Probabilistic stability analysis of embankments founded on clay

Rasmus Müller

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Division of Soil and Rock Mechanics
Department of Civil and Architectural Engineering
Royal Institute of Technology
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“Even the stupidest of men, by some instinct of nature, is convinced on his own that with more observations his risk of failure is diminished.”

Jacques Bernoulli (1654-1705)
Preface

The research project presented in this thesis was carried out between January 2008 and March 2013 at the Division for Soil and Rock Mechanics, Royal Institute of Technology (KTH). The first part of the research, presented in a licentiate thesis written by the author in 2010, considers the design of vertical drains in Swedish sulphide clay and the application of the observational method for the control of embankment stability. The second part, presented in this doctoral thesis, considers aspects related to probabilistic stability analysis of staged embankments founded on soft clay. The research was conducted under the supervision of Professor Stefan Larsson.

The funding for my research was provided in full by the Sven Tyréns Foundation. Sincere thanks are directed to the foundation for providing me with the opportunity to conduct PhD studies in the field of geotechnical engineering.

I am filled with gratitude to my supervisor, Stefan Larsson, for supporting and encouraging me throughout the project and for always managing to find the time for our fruitful discussions.

Sincere thanks are directed to Dr Lars Olsson, for giving me valuable guidance on many matters related to Bayesian statistics and probabilistic analysis, and also for his valuable comments on parts of this doctoral thesis.

I would also like to thank to my fellow PhD students at the Division for Soil and Rock Mechanics, in particular Johan Spross, Mohammed Al-Naqshabandy and Niclas Bergman for very rewarding intellectual discussions over the years. Thanks are also directed to my colleagues at Tyréns AB for their great collaboration and support during my time as a PhD student.

Finally, I shall always be grateful to my dear friend in life, Anna, for her patience and for her continuous encouragement.

Borlänge, April 2013

Rasmus Müller
Abstract

In practical geotechnical engineering, the art of making decisions in the presence of uncertainties must be mastered. Over the past 40 years, great progress has been made in the development of geostatistical methods and probabilistic design methods in this field of engineering. Probabilistic methods evolved from Bayesian statistics are particularly useful in geotechnical engineering, as the objective basis (e.g. results from site investigations) for analyses, statements and decisions is often rather limited, and the practitioner must therefore often rely on information of a more subjective character, such as judgments, experience and empirical knowledge. Bayesian approaches enable information from different sources to be combined in a systematic and rigorous manner.

A common task faced by the practicing geotechnical engineer is to analyse whether or not a structure, e.g. an embankment, is stable and also to judge how stable the structure is or, in other words, how certain is the result of a stability calculation. Several sources of uncertainty of fundamentally different natures contribute to the reliability of the calculation, and to the degree of certainty one can attach to the subsequent statements and decisions made based on the results. One of the most important quantities influencing the assessment of the stability is the shear resistance that the soil material in the ground can mobilize. In the assessment of the short-term stability of embankments founded on soft clay, the undrained shear strength, $S_u$, is often considered to determine the shear resistance. Constructing an embankment in stages and allowing the clay to consolidate between the stages triggers an increase in $S_u$ as a result of the increase in effective stress in the ground. The increase in $S_u$ in the clay can then be taken into account in the stability assessment for the subsequent construction stage(s). Assessments of the initial $S_u$ in the clay, and of the potential increase in $S_u$ due to consolidation, are afflicted with uncertainties. One way of reducing these uncertainties is to gather information from different sources, e.g. à priori predictions based on some empirical relationship and measurements, and process this information through a Bayesian updating procedure. The “multivariate approach” (Ching et al. 2010) is such a procedure, suitable for reducing the uncertainty in the determination of an average value, e.g. $\bar{S}_u$. In the process, a “multivariate average” value is also assessed. This average is a more objective assessment than the conventional arithmetic average, as it is biased towards the least uncertain source of information.

Within the frame of this thesis, important aspects related to probabilistic stability analysis of embankments founded on soft clay are considered. The contents of the introductory part, i.e. Ch. 1-6, form a broader basis for the three appended papers, Papers III-V, in which the main results of the research are presented. In Papers III and IV, an extension of the multivariate approach is proposed. This “extended multivariate approach” leads to a reduction in the total uncertainty in the assessment of a material property, from the pair-wise correlation with a measured quantity (variates) due to spatially averaging, before adopting the procedure suggested by Ching et al. (2010).
The extended multivariate approach is useful when data from several sources are available. It is a rational, mathematically sound methodology for combining measurement data and empirical knowledge for several variates into an objective assessment of the spatial average of a property and the associated uncertainty. In the process, the transformation errors inherited from the derivations of the property via empirical pair-wise correlations with measured variates are implicitly reduced. This approach may thus be an alternative to making more rigorous calibration tests. The use of the extended multivariate approach can also be of help when planning a site investigation. The most effective set of variates and the optimal number of boreholes can be estimated in advance. However, despite (and perhaps as a result of) the beneficial features described above, careful consideration must be given when evaluating the total uncertainty in the pair-wise correlation for the incorporated variates, since a low value of a single variate will greatly affect the results of the analysis.

In Paper IV, the positive influence of the extended multivariate approach (via the reduction of the uncertainty in the assessment of $\bar{S}_u$) on the results from probabilistic stability analyses is shown and discussed. However, it is also suggested that deterministic and probabilistic stability analyses complement each other and should be performed in parallel. This is a result of the difficulty in assessing all involved uncertainties probabilistically, besides those associated with the assessment of material properties, such as the adopted constitutive material models, the model describing the failure mechanism, or unknown failure mechanisms. It is argued that probabilistic approaches enable the relative probability of failure to be assessed and also the relative influences on the probability of failure of uncertainties in the relevant parameters. Deterministic approaches implicitly deal with uncertainties that are difficult to assess probabilistically and they provide measures of the expected performance of the embankment.

In Paper V, a number of mathematical formulations (models) for the design of vertical drains proposed in the literature are compared. The formulations vary in mathematical complexity and, although the more complex formulations capture the real behaviour of the clay-drain system more realistically, the impacts on the results of assessments via these models are insignificant for practical purposes. It is argued that the more simple models produce results which are as satisfactory as the more complex formulations, and that these models should be adopted in practical engineering projects because of their simplicity. Instead of putting effort into the assessment of all the parameters inherent in the more complex formulations (of which some are insignificant from a practical point of view), focus should be directed towards the assessment of the parameters that have the greatest influence on the result of an analysis.
Sammanfattning


En av de vanligaste uppgifterna som den praktiserande geoteknikern ställs inför är att bedöma om en konstruktion, t.ex. en bankfyllning, är stabil eller inte. Ofta krävs också en bedömning av hur stabil konstruktionen är, eller med andra ord, geoteknikern ska uttala sig om hur tillförlitliga resultaten från exempelvis en stabilitetsberäkning är. Tillförlitligheten i beräkningsresultaten beror av osäkerheterna i förutsättningarna för beräkningen, vilka härstammar från många olika källor av vitt varierande karaktär. En av de störheter som påverkar beräkningsresultatet mest är det skjuvmotstånd som jorden kan mobilisera. Vid analyser av kortidsstabiliteten för bankkonstruktioner på löss lera definieras ofta skjuvmotståndet av lerans odränerade skjuvhållfasthet, \( S_u \). Om banken fylls upp i steg, och lera tillåts konsolida mellan laststegen, ökar \( S_u \) till följd av att effektivspänningen i marken ökar. Den ökade skjuvhållfastheten kan sedan tillgodoräknas vid stabilitetsanalyser för de efterkommande laststegen. Utvärderingar av den initiala odränerade skjuvhållfastheten i marken, och den potentiella ökningen av \( S_u \) till följd av konsolidering, är behäftade med osäkerheter. Ett sätt att reducera dessa osäkerheter är att tillgodoräkna sig information från olika källor, t.ex. bedömningar baserade på empiri (s.k. \( \text{à priori kunskap} \)) och mätningar, och utnyttja någon procedur för Bayesiansk uppdatering. "Multivariabelanalys" (Ching m.fl. 2010) är en sådan procedur, vilken är lämplig för att reducera osäkerheten i utvärderingen av medelvärden i olika materialegenskaper, t.ex. \( S_{\text{avg}} \). Det medelvärde, som utvärderas via denna procedur, är mer objektivt än ett strikt aritmetiskt medelvärde eftersom informationen från de minst osäkra källorna ges större vikt än informationen från mer osäkra källor vid utvärderingen.

Inom ramen för denna avhandling behandlas aspekter som är viktiga i samband med sannolikhetsbaserad dimensionering av bankkonstruktioner på löss lera. Innehållet i den inledande delen av avhandlingen (kapitel 1-6) ger en bredare bakgrund till de tre bilagda artiklarna (Paper III-V) vilka redovisar de huvudsakliga forskningsresultatena. I Paper III och IV föreslås en utökad multivariabelanalys. Denna ”utökade multivariabelanalys” innebär att de totala osäkerheterna i de parvisa korrelationerna, dvs. osäkerheten i härledningarna av egenskapen från mätningssresultat från olika undersökningstyper (variabler), bestäms innan proceduren presenterad av Ching m.fl. (2010) utnyttjas. Vid bestämningen av den totala osäkerheten i en sådan parvis kombineras i princip fyra källor till osäkerhet, materialegenskapens naturliga spridning, den statistiska osäkerheten, mättfel och tranformationsfel. Eftersom den utökade multivariabelanalysen resulterar i ett medelvärde hos den sökta materialegenskapen, representativt för en viss jordvolym, kan de tre förstnämnda
Sammanfattning

Källorna till osäkerhet reduceras till följd av medelvärdesbildningen. Transformationsfelet kan dock inte reduceras på detta sätt, utan kräver i princip att mer rigorösa kalibreringsförsök utförs för att minska denna källa till osäkerhet.


I Paper IV redovisas den positiva effekt en tillämpning av den utökade multivariabelanalysen (via den reducerade osäkerheten i bestämningen av $\tilde{S}_n$) har på resultaten från sannolikhetsbaserade stabilitetsanalyser. Men det föreslås även att konventionella, deterministiska, analyser och sannolikhetsbaserade analyser kompletterar varandra och bör utföras parallellt. Detta beror på att flera av de osäkerheter som påverkar en stabilitetsanalys, utöver de som är kopplade till utvärderingen av materialegenskaper, kan vara svåra att uppskatta. Exempelvis är det svårt att uppskatta storleken på osäkerheten i de antagna konstitutiva materialmodellerna, i den matematiska formulering som används för att beskriva brottmekanismen (brottfunktionen) eller att vissa brottmekanismer inte beaktas. Sannolikhetsbaserade analysmetoder medger en utvärdering av den relativa sannolikheten för att brottfunktionen ska ha en ett visst värde, t.ex. 1, samt de ingående variablernas relativa bidrag till den totala osäkerheten hos brottfunktionen. I deterministiska analyser beaktas alla osäkerheter i den erforderliga säkerhetsfaktorn och dessa analyser kan även ge en uppskattning av en bankkonstruktions beteende.

I Paper V jämförs ett antal matematiska modeller som föreslås i litteraturen, avsedda för dimensionering av vertikaldräner. Modellerna skiljer sig åt i komplexitet och hur väl de beskriver den verkliga interaktionen mellan dräner och lera. Studien visar att, även om de mer komplexa modellerna är mer verklighetstrogna, så har detta en marginell betydelse ur praktisk synvinkel. I praktiken bör de mindre komplexa modellerna nyttjas för att de är enklare att använda och för att färre parametrar krävs. Istället för att bemöda sig med att bestämma alla parametrar som ingår i de mer komplexa modellerna (av vilka en del inte har någon praktisk betydelse), bör fokus riktas mot att bestämma de parametrar som har störst inverkan på analysresultatet.
List of publications

This thesis is based on the work presented in the following papers:


Paper V Müller, R and Larsson, S. Aspects on the modelling of smear zones around vertical drains. Accepted for presentation at the 18th International Conference on Soil Mechanics and Geotechnical Engineering, Paris, 2-6 September 2013.

In all five papers, the analyses of measurement data, calculations and writing were done by Müller. The co-author Larsson continuously supervised the work, thereby greatly contributing to the final products. The co-author Spross made significant intellectual contributions to papers III and IV and the co-author Westerberg contributed with his knowledge of sulphide soils in paper II.

Within the frame of the research project the following five publications were also established, but they are not part of this thesis:


Müller, R. Är det sannolikt att det är stabilt?. Presented at Grundläggningsdagen, Stockholm, 7 March 2013.
Notations and abbreviations

Latin uppercase letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>initial</td>
</tr>
<tr>
<td>3D</td>
<td>three-dimensional analysis</td>
</tr>
<tr>
<td>( \hat{\alpha} )</td>
<td>regression parameter</td>
</tr>
<tr>
<td>A</td>
<td>dimension A</td>
</tr>
<tr>
<td>ACS</td>
<td>autocorrelation structure</td>
</tr>
<tr>
<td>ACF</td>
<td>autocorrelation function</td>
</tr>
<tr>
<td>( \hat{\beta} )</td>
<td>regression parameter</td>
</tr>
<tr>
<td>B</td>
<td>dimension B</td>
</tr>
<tr>
<td>( C_c/C_k )</td>
<td>ratio between compression index and permeability change index</td>
</tr>
<tr>
<td>( C_{FC} )</td>
<td>empirical correction (transformation) factor for FC test</td>
</tr>
<tr>
<td>( C_{FS} )</td>
<td>empirical correction (transformation) factor for FV test</td>
</tr>
<tr>
<td>( COV_* )</td>
<td>coefficient of variation of *; or variability evaluated from a set of measurements of a quantity *</td>
</tr>
<tr>
<td>( COV_*</td>
<td># )</td>
</tr>
<tr>
<td>( Cov(\ast, #) )</td>
<td>covariance between * and #</td>
</tr>
<tr>
<td>CPT</td>
<td>cone penetration test</td>
</tr>
<tr>
<td>CRS</td>
<td>constant rate of strain oedometer test</td>
</tr>
<tr>
<td>CUDSS</td>
<td>consolidated undrained direct simple shear test</td>
</tr>
<tr>
<td>( E[\ast] )</td>
<td>the expected value of *</td>
</tr>
<tr>
<td>F</td>
<td>factor of safety</td>
</tr>
<tr>
<td>FC</td>
<td>laboratory fall cone test</td>
</tr>
<tr>
<td>FORM</td>
<td>first order reliability method</td>
</tr>
<tr>
<td>FOSM</td>
<td>first order second moment method</td>
</tr>
<tr>
<td>FV</td>
<td>field vane shear test</td>
</tr>
<tr>
<td>L</td>
<td>averaging length</td>
</tr>
<tr>
<td>( L(\ast) )</td>
<td>likelihood function</td>
</tr>
<tr>
<td>LN</td>
<td>log-normal distribution</td>
</tr>
<tr>
<td>LRFD</td>
<td>load and resistance factor design</td>
</tr>
<tr>
<td>M</td>
<td>safety margin</td>
</tr>
<tr>
<td>MC</td>
<td>Monte Carlo simulation</td>
</tr>
<tr>
<td>N</td>
<td>normal distribution; or number of realisations</td>
</tr>
<tr>
<td>( N_k )</td>
<td>empirical cone (transformation) factor for CPT</td>
</tr>
<tr>
<td>OCR</td>
<td>over consolidation ratio</td>
</tr>
<tr>
<td>PDF</td>
<td>probability density function</td>
</tr>
<tr>
<td>PE</td>
<td>point estimate method</td>
</tr>
<tr>
<td>( P_f )</td>
<td>probability of failure</td>
</tr>
<tr>
<td>PFM</td>
<td>partial factor method</td>
</tr>
<tr>
<td>PS</td>
<td>plane strain analysis</td>
</tr>
<tr>
<td>Q</td>
<td>stresses transferred from an applied load or load effect to the soil material</td>
</tr>
<tr>
<td>R</td>
<td>specified allowable stress or resistance</td>
</tr>
<tr>
<td>RBA</td>
<td>reliability-based analysis</td>
</tr>
<tr>
<td>( R(\tau) )</td>
<td>autocorrelation structure</td>
</tr>
<tr>
<td>S</td>
<td>sample space</td>
</tr>
<tr>
<td>( SHANSEP )</td>
<td>stress history and normalized soil engineering properties approach</td>
</tr>
<tr>
<td>SOSM</td>
<td>second order second moment method</td>
</tr>
<tr>
<td>( S_u )</td>
<td>undrained shear strength</td>
</tr>
<tr>
<td>( S_{u,i} )</td>
<td>( S_u ) at the base of slice ( i )</td>
</tr>
</tbody>
</table>
### Notations and abbreviations

**Latin lowercase letters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>time factor</td>
</tr>
<tr>
<td>$U$</td>
<td>degree of consolidation</td>
</tr>
<tr>
<td>$USA$</td>
<td>undrained strength analysis methodology</td>
</tr>
<tr>
<td>$\text{Var}(*)$</td>
<td>variance of $*$</td>
</tr>
<tr>
<td>$d\text{Var}_r$</td>
<td>contribution to the uncertainty in $*$</td>
</tr>
<tr>
<td>$W_i$</td>
<td>vertical stress acting at the base of slice $i$</td>
</tr>
<tr>
<td>$X$</td>
<td>measured quantity; or variate; or random variable; or event</td>
</tr>
<tr>
<td>$Y$</td>
<td>property; or population; or event</td>
</tr>
<tr>
<td>$Z$</td>
<td>limit state parameter</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>coefficient of consolidation</td>
</tr>
<tr>
<td>$\text{det}$</td>
<td>deterministic</td>
</tr>
<tr>
<td>$\text{det}r$</td>
<td>detrended value</td>
</tr>
<tr>
<td>$\text{equip}$</td>
<td>equipment and devices</td>
</tr>
<tr>
<td>$\text{err}$</td>
<td>total measurement error</td>
</tr>
<tr>
<td>$f(*)$</td>
<td>$PDF$ of $*$</td>
</tr>
<tr>
<td>$f'(*)$</td>
<td>$a$ priori $PDF$ of $*$</td>
</tr>
<tr>
<td>$f''(*)$</td>
<td>$a$ posteriori $PDF$ of $*$</td>
</tr>
<tr>
<td>$\tilde{f}(*)$</td>
<td>$PDF$ representing the Bayesian distribution for $*$</td>
</tr>
<tr>
<td>$g$</td>
<td>performance function</td>
</tr>
<tr>
<td>$h$</td>
<td>horizontal; or horizontal direction</td>
</tr>
<tr>
<td>$k$</td>
<td>ratio $S_u/\sigma'_v$; or empirical transformation factor; or constant; or hydraulic conductivity</td>
</tr>
<tr>
<td>$l$</td>
<td>location</td>
</tr>
<tr>
<td>$l_i$</td>
<td>base length of slice $i$</td>
</tr>
<tr>
<td>$m$</td>
<td>empirical constant</td>
</tr>
<tr>
<td>$\text{mod}$</td>
<td>model</td>
</tr>
<tr>
<td>$n$</td>
<td>number of test points</td>
</tr>
<tr>
<td>$\text{oper}$</td>
<td>operation of an equipment</td>
</tr>
<tr>
<td>$\text{prior}$</td>
<td>prior information</td>
</tr>
<tr>
<td>$q_{\text{net}}$</td>
<td>net corrected cone tip resistance obtained from $CPT$</td>
</tr>
<tr>
<td>$r$</td>
<td>degree of polynomial function; or distance (or interval) between measurement points</td>
</tr>
<tr>
<td>$\text{rand}$</td>
<td>random</td>
</tr>
<tr>
<td>$s$</td>
<td>extent of smear zone</td>
</tr>
<tr>
<td>$s^2_*$</td>
<td>point estimate of variance of $*$</td>
</tr>
<tr>
<td>$\text{spat}$</td>
<td>natural inherent (spatial) variability</td>
</tr>
<tr>
<td>$\text{stat}$</td>
<td>statistical uncertainty</td>
</tr>
<tr>
<td>$t(z)$</td>
<td>trend function</td>
</tr>
<tr>
<td>$\text{trans}$</td>
<td>transformation</td>
</tr>
<tr>
<td>$v$</td>
<td>vertical; or vertical direction</td>
</tr>
<tr>
<td>$w(z)$</td>
<td>fluctuation about the trend</td>
</tr>
<tr>
<td>$x$</td>
<td>sample value; or vector of variates</td>
</tr>
<tr>
<td>$y$</td>
<td>value of property $Y$</td>
</tr>
<tr>
<td>$z$</td>
<td>depth</td>
</tr>
</tbody>
</table>

