Theoretical Physics

# Experimental Tests of General Relativity

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**Abstract**

This paper treats Einstein's theory of General Relativity (GR), in particular three of the earliest experiments testing its validity. It covers the deflection of, and the redshift of light in a gravity field, two new phenomena predicted by GR. The perihelion precession of the planet Mercury and how GR matches observations of it more correctly than classic physics is also covered. In addition to the three older tests above, the more modern application in the GPS system is discussed, and how it can be regarded as a test of GR. Some theoretical questions are also discussed, including comparing classic physics to GR and the classical limit of GR.

Denna rapport handlar om Einsteins allmäna relativitetstori (GR), och mer specifikt om tre tidiga experiment som kan testa dess giltlighet. Här behandlas avböjning av, och rödförskjutning av ljus i gravitationsfält, två nya fenomen som förutsägs av GR. Planeten merkurius periheliumprecession och hur GR bättre matchar observationer av den täcks också upp. Förutom dessa tre experiment, diskuteras också tillämpningen av GR i GPS-systemet, och hur det kan betraktas som ett test av GR. Några teoretiska frågor diskuteras också; bland annat jämförs klassisk fysik med GR och den klassiska gränsen till GR tas upp.

# Contents

# Chapter 1

# Introduction

At the end of the 19:th century, it was a common opinion that the current knowledge of physics was a complete description of the world. There were however a few problems left to solve, which required completely new theoretical models, which were very different from all the earlier ones. One of the problems originated from Maxwell's equations [9] in the absence of charges or currents:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},
\end{aligned}$$

which can, taking the curl of the two last equations and applying vector algebra, be written as follows:

$$\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \nabla \cdot \mathbf{E} &= 0 \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{1}{\epsilon_0 \mu_0} \nabla \cdot \mathbf{B} &= 0.
\end{aligned}$$

Those are common wave equations, and according to them, electromagnetic waves (including light) propagates at speed

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

The issue was that, as with all other known waves, there was thought to be a medium, called the ether, which electromagnetic waves propagated through with speed $c$. Since Earth is orbiting around the Sun, it should then have been possible to measure Earth's motion relative to the ether. However, that motion appeared to be 0. This was one of the motivations behind Einstein's theory of Special Relativity (SR) [1], which states that there is no ether, and that light moves at the speed $c$ relative to any observer. SR is consistent with classic electromagnetic theory (in particular Maxwell's equations), but it is not consistent with Newton's laws of motion.

The theory of Special Relativity is incomplete, because unlike Newton's laws, it does not take acceleration into account. The theory of General relativity (GR) was introduced to address that problem, and as will be seen later in this paper, it does at the same time include the relativistic counterpart to Newton's universal law of gravity. In the same way as SR is based on the postulate that no inertial frame (reference frames moving at constant velocities) has a more central or important role than any other, GR postulates that all accelerating reference frames are on equal ground.

The theory of General Relativity modeled an already known phenomena, the perihelion precession of Mercury, better than Newtonian mechanics. In addition, it predicted several new phenomena, which made it possible to test the theory experimentally. Two of those have to do with how light is affected by gravity. Those tests, in addition to Mercury's perihelion precession, is sometimes referred to as "classic tests" of GR. This paper is about those tests, and the theory behind them. Also, the modern application of GR in GPS is also covered. It can be regarded as an experimental test, and the fact that the GPS system is working supports GR.

# Chapter 2

# Background

## 2.1 Special Relativity

The theory of Special Relativity (SR) was introduced by Albert Einstein in the paper "On the Electrodynamics of Moving Bodies"[1]. It builds on two postulates:

- In any two frames of reference moving at a constant velocity relative to each other, all physics are the same.
- The speed of light $c$ relative to any observer is constant.

From those two postulates, several important results can be derived, such as the relationship between the length of two time intervals in different inertial frames moving relative to each other with speed $v$:

$$\Delta t_1 = \gamma \Delta t_0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t_0.$$

This effect is called time dilatation. Consider an inertial frame A and two identical stopwatches, one stationary (relative to A), and one moving with speed $v$ (relative to A). When the stationary stopwatch shows time $\Delta t_1$, the other stopwatch will show time $\Delta t_0 = \Delta t_1/\gamma$, to an observer in A. The moving stopwatch will appear to tick at a slower rate (since $\gamma > 1$) and will fall behind.

## 2.2 General Relativity

General Relativity (GR) was introduced by Einstein in the paper "The Foundation of the Generalised Theory of Relativity" [2], and is a generalization of SR to be valid in the presence of gravity and acceleration. Thus, SR is the special case of GR when there is no gravity.

GR is based on the Equivalence Principle (EP), of which there are two variants: the weak EP and the strong EP. The weak principle can be stated as "**Gravitational mass is equivalent to inertial mass**". Gravitational mass is the mass appearing in Newton's law of Universal Gravity

$$F = G\frac{m_1 m_2}{r^2},$$

while inertial mass is the mass that appears in Newton's second law of motion

$$F = ma.$$

The strong EP, of which the weak EP is a special case, can be expressed as follows:

- Physics in a reference frame accelerating with acceleration $\mathbf{a}$ is equivalent to physics in a non-accelerating reference frame in a gravity field $\mathbf{g} = -\mathbf{a}$.

The strong EP has one important corollary:

- Physics in a freely-falling reference frame is equivalent to physics in an inertial frame with no gravity.

The strong EP can be used to predict some new phenomena, which will be discussed later on in this paper.

The strong EP leads to Einstein's field equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where $G_{\mu\nu}$ is a tensor related to the curvature of spacetime, and $T_{\mu\nu}$ is a tensor related to mass distribution. This paper will not treat this equation in detail, but one of its solutions will be studied in section 3.2.1. The equation plays the same role as the following equation from classic physics

$$\Delta\Phi = 4\pi G\rho$$

where $\Delta$ is the laplace operator, $\Phi$ the gravitation potential, and $\rho$ is the mass density.

