receding
Table 3.8: Percent savings of using receding horizon control, (R.H.C) with full information, with cost acquired from Table 3.4 compared to the time based control, (T.B.C) with cost acquired from Table 3.2. Cost acquired with receding horizon control is also compared, simulating the system without the weight of electricity tariff, with cost acquired in Table 3.5.

<table>
<thead>
<tr>
<th>Month</th>
<th>T.B.C vs. R.H.C</th>
<th>T.B.C vs. R.H.C, no tariff</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nord Pool</td>
<td>N.Y.Cρ</td>
</tr>
<tr>
<td>February</td>
<td>1.2 %</td>
<td>10.3 %</td>
</tr>
<tr>
<td>April</td>
<td>2.0%</td>
<td>31.9 %</td>
</tr>
<tr>
<td>June</td>
<td>3.0%</td>
<td>25.0%</td>
</tr>
</tbody>
</table>

Receding horizon control, output will stay near comfort constraints, given that the cost function is defined correctly.

Receding horizon control predicts demand on a rolling horizon principle. Assuming that electricity tariffs do not change after being given to the consumer, and the system does not suffer from any radical changes, a forecast of consumer demand will be available.
3.3 Receding Horizon Control - Cost Minimization of the use of a Heat Pump

This subsection considers the same household as before, but instead of directly converting electricity into heat, a heat pump is used. [HK02] provides with basic theory on heat pumps.

A heat pump operates under the laws of thermodynamics. Adding work to the system lifts thermal energy from a low temperature source to a high temperature sink, Figure 3.6. \( \text{COP}_{hp} \), the coefficient of performance, defines the heat pump gain which also defines the efficiency of the process. If the heat pump works under the Carnot cycle, equation (3.3.2) holds.

A simplified model, considering the summed added work for desirable effect is used, this would describe an ideal heat pump. Given that the ratio of the coefficient of performance of a theoretical Carnot cycle and the coefficient of performance of a high efficiency air source heat pump is approximately two at zero degrees Celsius, [Wik11] a factor of inefficiency \( \delta \), is multiplied with \( \text{COP}_{hp} \), as added work is calculated. This is to make results of cost more realistic. The same model of one consumer derived in (2.1.4) is applied. New parameters are introduced in 3.3.1.

![Figure 3.6: Schematic illustration of the basic principle of a heat pump. Added work lifts Q2 to Q1.](image)

\( \text{High Temperature Sink } T_{ia} \)

\( Q_1 \)

\( \text{work} \)

Heat Pump

\( Q_2 \)

\( \text{Low Temperature Source } T_{oa} \)
3.3.1 System Description

- $w_{el}$ [kW] is energy consumed in terms of added work. In order to transfer heat to a high temperature body, the Heat Pump requires work, defined by $work = Q_1 - Q_2$.
- $Q_1$ [kW] defines heat delivered as heating input. $Q_1$ is heat at the high temperature sink at $T_{ia}$.
- $Q_2$ [kW] is heat extracted at low temperature source at temperature $T_{oa}$.
- $\delta$ is the factor of inefficiency.

$$w_{el} = \frac{Q_1}{\delta \text{COP}_{hp}}$$ (3.3.1)

$$\text{COP}_{hp} = \frac{|Q_1|}{|Q_1| - |Q_2|}$$

$$\text{COP}_{hp} = \frac{T_{ia}}{T_{ia} - T_{oa}} = \frac{y_{it}}{y_{it} - v_{it}} \geq 1$$ (3.3.2)

To avoid the singularity in (3.3.2), no outdoor temperature above or equal to $y_{it}$ is considered neither the reverse effect i.e., to use the heat pump for cooling indoor temperature. If outdoor temperature exceeds indoor temperature, the heat pump is shut down, and supplied heat set to zero. Assuming that an estimate of outdoor temperature $v_{it}$ is available on a prediction horizon $N$, an estimation of the weight $\text{COP}_{hp}$ will also be available and vary as outdoor temperature changes. When outdoor temperature $v(t)$ approaches indoor temperature $y(t)$ or conversely, the heat pump will operate with a minimum of added work. A large $\text{COP}_{hp}$ is thus desirable. At absolute temperature zero Kelvin, no energy is to transfer and the $\text{COP}_{hp}$ is one. The coefficient of performance must always be larger or equal to one. The product of work and time-varying electricity tariff defines the running cost. The objective is to minimize added work simultaneously delivering required heat supply fulfilling constraints. The problem is formulated as follows.

$$\text{minimize} \quad \sum_{k=0}^{N-1} p_{t+k} \left( \frac{u_{t+k}(y_{t+k+1} - v_{t+k})}{y_{t+k+1}} \right) + \gamma \sum_{k=1}^{N} (\bar{e}_{t+k} + \underline{e}_{t+k})$$

subject to

$$y_{t+k+1} = Ay_{t+k} + Bu_{t+k} + Vv_{t+k}$$

$$y - \underline{e}_{t+k} \leq y_{t+k} \leq \bar{y} + \overline{e}_{t+k}$$

$$0 \leq u_{t+k} \leq \bar{u}$$

$$\underline{e}_{t+k} \geq 0$$

$$\overline{e}_{t+k} \geq 0$$

$$y_{it} = y(t)$$

(3.3.3)
Collecting all variables in a vector $z$, the problem is reformulated as (3.3.5).

$$z = \{ y_{t+1|t}, ..., y_{t+N|t}, u_{t|t}, ..., u_{t+N-1|t}, \varepsilon_{t|t}, ..., \varepsilon_{t+N|t}, \bar{\varepsilon}_{t|t}, ..., \bar{\varepsilon}_{t+N|t} \}$$

$$\begin{aligned}
&\text{minimize} & & J \\
&\text{subject to} & & Gz \leq S + Ey_{t|t} \\
& & & G \in \mathbb{R}^{8N \times 4N}, \quad S \in \mathbb{R}^{8N}, \quad E \in \mathbb{R}^{8N}
\end{aligned}$$ (3.3.4)

This is a nonlinear objective $J$ with affine constraints, and solved by calling \texttt{fmincon}. The solving algorithm is set to \texttt{active-set}. This optimization procedure provides with a local optimum and as opposed to a global optimum it may not be unique.

**Numerical Values:**

$\overline{y} = 19^\circ C$, $\overline{y} = 22^\circ C$, $\bar{u} = 3.8$ kW, $y_0 = 19^\circ C$, $\gamma = 1000$, $\delta = 0.5$.

