Robust Receive Beamforming with Interference and Channel Uncertainty

Jinghong Yang, Mats Bengtsson
ACCESS Linnaeus Center
KTH Royal Institute of Technology
SE-100 44 Stockholm, Sweden

Email: jinghong.yang@ee.kth.se, mats.bengtsson@ee.kth.se

Abstract—In this paper, we introduce a general convex framework for robust beamforming, which is valid for both deterministic and stochastic uncertainty models, and provides robustness against errors both in the channel and in the interference covariance matrix estimations. Furthermore, we extend our design to a multiple-state interference model [12] and show the performance gains obtained by exploiting the interference structure.

I. INTRODUCTION

Robust receive beamforming has attracted extensive studies for many years due to its practical significance in radar, wireless communications and many other fields. One basic idea about robust design is the worst-case performance optimization. Given an uncertainty region, many techniques are developed to maximize the receive SINR under a worst scenario.

There are mainly two kinds of uncertainty models for robust designs: *deterministic* uncertainty model and *stochastic* uncertainty model. The former leads to a beamformer that optimizes the worst scenario within the uncertainty region irrespective of its probability of occurrence; while the later designs a beamformer that optimizes the performance for a certain outage probability level.

Robust designs should preferably take both the channel estimation error and interference plus noise covariance matrix estimation error into account. Designs for *deterministic* uncertainty regions have been studied extensively [5]–[7], [9], [10], including algorithms that consider both channel distortion and interference covariance uncertainty [5]. However, under *stochastic* uncertainty model, existing algorithms are only robust against channel distortions [11] while neglecting the role of interference covariance matrix estimation errors.

In this paper, we extend the generic framework of [11] to provide robustness not only to channel mismatch but also to errors in the interference plus noise covariance matrix. Finally, we illustrate the performance on different interference models: traditional single-state interference model and the multiple-state interference model proposed in [12].

II. SYSTEM MODEL

Assume a multiple antenna receiver, and the received signal is modeled as

$$\mathbf{x}(k) = \mathbf{a}s(k) + \mathbf{i}(k) + \mathbf{n}(k),$$

where $\mathbf{a}$ is the *constant* channel vector, and $\mathbf{i}(k)$ and $\mathbf{n}(k)$ are the interference and noise components. The output signal after receive beamforming is

$$y(k) = \mathbf{w}^H \mathbf{x}(k).$$

The output SINR is defined as

$$\text{SINR} = \frac{|\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \mathbf{R}_{i+n} \mathbf{w}}$$

where we assume the symbol power $E\{s^2(k)\} = 1$, and $\mathbf{R}_{i+n} = E\{[\mathbf{i}(k) + \mathbf{n}(k)][\mathbf{i}(k) + \mathbf{n}(k)]^H\}$ is the interference plus noise covariance matrix.

The beamformer that maximizes the SINR is given by

$$\mathbf{w}^* = \alpha \mathbf{R}_{i+n}^{-1} \mathbf{a}, \quad (1)$$

for any scaling factor α that will not affect the SINR.

In practice, the interference plus noise covariance matrix $\mathbf{R}_{i+n}$ is difficult to obtain. However, this matrix can be replaced by the *data covariance* matrix

$$\mathbf{R} = \mathbf{a}\mathbf{a}^H + \mathbf{R}_{i+n}, \quad (2)$$

which results in the sample matrix inversion (SMI)-based minimum variance beamformer or Capon beamformer [1]

$$\mathbf{w}_c^* = \beta \mathbf{R}^{-1} \mathbf{a} \quad (3)$$

for some scalar β . In theory, $\mathbf{w}^*$ and $\mathbf{w}_c^*$ are equivalent, up to the scaling. In practice, the data covariance is estimated using sample mean

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^H = \frac{1}{N} \mathbf{X} \mathbf{X}^H, \quad (4)$$

where $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_N]$ is the array data matrix and N is the number of samples. Due to the limited number of samples, $\hat{\mathbf{R}} \neq \mathbf{R}$. Furthermore, the channel vector is estimated with the appearance of interference and noise, making $\hat{\mathbf{a}} \neq \mathbf{a}$. The Capon beamformer with estimated parameters is known to dramatically degrade the performance as compared to the optimal beamformer due to signal cancellation [2]–[4]. This has motivated a series of work known as robust designs.