## 2.3 Classic Physics as an Approximation of General Relativity

Newton's laws of motion and universal law of gravity are an accurate model for reality, when certain approximations can be made:

- Speeds are low ($v \ll c$).
- Gravity fields are weak ($r \gg 2GM/c^2$).
- Gravity fields are static. $\left(\frac{dg}{dt} = 0\right)$

Those approximations can be made during many circumstances and for many applications.

# Chapter 3

# Investigation

## 3.1 Terminology

Unless otherwise specified, "year" refers to Earth years. An arc-second is a measurement of angle. An arc-minute is defined as $1/60^{th}$ of 1 degree, and an arc-second is defined as a $1/60^{th}$ of an arc-minute. Thus, there are 1296000 arc-seconds in a circle.

When discussing light, both the terms "beam of light" and "photon" are used. This is done to make explaining easier. The wave-particle duality of light is not of interest in this paper, and therefore the terms can be used without considering it.

## 3.2 Comparisons with Classic Physics

### 3.2.1 The Schwarzschild Solution

One analytical solution to the Einstein field equation (see section 2.2) is the Schwarzschild solution[3]. It describes the curved spacetime around a spherical, non-moving, non-rotating source, and is valid at points outside the source body

$$c^2 d\tau^2 = \left(1 - \frac{r_*}{r}\right) c^2 dt^2 - \left(1 - \frac{r_*}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2.$$

The solution is expressed in Schwarzschild coordinates, which is a spherical coordinate system with an additional time component $t$, which is the time measured infinitely far away from the body. Here $\tau$ is the time measured at the position with space-coordinates $r$, $\theta$ and $\phi$. $r_*$ is the Schwarzschild radius, which is a function of the body's mass, given by the equation

$$r_* = \frac{2GM}{c^2}.$$

The terms $r^2$ and $r^2 \sin^2(\theta)$, appearing before $d\theta^2$ and $d\phi^2$ respectively, are just scale-factors.

Since the Sun has a relatively slow rotation [11], and is much more massive than nearby objects and therefore does not move a lot, the Schwarzschild solution is a good approximation for the space-time around it. The Sun's Schwarzschild radius is

$$r_{*\odot} = \frac{2GM_\odot}{c^2} \approx 2953\,\mathrm{m},$$

which is very small compared to its radius $r_\odot$. Therefore, the ratio $r_*/r$ appearing in the Schwarzschild solution is very small for points outside the sun.

### 3.2.2 Perihelion Precession of Mercury

The perihelion of the planet Mercury, the point in its orbit at which it is closest to the sun, has been observed to move at a rate of approximately 5.74 arc-seconds per year [8]. That is higher than the result expected from classical physics, approximately 5.31 arc-seconds per year [8], which is a substantially different from the observed value. The theory of GR predicts the value much more accurately.

According to Newtonian physics, planets have an elliptical orbit around the sun. If the influence from other planets is neglected, the ellipse is static, and therefore the perihelion does not move. In the case of Mercury, if the gravity force from the 7 other known planets is taken into account, the result obtained is 0.43 arc-seconds off from the observed value. To account for the error, the existence of a planet named Vulcan, with an orbit closer to the sun than Mercury, was suggested in the 19th century. However, no such planet has been observed. GR provides a more accurate prediction of the value, without requiring the existence of an undiscovered planet, and is therefore in this case a better description of reality.

First we consider the relativistic Kepler problem: a general case of a small body (the orbiting body) orbiting a massive body (the central body). Using the Schwarzschild solution, with the coordinates chosen such that $\theta \equiv \pi/2$ (and therefore $d\theta \equiv 0$), the distance $r$ between the bodies can be written as a function of the angle $\phi$ [5]:

$$r(\phi) = \frac{(1+e)r_{min}}{1 + e\cos\left(\left(1 - \frac{3r_*}{2(1+e)r_{min}}\right)\phi\right)},$$

where $r_*$ is the Schwarzschild radius of the central body, $e$ is the eccentricity of the orbit, and $r_{min}$ is the minimum distance between the bodies, which when the central body is the Sun, is the distance when the orbiting body is at perihelion. To simplify it, we introduce the factor

$$\beta = 1 - \frac{3r_*}{2(1+e)r_{min}}$$

which when used, the solution can be written as

$$r(\phi) = \frac{(1+e)r_{min}}{1 + e\cos(\beta\phi)}.$$

In the non-relativistic limit, when $r_*/r_{min} \ll 1$, $\beta$ approaches 1 and the equation is reduced to

$$r(\phi) = \frac{(1+e)r_{min}}{1 + e\cos(\phi)},$$

which is the same equation as obtained by the non-relativistic Kepler problem. We can easily see that this function is periodic with the period $2\pi$, which creates a perfect elliptical orbit. However, the period when $\beta \neq 1$ is $2\pi/\beta$, and the difference

$$2\pi/\beta - 2\pi = 2\pi(1 - \beta^{-1})$$

gives the rate of perihelion precession per revolution around the Sun. The formula can be expanded as

$$2\pi(1-\beta^{-1}) = 2\pi\left(\frac{2(1+e)r_{min}}{3r_*} - 1\right)^{-1} \approx \frac{3\pi r_*}{(1+e)r_{min}}.$$

When inserting the values for Mercury, $e = 0.2056$ and $r_{min} = 4.6 \cdot 10^{10}$ m [6], we get approximately $5 \cdot 10^{-7}$ radians per revolution, corresponding to (with Mercury's orbital period of 0.241 years[6]) 0.43 arc-seconds per year, which is consistent with observations. Using the parameters for Earth, $e = 0.0167$ and $r_{min} = 1.4709 \cdot 10^{11}$ m [6], gives a much smaller precession rate at 0.038 arc-seconds per year. Therefore, the orbit of Earth is much more consistent with Newton's laws than the orbit of Mercury.