**Greek letters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>inclination of the base of slice $i$; or sensitivity factor</td>
</tr>
<tr>
<td>$\beta$</td>
<td>safety index</td>
</tr>
<tr>
<td>$\Gamma^2$</td>
<td>variance function for $*$</td>
</tr>
<tr>
<td>$\delta_v$</td>
<td>vertical scale of fluctuation</td>
</tr>
<tr>
<td>$\varepsilon_*$</td>
<td>small increment in $*$</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>property (population)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>scale of fluctuation; or value of property $\Theta$</td>
</tr>
<tr>
<td>$\theta_*$</td>
<td>scale of fluctuation for $*$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>ratio of undisturbed to disturbed permeability</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>parameter in a log-normal $PDF$</td>
</tr>
</tbody>
</table>
### Notations and abbreviations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_*$</td>
<td>mean value of $*$</td>
</tr>
<tr>
<td>$\mu'_*$</td>
<td>à priori mean value of $*$</td>
</tr>
<tr>
<td>$\mu''_*$</td>
<td>à posteriori mean value of $*$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>uncertainty from assuming the statistical model; or parameter in a log-normal PDF</td>
</tr>
<tr>
<td>$\xi(\sigma)$</td>
<td>value of a property</td>
</tr>
<tr>
<td>$\rho_*$</td>
<td>autocorrelation coefficient for $*$</td>
</tr>
<tr>
<td>$\rho_{*,#}$</td>
<td>correlation coefficient between $*$ and $#$</td>
</tr>
<tr>
<td>$\sigma_*$</td>
<td>standard deviation of $*$</td>
</tr>
<tr>
<td>$\sigma^2_*$</td>
<td>variance of $*$</td>
</tr>
<tr>
<td>$\sigma'_{p}$</td>
<td>preconsolidation pressure</td>
</tr>
<tr>
<td>$\sigma^2_{v}$</td>
<td>effective stress</td>
</tr>
<tr>
<td>$\sigma^2_{\mu}$</td>
<td>à priori variance of $\mu'_*$</td>
</tr>
<tr>
<td>$\sigma^2_{\mu''}$</td>
<td>à posteriori variance of $\mu'_*$</td>
</tr>
<tr>
<td>$\bar{t}$</td>
<td>average shear stress</td>
</tr>
<tr>
<td>$\tau_{FC}$</td>
<td>shear strength value obtained from FC test</td>
</tr>
<tr>
<td>$\tau_{FV}$</td>
<td>shear strength value obtained from FV test</td>
</tr>
<tr>
<td>$\upsilon$</td>
<td>number of realisations rendering failure in MC</td>
</tr>
<tr>
<td>$\phi_i$</td>
<td>friction angle in the soil material at the base of slice $i$</td>
</tr>
<tr>
<td>$\psi_*$</td>
<td>Statistical reduction factor for $*$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>value of a variable</td>
</tr>
</tbody>
</table>

### Miscellaneous

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$\bar{z}$</td>
<td>average value of $<em>$; or point estimate of $\mu_</em>$</td>
</tr>
<tr>
<td>$*</td>
<td>#$</td>
</tr>
<tr>
<td>$\partial *</td>
<td>\partial#$</td>
</tr>
<tr>
<td>$f(<em>) = N(\mu_</em>, \sigma_*)$</td>
<td>$<em>$ is normally distributed with parameters $\mu_</em>$ and $\sigma_*$</td>
</tr>
<tr>
<td>$* \in N(\mu_<em>, \sigma_</em>)$</td>
<td>$<em>$ is normally distributed with parameters $\mu_</em>$ and $\sigma_*$</td>
</tr>
<tr>
<td>$P[*]$</td>
<td>probability of event $*$ occurring</td>
</tr>
<tr>
<td>$P[!*]$</td>
<td>probability of event $*$ not occurring</td>
</tr>
<tr>
<td>$P[*</td>
<td>#]$</td>
</tr>
<tr>
<td>$P[*</td>
<td>#]$</td>
</tr>
<tr>
<td>$P[*\cap#]$</td>
<td>probability of both event $*$ and event $#$ occurring</td>
</tr>
<tr>
<td>$x_1, x_2, ..., x_n$</td>
<td>a set of $n$ sample values</td>
</tr>
<tr>
<td>$X_1, X_2, ..., X_n$</td>
<td>a set of $n$ variates</td>
</tr>
</tbody>
</table>
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Chapter 1 – Introduction

Summary: This chapter gives some brief historical background in the field of probabilistic (reliability-based) design in geotechnical engineering. In addition, the outline, purpose and research contribution of the thesis is presented.

1.1 Development of probabilistic design in geotechnical engineering

Geotechnical engineering is to a large extent the art of making decisions in the presence of uncertainty, i.e. of managing risks. In comparison to other nearby fields of engineering, such as structural or mechanical engineering, materials provided by nature rather than man-made materials are treated. The soil deposits found on Earth are characterised by irregular layers of various soil materials with widely varying properties that must be inferred from a limited number of relatively costly observations (soundings, samplings etc.) made during geotechnical site investigations. In addition to the uncertainty arising from the randomness in the underlying natural phenomena (e.g. the soil property), the total uncertainty in a geotechnical analysis is also dependent on how well the real world (e.g. the failure mechanism) is modelled.

In the design and planning (the decision process) of a construction built on a soil material, the risks involved must be dealt with, i.e. the product of the probabilities of unfavourable deviations from the most likely subsoil conditions and the consequences thereof must be assessed. Traditionally, sufficient margins of safety have been introduced to satisfy the safety demands (handle the risks) of a construction. Establishing tolerable safety margins is a trade-off between the cost of reducing the probability of failure (e.g. extending the site investigation to obtain less uncertain assessments of the geotechnical conditions; or increasing the support of a construction) and the cost of accepting the risk of failure.

Some of the pioneers in the field of geotechnical engineering, such as Karl Terzaghi, Arthur Casagrande and Ralph Peck, recognised that it was not always financially or physically feasible to take fully into account all the uncertainties involved in a geotechnical design. The assessed values of loads and material properties would be too conservative if all the involved uncertainties were handled in this process, resulting in too expensive and too extensive supporting measures. Instead they proposed a structured methodology where evaluations of the probable conditions of the quantities involved in an analysis (the design) were supplemented by evaluations based on possible unfavourable deviations from these conditions. A treatment plan for any unfavourable deviations in the behaviour of a construction was also established. During the building phase, the real performance of the construction was observed and compared with the design estimates, possibly calling for action according to the predetermined plan. They called the methodology “the experimental method”, also known as “the observational method” as presented in the famous paper by Peck (1969).
Starting in the 1950s, significant work in the field of structural reliability, including the statistical description of material properties and the development of probabilistic (i.e. reliability-based) methods suitable for structural design, was done by Alfred Freudenthal and his successors. In the 1970s, and based on the findings for structural reliability, the problem of treating the uncertainties involved in geotechnical analyses based on statistics and probability theory was approached. Early reliability-based analyses in the field of geotechnical engineering were applied to off-shore constructions, surface mining and dams. [The above sections were freely interpreted by the author based on Baecher and Christian 2003; Ang and Tang 2007; and others]

Applications of probabilistic methods in geotechnical engineering have increased remarkably in recent years and major advances in this field of research have been made (Christian 2004). According to El-Ramly et al. (2002), probabilistic slope stability analysis was one of the first applications of reliability-based design in geotechnical engineering and can be dated back to the 1970s. Since then, a lot of research effort has been put into the probabilistic description of the uncertainty in soil properties (e.g. Lumb 1966, 1974; Vanmarcke 1977; Orchant et al. 1988; Alén 1998; Phoon and Kulhawy 1999a, 1999b) and the development of probabilistic calculation algorithms, both analytical (e.g. Tang et al. 1976; Low 1996, 2001; Low and Tang 1997a, 1997b); Xue and Gavin 2007) and numerical (e.g. Griffiths and Lane 1999; Griffiths and Fenton 2004; Xu and Low 2006).

The application of a probabilistic approach (i.e. a reliability-based approach) to a geotechnical problem does not eliminate the uncertainties in an analysis or remove the need for judgements, but it does provide a way of assessing the uncertainties and of handling them consistently. According to Baecher and Christian (2003, pp. 13-14) the geotechnical engineer of today must be able to deal with reliability-based analyses mainly because; 1) the decision processes are becoming increasingly dependent on the quantification of risks and benefits; 2) the regulatory codes are based on reliability, e.g. the Partial Factor Method (PFM) used in Europe (e.g. Olsson and Stille 1984; Bengtsson et al. 1991; CEN 2002) or the Load and Resistance Factor Design (LRFD) method used in North America (e.g. AASHTO 1997); and 3) it enables the engineer to express the confidence he/she has in the results of an analysis, e.g. how much do I need to investigate to reach a certain level of certainty?, etc.

Another reason for the geotechnical engineer to learn how to deal with reliability and statistics is the existence of valuable previous knowledge of correlations between different properties of soil materials or other empirical relations valid for certain soil types. Because of limitations in the available observations, i.e. the site investigations, the engineer is often forced to rely on such empirical knowledge for assessments and analyses. One way of combining information of a subjective (judgemental) character with more objective information (e.g. from measurements) in a robust way is via Bayesian statistics. As a consequence, the Bayesian approach has been adopted in numerous studies related to geotechnical engineering found in the literature. Some recently published papers on the subject of characterisation of geotechnical properties are Zhang et al. (2004, 2009), Cao and Wang (2012), Ching et al. (2010) and, on the subjects of stability analysis and embankment design, Cheung and Tang (2005), Wu et al. (2007), Wang et al. (2010), Zhang et al. (2010), Wu (2011), to mention just a few.
1.2 Outline of the thesis

This doctoral thesis is the second part of a research project with the intention of increasing the knowledge of some aspects related to embankments founded on clay, in particular the sulphide clay found along the north-east coast of Sweden (Fig 1.1). The first part of the research project was directed towards the design of vertical drains in this type of clay and towards the application of the observational method (Peck 1969) for the purpose of controlling the stability of embankments constructed in stages founded on vertically drained sulphide clay. The results of the first part of the research project were presented in a licentiate thesis (Müller 2010). In Papers I and II of the present doctoral thesis, the main research work conducted in the first part is presented. For further background to these two papers, i.e. a description of the origin, occurrence and classification of sulphide soils, the design of vertical drains and the application of the observational method, the reader is referred to Müller (2010).

The present doctoral thesis consists of an introductory section (Ch. 1-6), two appended papers published in peer-reviewed international journals (Papers I and II), two appended papers submitted to peer-reviewed international journals (Papers III and IV) and one appended conference paper (Paper V). The main purposes of the introductory section are; 1) to report the results of a literature survey, thereby providing the reader with a broader background to the field of probabilistic (reliability-based) stability analysis of embankments founded on soft clay and providing some more background to Papers III-V; 2) to present summaries of the research work presented in Papers I-V; and 3) to present the main conclusions from the research project and suggest further research on the use of the extended multivariate approach for other geotechnical applications.

In Ch. 2, the uncertainty in the determination of geotechnical properties and other uncertainties in geotechnical analyses are described and discussed. The sources contributing to the assessment of the total uncertainty in average values of geotechnical properties and how these sources can be treated in the context of random field theory are also presented and discussed.

In Ch. 3, some fundamentals of Bayesian statistics that have an application to practical geotechnical engineering problems are presented. Bayes’ theorem, Bayesian updating, the
Chapter 1 - Introduction

Bayesian distribution and Bayesian inference from sampling are introduced to the reader. Furthermore, the connection between the observational method and the Bayesian approach is discussed and an introduction is given to the multivariate approach (Ching et al. 2010) and the proposed extended multivariate approach, adopted in Papers III and IV.

In Ch. 4, the difference between deterministic and probabilistic stability analyses, including their respective strengths and weaknesses, is discussed. The most common probabilistic approaches are introduced briefly and the First Order Second Moment (FOSM) approach applied to stability calculations via the method of slices (adopted in Paper IV) is described in greater depth. The contributions from other sources (apart from the uncertainty in the geotechnical properties) to the uncertainty in probabilistic stability analyses are also discussed.

In Ch. 5, summaries of the contents and the most important findings presented in the appended papers are given. The main study object in the papers is the Veda embankment, the appearance of which is shown in Fig. 1.2 at various stages in the construction process.

In Ch. 6, the main conclusions from the research project are presented and suggestions for future research are proposed.

(a)  
(b)  
(c)
Fig. 1.21. The Veda embankment; a) during installation of the vertical drains (autumn 2006); b) during the early stages of the filling process (spring 2007); c) and d) after applying the preload (autumn 2008); e) after completion (spring 2013) Photos E Torstensson, H Henriksson and H Frelin

1.3 Purpose of the thesis and research contribution

As stated in the previous chapter, the purpose of the first part of the research project was directed towards the design of vertical drains, and towards the application of the observational method for the purpose of controlling the stability for stage-constructed embankments founded on vertically drained sulphide clay. This part was treated in the licentiate thesis by Müller (2010).

1 Note the concrete wells that were placed on the ground surface to protect the measurement devices
The purpose of the second part, the present doctoral thesis, was mainly to increase the knowledge of aspects related to probabilistic stability analysis of embankments founded on soft clay. In this sense, the assessment of total uncertainty in the evaluation of soil properties affecting the stability is vital and the possibility of reducing the uncertainty is appealing. The uncertainty can be reduced by averaging a property spatially over a certain soil volume, i.e. the volume affected by a given failure mechanism. For instance, for embankments founded on soft clay, the average value of the undrained shear strength in the clay, $\bar{S}_u$, over the volume defined by the critical shear surface found in stability analyses is of interest. The uncertainty can also be reduced when the value of a property is assessed from different sources and the information obtained is processed through a Bayesian updating procedure. In the present thesis, an extension of the multivariate approach (a form of Bayesian updating) suggested by Ching et al. (2010) is proposed. The extension involves a spatial averaging procedure before the multivariate approach is adopted.

In Papers III and IV, $\bar{S}_u$ in the Veda sulphide clay was assessed from the measurement results obtained from different investigation methods, e.g. from measured shear strength values, $\tau_{FV}$ and $\tau_{FC}$, via field vane shear tests, $FV$, and laboratory fall cone tests, $FC$, and from net corrected cone tip resistances, $q_{net}$, obtained from cone penetration tests, $CPTs$. In Paper III, the extended multivariate approach is illustrated for the assessments of $\bar{S}_u$ and the associated uncertainty, represented by the coefficient of variation, $COV_{\bar{S}_u}$, evaluated from site investigations prior to the construction of the embankment. In addition, some findings for the Veda sulphide clay related to probabilistic analyses, such as the inherent variability in the variates ($\tau_{FV}$, $\tau_{FC}$ and $q_{net}$) and the scales of fluctuation of the variates, are presented. The findings are compared with suggestions, for other types of clays, found in the literature with intention of increasing the empirical knowledge of these matters. In Paper IV, the extended multivariate approach is illustrated for the assessments of $\bar{S}_u$ and $COV_{\bar{S}_u}$ from measurements made during the construction of the embankment. The effects of the application of the approach on the results of probabilistic stability analyses for the Veda embankment are also presented and discussed.

In Paper V, attention is once again directed towards the design of vertical drains. Different models, with varying degrees of mathematical complexity, found in the literature and proposed for the design of vertical drains are compared. The relative influences of the variables in the models on the calculations of the average degree of consolidation and on the uncertainties in the results are evaluated via the FOSM approach and discussed from a practical point of view.

The research presented in the papers and in the introductory part of this thesis is aimed at direct applicability for probabilistic (reliability-based) design in practical engineering projects. The suggested extended multivariate approach has proved itself to be a strong tool for the purpose of evaluating average values of $S_u$ and the related uncertainties and it can be adopted for the assessment of other properties of soil materials or other quantities, e.g. hydrogeological or geometrical conditions, encountered in practical engineering projects. Together with the first part of this research project (presented in Müller 2010) the knowledge of the design of vertical drains in clay (particularly sulphide clay) is also increased.
Chapter 2 – Uncertainty in geotechnical properties

Summary: This chapter presents the sources of uncertainty contributing to the total uncertainty in the determination of a geotechnical property. Random field theory, autocorrelation, scale of fluctuation and the variance function are briefly described.

2.1 Sources of uncertainty

Within the field of geotechnical engineering, real world problems are treated and this is inevitably associated with uncertainties arising from various sources. Since the focus of the present study is on the modelling of natural (physical) phenomena and not on the subsequent decision process based on the results from the modelling, the sources of uncertainty may in principle be related to uncertainties of nature and uncertainties of the mind, often referred to as aleatory and epistemic uncertainties (e.g. Lacasse and Nadim 1996; Baecher and Christian 2003, pp. 21-26).

Natural phenomena encountered in geotechnical engineering, e.g. the shear strength in a soil or the ground water head, are constant from neither a spatial nor a temporal point of view. The geological conditions at different locations at a site may be similar but they are never exactly the same and the on-going geological, environmental, tectonic and physical-chemical processes continuously alter the conditions over time (e.g. Mitchell and Soga 2004, Ch. 2). If it were possible to observe a sought after phenomenon accurately everywhere at a site and at every point in time, there would be no uncertainty of nature, only a varying nature. However, this is practically impossible as geotechnical site investigations intended for the assessment of a natural phenomenon (a geotechnical property) are in most cases made at discrete locations at a site and at certain points in time, providing a sample from which the characteristics of the property are inferred. As a result, the assessments are associated with epistemic spatial and temporal uncertainties. In practical geotechnical engineering, however, it is customary to simplify the modelling and inference of measurement data by treating them as parts of a random process, thereby transferring the uncertainty to the aleatory branch (Baecher and Christian 2003, p. 25). Furthermore, the majority of geotechnical properties can in most cases be assumed to be temporally constant over the relevant periods of time, e.g. the life cycle of a geotechnical construction. Hence, the uncertainties of nature are assumed to be spatially random and temporally constant.

For geotechnical engineering applications the uncertainty of mind, or epistemic uncertainty, divides into three categories; 1) characterisation uncertainty; 2) model uncertainty; and 3) parameter uncertainty (Baecher and Christian 2003, p. 23-24). The first category is related to the adequacy of the interpretations drawn about the geology from the results of a site investigation. The uncertainty arises basically due to measurement errors, errors related to the handling and transcription of measurement data and errors due to the limited number of samples collected. Uncertainties in the second category arise as the true physical behaviour of the ground is simplified to fit into some model (perhaps mathematical or statistical). The third category arises when the parameters (properties) needed to describe the behaviour of the
The five terms to the right of the equality sign in Eq. 2.1 correspond to the sources of uncertainty discussed previously, see Fig. 2.1.

2.1.1 Measurement error

One source of random error in the estimation of a property lies in imperfections during the measurements. According to Orchant et al. (1988), measurement errors arise in principle due to imperfections during the measurements. All the uncertainties discussed above contribute to the total uncertainty in the description of a geotechnical property (e.g. for use in a mathematical model) and can be visualised as shown in Fig. 2.1.
to; 1) imperfections in the equipment and devices used for the measurements; 2) operational and procedural effects; and 3) random test effects.

Firstly, the equipment and devices employed for the measurements may not all have the same level of reliability or the equipment used may be damaged or not properly calibrated, thereby contributing to the random error in the measurements. Secondly, limitations in the standardisation of the testing procedures can mean that different persons perform a test differently, which could be a source of error. Thirdly, the scatter in test results not directly assignable to equipment or operational effects and not caused by spatial variability represents a random test effect.

In the study by Orchant et al. (1988), it is suggested that the variability evaluated from a set of measurements of a quantity $X$, $COV_X$, consists of the three aforementioned sources of measurement error in addition to natural variability in the soil:

$$COV_X^2 = COV_{spat,X}^2 + COV_{err,X}^2$$

(2.2a)

where

$$COV_{err,X}^2 = COV_{err,equip}^2 + COV_{err,oper}^2 + COV_{err,rand}^2$$

(2.2b)

and $COV_{err,equip}$ represents errors due to imperfect equipment and devices; $COV_{err,oper}$ represents errors due to imperfect operation of the equipment; and $COV_{err,rand}$ represents the random measurement errors. The measurement error is often treated as a random variable with zero mean. When an average value, i.e. $\bar{X}$, is evaluated, the random total measurement error tends to diminish as the number of test points, $n$, is increased:

$$COV_{err,X}^2 = \frac{COV_{err,X}^2}{n}$$

(2.3)

According to Orchant et al. (1988), the factors contributing the most to $COV_{err,equip}$ and $COV_{err,oper}$ for $FV$ tests and $CPT's$ (relevant for the studies presented in Papers III and IV) are presented in Table 2.1.