Denote the estimation errors by $\tilde{\mathbf{a}} = \hat{\mathbf{a}} - \mathbf{a}$, $\tilde{\mathbf{R}} = \hat{\mathbf{R}} - \mathbf{R}$. If $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{R}}$ are confined within a deterministic region, we have

a deterministic uncertainty model. The robust beamformer is the solution to the worst-case SINR maximization problem

$$\begin{aligned} & \underset{\mathbf{w}}{\text{maximize}} && \min_{\tilde{\mathbf{a}}, \tilde{\mathbf{R}}} \frac{|\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}} \\ & \text{subject to} && \tilde{\mathbf{a}} \in \mathcal{E}, \quad \tilde{\mathbf{R}} \in \Sigma \end{aligned} \quad (5)$$

If stochastic information about $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{R}}$ is available, we have a stochastic uncertainty model. The probability constrained robust beamformer is obtained by solving

$$\begin{aligned} & \underset{\mathbf{w}}{\text{maximize}} && \gamma \\ & \text{subject to} && \Pr\left\{\frac{|\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}} \geq \gamma\right\} \geq p. \end{aligned} \quad (6)$$

A. Multiple-state Interference Model

Apart from the afore mentioned single-state interference model where the interference is modeled as colored noise, in [12], we proposed a multiple-state interference model, which can better reflect the bursty nature of the interference generated from modern fast resource allocation schemes. Here the interference is generated from one of the K states, and each state is characterized by a covariance matrix $\mathbf{R}_{ik}$ associated with a probability of occurrence p_k , $k = 1, 2, \dots, K$, K being the number of states in the model.

If it is possible to separate the observations from different states, then the sample mean data covariance for state k is

$$\hat{\mathbf{R}}_k = \frac{1}{N_k} \mathbf{X}_k \mathbf{X}_k^H, \quad (7)$$

where $\mathbf{X}_k$ is the observed data matrix generated from state k , with column number N_k . Information about the current interference state could for example be obtained using low-rate backhaul communication from neighboring base stations, in a cellular system.

There are also other ways to estimate the state parameters when the observations are not separable, i.e. EM algorithm. In this paper, we just assume the observations are separable and the state covariance have been estimated using (7).

Notation:

- σ_n^2 : noise power
- $\mathbf{R}_i$: interference covariance
- $\mathbf{R}_{ik}$: interference covariance for state k
- $\mathbf{R}_{i+n} = \mathbf{R}_i + \sigma_n^2 \mathbf{I}$: interference plus noise covariance
- $\mathbf{R}_{ik+n} = \mathbf{R}_{ik} + \sigma_n^2 \mathbf{I}$: interference plus noise covariance for state k
- $\mathbf{R} = \mathbf{a}\mathbf{a}^H + \mathbf{R}_{i+n}$: data covariance
- $\mathbf{R}_k = \mathbf{a}\mathbf{a}^H + \mathbf{R}_{ik+n}$: data covariance for state k

III. GENERAL FORM OF ROBUST RECEIVE BEAMFORMING PROBLEM

In this section, we will derive a general convex form of the robust beamforming design problem. First, notice that (6) can be relaxed into

$$\begin{aligned} & \underset{\mathbf{w}, \tilde{\mathbf{a}}, \tilde{\mathbf{R}}, \eta}{\text{minimize}} && \eta \\ & \text{subject to} && \Pr\{|\mathbf{w}^H \mathbf{a}| \geq 1\} \geq p, \end{aligned} \quad (8)$$

$$\Pr\{|\mathbf{w}^H \mathbf{a}| \geq 1\} \geq p, \quad (9)$$

$$\Pr\{\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} \leq \eta\} \geq p. \quad (10)$$