### 3.3.2 Simulation of Receding Horizon Control - Cost Minimization of a Heat Pump

Minimum work is required when $COP_{hp}$ is maximum. The running cost is defined by the time varying electricity tariff and work $w_{el}$, which varies with outdoor temperature. As outdoor temperature increases, $w_{el}$ decreases. Outdoor temperature most probably peak around hour 15, which as seen also coincide with relative low electricity tariffs. The product $w_{el|k|t}$ and $p_{t+k|t}$ should be relative small in this time interval.

In the three simulaton that follows, Figure 3.7, 3.8 and 3.9, the left column represents simulations using the New York City tariff, and the right column represents simulations using the Nord Pool electricity tariff. The top row show indoor temperature, the second row shows electric power and supplied heat together. In the third row electric power is plotted with outdoor temperature, presented with a black dashed line. The last row illustrates the two different electricity tariffs considered.

It can be noted that changing the comfort constraints have a direct effect on the cost function. A smaller lower limit, makes the heat pump more efficient and changes the consumption pattern.

**Relative Small $COP_{hp}$ - simulating a day in February.**

Figure 3.7 simulating a day in February, shows that the need of preheating is large. For the New York City tariff case indoor temperature peaks around 20.4 degrees at
Figure 3.7: Simulation of the system, February 11th. Load profile is still quite similar as when using direct electric heating. Still, the effect of increasing $COP_{hp}$ in the middle of the day is visible, making the control to assign a larger amount of input between hour 14 and hour 17 compared to direct electric heating. For the less volatile electricity tariff, input is more spread out on the horizon, compared to the case where the New York City tariff is used.

hour 6 and then peaks again at hour 16 with 19.9 degrees. For the Nord Pool tariff case, temperature peaks around the same hours with about 20 degrees around hour 6 and reaches a maximum of 20 degrees at hour 16. The need of preheating is thus
larger with the highly volatile New York City tariff. Required work to supply the system with heat varies with outdoor temperature. Less work is needed at hour 15 compared to the time interval 1 – 8. The less volatile Nord Pool tariff cause the control to assign a second large input when outdoor temperature is relative large. As a result, the less volatile price shifts preheating peak.

\(COP_{hp}\) increases - simulating a day in April

When the system is simulated in April, Figure 3.8 \(COP_{hp}\) is quite high. For the New York City tariff the running cost is relatively small around time interval 1 – 7, and the control assigns input, causing an indoor temperature peak of 20.1 degrees at about hour 6. As temperature grow, \(COP_{hp}\) increases, making it preferable to assign a second input at hour 15. For the less volatile Nord Pool tariff, the impact of the time varying \(COP_{hp}\) is larger. The decisions of the control is thus more dependent upon outdoor temperature, resulting in a large input at hour around 15 and an increasing indoor temperature.

The effect of high volatility is weakened, large \(COP_{hp}\) - simulating a day in June

Figure 3.9 shows that the effect of the \(COP_{hp}\) is larger, simulating a day in June for both cases. The product defining the running cost is less at around hour 15 for both electricity tariffs, and output profiles are now quite similar.

Load curves for an increasing number of iterations

Assuming that new price updates are available on a rolling horizon principle, i.e., a new forecasted spot price is revealed for \(k = N - 1\), after optimal input at time \(t\) has been assigned, the algorithm is let to run for seven days in February and April, \(M = 168\). This makes the interchange between electricity tariff and outdoor temperature more visible. Simulating the heating system in February shows that input assigned vary heavily on electricity tariffs, see Figure 3.10 and Figure 3.11. Simulating the heating system in April, shows that shorter but larger input intervals with higher frequency are preferable for volatile prices, following the dynamics of the electricity tariff, Figure 3.12. Input assigned follows the variations of electricity tariff. Input generated with the Nord Pool tariff follows the dynamics of outdoor temperature. Input assigned follows the variations of outdoor temperature lows, see Figure 3.13.

38
Figure 3.8: Cost minimization of a heat pump, April 6th. Increasing outdoor temperature spreads load more continuously over the horizon, compared to direct electric heating.
Figure 3.9: Cost minimization of a heat pump, June 1st. The effect of high volatility is weakened, input assigned is quite similar for these both cases. Compared to direct electric heating in June, simulation shows that loads generated are more continuously spread out on the horizon. This argument holds for both electricity tariffs.
Figure 3.10: The use of a Heat Pump in February for the New York City tariff. Input is assigned with the variations of electricity tariffs valleys.

Figure 3.11: The use of a Heat Pump in February for New York City tariff. Input is assigned with the variations of electricity tariffs valleys.
Figure 3.12: The use of a Heat Pump in April for the New York City tariff. Input is assigned with the variations of electricity tariffs valleys.

Figure 3.13: The use of a Heat Pump in April for Nord Pool tariff. Input is assigned with the variations of outdoor temperature highs.
3.3.3 Results of Receding Horizon Control - Cost Minimization of a Heat Pump

Span of Achievable Savings

To compare how the span of possible savings change when implementing a heat pump, minimum cost and maximum cost is calculated. The result is shown in Table 3.9.

\[
\begin{align*}
\text{maximize } & \quad J \\
\text{subject to } & \quad Gz \leq S + E_{ytf}
\end{align*}
\]

(3.3.5)

<table>
<thead>
<tr>
<th>Month</th>
<th>min cost</th>
<th>max cost</th>
<th>min cost</th>
<th>max cost</th>
<th>span</th>
<th>span</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 11th</td>
<td>0.45</td>
<td>0.81</td>
<td>0.32</td>
<td>0.69</td>
<td>80%</td>
<td>116%</td>
</tr>
<tr>
<td>April 6th</td>
<td>0.10</td>
<td>0.17</td>
<td>0.04</td>
<td>0.19</td>
<td>70%</td>
<td>375%</td>
</tr>
<tr>
<td>June 1st</td>
<td>0.007</td>
<td>0.08</td>
<td>0.004</td>
<td>0.074</td>
<td>1043%</td>
<td>1750%</td>
</tr>
</tbody>
</table>

Comparing Impact of Volatility

As before minimum cost is calculated and compared when volatility is the only parameter differing the two electricity tariffs. The result is presented in Table 3.10, where the New York City tariff is scaled with \( \rho \).