In the study by Phoon and Kulhawy (1999a), findings regarding $COV_{err,X}$ for different types of laboratory tests and in situ tests presented in the literature until that date are presented. Relevant data from their study are presented in Table A1 in Appendix A.
Table 2.1. Contributing factors to $COV_{err,eqip}$ and $COV_{err,oper}$ (from Orchant et al. 1988)

<table>
<thead>
<tr>
<th>Method</th>
<th>Significance</th>
<th>$COV_{err,eqip}$</th>
<th>$COV_{err,oper}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FV$</td>
<td>minor</td>
<td>length of vane</td>
<td></td>
</tr>
<tr>
<td></td>
<td>minor to moderate</td>
<td></td>
<td>time delay between insertions and testing</td>
</tr>
<tr>
<td></td>
<td>moderate</td>
<td>ratio between height and width of vane; blade thickness</td>
<td>rate of vane rotation</td>
</tr>
<tr>
<td></td>
<td>moderate to serious</td>
<td>torque measurement device; damaged vane</td>
<td>method of vane insertion; rod friction calibration</td>
</tr>
<tr>
<td></td>
<td>serious</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$CPT$</td>
<td>minor</td>
<td>cone size; leaky seals</td>
<td>rate of penetration</td>
</tr>
<tr>
<td></td>
<td>minor to moderate</td>
<td>manufacturing defects; excessive cone wear</td>
<td>calibration error</td>
</tr>
<tr>
<td></td>
<td>moderate</td>
<td>mechanical or electrical cone; cone angle</td>
<td>telescoping vs. continuous penetration; inclined penetration</td>
</tr>
<tr>
<td></td>
<td>moderate to serious</td>
<td>compression of rods</td>
<td></td>
</tr>
</tbody>
</table>

2.1.2 Statistical inference and statistical uncertainty

A random quantity $X$, e.g. $q_{net}$ derived from CPTs in a certain soil volume, may be referred to as a real world population described by a probability distribution and the related probability density function (PDF). The parameters that describe the PDF are often assessed from the population mean, $\mu_X$ (or expected value $E[X]$), and the population variance, $\sigma_X^2$ (or standard deviation $\sigma_X$). As a geotechnical site investigation is limited in the number of test points and distribution of test points over the area, these parameters can only be approximately inferred from the results of the investigation.

2.1.2.1 Inference for the case when no deterministic trend is present

An investigation of a geotechnical quantity $X$, consisting of $n$ test points with a set of statistically independent sample values $x_1, x_2, ..., x_n$ that do not exhibit any deterministic trend (e.g. increasing values with increasing depth below the ground surface), can be considered as a single realisation from a set of random variables $X_1, X_2, ..., X_n$. From the sample, $\mu_X$ can be inferred from the average value evaluated from the test points, $\bar{x}$, and $\sigma_X^2$ can be inferred from the sample variance, $s_X^2$:

$$\mu_X \approx \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \quad (2.4a)$$

$$\sigma_X^2 \approx s_X^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \quad (2.4b)$$

where $\bar{x}$ and $s_X^2$ are the point estimates of the population mean and variance. The uncertainty in these estimates can be assessed by the statistical procedure described in e.g. Ang and Tang (2007, pp. 256-270) where the confidence intervals of the estimates are evaluated. In general, the uncertainty in the point estimations is decreased, i.e. the confidence intervals become narrower, as $n$ is increased.
The PDF of $X$ can also be estimated from the sample values. Some common statistical procedures for doing this are described in e.g. Ang and Tang (2007, Ch. 7). In practical geotechnical engineering, the two most common PDFs are the normal distribution and the log-normal distribution. Lacasse and Nadim (1996) suggest the distributions presented in Table 2.2 for some geotechnical properties.

However, as previously stated, $\bar{X}$ of a quantity over some soil volume is of interest in practical geotechnical engineering. Therefore it is the uncertainty in the determination of $\bar{X}$ rather than $\sigma_X$ (representing the population as a whole) that is of interest in a subsequent analysis. The statistical uncertainty in this determination can be evaluated as:

$$s_X^2 = \psi_X(n) \times s_{\bar{X}}^2$$

(2.5a)

or alternatively, in terms of the coefficient of variation:

$$COV_{stat,\bar{X}}^2 = \psi_X(n) \times COV_{spat,\bar{X}}^2$$

(2.5b)

where $\psi_X(n) = 1/n$ for the case when there is no deterministic trend.

The statistical uncertainty is a bias but, as seen in Eqs. 2.5a-b, it can be reduced if the size of the sample (e.g. the number of test points) is increased.

### 2.1.2.2 Inference in the presence of a deterministic trend

Often the quantity $X$ exhibits a (physically explainable) deterministic trend, e.g. a successively increasing value of $\xi$ with increasing depth below the ground surface, $z$, in Fig. 2.2. In such a case, the value of $\psi_X$ (i.e. the reduction factor for the statistical uncertainty) must take into account the additional uncertainty arising from the uncertainty in the assessment of the trend line. For the case in Fig. 2.2, the trend line, $\bar{\xi}(z)$, can be assessed as a linear regression from $n$ sample values $\xi_1, \xi_2, \ldots, \xi_n$:

$$\bar{\xi}(z) = \bar{a} + \bar{b} \times z$$

(2.6a)
where \( \hat{a} \) and \( \hat{b} \) are the regression parameters evaluated from the sample values as:

\[
\hat{a} = \bar{\xi} - \hat{b} \times \bar{z} \tag{2.6b}
\]

and

\[
\hat{b} = \frac{\sum_{i=1}^{n}(x_i \times \xi_i) - n \times \bar{x} \times \bar{\xi}}{\sum_{i=1}^{n}x_i^2 - n \times \bar{x}^2} \tag{2.6c}
\]

The variance in \( \xi \) can be determined from the sample values (e.g. Ang and Tang 2007, p. 373) as:

\[
s_{\xi}^2 = \frac{\sum_{i=1}^{n}(\xi_i - \bar{\xi})^2 - \hat{b} \sum_{i=1}^{n}(x_i - \bar{x})^2}{n-2} \tag{2.7}
\]

For the case when \( s_{\xi}^2 \) is evaluated from the sample values (Eq. 2.7), the statistical uncertainty in the assessment of \( \bar{\xi}(z) \), i.e. \( \psi_{\xi}(n, z) \), can be evaluated (Tang 1980) as:

\[
\psi_{\xi}(n, z) = \frac{n-1}{n-3} \left[ 1 + \frac{n}{n-1} \times \frac{(z-\bar{z})^2}{s_{\xi}^2} \right] \tag{2.8}
\]

It should be noted that Eq. 2.8 requires \( \xi \) to be normally distributed and a suitable transformation of the sample values may therefore be required. For the case with “known” variance, e.g. if \( s_{\xi}^2 \) is assessed from another source (such as empirical knowledge), the term \( (n-1)/(n-3) \) is omitted in Eq. 2.8. In Paper III, the above procedure is exemplified.

Sometimes the best fitted trend function is nonlinear, for instance assessed as a polynomial function of degree \( r \) (as in Paper IV):
\[ \bar{\xi}(z) = \sum_{j=0}^{r} \hat{a}_j \times z^j \]  \hspace{1cm} (2.9)

where \( \hat{a}_i = (\hat{a}_0, \hat{a}_1, ..., \hat{a}_r) \) are the regression parameters, e.g. evaluated based on the suggestions for nonlinear regression in Ang and Tang (2007, Ch. 8.6).

In such cases, \( \psi_\xi(n, z) \) for a normally distributed quantity with a known variance can be evaluated based on Raiffa and Schlaifer (1961, Ch. 13) as:

\[ \psi_\xi(n, z_k) = \left[ E \times (z^T_\xi \times z_\xi)^{-1} \times E^T \right] \]  \hspace{1cm} (2.10)

where \( E = \begin{bmatrix} 1 & z_k & z_k^2 & \cdots & z_k^r \end{bmatrix} ; \) \( z_k \) is an arbitrary chosen value of \( z \) for which \( \psi_\xi(n, z_k) \) is evaluated; \( z_\xi = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^r \\ 1 & z_2 & z_2^2 & \cdots & z_2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^r \end{bmatrix} \) ; and \( (z_1, z_2, ..., z_n) \) are the \( n \) depths at which the quantity \( \xi \) was measured.

2.1.3 Natural inherent variability

Assuming that the geotechnical properties are temporally constant over the relevant time period, the natural inherent variability depends on random spatial variations which, in a geotechnical context, arise due to the intricate geological, environmental and physical-chemical processes that have affected the soil material over the years. Therefore, soil properties encountered in situ will vary horizontally and vertically. As stated by e.g. Lacasse and Nadim (1996) and Phoon and Kulhawy (1999a), the spatial variation of a property can be decomposed into a smoothly varying trend and a fluctuating component, see Fig. 2.2. The trend function \( f(z) \) in the figure) can in most cases be explained physically and is therefore treated as a deterministic constant, e.g. the stress dependence of \( S_u \) that gives rise to a more or less linear increase with increasing depth. The fluctuating component \( w(z) \) in the figure) represents the spatial variability.

In order to assess the spatial variability, i.e. the variability in \( w(z) \), the following example can be followed:

1. Measured values of the quantities \( \tau_{FV} \) and \( \tau_{FC} \) (treated in Paper III) are compiled in Fig. 2.3a. In this process, it is vital that only data judged as coming from the same population are stacked together. In practice, geotechnical site investigation methods (e.g. borings, soundings or samplings) collect data more or less continuously through a geological deposit and, as a result, data from different types of soil material and with different stress histories etc. are retrieved. The data from each investigation point must therefore be divided into samples representing a certain soil type before compilation.

2. From the compiled data, the deterministic trend is subtracted. Often a linear trend function can be assumed. The detrended values \( \tau_{FV, detr} \) and \( \tau_{FC, detr} \) are presented in Fig. 2.3b.
3. According to Phoon and Kulhawy (1999a), the standard deviation in the measured values, $\sigma_{FV}$ (or $\sigma_{FC}$), can be evaluated as: $\sigma_{FV} = \sqrt{\frac{\sum (F_{V,\text{detr}})^2}{(n-1)}}$ (cf. Eq. 2.4b), and the coefficient of variation as: $COV_{FV} = \sigma_{FV}/\bar{F}$, where $\bar{F}$ is the average value of the trend function. However, as stated in Ch. 2.1.1, the evaluated $COV_{FV}$ consists of both the spatial variability in the soil and measurement error. The spatial variability is therefore obtained as: $COV_{spat,\tau_{FV}}^2 = COV_{\tau_{FV}}^2 - COV_{err,\tau_{FV}}^2$. Alternatively, the detrending procedure can be omitted and Eq. 2.7 used instead.

It is often rational to describe the natural inherent variability of a soil property as a homogeneous random field (Vanmarcke 2010), described subsequently in more detail, see Ch. 2.2.

In the study by Phoon and Kulhawy (1999a), findings are presented regarding natural inherent variability for different soil properties, derived from laboratory and in situ tests, reported in the literature until that date. Relevant data from their study are presented in Table A2 in Appendix A.

2.1.4 Transformation uncertainty

The interpretation of a geotechnical property from some measured quantity often involves an empirical transformation. For instance, during a $FV$ test in clay, $S_q$ is interpreted from a shear strength value evaluated from the maximum torque resistance registered during the test. The interpretation is made via the introduction of an empirical correction (transformation) factor (e.g. Bjerrum 1972; Pilot 1972; Aas et al. 1986). Values of the transformation factor valid for
different investigation methods in different types of clay have historically been obtained by studying real embankment or slope failures, or by calibrations versus laboratory tests where $S_u$ was measured directly. In Papers II, III and IV, values of $S_u$ in the sulphide clay of interest were interpreted from the results of $FV$ tests, $FC$ tests and $CPTs$:

\[
S_u | \tau_{FV} = C_{FV} \times \tau_{FV} \tag{2.11a}
\]
\[
S_u | \tau_{FC} = C_{FC} \times \tau_{FC} \tag{2.11b}
\]
\[
S_u | q_{net} = \frac{q_{net}}{N_k} \tag{2.11c}
\]

and via the empirical “Stress History and Normalized Soil Engineering Properties” (SHANSEP) approach (Ladd and Foot 1974; Ladd 1991) based on the over consolidation ratio, $OCR (= \sigma_p'/\sigma_v')$:

\[
S_u | OCR = k \times \sigma_v' \times OCR^m \tag{2.11d}
\]

where the quantities $\tau_{FV}$ and $\tau_{FC}$ are the uncorrected undrained shear strength values evaluated from the $FV$ and $FC$ tests respectively; $C_{FV}$ and $C_{FC}$ are the corresponding empirical correction (transformation) factors; the quantity $q_{net}$ is the net corrected cone tip resistance obtained from $CPTs$; $N_k$ is the corresponding empirical cone (transformation) factor; the factor $k$ is an empirical transformation factor; $\sigma_p'$ is the preconsolidation pressure; $\sigma_v'$ is the prevailing effective stress in the clay and; $m$ is an empirical constant.

Furthermore, based on the results of piezometer measurements, the prevailing effective stress $\sigma_v'$ in the clay was assessed and $S_u$ was interpreted via the empirical transformation factor $k$ (assuming a normally consolidated stress state, i.e. $OCR = 1$):

\[
S_u | \sigma_v' = k \times \sigma_v' \tag{2.11e}
\]

The results of calibration tests (vs. consolidated undrained direct simple shear tests, CUDSS) for the evaluations of $S_u$ from $FV$ tests, $FC$ tests and $CPTs$ in several Swedish sulphide clays are presented in a study by Larsson et al. (2007). Characteristics of $C_{FV}$, $C_{FC}$, $N_k$ and $k$ established in the study are presented in Table 2.3. In the Veda area (the study object in the appended papers), a trial embankment was built. Based on the results of $FV$ tests and $CPTs$ made during the consolidation of the sulphide clay beneath the trial embankment and on the results of CUDSS tests on undisturbed samples of the sulphide clay retrieved in the area, characteristics of the ratio $k$, specific to the sulphide clay in the Veda area, were evaluated (see Table 2.3). More information about the trial embankment can be found in Paper II.

Table 2.3. Characteristics of the transformation factors for some investigation methods relevant for Swedish sulphide clay

<table>
<thead>
<tr>
<th>Transformation factor</th>
<th>Expected value, $E(*)$</th>
<th>Coefficient of variation, $COV(*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{FV}$</td>
<td>0.65</td>
<td>0.17</td>
</tr>
<tr>
<td>$C_{FC}$</td>
<td>0.65</td>
<td>0.16</td>
</tr>
<tr>
<td>$N_k$</td>
<td>20</td>
<td>0.10</td>
</tr>
<tr>
<td>$k^1$</td>
<td>0.28</td>
<td>0.10</td>
</tr>
<tr>
<td>$k^2$</td>
<td>0.31</td>
<td>0.24</td>
</tr>
</tbody>
</table>

1 From Larsson et al. (2007), used in Paper III; 2 from measurements at the trial embankment, used in Paper IV.
2.2 Random field theory, autocorrelation and the variance function

2.2.1 Random field theory

As stated earlier, the properties of any soil vary spatially within the deposit, both vertically and horizontally; there are areas with somewhat higher values of the property and areas with somewhat lower values. As shown for the vertical case in Fig. 2.2, a deterministic trend, \( t(z) \) in the figure, may be present and make up one part of the variation. The rest of the variation, \( w(z) \) in the figure, is explained by the natural inherent variability of the property often treated as a random fluctuation about the trend (e.g. De Groot and Baecher 1993). Adopting the nomenclature in Fig. 2.2, the value of the property \( \xi(z) \) as a function of the depth \( z \) can be described as:

\[
\xi(z) = t(z) + w(z)
\]

The horizontal variation in a property is analogous to the vertical case. One rational way of treating the random fluctuation of the property in an analysis is to consider it as a random field (Vanmarcke 1977, 2010). In practical geotechnical engineering, the random field is often assumed to be homogeneous (sometimes referred to as stationary) and horizontally isotropic, according to the definitions in Vanmarcke (2010, p 24). These assumptions imply that a sample (test point) taken anywhere in the area of interest comes from one specific population with a certain PDF. This means that from a set of tests made in the area (the sample), the average value and variance in the property (population) can be estimated (cf. Ch. 2.1.2). However, as the random field is said to be horizontally isotropic (i.e. isotropic along a horizontal plane in the soil) it is vital that only results from the same elevation are compared, i.e. at the same level of effective stress. Another implication of these assumptions is that there exist both a horizontal and a vertical scale of fluctuation. This term is a measure of the spatial correlation structure of the property and is in principle the distance within which results from different investigation points are said to be correlated, i.e. dependent on one another. Closely neighbouring investigation points are more likely to be similar in value than points wider apart. In Fig. 2.2, the scale of fluctuation is denoted \( \delta_e \) and is roughly the vertical distance between two adjacent intersections of the trend line.

2.2.2 Autocorrelation and scale of fluctuation

The degree of interdependency, or correlation, between two variables \( X \) and \( Y \) can be estimated by the correlation coefficient, \( \rho_{XY} \):

\[
\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sqrt{E[(X-\mu_X)^2] \times E[(Y-\mu_Y)^2]}}
\]

(2.13)

where \( \text{Cov}(X,Y) \) is the covariance between the variables; and \( \mu_X, \sigma_X, \mu_Y \) and \( \sigma_Y \) are the respective mean values and standard deviations.

Correlation can arise due to causality, i.e. when one event causes another (such as the interdependency between \( S_u \) and OCR in Eq. 2.11d). Correlation can also arise when two uncertainties share a common dependency on a third. For instance, performing a FV test and a CPT in close vicinity, the measurement results \( \tau_{FV} \) and \( q_{net} \) are dependent on the actual value of \( S_u \) in the clay by cause and \( \tau_{FV} \) and \( q_{net} \) are therefore probabilistically correlated. Another type of correlation, statistical correlation, can arise if two variables are derived from the same set of data. For instance, fitting a Mohr-Coulomb envelope to a series of results from triaxial
tests can be done in many ways by assigning different values to the statistically correlated “cohesion-part” and “friction-part”. The spatial correlation described above is another type of correlation of great importance in the averaging of soil properties and is therefore described more closely.

The autocorrelation structure of a property is a measure of the spatial correlation and can be used to estimate the scale of fluctuation of the property based on a set of measurements (e.g. Baecher and Christian 2003, Ch. 9). In principle, the autocorrelation structure estimated from a set of measurements made at different locations at a site is obtained as follows:

1) Any deterministic trend is removed from the set of measured values of the property $X$ (cf. Ch. 2.1.3), i.e. the average value is subtracted from the data.
2) The distance $r$ between each pair of test points is evaluated. Sometimes, and preferably when the number of test points is limited, $r$ is defined as an interval and all pairs located at distances within the interval are then evaluated.
3) The autocorrelation structure as a function of $r$, $R(r)$, is evaluated by summing the covariance for each pair of detrended test points separated by a distance $r$ and dividing the sum by the number of pairs separated by a distance $r$, $n_r$, and by the sample variance, $s_X^2$, as (e.g. De Groot and Baecher 1993):

$$R(r) = \frac{\sum_{i=1}^{n_r} \text{cov}[x_i,x_{i+r}]}{n_r \times s_X^2} = \frac{\sum_{i=1}^{n_r} [x_i \times x_{i+r}]}{n_r \times s_X^2}$$

(2.14)

where $x_i$ is the detrended value of one of the measurements; $x_{i+r}$ is the detrended value of another measurement separated from $x_i$ by a distance $r$.
4) $R(r)$ is plotted in a diagram as a function of $r$.

An example of a horizontal autocorrelation structure evaluated for $q_{net}$ in the sulphide clay at Veda is presented in Fig. 2.4 (from Paper III). In the figure, the dots represent the autocorrelation structure evaluated according to the procedure described above with the interval of $r_h$ set to 20 metres.

![Figure 2.4. Example of an autocorrelation structure and an autocorrelation function](image)

Chapter 2 – Uncertainty in geotechnical properties
In order to estimate the scale of fluctuation, $\theta$, the autocorrelation structure is approximated by a mathematical autocorrelation function. Some common functions used for this purpose are presented in e.g. Vanmarcke (2010, Ch. 5). For the autocorrelation structure presented in Fig. 2.4, the triangular function was chosen, represented by the equation for $\rho_{h,\theta_{\text{net}}}$ in the figure.

As argued by e.g. De Groot and Baecher (1993) and Christian et al. (1994), there is often a spike in the autocorrelation function at $r = 0$. This spike represents the contribution to the total variance in the measurements of the property arising from small-scale fluctuations and from random measurement errors (cf. Ch. 2.1.1). In the example shown in Fig. 2.4, the variance due to small-scale fluctuations and measurement error (denoted nugget) was evaluated to about 0.07, corresponding to a $COV$ of about 0.04. This value is of the same order of magnitude as the suggested total measurement error for electrical $CPTs$ in Orchant et al. (1988), see also Table A.1 in Appendix A.