When $p = 1$, (8) reduces to (5). In the following, we will prove that (8) can be reformulated into the following convex optimization problem:

$$\begin{aligned} & \underset{\mathbf{w}, \eta}{\text{minimize}} && \eta \\ & \text{subject to} && \|\mathbf{A}^H \mathbf{w}\| - \text{Re}\{\hat{\mathbf{a}}^H \mathbf{w}\} + 1 \leq 0, \end{aligned} \quad (11)$$

$$\begin{aligned} & \text{Im}\{\hat{\mathbf{a}}^H \mathbf{w}\} && = 0, \\ & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \mathbf{B} \mathbf{w} - \eta && \leq 0. \end{aligned} \quad (12)$$

The key to prove (11) is equivalent to (8) is to find the proper $\mathbf{A}$ and $\mathbf{B}$ which make the constraints in (11) equivalent to the constraints in (8). The choice of $\mathbf{A}$ will depend on the uncertainty model assumed for the channel vector, and $\mathbf{B}$ will depend on the interference uncertainty model.

The guidelines on how to choose $\mathbf{A}$ are given in Theorem 1, where the results are mainly taken from [11] with modifications. The choice of $\mathbf{B}$ is provided in Theorem 2, which is novel and is a contribution of this paper.

Theorem 1. 1) If the channel estimation error $\tilde{\mathbf{a}}$ is assumed to be restricted to a deterministic ellipsoid region: $\mathcal{E} = \{\mathbf{C}_a^{\frac{1}{2}} \mathbf{u} \mid \|\mathbf{u}\| \leq 1\}$, then

$$\mathbf{A} = \mathbf{C}_a^{\frac{1}{2}}$$

makes (12) equivalent to (9) with $p = 1$.

2) If $\tilde{\mathbf{a}}$ is a stochastic variable with $\mathbf{0}$ mean and covariance $\mathbf{C}_a$, then for any distribution of $\tilde{\mathbf{a}}$, (9) is fulfilled if

$$\mathbf{A} = \frac{\mathbf{C}_a^{\frac{1}{2}}}{\sqrt{1-p}}.$$

3) As a special case of 2), if $\tilde{\mathbf{a}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_a)$, then

$$\mathbf{A} = \mathbf{C}_a^{\frac{1}{2}} \cdot \sqrt{-\ln(1-p_g)}, \quad p_g = 2p - 1 \quad (14)$$

guarantees (9).

Proof: See the Appendix. ■

Theorem 2. 1) If the covariance matrix estimation error $\tilde{\mathbf{R}}$ is bounded by some matrix norm $\|\tilde{\mathbf{R}}\|$, then

$$\mathbf{B} = \|\tilde{\mathbf{R}}\| \cdot \mathbf{I}$$

guarantees (10) with $p = 1$

2) If $\tilde{\mathbf{R}}$ is zero mean and the second-order statistics $E[\text{vec}(\tilde{\mathbf{R}})\text{vec}(\tilde{\mathbf{R}})^H] = \mathbf{Q}^c \otimes \mathbf{Q}$, then

$$\mathbf{B} = \frac{\mathbf{Q}}{\sqrt{1-p}}$$

fulfills (10) for any distribution of $\tilde{\mathbf{R}}$.

3) If $\tilde{\mathbf{R}}$ is estimated from (4) using i.i.d complex Gaussian interference and noise vectors with covariance matrix $\mathbf{R}_{i+n}$, which is bounded by some norm $\|\mathbf{R}_{i+n}\|$, then

$$\mathbf{B} = \max \left(1 - \frac{\Phi_{2N}^{-1}(0.5-p)}{2N}, \frac{\Phi_{2N}^{-1}(p-0.5)}{2N} - 1 \right) \|\mathbf{R}\| \cdot \mathbf{I}, \quad (15)$$

where $\Phi_{2N}^{-1}(p)$ is the inverse CDF of $2N$ degree chi-square distribution.