<table>
<thead>
<tr>
<th>Month</th>
<th>min cost</th>
<th>( N.Y.C \rho )</th>
<th>min cost</th>
<th>% saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 11th</td>
<td>0.45</td>
<td>0.39</td>
<td>13.3%</td>
<td></td>
</tr>
<tr>
<td>April 6th</td>
<td>0.10</td>
<td>0.051</td>
<td>49.0%</td>
<td></td>
</tr>
<tr>
<td>June 1st</td>
<td>0.007</td>
<td>0.005</td>
<td>28.6%</td>
<td></td>
</tr>
</tbody>
</table>

43
3.3.4 Conclusion of Receding Horizon Control - Cost Minimization of a Heat Pump

Simulations and tables of results show that the cost function defined, is minimized when the peak of outdoor temperature and low electricity tariff coincide. Input is assigned when this occur. Comparing simulations when applying different volatility, shows that minimizing input is more dependent on price, as volatility increases. Comparing the simulations when using the less volatile Nord Pool tariff, suggests that minimizing input is more dependent on the $COP_{hp}$. As outdoor temperature increases, minimizing input is more and more determined by the $COP_{hp}$ and when simulating a day in June, input generated look quite similar for the two electricity tariffs considered, the effect of volatility is weakened.

Letting the algorithm run for a week in February and April, and comparing the results, assigned input changes from following the pattern of tariff dynamics to following the dynamics of outdoor temperature. This result first occur for the less volatile Nord Pool electricity tariff.

Comparing the use of a heat pump and direct electrical heating

![Graphs showing load profiles](image)

Figure 3.14: Simulation showing a more continuously spread load profile, using a heat pump, instead of direct electrical heating. $u_Q$ is proportional to the load curve of $w_{el}$, acquired simulating the use of a heat pump.
Comparing input generated from electrically heated residential buildings and residential buildings supplied with heat by a heat pump, shows not only a considerably lower cost but also another consumption pattern. As outdoor temperature increases, input is spread more continuously over the horizon, Figure 3.14. For the volatile New York City tariff, the running cost is no longer always less in an interval between hour 1 – 7 where prices are low. In all considered cases using the New York City tariff, $COP_{hp}$ affects the running cost such that it is more beneficial to assign input in the middle of the day, where $COP_{hp}$ is large.
Chapter 4

Cost minimization - Distributed Setting

The thermal capacity of the building allows certain flexibility adjusting consumption to more economically preferable time intervals relative electricity tariffs without any significant or no loss of comfort. Problems (3.2.8) and (3.3.5) exploit price valleys, caused by the interchange of supply and demand, suggesting that heating cost can be reduced. So far this thesis has considered real-time pricing, such that electricity tariffs are directly passed to the end consumer and updated by the hour. This Chapter presents a third heating strategy, covering some considerations of cost minimization, using dual decomposition. The consumer addressed in Chapter 3 is now integrated in a defined system, sharing a resource with another consumer. Supply could be considered locally generated and stored, or charged and stored from an electricity grid where stability is guaranteed. Two consumers of slightly different thermal characteristics are considered. As opposed to the strategies presented before, energy is locally distributed via shadow pricing, [SSD07].

4.1 Decomposition - Three Separable Subsystems

Methods of decomposition are frequently used when solving large scale problems, decomposing complete system into separate subsystems, [SSD07]. When a problem has coupling constraints, dual decomposition is an appropriate method, relaxing the constraints by using the Lagrangian. The complete system will thus consist of subproblems of $i$ consumers and $j$ producers and a master dual problem, in charge of updating the dual variable $\lambda$, [DPM06].

In the distributed algorithm used, subsystems are accessed via interfaces, defined by coupling constraints and connected in a net. Producers and consumers are assumed price sensitive, subject to cost minimizing decisions, in units small enough to not have the power to dictate prices themselves. Information exchange bring local solutions into a globally optimal solution, given that the algorithm used converge, [SSD07].
4.1.1 System Description

In this section, an energy system of separate subsystems is considered, consisting of \( i \) consumers and a producer or generator supplying consumers with energy. Consumer \( i \) is governed by the same equations defined in (3.2.8), with the objective to minimize cost of energy consumption. The producer, offering resources of limited amount could be considered owned by the consumers, and may be viewed as an energy storage distributing resources, and capable of loading and storing energy under suitable conditions with respect to cost function. In this case, the producer is governed by the heat balance equation, (4.1.1), and is the model of a battery defined in [JYS10]. Dynamics of an energy storage can also be found and presented in [DK95]. In this case, following parameters are used to describe the battery. The producer should be easily exchanged with any other kind of energy storage technology.

- \( q(t) \) [kWh] is the amount of energy in the storage at time \( t \).
- \( u^{ch}(t) \) [kW] is the amount of energy charged from an outer source in to the energy storage during one time step \( \Delta t \).
- \( u^{dis}(t) \) [kW] is the amount of energy discharged from the energy storage during one time step \( \Delta t \).
- \( B_{q,ch} \) is the charge efficiency.
- \( B_{q,dis} \) is the discharge efficiency.

\( C_{q,ch} \) [kWh] is the capacity of the energy storage defining an upper bound of available energy \( q(t) \) at time \( t \). \( \bar{u}^{ch} \) and \( \bar{u}^{dis} \) [kW] (per time step \( \Delta t \)) are charging and discharging rates respectively and determines the upper bound for how fast energy can be transferred from one source to another. Leakage is not considered. This is a simplified linear model of an energy storage, (4.1.1).
\[ q(t + 1) = q(t) + B_{q,ch}u^{ch}(t) - B_{q,dis}u^{dis}(t) \] (4.1.1)

(4.1.2) defines cost function and constraints for the producer presented in (4.1.1). \( p_{t+k+j} \) is the weight on charging the storage over the prediction horizon \( N \), and defined by the sets of electricity tariffs used before in the thesis, which if given to the producer are assumed stable and predictable. \( p_{t+k+j}u^{ch}_{t+k+j} \) could thus be interpreted as the time-varying cost for charging energy from an outer source at time \( t \).

\[
\minimize \sum_{k=0}^{N-1} p_{t+k+j}(u^{ch}_{t+k+j} + u^{dis}_{t+k+j}) \\
\text{subject to } q_{t+k+1} = q_{t+k+j} + B_{q,ch}u^{ch}_{t+k+j} - B_{q,dis}u^{dis}_{t+k+j} \\
0 \leq q_{t+k+j} \leq C_{q,ch} \\
0 \leq u^{ch}_{t+k+j} \leq \bar{u}^{ch} \\
0 \leq u^{dis}_{t+k+j} \leq \bar{u}^{dis} \\
q_{jt} = q(t) \] (4.1.2)

The vector of local variables for the producer is given in (4.1.3).

\[ z = \{ q_{t+1}, \ldots, q_{t+N}, u^{ch}_{jt}, \ldots, u^{ch}_{t+N-1+j}, u^{dis}_{jt}, \ldots, u^{dis}_{t+N-1+j} \} \] (4.1.3)

### 4.1.2 Receding Horizon Control via Dual Decomposition

The following section suggests solving a system of three subsystems with receding horizon control via dual decomposition. The proposition can be derived from the general framework of dual decomposition presented in [SSD07] and recited in A.4. Notation throughout this section can be found and explained in A.4.