The horizontal scale of fluctuation for $q_{\text{net}}$ in the study presented in Paper III, $\theta_{h,\theta_{\text{net}}}$, was then calculated (e.g. Vanmarcke 2010, p. 193) as:

$$\theta_{h,\theta_{\text{net}}} = 2 \int_{0}^{\infty} \rho_{h,\theta_{\text{net}}} \, dr_{h} \quad (2.15)$$

In Table A3 in Appendix A, suggestions for $\theta$ for various soil properties are presented based on the compilation in Phoon and Kulhawy (1999a).

As illustrated in Fig. 2.2, the scale of fluctuation can be interpreted as the distance between zones in the soil with relatively high values of a property and zones with relatively low values, and this distance is important when establishing an average value of a property. This can be explained by introducing what is referred to as the observation scale, see Fig. 2.5 (from Baecher and Christian 2003, p 228). As seen in the figure, the “broader average value” of the property when the observation scale spans the whole 200 metres is about 30 and the scale of fluctuation is about 25 metres. However, if the observation scale is reduced, the “local average value” varies significantly with the location. If the observation scale is represented by the “local window” in the figure, the value of the “local average” observed differs depending on where along the x-axis the window is placed.

![Figure 2.5. Illustration of the observation scale (from Baecher and Christian 2003)](image-url)
This has a significant bearing on the choice of an appropriate value of the property for use in an analysis of a construction placed in the area. It may be intuitively realised that the shorter the value of the scale of fluctuation, the lower is the probability that a potential construction with a pre-determined size is erected at a spot within the area with a local average that deviates significantly from the global average over the area as a whole. One way of treating this matter in an analysis is by introducing the variance function (Vanmarcke 2010, pp. 189-193).

2.2.3 The variance function

As stated earlier, the average value of some property over a given soil volume is of interest in any geotechnical analysis. The size of this soil volume is determined by the type of construction and the type of failure that is being analysed. For instance, a single cohesion pile installed in an area with subsoil consisting of clay would basically fail if the average shear stress transmitted from the pile to the clay along its periphery, \( \bar{\sigma}_s \), exceeded the average value of \( S_u \) that the clay in the vicinity of pile was able to mobilize (\( S_u \)). In such a case, the location and the size of the potential failure mechanism are known, see Fig. 2.6a. On the other hand, an embankment built to span the area would basically fail if \( \bar{\sigma}_s \) transmitted from the embankment loading to the clay via a potential shear failure surface through the clay exceeds \( S_u \) along that surface, see Fig. 2.6b-c. In such a case, neither the location nor the exact extent of a potential failure mechanism (shear surface) would be known. However, the soil volume potentially affected by the failure surface in the embankment case would most certainly be several magnitudes larger than the soil volume affected by a failure of the pile. Hence, the probability that the value of \( S_u \) determining the bearing capacity of the pile deviates from the global average value (i.e. the pile is installed in an area with a somewhat higher or lower value of \( S_u \) than the global average) is higher than for the embankment case. This is analogous to the “local window” in Fig. 2.5. If the window is small (the pile case) there is a greater probability that the window is placed over a length span of the x-axis where a local average deviates from the global average than if the window is expanded (the embankment case). The variance function, \( \Gamma^2 \), (Vanmarcke 2010, pp.189-190) measures the reduction in the point variance under local averaging and is one way of dealing with this situation.

If a geotechnical quantity \( X \) in a given soil volume is considered as a random variable with average \( \bar{X} \) and spatial variance \( \sigma^2_X \), the value of \( X \) is dependent on the location, \( l \), in the considered soil volume, i.e. \( X = X(l) \). It can then be treated as a stationary random field giving rise to a family of moving average processes as:

\[
X_l(l) = \frac{1}{L} \int_{l-L/2}^{l+L/2} X(u) \, du
\]

(2.16)

where \( L \) denotes the averaging length. The averaging length is analogous to the “local window” in Fig. 2.5 and can be considered as the length within the soil volume that is affected by a geotechnical construction and \( l \) is the position along the x-axis. For instance, in an analysis of the bearing capacity of the single cohesion pile, \( l \) would be the vertical position in the clay (neglecting the extension of the pile in the horizontal direction) and \( L \) would be the length of the pile, see Fig 2.6a.
Figure 2.6. Illustrations of failure mechanisms; a) for a cohesion pile; b) for an embankment; c) overview of the embankment.

In the moving averaging process, the global average $\bar{X}$ is not affected but the variance may be expressed as:

$$\sigma^2_{\bar{X}_L} = \Gamma^2(L) \times \sigma^2_{\bar{X}}$$  \hspace{1cm} (2.17a)

or alternatively, in terms of the coefficient of variation:
\[ COV_{X_L}^2 = \Gamma^2(L) \times COV_{\text{spat},X}^2 \]  
\[ (2.17b) \]

where \( \sigma_{X_L}^2 \) is the variance of \( X \) locally averaged over the length \( L \); \( \Gamma^2(L) \) is the variance function; and \( COV_{X_L}^2 \) is the coefficient of variation of \( X \) locally averaged over the length \( L \).

The appearance of the variance function depends on the autocorrelation function chosen to approximate the autocorrelation structure. For the case described in Fig. 2.4, the variance function in one horizontal dimension \( \Gamma_{1D,\text{net}} \) is defined as:

\[ \Gamma_{1D,\text{net}}^2 = \begin{cases} 1 - \frac{L}{3 \theta_{h,\text{net}}} & L < \theta_{h,\text{net}} \\ \frac{\theta_{h,\text{net}}}{L} \left( 1 - \frac{\theta_{h,\text{net}}}{3L} \right) & L \geq \theta_{h,\text{net}} \end{cases} \]  
\[ (2.18a) \]

where \( L \) is the length of the failure mechanism in one dimension. For the embankment case described in Figs. 2.6b-c, and neglecting the extension in the vertical direction (the thickness of the failure surface), the failure mechanism spans both horizontal dimensions \( (L_A \text{ and } L_B) \) in Figs. 2.6b-c), and the total variance function is evaluated as:

\[ \Gamma_{2D,\text{net}}^2 = \Gamma_{1D,\text{net},A} \times \Gamma_{1D,\text{net},B} \]  
\[ (2.13b) \]

where the subscripts \( A \) and \( B \) represents the two horizontal dimensions.

### 2.3 Total uncertainty in the assessment of a property

In a probabilistic analysis of a geotechnical construction, an assessment of the uncertainty in the determination of an average value of a property \( Y \) over some soil volume is required. As discussed previously, the uncertainty arises from various sources that are fundamentally different in nature. In Paper III, a process for obtaining the total uncertainty in \( \bar{Y} \) based on a set of measurements of a quantity \( X \) for the analysis of a certain construction is suggested. In principle, it consists of the following steps (see also Fig. 2.7):

1. Representative values evaluated from data available from the site investigation are compiled (cf. Ch. 2.1.3).
2. From the compiled data, the presence of a deterministic trend and \( \psi_X \) are evaluated (cf. Ch. 2.1.2 and 2.1.3).
3. From the detrended data, \( COV_X \) is evaluated (cf. Ch. 2.1.3).
4. The autocorrelation structures, autocorrelation functions and scales of fluctuation in the vertical direction are then estimated from data obtained from each of the investigation points, and analogously in the horizontal direction from data obtained from the total set of investigation points. Often, a horizontally isotropic random field is assumed (cf. Ch. 2.2.2).
5. The variance function(s) are then established based on the scale(s) of fluctuation and the size of the structure that is being analysed (cf. Ch. 2.2.3).
6. Appropriate values of the uncertainty due to measurement errors and transformation uncertainty are chosen (cf. Ch. 2.1.1 and 2.1.4).
7. The total uncertainty in the determination of \( \bar{Y} \) based on a set of measurements of a quantity \( X \), \( COV_{\bar{Y}|X} \), can then be evaluated (Eq. 2.1).
Based on the discussions in the previous chapters, Eq. 2.1 can be rewritten on the form:

$$COV^2_{Y|X} \approx COV^2_{spat,X} \times \Gamma^2 + COV^2_{err,X} \times \frac{1}{n} + COV^2_{spat,X} \times \psi_X + COV^2_{trans,X} + \xi \Rightarrow$$

$$COV^2_{Y|X} \approx \left( COV^2_{spat,X} \right) \times (\psi_X + \Gamma^2_X) + COV^2_{err,X} \times \frac{1}{n} + COV^2_{trans,X} + \xi$$  \hspace{1cm} (2.19a)

As noted in Ch. 2.1.1 (Eq. 2.2a), $COV_X$ evaluated from the measurements consists of both $COV_{spat,X}$ and $COV_{err,X}$, and as a consequence we finally arrive at:

$$COV^2_{Y|X} \approx \left( COV^2_X - COV^2_{err,X} \right) \times (\psi_X + \Gamma^2_X) + COV^2_{err,X} \times \frac{1}{n} + COV^2_{trans,X} + \xi$$  \hspace{1cm} (2.19b)

However, for most of the analyses performed in practical engineering projects, the procedure for the evaluation of total uncertainty described above cannot be adopted for every single property involved in the analyses. It is not possible to investigate every property to the extent needed to enable meaningful evaluations and assessments to be made during the steps inherent in the procedure. For some of the properties, previous empirical knowledge must be
relied upon. Results from a vast number of studies of the total uncertainty for geotechnical properties normally encountered in practical engineering, are compiled in Ch. 8 of the textbook written by Baecher and Christian (2003). Other compilations can be found in Phoon and Kulhawy (1999b) and Duncan (2000). Selected parts of these compilations are presented in Table A4 in Appendix A. These may serve to “fill the gaps” in an analysis, or as an à priori estimation against which the relevance of findings from evaluations of a set of data can be checked.

2.4 Model uncertainty

In the previous chapters, the assessment of the total uncertainty in geotechnical properties is discussed at some depth. Other uncertainties affecting a probabilistic geotechnical analysis such as the groundwater conditions, external loads, boundary conditions and geometrical conditions are not treated explicitly within the framework of this thesis, but they should be assessable in more or less the same manner. However, uncertainties inherited from other sources affecting the geotechnical problem studied may be harder to assess, such as; 1) imperfectness of the assumed constitutive material models describing the behaviour of the material; 2) inadequacy of the mathematical model adopted to describe the problem; 3) unintended omission of possible failure modes; 4) human error; and 5) environmental changes; (based on suggestions in Lumb 1974; Tang et al. 1976; Wu et al. 1989; Lacasse and Nadim 1996; Whitman 2000; El-Ramly et al. 2002; Phoon and Kulhawy 2003; Christian 2004; and Wu 2009; Doorn and Hansson 2011). In the studies of e.g. Wu et al. (1989), Cheung and Tang (2005) and Cassidy et al. (2008), it is suggested that errors arising from the mathematical modelling of real soil behaviour (Nos. 1 and 2 in the above listing) may be calibrated via large-scale tests or by observing the real field behaviour of geotechnical structures.

These types of uncertainties should, as far as possible, be included in a probabilistic analysis. One rather convenient way of doing this is by combining them in a model uncertainty defined by a coefficient of variation, $COV_{mod}$, and adding this to the calculated uncertainty resulting from the probabilistic analysis (e.g. Christian et al. 1994; Cassidy et al. 2008):

$$COV_{Z_{mod}}^2 = COV_Z^2 + COV_{mod}^2$$  \hspace{1cm} (2.20)

where $COV_{Z_{mod}}$ is the uncertainty in the analysis of a limit state parameter $Z$ (e.g. the total factor of safety or the magnitude of settlement); and $COV_Z$ is the uncertainty in $Z$ without the introduction of model uncertainties (i.e. obtained from a probabilistic analysis). In Ch. 4.3, the model uncertainty related to probabilistic stability analyses is discussed further.
Chapter 3 – Bayesian updating and the multivariate approach

Summary: In this chapter, the Bayesian approach and some applications and utilities related to geotechnical engineering are briefly presented. The multivariate approach is also described and discussed.

3.1 The Bayesian approach

3.1.1 Bayesian vs. Frequentist approach

The difference between the two schools of thought in statistics and probability theory, the Bayesian approach and the Frequentist approach, lies in the definition of probability. The Bayesian approach treats the uncertainties in the probabilistic work as a measure of the degree of belief or confidence one puts in knowing the state of the world (i.e. probability is in the mind of the individual), and distinguishes itself from the Frequentist approach where probability is defined as the frequency of a long series of similar events (e.g. Glickman and van Dyk 2007, pp. 319-320; Baecher and Christian 2003, p. 16). In a probabilistic analysis, probability distributions and the related probability density functions, PDFs, with their associated parameters must be assigned to the quantities involved, e.g. the material properties, the geometrical conditions and the boundary values that define the problem. As stated in Ch. 2, accurate estimations of the distributions and their parameters require large amounts of data. In geotechnical engineering, the observed data are often limited and the statistical estimates have to be supplemented by judgemental information (Ang and Tang 2007, pp. 346-347). The traditional (Frequentist) approach does not allow judgemental and observed information to be combined, whereas the Bayesian approach (sometimes referred to as the subjective probability approach) allows all sources of information to be incorporated systematically. In addition, a Bayesian approach can describe the uncertainty of a statement about an unknown parameter in terms of a probability, whereas a Frequentist cannot (Glickman and van Dyk 2007, p. 320).

Hence, observing the world in a Bayesian way is rather appealing when working in the field of geotechnical engineering. In general, we must draw conclusions based on a limited number of tests and it is seldom possible to apply a geotechnical investigation method on exactly the same specimen more than once. The tested specimen is in most cases destroyed or at least disturbed to some degree by the test itself, ruling out exact duplication of the test on exactly the same specimen (as required in the Frequentist approach). In addition, a great deal of the uncertainty of the geological world is associated with a lack of knowledge of the definite arrangement of the geological materials and of the values of the geotechnical properties. However, the Bayesian approach should not be seen as an accessible route to be used only when the data is limited (thus precluding the Frequentist approach) but as a philosophically different definition of probability, making it possible to make a statement of the probability of the occurrence of a single event, e.g. the failure of an embankment.
3.1.2 Bayes’ theorem

The concept of conditional probability, i.e. the notion that the probability of an event \( Y \) may change if another event \( X \) occurs, forms the basis of Bayes’ theorem (Bayes 1763):

\[
P[Y|X] = \frac{P[Y \cap X]}{P[X]} = \frac{P[X|Y] \times P[Y]}{P[X]} = \frac{P[X|Y] \times P[Y]}{P[X|Y] \times P[Y] + P[X|\overline{Y}] \times P[\overline{Y}]} \tag{3.1}
\]

where \( P[Y|X] \) is the probability of event \( Y \) occurring given (conditional on) the occurrence of event \( X \); \( P[Y \cap X] \) is the probability of both event \( Y \) and event \( X \) occurring; \( P[X|Y] \) is the probability of event \( X \) occurring given the occurrence of event \( Y \); \( P[Y] \) and \( P[\overline{Y}] \) are the probabilities of \( Y \) occurring and not occurring respectively; \( P[X] \) is the probability of event \( X \) occurring; and \( P[X|\overline{Y}] \) is the probability of event \( X \) occurring if event \( Y \) does not occur. This is also depicted in the Venn diagram (Venn 1866) in Fig. 3.1 where the sample space \( (S) \) in the figure represents all possible outcomes, i.e. \( P[S] = 1 \). From the figure it can be understood that observing event \( X \) implies a specific probability that the event \( Y \) also occurs and that this probability is different from what it would be without the observation of \( X \). In this context it is important to remark that Bayes’ theorem is not dependent on the definition of probability, i.e. it is also valid for a Frequentist approach.

3.1.3 Bayesian updating

In practice, the geotechnical engineer often has some clue about the probability distribution, i.e. the PDF and the associated parameters, of the quantity of interest (the population). The quantity could be e.g. a specific soil property, the depth to bedrock or the elevation of the groundwater head. In the Bayesian world, this is referred to as prior or â priori knowledge. The â priori knowledge could for instance be based on intuition, experience, expert judgement, or on some empirical relationship. When measurements of the sought after quantity are available, better estimates of the PDF and the associated parameters can be made via Bayes’ theorem. The procedure of combining â priori knowledge with measurements is called Bayesian updating.

An illustration of the Bayesian updating procedure adopted for a geotechnical site investigation intended for evaluation of a property \( \Theta \) in the soil volume is presented in Ang and Tang 2007, pp. 352-353). The property \( \Theta \) represents a population with an arbitrary PDF.
Based on expert judgement, the à priori PDF, \( f'(\theta) \), is estimated (Fig. 3.2). The à priori probability that the value of the property lies within a certain interval, \( \theta_i \leq \theta < \theta_i + \Delta\theta \), is \( f'(\theta_i)\Delta\theta \). The result of a measurement of the property conducted in the site investigation, \( \varepsilon \), is a sample drawn from the population of \( \theta \). Applying Bayes’ theorem, the à posteriori probability that the value of the property lies within the interval, \( f''(\theta_i)\Delta\theta \), can be calculated taking both the à priori judgement and the observed measurement into account:

\[
f''(\theta_i)\Delta\theta = \frac{p[\varepsilon|\theta_i] \times f'(\theta_i)\Delta\theta}{\sum_{i=1}^{k} p[\varepsilon|\theta_i] \times f'(\theta_i)\Delta\theta}
\]  

(3.2)

or in the limit:

\[
f''(\theta_i) = \frac{p[\varepsilon|\theta] \times f'(\theta)}{\int_{-\infty}^{\infty} p[\varepsilon|\theta] \times f'(\theta)d\theta}
\]  

(3.3)

where \( P[\varepsilon|\theta] \) is the conditional probability or likelihood of observing \( \varepsilon \) under the assumption that the value of the property is \( \theta \). As seen, \( P[\varepsilon|\theta] \) is a function of \( \theta \), commonly referred to as the likelihood function \( L(\theta) \). Solving the integral in the denominator in Eq. 3.3, this term becomes independent of \( \theta \); hence the term is a constant \( (k) \). Equation 3.3 can then be written in the form:

\[
f''(\theta_i) = k \times L(\theta) \times f'(\theta)
\]  

(3.4)

In this way, both the prior judgemental assumption (via \( f'(\theta) \)) and the results of the measurement (via \( L(\theta) \)) are incorporated in the posterior PDF via a Bayesian updating procedure.

In geotechnical engineering, as stated in Ch. 2, we are often interested in the average value of a property. In the above example, the PDF for the population was updated. The same procedure can be adopted for updating the parameters of the distribution, e.g. the average value, \( \mu_\theta \), and the standard deviation, \( \sigma_\theta \), for a normal distribution (using the notations in the example).

![Continuous prior distribution of \( \Theta \) (from Ang and Tang 2007, p. 353)](image-url)
In the multivariate (Bayesian) analyses presented in Paper III, the property $\bar{S}_u$ (the average value) of the population $S_u$ in the Veda sulphide clay was studied, rather than the population itself. In the paper, the â priori knowledge was based on the empirical relationship between the preconsolidation pressure in the clay, $\sigma'_p$, and $S_u$ (Eq. 2.11d). From measurements of $\sigma'_p$ at the Veda site, an â priori estimate of the PDF for $\bar{S}_u$ was obtained. From measurements intended for the evaluation of $S_u$ (e.g. $\tau_{FV}$) preformed during the subsequent site investigation at the site, it was possible to update the à posteriori PDF of $\bar{S}_u$. In the form of Eq. 3.4, this can be written as:

$$f''[\bar{S}_u|\tau_{FV}] = k \times L[\tau_{FV}|\bar{S}_u] \times f'[\bar{S}_u]$$ (3.5)

where $f''[\bar{S}_u|\tau_{FV}]$ is the updated à posteriori PDF of $\bar{S}_u$ given the information retrieved from measurements of both $\sigma'_p$ and $\tau_{FV}$; $k$ is a constant (analogous to Eq. 3.4); $L[\tau_{FV}|\bar{S}_u]$ is the likelihood function (containing the information retrieved from the measurements of $\tau_{FV}$); and $f'[\bar{S}_u]$ is the prior PDF of $\bar{S}_u$ (based on measurements of $\sigma'_p$).

The procedure for processing initial prior information and observations through Bayes’ theorem is shown in Fig. 3.3, where the terms for the case described above are also incorporated.