Proof: See the Appendix. ■

A. Extension to Multiple-state interference model

In the multiple-state interference model, the current interference covariance is unknown. Of course we can choose not to exploit the multiple-state nature, and estimate the data covariance using (4) then solve (11). In this case,

$$\mathbf{B} \approx \sqrt{\frac{\sum_k p_k \|\mathbf{R}_k - \bar{\mathbf{R}}\|^2}{1-p}} \mathbf{I}, \quad (16)$$

where $\bar{\mathbf{R}}$ is the estimated covariance.

Proof: Notice that

$$\begin{aligned} \text{Var}(\mathbf{w}^H \bar{\mathbf{R}} \mathbf{w}) &\stackrel{N \rightarrow \infty}{=} \sum_k p_k (\mathbf{w}^H (\bar{\mathbf{R}} - \mathbf{R}_k) \mathbf{w})^2 \\ &\leq \sum_k p_k \|\mathbf{R}_k - \bar{\mathbf{R}}\|^2, \end{aligned}$$

inserting the above expression into (20), we obtain $\mathbf{B}$. ■

However, if it is possible to separate the observations generated from different states, then it is possible to estimate the data covariance for each state. In this case, we generalize (11) to the following multiple-state robust design problem.

$$\begin{aligned} &\underset{\mathbf{w}, \eta}{\text{minimize}} && \eta \\ &\text{subject to} && \|\mathbf{A}^H \mathbf{w}\| - \text{Re}\{\hat{\mathbf{a}}^H \mathbf{w}\} + 1 \leq 0, \\ & && \text{Im}\{\hat{\mathbf{a}}^H \mathbf{w}\} = 0, \\ & && \mathbf{w}^H \hat{\mathbf{R}}_k \mathbf{w} + \mathbf{w}^H \mathbf{B}_k \mathbf{w} - \eta \leq 0, k = 1, \dots, K. \end{aligned} \quad (17)$$

$\mathbf{B}_k$ are chosen according to Theorem 2, using the covariance error distribution of state k . For examples, when $\hat{\mathbf{R}}_k$ is estimated using sample mean from i.i.d complex Gaussian vectors generated from state k , which corresponds to case 3) in Theorem 2, then

$$\mathbf{B}_k = \max \left(1 - \frac{\Phi_{2N}^{-1}(0.5-p)}{2N}, \frac{\Phi_{2N}^{-1}(p-0.5)}{2N} - 1 \right) \|\mathbf{R}_k\| \cdot \mathbf{I}. \quad (18)$$

In principle, (17) is a worst-state optimization problem. Whatever the current interference state is, beamformer designed from (17) will always satisfy the constraints in (11).

IV. SIMULATION RESULTS

A. Performance on Single-state Interference Model

In this section, we will test the performance of the beamformer designed from (11). We assume the receiver has $M = 4$ antennas, and the noise power $\sigma_n^2 = 1$. The interference to noise ratio

$$\text{INR} = \frac{\rho(\mathbf{R}_i)}{\sigma_n^2} = 10 \text{ dB},$$

where $\rho(\cdot)$ denotes the spectral radius. $\mathbf{R}_i$ is randomly generated with rank 1.

The signal to noise ratio

$$\text{SNR} = \frac{|\mathbf{a}|^2}{\sigma_n^2}$$

can vary from 10 to 20 dB.

