On Coordination and Compression in Networks

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Abstract

The current trends in communications suggest that the transfer of information between machines will soon predominate over the traditional human-oriented exchange. The new applications in machine-to-machine communications demand for a new type of networks that are much larger and, especially, much denser. However, there are currently many challenges that hinder an efficient deployment of such networks. In this thesis, we study some fundamental and practical aspects of two of these challenges: coordination and compression.

The problem of coordination in a network is that of organizing the nodes to make them work together. The information-theoretic abstraction of this corresponds to generating actions with a desired empirical distribution. In this thesis, we construct polar codes for coordination for a variety of topologies. These codes combine elements of source coding, used to produce the actions, with elements of channel coding, used to obtain efficient descriptions. We show that our constructions achieve several fundamental coordination limits in a structured manner and with affordable complexity.

Then, we consider the problem of coordinating communications to control the interference created to an external observer, measured in terms of its empirical distribution. To study the relationship between communication and interference, we introduce the notion of communication-interference capacity region. We obtain a complete characterization of this region for the single user scenario and a partial solution for a multiple user case. Our results reveal a fundamental tradeoff between communication, coordination, and interference in this type of networks.

The second problem considered in this thesis, compression, involves capturing the essence of data and discarding the irrelevant aspects to obtain compact representations. This takes on a new dimension in networks, where the importance of data is no longer a local matter. In this thesis, we show that polar codes are also suitable for achieving information-theoretic bounds that involve compression in networks. More precisely, we extend our coordination constructions to realize compress-and-forward relaying with affordable complexity.

In the last part of the thesis, we take a network approach to the problem of compressive sensing and develop methods for partial support set recovery. We use these methods to characterize the tradeoff between the measurement rate and the mean square error. Finally, we show that partial support recovery is instrumental in minimizing measurement outages when estimating random sparse signals.
“Nothing has such power to broaden the mind as the ability to investigate systematically and truly all that comes under thy observation in life”

Marcus Aurelius, Meditations, III, 11
A mis padres
Writing this thesis has been a unique and unforgettable task. In many respects, it has also been an extreme challenge. Over the past years, I have experienced an indescribable mixture of curiosity, fascination, intellectual stimulation, and satisfaction, but also frustration and fear. A long list of people made me enjoy the good moments and overcome the bad ones. I want to take this opportunity to thank them.

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Chapter 1

Introduction

In the past twenty years, we have witnessed an unprecedented expansion of communication networks. Not only their size has increased, but also their range of uses [ITU05, Com09]. They are no longer exclusively oriented towards person-to-person communication; a new class of applications devoted to machine-to-machine communications has now emerged. We expect these new applications to be dominant in the future given that, in any reasonable sense, the number of machines can grow much faster than the number of humans [Eri10]. In addition, the potential uses in commercial and production processes and their consequences in terms of economy support this view. If the prediction holds true, communication networks will have to grow both at a macroscopic level (e.g., connecting people in different cities or countries) and at a much smaller scale (e.g., within machines). In fact, this process is already underway. For example, each mobile phone, in addition to being part of the cellular network, is a communication network in itself: vast amounts of information are exchanged between its different components.

The benefits of this expansion are countless: we have ubiquitous access to virtually unlimited amounts of data in all perceptible formats (e.g., text, video, audio, etc.), communication between geographically remote places takes place in a matter of milliseconds, machines have replaced humans in many tasks that formerly relied on interaction between humans or in places that were previously inaccessible to them, etc. [AIM10]. Above all, many of these advances have enabled an ever-increasing rise of industrial productivity on which modern societies have become dependent.

Arguably, we will need a very deep understanding of information networks to deploy them in an efficient manner. As of today, we lack much of this knowledge. To start with, our insights into the fundamental behavior of communication networks are quite limited. For example, we do not know how much information can be reliably conveyed even through the simplest networks; we have only a partial characterization of the tradeoffs between the different elements in the network; or
we do not know what is the minimal effort in terms of bits or energy that is necessary to coordinate the different nodes. Even for those cases in which we have satisfactory answers to these questions, our current implementations are far away from the limits. More critically, in all but a handful of cases, the gaps to these limits grow with the size of the network. That is, the larger the network, the less efficient we are. This hindrance is all the more severe given the constraints that energy consumption and spectrum availability place on communications, especially if they are wireless.

The list of issues that need to be addressed is long and a detailed discussion would take us too far afield from the matter at hand in this thesis. Therefore, we will only describe the two problems that are considered here: coordination and compression.

**Coordination**

One basic problem in networks is how to coordinate the behavior of the different nodes. That is, how to make all of them work together in an organized way [Cam13]. This was a minor issue while communication was an interpersonal matter. However, with the deployment of large scale machine-controlled networks that are in charge of critical tasks, coordination becomes as sensitive task.

In the simplest formulation, we are interested in characterizing the relationship between the communication resources in a network and the degrees of coordination that are possible. In other words, quantifying the amount of communication that is necessary to achieve a certain degree of coordination in a given network. Consider, for example, the hierarchical communication network in Figure 1.1. The topmost node (i.e., the ‘head’) wants to coordinate the rest of nodes in the network (i.e.,

![Figure 1.1: Hierarchical coordination network.](image_url)
the ‘subordinates’) in response to some external stimuli. If there is no or little communication between them, the reaction will have to be quite simple. In contrast, with large communication resources, the ‘head’ can give a more detailed description of the type of response that the ‘subordinates’ should provide. Establishing an explicit characterization of this tradeoff is of utmost importance because in an ever-growing network we wish to satisfy the coordination requirements with the minimum communication expenditure. Moreover, we would like our solutions to be scalable and adaptable to changing conditions.

Although these fundamental problems are important, they have been formalized only very recently and we know surprisingly little about them [CPC10]. Needless to say, other variants are also possible. Indeed, considerations about coordination are not exclusive of dedicated communication networks like the one in Figure 1.1. For example, we might be interested in using an existing infrastructure to coordinate wireless transmissions so that they make an efficient use of the spectrum. This corresponds to the situation depicted in Figure 1.2 where we have two different
groups of users. Communication is restricted to users within the same group. However, due to the broadcast nature of wireless communication, the behavior of users in the first group affects also those in the second group and vice versa. In an effort to mitigate this mutual disturbance, the users could coordinate their actions at a group level by establishing a communication link between two elected group ‘heads’. This represents a departure from the pure coordination scenario in Figure 1.1 for the tradeoff involves now the basic utility of the network (i.e., the communication between users in the same group) as well.

Compression

Another challenge in future communications is to process redundant data. A large body of research in the last century has been devoted to characterizing the essence of information and finding efficient ways of representing data. That is, ways of compressing them [CT06]. The problem of dealing with redundant data is certainly not exclusive to communication networks but, in combination with them, it acquires a new dimension. Large networks serving machine-to-machine communications will most likely process large amounts of data. Arguably, much of them will be irrelevant. Thus, the success of the networks will hinge on their ability to discriminate between essential and superficial aspects of the data.

It is important to emphasize that, large networks are inherently prone to the circulation of redundant data, in particular if they include wireless links. Even if the data first enters the network in a compressed form, intermediate nodes or links

![Figure 1.3: Cooperative network.](image-url)
can produce redundancy. If left uncontrolled, this redundancy yields an undesirable waste of communication resources. However, the presence of redundancy can also be exploited to improve the performance of communication networks. For example, it can be used to increase reliability by routing the information along several paths. Alternatively, redundancy can be used to increase capacity, for example, by having some of the nodes relay the information of their peers. Consider the wireless cooperative scenario depicted in Figure 1.3. A pair of nodes want to exchange more information than their direct link would support. To achieve their goal, they can use the service of some relay nodes. How should these nodes process the signals received? In general, we would like each of them to extract the relevant features contained in its observations and forward them to the interested parts. Observe, however, that the relevancy of the features is dictated by the needs of the nodes that are trying to communicate. The problem is even more involved, given that we would like the different relays to convey non-overlapping pieces of information. Thus, we see that compression is no longer a local matter but a network issue.

1.1 Outline and Contributions

This thesis is divided into eight chapters. As the title suggests, the present one is introductory. In the following, we summarize the contents of the remaining seven. For each of them, we enumerate the publications or manuscripts on which the chapter is based.

Chapter 2

In this chapter, we introduce the different models and problems studied in this thesis along with many of the mathematical tools used in the sequel. Most of the material included can be found in standard textbooks and reputed publications and is referenced accordingly.

Chapter 3

In this chapter, we consider the design of polar codes for coordination over a variety of network topologies. We show that polar codes achieve many of the known fundamental limits for coordination.

This contribution is based on:


To our knowledge, this was the first work that explicitly designed low-complexity codes for coordination.
Chapter 4

In this chapter, we consider the application of polar codes for communication over the relay channel. We show that a similar construction to the one used for coordination is optimal for compress-and-forward relaying.

This contribution is based on:


In addition, parts of the material in this chapter are included in the author’s licentiate thesis:


Chapter 5

In this chapter, we consider the problem of coordinating communications to control the interference created to an external observer. To described the tradeoff between communication, coordination, and interference, we introduce the notion of communication-interference capacity region. We characterize completely the single-user case and partially a multiple-user scenario. The material in this chapter has not been published previously; it is part of a manuscript in preparation.

Chapter 6

In this chapter, we study the relationship between two relevant quantities in compressive sensing: the measurement rate and the mean square error. In particular, we consider partial support recovery methods for estimating sparse signals.

This contribution is based on:

1.2 Contributions Outside the Scope of this Thesis

The author of this thesis has also contributed to several other publications that are outside the scope of this thesis. We give a short account of them in the following.

We designed codes for compress-and-forward relaying using on iterative processing methods in:


Parts of this material appear also in the author’s licentiate thesis [Bla10]. Although these works are related to the material included in this thesis (in particular, Chapter 4), the constructions are based on heuristic rules and their analysis is mostly based on numerical simulations. In addition, their performance is strictly suboptimal. We have therefore decided to exclude them from the present thesis.

In a different line of work, we studied transmission strategies for mixed multiple-input/multiple-output and multiple-input/single-output cognitive radio channels under message-learning constraints. The results have been published in:
Although the channel model is again related to the one in Chapter 4, the type of analysis and the goals are quite different from the ones presented in this thesis. Therefore, we have decided to exclude all this material from the thesis.

1.3 Notation and Acronyms

Notation

Throughout this thesis we use the following notation:

- \( X \) Real-valued random variable
- \( x \) Realization of the random variable \( X \)
- \( P_X(x), P_X, P(X) \) Probability distribution of the random variable \( X \)
- \( X \sim P_X \) \( X \) is a random variable with distribution \( P_X \)
- \( X, X^n \) Real-valued random vector
- \( x, x^n \) Realization of the random vector \( X \)
- \( x_j^i \) Sub-vector \([x_i, \ldots, x_j]^T\) (empty for \( j < i \))
- \( x^T \) Transpose of \( x \)
- \( \|x\| \) Frobenius norm of \( x; \sqrt{x^T x} \)
- \( T_{x^n}(x), T_{x^n} \) Type (or empirical distribution) of \( x^n \) (Definition 2.5)
- \( I, I_k \) Identity matrix (of size \( k \)
1.3 Notation and Acronyms

\( \text{tr} \{ \Phi \} \)  
Trace of a square matrix \( \Phi \)

\( \Phi \otimes \Sigma \)  
Kronecker product of \( \Phi \) and \( \Sigma \)

\( \Phi \otimes_p \)  
\( p^{th} \) Kronecker power of \( \Phi \); \( \Phi \otimes_p = \Phi \otimes \Phi \otimes (p-1) \), \( \Phi \otimes_0 \triangleq [1] \)

\( X - Y - Z \)  
Markov chain (Definition 2.2)

\( r \)  
Measurement rate (Definition 2.77)

\( E \{ X \} \), \( E_X \{ X \} \)  
Expectation of the random variable \( X \)

\( N(\mu, \sigma^2), N(\mu, \Sigma) \)  
Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \) (multivariate with mean vector \( \mu \) and covariance matrix \( \Sigma \))

\( \text{Unif}\{1,\ldots,n\} \)  
Uniform distribution over the set \( \{1,\ldots,n\} \)

\( \mathbb{R} \)  
The set of real numbers

\( \mathbb{R}^+ \)  
The set of non-negative real numbers

\( \mathbb{N} \)  
The set of natural numbers; \( \{1,2,3,\ldots\} \)

\( H(X) \)  
Entropy of \( X \)

\( H(X|Y) \)  
Conditional entropy of \( X \) given \( Y \)

\( I(X;Y) \)  
Mutual information between \( X \) and \( Y \)

\( I(X;Y|Z) \)  
Conditional mutual information between \( X \) and \( Y \) given \( Z \)

\( 1 \{ \cdot \} \)  
Indicator function

\( \sum_x \)  
Summation over all \( x \in X \)

\( \text{GF}(q) \)  
Galois field of size \( q \)

\( \oplus \)  
Sum over \( \text{GF}(q) \)

\( \mathcal{F} \)  
Set

\( \mathcal{F}^c \)  
Complement of \( \mathcal{F} \) (with respect to the universal set)

\( |\mathcal{F}| \)  
Cardinality (i.e., number of elements) of \( \mathcal{F} \)

\( \mathcal{P}(\mathcal{F}) \)  
Power set of \( \mathcal{F} \) (i.e., set containing all valid subsets of \( \mathcal{F} \))

\( \|P_X - Q_X\|_{TV} \)  
Total variation (Definition 2.3)

\( \text{Pr}(\mathcal{E}) \)  
Probability of the event \( \mathcal{E} \)

\( [a] \)  
The smallest integer that is not smaller than the scalar \( a \)

\( [a] \)  
The largest integer that is not larger than the scalar \( a \)

\( |a| \)  
Absolute value of \( a \)

\( [a]^+ \)  
\( \max(a,0) \)

\( f(x) = O(g(x)) \)  
There exist \( M > 0 \) and \( x_0 \in \mathbb{R} \) such that \( |f(x)| \leq M |g(x)| \) for all \( x \geq x_0 \)

\( f(x) = o(g(x)) \)  
\( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \)

Each theorem in this thesis will be contained in a gray box. We will use the symbol \( \square \) to mark the end of the statement of a lemma or a corollary. The symbol ■ will mark the end of a proof. Similarly, the symbol \( \diamond \) will denote the end of a definition, an example, or a remark. We use a unique numbering for all these items. For example, Theorem 3.2 is followed by Example 3.3 and Definition 3.4. In all cases, the first number identifies the chapter. That is, the theorem, the example, and the belong to Chapter 3.
Vectors and matrices. We will use the same notation for vectors and matrices and will reserve upper and lower case letter to distinguish between random elements and their realizations. Whether an element is a vector or a matrix will be clear from the context (e.g. by writing $\mathbf{\phi} \in \mathbb{R}^{m \times n}$).

The entries of vectors are numbered starting with 1. The $i^{th}$ entry (for $i \in \{1, 2, \ldots \}$) in vector $\mathbf{x}$ will be denoted by $x_i$. The column vector corresponding to the $j^{th}$ column of a matrix $\mathbf{\phi}$ is denoted by $\mathbf{\phi}_j$. The scalar in row $i$, column $j$ of a matrix $\mathbf{\phi}$ is denoted by $\phi_{i,j}$.

Given a set $\mathcal{S}$ with elements from $\mathbb{N}$ and a vector $\mathbf{x}$, $\mathbf{x}_\mathcal{S}$ will identify the subvector $[x_{s_1}, \ldots, x_{s_l}]^T$, where $s_1 < s_2 < \ldots < s_l$ corresponds to the natural ordering (i.e., increasing) of the elements in $\mathcal{S}$. Similarly, given a matrix $\mathbf{\phi}$, $\mathbf{\phi}_\mathcal{S}$ denotes the submatrix obtained by considering only the columns specified by the set $\mathcal{S}$, with the natural ordering of the elements in $\mathcal{S}$.

In general, we will take vectors to be column vectors. However, to be consistent with the literature, we will use row vectors in the context of polar codes (i.e., Section 2.3 in Chapter 2, Chapter 3, and Chapter 4). For large part of our analysis, the differentiation will be immaterial. This will only be important when multiplying vectors with matrices. That is, given a matrix $\mathbf{\phi} \in \mathbb{R}^{m \times n}$ and a column vector $\mathbf{x} \in \mathbb{R}^n$, only the multiplication $\mathbf{\phi}\mathbf{x}$ is well defined. On the obverse, if $\mathbf{x} \in \mathbb{R}^m$ is a row vector, we must have $\mathbf{x}\mathbf{\phi}$.

Acronyms

The abbreviations and acronyms used throughout this thesis are summarized in the following.

- **BER** Bit error rate
- **BSC** Binary symmetric channel
- **[bpa]** Bits per action
- **[bpcu]** Bits per channel use
- **CF** Compress-and-forward
- **CS** Compressive sensing
- **DMC** Discrete memoryless channel
- **DMS** Discrete memoryless source
- **DF** Decoded-and-forward
- **i.i.d.** Independent and identically distributed
- **LDPC** Low-density parity-check
- **MAC** Multiple access channel
- **ML** Maximum likelihood
- **MSE** Mean square error
- **PC** Polar code
- **pdf** Probability density function
- **pmf** Probability mass function
- **SC** Successive cancellation
- **s.t.** Such that
<table>
<thead>
<tr>
<th>SNR</th>
<th>Signal-to-noise ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>w.r.t.</td>
<td>with respect to</td>
</tr>
</tbody>
</table>
Chapter 2

Review

The purpose of this chapter is two-fold. It serves us to introduce the problems considered in this thesis and to give an account on previous studies on them. At the same time, we establish the basic terminology and notation that will be used in the coming chapters. The majority of the material included in this chapter can be found in standard textbooks or reputed publications. Therefore, we will only include the proofs of those results that are new or for which we could not find a convenient reference. For the rest, we will point the reader to the corresponding sources.

The chapter is divided in four sections. In the first section, we summarize the basic definitions and results on probability and information theory that appear throughout the thesis. In the second section, we review the problems of communication and coordination over networks. In the third section, we introduce the phenomenon of channel polarization and its applications to channel and source coding. Finally, in the last section, we review the problem of compressive sensing with an emphasis on the connections to the problem of channel coding for communication networks.

2.1 Mathematical Preliminaries

In this section, we establish the basic probability notation and review several basic results that are used later in the thesis. In addition, we introduce Shannon’s basic information measures along with some of their basic properties.

2.1.1 Discrete Random Variables

Let $X$ be a discrete random variable with alphabet $\mathcal{X}$. We denote the probability mass function of $X$ by $P_X(x)$ or, more compactly, by $P_X$ or $P(x)$. We will use the shorthand notation $X \sim P_X$ to mean that $X$ is a random variable with distribution
A realization of $X$ is represented using the lower case letter $x$. Occasionally, we will use other letters to denote probability mass functions, for example, $Q_X$. We denote pairs, triples, or vectors of random variables and their distributions using the same notation, for example, $P_{X,Y}$, $P_{X,Y,Z}$, and $P_{X^n}$, respectively.

In the following, we present some basic definitions and properties of random variables. Most of them have straightforward generalizations to an arbitrary number of random variables or vectors. Observe that vectors are written using bold face and have a super-index indicating their length, for example, $\mathbf{X}^n$. Whenever the length of the vector plays a minor role or is clear from the context, we will drop the super-index (mostly in Chapters 3-4 and 6). Our notation will not distinguish between vectors and matrices, although the latter will never have super-indices indicating their dimensions. Matrices will appear seldom in this thesis (mostly in Chapter 6) and their nature will be emphasized in the context, for example, by writing $\Phi \in \mathbb{R}^{m \times n}$.