In this case, three separate subsystems are considered, decoupled by two consistency constraints, Figure 4.2. Subsystem one and three are consumers of the same value function and subject to the same constrains presented in (3.2.8). Subsystem two is a producer with the same dynamics found in (4.1.1). The matrix \( E \in \mathbb{R}^{4N \times N} \) can be derived from the netlist presented in A.4, and describes the connectivity between producer and consumer. In this case the net will consists of two edges or links, with consistency constraints requiring equality between supply and demand. The matrix \( E \) consists of the matrices \( E_i, i = 1, 2, 3 \) associated with each
Figure 4.2: System of three separable subsystems, where subsystem two is a rechargeable generator of time varying supply. The system consists of two edges, requiring equality of supply and demand.

subsystem, $E_i \in \mathbb{R}^{N \times 2N}$, $i = 1, 3$ and $E_2 \in \mathbb{R}^{2N \times 2N}$, mapping the common values on corresponding links into the public variables, [SSD07].

$$E_1 = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Energy distributed via public variables is denoted with $(y)$. The variable $(y_1)$ and $(y_4)$ will thus denote demand for the consumers and $(y_2)$ and $(y_3)$ will denote distributed supply at time $t$. Subsystem one and three will have local and public variables given in $z_i$, $i = 1, 4$. Local and public variables of the producer is given in $z_2$.

$$z_i = \{(e_i), \cdot, (e_i)_{t+|\mathcal{I}|}, (e_i)_{t+N}, \cdot, (e_i)_{t+|\mathcal{I}|}, (y_i), \cdot, (y_i)_{t+|\mathcal{J}|}, (y_i)_{t+N-1}|\mathcal{J}|\}$$

$$z_2 = \{q_{t+1}|\mathcal{J}|, \cdot, q_{t+N}|\mathcal{J}|, u_{t+N-1}|\mathcal{H}^h, \cdot, u_{t+|\mathcal{J}|}, (y_2), \cdot, (y_2)_{t+|\mathcal{J}|}, (y_3), \cdot, (y_3)_{t+N-1}|\mathcal{J}|\}$$

Capital letters are used defining parameters and variables over the prediction horizon $N$. The public variables over the prediction horizon are collected in a vector $\mathbf{Y} \in \mathbb{R}^{pN}$, and the associated dual variables over the prediction horizon are collected in vector $\Lambda \in \mathbb{R}^{pN}$, where $p = 4$ denotes the number of scalar public variables, derived from subsystem interaction. Here $i = 1, 4$ are public variables of consumer one and two respectively, and $i = 2, 3$ are public variables of the producer.

$$\mathbf{Y}_i = \{(y_i), \cdot, (y_i)_{t+|\mathcal{I}|}\} \quad i = 1, \ldots, 4$$
\[ \Lambda_i = \{ (\lambda_i)_t, \ldots, (\lambda_i)_{t+N-1} \} \quad i = 1, \ldots, 4 \]

\[ \bar{S} \in \mathbb{R}^{2N} \] denotes the common value of the resource on the links between producer and consumers. The value on each link of the interfacing variables are given in the vector \( \bar{S}_i \), where \( \bar{S}_1 \) denotes common value of the first link at time \( t \) and \( \bar{S}_2 \) denotes the common value of the second link, at time \( t \).

The constraints are expressed as \( \bar{Y} = \bar{E} \bar{S} \), where \( \bar{E} \) is the matrix defined above. Lagrange multipliers or dual variables are introduced for the coupling constraints, \( \bar{Y} - \bar{E} \bar{S} \), and allows solving each subsystem as a separate optimization problem. The problem is set on compact form, (4.1.4). The term \( \Lambda^T \bar{E} \bar{S} \) equals zero when optimizing with respect to \( \bar{S} \), defining the dual. \( \Lambda^T \bar{E} \bar{S} \) equals zero, stating that the sum of the Lagrange multipliers over the links is zero. In (4.1.4), \( f(z) \) denotes the summed local cost function of all subsystems.

\[
L(z, \bar{S}, \Lambda) = \sum_{i=1}^{K} f(z) + \Lambda^T (\bar{Y} - \bar{E} \bar{S}) = \sum_{i=1}^{K} (f(z) + \Lambda^T \bar{Y}_i) + \Lambda^T \bar{E} \bar{S} \quad (4.1.4)
\]

(4.1.4) is now separable, such that each subsystem may be solved independently and minimized with respect to local and public variables. For the \( i \)th consumer \( i = 1, 4 \), there is an associated Lagrangian function subject to local constraints, (4.1.5).

\[
\text{minimize} \quad L_i = \gamma \sum_{k=1}^{N} ((\varepsilon_i)_t + (\varepsilon_i)_{t+k}) + \sum_{k=0}^{N-1} (\lambda_i)_t^T (y_i)_{t+k} \quad (4.1.5)
\]

subject to \quad Constraints given in (3.2.8) for the \( i \)th consumer
The Lagrangian for the producer is given in (4.1.6).

\[
\begin{align*}
\text{minimize} & \quad L_2 = \sum_{k=0}^{N-1} p_{t+k|t} u_{t+k|t}^h + \sum_{k=0}^{N-1} (\lambda_2)_{t+k|t}^T (y_2)_{t+k|t} + \sum_{k=0}^{N-1} (\lambda_3)_{t+k|t}^T (y_3)_{t+k|t} \\
\text{subject to} & \quad q_{t+k+1|t} = q_{t+k|t} + B_{q,ch} u_{t+k|t}^h - B_{q,dis} (y_2)_{t+k|t} - B_{q,dis} (y_3)_{t+k|t} \\
& \quad 0 \leq q_{t+k|t} \leq \bar{C}_{q,th} \\
& \quad 0 \leq u_{t+k|t}^h \leq \bar{u}^h \\
& \quad 0 \leq (y_2)_{t+k|t} \leq \bar{y}_{dis}^{y_2} \\
& \quad 0 \leq (y_3)_{t+k|t} \leq \bar{y}_{dis}^{y_3} \\
& \quad q_{t|t} = q(t)
\end{align*}
\]  

(4.1.6)

**Numerical values:**

\[\bar{y}_{dis} = \bar{u}^h = 3.8 \text{ [kW]}, \quad \bar{C}_{q,th} = 20 \text{ [kWh], } B_{q,ch} = 0.8, \quad q_0 = 10 \text{ [kWh], } C_{1,th} = 2 \cdot 10^4 \text{ [kJK}^{-1}], \quad C_{2,th} = 3 \cdot 10^4 \text{ [kJK}^{-1}], \quad D_1 = D_2 = 0.065 \text{ [kWK}^{-1}], \quad \Delta t = 1 \text{ [h], } N = 12.\]

Other numerical values can be found and derived in sections presented before.