As can be seen, it is suitable to adopt the Bayesian approach and Bayes’ theorem when characterising geological conditions, including the geotechnical properties of the soil materials. Among the first to imply that it was possible to combine subjective information from intuition or experience, expert judgement or empiricism, with more objective information from measurements for engineering purposes were for instance Ang and Tang (1975), and it still remains an attractive feature to the geotechnical engineer. As a consequence, the Bayesian approach has been adopted in numerous studies related to geotechnical engineering over the years. The publications by Olsson (1986) and Stille et al. (2003) are two examples of Swedish research studies on this topic. Some recently published papers on the subjects of stability analysis and embankment design are Cheung and Tang (2005), Wu et al. (2007), Wang et al. (2010), Zhang et al. (2010) and Wu (2011) and on the subject of characterisation of geotechnical properties, Zhang et al. (2004, 2009), Cao and Wang (2012), Ching et al. (2010), to mention just a few.
3.1.4 The Bayesian distribution and Bayesian inference from sampling

Regardless of the type of PDF that represents a certain population (e.g. an engineering property), the mean (average) value is usually one of the parameters describing the PDF, e.g. the normal distribution which is described by the mean value and the standard deviation. As stated in Ch. 2 and above, the mean value is often itself a random variable. As a result, the uncertainty in the PDF representing the population consists of both the randomness inherent in the PDF representing the model distribution of the population and the uncertainty in the determination of the mean value (we assume that any other parameter needed to describe the PDF is constant). This leads to a Bayesian distribution (a.k.a. a compound distribution or a predictive distribution) representing the population which incorporates both sources of uncertainty simultaneously (e.g. Benjamin and Cornell 1970, p. 632):

\[ f_\theta(\theta) = \int_{-\infty}^{\infty} f_\theta(\theta|\omega)f(\omega)\,d\omega \]  

(3.6)

where \( f_\theta(\theta) \) is the PDF representing the Bayesian distribution of the population of \( \Theta \); \( f_\theta(\theta|\omega) \) is the PDF representing the model distribution; and \( f(\omega) \) is the PDF representing the distribution of the parameter describing \( f_\theta(\theta|\omega) \). The Bayesian distribution can be interpreted as a weighted average of all possible distributions \( f_\theta(\theta|\omega) \) as a function of the value of \( \omega \). Equation 3.6 can be used to assess both the à priori PDF \( (f_\theta^0(\theta)) \) and the à posteriori PDF \( (f_\theta^1(\theta)) \) of \( \Theta \) depending on whether the adopted \( f(\omega) \) is the à priori or à posteriori PDF of \( \omega \). As seen in Eq. 3.6, \( \omega \) will be integrated out and, as more information on \( \omega \) is incorporated (i.e. as the uncertainty in \( \omega \) is reduced), \( f_\theta(\theta) \) approaches the true PDF of \( \Theta \) (e.g. Benjamin and Cornell 1970, p. 633; Glickman and van Dyk 2007, pp. 330-331).

In this context, it is worth mentioning that it is much easier to solve Eqs. 3.4-3.6 if appropriate PDFs are chosen for the parameters (\( \omega \) in Eq. 3.6), with respect to that of the population (\( \Theta \) in Eq. 3.6). In the Bayesian terminology, such pairs are called conjugate distributions and the resulting updated à posteriori PDF of the parameters will have the same form (see e.g. Ang and Tang 2007, pp 365-367).

A special case of a Bayesian updating procedure is when both the population and the parameter \( \mu \) of the PDF are normally distributed (\( \sigma \) is known and constant). The resulting prior Bayesian distribution \( f^r_\theta(\theta) \) for a population \( \Theta \) can then be written as:

\[
\hat{f}^r_\theta(\theta) = N\left(\mu_\theta', \sqrt{\sigma_\theta^2 + \sigma^2_{\mu}}\right)
\]  

(3.7)

where \( \mu_\theta' \) is the à priori mean value of \( \Theta \); \( \sigma_\theta^2 \) is the (known) variance of \( \Theta \); and \( \sigma^2_{\mu} \) is the à priori variance of \( \mu_\theta' \). If \( n \) measurements from the distribution \( (\theta_1, \theta_2, ..., \theta_n) \) with an average \( \bar{\theta} \) are added, the mean value and the associated variance can be updated (e.g. Ang and Tang 2007, pp 361-362):

\[
\mu^\prime = \frac{\bar{\theta} \times \sigma^2_{\theta} + \mu_\theta' \times \sigma^2_{\mu}}{\sigma^2_{\theta} + \sigma^2_{\mu} / n}
\]  

(3.8a)

\[
\sigma^2_{\mu} = \frac{\sigma^2_{\theta} \times \sigma^2_{\mu} / n}{\sigma^2_{\theta} + \sigma^2_{\mu} / n}
\]  

(3.8b)
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The resulting updated PDF for the population \( \Theta \) can then be updated:

\[
\tilde{f}''_{\theta}(\theta) = N\left(\mu''_{\theta}, \sigma''_{\theta} + \sigma''_{\tilde{\mu}}\right)
\]  

(3.8c)

This is one example of the utilization of conjugate distributions, i.e. if the population \( (\Theta) \) is normally distributed (with known \( \sigma_{\theta}^2 \)) the conjugate distribution for \( \mu_{\theta} \) is also a normal distribution and as a result, the updated à posteriori distributions of both \( \Theta \) and \( \mu_{\theta} \) will be normally distributed. As illustrated in Example 3.1 below, this special case is relevant for geotechnical engineering applications. More information on the topic can be found in e.g. Ang and Tang (2007, pp. 360-365) or Baecher and Christian (2003, pp 78-83).

**Example 3.1**

Estimations of the PDFs of the population \( (S_u) \) and of the mean value \( (\mu_{S_u}) \) for use in the subsequent probabilistic analysis of a planned construction are intended. For reasons of simplicity, \( S_u \) and \( \mu_{S_u} \) are both assumed to be normally distributed. The influences of measurement and transformation errors are neglected and the intended construction will be much larger than the scale of fluctuation of \( S_u \) (i.e. \( \Gamma \approx 0 \)). The notations adopted in this example are presented in Table E-3.

### Table E-3. Notations adopted in the example

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_u )</td>
<td>population of the undrained shear strength</td>
<td>( \mu_{S_u} )</td>
<td>mean of ( S_u )</td>
</tr>
<tr>
<td>( S'_{u} )</td>
<td>à priori estimate of the population ( S_u )</td>
<td>( \mu'_{S_u} )</td>
<td>à priori estimate of the mean of ( S_u )</td>
</tr>
<tr>
<td>( S''_{u} )</td>
<td>à posteriori estimate of the population ( S_u )</td>
<td>( \mu''_{S_u} )</td>
<td>à posteriori estimate of the mean of ( S_u )</td>
</tr>
<tr>
<td>( COV_{S_u} )</td>
<td>coefficient of variation of ( S_u )</td>
<td>( \sigma_{\mu u}^2 )</td>
<td>à priori estimate of the variance of ( S_u )</td>
</tr>
<tr>
<td>( \sigma_{S_u}^2 )</td>
<td>variance of ( S_u )</td>
<td>( \sigma''_{\mu u}^2 )</td>
<td>à posteriori estimate of the variance of ( S_u )</td>
</tr>
<tr>
<td>( n )</td>
<td>number of independent test points</td>
<td>( \sigma'_p )</td>
<td>preconsolidation pressure</td>
</tr>
<tr>
<td>( q_{net} )</td>
<td>net cone resistance from a CPT</td>
<td>( S_{u} )</td>
<td>average value of ( S_u ) from measurements</td>
</tr>
</tbody>
</table>

An à priori estimate of the mean value of \( S_u \), based on the empirical relationship between \( s'_p \) and \( S_u \) (Eq. 2.11d), gave \( \mu'_{S_u} = 14 \text{ kPa} \). The variance of \( S_u \) \( (\sigma_{S_u}^2) \) was evaluated based on the suggested \( COV_{S_u} = 0.3 \) presented by Lee et al. (1983) and the estimated \( \mu'_{S_u} \), which lead to:

\[
\sigma_{S_u}^2 = (0.3 \times 14)^2 = 17.64 \text{ kPa}^2
\]

In this case, \( \sigma'_p \) was determined in \( n = 2 \) test points. Hence, the uncertainty in the determination of \( \mu'_{S_u} \) \( (\sigma''_{\mu u}^2) \) was evaluated as (cf. Eq. 2.5a):

\[
\sigma''_{\mu u}^2 = \frac{\sigma_{S_u}^2}{2} = \frac{17.64}{2} = 8.82 \text{ kPa}^2
\]

The variance in \( S_u \) arises from two sources; 1) the variation in \( S_u \) itself, i.e. \( \sigma_{S_u}^2 \); and 2) the uncertainty regarding the mean as described by the prior distribution, i.e. \( \sigma''_{\mu u}^2 \). This leads to the prior Bayesian distribution (cf. Eq. 3.7):
\[ \tilde{S}_u \in N \left( \mu_\tilde{S}_u', \sqrt{\sigma^2_{\tilde{S}_u} + \sigma^2_\mu} \right) \in N(17.64 + 8.82) \in N(14, 5.14) \]  
(Fig. E3a)

The distribution for the à priori estimation of the mean value of \( S_u \) (\( \mu_\tilde{S}_u' \)) is:

\[ \mu_\tilde{S}_u' \in N \left( \mu_{\tilde{S}_u}, \sigma_\mu \right) \in N(14, \sqrt{8.82}) \in N(14, 2.97) \]  
(Fig. E3b)

From measurements of \( q_{net} \) via 14 CPTs, the average \( \bar{S}_u = 17 \) kPa was evaluated (cf. Eq. 2.11c). Thereafter the data was processed via the following equations (Eqs. 3.8a-b):

\[
\begin{align*}
\mu_\tilde{S}_u'' &= \frac{\bar{S}_u \times \sigma^2_{\tilde{S}_u} + \mu_{\tilde{S}_u} \times \sigma^2_{\mu}}{\sigma^2_{\mu} + \sigma^2_{\tilde{S}_u}} = \frac{17 \times 8.82 + 14 \times 17.64/14}{8.82 + 17.64/14} \approx 16.6 \text{ kPa} \\
\sigma_\mu'' &= \sqrt{\frac{\sigma^2_{\tilde{S}_u}}{\sigma^2_{\mu} + \sigma^2_{\tilde{S}_u}}} = \sqrt{\frac{8.82 \times 17.64/14}{8.82 + 17.64/14}} = 1.05 \text{ kPa}
\end{align*}
\]

leading to the updated PDFs:

\[ S_u'' \in N \left( \mu_\tilde{S}_u'' \sqrt{\sigma^2_{\tilde{S}_u} + \sigma^2_\mu''} \right) \in N(16.6, 4.33) \]  
(Fig. E3a)

and

\[ \mu_\tilde{S}_u'' \in N \left( \mu_{\tilde{S}_u''}, \sigma_\mu'' \right) \in N(16.6, 1.05) \]  
(Fig. E3b)

As seen in Fig. E3a, the à posteriori population mean is shifted from the à priori estimate (\( \mu_{\tilde{S}_u}' = 14 \) kPa) towards the average value of the measurements (\( \bar{S}_u = 17 \) kPa). The Bayesian estimate of the à posteriori mean value (\( \mu_\tilde{S}_u'' \)) is an average of \( \mu_{\tilde{S}_u}' \) and \( \bar{S}_u \) weighted inversely with respect to their respective variances. Furthermore, the à posteriori distribution function for \( S_u \) becomes “narrower”, i.e. less uncertain after the introduction of the measured values (Fig E3a). The same applies for the estimation of the mean, where the introduction of the measurements has the effect of considerably decreasing the uncertainty in \( \mu_{S_u} \) (Fig. E3b).
3.2 The observational method and the Bayesian approach

The process of Bayesian updating as previously described allows the uncertainty in the determination of a geotechnical property or in the behaviour of a geotechnical construction to be decreased as additional information is included in the analysis. For instance, the actual behaviour of a construction and/or the subsoil on which the construction is founded can be controlled (measured) during the building phase, and this will successively increase the knowledge and hence reduce the associated uncertainties. The methodology of performing measurements during construction to verify design assumptions is linked with what is commonly known as the observational method (Peck 1969), and this method was adopted in the Veda embankment case. The analogy between the observational Method and the Bayesian approach is obvious and described by e.g. Baecher and Ladd (1997), Wu (2009, 2011) and Christian and Baecher (2011). The observational method is also one of the applicable design approaches in the Eurocode for Geotechnical Design, EN 1997-1 (CEN 2004), which states: “When prediction of geotechnical behaviour is difficult, it can be appropriate to apply the approach known as the ‘observational method’, in which the design is reviewed during construction.” The requirements for using the observational method are based on the ingredients suggested by Peck (1969). The basic ingredients (as interpreted by the author) are presented below and schematically in Fig. 3.4. In the figure, the analogy with the Bayesian approach is also highlighted.

1) Site investigation
   a. Assess the most probable conditions.
   b. Assess unfavourable deviations from these conditions.

2) Design phase
   a. Establish a design of the construction based on the most probable conditions using the estimated à priori values of properties etc. Show that there is an acceptable probability that the behaviour of the construction lies within acceptable limits.
   b. Calculate the possible behaviour of the construction assuming the unfavourable deviations from the probable conditions, i.e. by altering the estimated à priori values in an unfavourable manner.
   c. Select measurable quantities to be observed during construction to validate the prognosticated behaviour of the construction
   d. Select a course of action or modification of design to be adopted if the observations deviate significantly (unfavourable) from the prognosticated behaviour.

3) Construction phase
   a. Perform measurements during construction
   b. Analyse the actual conditions and/or behaviour using à posteriori values of properties etc.
   c. Modify the design accordingly.

The Veda embankment case (the main study object in the appended papers) was built adopting the observational method. Some aspects related to the adoption of the method for the purpose of controlling the stability of the embankment are presented in Paper II. In Paper IV, the Bayesian approach (in the form of the extended multivariate approach) for the evaluation of $S_u$ and its implications for the stability assessments of the embankment are discussed. A more thorough description of the adoption of the observational method in the Veda embankment case, including measurement results, can also be found in Müller (2010).
Figure 3.4: Illustration of the observational method in geotechnical engineering and the analogy with the Bayesian approach.
3.3 The multivariate approach

3.3.1 Pair-wise correlations and the multivariate approach

Practitioners in the field of geotechnical engineering often have access to valuable empirical knowledge of pair-wise correlations between quantities obtained from common site investigation methods and soil properties and between different properties of a soil material. These pair-wise correlations are in principle examples of regressions between two variables and they are thus associated with uncertainties in these regressions. For instance, and relevant to the studies presented in Papers II, III and IV, Mesri (1993) proposed that there is a correlation between the properties $S_u$ and $\sigma'_p$ in clays ($S_u \approx 0.26 \times \sigma'_p$), and Larsson et al. (2007) proposed that there are correlations between $S_u$ and quantities obtained by different site investigation methods for Swedish sulphide clays presented in Eqs. 2.11a-e. In a study by Ching and Phoon (2012), pair-wise data are compiled for $S_u$, $\sigma'_p$ and $\sigma'_v$ from clays found at 37 sites around the world. From a total of 345 data points, the Pearson correlation coefficient, $\rho$, between $S_u$ and $\sigma'_p$ was evaluated to 0.915, confirming the suggestion made by Mesri (1993). The well-established $SHANSEP$ concept (Eq. 2.11d) was also verified. The strong correlations $S_u \propto \sigma'_p$ and $S_u/\sigma'_v \propto OCR^m$ enable $S_u$ to be predicted in the absence of measurement data and these data can serve as a priori estimates in Bayesian analyses.

However, there are at least two limitations in many of the pair-wise correlations presented in the literature; 1) the correlations do not recognise any available additional information, i.e. from other sources; and 2) the correlations often permit only point estimates of the soil property and the uncertainty in the correlations is seldom provided (Ching and Phoon 2012). In cases where additional information that could be used for inference of the value of a property from other sources is available, e.g. from other site investigation methods, the Bayesian approach is useful. However, a general Bayesian analysis intended for such an inference based on multivariate test data (i.e. data arising from several sources) requires that data is available for all the variates in the same test point, or at least that the points are located in close proximity (Ching et al. 2010). In many practical engineering projects, this means that information from only a few of the available test points can be referred to in this context.

One way of overcoming the above limitations is by adopting the multivariate approach (Ching et al. 2008, 2010) where multivariate information is combined in a systematic way using Bayesian analysis strategies. If the property under investigation is $S_u$ (as in Papers III and IV), the graphical model (the Bayesian network) presented in Fig. 3.5 represents the interdependency between $S_u$ and the measured quantities (variates). The figure should be interpreted as follows: 1) the main factor influencing $S_u$ in clay is OCR (i.e. $\sigma'_p/\sigma'_v$) as indicated by the $SHANSEP$ concept; 2) the measured values of the variates $X_i$ (e.g. $\tau_{FY}$, $\tau_{EC}$ and $q_{net}$ in Paper III) are a consequence of $S_u$ and independent of OCR, i.e. once $S_u$ is known, OCR provides no further information on the variates; and 3) the variates are mutually independent and treated as several pieces of independent information on $S_u$. As will be explained in the next chapter, the pair-wise correlations between $S_u$ and OCR and between $S_u$ and $X_i$ can be combined via the multivariate approach and this will lead to updated PDFs of $S_u$ and $\tilde{S}_u$. 


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Figure 3.5. Graphical model of the interdependency between $S_u$ and OCR and between $S_u$ and the measured quantities (variates), based on Ching et al. (2010)

3.3.2 Formulation of the simplified multivariate approach

In the rigorous formulation of the multivariate approach, arbitrary PDFs representing the variates may be used. Solving the required equations is rather cumbersome and the simplified approach presented by Ching et al. (2010) is therefore attractive. However, the simplified approach assumes that all the variates are normally distributed, which is not always a valid assumption for geotechnical properties or for quantities derived from geotechnical site investigation methods as indicated in Table 2.2. For the cases presented in Papers III and IV, where the multivariate approach was used for the derivation of $S_u$, log-normal distributions were assumed for $S_u$ and for the involved variates. Hence, by taking the logarithm of the data of the variates, the simplified approach could be adopted (as the logarithms of log-normally distributed variates are normally distributed).

The simplified multivariate approach makes use of Eqs. 3.9a-b (cf. Ang and Tang 2007, p. 375; Fenton and Griffiths 2008, p. 55), which are valid for normally distributed multivariate data. The updated posterior average (expected) value and the corresponding variance of a property $Y$ are estimated on the basis of prior information $X_{prior}$ and measurements from a set of variates $X_1, X_2, \ldots, X_n$. Following the notations in Ching et al. (2010); $y = Y|X_{prior}$; and $x = [X_1, X_1, \ldots, X_n]^T$; the following equations are valid provided that all the variates are jointly normally distributed:

$$E[y|x] = E[y] + Cov(y,x) \times Var(x)^{-1} \times (x - E[x]) \quad (3.9a)$$

$$Var[y|x] = Var[y] - Cov(y,x) \times Var(x)^{-1} \times Cov(y,x)^T \quad (3.9b)$$

where $E[y|x]$ and $Var[y|x]$ are the posterior expected value and variance of $Y$ conditional on $X_{prior}$ and $x$; $E[y]$ and $Var[y]$ are the prior expected value and variance of $Y$ conditional on $X_{prior}$; $Cov(y,x)$ is the covariance vector between $y$ and $x$; and $Var(x)$ and $E[x]$ are respectively the covariance matrix and expected value of $x$.

These equations are solved rather easily and allow for an unlimited number of variates to be introduced into the analysis (cf. the appendices in Paper III and IV). It should be noted that the uncertainty defined by $Var[y]$ represents the total uncertainty in the pair-wise correlation
between $Y$ and $X_{prior}$ and that the uncertainty represented by the elements in $Var[x]$ represents the total uncertainties in the correlations between $Y$ and the $X_i$s (as defined by Eq. 2.19b).

The procedure for assessing the total uncertainties in the pair-wise correlations between $S_u$ and the variates involved (Ch. 2) before adopting the simplified multivariate approach for the evaluation of $\tilde{S}_u$ and $COV_{\tilde{S}_u}$, i.e. the proposed extended multivariate approach, is described in Paper III for the site investigation in the Veda area and for the control of the stability during construction of the staged Veda embankment in Paper IV. In Fig. 3.6, the procedure is shown schematically as an extension of Fig. 2.7.

If the standard deviation of $S_u$ (i.e. the population) and the evaluated $\tilde{S}_u$ and $COV_{\tilde{S}_u}$ are known, it is possible to assess an updated $PDF$ of $S_u$ based on Eq. 3.6.

3.3.3 Comments on the multivariate approach

Intuitively, the uncertainty in the determination of a geotechnical property should decrease if assessments based on information from several sources are available. The multivariate approach enables information from an unlimited number of variates to be combined in the derivation of a representative $PDF$ for a material property (or some other quantity). In the process, the average values derived from the pair-wise correlations between a property $Y$ and the involved variates $X_1, X_2, \ldots, X_n$, and the uncertainty in these correlations are implicitly treated and weighted on the basis of the relative levels of uncertainty in the correlations between $Y$ and the variates. As a result, objective values of $\bar{Y}$ and $COV_{\bar{Y}}$ are assessed, and in the process $COV_{\bar{Y}}$ is also reduced. This is discussed in Papers III and IV, and also in Ch. 5.3 and 5.4.