**Definition 2.1** (Independence). Let $(X,Y) \sim P_{X,Y}$. The random variables $X$ and $Y$ are statistically independent (or independent for short) if

$$P_{X,Y}(x,y) = P_X(x)P_Y(y)$$  \hspace{1cm} (2.1)

for every $(x,y) \in \mathcal{X} \times \mathcal{Y}$.

Unless otherwise stated, given a joint distribution $P_{X,Y}$, the distributions $P_X$ and $P_Y$ represent the marginals of $X$ and $Y$ respectively. Another special structure of random variables that we will encounter quite often is the Markov chain.

**Definition 2.2** (Markov Chain). Let $(X,Y,Z) \sim P_{X,Y,Z}$. The random variables $X,Y$, and $Z$ form a Markov chain $X \rightarrow Y \rightarrow Z$ if

$$P_{X,Y,Z}(x,y,z)P_Y(y) = P_{X,Y}(x,y)P_{Y,Z}(y,z)$$  \hspace{1cm} (2.2)

for every $(x,y,z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$.

Quite often we are interested in assessing the difference between two random variables. There exist many measures of (dis)similarity but the one that we will encounter most often in this thesis is the total variation (or variational distance) between their distributions.

**Definition 2.3** (Total Variation). Let $P_{X,Y}$ and $Q_{X,Y}$ be two probability distributions defined on $\mathcal{X} \times \mathcal{Y}$. The total variation (or variational distance) between them is defined as

$$\|P_{X,Y} - Q_{X,Y}\|_{TV} \triangleq \frac{1}{2} \sum_{x,y} |P_{X,Y}(x,y) - Q_{X,Y}(x,y)|.$$  \hspace{1cm} (2.3)

\[\diamond\]
Unless otherwise stated, all the summations are taken over all the elements in the corresponding set. For example, the summation in (2.3) is over \((x, y) \in \mathcal{X} \times \mathcal{Y}\). There are two reasons why we have chosen the total variation to measure the dissimilarity between random variables. The first one is the existence of the following lemma.

**Lemma 2.4** (Optimal coupling ([Ald83], Lemma 3.6)). Given two distributions \(P_X\) and \(Q_Y\), we can construct a distribution \(P_{\tilde{X}, \tilde{Y}}\) such that

\[
\Pr(\tilde{X} \neq \tilde{Y}) = \|P_X - Q_Y\|_{TV},
\]

and with marginals \(P_{\tilde{X}} = P_X\) and \(P_{\tilde{Y}} = Q_Y\). 

Throughout this thesis, we refer to the joint distribution of \((\tilde{X}, \tilde{Y})\), whose existence is ensured by the preceding lemma, as the optimal coupling between \(P_X\) and \(Q_Y\). We will often use the notation \(C_{PQ}(\tilde{x}, \tilde{y})\) to denote this distribution.

The second reason for choosing total variation is its role in the definition of one fundamental concept in information theory: strong typicality. Strong typicality, or typicality for short, is a characterization of sequences in terms of the frequency of appearance of the different letters. This frequency is measured by means of the type or empirical distribution.

**Definition 2.5** (Type). Let \(x^n \in \mathcal{X}^n\) and \(y^n \in \mathcal{Y}^n\). The type (or empirical distribution) of the tuple \((x^n, y^n)\) is defined as

\[
T_{x^n, y^n}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbbm{1}\{(x_i, y_i) = (x, y)\}
\]

for all \((x, y) \in \mathcal{X} \times \mathcal{Y}\), where \(\mathbbm{1}\{\cdot\}\) is the indicator function.

**Definition 2.6** (Typical sequence). Let \(x^n \in \mathcal{X}^n\). We say that \(x^n\) is \(\epsilon\)-typical (or just typical, for short) with respect to a distribution \(P_X\) if its type is at variational distance less than \(\epsilon\) from \(P_X\), that is, if

\[
\|T_{x^n} - P_X\|_{TV} < \epsilon.
\]

**Definition 2.7** (Typical Set). The typical set (or set of typical sequences) \(\mathcal{T}_{\epsilon}(P_X)\) with respect to a distribution \(P_X\) is the set of all length-\(n\), \(\epsilon\)-typical sequences, that is,

\[
\mathcal{T}_{\epsilon}(P_X) \triangleq \{x^n : \|T_{x^n} - P_X\|_{TV} < \epsilon\}.
\]
Lemma 2.8 (Consistency [Ber78]). Let $\epsilon > 0$ and $(x^n, y^n) \in T^{(n)}_\epsilon(P_{X,Y})$ for some distribution $P_{X,Y}$. Then,
\begin{align*}
x^n &\in T^{(n)}_\epsilon(P_X), \tag{2.8} \\
y^n &\in T^{(n)}_\epsilon(P_Y), \tag{2.9}
\end{align*}
where $P_X$ and $P_Y$ are the marginals of $P_{X,Y}$.

The study of typical sequences and their properties is central to information theory. We note here that there exist different notions and definitions of typicality. Many of them are roughly equivalent but some are not. We have chosen to use Definition 2.6 because it suits naturally the context of coordination [CPC10]. Before closing this section, we state some basic properties of types and typical sequences that will be used in the sequel. For convenience, we use the notation $\text{Unif}\{1, \ldots, n\}$ to denote the uniform distribution over the set $\{1, \ldots, n\}$, as in $Q \sim \text{Unif}\{1, \ldots, n\}$.

Lemma 2.9. Consider a pair of random sequences $(X^n, Y^n) \sim P_{X^n,Y^n}$ and an independent random variable $Q \sim \text{Unif}\{1, \ldots, n\}$. The expectation of the type $T^{(n)}_{X^n,Y^n}(x,y)$ satisfies:
\begin{align*}
\mathbb{E}\{T^{(n)}_{X^n,Y^n}(x,y)\} &= \frac{1}{n} \sum_{i=1}^{n} P_{X_i,Y_i}(x,y) \tag{2.10} \\
&= P_{X_Q,Y_Q}(x,y). \tag{2.11}
\end{align*}

Proof. The first equality is readily obtained using the definition of type. The second equality is proved in [CPC10] Section VII.B.2.

Lemma 2.10. Let $(X^n, Y^n) \sim \prod_{i=1}^{n} P_{Y|X}(y_i|x_i)P_{X^n}(x^n)$, where $P_{X^n}$ is an arbitrary distribution. The expectation of the type $T^{(n)}_{X^n,Y^n}(x,y)$ satisfies:
\begin{align*}
\mathbb{E}\{T^{(n)}_{X^n,Y^n}(x,y)\} &= P_{Y|X}(y|x) \mathbb{E}\{T^{(n)}_{X^n}(x)\}. \tag{2.12}
\end{align*}

Proof. From Lemma 2.9 we know that
\begin{align*}
\mathbb{E}\{T^{(n)}_{X^n,Y^n}(x,y)\} &= P_{Y_Q|X_Q}(y|x)P_{X_Q}(x), \tag{2.13}
\end{align*}
where $Q \sim \text{Unif}\{1, \ldots, n\}$ is independent of $(X^n, Y^n)$. Observe that this yields the Markov chain $Q - X_Q - Y_Q$ and that $P_{Y_Q|X_Q}(y|x) = P_{Y|X}(y|x)$. Using this and Lemma 2.9 again, we obtain the desired result:
\begin{align*}
\mathbb{E}\{T^{(n)}_{X^n,Y^n}(x,y)\} &= P_{Y|X}(y|x)P_{X_Q}(x) \tag{2.14} \\
&= P_{Y|X}(y|x) \mathbb{E}\{T^{(n)}_{X^n}(x)\}. \tag{2.15}
\end{align*}
Lemma 2.11 ([Yeu08]). Let \( \epsilon > 0 \) and consider a distribution \( P_X \). Let \( X^n \sim \prod_{i=1}^n P_X(x_i) \). Then, for sufficiently large \( n \), there exists \( \varphi(\epsilon) > 0 \) such that
\[
\Pr(X^n \not\in \mathcal{T}_\epsilon^{(n)}(P_X)) < 2^{-n\varphi(\epsilon)}.
\] (2.16)

Lemma 2.12 (Conditional Typicality Lemma ([CK81], Lemma 2.12)). Consider a joint distribution \( P_{X,Y} \) and let \( \epsilon_2 > \epsilon_1 > 0 \). Let \( Y^n \sim \prod_{i=1}^n P_{Y|X}(y_i|x_i) \) for given \( x^n \). For every \( x^n \in \mathcal{T}_{\epsilon_1}^{(n)}(P_X) \), we have that
\[
\Pr((x^n, y^n) \in \mathcal{T}_{\epsilon_2}^{(n)}(P_{X,Y})) \geq 1 - \delta_{\epsilon_1, \epsilon_2}(n),
\] (2.17)
where \( \delta_{\epsilon_1, \epsilon_2}(n) \doteq \frac{1}{4n} \left( \frac{|X||Y|}{\epsilon_2 - \epsilon_1} \right)^2 \).

\[\square\]

Corollary 2.13. Consider a joint distribution \( P_{X,Y} \) and let \( \epsilon_2 > \epsilon_1 > 0 \). Let \( Y^n \sim \prod_{i=1}^n P_{Y|X}(y_i|x_i) \) for given \( x^n \). For every \( x^n \in \mathcal{T}_{\epsilon_1}^{(n)}(P_X) \), we have that
\[
\Pr(Y^n \in \mathcal{T}_{\epsilon_2}^{(n)}(P_Y)) \geq 1 - \delta_{\epsilon_1, \epsilon_2}(n),
\] (2.18)
where \( \delta_{\epsilon_1, \epsilon_2}(n) \doteq \frac{1}{4n} \left( \frac{|X||Y|}{\epsilon_2 - \epsilon_1} \right)^2 \).

\[\square\]

Proof. The proof follows immediately by applying the consistency condition (i.e., Lemma 2.8) to the conditional typicality lemma (i.e., Lemma 2.12).

\[\square\]

Lemma 2.14 (Packing Lemma [GK11]). Consider a joint distribution \( P_{U,X,Y} \). Let \( (U^n, Y^n) \sim Q_{U^n,Y^n} \) with arbitrary \( Q_{U^n,Y^n} \) and let \( X^n(m) \), for \( m \in \{1, \ldots, [2^{nR}] \} \), be random sequences, each distributed according to \( \prod_{i=1}^n P_{X|U}(x_i|u_i) \). Assume that \( X^n(m) \) is pairwise conditionally independent of \( Y^n \) given \( U^n \) for every \( m \in \{1, \ldots, [2^{nR}] \} \). Then, there exists \( \delta(\epsilon) > 0 \) such that \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \) and such that
\[
\lim_{n \to \infty} \Pr\left( (U^n, X^n(m), Y^n) \in \mathcal{T}_\epsilon^{(n)}(P_{U,X,Y}) \text{ for some } m \in \{1, \ldots, [2^{nR}] \} \right) = 0,
\] (2.19)
if \( R < I(X; Y|U) - \delta(\epsilon) \).

The quantity \( I(X; Y|U) \), which is the mutual information between \( X \) and \( Y \) given \( U \), will be formally introduced Section 2.1.3.

Lemma 2.15 (Covering lemma [GK11]). Let \( \epsilon_2 > \epsilon_1 > 0 \) and consider a joint distribution \( P_{U,X,Y} \). Let \( (U^n, X^n) \sim Q_{U^n,X^n} \) with \( Q_{U^n,X^n} \) such that
\[
\lim_{n \to \infty} \Pr\left( (U^n, X^n) \in \mathcal{T}_{\epsilon_1}^{(n)}(P_{U,X}) \right) = 1.
\] (2.20)
Let $Y^n(m)$, for $m \in \{1, \ldots, [2^n R]\}$, be random sequences, conditionally independent of each other and of $X^n$ given $U^n$, each of them distributed according to $\prod_{i=1}^{n} P_{Y|U}(y_i|u_i)$. Then, there exists $\delta(\epsilon_2) > 0$ such that $\delta(\epsilon_2) \to 0$ as $\epsilon_2 \to 0$ and such that

$$\lim_{n \to \infty} \Pr \left((U^n, X^n, Y^n) \notin T^{(n)}_{\epsilon_2}(P_{U,X,Y}) \text{ for all } m \in \{1, \ldots, [2^n R]\}\right) = 0,$$

if $R > I(X; Y|U) + \delta(\epsilon_2)$.

**Definition 2.16** (Permutation-invariant distribution). Let $Z^n \sim P_{Z^n}$. The distribution of $Z^n$ is permutation-invariant with respect to $y^n$ if any two sequences $z^n$ and $\tilde{z}^n$ such that

$$T_{y^n, z^n}(y, z) = T_{y^n, \tilde{z}^n}(y, \tilde{z})$$

have the same probability, that is,

$$P_{Z^n}(z^n) = P_{Z^n}(\tilde{z}^n).$$

**Lemma 2.17** (Strong Markov Lemma ([CPC10], Theorem 12)). Consider a joint distribution $P_{X,Y,Z}$ that yields a Markov chain $X \leftarrow Y \rightarrow Z$. Let $(x^n, y^n) \in T^{(n)}_{\epsilon}(P_{X,Y})$. Let $Z^n$ be chosen randomly from the set of sequences $z^n$ such that $(y^n, z^n) \in T^{(n)}_{\epsilon}(P_{Y,Z})$, according to a distribution that is permutation-invariant with respect to $y^n$. Then,

$$\Pr \left((x^n, y^n, Z^n) \in T^{(n)}_{\epsilon}(P_{X,Y,Z})\right) \to 1$$

exponentially fast as $n \to \infty$.

### 2.1.2 Continuous Random Variables

In Chapter 6 we will encounter continuous random variables. Continuous random variables are defined on continuous alphabets and share many properties with discrete random variables.

Let $X$ be a continuous random variable defined on $\mathcal{X}$. We will use the shorthand notation $X \sim F_X$ to mean that $X$ is a random variable with distribution $F_X$. Similarly, we will use the shorthand notation $X \sim f_X$ to mean that $f_X$ is the probability density function (pdf) of $X$, provided that such a density exists (that is, the variable is absolutely continuous).

The Gaussian distribution will play a prominent role in our discussions. We use the notation $X \sim N(\mu, \sigma^2)$ to mean that $X$ is Gaussian distributed with mean $\mu$ and variance $\sigma^2$. Similarly, we use $X \sim N(\mu, \Sigma)$ to mean that $X$ is a Gaussian distributed random vector with mean vector $\mu$ and covariance matrix $\Sigma$. In Chapter 6 we will make use of the following results concerning Gaussian random vectors.
Lemma 2.18 (Chernoff bound \cite{Che52}). Let $X$ be any random variable with moment-generating function. Let $\psi(s)$ denote this function. We have that
\[ \Pr(X \geq a) \leq \min_{s \geq 0} e^{-sa}\psi(s). \quad (2.25) \]
\[ \square \]

Corollary 2.19 (Chernoff bound for $\chi^2_n$). Let $X \in \mathbb{R}^n$ be a vector with independent and identically distributed (i.i.d.) $X_i \sim N(0,P_x)$ and let $\mathcal{I}_\epsilon \triangleq (-\epsilon, \epsilon)$. We have that
\[ \Pr\left( \frac{X^T X}{n} - P_x \notin \mathcal{I}_\epsilon \right) \leq 2 \max\left\{ e^{-n(ab + \frac{1}{2} \ln(1-2b))}, e^{-n(\frac{1}{2} \ln(1+2d) - cd)} \right\} \quad (2.26) \]

with
\[ a \triangleq \left( 1 + \frac{\epsilon}{P_x} \right), \quad (2.27) \]
\[ b \triangleq \frac{\epsilon}{2(P_x + \epsilon)}, \quad (2.28) \]
\[ c \triangleq \left( 1 - \frac{\epsilon}{P_x} \right), \quad (2.29) \]
\[ d \triangleq \frac{\epsilon}{2(P_x - \epsilon)}. \quad (2.30) \]

Moreover, both terms decay exponentially with $n$ (i.e., the exponents are negative).
\[ \square \]

Proof. Let
\[ P_1 \triangleq \Pr\left( \frac{X^T X}{P_x} \geq n \left( 1 + \frac{\epsilon}{P_x} \right) \right), \quad (2.31) \]
\[ P_2 \triangleq \Pr\left( \frac{X^T X}{P_x} \leq n \left( 1 - \frac{\epsilon}{P_x} \right) \right). \quad (2.32) \]

Clearly, we have
\[ \Pr\left( \frac{X^T X}{n} - P_x \notin \mathcal{I}_\epsilon \right) \leq P_1 + P_2 \quad (2.33) \]
\[ \leq 2 \max\{P_1, P_2\}. \quad (2.34) \]

By the Chernoff bound (Lemma 2.18), we have that
\[ P_1 \leq \min_{s \geq 0} e^{-sna}\psi(s) \quad (2.35) \]
\[ = \min_{s \geq 0} e^{-n(sa + \frac{1}{2} \ln(1-2s))} \quad (2.36) \]
where \( \psi(s) = (1 - 2s)^{-\frac{n}{2}} \) is the moment-generating function of a \( \chi \)-square distribution with \( n \) degrees of freedom, which is defined for \( s < \frac{1}{2} \). Taking the first and second derivatives of the exponent term, we see that

\[
s = \frac{1}{2} a - \frac{1}{2} b \quad (2.37)
\]
\[
= b \quad (2.38)
\]
yields the tightest bound. Observe that \( 0 < c < \frac{1}{2} \). Proceeding similarly, we obtain

\[
P_2 \leq \min_{t \geq 0} e^{-n\left(\frac{1}{2} \ln(1+2t) - tc\right)} \quad (2.39)
\]
and the optimal value is \( t = d \).

\[\square\]

**Corollary 2.20** (Chernoff bound for product-normal variables). Let \( X \in \mathbb{R}^n, Y \in \mathbb{R}^n \) be two independent vectors, each of them with i.i.d. entries \( X_i \sim \mathcal{N}(0, P_x) \) and \( Y_i \sim \mathcal{N}(0, P_y) \), and let \( \mathcal{I}_\epsilon \triangleq (\epsilon, \epsilon) \). We have that

\[
\Pr\left(\frac{X^T Y}{n} \notin \mathcal{I}_\epsilon\right) \leq 2 \max \left\{ e^{-n\left(ab + \frac{1}{2} \ln(1-b^2)\right)}, e^{-n\left(\frac{1}{2} \ln(1-c^2) - ac\right)} \right\} \quad (2.40)
\]

with

\[
a \triangleq \frac{\epsilon}{\sqrt{P_x P_y}}, \quad (2.41)
\]
\[
b \triangleq -1 + \frac{1 + 4\epsilon^2}{2\epsilon}, \quad (2.42)
\]
\[
c \triangleq 1 + \frac{1 + 4\epsilon^2}{2\epsilon}. \quad (2.43)
\]

Moreover, both terms decay exponentially with \( n \) (i.e., the exponents are negative).

\[\square\]

**Proof.** The proof follows the same lines of that for Lemma 2.19. In this case, we need the moment-generating function of the sum of \( n \) products of two independent Gaussian \( \mathcal{N}(0, 1) \) random variables, which is given by [Cra36]

\[
\psi(s) = (1 - s^2)^{-\frac{n}{2}} \quad (2.44)
\]

\[\square\]

In the preceding corollaries, the interval \( \mathcal{I}_\epsilon \) was fixed (i.e., it did not depend on \( n \)). We show that it is possible to achieve a vanishing error probability even if the size of the interval decreases with \( n \).
Corollary 2.21. Let $X \in \mathbb{R}^n$ be a vector with i.i.d. $X_i \sim \mathcal{N}(0, P_x)$. Then, there exists a non-increasing sequence $\epsilon_n$ of positive numbers such $\epsilon_n \to 0$ as $n \to \infty$ and such that

$$\Pr \left( \frac{X^T X}{n} - P_x \notin I_{\epsilon_n} \right) \leq o\left(\frac{1}{n}\right).$$

(2.45)

Proof. By Corollary 2.19 we know that

$$\Pr \left( \frac{X^T X}{n} - P_x \notin I_{\epsilon_n} \right) \leq 2e^{-ng(\epsilon_n)}$$

(2.46)

where $g(\epsilon_n)$ is a positive function of $\epsilon_n$ such that $g(\epsilon_n) \to 0$ as $\epsilon_n \to 0$. Choose $\epsilon_n$ so that $g(\epsilon_1) = 1$ and $g(\epsilon_n) = \frac{1}{\log n}$ for $n > 1$. Then,

$$\lim_{n \to \infty} 2e^{-ng(\epsilon_n)} = \lim_{n \to \infty} 2e^{-\frac{n}{\log n}}$$

(2.47)

$$= 0.$$ 

(2.48)

This proves the claim. \qed

We have chosen to show that the bound decays as $o(1/n)$ because this result will be used in Chapter 6 but stronger results can be proved in the same way. A similar corollary can be established for the product of Gaussian random variables.