### 4.1.3 Solving with The Price Adjustment Algorithm

The following section suggests solving the problem described in section 4.1.2 with receding horizon control, via dual decomposition and the *price adjustment algorithm*, Algorithm 2. The algorithm is decentralized, using forecast values of available information planning for global optimality. Consumers and producers have locally stored public variables and associated dual variables, Figure 4.3. Dual variables are updated and if the algorithm converge, consistency of supply and demand is maintained.

Using shadow prices (language multipliers) as a balancing tool, the subsystems state how much of the resource they would like to supply, or demand given announced price. Prices are reduced for resources with excess supply and raised for resources with excess demand. As in [SSD07], a solution can be found using the projected subgradient method. By computing the net variables \( S \) on the links, the dual variables, \( \Lambda \) are adjusted. When price equilibrium or near price equilibrium is reached, energy is distributed accordingly and the dynamic equations updated with corresponding input. [DPM06] covers some concepts of using model predictive control via dual decomposition.
Algorithm 2 Algorithm, receding horizon control via dual decomposition

for \( t = 1 \rightarrow M \) do
  Initialize dual variable \( \Lambda(t) = 0 \) or \( \Lambda(t - 1) \).
  loop
    Find current distributed forecast values of tariff and outside temperatures.
    For the control period \( N \), solve subproblems separately to obtain
    optimal local and public variables.
    Compute average value of public variables over each net.
    \( \hat{S} := (E^T E)^{-1} E^T Y^* \).
    Update prices on public variables.
    \( \Lambda := \Lambda - \alpha(-Y^* + E\hat{S}) \).
  end loop
  Apply only \( (y_i^*)_{[0]} \), \( i = 1, \ldots, 4 \).
  Wait for the new sampling time \( t + 1 \).
end for

4.1.4 Simulating Distributed Setting - System of Three Subsystems

Following simulations show three subsystems consisting of two consumers of different thermal characteristics and a producer charging and discharging energy. In this set up the producer charges the storage when suitable with respect to the cost function. The producer starts with an inventory of \( q_0 = 10 \) [kWh]. Convergence for each separate value function of the subsystems without considering inventory, for \( M = 1 \) is ensured, and thereafter an optimal or near optimal solution is assumed.

Figure 4.4 and Figure 4.6 illustrates system behavior simulating a week in April for corresponding electricity tariff. The price adjustment algorithm is set to a fixed
number of iterations, $l = 50$. $\alpha$ is set to a constant step-size, $\alpha = 0.03$. Figure 4.5 and 4.7 illustrates shadow prices for corresponding process.

In the algorithm used, energy of an existing inventory could be considered as free, invoking cost minimizing decision of the consumers to preheat, reaching a maximum of approximately 22 degrees Celsius of indoor temperature. As in subsection 3.2, applying receding horizon control for electrical heating for a consumer, the algorithm successfully utilize price valleys by exploiting the thermal capacity of the building. Consumer two, with a higher thermal capacity has a larger storage capacity compared to consumer one, with a lower thermal capacity, which is illustrated in resulting indoor temperature. As shadow prices converges, indoor temperature of the consumers stays at the lower bound of approximately 19 degrees Celsius.

Comparing shadow prices when charging the storage with the an electricity tariff of low volatility and an electricity tariff of high volatility, shows that shadow prices are not completely stabilized for the latter case.
Charging storage from a less volatile source

Figure 4.4: Generated output and input for two consumers. The last row shows the amount of storage available and amount charged from a source with small variance to fulfill demand of the consumers.

Figure 4.5: Generated shadow prices for the two consumers, when charging form a source of small variance.
Charging storage from a highly volatile source

Figure 4.6: Generated output and input for two consumers. The last row shows the amount of storage available and amount charged from a source with large variance to fulfill demand of the consumers.

Figure 4.7: Generated shadow prices for the two consumers, when charging form a source of large variance.
Changing the step-size $\alpha$

Given a small enough constant step-size, the algorithm is guaranteed to converge, [SLAJ08]. Figure 4.8 shows shadow prices of consumer one, corresponding to $\lambda_1$. Simulating the same system as before, charging the storage with a heavily volatile electricity tariff, the step-size $\alpha$ is changed. As seen, shadow prices generated when charged from a more volatile electricity tariff are smoothed for a decreased step-size $\alpha$. Simulations could imply that choosing step-size $\alpha$ is an important parameter when designing a system dependent on a time varying parameter.

![Figure 4.8: Generated shadow prices for the consumer one, when charging from a source of large variance, with a decreased step-size, $\alpha$.](image)

4.1.5 Conclusion of Distributed Setting

Simulations show the possibility of using dual decomposition as a strategy for energy management. Consistency of supply and demand is maintained via shadow pricing in a defined system. Simulations also illustrate outcome, when charging from a highly volatile source, increasing requirement on the features of the solving method, i.e., the price adjustment algorithm. Correct step-size $\alpha$ with respect to determining parameters of subsystems should be important.

Dual decomposition as presented and used in this thesis should increase requirements on the subsystems in the defined system. Accurately enough mathematical models must be provided for all subsystems as well as a sufficient solving method. Considering system implementation, subsystems must be able to store
forecast values locally, and accessed for updates for each iteration of the price ad-
justment algorithm. Complexity should increase, implementing some kind of co-
ordinating software. For an appropriate solution of fast convergence, system setup
and the price adjustment algorithm presented in section 4.1 should be investigated
further.