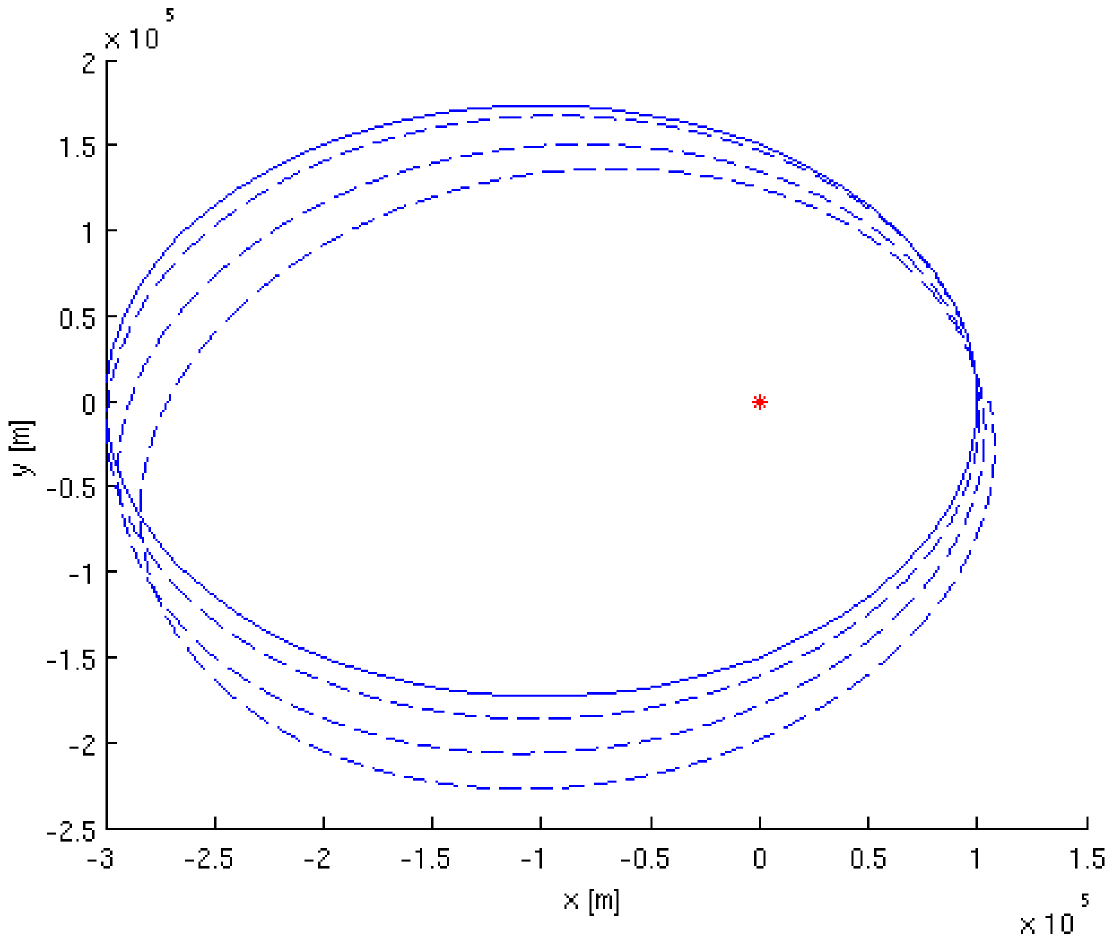


Figure 3.1: A theoretical orbit around the Sun with parameters $e = 0.6$ and $r_{min} = 1 \cdot 10^5$.

## 3.3 New Phenomena Predicted by GR

### 3.3.1 Gravitational Redshift

Gravitational Redshift is a new phenomena predicted by GR. It means that the wavelength (and therefore frequency) of electromagnetic radiation will change, when it travels through a gravitational field. When traveling from a lower gravitational potential to

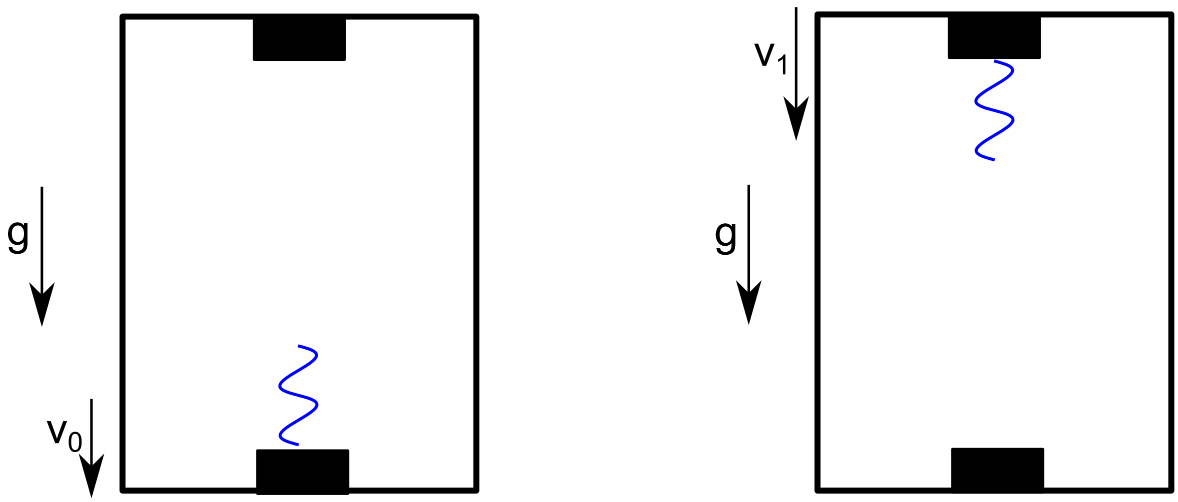


Figure 3.2: A light pulse emitted at the bottom, and received at the top, in a freely-falling box.