Corollary 2.22. Let $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^n$ be two independent vectors, each of them with i.i.d. entries $X_i \sim \mathcal{N}(0, P_x)$ and $Y_i \sim \mathcal{N}(0, P_y)$. Then, there exists a non-increasing sequence $\epsilon_n$ of positive numbers such $\epsilon_n \to 0$ as $n \to \infty$ and such that

$$\Pr \left( \frac{X^T X}{n} - P_x \notin I_{\epsilon_n} \right) \leq o\left(\frac{1}{n}\right).$$

(2.49)

Proof. The proof is identical to that for Corollary 2.21. \qed

Lemma 2.23 (\cite{JFR10}, Lemma 1). Let $0 < \beta < \alpha$. Let $u^m \in \mathbb{R}^m$ be a vector such that $\frac{1}{m}\|u^m\|^2 \in (\alpha - \beta, \alpha + \beta)$. Let $V^m \in \mathbb{R}^m$ be a random vector with i.i.d. $V_i \sim \mathcal{N}(0, \sigma^2_v)$. Then, for any $\lambda \in (0, \alpha - \beta)$,

$$\Pr \left( \frac{1}{m} \|u^m - V^m\|^2 \leq \lambda \right) \leq 2^{-\frac{m}{2} \log \left( \frac{\alpha - \beta}{\lambda} \right)}.$$ 

(2.50)

\qed
**Lemma 2.24.** Let \( \Phi \in \mathbb{R}^{m \times \ell} \) with i.i.d. \( \Phi_{i,j} \sim \mathcal{N}(0,1) \). For \( m > \ell + 1 \), we have that

\[
E \left\{ \text{tr} \left\{ (\Phi^T \Phi)^{-1} \right\} \right\} = \frac{\ell}{m-\ell-1}.
\] (2.51)

**Proof.** The product \( \Phi^T \Phi \) follows a Wishart distribution and thus \( (\Phi^T \Phi)^{-1} \) follows an inverse Wishart distribution. The result is obtained by computing the trace of the first moment of the inverse Wishart distribution (see, for example, [Pre05]).

\( \square \)

### 2.1.3 Information Measures

In this section, we introduce Shannon’s fundamental measures of information for discrete random variables. These measures and their basic properties are used repeatedly throughout the thesis. In their definitions, we adopt the convention that \( 0 \log 0 = 0 \).

**Definition 2.25 (Entropy).** Let \( X \sim P_X \). The entropy of the random variable \( X \) is defined as

\[
H(X) \triangleq - \sum_x P_X(x) \log P_X(x).
\] (2.52)

The entropy measures the average amount of information contained in a random variable, or equivalently, the average amount of uncertainty that is removed when the outcome of the random variable is revealed. The base of the logarithm determines the units of the entropy: bits (base-2 logarithm), nats (base-e), or, in general, \( q \)-ary units (base-\( q \)).

**Definition 2.26 (Conditional entropy).** Let \( (X,Y) \sim P_{X,Y} \). The conditional entropy of the random variable \( X \) given \( Y \) is defined as

\[
H(X|Y) \triangleq - \sum_{x,y} P_{X,Y}(x,y) \log P_{X|Y}(x|y).
\] (2.53)

Conditional entropy is a measure of the uncertainty left in \( X \) on average after the observation of \( Y \). A common and more intuitive name for conditional entropy is equivocation.

**Definition 2.27 (Mutual information).** Let \( (X,Y) \sim P_{X,Y} \). The mutual information between the random variables \( X \) and \( Y \) is defined as

\[
I(X;Y) \triangleq \sum_{x,y} P_{X,Y}(x,y) \log \frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}.
\] (2.54)
Mutual information measures the amount of information that $X$ contains about $Y$. Observe that the definition of mutual information is symmetric (i.e., $I(X; Y) = I(Y; X)$). That is, $I(X; Y)$ also measures the amount of information that $Y$ contains about $X$ and the two measures coincide.

**Definition 2.28 (Conditional mutual information).** Let $(X, Y, Z) \sim P_{X,Y,Z}$. The conditional mutual information between the random variables $X$ and $Y$ given $Z$ is defined as

$$I(X; Y | Z) \triangleq \sum_{x,y,z} P_{X,Y,Z}(x,y,z) \log \frac{P_{X,Y,Z}(x,y,z)}{P_{X|Z}(x|z)P_{Y|Z}(y|z)}.$$  

The conditional mutual information measures the reduction in uncertainty left in $X$ on average after the observation of $Y$ when $Z$ is given.

The definitions of these basic information measures are generalized to more random variables in a straightforward manner (e.g., joint entropy of a pair of random variables, etc). The following lemma summarizes some of their basic properties, which are used repeatedly through the thesis.

**Lemma 2.29 (Basic properties of entropy and mutual information [Sha48]).** Let $(X, Y) \sim P_{X,Y}$.

- $0 \leq H(X | Y) \leq H(X) \leq \log |\mathcal{X}|$.
- $I(X; Y) = H(X) - H(X | Y) = H(Y) - H(Y | X)$.

Let $(X^n, Y) \sim P_{X^n,Y}$.

- **Chain rule for entropy:** $H(X^n) = \sum_{i=1}^n H(X_i | X_{1}^{i-1})$.
- **Chain rule for mutual information:** $I(X^n; Y) = \sum_{i=1}^n I(X_i; Y | X_{1}^{i-1})$.

These properties were originally described by Shannon. A proof using modern notation can be found in most information theory books (e.g., [CT06]). We conclude this section with the statement of Fano’s inequality.

**Lemma 2.30 (Fano’s inequality [Fan52]).** Let $(X, Y) \sim P_{X,Y}$ and $P_e \triangleq \Pr(X \neq Y)$. Then,

$$H(X | Y) \leq 1 + P_e \log |\mathcal{X}|.$$  

(2.56)
Fano’s inequality connects the probability of error in guessing the value of the random variable $X$ from a related observation $Y$ to the conditional entropy $H(X|Y)$. The inequality describes a fundamental limit of statistical inference for our probabilistic model and is arguably one of the greatest results in information theory, central to most capacity results. A proof can be found in most textbooks (e.g., [CT06]).

2.2 Communication and Coordination in Networks

In this section, we review the problems of communication and coordination with an emphasis on the network models that are considered in this thesis.

2.2.1 Communication

The problem of communication is, as Shannon described it in his landmark paper, “that of reproducing at one point either exactly or approximately a message selected at another point” [Sha48]. In this thesis we consider two variations of this problem: point-to-point communication and communication over a channel with a relay.

Point-to-Point Communication

The simplest example of communication over noisy channels is the point-to-point channel in Figure 2.1. The source node wants to communicate reliably a message $M$ to the destination over the noisy discrete memoryless channel $(\mathcal{X}, \mathcal{P}_{Y|X}, \mathcal{Y})$.

Definition 2.31 (Discrete memoryless channel). A discrete memoryless channel $(\mathcal{X}, \mathcal{P}_{Y|X}, \mathcal{Y})$ consists of a finite input alphabet $\mathcal{X}$, a finite output alphabet $\mathcal{Y}$, and a family of conditional probability distributions $\mathcal{P}_{Y|X}(y|x)$ for every $x \in \mathcal{X}$.

For economy of notation, we will refer to the discrete memoryless channel (DMC) simply by $\mathcal{P}_{Y|X}$ or $P(y|x)$. The channel is inherently unreliable in the sense that, for every single output symbol there is some uncertainty regarding the corresponding input. In order to improve reliability, source and destination use a protocol that employs a block of $n$ channel uses to transmit a single message chosen uniformly at random from a set $\mathcal{M}$. This protocol is an $(n, 2^{nR})$-code.

Definition 2.32 (Code). An $(n, 2^{nR})$-code for the point-to-point channel consists of:

- a message set $\mathcal{M} \triangleq \{1, \ldots, \lfloor 2^{nR} \rfloor\}$,
- an encoding function $x^n : \mathcal{M} \rightarrow \mathcal{X}^n$,
- a decoding function $\hat{m} : \mathcal{Y}^n \rightarrow \mathcal{M} \cup \{e\}$.
The quantity $R$ is known as the communication rate. Roughly speaking, it quantifies the average amount of information that the source puts into the channel per channel use.

**Definition 2.33** (Induced distribution). The code, in conjunction with the uniform distribution of the messages and the effect of the channel, induces the distribution

$$
\frac{1}{|M|} P_{X^n|M}(x^n|m) \prod_{i=1}^{n} P_{Y|X}(y_i|x_i) P_{\hat{M}|Y^n}(\hat{m}|y^n).
$$

(2.57)

**Definition 2.34** (Achievability). A rate $R$ is achievable if there exists a sequence of $(n, 2^{nR})$-codes, indexed by the codeword length $n$, such that

$$
\lim_{n \to \infty} \Pr(\hat{M} \neq M) = 0
$$

under the distribution induced by the codes.

Note that, formally speaking, the achievability statement refers to the sequence of distributions induced by the sequence of codes. However, to avoid making cumbersome statements, we will simply talk about the distribution induced by the codes.

**Definition 2.35** (Capacity). The capacity $C$ of the point-to-point channel is the supremum of all rates that are achievable.

The most celebrated result in information theory is Shannon’s characterization of the capacity of the point-to-point channel.¹

**Theorem 2.36** (Channel coding theorem [Sha48]). The capacity of the point-to-point discrete memoryless channel is given by

$$
C = \max_{P_X} I(X;Y).
$$

(2.59)

¹Strictly speaking, Shannon did not provide a complete proof of the result. The result was subsequently formalized by several information theorists. A rigorous proof with modern notation and a detailed account of the history of the proof can be found in many textbooks (e.g., [CT06]).
Thus, finding the capacity of a discrete memoryless channel is tantamount to finding the distribution that maximizes the input-output distribution. For the class of symmetric channels (as defined in [Gal68]), which are considered often in this thesis, the capacity corresponds is given by the uniform input distribution.

**Definition 2.37** (Symmetric discrete memoryless channel [Gal68]). Let $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ be discrete memoryless channel. Let $P$ be the matrix of transition probabilities (with the inputs determining rows and the outputs determining columns). For any subset $Y_i \subseteq \mathcal{Y}$, we construct its submatrix of transition probabilities $P_{Y_i}$ obtained by considering only the columns of $P$ whose symbols are contained in $Y_i$. The channel is symmetric if $\mathcal{Y}$ can be partitioned into subsets $Y_1, \ldots, Y_k$ in such a way that, for each $Y_i$, the submatrix $P_{Y_i}$ satisfies that:

- Each row is a permutation of each other row.
- Each column (if there are more than one) is a permutation of each other column.

**Lemma 2.38** ([Gal68], Theorem 4.5.2). For every symmetric discrete memoryless channel $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$, the uniform distribution on $\mathcal{X}$ maximizes the input-output mutual information $I(X; Y)$. □

We conclude our review on point-to-point channels by introducing channel degradation, which is one way of formalizing the notion that some channels are better suited for communication than others.

**Definition 2.39** (Stochastic degradation). Consider two DMCs $P_{Y_1|X}$ and $P_{Y_2|X}$. The channel $P_{Y_2|X}$ is stochastically degraded with respect to $P_{Y_1|X}$ if there exists a distribution $P_{Y_2|Y_1}$ such that

$$P_{Y_2|X}(y_2|x) = \sum_{y_1} P_{Y_1|X}(y_1|x)P_{Y_2|Y_1}(y_2|y_1)$$

(2.60)

for every $(x,y_2) \in \mathcal{X} \times \mathcal{Y}_2$. □

**Relay Channel**

The second model for communication considered in this thesis is the relay channel with orthogonal receivers. We refer to it as the relay channel for short, although it is only an instance of a more general model introduced in [vdM71]. The relay channel is a multi terminal problem in which a source node wants to convey reliably a message to a destination. There is a third terminal, known as the relay, that can help the source-receiver pair to carry their communication.
2.2 Communication and Coordination in Networks

Definition 2.40 (Relay channel with orthogonal receivers). A discrete memoryless relay channel with orthogonal receivers $(X \times X_R, P_{Y_{SD,Y_{SR}}|X}, Y_{SD} \times Y_{SR} \times Y_{RD})$ consists of two finite input alphabets $X$ and $X_R$, three finite output alphabets $Y_{SD}, Y_{SR}$, and $Y_{RD}$, and a family of conditional product probability distributions

$$P_{Y_{SD,Y_{SR}}|X}(y_{sd},y_{sr}|x)P_{Y_{RD}|X_R}(y_{rd}|x_R)$$

for every $(x,x_r) \in X \times X_R$.

The protocol used by source, relay, and destination to carry out the communication is an $(n, 2^{nR})$-code.

Definition 2.41 (Code). An $(n, 2^{nR})$-code for the relay channel consists of:

- a message set $\mathcal{M} \triangleq \{1, \ldots, [2^{nR}]\}$,
- an encoding function $x^n : \mathcal{M} \rightarrow X^n$,
- a set $\{x_{r,i}\}$ of relaying functions $x_{r,i} : Y_{SR}^i \rightarrow X_R$, defined for $1 \leq i \leq n$,
- a decoding function $\hat{m} : Y_{SD}^n \times Y_{RD}^n \rightarrow \mathcal{M} \cup \{e\}$.

We assume that the message is uniformly distributed over the message set. The notions of achievability and capacity for the relay channel have an identical meaning as their point-to-point counterparts. However, the capacity of the relay channel remains unknown except for a few specific classes, which do not include the relay channel with orthogonal receivers (see, e.g., [GK11]).

Cover and El Gamal established in [CG79] two fundamental strategies for communication based on two different relaying philosophies: decode-and-forward and compress-and-forward. In decode-and-forward, as the name suggests, the relay decodes the source transmission and forwards some information that allows the destination to determine the message transmitted by the source. Using this strategy, reliable communication is possible at any rate up to:

![Relay channel with orthogonal receivers](image-url)
Definition 2.42 (Decode-and-forward [CG79]).

\[
R_{DF} \triangleq \max_{P_X, P_{X_R}} \min \{I(X; Y_{SR}), I(X; Y_{SD}) + I(X_R; Y_{RD})\}. \tag{2.62}
\]

In contrast, in compress-and-forward, the relay only describes its observation to the destination. Using this strategy, reliable communication is possible at any rate up to:

Definition 2.43 (Compress-and-forward [CG79]).

\[
R_{CF} \triangleq \max_{P_X, P_{X_R}, P_{Y_Q|Y_{SR}}} \{I(X; Y_Q Y_{SD}) : I(Y_Q; Y_{SR}|Y_{SD}) \leq I(X_R; Y_{RD})\} \tag{2.63}
\]

with auxiliary random variable \(Y_Q\) such that \(|Y_Q| \leq |Y_{SR}| + 1\).

In the formulation in (2.63), the auxiliary random variable \(Y_Q\) plays the role of the compressed observation at the relay. Roughly speaking, the distribution \(P_{Y_Q|Y_{SR}}\) in the maximization determines the fidelity of this compression [SW73, WZ76]. Observe that, when compressing the observation, the relay can exploit the correlation between the observations \(Y_{SR}\) and \(Y_{SD}\), and that the fidelity is limited by the capacity of the relay-destination link through the constraint \(I(Y_Q; Y_{SR}|Y_{SD}) \leq I(X_R; Y_{RD})\).

Together with these two transmission strategies, Cover and El Gamal presented an upper bound on the capacity of the relay channel. This bound is based on Fano’s inequality (Lemma 2.30) and was later generalized to arbitrary networks. It is now known as the cut-set bound and its expression for relay channels with orthogonal receivers is the following:

Definition 2.44 (Cut-set bound [CG79]).

\[
R_{CS} \triangleq \max_{P_X, P_{X_R}} \min \{I(X; Y_{SD}) + I(X_R; Y_{RD}), I(X; Y_{SD} Y_{SR})\}. \tag{2.64}
\]

Theorem 2.45 ([CG79]). The capacity \(C\) of the relay channel satisfies

\[
\max \{R_{DF}, R_{CF}\} \leq C \leq R_{CS}. \tag{2.65}
\]

\[\text{In general, the optimization in the cut-set bound is over joint distributions of the inputs } P_{X, X_R}. \] However, due to the orthogonality of the receivers, the bound reduces to (2.64), see [Kim07].
2.2 Communication and Coordination in Networks

Stochastic degradation, which was introduced for point-to-point channels, can be used to compare the channels from source to destination and from source to relay. In particular, we say that the relay channel is stochastically degraded when the former (i.e., the marginal $P_{Y_{SD} | X}$) is stochastically degraded with respect to the latter (i.e., the marginal $P_{Y_{SR} | X}$). In addition, in the context of relay channels, it is also interesting to consider the stronger notion of physical degradation.

**Definition 2.46** (Physical degradation). We say that the relay channel is degraded if $X, Y_{SR},$ and $Y_{SD}$ form a Markov chain $X - Y_{SR} - Y_{SD}$.

Note that these two notions of degradation are not equivalent. Every physically degraded relay channel is also stochastically degraded but the reverse is not true. Moreover, the capacity of the physically degraded relay channel and of the stochastically degraded relay channel are different. For the former, it coincides with the decode-and-forward lower bound. For the latter, a single letter characterization of the capacity has not been obtained yet, but it is well-known that, in general, it does not coincide with the decode-and-forward lower bound. Neither does it coincide with the compress-and-forward lower bound (see, e.g., [GK11]).

**Practical Codes for the Relay Channel**

Little research was carried out on the relay channel and on network information theory in general for more than a decade after the initial studies described in the previous section. In the early nineties, with the advent of wireless communication and the Internet, researchers focused again on network information theory. This coincided in time with the rediscovery of iterative decoding, which allowed for constructing codes that came very close to the capacity of the point-to-point channel.