Using the price adjustment algorithm, the defined system is solved with respect
to locally known information in a collaborative manner. Considering forecast val-
ues of electricity tariffs and the dynamics of such, discussed before in this thesis,
the distributed setting proposed has some mentionable advantages. Suppliers and
consumers set their level of supply and demand, based on knowing only their own
cost function, which should remain unexposed to other participants than them-

selves. Given that producers and consumers are presented as price sensitive in
the net and that no participant is large enough to have the power to dictate prices,
energy should be fairly distributed, satisfying defined consistency constraints. Par-
ticipants of the net are thus not in the control of a profit seeking auctioneer, which
could be the case with the structural model of centrally available and managed
information.
Chapter 5

Conclusion and Future Work

5.1 Conclusion

This thesis has discussed associated values reducing cost of electrically heated buildings in the smart electrical grid. Taking off from a consumer perspective, reducing cost of electrically heated buildings, percent savings of a time based control was compared to applying receding horizon control with a linear time varying cost function, dependent on electricity tariff dynamics. Results show an economic gain of using receding horizon control instead of using the manual control presented in this thesis. Secondly, cost minimization of the use of a heat pump was investigated, using receding horizon control with a nonlinear cost function of two time varying parameters, electricity tariff and outdoor temperature. It was shown that cost minimization of the use of a heat pump not only reduces cost, but also smoothes load curve on the prediction horizon. Continuing with a consumer and producer perspective, dual decomposition was proposed, showing consumers and a producer in a defined system, sharing a resource via shadow pricing.

For electricity tariffs similar to the one of Nord Pool, which are assumed well behaved, revealing a well-known pattern of electricity tariff peaks and valleys, an algorithm such as the time based control, of fixed predetermined time intervals could be sufficient. Adjustment to electricity tariffs and outdoor temperature can be executed in any environment of measurable indoor and outdoor temperatures, where forecast values could be available on-line. Software should not be necessary. For an increasing share of renewables on the energy market, electricity tariffs are assumed to change. Manual controls, such as the time based control may not be sufficient in terms of economy and manageability. Receding horizon control solves an optimization problem on a rolling horizon principle, accounting for changes on the prediction horizon which should be of great advantage if the pattern of electricity tariffs are not so easily forecasted.

Complexity of implementation, applying receding horizon control should increase since software and a sufficiently good mathematical model of the building is required. Realization of the solution should be a consideration between investment
in software and hardware building the solution, and gain of implementing receding horizon control compared to any other manual control. Furthermore, the consumer should be dependent on the forecast value updates provided. If, as in Chapter 3 of this thesis, forecast values on the prediction horizon is centrally accessible, consumer cost functions could be described as centrally managed. The weights from which cost minimizing decisions are determined, are thus not in hands of the individual consumer, which could be of disadvantage considering a profit seeking supplier.

Receding horizon control via dual decomposition should be preferable in an environment were an energy resource is shared between different subsystems. As opposed to the heating strategies presented before, dependency of centrally accessible information is relaxed and cost function of system participants are left unexposed. Complexity implementing the solution should be higher than the cases presented before. Mathematical models of consumers and producer are required as well as a solution for storing forecast values locally. Realization of the solution should be a consideration between the gain of using the common resource, the advantage of distributing energy via shadow pricing and cost of the actual solution in terms of software and hardware.

5.2 Future Work

Cost reducing decisions with respect to electrically heated buildings may have potential, shifting load to decrease cost for the consumer. For a more thorough analysis and for real implementations, a more realistic thermal model of a building with associated cost functions should be investigated. A suitable solution of software and hardware for cost minimization of electrically heated buildings, with an algorithm such as receding horizon control or receding horizon control via dual decomposition, including finding the correct step-size alpha, should also be considered.

This thesis makes use of the concept of two structural models. In Chapter 3 consumers construct cost functions out of centrally available forecast values, from which demand is determined by cost minimizing decisions. Given centrally available information, aggregate cost minimizing decisions of a large group of consumers, may themselves have great impact on electricity tariff dynamics. A large share of real time services in the electrical grid where real time price updates are directly passed to end consumer has shown to cause highly volatile or even unstable electricity tariffs. Very interesting contributions on the scalability of real time services can be found in [FT98]. Unstable electricity tariffs for aggregate demand of identical cost functions should hold for any device, minimizing cost with respect to some resource where price is determined upon the mechanisms of trading, given that demand of these price sensitive consumers, utilizing real time services is large enough relative supply.

In Chapter 4 a structure is discussed using methods of dual decomposition. A
structural model is used distributing forecast values locally. Dependency of centrally accessible information should be relaxed. Either by the fact that production is locally generated, stored and fairly distributed with respect to consistency constraints, or that consumers are attached to a producer, storing and distributing supply fairly with respect to consistency constraints by charging energy from an electricity grid where stability of electricity tariffs is guaranteed. The proposition is by no means completed.

The two types of structures presented in this thesis distinguish in the way they distribute and process forecast values. How they fully respond on exposure of the behavior causing extremely volatile or unstable electricity tariffs, discussed in [FF98] remains to be investigated.
Appendix A

Optimization Problems and Receding Horizon Control

Different optimization techniques can be found in [SL04], setting ground for optimization in real time. A good and thorough introduction of predictive control with relevant examples can be found in [JM02] and [FAM11]. [AM] also offers good insight in the different techniques used throughout this thesis. In this section, some basic definitions, necessary for solving posed problems are presented.

All optimization problems may be written in the following general form,

\[
\begin{align*}
\min/\max & \quad f(z) \\
\text{subject to} & \quad z \in S \subseteq Z
\end{align*}
\]  

(A.0.1)

The vector \( z \) defines the decision variables of the problem. \( S \) is the set of feasible decision on a domain \( Z \). To each possible decision \( f \) assigns a cost. Solving (A.0.1) yields the optimal cost \( J^* \).

\[
\arg\min_{z \in S} f(z) \triangleq \{ z \in S : f(z) = J^* \}
\]

(A.0.2)

A.1 Convexity

An optimization problem is said to be a convex optimization problem if the constraint set and defined cost function to be minimized are convex. For corresponding maximization problem, a concave cost function is required. This is a class of optimization problems, including least-squares and linear programming problems.

\( x^2, x^4, \exp^x, \exp^{-x}, \) and \( -\log x \) are all convex functions, multiplying them with minus will give a concave function. Or, if \( f \) is convex, \( -f \) is concave. Affine
functions are both convex and concave. If the optimization problem is convex, then a local optima is also a global optima.

A.2 Nonlinear Programming

Nonlinear problems can be addressed by using first and second order necessary conditions on an extremum point, constructing the Lagrangian. The Lagrangian is formulated with the constraints as a weighted sum.