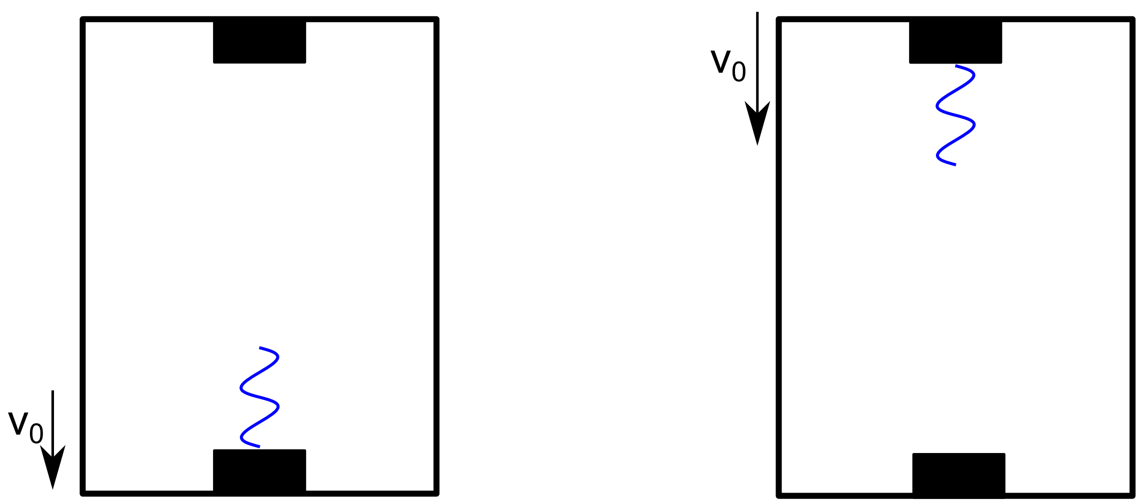


Figure 3.3: A light pulse in a stationary box. According to the equivalence principle, the result must be the same as in figure 3.2.

a higher, the frequency will decrease (which is called redshift), while traveling from a higher potential to a lower causes the opposite effect (called blueshift).

It is possible to make an argument for the fact that this must be the case, using the equivalence principle. Consider as in figure 3.2 a free-falling box of length $l$ in a uniform gravity field of strength $g$. At the bottom of the box, there is an emitter, emitting electromagnetic radiation at frequency $f$. The time it takes for a pulse to reach a receiver at the top of the box is $\Delta t = l/c$. During that time interval, the box's speed has increased by $\Delta v = g\Delta t$. Therefore, the Doppler effect causes a blueshift at the receiver:

$$\frac{\Delta f_{Doppler}}{f} = \frac{\Delta v}{c} = \frac{gl}{c^2}.$$

However, the equivalence principle requires that all physical laws in a free-falling frame in a gravity field, are the same as in a stationary frame unaffected by any gravity. Without gravity, i.e. $g = 0$ above, as in figure 3.3, there would not be any Doppler effect, and the observed frequency would be the same as the emitted frequency. That must also be true, when $g \neq 0$, which means that there must be another phenomenon, that compensates for the Doppler blueshift. This is the gravitational redshift.

$$\frac{\Delta f_{grav}}{f} = -\frac{\Delta f_{Doppler}}{f} = -\frac{gl}{c^2}.$$

Since the gravity field is uniform, the gravitational potential difference between the bottom and the top of the box is $\Delta\phi = gl$. The equation above can be rewritten as

$$\frac{\Delta f_{grav}}{f} = -\frac{\Delta\Phi}{c^2},$$

which is the formula for redshift as a result of gravity. This effect is often very small - for light emitted at the surface of the sun observed infinitely far away, the factor is:

$$\frac{GM_\odot}{c^2 r_\odot} \approx 2.123 \cdot 10^{-6}.$$

An alternative way to obtain this result is using the Schwarzschild solution (see section 3.2.1) to find out at which rates time passes at the point of emission and infinitely far away. Consider two clocks, one placed where the photon is emitted and one infinitely far away. Then, $\tau$ is the time at the first clock and $t$ is the time at the second clock. Since nothing is moving through space, it holds that $dr = d\theta = d\phi = 0$. Therefore, the following equation is obtained, after dividing by $c^2$:

$$d\tau^2 = \left(1 - \frac{r_*}{r}\right) dt^2.$$

Taking the square root of it yields:

$$d\tau = \sqrt{1 - r_*/r}\,dt$$

Since $r$ does not depend on either $t$ or $\tau$, it can be integrated to

$$\Delta\tau = \sqrt{1 - r_*/r}\,\Delta t.$$

Since frequency is the inverse of time, the redshift can be calculated as follows:

$$\frac{\Delta f_{grav}}{f} = \frac{1/\Delta t - 1/\Delta\tau}{1/\Delta\tau} = \frac{\Delta\tau}{\Delta t} - 1 = \sqrt{1 - r_*/r} - 1.$$

Using the first-order Taylor expansion (see Appendix C) we get the same result as above:

$$\frac{\Delta f_{grav}}{f} = -\frac{r_*}{2r} = -\frac{GM}{c^2 r} = -\frac{\Delta\Phi}{c^2}.$$

This effect was measured in the Pound-Rebka experiment in 1959 [7]. The height difference between the emitter and the receiver was $h = 22.5$ m, and since it took place near the surface of Earth, the gravity field can be approximated as homogeneous, and the potential difference as $gh$, where $g \approx 9.82$. The theory predicts a redshift of

$$\frac{\Delta f_{grav}}{f} = -\frac{\Delta\Phi}{c^2} = -\frac{gh}{c^2} \approx 2.5 \cdot 10^{-15},$$

which is a very small value. However, with high-precision instruments, the experiment obtained the result predicted by the theory.

### 3.3.2 Gravitational Light Deflection

When the direction of light is transversal to a gravity field, instead of parallel as above, the direction of light is changed instead of the frequency. The equivalence principle can be used to conclude the fact that this must be the case, but a more advanced approach is required to obtain a valid quantitative result.