Iterative methods have also been instrumental in developing practical codes that approach the information-theoretic limits of other problems, including the relay channel. Examples of constructions for decode-and-forward are: distributed turbo coding [ZV03], soft decode-and-forward [SV05], block LDPC codes [CdSA07], or spatially-coupled LDPC codes [UKS11, STS13]. In comparison, there are fewer available constructions for compress-and-forward relaying. In part, this is motivated by the smaller interest for application. These constructions usually combine elements of source and channel coding [JTGG09], and rely on schemes for Slepian-Wolf [SW73] or Wyner-Ziv [WZ76] coding [HT06], often using iterative processing as well [ULSX09, BTS10, BTS11].

Although many of these constructions have empirically been shown to perform well, none of them achieves, in a strict sense, any of the basic rates introduced previously (except for the new spatially-coupled LDPC codes). Polar codes, which have been developed recently and will be discussed in Section 2.3, constitute a leap forward in the search for efficient codes for the relay channel, since they achieve the decode-and-forward [ART+10, Kar12, Bra13] and compress-and-forward rates [BTRS10, BTA+12] with affordable complexity.
2.2.2 Coordination

In a broad sense, coordination can be described as “the act of making agents work together in an organized way” [Cam13]. Several formulations of the problem of coordination can be accommodated under this definition. In this thesis, we are concerned with the interplay between communication and coordination on simple abstractions of networks. In other words, we are interested on the amount of information that has to be exchanged in a network to achieve a certain degree of coordination. This formulation was introduced by Cuff, Permuter, and Cover in [CPC10].

Cuff et al. proposed two different notions of coordination based on alternative characterizations of the actions. In the first notion, known as strong coordination, the actions are characterized in terms of their probability distribution (e.g., $P_{X^n,Y^n}(x^n,y^n)$). Strong coordination is achieved if the actions produced by the network are statistically indistinguishable from those obtained by sampling a fixed distribution. That is, if the distribution of the actions is arbitrarily close (in total variation, cf. Definition 2.3) to a given distribution. In contrast, in empirical coordination, the actions are characterized in terms of their type or empirical distribution (e.g., $T_{X^n,Y^n}(x,y)$, cf. Definition 2.5). In this way, we say that empirical coordination is achieved if the type of the actions is arbitrarily close (in total variation) to a certain (single-letter) distribution. This is a weaker notion than strong coordination but it has two features that make it particularly appealing: i) it is general enough to cover a wide range of applications (e.g., source coding, control of interference, etc.), and ii) it is more tractable from a mathematical point of view. For these reasons, in this thesis, we only consider this latter notion and, thus, we drop the qualifier ‘empirical’.

Although [CPC10] gave the first formulation of the problem in terms of coordination, some aspects had already been studied in [KS07] in a different context. Interestingly, Cuff et al. showed that there are deep connections to the theory of rate-distortion, which was introduced by Shannon [Sha48, Sha59]. We will review this relationship in Section 2.2.3.

In the following, we use a simple two-node network to introduce the basic terminology and give a precise statement of the problem. Then, we summarize the results for more complex network topologies. In the next section, we will review the aforementioned connections to the theory of rate-distortion. In all the discussion, it is implicit that the number of possible actions is finite.

Problem Formulation

The first component in the class of coordination problems that we consider in this thesis is the source.

Definition 2.47 (Discrete memoryless source). A discrete memoryless source $(\mathcal{X}, P_X)$ consists of a finite alphabet $\mathcal{X}$ and a probability mass function $P_X(x)$ defined on $\mathcal{X}$.
2.2 Communication and Coordination in Networks

![Two-node network diagram]

Figure 2.3: Two-node network.

For economy of notation, we refer to the discrete memoryless source (DMS) \((\mathcal{X}, P_X)\) simply by \(P_X\) or \(P(x)\).

In the two-node network in Figure 2.3, Node \(X\) observes a sequence \(X^n\) of i.i.d. actions that are generated externally by the DMS \(P_X\). The node can communicate to Node \(Y\) over a noiseless channel of capacity \(R\) bits per action [bpa]. The purpose of this communication is to have Node \(Y\) generate a sequence \(Y^n\) such that the joint type \(T_{X^n,Y^n}\) is close to a desired distribution \(P_{X,Y}\) with high probability. An important aspect of our approach is that the nodes are allowed to process the actions in blocks. To implement the coordination, the network uses an \((n, 2^{nR})\)-code.

**Definition 2.48 (Code).** An \((n, 2^{nR})\)-code for coordination in the two-node network consists of:

- a message set \(\mathcal{M} \triangleq \{1, \ldots, 2^{nR}\}\),
- an encoding function \(i : \mathcal{X}^n \times \Omega \to \mathcal{M}\),
- a decoding function \(y^n : \mathcal{M} \times \Omega \to \mathcal{Y}^n\),

where \(\Omega\) is a source of common randomness, independent of the external actions, and shared by both nodes.

**Definition 2.49 (Induced distribution).** The code for coordination, in conjunction with the distribution \(P_X\) of the external actions, induces a distribution \(P_{X^n,Y^n}(x^n, y^n)\) on the tuple of actions \((X^n, Y^n)\).

As discussed before, it would be more appropriate to talk about the sequence of distributions induced by the sequence of coordination codes. However, to avoid cumbersome statements like the preceding one, we will simply talk about the distribution induced by the codes. Similarly, we will use the notation \(P_{X^n,Y^n}(x^n, y^n)\) to refer to the sequence \(\{P_{X^n,Y^n}(x^n, y^n)\}\).

**Definition 2.50 (Achievability).** A distribution \(P_{Y|X}P_X\) is achievable for coordination with rate \(R\) if there exists a sequence of \((n, 2^{nR})\)-codes, indexed by the
codeword length \( n \), and a choice of distribution \( P_\Omega \) for common randomness such that, for any \( \epsilon > 0 \),
\[
\lim_{n \to \infty} \Pr(\|T_{X^n,Y^n} - P_{Y|X}P_X\|_{TV} \geq \epsilon) = 0 \tag{2.66}
\]
under the distribution induced by the codes.

For obvious reasons, we will refer to the probability
\[
\Pr(\|T_{X^n,Y^n} - P_{Y|X}P_X\|_{TV} \geq \epsilon) \tag{2.67}
\]
(for given \( \epsilon \)) as the probability of coordination error. In addition, we will use the shorthand notation
\[
\|T_{X^n,Y^n} - P_{Y|X}P_X\|_{TV} \to 0 \text{ in probability} \tag{2.68}
\]
to mean that (2.66) holds for any \( \epsilon > 0 \). One important result about empirical coordination is that common randomness is not necessary at all.

**Lemma 2.51** (Common randomness is not necessary ([CPC10], Theorem 2)). Any distribution \( P_{Y|X}P_X \) that is achievable for coordination with rate \( R \) can also be achieved with \( \Omega = \emptyset \). \( \square \)

**Definition 2.52** (Coordination capacity region). Given a source distribution \( P_X \), the coordination capacity region \( C_{P_X} \) is the closure of the set of achievable rate-coordination tuples \((R, P_{Y|X})\). \( \diamond \)

The coordination capacity region is known only for a few network topologies. In the following, we review the cases that are relevant for this thesis. By virtue of Lemma 2.51, we omit any reference to common randomness in the definitions and characterizations that follow.

**Two-Node Network**

Consider the two-node network in Figure 2.3 along with the definitions introduced in the previous section (with \( \Omega = \emptyset \)).

**Theorem 2.53** ([CPC10], Theorem 3). The coordination capacity region \( C_{P_X} \) of the two-node network is given by
\[
C_{P_X} = \{(R, P_{Y|X}) : R \geq I(X;Y)\}. \tag{2.69}
\]
Consider the three-node cascade network in Figure 2.4. Node $X$ observes a sequence $X^n$ of i.i.d. actions that are generated externally by the DMS $P_X$. The node can communicate to Node $Y$ over a noiseless channel of capacity $R_1$ [bpa]. In turn, Node $Y$ can communicate to Node $Z$ at rate $R_2$ [bpa]. The purpose of these communications is to have Nodes $Y$ and $Z$ generate tuples of actions, $Y^n$ and $Z^n$, respectively, such that the joint type $T_{X^n,Y^n,Z^n}$ is close to a desired distribution $P_{X,Y,Z}$ with high probability. To implement this coordination, the network uses an $(n, 2^{nR_1}, 2^{nR_2})$-code.

**Definition 2.54 (Code).** An $(n, 2^{nR_1}, 2^{nR_2})$-code for coordination in the three-node cascade network consists of:

- two message sets
  \[
  M_1 \triangleq \{1, \ldots, \lfloor 2^{nR_1} \rfloor\},
  \]
  \[
  M_2 \triangleq \{1, \ldots, \lfloor 2^{nR_2} \rfloor\},
  \]

- an encoding function $i : X^n \to M_1$,

- a recoding function $j : M_1 \to M_2$,

- two decoding functions
  \[
  y^n : M_1 \to Y^n,
  \]
  \[
  z^n : M_2 \to Z^n.
  \]

The notions of induced distribution, achievability, and coordination capacity region are straightforward extensions of their two-node network counterparts.
Theorem 2.55 (CPC10, Theorem 4). The coordination capacity region $C_{PX}$ of the three-node cascade network is given by

$$C_{PX} = \left\{ (R_1, R_2, P_{Y,Z|X}) : R_1 \geq I(X;YZ), R_2 \geq I(X;Z) \right\}. \quad (2.75)$$

Consider now the $(k + 1)$-node cascade network in Figure 2.5. The first node observes a sequence of external actions and uses a chain of communication links to enforce coordination with the remaining $k$ nodes. The definitions of $(n, 2^nR_1, \ldots, 2^nR_k)$-code, induced distribution, achievability, and coordination capacity region are straightforward generalizations of the ones for the three-node cascade network.

Corollary 2.56 (CPC10, Section 4). The coordination capacity region $C_{PX}$ of the $(k + 1)$-node cascade network is given by

$$C_{PX} = \left\{ (R_1, \ldots, R_k, P_{Y_1,\ldots,Y_k|X}) : R_i \geq I(X;Y_i^k) \text{ for every } i \in \{1, \ldots, k\} \right\}. \quad (2.76)$$

The strategy that yields the coordination capacity region was first introduced in [Yam81] in the context of lossy source coding.

Broadcast Network

Consider the broadcast network in Figure 2.6. Node $X$ observes a sequence $X^n$ of i.i.d. actions that are generated externally by the DMS $P_X$. The node can communicate to Nodes $Y$ and $Z$ over independent noiseless channels of capacities $R_1$ [bpa] and $R_2$ [bpa], respectively. The purpose of these communications is to have Nodes $Y$ and $Z$ generate tuples of actions, $Y^n$ and $Z^n$, respectively, such that the
The joint type $T_{X^n,Y^n,Z^n}$ is close to a desired distribution $P_{X,Y,Z}$ with high probability. To implement this coordination, the network uses an $(n, 2^{nR_1}, 2^{nR_2})$-code.

**Definition 2.57 (Code).** An $(n, 2^{nR_1}, 2^{nR_2})$-code for coordination in the broadcast network consists of:

- two message sets
  \[ M_1 \triangleq \{1, \ldots, [2^{nR_1}]\}, \quad (2.77) \]
  \[ M_2 \triangleq \{1, \ldots, [2^{nR_2}]\}, \quad (2.78) \]

- two encoding functions
  \[ i : \mathcal{X}^n \to M_1, \quad (2.80) \]
  \[ j : \mathcal{X}^n \to M_2, \quad (2.81) \]

- two decoding functions
  \[ y^n : M_1 \to \mathcal{Y}^n, \quad (2.82) \]
  \[ z^n : M_2 \to \mathcal{Z}^n. \quad (2.83) \]

The notions of induced distribution, achievability, and coordination capacity region are straightforward extensions of their two-node network counterparts. Consider the following two sets of rate-coordination tuples:

\[
\mathcal{R}_{P_X}^{in} \triangleq \left\{ \left( R_1, R_2, P_{Y,Z|X} \right) : \exists P_{V|X,Y,Z} \text{ s.t.} \right. \\
R_1 \geq I(X;YV) , \\
R_2 \geq I(X;VZ) , \\
R_1 + R_2 \geq I(X;YV) + I(X;VZ) + I(Y;Z|XV) \right\} \quad (2.84)
\]
and

\[
\mathcal{R}_{P_X}^{\text{out}} \triangleq \left\{ (R_1, R_2, P_{Y,Z|X}) : \\
R_1 \geq I(X;Y), \\
R_2 \geq I(X;Z), \\
R_1 + R_2 \geq I(X;YZ) \right\}.
\] 

(2.85)

**Theorem 2.58** ([CPC10], Theorem 7). The coordination capacity region \( \mathcal{C}_{P_X} \) of the broadcast network satisfies

\[
\mathcal{R}_{P_X}^{\text{in}} \subset \mathcal{C}_{P_X} \subset \mathcal{R}_{P_X}^{\text{out}}.
\]

(2.86)

The two bounds are not equal in general. However, they coincide for some distributions, for example, when \( X \rightarrow Y \rightarrow Z \) form a Markov chain, when \( Y \rightarrow X \rightarrow Z \) form a Markov chain, or when \( Z \) is a deterministic function of \((X,Y)\) [CPC10].

### 2.2.3 Coordination and Rate-Distortion Theory

We conclude our review on coordination by describing its connections to the theory of rate-distortion. Rate-distortion theory studies the minimum rate required to describe a source within a certain distortion [CT06]. For simplicity, we present the basic definitions using a single discrete memoryless source (i.e., the two-node network in Figure 2.3), but these are readily extended to arbitrary topologies. We restrict our attention to the specific class of single-letter distortion measures. This restriction entails a loss of generality in the sense that the results obtained for this class of distortions do not extend trivially to other measures. This is important because many relevant distortion measures are not single-letter.

**Definition 2.59** (Single-letter distortion measure). A single-letter distortion measure is a non-negative real-valued function

\[
d : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}^+.
\]

(2.87)

When using single-letter measures, the distortion between two vectors \( \mathbf{x}^n \) and \( \mathbf{y}^n \) is given by

\[
d(\mathbf{x}^n, \mathbf{y}^n) \triangleq \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i).
\]

(2.88)

Consider a DMS \( X \sim P_X \). In rate-distortion or lossy source coding problems, we are interested in obtaining a compressed version \( Y \) of the source output \( X \)
that, approximately, can be described with $R$ bits (per source output) and is at distortion $D$, on average. We use an $(n, 2^{nR})$-code that processes a large block of source outputs $X^n$ to generate the corresponding compressed symbols $Y^n$. The formal definition of the code is identical to Definition 2.48 (with $\Omega = \emptyset$). The distribution of the source and the code induce a joint distribution on $(X^n, Y^n)$. Based on this distribution, we define achievability and the rate-distortion region in a similar way as we did for coordination.

**Definition 2.60** (Achievability). A rate-distortion pair $(R, D)$ is achievable if there exists a sequence of $(n, 2^{nR})$-codes, indexed by the codeword length $n$, such that

$$
\limsup_{n \to \infty} E\{d(X^n, Y^n)\} \leq D
$$

under the distribution induced by the codes. ♦

**Definition 2.61** (Rate-distortion region). Given a source distribution $P_X$, the rate-distortion region $D_{P_X}$ is the closure of the set of achievable rate-distortion pairs. ♦

The notion of distortion is less abstract than that of coordination and it has a well-defined intuitive ordering: the less distortion the better. For this reason, much of the interest in rate-distortion theory focuses on the rate-distortion function.

**Definition 2.62** (Rate-distortion function). Given a source distribution $P_X$, the rate-distortion function $R(D)$ is the infimum of rates $R$ such that $(R, D) \in D_{P_X}$. ♦

We can use coordination codes to achieve rate-distortion pairs in the following simpler manner. Select a sequence of coordination codes that achieve a rate-coordination tuple $(R, P_{Y|X})$. This sequence also achieves the rate-distortion pair $(R, D)$, where $D$ is the average distortion induced by $P_{Y|X} P_X$ (i.e. $E_{X,Y} \{d(X,Y)\}$). The following theorem, due to Cuff et al., expounds the relationship between the rate-distortion and the coordination capacity regions for any memoryless source $P_X$ and any arbitrary network topology, and establishes that coordination codes are indeed optimal for rate-distortion problems.

**Theorem 2.63** (Coordination and rate-distortion ([CPC10], Theorem 9)).

Given a memoryless source with distribution $P_X$, the rate-distortion region $D_{P_X}$ in an arbitrary rate-limited network is a linear projection of the coordination capacity region $C_{P_X}$. That is, $v_d \in D_{P_X}$ if and only if

$$
v_d = \Pi w_c
$$

for some $w_c \in C_{P_X}$, where $\Pi$ is a fixed and known projection matrix that contains information about the source distribution and the distortion measure.
This fundamental connection between both theories is also visible when comparing Theorem 2.53 to Shannon’s lossy source coding theorem.

**Theorem 2.64** (Lossy source coding theorem [Sha59]). The rate-distortion function for a discrete memoryless source $X \sim P_X$ and a single-letter distortion measure $d$ is

$$
R(D) = \min_{P_Y|X : \mathbb{E}\{d(X,Y)\} \leq D} I(X;Y).
$$

The minimization in (2.91) illustrates our description on how to use coordination codes for rate-distortion problems. To achieve the rate-distortion function, we can use a sequence of codes that achieves the rate-coordination tuple $(R(D), P_{Y|X})$, where $P_{Y|X}$ correspond to optimal distribution in (2.91). As we will see, when designing codes for rate-distortion problems, the distribution $P_{X|Y}$, which is obtained by conditioning and marginalizing the product $P_X P_{Y|X}$ (of source and minimizing conditional distributions), plays an important role. For reasons that will later become clear, we refer to this distribution as the **test channel**.

### 2.3 Polar Codes

In this section, we review the method of channel polarization and its application for constructing sequences of codes that are optimal for channel and source coding.

Consider the following two basic parameters that characterize any discrete memoryless channel $W(y|x)$: the Bhattacharyya parameter and the symmetric capacity.\(^3\)

**Definition 2.65** (Bhattacharyya parameter). The Bhattacharyya parameter of a $q$-ary discrete memoryless channel $W(y|x)$ is defined as

$$
Z(W) \triangleq \frac{1}{q(q-1)} \sum_{x,x' \in X} \sum_{y \in Y} \sqrt{W(y|x)W(y|x')}.
$$

The Bhattacharyya parameter is an upper bound on the probability of error for uncoded transmission of a single symbol over $W(y|x)$ when maximum likelihood decoding is used.

---

\(^3\)For economy of notation, in the context of polar codes, we will often refer to the DMC by $W(y|x)$ rather than $W_{Y|X}$. 
Definition 2.66 (Symmetric capacity). The symmetric capacity of a \(q\)-ary discrete memoryless channel \(W(y|x)\) is defined as

\[
I(W) \triangleq \sum_{x,y} \frac{1}{q} W(y|x) \log_q \left( \frac{W(y|x)}{\frac{1}{q} \sum_{\tilde{x}} W(y|\tilde{x})} \right).
\]

The symmetric capacity is the highest rate at which reliable communication over \(W(y|x)\) is feasible using the input symbols with equal frequency. Note that our choice of logarithm base ensures that \(0 \leq I(W) \leq 1\) (cf. Lemma 2.29). In the following, we exclude the cases \(I(W) \in \{0,1\}\) from our discussion, as achieving the capacity of such channels is trivial.