A.2.1 First Order Necessary Conditions

Nonlinear optimization problems with equality constraints can be formulated as

\[
\begin{align*}
\text{minimize} & \quad f(x_1, \ldots, x_n) \\
\text{subject to} & \quad G(g_1(x_1), \ldots, g_k(x_n)) = 0
\end{align*}
\]  

(A.2.1)

\(G\) is a vector of constraints. Assuming that \(f\) and \(G\) are differentiable, the Jacobian can be defined as,

\[
\nabla G = \frac{\partial g_m(X)}{\partial x_i} = 0, \quad i = 1, \ldots, n
\]

Lagrange Theorem: Assume \(\bar{x}\) is a local minimizer of (A.2.1). If the functions \(f\) and \(G\) are differentiable at \(\bar{x}\) and the Jacobian, \(\nabla G\) have full rank, then there exists unique scalars \(\lambda\), such that,

\[
\frac{\partial f(X_0)}{\partial x_i} - \sum_{m=1}^{k} \lambda^*_m \frac{\partial g_m(X^*)}{\partial x_i} = 0, \quad i = 1, \ldots, n
\]

(A.2.2)

or

\[
\nabla L = 0
\]

(A.2.3)

(A.2.2) yield the optimal solution for all decision variables, including \(\lambda\).

where Lagrange function is defined as follows.

\[
L(X, \lambda) = f(X) - \sum_i \lambda_i g_i(X)
\]

(A.2.4)

\[
X = (x_1, \ldots, x_n)
\]

\[
\lambda = (\lambda_1, \ldots, \lambda_n)
\]

Nonlinear optimization problem with inequality constraints, are solved in a similar manner.
minimize \[ f(x_1, \ldots, x_n) \] subject to \[ H(h_1(x_1), \ldots, h_m(x_n)) \geq 0 \] (A.2.5)

Inequality conditions may be written as \( h(x) \geq 0 \), or \( -h(x) \leq 0 \). For any given \( x \), \( h(x) \) has either \( h(x) = 0 \) or \( h(x) > 0 \). If \( h_i(x) = 0 \), the set is active at \( x \), otherwise it is inactive and removed by setting \( \mu_i = 0 \).

**The Karush Kuhn Tucker Theorem:** If \( \bar{x} \) is a local minimizer for a nonlinear optimization problem with inequality constraints, and the constraints are satisfied at \( \bar{x} \) then there exists scalars \( \mu \) such that,

\[
\frac{\partial f(X_0)}{\partial x_i} - \sum_{m=1}^{k} \mu_m \frac{\partial h_m(X^*)}{\partial x_i} = 0, \quad i = 1, \ldots, n
\]

\[
\mu_i \geq 0
\]

\[
\mu h_i = 0
\]

or

\[
\nabla L = 0
\]

\[
\mu \geq 0
\]

\[
\mu_i h_i = 0
\]

(A.2.7)

The *Lagrange function* is defined as follows.

\[
L(X, \mu) = f(X) - \sum_i \mu_i h_i(X)
\]

\[
X = (x_1, \ldots, x_n)
\]

\[
\mu = (\mu_1, \ldots, \mu_n)
\]

(A.2.8)

The Karush Kuhn Tucker, *KKT*, provides with first order necessary conditions for optimality, where the \( \bar{x} \) is a *KKT* stationary point. This implies that there exist a \( \mu \) such that the pair \( h_i, \mu \) satisfies the first order necessary conditions for local optimality.

**A.2.2 Lagrange Function and Convexity**

If the optimization problem is convex, with affine equality constraints and convex inequality constraints, then \( \bar{x} \), as a local optimum, is also a global optimum.
A.3 Receding Horizon Control

Receding horizon control has a wide range of different successful implementations in diverse areas. It is a real-time optimization procedure, solving an optimization problem at each time step. The algorithm predicts future behavior of a dynamic process. It computes corrective control actions, fulfilling constraints and minimizing/maximizing a value function, simultaneously. [FAM11] provides with a good and thorough overview.

All Predictive Control algorithms share some common elements,

- A dynamic model.
- Constraints.
- Objective function.

Following subsections gives a short introduction to these elements.

A.3.1 Dynamic Model and Constraints

A dynamic discrete-time model can be described as,

\[ x(t + 1) = Ax(t) + Bu(t) \]  \hspace{1cm} (A.3.1)

\( A \in \mathbb{R}^{n\times n} \), and \( B \in \mathbb{R}^{n\times m} \). \( x(t) \in \mathbb{R}^{n} \) are states and \( u(t) \in \mathbb{R}^{m} \), are inputs or decision variables. \( n \) denotes the order or the state-space dimension of the system, while \( m \) defines the number of inputs.

In some cases, it might me desirable to limit the solution of a given problem, these limits are defined as constraints and can be formulated as equality constraints or inequality constraints. The more restrictive equality can be replaced with inequalities. For any function \( h_i = 0 \), can be replaced with \( h_i \leq 0 \) and \( -h_i \leq 0 \). The same relation hold. The sets of states and inputs are defined as:

\[ x(t) \in X, \ u(t) \in U, \ \forall t \geq 0 \]

\( \mathbb{R}^{n} \) and \( \mathbb{R}^{m} \) defines the euclidean space and \( X \) and \( U \) defines the polyhedral of feasible states and inputs. That is,

\[ X \subseteq \mathbb{R}^{n}, \ U \subseteq \mathbb{R}^{m} \]

A.3.2 Objective function

To each decision variable the value function assigns a cost. The cost function is designed in such a way that the correct control law is achieved on a given horizon, for the specified task to solve. There are many ways of designing an appropriate cost function to acquire best possible result.
A.3.3 Problem Formulation, Standard Form

The predictive controller solves an optimal control problem in following standard form at time $t$, for $k = 0, ..., N - 1$.

\[
\begin{align*}
\text{minimize} & \quad J(x_{t+k|t}, u_{t+k|t}, N, N_m) \\
\text{subject to} & \quad x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} \\
& \quad x_{t+k|t} \in X \\
& \quad u_{t+k|t} \in U \\
& \quad x_{t|t} = x(t)
\end{align*}
\] (A.3.2)

The predicted states can be derived from solving a set of $N$ equations where $x_{t+k|t}$ will denote the prediction obtained from forward recursion. This procedure will yield the correct matrices for solving (A.3.2). Solving the optimal control problem, reveals all decision variables defined in the cost function. $x_{t+k|t}$ denotes the state estimated for $k = 0, ..., N - 1$, solved at time $t$. The initial and measured state $x_{t|t}$, and $u_{t+k|t}$ denotes all planned future corrective actions. Given the vector of planned inputs, at time $t$, $U^* = \{u_{t|t}^*, ..., u_{t+N-1|t}^*\}$ only the first input $u_{t|t}^*$ is applied.

\[
 u(t) = u_{t|t}^*
\] (A.3.3)

This optimization procedure is repeated for the next time step, $t + 1$, now with a new measured initial state, $x_{t+1|t+1}$. The fixed prediction horizon $N$ is shifted one time-step forward for every obtained solution, yielding a receding horizon control strategy. As new measurements are obtained, for every shifted time-step, the difference between the predicted output and desired output, is minimized over a future horizon.