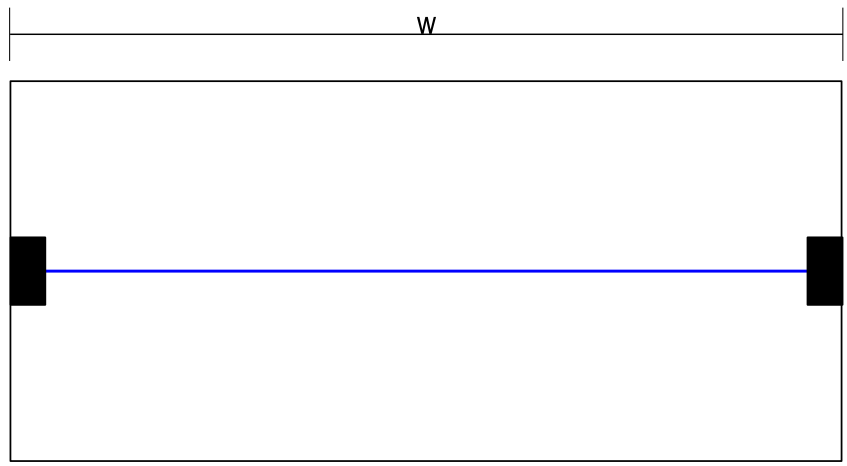


Figure 3.4: Light traveling across a box in an inertial frame.

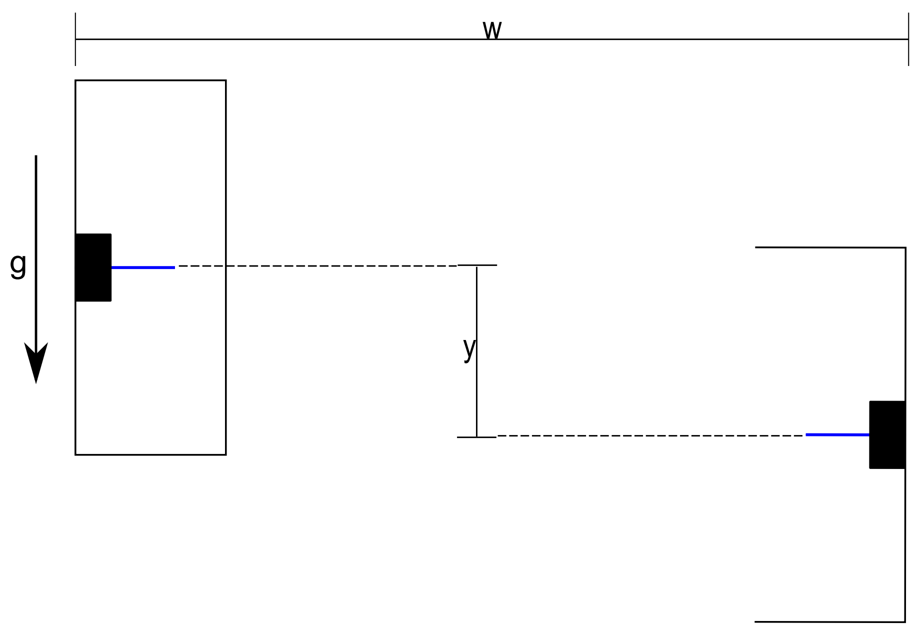


Figure 3.5: The situation in figure 3.4, as seen in a different reference frame.

Imagine a box of width $w$, freely falling in a homogeneous gravitation field (as in figure 3.4). Since it is freely-falling, EP tells us that it is an inertial frame. Therefore, light emitted at one edge will travel along a straight line to the other edge, and the time it takes for light to travel from one end of the box to the other is $\Delta t = w/c$. Now, consider a second frame of reference, in which the box is accelerating with acceleration $g$. At time $t_0$, the box has the velocity 0, and a photon is emitted. Then, by the time $t_1 = t_0 + \Delta t = t_0 + w/c$ a photon has traveled across the box, the box accelerated and moved a distance $y = g\Delta t^2$ vertically. However, it must still be true that the photon arrives at the receiver, which now is located below the height where the emitter was at $t_0$. Therefore, in this frame of reference, the direction of the light beam must have changed. Quantitatively, one can guess that the difference in angle is

$$\Delta\theta = \arctan\left(\frac{y}{w}\right) = \arctan\left(\frac{g\Delta t^2}{w}\right) = \arctan\left(\frac{gw}{c^2}\right).$$

Using the Schwarzschild solution, one can calculate the deflection angle of light passing near the Sun. We consider the fact that, due to gravitational time dilatation, an observer infinitely far away will observe that light is moving at a different speed than $c$, at positions not infinitely far away. We will first find this speed in the Schwarzschild solution. For light, $d\tau \equiv 0$ [5], and we can assume, due to symmetry reasons, that the speed of light does not depend on $\theta$ or $\phi$, hence $d\theta = d\phi = 0$. We then get the equation:

$$\left(1 - \frac{r_*}{r}\right) c^2 dt^2 - \left(1 - \frac{r_*}{r}\right)^{-1} dr^2 = 0.$$