Consider a block of \(n\) channel uses of the \(q\)-ary input discrete memoryless channel \(W(y|x)\):

\[
W^n(y|x) \triangleq \prod_{i=1}^{n} W(y|x).
\]

Consider also the linear transformation defined by the matrix \(G_n = B_n F^\otimes p\), where \(B_n\) is a permutation matrix (bit-reversal) and \(F^\otimes p\) is the \(p^{th}\) Kronecker power of

\[
F \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.
\]

Observe that this transformation can only be applied to blocks of \(n = 2^p\) symbols with \(p \in \mathbb{N}\). There exists generalizations of the construction (see e.g., [Ari09]) but these will not appear in this thesis.

Channel polarization is a method for synthesizing a set of \(n\) channels with extremal properties, out of \(n\) identical copies of a discrete memoryless channel (i.e., \(W^n(y|x)\)). The new channels have extremal properties in the sense that, except for a vanishing fraction, all of them either become asymptotically noiseless or their outputs become independent from their inputs. The method is based on two basic operations: channel combining and channel splitting. Channel combining consists of applying the transformation \(G_n\) to a vector \(u\) of \(n\) uniform i.i.d. \(q\)-ary symbols prior to transmission over \(W^n(y|x)\). That is, the input to the channel is \(x = uG_n\). This creates a statistical relationship between \(U\), which is the input to the transformation, and the output of the channel \(Y\). This relationship is described by

\[\text{[Footnote]}\]
the conditional distribution

\[ W_n(y|u) \triangleq W^n(uG_n). \]  

(2.96)

Given the combined channel in (2.96), we define a new set of \( n \) channels:

\[ W_n^{(i)}(y, u_i^{i-1}|u_i) \triangleq \sum_{u_{i+1}^n} \frac{1}{q^n-1} W_n(y|u), \]  

(2.97)

for \( i \in \{1, \ldots, n\} \). This operation is known as channel splitting. Note that each of the \( n \) channels obtained in this manner is different. In particular, the \( i^{th} \) channel has an output that consists of the vector \( y \) as well as the values of the symbols \( u_i^{i-1} \).

It is the channels in (2.97) that show the aforementioned polarization behavior, as described by the following result.

**Theorem 2.67** (Polarization Theorem [Arı09, AT09]). For any discrete memoryless channel \( W(y|x) \) and any \( 0 < \beta < \frac{1}{2} \),

\[ \lim_{n \to \infty} \frac{|\{i \in \{1, \ldots, n\} : Z(W_n^{(i)}) < 2^{-n^\beta}\}|}{n} = I(W). \]  

(2.98)

As we will shortly see, this result is instrumental in achieving many fundamental limits of information theory with affordable complexity. Observe that the statement of the polarization theorem in (2.98) uses the characterization of the synthetic channels in terms of the Bhattacharyya parameters. Quite often, the result is stated in terms of the symmetric capacities (e.g., replacing \( Z(W_n^{(i)}) < 2^{-n^\beta} \) by \( I(W_n^{(i)}) > 1 - 2^{-n^\beta} \)). The following lemma allows for the translation of the results:

**Lemma 2.68** ([STA09], Proposition 3). The symmetric capacity and the Bhattacharyya parameter of an arbitrary discrete memoryless channel \( W(y|x) \) satisfy the following relations:

\[ I(W) \geq \log_q \frac{q}{1 + (q-1)Z(W)}, \]  

(2.99)

\[ I(W) \leq \log_q \frac{q}{2} + \sqrt{1-Z(W)^2} \log_q 2, \]  

(2.100)

\[ I(W) \leq \sqrt{1-Z(W)^2} 2(q-1) \log_q e. \]  

(2.101)

\[ \square \]

We will also use the following related result.

---

5Observe that we use row vectors when working with polar codes (see Chapter 1, Section 1.3).
Lemma 2.69. Consider an arbitrary q-ary discrete memoryless channel \( W(y|x) \). Let \( \mathcal{F} \) and \( \mathcal{F}' \) be the sets defined by

\[
\mathcal{F} \triangleq \left\{ i \in \{1, \ldots, n\} : I(W_n^{(i)}) \leq \delta_n^2 \right\},
\]

\[
\mathcal{F}' \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W_n^{(i)}) \geq 1 - \delta'_n \right\}
\]

with \( \delta'_n = \frac{q - q^{1 - \delta_n^2}}{q - 1} \). Then \( \mathcal{F} \subseteq \mathcal{F}' \). Moreover, for \( \delta_n \in [0, 1] \) and given \( 0 < \beta < \frac{1}{2} \), if \( \delta_n \leq O(2^{-n^\beta}) \) then we have that \( \delta_n^2 \leq \delta'_n \leq n \delta'_n \leq O(2^{-n^\beta}) \).

**Proof.** The fact that \( \mathcal{F} \subseteq \mathcal{F}' \) follows from Lemma 2.68. To prove that \( \delta_n^2 \leq \delta'_n \) note that \( \delta'_n = \delta_n^2 \) for \( \delta_n \in \{0, 1\} \) and that \( \delta'_n \) is an increasing concave function of \( \delta_n^2 \) in \([0, 1]\). Therefore \( \delta_n^2 \leq \delta'_n \leq n \delta'_n \) in \([0, 1]\). Now, note that \( \delta_n \leq O(2^{-N^\beta}) \) means that there exists \( K > 0 \) such that \( \delta_n \leq K 2^{-N^\beta} \) for all sufficiently large \( n \). Consider the function \( \delta'_n(\delta_n) \). We have that

\[
\lim_{n \to \infty} \frac{n \delta'_n(K 2^{-n^\beta})}{2^{-n^\beta}} = 0
\]

so that \( n \delta'_n(K 2^{-n^\beta}) \leq O(2^{-n^\beta}) \). Since for all sufficiently large \( n \) we have that \( \delta'_n(\delta_n) \leq \delta'_n(K 2^{-n^\beta}) \), we conclude that \( n \delta'_n(\delta_n) \leq O(2^{-n^\beta}) \).

Before discussing the applications of channel polarization to channel coding, we state a fundamental property of polarization for stochastically degraded channels.

**Lemma 2.70 ([KU10], Lemma 13).** Let \( W_1(y|x) \) and \( W_2(y|x) \) be two q-ary discrete memoryless channels such that \( W_1(y|x) \) is stochastically degraded with respect to \( W_2(y|x) \). Let \( W_1^{(i)} \) and \( W_2^{(i)} \) be the \( i \)-th synthetic channels generated from \( W_1(y|x) \) and \( W_2(y|x) \), respectively \( (i \in \{1, \ldots, n\}) \). For \( j \in \{1, 2\} \), consider the sets

\[
\mathcal{F}_j \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W_j^{(i)}) \geq \delta_n \right\}
\]

defined for some fixed \( \delta_n > 0 \). Then, for all \( i \in \{1, \ldots, n\} \), the following properties hold:

1. The Bhattacharyya parameters of the synthetic channels satisfy: \( Z(W_1^{(i)}) \geq Z(W_2^{(i)}) \).

2. If \( i \in \mathcal{F}_2 \) then \( i \in \mathcal{F}_1 \) as well. That is, \( \mathcal{F}_2 \subseteq \mathcal{F}_1 \).

**Proof.** The proof of the first claim follows the same lines as in the binary case [KU10]. The second claim follows directly from the first one.
2.3.1 Channel Coding

Arikan proposed a new class of codes, which he named polar codes, together with encoding and decoding algorithms, and showed that they achieve the capacity of any binary-input symmetric discrete memoryless channel. His construction was subsequently extended to $q$-ary discrete memoryless channels in [STA09]. Polar codes exploit the result in Theorem 2.67 by synthesizing $n$ channels $W^{(i)}_n$ and transmitting uncoded information only through the good channels. That is, information is put only into $W^{(i)}_n$ with $i \in F_c$, where

$$
F_c \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W^{(i)}_n) < \frac{1}{n} 2^{-n^\beta} \right\}.
$$

(2.106)

It is customary to refer to the symbols put into the channels in $F_c$ as the information symbols. For the rest of the channels, that is, $W^{(i)}_n$ with $i \in F$, where

$$
F \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W^{(i)}_n) \geq \frac{1}{n} 2^{-n^\beta} \right\},
$$

(2.107)

we use a sequence of symbols known by the decoder. $F$ is usually referred to as the frozen set of channels and the symbols put to these channels are known as the frozen symbols. Consequently, a polar code for channel coding is defined by the following three parameters: codeword length $n$, frozen set $F$, and values of the frozen symbols $u_F$.

**Definition 2.71.** A polar code $(n, F, u_F)$ consists of all the $q$-ary codewords $x$ of length $n$ of the form

$$
x = u_{F^c}(G_n)_{F^c} \oplus u_F(G_n)_F,
$$

(2.108)

where $u_{F^c}$ correspond to the information symbols.

Given a frozen set $F$, the set of all possible polar codes (one for each choice of the frozen symbols) is sometimes referred to as the ensemble of polar codes. A usual approach to prove polar coding results is to show that the ensemble of polar codes averaged over a random distribution of the frozen symbols has a certain property. This in turn implies the existence of a deterministic choice of the frozen symbols with the desired property. In fact, under some conditions, it is sometimes possible to show that all choices of the frozen symbols have the desired property. For this reason, whenever we say that we design a polar code we mean that we define a rule for choosing the frozen set.

Out of the length-$n$ vector $u$, only the symbols $u_i$ with $i \in F^c$ carry information. Thus, the rate $R$ of a polar code is

$$
R = \frac{|F^c|}{n} = 1 - \frac{|F|}{n}.
$$

(2.109)
2.3 Polar Codes

To decode polar codes, Arikan introduced a successive cancellation (SC) algorithm that sequentially estimates the information symbols

\[
\hat{u}_i = \begin{cases} 
  u_i & \text{if } i \in \mathcal{F}, \\
  u = \arg \max_{u \in \mathcal{U}} W_n^{(i)}(\mathbf{y}_1^n, \hat{u}_i^{i-1}|u) & \text{otherwise},
\end{cases}
\] (2.110)

in order from \(i = 1\) to \(n\). Observe that, in decoding the \(i^{th}\) symbol \(u_i\), the SC algorithm uses the \(i - 1\) estimates \(\hat{u}_i^{i-1}\) obtained previously. Thus, this differs from the characterization of the synthetic channels in (2.97) that assumes that the symbols \(u_i^{i-1}\) are available at the output. Fortunately, this poses no problem because polarization happens at sufficiently high rate as to compensate for the possibility of error propagation (cf. Theorem 2.67).

**Theorem 2.72** (Polar codes achieve the symmetric capacity [Ari09, STA09]). For any given discrete memoryless channel \(W(\mathbf{y}|\mathbf{x})\), any \(0 < \beta < \frac{1}{2}\), and fixed rate \(R < I(W)\), there exists a sequence of polar codes of length \(n\) (with \(n = 2^p\), \(p \in \mathbb{N}\)) and rate \(R_n > R\) with error probability

\[
\Pr(\hat{U} \neq U) = O(2^{-n^\beta})
\] (2.111)

under successive cancellation decoding. The complexity of the encoding and decoding operations is \(O(n \log n)\) for each.

Theorem 2.72 states that polar coding is sufficient for achieving the symmetric capacity of a channel. This coincides with the Shannon capacity for the class of symmetric channels (cf. Lemma 2.38) but is strictly lower for many other channels. There exist simple methods that allow polar codes to achieve the Shannon capacity for non-symmetric channels, too [Ari09, STA09, Gal68]. These methods, which will be briefly discussed in Section 3.3, can be used for most of the results presented in the coming chapters. However, in general, they bring no insight into the problems under consideration.

### 2.3.2 Source Coding

Shortly after the introduction of polar codes, Korada and Urbanke showed that they are also optimal for lossy source compression with equally used binary reproduction letters [KU10, Kor09]. This was later extended in [KT10] to arbitrary alphabets. We describe their coding method in the following.

The main idea to show the optimality of polar codes for source coding is to exploit the duality between the source and channel coding problems. Consider a DMS \(Y \sim P_Y\) and let \(P_{Y|X}\) denote the test channel corresponding to the desired
rate-distortion pair \((R, D)\). Roughly speaking, compression of the DMS \(Y \sim P_Y\) into \(X\) with polar codes can be seen as decoding a transmission over \(P_{Y|X}\). Given this test channel, consider the joint distribution

\[
P_{U, X, Y}(u, x, y) = \frac{1}{q^n} \mathbb{I}\{x = uG_n\} \prod_{i=1}^{n} P_{Y|X}(y_i|x_i).
\] (2.112)

Observe the connections to (2.96). Observe also that (2.112) implicitly assumes a uniform i.i.d. distribution of the symbols in \(U\) and \(X\). We shall refer to \(P_{U, X, Y}\) and the marginal \(P_{X, Y}\) as the design distributions. Using simple manipulations, we obtain:

\[
P_{U|U_1^{i-1}, Y}(u_i|u_1^{i-1}, y) = \frac{1}{q P_{U_1^{i-1}, Y}(u_1^{i-1}, y)} W_n(i)(u_1^{i-1}, y|u_i).
\] (2.113)

Observe the connections between (2.113) and (2.110). The marginal distribution in (2.113) is used to specify the following variation of the SC algorithm (cf. (2.110)) that is used to compress a source realization \(y\) into a vector \(\hat{u}\) of \(q\)-ary symbols. Given a set \(F\) (to be specified later), and a vector \(u_F\) of \(q\)-ary symbols, let

\[
\hat{u}_i = \begin{cases} u_i & \text{if } i \in F, \\ j & \text{with probability } P_{U|U_1^{i-1}, Y}(j|u_1^{i-1}, y) \end{cases} \quad \text{if } i \in F^c,
\] (2.114)

sequentially, in order from \(i = 1\) to \(n\). To distinguish this variant of the SC algorithm from the one used in channel decoding, we will refer to (2.114) as the SC encoding or SC compression algorithm. The reconstruction vector \(x\) is readily obtained from \(\hat{u}\) by multiplying with the matrix \(G_n\) (i.e., \(x = \hat{u}G_n\)). Note that the values \(\hat{u}_i\) for \(i \in F\) are chosen prior to compression (i.e., \(\hat{u}_i = u_i\)) and are known by the compressing and reconstructing parts. Thus, it is only necessary to convey \(\hat{u}_i\) for \(i \in F^c\) for describing \(x\). The compression rate is therefore

\[
R = \frac{|F^c|}{n}.
\] (2.115)

Observe also that the encoding rule in (2.114) is random. That is, each \(\hat{u}_i\) with \(i \in F^c\) is chosen at random. Thus, we need to speak about the probability that a given source realization \(y\) is compressed into \(\hat{u}_{\bar{F}}\) when the frozen symbols are set to \(\hat{u}_F = u_F\). This probability is

\[
Q_{U_{\bar{F}}|Y, U_F}(\hat{u}_{\bar{F}}|y, \hat{u}_F) = \prod_{i \in F^c} P_{U_i|U_1^{i-1}, Y}(\hat{u}_i|u_1^{i-1}, y).
\] (2.116)

From (2.116) and the source distribution \(P_Y\), we can readily obtain the induced (joint) distribution on \((U, X, Y)\) when the frozen symbols are drawn independently at random from a uniform distribution:

\[
Q_{U, X, Y}(\hat{u}, x, y) = \mathbb{I}\{x = \hat{u}G_n\} \frac{1}{q^{|\bar{F}|}} P_Y(y) \prod_{i \in F^c} P_{U_i|U_1^{i-1}, Y}(\hat{u}_i|u_1^{i-1}, y).
\] (2.117)
The expression in (2.117) corresponds to the distribution induced by the ensemble of codes. As in the case of channel coding, designing a polar code for source compression amounts, almost exclusively, to defining the frozen set $\mathcal{F}$. In the context of source coding, the basic property that characterizes the selection of $\mathcal{F}$ is implicitly described by the following result.

**Theorem 2.73** (Convergence of design and induced distributions [KU10, KT10]). For any $0 < \beta < \frac{1}{2}$, let $\delta_n = \frac{1}{n}2^{-n\beta}$ and

$$
\mathcal{F} = \left\{ i \in \{1, \ldots, n\} : I(P^{(i)}) \leq \delta_n^2 \right\}.
$$

(2.118)

The design distribution and the induced distribution satisfy

$$
\|P_{X,Y} - Q_{X,Y}\|_{TV} \leq O(2^{-n\beta}).
$$

(2.119)

Again, we note that we can make an equivalent statement in terms of the Bhattacharyya parameters of the synthetic channel (e.g., using $Z(P^{(i)}) \geq 1 - \delta_n^2$ in (2.118)). We will make use of both characterizations in the sequel. Observe also that the characterization of the synthetic channels in the frozen set is slightly different from the one used in channel coding (cf. (2.107)). Roughly speaking, for channel coding the frozen set consist of all those synthetic channels that do not have very high symmetric capacity (i.e., not close to 1). In contrast, for source coding, it consists of those synthetic channels with very low symmetric capacity (i.e., close to 0). For the new type of frozen set we have the following polarization result:

**Theorem 2.74** (Polarization theorem for source coding [KU10, KT10]). For any $0 < \beta < \frac{1}{2}$

$$
\lim_{n \to \infty} \frac{|\left\{ i \in \{1, \ldots, n\} : I(W_n^{(i)}) \leq 2^{-n\beta} \right\}|}{n} = 1 - I(W).
$$

(2.120)

These two results are the basis for establishing the optimality of polar coding for source coding. First, by choosing the frozen set as defined in (2.118), we ensure that the distribution induced by the code is arbitrarily close to the design distribution. Consequently, the average distortion is also close to the design distortion. Finally, Theorem 2.74 states the size of the frozen set and, thus, determines the rate of transmission. For the same reason that polar codes achieved only the symmetric
capacity $I(W)$ of a channel, in the context of source coding, polar codes only achieve the symmetric rate-distortion function $R_s(D)$. The function $R_s(D)$ is obtained by solving the minimization in (2.91) under the additional constraint that the uniform distribution must be induced on $X$.

**Theorem 2.75** (Polar codes are optimal for source coding). Consider any discrete memoryless source $Y \sim P_Y$ and any bounded distortion measure $d: Y \times X \to \mathbb{R}^+$. For any fixed rate $R > R_s(D)$ and any $0 < \beta < \frac{1}{2}$, there exists a sequence of polar codes of length $n$ (with $n = 2^p$, $p \in \mathbb{N}$) and rate $R_n < R$ with expected distortion

$$E\{d(X, Y)\} \leq D + O(2^{-n\beta}) \tag{2.121}$$

under successive cancellation compression. The complexity of the compression and reconstruction algorithms is $O(n \log n)$ for each.

The same methods that allow for extending polar codes for channel coding to achieve the Shannon capacity of a channel can also be used to achieve the Shannon rate-distortion function (cf. Theorem 2.64).

### 2.3.3 Polar Codes for Other Problems

The polarization phenomena and polar codes have found applications in many problems of information theory. For example, the polarization of multiple access channels was studied in [AT12]. Polar codes for degraded broadcast channels were first constructed in [Kor09] and studied in detail in [GAG13]. Polar codes have also been used in information-theoretic security contexts, for example, wiretap channel [ART+10, HS10, KE10, MV11] or bidirectional broadcasting with confidential messages [AWOS12]. Polar codes for relaying were first considered in [ART+10]. Also important for this thesis, the optimality of polar codes for some multi-terminal source coding problems, including the binary versions of the Wyner-Ziv, Slepian-Wolf, and Gel’fand-Pinsker problems, was established in [KU10]. Strong coordination with polar codes was recently considered in [BLK12].