We generate the channel estimation error $\tilde{\mathbf{a}} \sim \mathcal{CN}(\mathbf{0}, \text{INR} \cdot \mathbf{I})$. The interference plus noise vectors are also generated

Figure 1. Distribution of the signal power given a certain beamformer

Figure 2. Distribution of the data power given a certain beamformer

according to complex Gaussian distribution $\mathcal{CN}(\mathbf{0}, \mathbf{R}_i + \sigma_n^2 \mathbf{I})$ and the covariance is estimated using sample mean (4) with $N = 100$. We choose $p = 0.9$ in the probability constraints.

1) *The tightness of probability constraints:* We first illustrate the tightness of the bound in (8) for Gaussian cases, i.e. using $\mathbf{A}$ from (14) and $\mathbf{B}$ from (15), to see how they will fulfill the constraints (9) and (10) respectively. To get the CDF curves of Figure 1 and Figure 2, we first randomly generate an estimate of channel vector $\hat{\mathbf{a}}$ (SNR = 10 dB) and interference covariance $\hat{\mathbf{R}}_i$ (INR = 10 dB), and a beamformer is solved from (11). Then we test the performance of this beamformer on 1000 possible values of the true $\mathbf{a}$ and $\mathbf{R}_i$ that can give the estimates. For reference, we compare to the beamformer proposed in [11].

In Figure 1, we plot the CDFs of the signal power $|\mathbf{w}^H \mathbf{a}|$, whose value should exceed 1 with probability $p = 0.9$ according to (9). We can see that the modified channel probability constraint (14) is tighter than the one proposed in [11], but both fulfill the probability constraint (9).

Figure 2 shows a typical CDF for the data power $\mathbf{w}^H \mathbf{R} \mathbf{w}$, whose value should be less than η with probability $p = 0.9$ according to (10). We can see that the probability constraint (10) is as fulfilled.

2) *Performance of the robust design:* To study the performance of the robust design in the sense of output SINR, we first fix an SNR value, randomly generating a channel vector $\mathbf{a}$ and a interference covariance $\mathbf{R}_i$ accordingly, and

Figure 3. Robust beamformer performance under *single* state interference model

Figure 4. Robust beamformer performance under *Multiple* state interference model

then generate 1000 possible estimates. For each estimate, a beamformer is solved from (11), and an output SINR value is obtained. The mean and 90% percentile of these 1000 output SINRs are computed. We test the following beamformers: the Capon beamformer (3) with estimated parameters, channel robust beamformer (11) with $\mathbf{B} = \mathbf{0}$, and channel & Covariance robust beamformer (11). From Figure 3 we can see:

- 1) The robust designs not only increase the 90% percentile (guaranteed) SINR, but also improve the mean SINR.
- 2) Taking interference uncertainty into account will increase both the guaranteed and the mean SINR performance.

B. Performance on Multiple-state Interference Model

We consider a two-state interference model, where both states have equal probability. Two beamformers are tested: the single-state robust design (11) with $\mathbf{B}$ from (16) and the multiple-state robust design (17) with $\mathbf{B}$ from (18). The way of generating channel and covariance as well as their estimates are the same as in subsection A above for single-state model, but here we have to do this for each state. From figure 4 we can see both mean and guaranteed performance are improved by exploiting the multiple-state structure of the interference.

V. CONCLUSIONS

A general convex formulation for robust beamforming design, and an extension to a multiple-state interference model, are proposed. Simulations show that our design is superior not only in the sense of improving guaranteed output SINR, but also increasing mean output SINR. Furthermore, exploiting the multiple-state interference structure and thus applying the extended version of the general robust design problem brings further performance gains. The results were formulated and derived in terms of a receiving beamformer, for simplicity, but may be even more relevant for a transmitting beamformer based on the Signal to Leakage and Noise ratio (SLNR) [8], where the beamformer cannot be adapted as quickly to varying interference situations.