The speed of light at a point with the radial coordinate $r$, according to an observer infinitely far away, is:

$$C(r) = \frac{dr}{dt} = \left(1 - \frac{r_*}{r}\right) c.$$

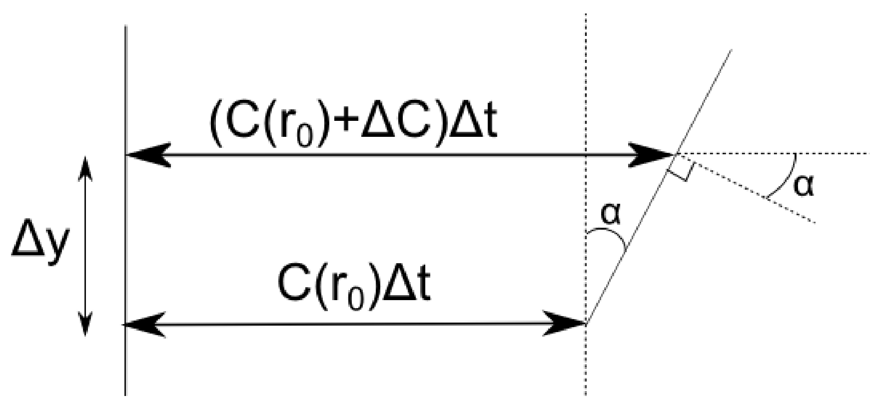


Figure 3.6: A wave is traveling to the right. The vertical line to the left is a wave front at one time point, and the diagonal line to the right is the same wave front after time $\Delta t$ has passed.

Now, consider a wave-front, as in figure 3.6. The angle $\alpha$ can be calculated as follows

$$\alpha \approx \tan(\alpha) = \frac{\Delta C \Delta t}{\Delta y}$$

In the limit when all the values are infinitesimal, using $dx = cdt$, we get:

$$d\alpha = \frac{1}{c}\frac{dC}{dy}dx$$

We will use this in a situation as in figure 3.7 (with the center of the sun as the origin). We define $r_{min}$ to be the shortest distance between the light beam and the origin. To

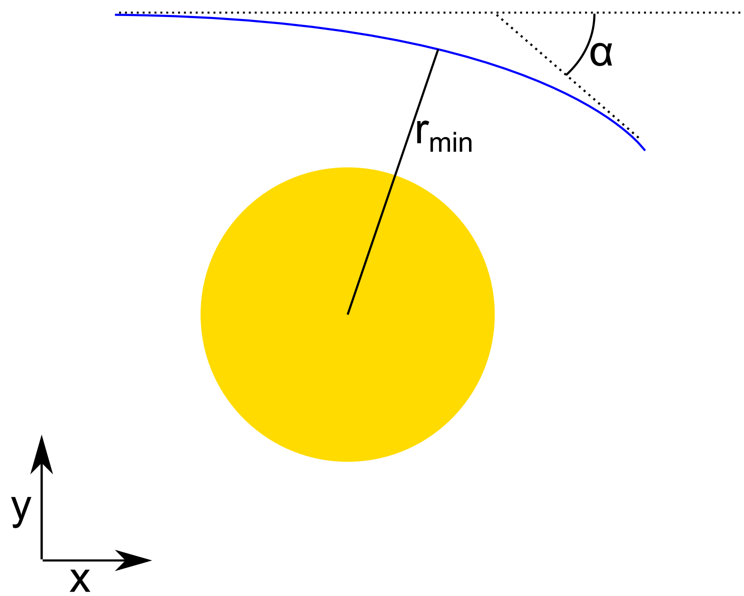


Figure 3.7: A light beam (blue) bending around the Sun (yellow). The bending is strongly exaggerated.

calculate $dC/dy$, we expand $C(r)$ and derive it with respect of $y$.

$$C(r) = \left(1 - \frac{r_*}{r}\right) c = c - \frac{2GM}{cr} = c - \frac{2GM}{c\sqrt{x^2+y^2}} = c - \frac{2GM}{c}(x^2+y^2)^{-1/2}$$

$$\frac{dC}{dy} = 2y\frac{GM}{c}(x^2+y^2)^{-3/2} = \frac{2GM}{c}\frac{y}{r^3}$$

Now, we wish to find the total $\alpha$ as in figure 3.7. We therefore integrate the expression for $d\alpha$ we obtained earlier:

$$\alpha = \int_{-\infty}^{\infty} \frac{1}{c}\frac{dC}{dy}dx = \frac{2GM}{c^2}\int_{-\infty}^{\infty}\frac{y}{r^3}dx$$

We will make an additional approximation. Since the deflection angle is very small, the light trajectory is almost a straight line, and therefore we can do the approximation $y \approx r_{min}$. The integral can then be calculated:

$$\alpha = \frac{2GM}{c^2}\int_{-\infty}^{\infty}\frac{r_{min}}{(x^2+r_{min}^2)^{3/2}}dx = \frac{4GM}{c^2 r_{min}}$$

The maximum angle light passing the sun can be bent is received by putting $r_{min} = r_{\odot}$ (since light cannot pass through the Sun) and $M = M_{\odot}$. Doing so yields the value $\alpha = 8.49 \cdot 10^{-6}$ radians, or 1.75 arc-seconds. This was experimentally confirmed in 1919 [10], during a solar eclipse, when it could be observed that the light from a distant star was deflected by the predicted value.

## 3.4 GPS - A Modern Application

GPS and similar systems are used to determine the coordinates of a given point on or near Earth, through the usage of satellites. It is based on the fact that the speed of light is (a known) constant, and that if it takes a certain amount of time for light to travel from one point to another, the distance is equal to that time multiplied by the speed of light.

Each GPS satellite continuously broadcasts a signal containing information about its position and the time at which it was sent. When a receiver receives the signal, it will know how long time it has been since the signal was broadcasted (through comparison with its own clock). By multiplying that with $c$, the distance to the satellite is found. Knowing the distance to at least four different satellites, the exact position can be computed, which is described in Appendix B.

It becomes more complicated by the fact that, as predicted by SR and GR, time passes at different rates at the satellite and at the receiver. Even though the difference is small, it has a significant impact, because if the time measured is off by $\Delta t$, the calculated position will be off by a distance at the order of magnitude $c\Delta t$, which is large even for small $\Delta t$. For example, a time error of one microsecond gives a position error of about 300 meters. Because of this, the clocks in the satellites are configured to run at a different rate, so it becomes right when they are observed from Earth [4].