### 2.4 Compressive Sensing

In this section, we review the problem of compressive sensing, which is the subject of Chapter 6. Compressive sensing deals with the simultaneous sensing and compression of signals and predicts that signals with a special structure can be recovered from underdetermined sets of measurements [Don06, EK12]. As an illustration of this problem, consider recovering a vector $x \in \mathbb{R}^n$ from the linear measurements
\( y \in \mathbb{R}^m \) obtained using the measurement matrix \( \phi \in \mathbb{R}^{m \times n} \):

\[
y = \phi x. \tag{2.122}
\]

If \( x \) is arbitrary, then only a large number of measurements \( m \geq n \) can ensure unambiguous identification. However, if \( x \) has some structure that can be exploited, then it might be possible to reduce the number of measurements required for identifying it. In particular, if all but \( k \ll n \) components of \( x \) are zero (i.e., \( x \) is \( k \)-sparse), then it is possible to recover it from significantly fewer measurements (i.e., \( m \ll n \)).

The rationale behind the sparse-signal model is that many signals of interest concentrate most of their features in a low-dimensionality space when represented in some appropriate basis. The acquisition of such signals (i.e., the model in (2.122)) is an interesting topic whose discussion would take us too far afield from the matter at hand in this thesis (see, e.g., [CW08, EK12] and references therein). Thus, without further considerations, we will assume a linear measurement model and concentrate on inference of sparse signals. In practice, the measurements are contaminated by noise, for example,

\[
y = \phi x + z, \tag{2.123}
\]

and thus exact recovery of \( x \) is impossible. Instead, a common goal is to obtain an approximation \( \hat{x} \) that is close to \( x \) in some sense, for example in mean square error (MSE).

In the following, we introduce the basic signal model for sparse signals used in this thesis. Then, we review the problem of complete support recovery and discuss its similarities with the problem of channel coding. Finally, we address briefly the general problem of estimation of sparse signals.

### 2.4.1 System Model

There are several different approaches for modeling sparse signals. In this thesis, we will consider mainly the following model, whose central element is the support set.

**Definition 2.76 (Support set).** The support set \( \mathcal{S} \) is a subset of \( \{1, \ldots, n\} \) that determines the positions of the non-zero entries of a sparse vector of dimension \( n \).

Using a support set \( \mathcal{S} \) of size \( k \), we construct the \( k \)-sparse vector \( x \in \mathbb{R}^n \) in the following way: let \( [s_1, \ldots, s_k]^T \) contain the elements in \( \mathcal{S} \) in some arbitrary order and let \( w \in \mathbb{R}^k \) have \( k \) non-zero entries sorted by decreasing magnitude. \( x \) is defined component-wise as

\[
x_i = \begin{cases} 
  w_j & \text{if } i = s_j, \\
  0 & \text{if } i \notin \mathcal{S}
\end{cases} \tag{2.124}
\]
for $i = \{1, \ldots, n\}$. Unless otherwise stated, we assume that $w, S$, and the ordering of the elements in $S$ used to construct $x$ are deterministic but unknown. The sparse vector is measured using the matrix $\Phi \in \mathbb{R}^{m \times n}$, that is,

$$Y = \Phi x + Z.$$  \hfill (2.125)

The measurements are contaminated by an additive random noise vector $Z$ with Gaussian i.i.d. entries, $Z_i \sim \mathcal{N}(0, P_z)$. Two alternative models for the measurement matrix are usually considered: deterministic and random. A deterministic measurement matrix is reasonable if we assume that we have full control of the measurement process and that we can use the same matrix for measuring different vectors. In contrast, it seems more appropriate to use the random model when the properties of the measurements cannot be controlled. We will consider, almost exclusively, the latter type; in particular, random measurement matrices with Gaussian i.i.d. entries, $\Phi_{i,j} \sim \mathcal{N}(0, P_\Phi)$.

In this thesis, we will refer to $n, m$, and $k$ as the signal dimension, the number of measurements, and the sparsity level, respectively.

### 2.4.2 Support Set Recovery

Given the model in (2.124), it is clear that the support set $S$ is a central component for inference of sparse signals from an underdetermined relation of linear measurements in noise. For example, if the support set is known, then minimum square error estimation of the sparse vector $x$ amounts to solving a least-squares problem. However, in general, it is not possible to detect the support set of a sparse vector from any collection of measurements. A large body of research has been devoted to establishing frameworks for characterizing the fundamental tradeoffs between $n, m$, and $k$ for detection of the support set of sparse vectors. In the following, we introduce the framework that we use in this thesis for studying complete support set recovery. It is largely based on [JKR11], although we redefine some quantities for convenience.

To estimate the support set of a sparse signal, we will use a support recovery map $d$, which is a function that assigns a subset of $\{1, \ldots, n\}$ to every pair of measurement vector and measurement matrices $(y, \phi)$. That is,

$$d : \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathcal{P}(\{1, \ldots, n\}),$$  \hfill (2.126)

where $\mathcal{P}(\{1, \ldots, n\})$ is the power set of $\{1, \ldots, n\}$.

For every sparse vector $x$ and support recovery map $d$, we define the error probability with respect to the distribution of the noise and the measurement matrix as

$$P_e(x, d) \triangleq \Pr(d(Y, \Phi) \neq S).$$  \hfill (2.127)

It would be desirable to characterize (2.127) for every pair $(x, d)$. Unfortunately, it is probably impossible to obtain such a complete description. A less ambitious but
more tractable problem is to characterize the regimes of \((k, m, n)\) for which it is possible to construct support recovery maps with arbitrarily low error probability. In this thesis, we consider only one such regime, namely \(m\) and \(n\) are allowed to go to infinity while keeping \(k\), \(S\), and \(w\) fixed. That is, formally, we consider sequences \(\{x \in \mathbb{R}^n\}\) of vectors with common support set \(S\) and non-zero entries \(w\), and sequences of random measurements \(\{Y \in \mathbb{R}^{mn}\}\). We index the sequences by \(n\) and, thus, refer to the number of measurements by \(m_n\). For this regime, the aforementioned characterization can be expressed in terms of one single parameter, the measurement rate.

**Definition 2.77 (Measurement rate).** The measurement rate \(r\) of a sequence of measurements \(\{Y \in \mathbb{R}^{mn}\}\) of a sequence of vectors \(\{x \in \mathbb{R}^n\}\) is defined as

\[
r \triangleq \limsup_{n \to \infty} \frac{m_n}{\log_2 n}.
\]  

(2.128)

The measurement rate is a deterministic quantity that relates the scaling of the signal dimension \(n\) and the number of measurement \(m_n\) or, equivalently, describes the asymptotic growth of the dimensions of the measurement matrix. Before reviewing some known characterizations of complete support recovery, we discuss the connections to the problem of communication over noisy channels.

**Connections to Channel Coding**

The problem of recovering the complete support set of a sparse signal and the problem of communicating over noisy channels have some fundamental connections, in spite of the differences in appearance. These connections have been reported in several works, e.g \[\text{SBB06, Tro06, WWR10, AT10}\]. Of particular interest for us are the connections to the Gaussian multiple-access channel (MAC) unveiled in \[\text{JR08, JKR10, JKR11}\], which we describe in the following.

First, consider the \(k\)-user Gaussian MAC in Figure 2.7. The goal is to design codebooks, encoders, and decoders subject to a power constraint so that, given a vector of channel observations

\[
y_{MAC} = \sum_{i=1}^{k} h_i \theta^{(i)} + z,
\]  

(2.129)

it is possible to detect the codeword \(\theta^{(i)}\) transmitted by each of the \(k\)-users with low error probability. In general, each of the users has a different codebook \(C^{(i)}\). However, if we assume that all users use the same codebook (and hence communicate at the same rate), and exclude the possibility that two users choose the same codeword, the vector of channel observations becomes

\[
y_{MAC} = \sum_{i=1}^{k} h_i \theta_{j(i)} + z,
\]  

(2.130)
where $j(i)$ is the function that maps a user number to the corresponding transmitted codeword. If we envision the common codebook as a matrix containing all possible codewords, detecting all the transmitted $\theta_{j(i)}$ is tantamount to identifying columns of this matrix.

In contrast, as we have just described, in support recovery the goal is to design sensing matrices and support recovery maps so that it is possible to detect reliably the set $S$ from a given a vector of noisy linear measurements

$$y_{CS} = \sum_{i \in S} x_i \phi_i + z. \quad (2.131)$$

In view of the formulations in (2.130) and (2.131), the similarities between the two problems are self-evident. To make the analogy complete, it is necessary to define the support set as a random entity (with a uniform distribution over the set of all size-$k$ subsets of \{1, \ldots, n\}), in the same way as the messages in the MAC channel are randomly selected by the users.\(^6\) Nevertheless, there is, apparently, still one important difference between the two problems. In detecting the users in the $k$-MAC, we usually assume that the receiver has perfect knowledge of the channel coefficients $h_i$. In contrast, in the context of support recovery and compressive sensing, such an assumption is hard to justify. One of the striking results in [JKR11, Section IV-A] is that the advantage of knowing the values of the non-zero coefficients $x_i$ is immaterial in terms of the asymptotic characterizations of complete support recovery discussed here.

---

\(^6\)A random support set is crucial for having a well defined problem with a converse result, as in Theorem 2.78. In contrast, if one handles with care the model of deterministic $S$ (e.g., to avoid trivial solutions, etc), the achievability arguments remain valid. In this thesis, we only consider achievability results.
Using this analogy between the two problems, Jin, Kim, and Rao established a full characterization for complete support recovery for a fixed and known sparsity level $k$ in terms of the quantity\footnote{In this thesis we use the inverse expressions to those in [JKR11]. That is, we study support recovery in terms of the ratio $\frac{m}{\log_2 n}$ and compare them to the quantity $\frac{1}{c(w)}$. The reason for these changes is mere convenience. We find more intuitive to express more demanding requirements (i.e., more measurements per dimension) by saying that the measurement rate is larger.}

$$c(w) \triangleq \min_{i \in \{1,...,k\}} \frac{1}{2i} \log_2 \left( 1 + \frac{P_\Phi}{P_z} \sum_{j=k+1-i}^{k} w_j^2 \right). \tag{2.132}$$

\begin{theorem} ([JKR11], Theorems 1 and 2). For every measurement rate $r > \frac{1}{c(w)}$, there exists a sequence of support recovery maps $\{d(n)\}$ with

$$\lim_{n \to \infty} \Pr(d(n)(Y, \Phi^{(n)})) \neq S) = 0. \tag{2.133}$$

Conversely, if $r < \frac{1}{c(w)}$, any sequence of recovery maps $\{d(n)\}$ has

$$\lim_{n \to \infty} \Pr(d(n)(Y, \Phi^{(n)})) \neq S) > 0. \tag{2.134}$$
\end{theorem}

Note that the characterization in Theorem 2.78 only depends on the sparse vector $x$ through its non-zero components; that is, through $w$. Another interesting consequence of this result is that any attempt to retrieve the support set of random signals (i.e., with $W$ and $X$ drawn according to some distribution) incurs in a non-vanishing error probability (i.e., measurement outages) [JKR11, Theorem 5]. The reason is that, for every fixed $r$, there is always a positive probability that the realization of the sparse signal has some small, non-zero entries that push the measurement rate requirements beyond $r$.

In addition to the case of fixed $k$, other regimes of sparsity have been discussed in the literature. Necessary and sufficient conditions for complete support recovery for different scaling regimes of $k$ were established in [Wai09, RR09, WWR10, AT10, JKR11]. Conditions for other recovery criteria were also discussed in [AT10]. Of especial relevance for us, they considered the recovery of a large fraction of the energy in the sparse signal. The notion of partial support recovery in the large system setting was also studied in [RG08], where it was shown that the conditions for partial recovery are less stringent than those for complete support recovery.
2.4.3 Estimation of Sparse Signals

Recall the measurement model introduced in (2.125), namely

\[ Y = \phi x + Z, \quad (2.135) \]

where \( \phi \in \mathbb{R}^{m \times n} \) is now a fixed measurement matrix. As we discussed before introducing the problem of support recovery, we are often interested in estimating the sparse vector \( x \). It is therefore necessary to define a systematic way of gauging the performance of different estimators \( \hat{X} \). A usual approach is to rank estimators based on their mean square error performance:

\[ \text{mse}(x) \triangleq E\left\{ \|x - \hat{X}\|^2 \right\}. \quad (2.136) \]

Given (2.136), it is natural to ask what is the minimum attainable MSE for each particular \( x \). The most common way of studying this problem is by establishing lower bounds on the achievable MSE. In the context of compressive sensing, the most prominent ones are the oracle bound and the Cramér-Rao bound.

The oracle bound corresponds to the best performance attainable by an unbiased estimator \( \hat{X}_{or} \) that has non-causal knowledge of the support set \( S \) of the sparse vector \( x \). As we mentioned before, given this additional knowledge, the estimation reduces to a least-squares problem.

**Lemma 2.79** (Oracle bound, [CT07]).

\[ E\left\{ \|x - \hat{X}_{or}\|^2 \right\} = P_z \text{tr}\left\{ (\phi_S^T \phi_S)^{-1} \right\}. \quad (2.137) \]

This bound is impractical in the sense that it is impossible to construct an oracle bound. Nevertheless, it was shown in [BKT09] that, for the ensemble of Gaussian matrices, the bound is asymptotically achievable even without explicit knowledge of the support set. This result was extended in [NBZJ12] to wider ensembles of measurement matrices characterized in terms of a concentration inequality.

The Cramér-Rao bound makes use of the statistical characterization of the relationship between the parameter of interest and the vector of observations to provide a fundamental limit on the attainable mean square error performance. It useful for a much wider class of problems than the Oracle bound and it can also be specialized to incorporate constraints on the parameter space [BHE10]. This is relevant in compressive sensing, because given the model in (2.125) and (2.124), we can include the constraint \( |S| \leq k \). It was shown in [BHE10] that, for the case under consideration here (i.e., an underdetermined relation of measurements) no estimator is unbiased for all \( x \) that satisfy the sparsity (inequality) constraint. However, unbiasedness is possible for \( x \) with \( |S| = k \):
Lemma 2.80 (Crámer-Rao bound for sparse estimation ([BHE10], Theorem 2)). For sparse vectors with maximal support (i.e., $|S| = k$), we have that

$$
E\left\{ \| \mathbf{x} - \hat{\mathbf{x}} \|^2 \right\} \geq P_z \operatorname{tr}\left\{ \left( \phi_S^T \phi_S \right)^{-1} \right\}.
$$

(2.138)

Thus, for parameters with maximal support, the Crámer-Rao bound coincides with the oracle bound, even in the non-asymptotic regime.
Chapter 3

Coordination Using Polar Codes

In this chapter, we construct polar codes (PCs) for coordination for a variety of scenarios. First, we consider the two-node case and introduce a construction that achieves all points in the symmetric rate-coordination capacity region. Building on this construction, we design codes to achieve all points in the symmetric rate-coordination capacity region for the cascade network. Finally, we consider the broadcast network. Using the results for the two-node and cascade networks, we construct polar codes that yield a symmetric version of the rate-coordination inner bound. We conclude the chapter by discussing the extension to non-binary and non-symmetric scenarios. All the proofs are provided in the appendix.

3.1 Preliminaries

The constructions of polar codes for coordination presented here combine elements of source coding with elements of channel coding (see Sections 2.3.2 and 2.3.1, respectively). The source coding components are employed to generate sequences of actions with the desired type. They will always be used in the same way: we will design a polar code based on a certain distribution, e.g., $P_{X|Y}$, by specifying the frozen set, e.g., $\mathcal{F}$. A sequence $X$ will be compressed into a sequence of bits $U$ using the SC encoding algorithm. Part of these bits will be frozen (i.e., $U_{\mathcal{F}}$) and thus, only the information bits (i.e., $U_{\mathcal{F}^c}$) will be conveyed to the node interested in constructing the sequence $Y = UG_n$. Whenever it is clear from the context how the bits $U_{\mathcal{F}^c}$ are conveyed, we will simply say that $X$ is compressed into $Y$.

In contrast, the channel coding components are used to reduce the communication rate by exploiting the correlations in the network. Recall that, in the case of channel coding, the matrix $G_n$ is used to encode the information bits and the SC decoding algorithm is used to estimate these bits from the channel output.

1Parts of the material presented in this chapter are based on a transcript of our work previously published in [BTS11].
The first constructions combining these two elements were introduced in [KU10] for source coding and multi terminal source coding, in particular for Wyner-Ziv coding [WZ76].

3.1.1 Notation

In this chapter, most expressions involve probability distributions that belong to one of the following three classes. The first one is that of the design distributions. These are the distributions of the actions that, ideally, our network should produce. We reserve the letter \( P \) for them and derived distributions (e.g., \( P_{X,Y} \) or \( P_{Y|X} \)). In the latter case, the relationship will be straightforward, for example, if \( P_{X,Y} \) is a design distribution then \( P_{Y|X} = \frac{P_{X,Y}}{P_X} \). If \( P \) is the distribution of a vector, e.g. \( P_{X,Y} \), then it corresponds to a product of design (or related) distributions, i.e.

\[
P_{X,Y}(x, y) \triangleq \prod_{i=1}^{n} P_{X,Y}(x_i, y_i).
\] (3.1)

The second class is that of the distributions induced by the code. These are the actual distributions of the sequences of actions in the network. They will always be denoted using the letter \( Q \), e.g. \( Q_{X,Y} \). Finally, the last class corresponds to the types (i.e., empirical distributions) of the actions in the network, which will be denoted using the usual notation \( T_{X,Y}(x,y) \) (cf. Definition 2.5).

In addition to the visible actions in the network (i.e., the actions specified in the description of each model), our codes will sometimes generate auxiliary sequences. Whenever these are directly connected with a visible sequence in the network, we will use the tilde notation. For example, \( \tilde{Y} \) is an auxiliary sequence used to construct \( Y \). Otherwise we will use a different letter, e.g. \( V \) in the broadcast network.

3.1.2 Common Randomness

As we discussed in Section 2.2.2, common randomness is not necessary for achieving empirical coordination (cf. Lemma 2.51). However, it is useful for establishing achievability results with polar codes. Our approach is, thus, to prove every statement assuming that the nodes have access to an unlimited source of common randomness independent of the external actions, and then, strengthen the result showing that common randomness is indeed unnecessary for our constructions.

3.2 Main Results

3.2.1 Two-Node Network

In this section, we study the two-node network introduced in Section 2.2.2 and depicted in Figure 3.1. Recall the definitions of an \((n, 2^{nR})\)-code (Definition 2.48).
and achievability (Definition 2.50) for this network. The coordination capacity region is given by Theorem 2.53.

When designing polar codes for this network, we restrict our attention to the following subset of the coordination capacity region:

\[ C^s_{P_X} = \left\{ (R, P_{Y|X}) : R > I(X;Y), \quad Y \sim \text{Unif}\{0,1\} \right\} \]

(3.2)

That is, we only consider binary actions \( Y \) and conditional distributions \( P_{Y|X} \) that induce a uniform distribution on \( Y \). These limitations are analogous to those described in Sections 2.3.1 and 2.3.2 for channel and source coding with polar codes; extensions are discussed in Section 3.3. Note also that the definition of \( C^s_{P_X} \) excludes those pairs with \( R = I(X;Y) \). In the characterization of the coordination capacity region, these pairs are part of the closure of the set of achievable rates and their achievability is usually not discussed. We refer to \( C^s_{P_X} \) as the symmetric coordination capacity region.