VI. APPENDIX

A. Proof of Theorem 1

See [11] for 1) and 2). In 3), differing from [11], we use p_g instead of p in the expression of $\mathbf{A}$. In [11], the search for the optimal beamformer is restricted to the region that $\Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \leq |\mathbf{w}^H \hat{\mathbf{a}}| - 1\} = p_g$, which means the eligible beamformer must satisfy $|\mathbf{w}^H \hat{\mathbf{a}}| \geq 1$. For such a beamformer,

$$\begin{aligned} & \Pr\{|\mathbf{w}^H \mathbf{a}| \geq 1\} \\ &= \Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \leq |\mathbf{w}^H \hat{\mathbf{a}}| - 1\} + \\ & \quad + \Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \geq |\mathbf{w}^H \hat{\mathbf{a}}| - 1, |\mathbf{w}^H \hat{\mathbf{a}} - \mathbf{w}^H \tilde{\mathbf{a}}| \geq 1\} \\ &\geq \Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \leq |\mathbf{w}^H \hat{\mathbf{a}}| - 1\} + \\ & \quad + \Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \geq |\mathbf{w}^H \hat{\mathbf{a}}| - 1, |\mathbf{w}^H \hat{\mathbf{a}} - \mathbf{w}^H \tilde{\mathbf{a}}| \geq |\mathbf{w}^H \hat{\mathbf{a}}|\} \end{aligned}$$

Since $\tilde{\mathbf{a}}$ is symmetrically distributed ($\Pr\{\tilde{\mathbf{a}} = \mathbf{a}_0\} = \Pr\{\tilde{\mathbf{a}} = -\mathbf{a}_0\}$), $\Pr\{|\mathbf{w}^H \hat{\mathbf{a}} - \mathbf{w}^H \tilde{\mathbf{a}}| \geq |\mathbf{w}^H \hat{\mathbf{a}}|\} = 0.5$. Furthermore, $\Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \geq |\mathbf{w}^H \hat{\mathbf{a}}| - 1\} = 1 - p_g$. (19) becomes $\Pr\{|\mathbf{w}^H \mathbf{a}| \geq 1\} \geq p_g + \frac{1}{2}(1 - p_g)$. This implies that for any symmetrically distributed errors, if we want to design a beamformer that guarantees $\Pr\{|\mathbf{w}^H \mathbf{a}| \geq 1\} \geq p$, we only have to make $\Pr\{|\mathbf{w}^H \tilde{\mathbf{a}}| \leq |\mathbf{w}^H \hat{\mathbf{a}}| - 1\} \geq p_g$ satisfied. The relationship between p and p_g is

$$p = p_g + \frac{1}{2}(1 - p_g)$$

B. Proof of Theorem 2

- 1) First we want to prove if $\mathbf{B} = \|\tilde{\mathbf{R}}\| \cdot \mathbf{I}$, (13) guarantees $\mathbf{w}^H \mathbf{R} \mathbf{w} \leq \eta$. Let $\tilde{\rho}$ denote the spectral radius of $\tilde{\mathbf{R}}$,

$$\mathbf{w}^H \mathbf{B} \mathbf{w} = \mathbf{w}^H \mathbf{w} \|\tilde{\mathbf{R}}\| \geq \mathbf{w}^H \mathbf{w} \tilde{\rho} \geq \mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}. \quad (19)$$

Besides, notice that

$$\mathbf{w}^H \mathbf{R} \mathbf{w} = \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}.$$

It is easy to see (13) guarantees (10).

- 2) We start from (10),

$$\begin{aligned} \Pr\{\mathbf{w}^H \mathbf{R} \mathbf{w} \leq \eta\} &= \Pr\{\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} \leq \eta\} \\ &\geq \Pr\{\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + |\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}| \leq \eta\} \\ &\text{(Chebyshev inequality)} \geq 1 - \frac{\text{Var}(|\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}|)}{(\eta - \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w})^2}, \end{aligned}$$

which means if

$$1 - \frac{\text{Var}(|\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}|)}{(\eta - \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w})^2} \geq p, \quad (20)$$

(10) will be fulfilled. Utilizing the second order statistics of $\tilde{\mathbf{R}}$, we have $\text{Var}(\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}) = (\mathbf{w}^H \mathbf{Q} \mathbf{w})^2$. Inserted into (20), we obtain

$$\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \frac{\mathbf{Q}}{\sqrt{1-p}} \mathbf{w} - \eta \leq 0.$$

Comparing with (13) gives $\mathbf{B} = \frac{\mathbf{Q}}{\sqrt{1-p}}$.