A feature of the extended multivariate approach is that it implicitly reduces the individual transformation uncertainties, i.e. the uncertainties in the pair-wise correlations. As seen in Eq. 2.19b, the transformation uncertainty is a bias and cannot be reduced by increasing the number of test points or by spatial averaging. For the cases presented in Paper III (Fig. 7 in the paper) and Paper IV (Fig. 6 in the paper), a vast part of the total uncertainty in the determination of $\tilde{S}_u$ from the separate variates arises from the transformations of the measured quantities. In the papers, it is shown that the evaluated $COV_{\tilde{S}_u}$ will always decrease as the number of variates taken into account is increased, regardless of the total uncertainty in the correlations based on the separate variates. Hence, evaluations via the extended multivariate approach reduces the $COV_{\tilde{S}_u}$ below the $COV_{trans,X}$ of the individual variates.

The extended multivariate approach can also be of help when a site investigation is planned. Paper III shows that the effect on the level of uncertainty in the determination of a property of different designs of a site investigation can be assessed. The optimal design of a site investigation, i.e. the number of test points and the number of test methods (variates) employed to reach a certain level of uncertainty with the lowest effort, can be derived based on the type of analysis presented in the paper.

However, care should be taken when assigning the individual uncertainties in the pair-wise correlations, as the result of a multivariate analysis is biased towards the relatively least uncertain correlation (i.e. site investigation method).
Chapter 3 – Bayesian updating and the multivariate approach

Figure 3.6. Illustration of the extended multivariate approach
Chapter 4 – Probabilistic analysis of embankment stability

Summary: This chapter discusses the different characteristics, strengths and weaknesses of deterministic and probabilistic stability analyses. Some common probabilistic approaches are briefly presented and the FOSM approach and Taylor series approximation are described in greater depth. The application of model uncertainties to probabilistic stability analyses is also touched upon.

4.1 Probabilistic vs. deterministic analyses

The reasons for performing analyses of geotechnical problems are basically to assess the risks involved and to assist in the process of decision-making, i.e. preparing a design of a construction in order to avoid failure (interpreted from statements made by Whitman 2000). In this context, failure does not necessarily mean a dramatic event such as a landslide or a collapsed sheet pile wall, but can also be interpreted as a detrimental deformation or, as stated by Leonards (1975), any “unacceptable difference between expected and observed performance”. In general terms, failure occurs in a soil material if the stresses transferred to the soil material from an applied load or load effect, \( Q \), are higher than a specified allowable stress (i.e. resistance), \( R \). Because of the inevitable uncertainties involved in a geotechnical analysis, a required safety margin, \( M \), or a required factor of safety, \( F \), against failure must be present:

\[
M = R - Q \tag{4.1a}
\]

\[
F = \frac{R}{Q} \tag{4.1b}
\]

The preceding chapters focus on explaining the nature of the uncertainty in the determination of soil properties and on how to assess this source of uncertainty in an appropriate manner. This is an important part of any engineering project involving soil material, particularly when a probabilistic approach to the problem is adopted, i.e. when a reliability-based analysis, RBA, is performed. In many cases, the uncertainties related to the soil properties constitute a major part of the total uncertainty in the results of an analysis, but uncertainties arising from other sources may also have a noteworthy impact (discussed in Ch. 2.4 and 4.3). In an analysis, these uncertainties must be managed in some way. Basically, two different strategies serve this purpose:

1) The traditional deterministic approach, also denoted allowable stress design (Baecher and Christian 2003, pp 433-436), where the respective average values, often referred to as characteristic values, are assigned to \( R \) and \( Q \). All of the uncertainties involved in the analysis are combined in \( M \) or \( F \). The magnitudes of the required values of \( M \) and \( F \) typically depend on the type of construction (embankment, sheet pile wall etc.), the type of soil material (clay, sand etc.) and the type of ultimate limit state analysis
performed (Spencer’s method of slices, Terzaghi’s bearing capacity equation etc.). The required values of $M$ or $F$ should preferably reflect the uncertainties involved in an analysis and the consequences of failure. In reality, design codes or similar documents often state the applicable values for use in deterministic analyses, and do not always consider the degree of uncertainty.

2) The probabilistic approach, where the average values of the quantities involved in an analysis are accompanied by the corresponding uncertainties, i.e. the PDFs and any correlations between the quantities. The statistical characteristics of the involved quantities are then processed through an appropriate probabilistic method (discussed in Ch. 4.2), resulting in a probability of failure, $P_f$, generally $P_f = P(M < 0)$ or $P_f = P(F < 1)$, or a value of the safety index, $\beta$:

$$\beta = \frac{\mu_M}{\sigma_M} \quad (4.2a)$$

$$\beta = \frac{\mu_F^{-1}}{\sigma_F} \quad (4.2b)$$

where $\mu_M$, $\sigma_M$, $\mu_F$ and $\sigma_F$ are the respective mean values and standard deviations of $M$ and $F$ obtained from the probabilistic analysis. Both $M$ and $F$ describe the performance of a geotechnical structure, so they are referred to as the performance functions. If the form of the PDF of the performance function is known or can be assumed to have a certain form, $P_f$ can be derived from $\beta$ and vice versa. The connections between $\beta$ and $P_f$ for three common distributions are presented in Fig. 4.1 (from Baecher and Christian 2003, p. 307).

![Figure 4.1. Probability of failure vs. safety index (from Baecher and Christian 2003, p. 307)](image-url)
As seen, the fundamental difference between the two strategies is that only the average values (or characteristic values) of \( R \) and \( Q \) are involved in the deterministic approach, whereas the other statistical characteristics are also incorporated in a probabilistic approach. As previously stated, the required value of \( M \) or \( F \) for use in a deterministic analysis should be project-specific, but it is more likely that the engineer is obliged to refer to some kind of design code or similar document, e.g. TKGeo 11 (STA 2011). Therefore, the deterministic approach basically enables relative comparisons between different designs to be applied to a specific problem in a specific project, e.g. for comparisons of the different widths of supporting berms in an embankment case. Such an analysis says very little about the probability of failure within the construction’s design life, or about the relative influences of the various uncertainties incorporated in \( R \) and \( Q \) on the calculated \( M \) or \( F \). As pointed out by Duncan (2000), it is not logical to apply the same value of \( F \) for conditions with widely varying degrees of uncertainty, e.g. in different projects. In order to improve the basis for decision-making and the evaluation of risks, an approach based on reliability is preferable. However, as argued by e.g. Duncan (2000) and El-Ramly et al. (2002), reliability-based approaches are not routinely adopted in the geotechnical engineering profession, for four main reasons; 1) engineers are not very familiar with the concepts of reliability theory and statistics; 2) there is a common misinterpretation that a RBA (i.e. probabilistic approach) requires more data, time and effort than a deterministic analysis; 3) there are only a few published studies illustrating the implementation and benefits of a RBA; and 4) acceptable probabilities of failure are ill-defined and the link between probabilistic and deterministic strategies is absent. Furthermore, as discussed in Ch. 4.3 and as argued by e.g. Duncan (2000), Whitman (2000), Christian (2004) and El-Ramly et al. (2005), both deterministic and reliability-based analyses should be made, mainly because there are uncertainties contributing to the total uncertainty of the performance function that arise from sources that may be difficult to assess in a RBA (cf. Ch. 2.4 and 4.3) but are more readily incorporated in the required values of \( M \) or \( F \).

In this context, two hybrids of the two strategies commonly used in geotechnical design are worth mentioning, the Partial Factor Method, PFM (e.g. Olsson and Stille 1984; Bengtsson et al. 1991; CEN 2002) and the Load and Resistance Factor Design, LRFD (e.g. AASHTO 1997). Analyses based on either of these two methods are essentially performed in a deterministic manner, but both \( R \) and \( Q \) are factorised before the analysis is carried out. In principle, the load or load effect is increased and the resistance is decreased to a level meeting the safety demands. The factors applied to \( R \) and \( Q \) can be chosen based on the uncertainty in the respective quantity, on the required level of safety (i.e. what probability of failure can be accepted) and on how sensitive the analysis is to the uncertainty in the respective variables.

The following example illustrates the fundamentals and differences between a deterministic approach and a probabilistic approach.

**Example 4.1**

The safety factor against shear failure of a slope is to be estimated (Eq. 4.1b). It is assumed that the characteristics of the resistance \( R \) and the load effect \( Q \) along a potential shear surface have been determined. Both variables are assumed to be log-normally distributed with parameters according to Table E4 and uncorrelated. The distribution functions are shown in Fig. E4a.
Table E4. Parameters for R and Q

<table>
<thead>
<tr>
<th>Parameter</th>
<th>R</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>25 kPa</td>
<td>15 kPa</td>
</tr>
<tr>
<td>COV</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>$\lambda^*$</td>
<td>3.2</td>
<td>2.7</td>
</tr>
<tr>
<td>$\xi^*$</td>
<td>0.20</td>
<td>0.10</td>
</tr>
</tbody>
</table>

$^* \lambda = \ln(\mu) - 0.5\xi^2$

$^* \xi = \ln(1 - \text{COV}^2)$

![Diagram](image)

Figure E4. Probability density functions for; a) R and Q; b) ln(R) and ln(Q); c) F; d) ln(F)

The deterministic factor of safety $F$ is simply $F = \mu_R / \mu_Q = 25/15 \approx 1.67$, which is considered a sufficient factor of safety in most practical cases. However, according to the probabilistic approach, it can be seen in Fig. E4a that there is a possibility for $Q > R$ (the dashed area in the figure), i.e. $F < 1$. As argued by e.g. Baecher and Christian (2003, p 306), it is easier to assess $\beta$ via Eq. 4.1a than via Eq. 4.1b. Taking the logarithms of $R$ and $Q$ (which are then normally distributed, Fig. E4b), the performance function can be written as:
\[
\ln(F) = \ln(R) - \ln(Q) \quad \text{(cf. Eq. 4.1a)}
\]

and the mean value of \(\ln(F)\), \(\mu_{\ln(F)}\), can then be calculated as:

\[
\mu_{\ln(F)} = \mu_{\ln(R)} - \mu_{\ln(Q)} = \lambda_R - \lambda_Q = 3.2 - 2.7 = 0.5
\]

and the standard deviation, \(\sigma_{\ln(F)}\), as (e.g. Ang and Tang 2007, p.161):

\[
\sigma_{\ln(F)} = \sqrt{\sigma_{\ln(R)}^2 + \sigma_{\ln(Q)}^2} = \sqrt{\xi_R^2 - \xi_Q^2} = \sqrt{0.2^2 + 0.1^2} \approx 0.224
\]

and \(\beta\) as (see also Fig. E4d):

\[
\beta = \frac{\mu_{\ln(F)}}{\sigma_{\ln(F)}} = \frac{0.5}{0.224} \approx 2.23 \quad \text{(cf. Eq. 4.2a)}
\]

From Fig. 4.1, \(P_f\) is approximately 0.015, i.e. \(P[\ln(F) < 0] \approx 0.015\), which is a relatively high probability of failure. In this context it is interesting to note that an evaluation based on Eq. 4.1b (i.e. not taking the logarithms of \(R\) and \(Q\)) via Monte Carlo simulation (see Ch. 4.2.1), would lead to the PDF of \(F\) (rather than \(\ln(F)\)) shown in Fig. E4c. From these simulations, \(P[F < 1] \approx 0.028\) and, assuming a log-normal distribution for \(F\) with \(\text{COF}=0.15\), \(\beta_{MC}=1.7\) was obtained.

In this example the fundamental differences between the deterministic and the probabilistic approaches are illustrated and, as seen, a relatively high value of \(F\) does not automatically lead to a relatively high value of \(\beta\) (or a low \(P_f\)). It can also be noted that different probabilistic approaches led to different values on \(\beta\) and \(P_f\), cf. the areas to the left of the y-axes (at respectively \(F = 1\) and \(\ln(F) = 0\)) and below the curves describing the PDFs of \(\ln(F)\) and \(F\).

4.2 Probabilistic stability analysis

4.2.1 General

In order to analyse an engineering problem via calculations, the problem needs to be described in terms of a suitable mathematical algorithm. In a probabilistic analysis, these equation(s) are often denoted the performance function, \(g\), dependent on several input variables \((X_1, X_2, ..., X_n)\) evaluated at some point \((x_1, x_2, ..., x_n)\):

\[
g = g(x_1, x_2, ..., x_n) \quad \text{(4.3)}
\]

Equations 4.1a-b represent two examples of simple performance functions. In order to calculate \(\beta\), the mean, \(\mu_g\), and the standard deviation, \(\sigma_g\), must be assessed. The approach described in Example 4.1 is an illustration of these assessments where the solutions for \(\beta\) and \(P_f\) were derived rather easily. The average values and the uncertainties in \(R\) and \(Q\) were propagated through the performance function, \(\ln(F)\) (or \(F\)), and the mean value and standard deviation were evaluated. This is a simple case of error propagation which, in principle, is
the purpose of performing a probabilistic analysis, i.e. to estimate the mean value and the uncertainty in the performance function \( g \) given the average values and uncertainties in \( x_1, x_2, ..., x_n \). In most cases however, the performance function is not as simple as in this example. As a consequence, some probabilistic approach must be adopted in order to solve more complex formulations. According to Christian (2004), the most commonly used approaches are:

- **The First order Second Moment (FOSM) method.** In this approach, \( \mu_g \) and \( \sigma_g \) are approximated using the first order terms of a Taylor series. Since the FOSM method was adopted in the probabilistic stability analyses in Paper IV, the procedure is described in more detail in Ch. 4.2.2.

- **The Second order Second Moment (SOSM) method.** Similar to FOSM with the addition of the second order terms of a Taylor series.

- **The Point Estimate (PE) method.** The method proposed by Rosenblueth (1975) obtains the statistical moments (i.e. mean, variance) of \( g \) by evaluating \( g \) at a set of discrete points. In principle, the PE method omits the need for the sometimes cumbersome partial derivations of \( g \) needed in the FOSM and SOSM approaches but the calculation effort becomes heavy when the number of uncertain variables in \( g \) is large.

- **The Hasofer-Lind or First Order Reliability Method (FORM) method.** The principal difference between the FOSM and the FORM approaches (Hasofer and Lind, 1974) is the definition of \( \beta \). In the former, \( \beta \) is defined as the quotient between \( \mu_g \) and \( \sigma_g \) whereas in the latter, \( \beta \) is interpreted geometrically. Two of the drawbacks of the FOSM approach are omitted by using the FORM approach; 1) the FOSM approach assumes that the partial derivative \( \partial g / \partial x_i \) is constant and independent of the value of \( x_i \) at which the derivative is taken, which may not always be the case; and 2) the PDF of \( g \) is not assessed and it therefore has to be assumed in order to evaluate \( P_f \) from \( \beta \). More information can be found in e.g. Baecher and Christian (2003, Ch. 16) and Al-Naqqashabandy (2012, pp. 21-24).

- **Monte Carlo (MC) simulation.** A MC simulation primarily consists of the creation of a large number of realisations, \( N \), of the involved uncertain variables \( (X_1, X_2, ..., X_n) \) and processing these realisations, one at a time, through the algorithm for \( g \). From the results of the realisations, the values of \( P_f, \mu_g, \sigma_g \) and the PDF of \( g \) are obtained. In contrast to the other methods, a MC simulation requires a knowledge of the PDFs of all of the involved variables, which can be a potential obstacle, but which in return yields the PDF for the performance function which can be important for low probabilities of failure (El-Ramly et al. 2002). Drawbacks of the method are that it does not evaluate the relative contributions of the different variables and that it requires a large number of realisations. The average probability of failure is evaluated as the ratio of the number of realisations leading to failure \( (\nu) \), i.e. \( g \leq 0 \), to the total number of realisations, \( N \):

\[
P_f = \frac{\nu}{N} \quad (4.4a)
\]

and the uncertainty in the evaluation of \( P_f \) can be derived as:
4.2.2 Taylor series and the FOSM approach applied to the method of slices formulations

The FOSM approach makes use of the first order terms in a Taylor series (e.g. Li and Lumb 1987; Baecher and Christian 2003, pp. 311-313; Fenton and Griffiths 2008, pp. 30-32) for the assessments of $\mu_g$ and $\sigma_g$ of the performance function defined by the input variables $(X_1, X_2, \ldots, X_n)$:

$$
\mu_g \approx g(\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n}) \quad \text{(4.5a)}
$$

and

$$
\sigma_g^2 \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{X_iX_j} \sigma_{X_i} \sigma_{X_j} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \quad \text{(4.5b)}
$$

where $\mu_{x_i}$ is the mean value of $X_i$; $\rho_{X_iX_j}$ is the correlation coefficient between $X_i$ and $X_j$; and $\sigma_{X_i}$ is the standard deviation of $X_i$. If the variables are uncorrelated, Eq. 4.5b can be simplified:
\[ \sigma_y^2 \approx \sum_{i=1}^{n} \sigma_{N_i}^2 \left( \frac{\partial N_i}{\partial x_i} \right)^2 \]  \hspace{1cm} (4.5c)

In any analysis of slope stability or embankment stability, Eq. 4.1b is usually adopted, where the formulation of the factor of safety \( F = F(R, Q) \) represents the performance function. In analytical calculations, any of the existing formulations of the method of slices is typically used to calculate \( F \), see e.g. Duncan and Wright (2005) for details of the formulations. The performance functions defined by most of the method of slices formulations do not allow a direct calculation of \( F \), since the equations are nonlinear. In Paper IV, stability analyses of the non-circular slip surface (Fig. 1c in the paper) were performed in terms of total stress assuming undrained conditions, using Janbu’s simplified formulation (Janbu 1954). Janbu’s simplified formulation considers only force equilibrium of the slices, see the generalised sketch in Fig. 4.2. The performance function \( F \) can then be written as:

\[ F = \sum_{i=1}^{n} \frac{(S_{u,i} \times l_i \times \cos \alpha_i + N_i \times \tan \phi_i \cos \alpha_i)}{\sum_{i=1}^{n} N_i \times \sin \alpha_i} \]  \hspace{1cm} (4.6a)

where

\[ N_i = \frac{W_i - S_{u,i} \times l_i \times \sin \alpha_i}{\cos \alpha_i + \tan \phi_i \sin \alpha_i} \]  \hspace{1cm} (4.6b)

and \( S_{u,i} \) is the undrained shear strength in the soil material at the base of slice \( i \); \( l_i \) is the base length of slice \( i \); \( \alpha_i \) is the inclination of the base of slice \( i \); \( W_i \) is the vertical stress acting at the base of slice \( i \) induced by the “weight” of the soil material; and \( \phi_i \) is the friction angle in the soil material at the base of slice \( i \). Since the expression in Eq. 4.6a is not linear, an iterative procedure must be adopted in order to solve for \( F \).

The assessment of \( \mu_F \) is rather straightforward, being based on the mean values of the variables (cf. Eq. 4.5a), and it is therefore equal to the deterministic factor of safety, \( F_{det} \). The assessment of \( \sigma_F \) is rather cumbersome, since the expression for \( F \) is nonlinear and iteration is required, \( F \) can therefore not be differentiated directly. The partial derivatives needed must be found numerically. One way of doing this is explained by Baecher and Christian (2003, p.
where each variable \( x_i \) in Eqs. 4.5b-c is increased and reduced by a small increment, \( \varepsilon_i \), and the partial derivative \( \frac{\partial F}{\partial x_i} \) at the mean values \( (\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_n}) \) is assessed:

\[
\frac{\partial F}{\partial x_i} = \frac{1}{2\varepsilon_i} \left[ F\left(\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_i} + \varepsilon_i, ..., \mu_{x_n}\right) - F\left(\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_i} - \varepsilon_i, ..., \mu_{x_n}\right) \right]
\]  

(4.7)

Thereafter \( \sigma_F \) is assessed via Eq. 4.5b (or 4.5c) and \( \beta \) can be calculated (Eq. 4.2b). This procedure was adopted in the FOSM analyses presented in Paper IV.

One advantage of the FOSM approach is that the influence (sensitivity factor), \( \alpha_i \), of each variable on the performance function can easily be assessed as:

\[
\alpha_i = \frac{\frac{\partial F}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\right)^2}}
\]  

(4.8)

and the contribution to the uncertainty in the performance function, \( d\text{Var}_F \), can be assessed (suggested by e.g. Christian et al. 1994) as:

\[
d\text{Var}_F = \frac{\left(\frac{\partial F}{\partial x_i}\right)^2 \times \sigma_{x_i}^2}{\sum_{i=1}^{n} \left[\left(\frac{\partial F}{\partial x_i}\right)^2 \times \sigma_{x_i}^2\right]}
\]  

(4.9)

This procedure was adopted in the analyses presented in Paper V.