Our main result for the two-node network is the following:

**Theorem 3.1.** All pairs \((R, P_{Y|X}) \in C^s_{P_X}\) are achievable using a sequence of \((n, 2^{nR})\)-polar codes (with \( n = 2^p, \ p \in \mathbb{N} \)), with probability of coordination error

\[ P_e \leq O(2^{-n^\beta}) \]

(3.3)

for any \( 0 < \beta < \frac{1}{2} \). The complexity of generating the actions is \( O(n \log n) \).

**Proof.** The proof is provided in Appendix 3.A.1.

The above proof requires both nodes to generate the frozen bits using common randomness. The following corollary shows that there exists a fixed choice of the values of these bits such that the sequence of PCs achieves empirical coordination.
Corollary 3.2. All pairs \((R, P_{Y|X}) \in C_{P_X}^s\) are achievable using a sequence of \((n, 2^{nR})\)-polar codes (with \(n = 2^p, p \in \mathbb{N}\)) and a fixed choice of frozen bits, with probability of coordination error \(P_E \leq O(2^{-n^\beta})\) for any \(0 < \beta < \frac{1}{2}\). The complexity of generating the actions is \(O(n \log n)\).

Proof. The proof is provided in Appendix \[3.4\].

3.2.2 Cascade Network

In this section, we study the cascade network introduced in Section \[2.2.2\]. First, consider the three-node cascade network depicted in Figure 3.2 and its corresponding definitions of \((n, 2^{nR_1}, 2^{nR_2})\)-code (Definition \[2.54\]) and achievability. The coordination capacity region is given by Theorem \[2.55\].

When designing polar codes for this network, we restrict our attention to the following subset of the coordination capacity region:

\[
C_{P_X}^s \triangleq \left\{ (R_1, R_2, P_{Y,Z|X}) : \\
R_1 > I(X; YZ), \\
R_2 > I(X; Z), \\
Y, Z \sim \text{Unif}\{0,1\} \right\}. 
\]  

That is, we only consider binary actions \(Y\) and \(Z\) and conditional distributions \(P_{Y,Z|X}\) that induce a uniform distribution on \(Y\) and a uniform distribution on \(Z\). These limitations are analog to those described in Sections \[2.3.1\] and \[2.3.2\]; extensions are discussed in Section \[3.3\]. Note also that the definition of \(C_{P_X}^s\) excludes those triples with \(R_1 = I(YZ; X)\) or \(R_2 = I(Z; X)\), as before. In the characterization of the coordination capacity region, these triples are part of the closure of the set of achievable rates and their achievability is usually not discussed. We refer to \(C_{P_X}^s\) as the symmetric coordination capacity region.

Our main result for the cascade network is the following:
Figure 3.3: \((k + 1)\)-node cascade network.

**Theorem 3.3.** All triples \((R_1, R_2, P_{Y,Z|X}) \in C_{P_X}^s\) are achievable using a sequence of \((n, 2^{nR_1}, 2^{nR_2})\)-polar codes (with \(n = 2^p, p \in \mathbb{N}\)), with probability of coordination error

\[
P_e \leq O(2^{-n^\beta})
\]  

(3.5)

for any \(0 < \beta < \frac{1}{2}\). The complexity of generating the actions is \(O(n \log n)\).

**Proof.** The proof is provided in Appendix 3.A.2.

\[\square\]

As for the two-node network, there exists a choice of the values of the frozen bits such that the sequence of PCs achieves empirical coordination.

**Corollary 3.4.** All triples \((R_1, R_2, P_{Y,Z|X}) \in C_{P_X}^s\) are achievable using a sequence of \((n, 2^{nR_1}, 2^{nR_2})\)-polar codes (with \(n = 2^p, p \in \mathbb{N}\) and a fixed choice of frozen bits, with probability of coordination error \(P_e \leq O(2^{-n^\beta})\) for any \(0 < \beta < \frac{1}{2}\). The complexity of generating the actions is \(O(n \log n)\).

**Proof.** The proof is provided in Appendix 3.A.4.

\[\square\]

The preceding results can be easily extended to the general \((k + 1)\)-node cascade in Figure 3.3. As before, when designing polar codes for this network, we restrict our attention to the following subset of the coordination capacity region (cf. Corollary 2.56):

\[
C_{P_X}^s \triangleq \left\{ (R_1, \ldots, R_k, P_{Y_1 \ldots Y_k|X}) : R_i > I(X; Y_i^k) \text{ for every } i \in \{1, \ldots, k\}, Y_1, \ldots, Y_k \sim \text{Unif}\{0,1\} \right\}. \tag{3.6}
\]

The additional constraints are straightforward generalizations of those in the three-node cascade network.
Corollary 3.5. All tuples \((R_1, \ldots, R_k, P_{Y_1, \ldots, Y_k|X}) \in C_{P_X}^s\) are achievable using a sequence of \((n, 2^{nR_1}, \ldots, 2^{nR_k})\)-polar codes (with \(n = 2^p\), \(p \in \mathbb{N}\)), with probability of coordination error \(P_e \leq O(2^{-n\beta})\) for any \(0 < \beta < \frac{1}{2}\). The complexity of generating the actions is \(O(n \log n)\).

Proof. The proof is provided in Appendix 3.A.2.

Corollary 3.6. All tuples \((R_1, \ldots, R_k, P_{Y_1, \ldots, Y_k|X}) \in C_{P_X}^s\) are achievable using a sequence of \((n, 2^{nR_1}, \ldots, 2^{nR_k})\)-polar codes (with \(n = 2^p\), \(p \in \mathbb{N}\)) and a fixed choice of frozen bits, with probability of coordination error \(P_e \leq O(2^{-n\beta})\) for any \(0 < \beta < \frac{1}{2}\). The complexity of generating the actions is \(O(n \log n)\).

Proof. The proof is provided in Appendix 3.A.4.

3.2.3 Broadcast Network

In this section, we study the broadcast network introduced in Section 2.2.2 and depicted in Figure 3.4. Consider the definition of an \((n, 2^{nR_1}, 2^{nR_2})\)-code for this network (Definition 2.57) and the definition of achievability. Consider the inner bound \(R_{P_X}^s\) on the coordination capacity region in (2.84). When designing polar codes for this network, we restrict our attention to the following subset of this bound:

\[
R_{P_X}^s \triangleq \left\{ \left( R_1, R_2, P_{Y,Z|X} \right) : \exists P_{V|X,Y,Z} \text{ s.t.} \right. \\
R_1 > I(X; VY), \\
R_2 > I(X; VZ), \\
R_1 + R_2 > I(X; VY) + I(X; VZ) + I(Y; Z|XV), \\
V, Y, Z \sim \text{Unif}\{0, 1\} \right\}. 
\]

(3.7)

That is, we only consider binary actions \(Y\) and \(Z\) and points that can be achieved with a binary auxiliary random variable \(V\), and conditional distributions \(P_{Y,Z|X}\) and \(P_{V|X,Y,Z}\) that induce (individual) marginal uniform distributions on \(V, Y,\) and \(Z\). Extensions are discussed in Section 3.3.

Our main result for the broadcast network is the following:

Theorem 3.7. All triples \((R_1, R_2, P_{Y,Z|X}) \in R_{P_X}^s\) are achievable by time-sharing between sequences of \((n, 2^{nR_1}, 2^{nR_2})\)-polar codes (with \(n = 2^p\), \(p \in \mathbb{N}\)), with probability of coordination error

\[
P_e \leq O(2^{-n\beta})
\]

(3.8)

for any \(0 < \beta < \frac{1}{2}\). The complexity of generating the actions is \(O(n \log n)\).
3.3 Extensions

So far, our results have been restricted to binary induced actions with (individual) uniform marginal distributions. We now discuss the extension to arbitrary alphabets and non-uniform distributions.

The constructions of polar codes used in the proofs of our results rely fundamentally on the polarization theorems (Theorems 2.67 and 2.74), the theorem on the vanishing distance between design and induced distributions (Theorem 2.73), and the lemma on the nesting of frozen sets (Lemma 2.70). All these results apply for larger classes of alphabets than binary alphabets. It is therefore straightforward to extend the results to non-binary alphabets with the aid of the Lemmas 2.68 and 2.69. In the next chapter one such construction is used for compress-and-forward relaying.

An extension to non-uniform distributions is also possible, albeit less elegant, by using the simple method from [Gal68, p. 208]. This method consists of augmenting the channel using a deterministic function so that the uniform input distribution yields the capacity of the augmented channel (strictly speaking, sometimes we cannot create a function that induces the desired \( P_Y \) exactly, but we can always get arbitrarily close to it). For example, consider the channel \( P_{X|Y} \) in Figure 3.5 and its

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**Proof.** The proof is provided in Appendix 3.A.3.

**Corollary 3.8.** All triples \((R_1, R_2, P_{Y,Z|X}) \in \mathcal{R}_{P_X}^*\) are achievable using a sequence of \((n, 2^{nR_1}, 2^{nR_2})\)-polar codes (with \( n = 2^p \), \( p \in \mathbb{N} \)) and a fixed choice of frozen bits, with probability of coordination error \( P_e \leq O(2^{-n^\beta}) \) for any \( 0 < \beta < \frac{1}{2} \). The complexity of generating the actions is \( O(n \log n) \).

**Proof.** The proof is provided in Appendix 3.A.4.
corresponding extension $P_{X|Y'}$. The deterministic function $F_{Y'|Y'}$ is designed such that a uniform distribution on $Y'$ induces the desired (non-uniform) distribution on $Y$. In our example, this corresponds to $P_Y(y_0) = \frac{2}{3}, P_Y(y_1) = \frac{1}{3}$, which is trivially induced by a uniform distribution on $Y'$ and the deterministic function $F_{Y'|Y'}$ in Figure 3.5.

Now, we show how this method is useful for coordination. Consider the two-node network and assume that the target distribution $P_{Y|X}$, combined with the distribution of the external actions $P_X$, induces a non-uniform distribution $P_{Y'}$. We construct an augmented version $P_{X|Y'}$ of the distribution $P_{X|Y}$ and design the coordination code using $P_{X|Y'}$. Thus, we know that

$$\|P_{X,Y'} - Q_{X,Y'}\|_{TV} \leq O(2^{-n^a}).$$

(3.9)

The deterministic relationship between $Y'$ and $Y$ is the same for both design and induced distributions and is given by

$$F_{Y|Y'} = \prod_i F_{Y_i|Y'_i}.$$  \hfill (3.10)

Using this relationship and the triangle inequality, we see that

$$\|P_{X,Y'} - Q_{X,Y'}\|_{TV} = \frac{1}{2} \sum_{x,y'} |P_{X,Y'}(x,y') - Q_{X,Y'}(x,y')|$$  \hfill (3.11)

$$= \frac{1}{2} \sum_{x,y,y'} F_{Y|Y'}(y|y') |P_{X,Y'}(x,y') - Q_{X,Y'}(x,y')|$$ \hfill (3.12)

$$\geq \frac{1}{2} \sum_{x,y} \left| \sum_{y'} (F_{Y|Y'}P_{X,Y'}(x,y') - F_{Y|Y'}Q_{X,Y'}(x,y')) \right|$$  \hfill (3.13)

$$= \|P_{X,Y} - Q_{X,Y}\|_{TV}.$$ \hfill (3.14)
Thus,

\[ \|P_{XY} - Q_{XY}\|_{TV} \leq O(2^{-n^{\beta}}) \] (3.15)

as well. Observe that, although we have changed the distribution used in the design of the code, the coordination rate does not vary:

\[ I(Y'; X) = I(YY'; X) = I(Y; X). \] (3.16) (3.17)

The equality in (3.16) is given by the fact \( Y \) is a deterministic function of \( Y' \), whereas the equality in (3.17) is due to the Markov chain \( Y' - Y - X \). Having proved this, it follows from the rest of the proof of Theorem 3.11 that this implies empirical coordination. Similarly, we can show that this strategy works also for cascade and broadcast networks. Note that in those cases, the channel coding components also need to be designed for the augmented channels but this poses no problem (see [Kor09]).

3.4 Summary and Concluding Remarks

In this chapter, we have constructed polar codes for achieving rate-coordination tuples in the cascade and broadcast networks. Our constructions combine elements of source coding and channel coding in a nested way. The polar codes for source coding are used to generate actions with the desired type, whereas the polar codes for channel coding are used to reduce the rate by exploiting the correlations in the network.

We expect the basic structures presented in this chapter to play an important role in achieving coordination with affordable complexity in larger networks. However, they provide only limited tools and, therefore, it is unlikely that these constructions alone will suffice for the general case.
3.A Proofs

3.A.1 Two-Node Network

Proof of Theorem 3.1. Fix $\epsilon > 0$, fix $0 < \beta < \frac{1}{2}$, and let $\delta_n = \frac{1}{n}2^{-n\beta}$. Choose $(R, P_{X|Y})$ in $C_{P_X}$ and let

$$
P_{X|Y} = \frac{P_{X,Y}}{P_Y} = \frac{P_{Y|X}P_X}{P_Y}$$

(3.18)

where $P_Y$ is, by assumption, the uniform distribution. We assume that both nodes have access to an unlimited source of common randomness that is used to generate i.i.d. random frozen bits.

**Code construction and generation of the actions.** To generate $Y$, we use a polar code with design based on $P_{X|Y}$. Consider the following definition of the frozen set for a sequence of polar codes:

$$
\mathcal{F} \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P^{(i)}_{X|Y}) \geq 1 - \delta_n^2 \right\}.
$$

(3.19)

By the polarization theorem for source coding (Theorem 2.74), we know that $\frac{|\mathcal{F}|}{n} \leq R$ if $n$ is sufficiently large.

Using the source of common randomness, both nodes generate $|\mathcal{F}|$ i.i.d. random bits according to a uniform distribution. Using them as frozen bits, Node $X$ compresses its observation $X$ with the SC encoding algorithm. The output of the algorithm is the vector $U$, which contains $|\mathcal{F}^c|$ information bits (i.e., $U_{\mathcal{F}^c}$) and $|\mathcal{F}|$ frozen bits (i.e., $U_{\mathcal{F}}$). Therefore, a rate of $R$ [bpa] suffices to communicate the information bits to Node $Y$ if $n$ is sufficiently large. This allows Node $Y$ to generate the sequence $Y = UG_n$.

The claim on the complexity follows from the fact that we have only used the encoding and decoding algorithms for polar codes.

**Analysis of the type and coordination error probability.** We now show that this construction achieves empirical coordination under the distribution induced by the external actions and the code. This distribution is

$$
Q_{X,Y} = P_XQ_{Y|X}.
$$

(3.20)

From Theorem 2.73, we know that $Q_{X,Y}$ is close in total variation to the desired $n$-product distribution $P_{X,Y} = \prod P_{Y|X}P_X$, i.e.

$$
\|P_{X,Y} - Q_{X,Y}\|_{TV} \leq O(2^{-n\beta}).
$$

(3.21)
Now, construct the optimal coupling (cf. Lemma 2.4) $C_{PQ}(x_P, y_P, x_Q, y_Q)$ between $P_{X,Y}(x_P, y_P)$ and $Q_{X,Y}(x_Q, y_Q)$. Consider the events
\[
\mathcal{E} \triangleq \{ \| P_{X,Y} - T_{X_Q,Y_Q} \|_{TV} \geq \epsilon \},
\]
\[
\mathcal{E}_{XY} \triangleq \{ (X_P, Y_P) \neq (X_Q, Y_Q) \},
\]
and their complements $\mathcal{E}^c$ and $\mathcal{E}_{XY}^c$. Observe that the probability of coordination error corresponds to $\Pr(\mathcal{E})$ evaluated over $C_{PQ}$. We write
\[
\Pr(\mathcal{E}) = \Pr(\mathcal{E}|\mathcal{E}_{XY}) \Pr(\mathcal{E}_{XY}) + \Pr(\mathcal{E}|\mathcal{E}_{XY}^c) \Pr(\mathcal{E}_{XY}^c).
\]
For the first term in (3.24) we have that $\Pr(\mathcal{E}_{XY}) \leq O(2^{-n^\beta})$ by Lemma 2.4 and (3.21). We upper bound the second term in (3.24) as follows:
\[
\Pr(\mathcal{E}|\mathcal{E}_{XY}) \Pr(\mathcal{E}_{XY}^c) = \Pr(\| P_{X,Y} - T_{X_P,Y_P} \|_{TV} \geq \epsilon | \mathcal{E}_{XY}^c) \Pr(\mathcal{E}_{XY}^c)
\]
\[
\leq \Pr(\| P_{X,Y} - T_{X_P,Y_P} \|_{TV} \geq \epsilon).
\]
The term in (3.26) is just the probability that a tuple $(X, Y) \sim P_{X,Y}$ has a type with total variation with respect to $P_{X,Y}$ larger than or equal to $\epsilon$. By Lemma 2.11, we know that this probability goes to zero exponentially fast with $n$. Therefore, the bound for the first term in (3.24) dominates and we have:
\[
\Pr(\mathcal{E}) \leq O(2^{-n^\beta}).
\]
Hence, we conclude that the sequence of PCs achieves empirical coordination. ■

3.3.2 Cascade Network

Proof of Theorem 3.3. Let $P_{Y,Z|X}$ be the desired distribution for coordination. Fix $R_1 = I(X; Y|Z) + \epsilon_1$ and $R_2 = I(X; Z) + \epsilon_2$ for $\epsilon_1$ and $\epsilon_2$ such that $\epsilon_1 > 3\epsilon_2 > 0$. Let $0 < \beta < \frac{1}{2}$ and $\delta_n = \frac{1}{n}2^{-n^\beta}$ and fix $\epsilon > 0$. We assume that all nodes have access to an unlimited source of common randomness that is used to generate i.i.d. random frozen bits.

**Code construction and generation of the actions.** Generation of $Z$ is identical to that in the two-node scenario. That is, Node $X$ uses a PC with design based on $P_{X|Z}$ (obtained from $P_{Y,Z|X} P_X$ by conditioning and marginalizing) to compress $X$ into $Z$. Consider the following definition of the frozen set for a sequence of polar codes:
\[
\mathcal{F}_1 \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P^{(i)}_{X|Z}) \geq 1 - \delta^2_n \right\}.
\]
By Theorem 2.74 for sufficiently large $n$, we have that $|\mathcal{F}_1| \leq R_2$. The vector resulting from source compression is sent to Node $Y$ and then forwarded to Node $Z$. 
Thus, for sufficiently large $n$, a communication rate equal to $R_2$ [bpa] suffices to allow both nodes to generate $Z$. This exhausts all communication resources to Node Z and leaves the link to Node Y with only $R_1 - R_2$ [bpa] available, where

$$
R_1 - R_2 = I(X;YZ) - I(X;Z) + \epsilon_1 - \epsilon_2 
$$

(3.29)

$$
> I(X;Y|Z) + 2\epsilon_2
$$

(3.30)

$$
\triangleq R'_1.
$$

(3.31)

We generate the sequence of actions $Y$ in two consecutive steps: i) Node $X$ generates a sequence $\tilde{Y}$ that satisfies the type constraints with high probability. ii) Node $Y$ generates $Y$, which is equal to $\tilde{Y}$ with high probability. In addition, the code exploits the correlation between $\tilde{Y}$ and $Z$ to minimize the communication rate requirements. We describe the two steps in detail in the following.