3) The estimation error is

$$\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} = \mathbf{w}^H \mathbf{R} \mathbf{w} - \frac{1}{N} \mathbf{w}^H \mathbf{X} \mathbf{X}^H \mathbf{w}.$$

Since $\mathbf{X}^H \mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \mathbf{w}^H \mathbf{R} \mathbf{w})$, $\frac{2\mathbf{w}^H \mathbf{X} \mathbf{X}^H \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}} \sim \chi^2(2N)$ which is chi-square distributed with degree $2N$, resulting in

$$2N(1 - \frac{\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}}) \sim \chi^2(2N).$$

As shown in (20), the constraint (10)

$$\Pr \{ \mathbf{w}^H \mathbf{R} \mathbf{w} \leq \eta \} \geq \Pr \{ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + |\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}| \leq \eta \}. \quad (21)$$

When $\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} \geq 0$,

$$\begin{aligned} & \Pr \{ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + |\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}| \leq \eta \} \\ &= \Pr \{ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} \leq \eta \} \\ &= \Pr \left\{ \underbrace{2N(1 - \frac{\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}})}_{\chi^2(2N)} \geq 2N(1 - \frac{\eta - \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}}) \right\} \\ &\geq p_g. \end{aligned}$$

Notice that due to the symmetric property of the error, we only have to consider the region with probability p_g instead of p . Letting $\Phi_{2N}^{-1}(p)$ be the inverse CDF of $2N$ degree chi-square distribution, (22) becomes

$$\Phi_{2N}(2N(1 - \frac{\eta - \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}})) \leq 1 - p$$

which can be rewritten as

$$\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \left(1 - \frac{\Phi_{2N}^{-1}(1-p)}{2N} \right) \mathbf{R} \mathbf{w} - \eta \leq 0. \quad (22)$$

When $\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} \leq 0$,

$$\begin{aligned} & \Pr \{ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + |\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}| \leq \eta \} \\ &= \Pr \{ \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} - \mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w} \leq \eta \} \\ &= \Pr \left\{ \underbrace{2N(1 - \frac{\mathbf{w}^H \tilde{\mathbf{R}} \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}})}_{\chi^2(2N)} \leq 2N(1 + \frac{\eta - \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}}) \right\} \\ &\geq p_g \end{aligned}$$

which similarly can be rewritten as

$$\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \mathbf{w}^H \left(\frac{\Phi_{2N}^{-1}(p)}{2N} - 1 \right) \mathbf{R} \mathbf{w} - \eta \leq 0. \quad (23)$$

We need both (22) and (23) to hold, which means that

$$\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \max \left(1 - \frac{\Phi_{2N}^{-1}(1-p)}{2N}, \frac{\Phi_{2N}^{-1}(p)}{2N} - 1 \right) \cdot \mathbf{w}^H \mathbf{R} \mathbf{w} - \eta \leq 0.$$

As proved in (19), $\mathbf{w}^H \mathbf{R} \mathbf{w} \leq \mathbf{w}^H \mathbf{w} \|\mathbf{R}\|$, the above constraint can be further relaxed as

$$\mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + \max \left(1 - \frac{\Phi_{2N}^{-1}(1-p)}{2N}, \frac{\Phi_{2N}^{-1}(p)}{2N} - 1 \right) \cdot \|\mathbf{R}\| \mathbf{w}^H \mathbf{w} - \eta \leq 0$$

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