### 4.3 Model uncertainties applied to probabilistic stability analysis

In Ch. 2.4, it was stated the uncertainties inherited from other factors than the determination of the geotechnical properties affect geotechnical problems, such as the mathematical models adopted for the description of soil behaviour and the failure mechanism, environmental changes and human errors.

For the case of embankment stability, Azzouz et al. (1983) suggest that the error in calculated safety factors assuming a plane strain situation (as in the method of slices formulations), \( F_{PS} \), rather than a full 3D analysis, \( F_{3D} \), based on 18 cases of embankment failures, is on average 1.11 with a standard deviation of 0.06. In the papers by e.g. Li and Lumb (1987), Chowdhury and Tang (1987) and Hassan and Wolff (1999, 2000), the uncertainty related to the choice of the method of slices formulation is discussed and it is also suggested that the location of the critical shear surface may not be the same when assessed in a deterministic stability analysis as when it is assessed in a probabilistic analysis. Hassan and Wolff (2000) compared results from slope stability analyses for three embankment dams and found that the location of the critical shear surface had a greater impact on the results of the calculations than the choice of formulation for the method of slices. Low (1996, 2001, 2003), Low and Tang (1997a, 1997b, 2007) and Xue and Gavin (2007) present analytical algorithms solving for this shortcoming, where the search for the critical shear surface and the evaluation of \( \beta \) are coupled. Another way of overcoming this problem is by coupling the probabilistic approach with numerical methods such as the finite element method, where the constraint of choosing a critical shear surface is obviated (e.g. Griffiths and Lane 1999; Griffiths a Fenton 2004; Xu and Low 2006; Griffiths et al. 2009; Huang et al. 2010).
In the studies by Christian et al. (1994), Liang et al. (1999) and Cassidy et al. (2008), model uncertainties (errors) were incorporated in the probabilistic stability analyses which they presented. In their analyses they treated: 1) the plane strain/3D effect, according to the suggestions of Azzouz et al. 1983; 2) the uncertainty in the location of the critical shear surface (overestimate $F$ by 5% with a standard deviation of 5%); and 3) the rounding error (contributing to the uncertainty by 2%). The model error was then evaluated statistically as, $\mu_{mod} = 1.11 \times 0.95 \approx 1.05$; $COV_{mod} = \sqrt{(0.06/1.11)^2 + (0.05/0.95)^2 + 0.02^2} \approx 0.078$. In these studies, $\mu_F$ assessed from the probabilistic analyses was therefore increased by a factor of 1.05, i.e. $\mu_{F|mod} = \mu_F \times 1.05$, and the uncertainty, in the form of $COV_F$, was increased according to Eq. 2.20, i.e. $COV_{F|mod}^2 = COV_F^2 + COV_{mod}^2$. Thereafter, $\beta$ was evaluated as:

$$\beta = \frac{\mu_{F|mod}^{-1}}{\mu_F|mod \times COV_{F|mod}} \quad (4.10)$$

However, care should be taken when adopting this procedure as both $\mu_F$ and $COV_{F|mod}$ are increased, which may lead to unexpected results. In the studies by Christian et al. (1994) and Cassidy et al. (2008), the introduction of model errors led to higher values of $\beta$ in some cases.

In Paper IV, $\bar{S}_u$ at different stages of construction and the associated uncertainty in the determination, $COV_{\bar{S}_u}$, were assessed for different sets of variates, i.e. for different site investigation methods. Probabilistic stability analyses were made at two stages in the construction process of the Veda embankment adopting the values of $\bar{S}_u$ and $COV_{\bar{S}_u}$. Some of the results are presented in Fig. 4.3 (Fig. 9b in Paper IV). For the analyses of $\beta$ based on the variate set ‘$\sigma_u', P$”, it can be seen that the value of $\beta$ is about 5, which can be interpreted as meaning that $P(F < 1) = 2.9 \times 10^{-6}$ (Fig. 4.1 assuming that $F$ is normally distributed). From the point of view of embankment stability, this is a rather low and fully satisfactory probability of failure. In the analyses, only uncertainties in the load (i.e. the unit weight of the embankment fill), the friction angle of the fill material and $S_u$ in the clay were treated stochastically, hence no model uncertainty was incorporated. However, the introduction of a rather modest value of $COV_{mod} = 0.05$ would mean that the value of $\beta$ would drop to approximately 4.2, corresponding to $(F < 1) = 1.5 \times 10^{-5}$, i.e. the assessed probability of failure would increase by a factor of 50. In Fig. 4.3 the effect on $\beta$ of introducing $COV_{mod} = 0.05$ is represented by the lines placed across the bars.

This finding agree with the arguments of e.g. Duncan (2000), Whitman (2000), Christian (2004) and Doorn and Hansson (2010), who suggest that probabilistic and deterministic analyses should be performed in parallel, as the uncertainties arising from sources that contribute to the model uncertainty are difficult to assess and they may have a significant influence on the results of a probabilistic analysis, whereas these types of uncertainties are implicitly treated in the required $F_{det}$. 

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Figure 4.3. Effect on the calculated $\beta$ of introducing $COV_{mod} = 0.05$ (indicated by the lines across the bars)
Chapter 5 – Summary of the appended papers

Summary: In this chapter a synopsis of each of the appended papers is presented in chronological order. The first two papers were also appended to the licentiate thesis presented by the author in 2010 (Müller 2010).

5.1 Paper I – Hydraulic conductivity and the coefficient of consolidation of two sulphide clays in Sweden

The theories used for the design of vertical drains described in e.g. Müller (2010, Ch. 3) require a knowledge of the hydraulic conductivity and of the coefficient of consolidation in the horizontal direction. Conventional laboratory testing of clay, e.g. constant rate of strain oedometer tests, CRS, offers an interpretations of these properties in the vertical direction only. Hence, one is often left to make assumptions, based on empirical knowledge of a certain type of clay or simply based on a more or less qualified guess, when a design of PVDs are made.

This paper presents results from a series of CRS tests performed on samples of sulphide clay retrieved from two different sites where large infrastructure projects involving embankments on vertically drained sulphide clay were being carried out. A methodology to investigate the hydraulic conductivity in both the vertical \( k_v \) and horizontal \( k_h \) directions via conventional CRS tests with vertical drainage of excess pore pressures is proposed. In order to evaluate \( k_h \), samples were trimmed from the sides of the sample cylinders into oedometer rings with a slightly smaller diameter (45 mm) than those conventionally used (50 mm), see Fig. 5.1. On samples trimmed from the same testing tubes, both conventional vertical tests and horizontal tests were conducted. The main purpose of the tests was to investigate the anisotropy of the hydraulic conductivity and consequently the coefficient of consolidation (\( c_h \) and \( c_v \)) via the ratios \( k_h/k_v \) and \( c_h/c_v \).

The laboratory-evaluated values of \( c_h \) were also compared with evaluations based on in situ measurements of the consolidation rates. In addition, as a number of parallel tests were performed on samples of the “same” clay, it was possible to study the variations in the evaluated properties. The validity of previously presented empirical correlations, linking the hydraulic conductivity with other properties of clay, was also investigated.

The results of the laboratory tests show that the anisotropy in both the hydraulic conductivity and the coefficient of consolidation was low. For the sulphide clays tested, an average value of \( k_h/k_v \approx 1.3 \) was found (Fig. 5.2a).

The scatters in the values of the initial hydraulic conductivities (\( k_{v0} \) and \( k_{h0} \)) obtained were significant. The quotients of maximum to minimum values from corresponding tests were between 1.7 and 3.3 with maximum deviations from the mean of 34% to 110% (Fig. 5.2b). Regarding the coefficients of consolidation, the scatter was even larger. Quotients of
maximum to minimum values from corresponding tests between 1.6 and 5.0 were obtained. These findings underline the virtually impossible task of determining these properties for design purposes based on results from a single test or only a few tests. The paper concludes that several parallel tests should be made or that partial factors of safety should be introduced in the design.

Figure 5.1. Preparing the horizontal samples in the laboratory; a) the cutting device; b) trimming; c) the cutting device with the clay sample; d) mounting the clay sample in the oedometer ring       Photos I-M Kaller

Figure 5.2. a) Measured initial horizontal hydraulic conductivity vs. initial vertical hydraulic conductivity; b) Measured initial hydraulic conductivities vs. initial void ratio
5.2 Paper II – Stability for a high embankment founded on sulphide clay

Controlling the stability against shear failure is a very important issue to address in both the design and the building phase of an embankment construction. A case study describing how the stability was handled in the Veda embankment project is presented. The embankment was built in stages on vertically drained sulphide clay, taking into account the increase in undrained shear strength, $S_u$, as a result of consolidation between the stages. As previous experience from building high embankments on sulphide clay and empirical knowledge of vertical drains in sulphide clay was limited, a trial embankment was built and the observational method was employed in the project.

Building an embankment in stages allows some consolidation to occur during construction and this increases the effective stresses in the clay. If the effective stresses supersede the maximum past preconsolidation pressure in the clay, the shear strength in the clay increases. In the project, this was taken into account in the design phase by using the undrained strength analysis (USA) methodology proposed by Ladd (1991). In order to adopt the USA methodology, reliable predictions of the increase in effective stress during the building stages are required, i.e. the rate of the dissipation of excess pore pressure must be anticipated. Furthermore, reliable predictions of the increase in $S_u$ in relation to the present preconsolidation pressure $\sigma'_p$ must be made, i.e. the ratio $S_u/\sigma'_p$ must be established for the clay. Neither of these prerequisites was entirely fulfilled due to uncertainties in predicting the real behaviour of the sulphide clay. Based on predictions of $S_u$, the stability of the Veda embankment at the various construction stages was evaluated. In accordance with the observational method, the design predictions were controlled during the building phase of the embankment via continuous measurements of pore pressures and in situ tests of $S_u$ at two construction stages.

The paper discusses the importance of assessing a certain quantity, e.g. a material property, from several independent sources of measurement. In this case, the short-term stability of the embankment was of concern; hence, $S_u$ and the increase in $S_u$ in the sulphide clay due to consolidation were of paramount importance. During construction, assessments of $S_u$ at various stages were made by three different methods, directly via $FV$ tests and CPTs and indirectly via piezometer measurements. The measurements were also made at six different locations (Fig. 9 in Paper II). Comparison of the $S_u$ profiles evaluated in this way with the design predictions made it possible to evaluate the stability of the construction during the building process. The large number of measurements made during construction showed that the ratio $S_u/\sigma'_p = 0.25$ is valid for a large stress interval (Fig. 5.3). As shown, the scatter is considerable; hence, a single test or a few tests can be very misleading.
5.3 Paper III – Extended multivariate approach for uncertainty reduction in the assessment of undrained shear strength in clays

Predicting the probability of failure (i.e. the stability) of an embankment founded on soft soil is a common task in practical geotechnical engineering, and probabilistic methods allow rational assessments to be made of the uncertainties involved in such analyses. A major task in this context is to characterise and assess the undrained shear strength, $S_u$, in the ground and the associated uncertainty. Evaluations from site investigations usually involve significant assumptions and empiricism, so it is beneficial to cross-validate interpretations from different site investigation methods in order to reduce the uncertainty. In this paper, the average value, $\bar{S}_u$, for the Veda sulphide clay was assessed on the basis of an empirical relationship between $S_u$ and $\sigma'_p$ and from measurements via FV tests, FC tests and CPTs and the empirical transformations (i.e. pair-wise correlations) presented in Eqs. 2.11a-d.

The extended multivariate approach (Ch. 3.3) was adopted for the assessments of $\bar{S}_u$ and the associated $COV_{\bar{S}_u}$, from different combinations of the a priori prediction (based on OCR) and the measured variates ($\tau_{FV}$, $\tau_{FC}$ and $q_{net}$). In the light of the Veda embankment case, some important features of the extended multivariate approach are discussed and an extension to this approach is proposed, whereby the total uncertainty in the variates due to spatial averaging is assessed prior to its adoption. The influence of the number of boreholes, the number of different test methods (variates) and the uncertainty in the prior prediction used in the extended multivariate approach on the evaluated $COV_{\bar{S}_u}$ is analysed.

In addition, some specific findings for the variates in Veda sulphide clay related to probabilistic stability analysis, such as the autocorrelation structure, the scales of fluctuations and the coefficient of variation for the variates, $\tau_{FV}$, $\tau_{FC}$ and $q_{net}$, are presented, discussed and compared with previous experience from other types of clay. In this regard, the study aims at increasing the empirical knowledge of clays in general and of sulphide clay in particular.
Figure 5.4. a) \( COV_{S_u} \); and b) \( S_u \) assessed for each variate and from multivariate analyses with different sets of variates

From the analyses of the variate \( q_{net} \), the vertical scale of fluctuation was evaluated as \( \theta_v q_{net} \approx 0.4 \) metres and from \( \tau_{FY} \) and \( q_{net} \), the horizontal scales of fluctuation were evaluated as \( \theta_h \approx 20 \) metres (similar for both variates). From measurements made in the sulphide clay at the Veda site, \( COV_{\tau_{FY}} = 0.23 \), \( COV_{q_{net}} = 0.19 \) and \( COV_{q_{net}} = 0.16 \) were evaluated.

The assessments of \( COV_{S_u} \) and \( S_u \) from different combination of variates are shown in Fig. 5.4. As seen, \( COV_{S_u} \) is significantly reduced as more information (i.e. variates) is incorporated in the analyses and the \( S_u \) values assessed from each variate (the black bars) differ significantly. The weighting procedure inherent in the multivariate approach is also shown. The lines placed across the bars representing the “multivariate” cases, i.e. the bars representing 2, 3 or 4 variates, correspond to the arithmetic averages. Considering the set “\( q_{net}, P \)”, where \( q_{net} \) is considerably less uncertain than \( P \) (cf. Fig 5.4a), the “multivariate
average” deviated from the arithmetic average towards $\bar{S}_u$ assessed from the less uncertain variate, i.e. $q_{net}$.

Figure 5.5 shows the influence of the number of boreholes (influencing $\psi_X$ and $n$ in Eq. 2.19b) on the assessment of $COV_{\bar{S}_u}$. In this case, under the assumptions adopted in the calculations, there seems to be a threshold value of about 12 boreholes (dependent on the number of variates adopted in the analysis) beyond which additional test points do not lead to any significant decrease in the uncertainty in the determination of $\bar{S}_u$. This form of presentation can be made already in the planning stage of a project, whereby an optimal design of a site investigation (i.e. the combination of test methods and number of boreholes) is achievable.

In conclusion, the extended multivariate approach is useful when data from several sources are available and are included in the evaluation of the average value and the corresponding uncertainty of a material property. However, careful consideration must be given to the evaluation of $COV_{\bar{S}_u|\Delta}$ for the incorporated variates, since a low value for a single variate will greatly affect the results of the analysis.

5.4 Paper IV – Multivariate stability assessment during staged construction

A reliable assessment of the stability is essential when designing and constructing a staged embankment founded on soft clay. The stability is to a large extent dependent on the undrained shear strength in the clay, $S_u$, and the increase in $S_u$ that develops as a result of consolidation between the stages. Predicting the magnitude of the increase in $S_u$ is associated with uncertainties and, as a consequence, site measurements are usually performed during construction. The extended multivariate approach (Ch. 3.3) is a rational methodology for assessment of the average value $\bar{S}_u$ and the associated uncertainty, represented for example by the coefficient of variation $COV_{\bar{S}_u}$. Information from the predictions and from the measurements are processed via a Bayesian updating procedure, resulting in representative values of the average value $\bar{S}_u$ and $COV_{\bar{S}_u}$.

In this paper, the extended multivariate approach (proposed in Paper III) is illustrated for the assessment of $\bar{S}_u$ and $COV_{\bar{S}_u}$ during the construction of a staged embankment. The implications of the procedure on deterministic and probabilistic stability analyses are presented and discussed in the light of the Veda embankment case. At first, à priori predictions of the effective stress, $\sigma'\psi$, in the clay at different stages during construction were established on the basis of consolidation analyses. From these estimates, à priori predictions of $\bar{S}_u$ (via Eq. 2.11e) and $COV_{\bar{S}_u}$ were assessed. From results of measurements via $FV$ tests, $CPT$s and piezometers during construction, values of $\bar{S}_u$ and $COV_{\bar{S}_u}$ were derived (Eqs. 2.11a,e,c) and incorporated in the multivariate analyses forming à posteriori assessments of $\bar{S}_u$ and $COV_{\bar{S}_u}$ via Eqs. 3.9a-b. Deterministic and probabilistic stability analyses adopting the Janbu formulation for the method of slices (see Ch. 4.2.2) were made using these à posteriori assessments. The FOSM approach (see Ch. 4.2.2) was chosen for the probabilistic analyses and safety indices $\beta$ were assessed.

The results of the deterministic and probabilistic stability analyses for one of the construction stages are shown in Fig. 5.6. As seen, the number of variates adopted in the multivariate
analyses (i.e. the reduced uncertainty, analogous to Fig. 5.4a), affects the assessed $\beta$. The more information gathered in the evaluation of $\tilde{S}_u$, the smaller is the uncertainty in the property and; hence, the higher is the value of $\beta$. However, introducing a small model uncertainty ($\text{COV}_{\text{mod}} = 0.05$) in accordance with the procedure described in Ch. 4.3, the value of $\beta$ drops significantly (the lines across the bars). This indicates the sensitivity to model uncertainties in this type of calculation (cf. Ch. 2.4 and 4.3). As expected, the decreased uncertainty in the determination of $\tilde{S}_u$ had no significant effect on the calculated deterministic factor of safety, $F_{\text{det}}$. The only effect was the weighing procedure in the assessments of $\tilde{S}_u$, as previously stated.

In this paper, it was concluded that the extended multivariate approach takes into account information from several sources (variates), and this yields a more objective assessment of the average of a property and the associated uncertainty than an assessment based on the strict arithmetic average. In this process, the transformation uncertainties related to the assessment of a property from the variates can be reduced. However, the results obtained by this methodology are biased towards the relatively least uncertain variate, and this should be taken into account when assigning the individual uncertainties to the different variates. Assessing the stability of an embankment, deterministic and probabilistic approaches complement each other and should be employed in parallel. Probabilistic approaches enable assessments of the relative probability of failure and of the relative influences of the uncertainties in the relevant properties. Some sources of uncertainty may nevertheless be difficult to assess (the model uncertainties treated in Ch. 2.4 and 4.3), and thus difficult to incorporate in a probabilistic analysis. Deterministic approaches implicitly deal with these uncertainties and can also provide measures of the expected performance of the embankment.

![Figure 5.6. Results of stability analyses for stage 6](image-url)
5.5 Paper V – Aspects on the modelling of smear zones around vertical drains

When vertical drains are installed in soft clay, the original soil structure is disturbed. This reduces the permeability in the clay around the drain to some extent, thereby affecting the subsequent consolidation process. Several analytical formulations suggesting how to model the disturbed (smear) zone can be found in the literature. Some of them describe the smear zone in a rather simple manner while others describe the smear zone more realistically, via more complex mathematical formulations.

This paper investigates the differences between six of the models available in the literature and the influences of the variables in the mathematical formulations on the results of the calculations, i.e. the calculated degree of consolidation, $U$. The primary variables of the models are in principle; 1) the coefficient of consolidation, $c_h$, in the undisturbed clay, represented by the variable $T_h$; 2) the degree of disturbance in the smear zone, i.e. the ratio between the undisturbed and the disturbed permeability, represented by the variable $\kappa$; and 3) the extent (size) of the smear zone, represented by the variable $s$.

The sensitivity factor, $\alpha$, (Eq. 4.8) for each variable on the assessment of $U$ was evaluated for the models. In addition, the relative influences of the uncertainties in each variable on the uncertainty in the assessment of $U$ (defined by the variance contribution, $d\text{Var}_u$) was evaluated (Eq. 4.9).

![Figure 5.7](image)

Figure 5.7. a) The influence on $U$ of $T_h$ and $\kappa$ ($\alpha$ from the other variables were <0.045); b) The contribution to $\text{Var}_u$ of the variances in $T_h$, $s$, $\kappa$ and $C_c/C_k$
The results shown in Fig. 5.7 indicate that the variable $c_h$ (represented by $T_h$ in the figures) was predominant in the assessment of both $U$ and the associated uncertainty $Var_U$. The variable $\kappa$ was found to influence the assessment of $U$ in the latter stages of the consolidation process (i.e. at high values on $U$). In addition to $T_h$, $\kappa$ made up for most of the remainder of $Var_U$.