We are interested in computing the factors $\alpha_{SR}$ and $\alpha_{GR}$, which are the ratios $(\Delta t_s - \Delta t_r)/\Delta t_r$ (with $\Delta t_s$ being a time interval at the satellite and $\Delta t_r$ is a time interval at the receiver), due to effects in SR and GR, respectively. We will determine them as a function of the satellites' orbital radius $r_s$.

### 3.4.1 GPS in Special Relativity

As predicted by SR, time passes at different rates in the satellites compared to at the surface of Earth, since the satellites orbit Earth at a high speed (see section 2.1). The speed can be calculated using classic physics, given the orbital radius $r_s$. The orbits are circular, which is described by the equation

$$v_s^2 = a \cdot r_s,$$

where the acceleration is due to gravity from earth, and must therefore be equal to Earth's gravity field

$$g(r) = \frac{GM_E}{r^2}.$$

Inserting it into the equation above, the speed can be determined as a function of the orbital radius:

$$v_s^2 = g(r_s) \cdot r_s,$$
$$v_s^2 = \frac{GM_E}{r_s},$$
$$v_s = \sqrt{\frac{GM_E}{r_s}},$$

For reasons of simplicity, the speed of the receiver (due to motion relative to Earth's surface, and due to Earth's rotation) is neglected, since its effect is small. The satellite can then be assumed to travel at the speed $v_s$ relative to the receiver. The $\gamma$ factor then becomes

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_s^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{GM_E}{c^2 r_s}}}.$$

As seen in section 2.1, the $\alpha_{SR}$ value must be

$$\alpha_{SR} = \frac{\Delta t_s - \Delta t_r}{\Delta t_r} = \frac{\Delta t_s - \gamma \Delta t_s}{\gamma \Delta t_s} = \frac{1}{\gamma} - 1$$

The orbital radius of GPS satellites is approximately 26 600 km[4], which yields a $\gamma$ factor close to, but slightly greater than 1. Computing the value numerically, for example by using a calculator, is non-trivial due to the way (which is outside the scope of this paper). A simple and efficient workaround, described in appendix C, gives the result

$$\alpha_{SR} = \frac{1}{\gamma} - 1 \approx -\frac{GM_E}{2c^2 r_s}$$

### 3.4.2 GPS in General Relativity

There is another significant relativistic effect that must be taken into account. According to GR, time passes at different rates at different gravitational potentials. In the case of GPS, only Earth's gravitation matters, which can be modeled with Newton's universal law of gravity. Since it is only the difference between two potentials that matters, an arbitrary constant can be added to it. In this case, the potential is defined to be 0 at an infinite distance

$$\phi(r) = -\int_r^\infty g(r) dr = -\frac{GM_E}{r}.$$

The time dilatation due to potential difference is given by the following formula [5]

$$\alpha_{GR} = \frac{\Delta \phi}{c^2}.$$

Assuming that the observer is located at Earth's surface, the potential difference is

$$\phi(r_s) - \phi(r_E) = -GM_E \left( \frac{1}{r_s} - \frac{1}{r_E} \right)$$

and the time dilatation factor

$$\alpha_{GR} = -\frac{GM_E}{c^2} \left( \frac{1}{r_s} - \frac{1}{r_E} \right).$$

With both $\alpha_{SR}$ and $\alpha_{GR}$ known, we can add them together to obtain

$$\alpha_{SR} + \alpha_{GR} = -\frac{GM_E}{c^2} \left( \frac{1}{2r_s} + \frac{1}{r_s} - \frac{1}{r_E} \right) = \frac{GM_E}{c^2} \left( \frac{1}{r_E} - \frac{3}{2r_s} \right)$$

Inserting $r_s = 2.66 \cdot 10^7$ m, we get $\alpha_{SR} + \alpha_{GR} \approx 4.45 \cdot 10^{-10}$. This means that during one day, the clocks in the GPS satellites would be 38 $\mu$s ahead, giving an error in the measured position at the order of magnitude $38\,\mu\text{s} \cdot c = 11.5$ km.