Figure A.1 illustrates the concept of model predictive control, where the manipulated inputs are defined by $U^*$. As indicated in the picture, only the first encircled input is applied. $N$ denotes the prediction horizon and $N_m$ denotes the length of the control horizon. When $N = \infty$, it is referred to as an infinite horizon problem, similarly when $N$ is finite it is referred to as a finite horizon problem. A large prediction horizon is often desirable, if sufficient information can be provided.

A.3.4 Feasibility

By choosing cost function and stability constraints correctly, feasibility can be preserved for all time steps. An infeasible solution may occur if imposed constraints cannot satisfy desired output constraints. If the problem is infeasible, the constraints involving states should be softened. This can be done by adding slack variables. As the inputs are generated by the optimization procedure, the input constraints can always remain as hard. The slack variables are non-zero, non-negative only when the output constraints are violated. In order to keep the slack variables close to zero, they are included in the cost function and penalized hard, in this thesis denoted with $\gamma$.  

65
Fig. 2. Receding horizon strategy: only the first one of the computed moves $u(t)$ is implemented as well as specific stability and performance criteria. Although a rich theory has been developed for the robust control of linear systems, very little is known about the robust control of linear systems with constraints. Recently, this type of problem has been addressed in the context of MPC. This paper will give an overview of these attempts to endow MPC with some robustness guarantees. The discussion is limited to linear time invariant (LTI) systems. While the use of MPC has also been proposed for LTI systems without constraints, MPC does not have any practical advantage in this case. Many other methods are available which are at least equally suitable.

2 MPC Formulation

In the research literature MPC is formulated almost always in the state space. Let the model $\Sigma$ of the plant to be controlled be described by the linear discrete-time difference equations

$$ x(t+1) = Ax(t) + Bu(t) + Vv(t) \quad (A.3.4) $$

Here $v(t)$ is the measured value of the disturbance at time $t$. If $v(t)$ is a set of known parameters over a prediction horizon, the effect of the disturbance can be predicted and included in the dynamics, to design an appropriate control. For unmeasurable disturbances, see [FAM11].

A.3.5 Disturbances

Measured and unmeasured disturbances can be included in the model, as an addition, in this case with $v(t)$.

$$ x(t+1) = Ax(t) + Bu(t) + Vv(t) $$

Here $v(t)$ is the measured value of the disturbance at time $t$. If $v(t)$ is a set of known parameters over a prediction horizon, the effect of the disturbance can be predicted and included in the dynamics, to design an appropriate control. For unmeasurable disturbances, see [FAM11].

A.3.6 Stability

Stability is not guaranteed when applying receding horizon control, but there are some techniques to enforce it. [FAM11] offers sufficient conditions for closed loop stability of receding horizon control. The proof can be derived from the arguments of Lyapunov and the Value function $V$.

A.4 Distributed setting, general framework

This section covers some theory of receding horizon control via dual decomposition, recited from [SSD07] and [DPM06]. A system set up to comprise $K$ subsystems may be formulated considering subsystem $i$ and a set of private variables, here denoted as $x_i \in \mathbb{R}^n$ and a set of public variables $y_i \in \mathbb{R}^n$. Each subsystem has

Figure A.1: The concept of receding horizon control.

\[\text{past} \quad \text{future} \]

Predicted outputs $y(t+k|t)$

Manipulated $u(t+k)$

Inputs

$t \quad t+1 \quad t+N_m \quad t+N_p$

input horizon

output horizon
its own cost function $f_i$, subject to local constraints and interact with one another via these public or interfacing variables. The overall cost function for the complete system is a sum of these local cost functions. Optimality for the complete system is obtained when interfacing variables are in consistency.

As in [SSD07], following notation is used to describe system setup. For $K$ subsystems, public or interfacing variables are collected in a vector $y = (y_1, ..., y_K) \in \mathbb{R}^p$. $p = p_1 + ... + p_k$ is the total number of scalar public variables, and is thus associated with all interaction of the system. $(y)_i$ is used to denote the $i$th scalar component of interfacing variable $y$, for $i = 1, ..., p$. For $N_c$ consistency constraints there exists a vector $z \in \mathbb{R}^{N_c}$, yielding the actual values of the interfacing variables for each constraint.

Matrix $E$ gives the netlist for the decomposition structure and maps the vector of net variables $z$ onto the public variables of the subsystems.

$$E_{ij} = \begin{cases} 1 & (y)_i \text{ is in constraint } j \\ 0 & \text{otherwise} \end{cases} \quad (A.4.1)$$

### A.4.1 The Price Adjustment Algorithm

The price adjustment algorithm, Algorithm 3, simultaneously solve given local optimization problems with respect to demand and supply. Convergence depends on the step-size $\alpha$. There are many ways of choosing correct step-size $\alpha$ to ensure sufficiently fast convergence of the solution. Some conditions for setting the step-size $\alpha$ can be found in [DPM06].

At each step the value of the net variables are updated using the optimal value of the public variables of the neighboring subsystem. The dual variable $\lambda$ is updated by using the step-size $\alpha$ such that consistency of supply and demand is obtained.

Implementing receding horizon control via dual decomposition, states of the subsystems are updated when the dual variables have converged, running Algorithm 3.

**Algorithm 3** Algorithm dual decomposition

Initialize dual variable $\lambda(k) = 0$.

```
loop
    Solve subproblems separately to obtain $x_i^*, (y)_i^*$. 
    Compute average value of public variables over each net. 
    $\bar{z} := (E^T E)^{-1} E^T y^*$. 
    Update prices on public variables. 
    $\lambda := \lambda - \alpha (-y^* + E\bar{z})$. 
end loop
```
Bibliography