The difference between the models lies in the formulations for the description of the smear zone which is independent of $c_h$. For the cases studied, the different models gave more or less the same results in the assessment of $U$; hence, the choice of model was unimportant for practical purposes. The study concludes that, in practical engineering projects, efforts should be directed towards determining $c_h$ (i.e. reducing the uncertainty in $c_h$) rather than towards investigating the characteristics of the smear zone (i.e. the degree of disturbance defined by $\kappa$ and the extent of the smear zone defined by $s$).
Chapter 6 – Concluding remarks and proposed future research

Summary: In this chapter the main conclusions from the research project are stated. Suggestions for further research related to the practical use of the extended multivariate approach are also proposed.

6.1 Concluding remarks

In the appended Papers III and IV, the extended multivariate approach is presented. The extension involves assessing the total uncertainty in the determination of a property, including the reduction in the uncertainty due to spatial averaging, before adoption of the multivariate approach. In Paper III, the extended multivariate approach was illustrated for the evaluation of \( \bar{S}_u \) and the associated uncertainty in sulphide clay, from a site investigation in the Veda area. In Paper IV, the methodology was illustrated for the assessment of \( \bar{S}_u \) and the associated uncertainty at different stages during construction, with the intention of controlling the stability of the Veda embankment. The effect of utilising the methodology on the results of deterministic and probabilistic stability analyses was also demonstrated.

From the results presented in the papers, it can be concluded that the methodology provides a rational and robust way of assessing spatially averaged values and associated uncertainties in soil properties based on information retrieved from several sources (investigation methods). As shown in the papers, the total uncertainty in the assessment (i.e. the pair-wise correlation) of a property, e.g. \( \bar{S}_u \), from a measured quantity (variates), e.g. \( q_{\text{net}} \), is to a large extent dependent on the uncertainty in the correlation (i.e. the transformation uncertainty). This type of uncertainty is a bias and cannot be reduced explicitly unless more thorough calibration tests are undertaken. However, combining information from several sources and processing it through the proposed methodology implicitly enables these transformation uncertainties to be reduced. As shown in Papers III and IV, this can have a significant impact on the evaluated uncertainty in the determination of the average value and on the subsequent probabilistic analyses. The methodology can also be of help when planning a site investigation or when planning complementary tests in order to reduce the uncertainty in the assessment of the value of a property. Paper III shows the effect on the evaluation of the uncertainty of the number of test points and of the number of variates employed. This type of analysis can be carried out during the planning of the investigation, instructing the engineering on how to design the investigation in an optimal way and thereby be of help in the decision process.

From the study presented in Paper III, some important findings related to the assessment of \( \bar{S}_u \) in sulphide clay from measurements by \( FV, FC \) and \( CPT \) are presented. From the measurements, the scales of fluctuation were assessed as; \( \theta_v \approx 0.4 \) metres; and \( \theta_h \approx 20 \) metres in the vertical and horizontal directions, and the coefficients of variation evaluated from the measurements were; \( COV_{\tau_{pv}} \approx 0.23 \); \( COV_{\tau_{FC}} \approx 0.19 \); and \( COV_{q_{\text{net}}} \approx 0.16 \). These findings can be used as à priori estimates in future analyses involving sulphide clay.
In Paper IV, the effects of the adoption of the extended multivariate approach on the results of the probabilistic stability analyses of the Veda embankment are presented. Different combinations of variates were employed for the evaluation of \( \bar{S}_u \), leading to different assessments of \( COV_{s_u} \). The performance function, i.e. \( F \), was highly dependent on \( \bar{S}_u \), and, as a consequence, the calculated \( \beta \) was dependent on the combinations of variates. As expected, the uncertainty in the determination of \( \bar{S}_u \) did not reflect on \( F_{det} \) obtained from the deterministic stability analyses. This may imply that a probabilistic analysis is superior to a deterministic analysis. However, uncertainties from other sources that might be difficult to assess probabilistically contribute to the total uncertainty in \( F \). In this paper, it was shown that the introduction of a rather modest model uncertainty \( (COV_{mod} = 0.05) \) significantly reduced the evaluated \( \beta \). In a deterministic analysis, all sources of uncertainty are contained in the required \( F_{det} \). It was therefore suggested that the two types of analyses complement each other and should be performed in parallel.

In Paper V, six mathematical models available in the literature with varying degrees of complexity intended for the design of vertical drains were investigated. The differences between the investigated models primarily lie in the way they describe the smear zone, i.e. the disturbed zone induced in the soil during installation of the drains. It was shown that the degree of complexity in the models, i.e. how rigorous the smear zone is modelled, had very little influence on the results of the calculations. It was concluded that more simple models can be adopted for practical design purposes.

6.2 Future research

During the work leading to this doctoral thesis, a number of questions and suggestions for future research related to the practical use of the extended multivariate approach arose, such as:

- How do we make the extended multivariate approach available to the practicing engineer not very familiar with statistics and probabilistic analyses? One of the strengths of the approach is that it can handle the reduction in uncertainty in the determination of a quantity, e.g. a material property, when information from different sources is available. The amount of knowledge obtained should match the required safety margins or factors of safety in practical projects. Much of the present geotechnical design conducted in Sweden and in other parts of Europe employs the partial factor method as suggested in the Eurocode. One way of taking advantage of the approach could be to work out a structured methodology, based on a multivariate analysis, for how to choose partial factors for different quantities and for different types of constructions depending on the amount of information available.

- Another advantage of the extended multivariate approach is that it makes it possible to assess representative average values of quantities involved in an analysis, when information from different sources is available. The least uncertain source of information is given the most weight in such evaluations. This is a more objective way of choosing average values for use in design than, for example, evaluations of strict arithmetic averages or judgemental assessments. How can we introduce this feature into everyday engineering practice? Could we incorporate this feature in the aforementioned methodology for choosing partial factors?
In this research project, the extended multivariate approach was limited to applications for the determination of \( \bar{S}_u \), with the intention of performing undrained short-term stability analyses of embankments. In such cases, the stability is to a large extent dependent on \( \bar{S}_u \). In other closely related cases such as the stability of natural slopes or the stability of excavations or cuts, or less related cases such as the design of supporting structures (sheet pile walls, piles, etc.), other quantities play major roles and should therefore be focused on in an analysis. It would be valuable to extend the use and test the usefulness of the approach beyond the rather limited application treated within the frame of this thesis.

The link between the extended multivariate approach and the observational method is another interesting area of research that could be beneficial for practical engineering purposes. The development of a structured methodology for how to take advantage of both the extended multivariate approach and the observational method in practice is proposed.
References


Appendix A

In this appendix suggested values of the measurement error, spatial (naturally inherent) variability, scale of fluctuation and total uncertainty for a selection of soil properties, retrieved from compilations found in the literature, are presented. In the tables, the abbreviations and notations are explained when they appear for the first time.

Table A1. Suggested values of the measurement error (from compilation in Phoon and Kulhawy 1999a)

<table>
<thead>
<tr>
<th>Property, measured in the laboratory</th>
<th>Soil type</th>
<th>No. of data groups</th>
<th>Range of $COV_{err}$ (%)</th>
<th>Average of $COV_{err}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$, TC (friction angle, triaxial compression test)</td>
<td>Clay, silt</td>
<td>4</td>
<td>7-56</td>
<td>24</td>
</tr>
<tr>
<td>$\phi$, DSS (direct simple shear test)</td>
<td>Clay, silt</td>
<td>5</td>
<td>3-29</td>
<td>13</td>
</tr>
<tr>
<td>$\phi$, DSS</td>
<td>Sand</td>
<td>2</td>
<td>13-14</td>
<td>14</td>
</tr>
<tr>
<td>tan $\phi$, TC</td>
<td>Sand, silt</td>
<td>6</td>
<td>2-22</td>
<td>8</td>
</tr>
<tr>
<td>tan $\phi$, DSS</td>
<td>Clay</td>
<td>2</td>
<td>6-22</td>
<td>14</td>
</tr>
<tr>
<td>$\gamma$ (unit weight)</td>
<td>Fine grained</td>
<td>3</td>
<td>1-2</td>
<td>1</td>
</tr>
<tr>
<td>$PI$ (plasticity index)</td>
<td>Fine grained</td>
<td>10</td>
<td>5-51</td>
<td>24</td>
</tr>
<tr>
<td>$S_u$, TC (undrained shear strength)</td>
<td>Clay, silt</td>
<td>11</td>
<td>8-38</td>
<td>19</td>
</tr>
<tr>
<td>$S_u$, DSS</td>
<td>Clay, silt</td>
<td>2</td>
<td>19-20</td>
<td>20</td>
</tr>
<tr>
<td>$S_u$, LV (laboratory vane shear test)</td>
<td>Clay</td>
<td>15</td>
<td>5-37</td>
<td>13</td>
</tr>
<tr>
<td>$w_a$ (water content)</td>
<td>Fine grained</td>
<td>3</td>
<td>6-12</td>
<td>8</td>
</tr>
<tr>
<td>$w_l$ (liquid limit)</td>
<td>Fine grained</td>
<td>26</td>
<td>3-11</td>
<td>7</td>
</tr>
<tr>
<td>$w_p$ (plasticity limit)</td>
<td>Fine grained</td>
<td>26</td>
<td>7-18</td>
<td>10</td>
</tr>
</tbody>
</table>

Test method in the field

<table>
<thead>
<tr>
<th>Method</th>
<th>Range of $COV_{err}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mechanical CPT (cone penetration test)</td>
<td>15-22</td>
</tr>
<tr>
<td>Electrical CPT</td>
<td>5-15</td>
</tr>
<tr>
<td>FVS test (field vane shear test)</td>
<td>10-20</td>
</tr>
</tbody>
</table>

Table A2. Suggested values of the natural inherent variability (compiled in Phoon and Kulhawy 1999a)

<table>
<thead>
<tr>
<th>Property, measured in the laboratory</th>
<th>Soil type</th>
<th>No. of data groups</th>
<th>Range of $COV_{spat}$ (%)</th>
<th>Average of $COV_{spat}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>Sand</td>
<td>7</td>
<td>5-11</td>
<td>9</td>
</tr>
<tr>
<td>$\phi$ ($\phi \approx 15^\circ$)</td>
<td>Clay, silt</td>
<td>12</td>
<td>10-50</td>
<td>21</td>
</tr>
<tr>
<td>$\phi$ ($\phi \approx 33^\circ$)</td>
<td>Clay, silt</td>
<td>9</td>
<td>4-12</td>
<td>9</td>
</tr>
<tr>
<td>tan $\phi$, TC</td>
<td>Clay, silt</td>
<td>4</td>
<td>6-46</td>
<td>20</td>
</tr>
<tr>
<td>tan $\phi$, DSS</td>
<td>Clay, silt</td>
<td>3</td>
<td>6-46</td>
<td>23</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Fine grained</td>
<td>6</td>
<td>3-20</td>
<td>9</td>
</tr>
<tr>
<td>$PI$ (plasticity index)</td>
<td>Fine grained</td>
<td>33</td>
<td>9-57</td>
<td>29</td>
</tr>
<tr>
<td>$LI$ (liquidity index)</td>
<td>Clay, silt</td>
<td>2</td>
<td>60-88</td>
<td>74</td>
</tr>
<tr>
<td>$S_u$, UC (unconfined compression test)</td>
<td>Fine grained</td>
<td>38</td>
<td>6-56</td>
<td>33</td>
</tr>
<tr>
<td>$S_u$, UU (unconsolidated undrained triaxial test)</td>
<td>Clay, silt</td>
<td>13</td>
<td>11-49</td>
<td>22</td>
</tr>
<tr>
<td>$S_u$, CIUC (consolidated isotropic undrained triaxial test)</td>
<td>Clay</td>
<td>10</td>
<td>18-42</td>
<td>32</td>
</tr>
<tr>
<td>$S_u$ general</td>
<td>Clay</td>
<td>42</td>
<td>6-80</td>
<td>32</td>
</tr>
<tr>
<td>$w_a$</td>
<td>Fine grained</td>
<td>40</td>
<td>7-46</td>
<td>18</td>
</tr>
<tr>
<td>$w_L$</td>
<td>Fine grained</td>
<td>38</td>
<td>7-39</td>
<td>18</td>
</tr>
</tbody>
</table>
Appendix A

Table A2. Continued

<table>
<thead>
<tr>
<th>Property, measured in the field</th>
<th>Soil type</th>
<th>No. of data groups</th>
<th>Range of ( COV_{spt} (%) )</th>
<th>Average of ( COV_{spt} (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_p )</td>
<td>Fine grained</td>
<td>23</td>
<td>6-34</td>
<td>16</td>
</tr>
<tr>
<td>( q_c ) (uncorrected cone tip resistance from CPT)</td>
<td>Sand</td>
<td>57</td>
<td>10-81</td>
<td>38</td>
</tr>
<tr>
<td>( q_c )</td>
<td>Silty clay</td>
<td>12</td>
<td>5-40</td>
<td>27</td>
</tr>
<tr>
<td>( q_r ) (corrected cone tip resistance from CPT)</td>
<td>Clay</td>
<td>9</td>
<td>2-17</td>
<td>8</td>
</tr>
<tr>
<td>( S_u, FVS ) (corrected value from field vane shear tests)</td>
<td>Clay</td>
<td>31</td>
<td>4-44</td>
<td>24</td>
</tr>
</tbody>
</table>

Table A3. Suggested values of the scale of fluctuation (compiled in Phoon and Kulhawy 1999a)

<table>
<thead>
<tr>
<th>Property</th>
<th>Soil type</th>
<th>No. of data groups</th>
<th>Range of ( \theta (m) )</th>
<th>Average of ( \theta (m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Clay, loam</td>
<td>2</td>
<td>2.4-7.9</td>
<td>5.2</td>
</tr>
<tr>
<td>( q_c )</td>
<td>Sand, clay</td>
<td>7</td>
<td>0.1-2.2</td>
<td>0.9</td>
</tr>
<tr>
<td>( q_r )</td>
<td>Clay</td>
<td>10</td>
<td>0.2-0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>( S_u, ) laboratory test</td>
<td>Clay</td>
<td>5</td>
<td>0.8-6.1</td>
<td>2.5</td>
</tr>
<tr>
<td>( S_u, FVS )</td>
<td>Clay</td>
<td>6</td>
<td>2.0-6.2</td>
<td>3.8</td>
</tr>
<tr>
<td>( w_a )</td>
<td>Clay, loam</td>
<td>3</td>
<td>1.6-12.7</td>
<td>5.7</td>
</tr>
<tr>
<td>( w_l )</td>
<td>Clay, loam</td>
<td>2</td>
<td>1.6-8.7</td>
<td>5.2</td>
</tr>
<tr>
<td>Horizontal</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( q_c )</td>
<td>Sand, clay</td>
<td>11</td>
<td>3.0-80.0</td>
<td>47.9</td>
</tr>
<tr>
<td>( q_r )</td>
<td>Clay</td>
<td>2</td>
<td>23.0-66.0</td>
<td>44.5</td>
</tr>
<tr>
<td>( S_u, FVS )</td>
<td>Clay</td>
<td>3</td>
<td>46.0-60.0</td>
<td>50.7</td>
</tr>
<tr>
<td>( w_a )</td>
<td>Clay, loam</td>
<td>1</td>
<td>170.0</td>
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</tr>
</tbody>
</table>

Table A4. Suggested values for the total uncertainty (based on Lumb 1974 (A); Lee et al. 1983 (B); Lacasse and Nadim 1996 (C); Phoon and Kulhawy 1999b (D); Duncan 2000 (E))

<table>
<thead>
<tr>
<th>Property, test type</th>
<th>Soil type</th>
<th>Range of ( COV_{total} )</th>
<th>Standard ( COV_{total} )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_c ) (compression index)</td>
<td></td>
<td>25-30</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( C_c )</td>
<td></td>
<td>18-73</td>
<td>30</td>
<td>B</td>
</tr>
<tr>
<td>( C_c )</td>
<td></td>
<td>10-37</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>( c_v ) (coefficient of consolidation)</td>
<td></td>
<td>25-50</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( c_v )</td>
<td></td>
<td>25-100</td>
<td>50</td>
<td>B</td>
</tr>
<tr>
<td>( c_v )</td>
<td></td>
<td>33-68</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>( E ) (elastic modulus)</td>
<td></td>
<td>2-42</td>
<td>30</td>
<td>B</td>
</tr>
<tr>
<td>( e ) (void ratio)</td>
<td></td>
<td>15-30</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( e )</td>
<td></td>
<td>13-42</td>
<td>25</td>
<td>B</td>
</tr>
<tr>
<td>( e )</td>
<td></td>
<td>7-30</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Sand</td>
<td>5-15</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Sand</td>
<td>5-15</td>
<td>10</td>
<td>B</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Clay</td>
<td>12-56</td>
<td></td>
<td>B</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Sand</td>
<td>2-5</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( \phi ) evaluated in the laboratory</td>
<td>Clay, sand</td>
<td>7-20</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( \phi_{TC} ) evaluated from ( q_r )</td>
<td>Sand</td>
<td>10-15</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( \phi )</td>
<td></td>
<td>2-13</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>( \gamma )</td>
<td></td>
<td>1-10</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( \gamma )</td>
<td></td>
<td>1-10</td>
<td>3</td>
<td>B</td>
</tr>
<tr>
<td>( \gamma' ) (buoyant unit weight)</td>
<td></td>
<td>0-10</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( \gamma' )</td>
<td></td>
<td>3-7</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>( K_0 ) (coefficient of lateral earth pressure at rest)</td>
<td>Clay</td>
<td>20-45</td>
<td></td>
<td>D</td>
</tr>
</tbody>
</table>
### Table A4. Continued

<table>
<thead>
<tr>
<th>Property, test type</th>
<th>Soil type</th>
<th>Range of ( \text{COV}_{\text{total}} )</th>
<th>Standard ( \text{COV}_{\text{total}} )</th>
<th>Reference</th>
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</thead>
<tbody>
<tr>
<td>( K_0 )</td>
<td>Sand</td>
<td>20-55</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( k ) (permeability)</td>
<td></td>
<td>200-300</td>
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<td>A</td>
</tr>
<tr>
<td>( k )</td>
<td></td>
<td>200-300</td>
<td>300</td>
<td>B</td>
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<tr>
<td>( k )</td>
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<td>68-90</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>OCR (over consolidation ratio)</td>
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<td>10-35</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( P_l )</td>
<td>Clay</td>
<td>7-79</td>
<td>30</td>
<td>B</td>
</tr>
<tr>
<td>( P_l )</td>
<td>Sand, gravel</td>
<td>7-79</td>
<td>70</td>
<td>B</td>
</tr>
<tr>
<td>( S_u )</td>
<td>Clay</td>
<td>20-50</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( S_u )</td>
<td>Clay</td>
<td>25-30</td>
<td>30</td>
<td>B</td>
</tr>
<tr>
<td>( S_u )</td>
<td>Clayey silt</td>
<td>10-30</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( S_{u, \text{in general}} )</td>
<td></td>
<td>13-40</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>( S_{u, CIUC} )</td>
<td>Clay</td>
<td>5-20</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( S_{u, \text{from index tests}} )</td>
<td>Clay</td>
<td>10-35</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( S_{u, UC} )</td>
<td>Clay</td>
<td>20-55</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( S_{u, UU} )</td>
<td>Clay</td>
<td>10-35</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( S_{u, CIUC} )</td>
<td>Clay</td>
<td>20-45</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( S_{u, FVS} )</td>
<td>Clay</td>
<td>15-50</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( S_{u, UU \text{ evaluated from } q_T} )</td>
<td>Clay</td>
<td>30-40</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( S_{u, CIUC \text{ evaluated from } q_T} )</td>
<td>Clay</td>
<td>35-50</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>( S_u / \sigma'_p ) (ratio of ( S_u ) to in situ effective stress)</td>
<td>Clay</td>
<td>5-15</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( \sigma'_p ) (preconsolidation pressure)</td>
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<td>10-35</td>
<td></td>
<td>E</td>
</tr>
<tr>
<td>( w' )</td>
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<td>2-48</td>
<td>10</td>
<td>B</td>
</tr>
<tr>
<td>( w_L )</td>
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<td>3-20</td>
<td></td>
<td>C</td>
</tr>
<tr>
<td>( w_T )</td>
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<td>6-63</td>
<td>15</td>
<td>B</td>
</tr>
<tr>
<td>( w_p )</td>
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<td>9-29</td>
<td>1</td>
<td>B</td>
</tr>
<tr>
<td>( w_p )</td>
<td></td>
<td>3-20</td>
<td></td>
<td>C</td>
</tr>
</tbody>
</table>