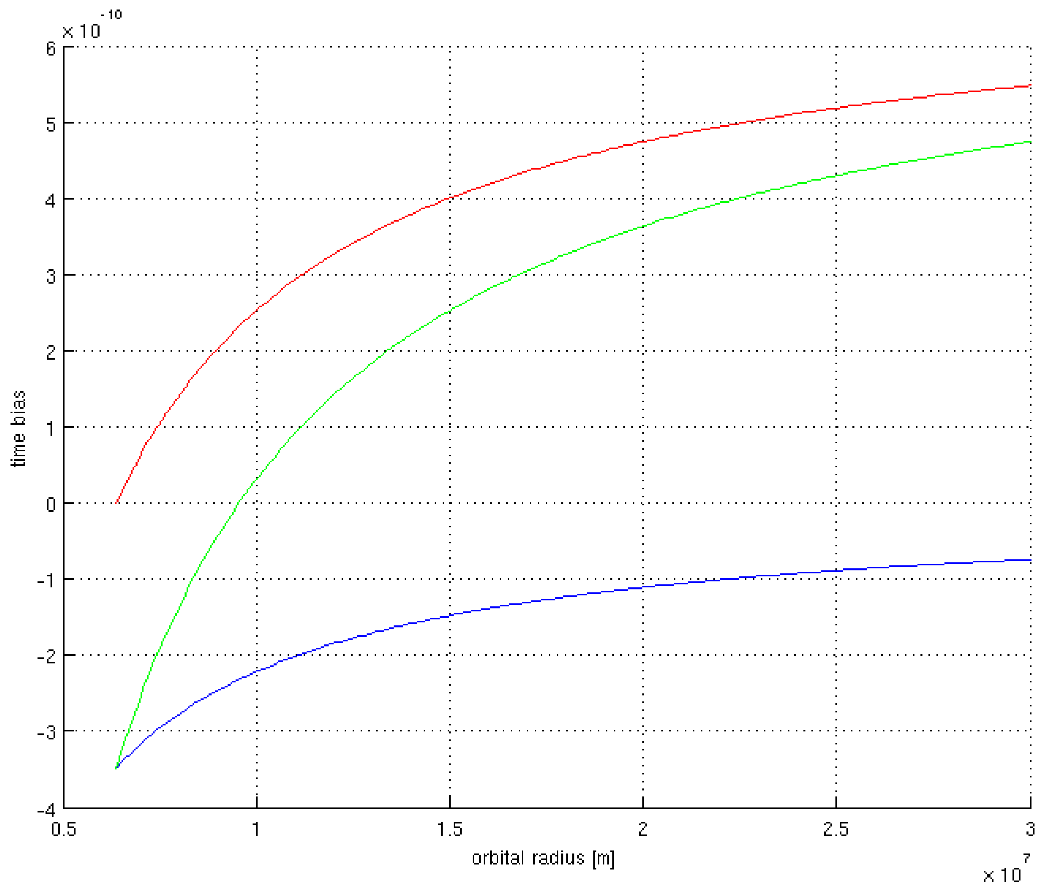


Figure 3.8: Time dilatation factors for SR (blue), GR (red) and total (green), as a function of orbital radius, for an object in a circular orbit around Earth.

# Chapter 4

# Summary and Conclusions

The theory of General Relativity was conceived to describe gravity and acceleration in a way that was consistent with Special Relativity, which in turn was conceived to accommodate for flaws in earlier theories. Both SR and GR are based on simple postulates, that do not agree with classic physics. The theories did however offer better descriptions of reality, than theories without the postulates.

In the case of Mercury's perihelion precession (see section 3.2.2), GR was a better description of previously existing observations than Newton's laws, because it did not require the existence of another, unobserved planet to fit. In the case of the orbits of the other planets, GR gives almost exactly the same result as Newton's laws, because in the limit when gravity is weak, as it is far away from the Sun, Newton's laws is a good approximation of GR.

In the cases with electromagnetic radiation propagating through a gravity field, GR provided the theoretical arguments that there must be phenomena, that could be tested experimentally. Both the predictions in case with propagation parallel (gravitational redshift) and perpendicular (gravitational light deflection) to the gravity field, were experimentally tested, and the result were agreeing with GR.

The GPS system is affected by relativistic effects, which even though they are small, still are important since a small error in time gives a large error in distance. GPS is used on a large scale, and each successful usage can be regarded as evidence that the GPS satellites are configured to compensate for relativistic effects, in a correct way that can be calculated through GR.

Often, a formula obtained from GR can be Taylor expanded to get the corresponding formula as when using classic physics. For example, in section 3.3.1, there was an equation obtained from the Schwarzschild solution, that when Taylor expanded resulted in the same formula as if Newton's gravity law had been used instead of the Schwarzschild solution. This is not a coincidence; in the limit in which GR-specific effects are small (see section 2.3), classic physics agree with GR, and in the same limits, a Taylor expansion is a good approximation of a function. More generally, for any GR formula to be consistent with observations, it must be possible to approximate it with classic physics when the approximations in section 2.3 can be made. This is because classic physics agree with observations and experiments under those conditions. Sometimes, as in section 3.3.1, a Taylor expansion is a good approximation if, and only if, the approximation can be made.

General Relativity is a flexible theory; it can be adopted to more recent observations, such as dark matter and energy, by adding extra terms to Einstein's field equations. The question of whether or not a newer theory could be a better fit for those observations still remain. In addition, there is not yet any generally accepted theory that both includes GR and quantum physics. In the same way as GR once replaced Newton's laws as the most general theory of its area, there is a possibility that GR will be replaced by a more general theory in the future.

# Appendix A

# Constants

| | | | |
|---|---|---|---|
| $c$ | 299792458 | m/s | Speed of Light |
| $G$ | $6.67384 \cdot 10^{-11}$ | $\mathrm{m^3 kg^{-1} s^{-2}}$ | Newton's Universal Gravity Constant |
| $M_E$ | $5.97224 \cdot 10^{24}$ | kg | Earth's Mass |
| $r_E$ | $6.371 \cdot 10^{6}$ | m | Earth's Radius |
| $M_\odot$ | $1.989 \cdot 10^{30}$ | kg | Sun's Mass |
| $r_\odot$ | $6.955 \cdot 10^{8}$ | m | Sun's Radius |

# Appendix B

# Calculating Position via GPS

If an electromagnetic signal was emitted at time $t_s$ and received at time $t_r$, the distance it has traveled is $c(t_r - t_s)$. Using the position of the emitter $\mathbf{x_s}$, we get the following equation, where $\mathbf{x}$ is the position of the receiver:

$$\|\mathbf{x} - \mathbf{x_s}\| = c(t_r - t_s).$$

In practice, there is another term, the time bias $\Delta t$, because the clock at the receiver might not be accurate.

$$\|\mathbf{x} - \mathbf{x_s}\| - c\Delta t = c(t_r - t_s)$$

This equation has four unknowns, $\Delta t$ and the three components of $\mathbf{x}$. With four equations, obtained from four different satellites, the position and the time bias can be calculated.

# Appendix C

# Approximating the Gamma Factor

The gamma factor used in SR is often close to 1 (in particular for small velocities). Due to limited precision, it can be hard to work with it using numeric calculations, such as when using a calculator. The problem can be worked around by using a Taylor expansion. Consider the following function and its derivative:

$$f(x) = \sqrt{1-x} = (1-x)^{1/2}$$
$$f'(x) = -\frac{1}{2}(1-x)^{-1/2}$$

When $x$ (which in this application is $v^2/c^2$) is very small, the first-order expansion around 0 gives us a very good approximation.

$$f(x) \approx f(0) + f'(0) \cdot x = 1 - x/2$$

If we are interested in knowing the value of $f(x)-1$, the computation becomes very easy: $f(x) - 1 \approx -x/2$.

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