For step i), Node $X$ uses a PC with design based on $P_{X,Z|Y}$ to compress $(X,Z)$ into $\tilde{Y}$. Consider the following definition of the frozen set for a sequence of polar codes:

$$
F_2 \triangleq \{ i \in \{ 1, \ldots, n \} : Z(P^{(i)}_{X,Z|Y}) \geq 1 - \delta^2_n \}.
$$

(3.32)

By Theorem 2.74 we know that

$$
\frac{|F_2^c|}{n} \leq I(XZ;Y) + \epsilon_2
$$

(3.33)

if $n$ is sufficiently large. As we will see later, this yields sequences of actions $(X, \tilde{Y}, Z)$ with the desired type with high probability. Recall that $\tilde{Y} = UG_n$, where $U$ is the bit-vector put out by the SC encoding algorithm. The vector $U$ includes the frozen bits $U_{F_2}$ used by the algorithm, which are also available at Node $Y$ (through the source of common randomness). Observe that the direct transmission of $U_{F_2}$ from Node $X$ to Node $Y$ would require a too high rate (i.e., $I(XZ;Y)$).

In step ii), Node $Y$ generates $Y$. To accomplish this while communicating at the desired rate (i.e., $R'_1$), the code takes advantage of the fact that $Z$, which is correlated with $\tilde{Y}$, is already available at Node $Y$. We model this correlation as if $\tilde{Y}$ were transmitted through the DMC $P_{Z|Y}$. That is, we treat $\tilde{Y} = UG_n$ (i.e., a codeword from a PC) as the input to the DMC and $Z$ as the output. Consider the following definition of a sequence of frozen sets for channel coding:

$$
F_3 \triangleq \{ i \in \{ 1, \ldots, n \} : Z(P^{(i)}_{Z|Y}) \geq \delta_n \}.
$$

(3.34)

As if it were a channel (de)coding problem, Node $Y$ obtains $Y$, which is an estimate of $\tilde{Y}$, using the SC decoding algorithm. To work properly, this decoder needs to know the values of the corresponding frozen bits (i.e., $U_{F_3}$). Observe that the bits $U_{F_2}$ are already known at Node $Y$ (cf. step i)). Moreover, the channel $P_{Z|Y}$ is
Figure 3.6: Nesting of the frozen sets.

degraded with respect to $P_{X,Z|Y}$ and, thus, by virtue of Lemma 2.70, $\mathcal{F}_2 \subseteq \mathcal{F}_3$ for sufficiently large $n$ (so that $\delta_n \leq 1 - \delta_n^2$). Therefore, only the bits in the set $\mathcal{F}_3 \cap \mathcal{F}_2^c$ need to be transmitted (see Figure 3.6). This requires a communication rate equal to $\frac{|\mathcal{F}_3| - |\mathcal{F}_2|}{n}$ [bpa]. By (3.33) and Theorem 2.67 for sufficiently large $n$, we have that

$$\frac{|\mathcal{F}_3| - |\mathcal{F}_2|}{n} = \frac{|\mathcal{F}_2^c| - |\mathcal{F}_3^c|}{n} \leq I(XZ;Y) + \epsilon_2 - I(Y;Z) + \epsilon_2 \leq R_1', \tag{3.35}$$

$$= I(Y;Z) + \epsilon_2 \leq R_1', \tag{3.36}$$

as desired. Since $\epsilon_1$ can be chosen arbitrarily small, it follows that any rate $R_1 > I(X;YZ)$ is sufficient (for sufficiently large $n$). Similarly, any rate $R_2 > I(X;Z)$ suffices.

The claim on the complexity follows from the fact that we have only used the encoding and decoding algorithms for polar codes.

**Analysis of the type and coordination error probability.** We now show that this construction achieves empirical coordination under the distribution induced by the external actions and the code. This distribution is

$$Q_{X,Y,\tilde{Y},Z} = P_X Q_{Z|X} Q_{\tilde{Y}|X,Z} Q_{Y|\tilde{Y},Z}. \tag{3.38}$$

First observe that, for our choice of $\mathcal{F}_1$, Theorem 2.73 ensures that

$$\|P_{X,Z} - P_{X} Q_{Z|X}\|_{TV} \leq O(2^{-n^\alpha}), \tag{3.39}$$

where $P_{X,Z} = \prod P_{Z|X} P_X$. Similarly, our choice of the frozen set $\mathcal{F}_2$ ensures that

$$\|P_{Y|X,Z} P_{X,Z} - Q_{\tilde{Y}|X,Z} P_{X,Z}\|_{TV} \leq O(2^{-n^\alpha}), \tag{3.40}$$

where $P_{Y|X,Z} = \prod P_{Y|X,Z}$. Using the triangle inequality, it is easy to show that (3.39) and (3.40) imply that

$$\|P_{Y|X,Z} P_{X,Z} - Q_{\tilde{Y}|X,Z} P_{X,Z}\|_{TV} \leq O(2^{-n^\alpha}). \tag{3.41}$$
As in Section 3.2.1, we use this result to build the optimal coupling \( \tilde{C}_{PQ} \) between \( P_{X,Y,Z} \) and \( Q_{X,\tilde{Y},Z} \). Under this coupling, the event

\[
\tilde{E}_C \triangleq \{ (X_P, Y_P, Z_P) \neq (X_Q, \tilde{Y}_Q, Z_Q) \}
\]  

has probability \( \Pr(\tilde{E}_C) \leq O(2^{-n^a}) \). We use \( \tilde{C}_{PQ} \) to evaluate the probability of the event

\[
\tilde{E} \triangleq \{ \| P_{X,Y,Z} - T_{X,\tilde{Y},Z} \|_{TV} \geq \epsilon \}
\]  

in a similar way to (3.24)-(3.27), obtaining that

\[
\Pr(\tilde{E}) \leq \Pr(\tilde{E}_C) + \Pr(\tilde{E}|\tilde{E}_C^c) \Pr(\tilde{E}_C^c) \leq O(2^{-n^a}).
\]  

This means that our construction will yield sequences of actions \((X, \tilde{Y}, Z)\) with the desired type with high probability.

Now, we show that the sequence \( Y \) constructed by Node \( Y \) is equal to \( \tilde{Y} \) with very high probability. That is, that the probability of the event

\[
\mathcal{E}_Y \triangleq \{ Y \neq \tilde{Y} \}
\]  

evaluated over the distribution induced by the code (i.e., (3.38)) is arbitrarily small. Note that only the marginal

\[
Q_{Y,\tilde{Y},Z} = Q_{\tilde{Y},Z}Q_{Y|\tilde{Y},Z}
\]  

of (3.38) is relevant here. Observe also that, if \((\tilde{Y}, Z)\) were sampled from the design distribution (i.e., \((\tilde{Y}, Z) \sim P_{Y,Z}\)) then, our model for the correlation between \( \tilde{Y} \) and \( Z \) would be correct. Thus, for our choice of \( \mathcal{F}_3 \), the basic results on polar codes for channel coding (Theorem 2.67) would ensure that

\[
\Pr(\mathcal{E}_Y) \leq O(2^{-n^a}).
\]  

It is easy to show that the behavior is the same under the distribution induced by the code (i.e., when \((\tilde{Y}, Z) \sim Q_{Y,\tilde{Y},Z}\)). To this end, consider the optimal coupling \( C_{PQ}(x_p, y_p, x_q, y_q) \) between \( P_{Y,Z} \) and \( Q_{Y,\tilde{Y},Z} \). Recall that the marginal of \( C_{PQ} \) is equal to \( Q_{Y,\tilde{Y},Z} \). Thus, evaluating \( \Pr(\mathcal{E}_Y) \) over (3.47) is equivalent to evaluating \( \Pr(Y_Q \neq \tilde{Y}) \) over \( C_{PQ}Q_{Y|\tilde{Y}} \). Using the event

\[
\mathcal{E}_C \triangleq \{ (X_P, Y_P) \neq (X_Q, Y_Q) \},
\]  

we obtain

\[
\Pr(Y_Q \neq \tilde{Y}) \leq \Pr(\mathcal{E}_C) + \Pr(Y_Q \neq \tilde{Y}|\mathcal{E}_C^c) \Pr(\mathcal{E}_C^c)
\]
\[
= \Pr(\mathcal{E}_C) + \Pr(Y_P \neq \tilde{Y}|\mathcal{E}_C^c) \Pr(\mathcal{E}_C^c)
\]
\[
\leq \Pr(\mathcal{E}_C) + \Pr(Y_P \neq \tilde{Y}).
\]
By the properties of the optimal coupling (Lemma 2.4), we know that
\[ \Pr(\mathcal{E}_C) = \| P_{Y,Z} - Q_{\tilde{Y},Z} \|_{TV}. \]  
(3.53)
It is easy to show that (3.41) implies
\[ \| P_{Y,Z} - Q_{\tilde{Y},Z} \|_{TV} \leq O(2^{-n^\beta}) \]  
(3.54)
by applying the triangle inequality and the definition of total variation (Definition 2.3). Observe also that, the second term in (3.52) corresponds to (3.48). Thus,
\[ \Pr(\mathcal{E}_Y) \leq O(2^{-n^\beta}) \]  
(3.55)
under the distribution (3.47), as well.

Finally, we show that (3.45) and (3.55) imply that the probability of the event
\[ \mathcal{E} \triangleq \{ \| P_{X,Y,Z} - T_{X,Y,Z} \|_{TV} \geq \epsilon \}, \]  
(3.56)
evaluated over the distribution induced by the code \( Q_{X,Y,Z} \) is arbitrarily low:
\[ \Pr(\mathcal{E}) \leq \Pr(\mathcal{E}_Y) + \Pr(\mathcal{E} | \mathcal{E}_Y) \Pr(\mathcal{E}_Y) \]  
(3.57)
\[ \leq \Pr(\mathcal{E}_Y) + \Pr(\tilde{\mathcal{E}}) \]  
(3.58)
\[ \leq O(2^{-n^\beta}). \]  
(3.59)
Hence, the sequence of PC achieves empirical coordination. □

**Proof of Corollary 3.5** The proof follows by a repeated application of the construction used for the three-node cascade network. That is, first, we compress \( X \) into \( Y_k \). Then, we compress \((X, Y_k)\) into \( Y_{k-1} \) and reduce the requirement on the communication rate by exploiting the correlation. Then, we compress \((X, Y_{k-1}, Y_k)\) into \( Y_{k-2} \) and reduce the requirement on the communication rate by exploiting the new correlation, etc. □

### 3.A.3 Broadcast Network

The structure of the coordination code for the broadcast network is very similar to that of a pair of codes for the three-node cascade network that share a common message. This shared message corresponds to the auxiliary sequence \( V \) and is used to strengthen the correlation between \( Y \) and \( Z \).

**Proof.** As in the general proof in [CPC10], we focus on achieving one of the corner points in \( R_{P_X}^s \). The other one can be achieved by reversing the roles of Nodes \( Y \) and \( Z \). Time sharing between the two corner points, we achieve the rest of the points in the region.

Fix \( \epsilon > 0, 0 < \beta < \frac{1}{2} \), and let \( \delta_n = \frac{1}{n}2^{-n^\beta} \). Let \( P_{Y,Z|X} \) be the desired pmf and \( P_{Y|X,Y,Z} \) be the conditional distribution of the auxiliary RV. We assume that all nodes have access to a source of common randomness that is used to generate i.i.d. random frozen bits.
**Code construction and generation of the actions.** First, Node $X$ generates the common message $V$ using a PC with design based on $P_{X|V}$. Consider the following definition of the frozen set for a sequence of polar codes:

$$\mathcal{F}_1 \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P_{X|V}^{(i)}) \geq 1 - \delta^2_n \right\}. \tag{3.60}$$

By similar arguments to the ones used in the two-node network, communication from Node $X$ to Nodes $Y$ and $Z$ at any rate $R_V > I(X;V)$ suffices to allow both them to reconstruct $V$ (provided $n$ is large enough).

To generate the sequence $Y$, Nodes $X$ and $Y$ follow the two-step strategy of the cascade network. That is, Node $X$ uses a PC with design based on $P_{V,X|Y}$ to compress $(V,X)$ into $\tilde{Y}$. Consider the following definition of the frozen set for a sequence of polar codes:

$$\mathcal{F}_2 \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P_{V,X|Y}^{(i)}) \geq 1 - \delta^2_n \right\}. \tag{3.61}$$

To reduce the rate of transmission, Node $Y$ exploits the correlation between $V$ and $\tilde{Y}$. For this purpose, we consider the following definition of the frozen set for a sequence of polar codes:

$$\mathcal{F}_3 \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P_{V|Y}^{(i)}) \geq \delta_n \right\}. \tag{3.62}$$

As in the cascade network, transmission of the bits in $\mathcal{F}_3 \cap \mathcal{F}_2$ suffices to allow Node $Y$ to construct a sequence $Y$ that is equal to $\tilde{Y}$ with high probability. This can be done at any rate $R_Y > I(X;Y|V)$ by choosing a sufficiently large $n$. The sum of the two rates yields the rate requirement

$$R_1 = R_V + R_Y > I(X;VY). \tag{3.63}$$

We use the same two-step strategy to generate $Z$: Node $X$ uses a PC with design based on $P_{V,X,Y|Z}$ to compress $(V,X,\tilde{Y})$ into $\tilde{Z}$. Consider the following definition of the frozen set for a sequence of polar codes:

$$\mathcal{F}_4 \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P_{V,X,Y|Z}^{(i)}) \geq 1 - \delta^2_n \right\}. \tag{3.64}$$

Consider also the following definition of the frozen set for a sequence of polar codes:

$$\mathcal{F}_5 \triangleq \left\{ i \in \{1, \ldots, n\} : Z(P_{V|Z}^{(i)}) \geq \delta_n \right\}. \tag{3.65}$$

Transmission of the bits in $\mathcal{F}_5 \cap \mathcal{F}_4$ suffices to allow Node $Z$ to construct a sequence $Z$ that is equal to $\tilde{Z}$ with high probability. This can be done at any rate $R_Z > I(Z;X|V)$ by choosing a sufficiently large $n$. The sum of the two rates yields the rate requirement

$$R_2 = R_V + R_Z > I(X;V) + I(Z;XY|V). \tag{3.66}$$

The claim on the complexity follows from the fact that we have only used the encoding and decoding algorithms for polar codes.
Analysis of the type and coordination error probability. We now show that this construction achieves empirical coordination under the distribution induced by the external actions and the code. This distribution is

\[ Q_{V,X,Y,\tilde{Y},Z,\tilde{Z}} = P_X Q_{V|X} Q_{\tilde{Y}|V,X} Q_{\tilde{Z}|V,X,Y} Q_{Y|V,\tilde{Y}} Q_{Z|V,\tilde{Z}}. \]  

(3.67)

Let \( Q_{V,X,\tilde{Y}} = P_X Q_{V|X} Q_{\tilde{Y}|V,X} \). From our analysis for the cascade network, we know that the design distribution \( P_{V,X,Y} \) and the distribution induced by the code \( Q_{V,X,\tilde{Y}} \) satisfy

\[ \|P_{V,X,Y} - Q_{V,X,\tilde{Y}}\|_{TV} \leq O(2^{-n^\beta}). \]  

(3.68)

Moreover, the error probability in constructing \( Y \) satisfies

\[ \Pr(Y \neq \tilde{Y}) \leq O(2^{-n^\beta}). \]  

(3.69)

Similarly,

\[ \|P_{V,X,Y} P_{V,X,Y} - Q_{\tilde{Z}|V,X,Y} P_{V,X,Y}\|_{TV} \leq O(2^{-n^\beta}). \]  

(3.70)

Now, using the triangle inequality, it is easy to show that (3.68) and (3.70) imply that

\[ \|P_{V,X,Y} P_{V,X,Y} - Q_{\tilde{Z}|V,X,Y} P_{V,X,Y}\|_{TV} \leq O(2^{-n^\beta}). \]  

(3.71)

In addition, by the same arguments used in the cascade network, we have that

\[ \Pr(Z \neq \tilde{Z}) \leq O(2^{-n^\beta}). \]  

(3.72)

Consider the events

\[ E \triangleq \{\|P_{X,Y,Z} - T_{X,Y,Z}\|_{TV} \geq \epsilon\}, \]  

(3.73)

\[ E_{Y,Z} \triangleq \{Y \neq \tilde{Y}\} \cup \{Z \neq \tilde{Z}\}. \]  

(3.74)

We have that

\[ \Pr(E) \leq \Pr(E_{Y,Z}) + \Pr(E|E_{Y,Z}^c) \Pr(E_{Y,Z}^c). \]  

(3.75)

For the first term, we have that

\[ \Pr(E_{Y,Z}) \leq \Pr(Y \neq \tilde{Y}) + \Pr(Z \neq \tilde{Z}). \]  

(3.76)

\[ \leq O(2^{-n^\beta}). \]  

(3.77)

For the second term, we see that

\[ \Pr(E|E_{Y,Z}^c) \Pr(E_{Y,Z}^c) = \Pr(\|P_{X,Y,Z} - T_{X,Y,Z}\|_{TV} \geq \epsilon|E_{Y,Z}^c) \Pr(E_{Y,Z}^c) \]  

(3.78)

\[ = \Pr(\|P_{X,Y,Z} - T_{X,\tilde{Y},\tilde{Z}}\|_{TV} \geq \epsilon|E_{Y,Z}^c) \Pr(E_{Y,Z}^c) \]  

(3.79)

\[ \leq \Pr(\|P_{X,Y,Z} - T_{X,\tilde{Y},\tilde{Z}}\|_{TV} \geq \epsilon). \]  

(3.80)
Similarly to (3.24)-(3.27), we can use Lemma 2.4 and (3.71) to create an optimal coupling between
\[ P_{Z|V,X,Y} P_{V,X,Y} \] and
\[ Q_{Z|V,X,Y} Q_{V,X,Y} \], and then use Lemma 2.11 to show that
\[ \Pr(\|P_{X,Y,Z} - T_{X,Y,Z}\|_{TV} \geq \epsilon) \leq O(2^{-n^\beta}). \] (3.81)
Collecting all terms we obtain the desired bound
\[ \Pr(E) \leq O(2^{-n^\beta}). \] (3.82)

**Time sharing:** Reversing the roles of Nodes \( Y \) and \( Z \), we establish achievability of empirical coordination for the other corner point. The rate constraints are:
\[ R_1 > I(X;V) + I(Y;XZ|V), \] (3.83)
\[ R_2 > I(X;VZ). \] (3.84)
Any other point in \( \mathcal{R}_{P_X}^s \) is achieved by time sharing between the two corner points. \( \blacksquare \)

### 3.A.4 Fixed Frozen Bits

We provide a generic proof that applies to Corollaries 3.2, 3.4, and 3.8.

**Proof.** Let \( X \) be the vector of external actions and let \( W \) and \( \tilde{W} \) be vectors containing the action sequences and *all* sequences (including auxiliary sequences) generated by the code, respectively. For example, for the cascade network \( W = [Y, Z] \) and \( \tilde{W} = [W, \tilde{Y}] \). Observe that all sequences of actions produced by the code have the form \( UG_n \) where \( U \) is a bit vector containing frozen and information bits. Let \( V \) be the concatenation of all such bit vectors. The translation between \( V \) and \( W \) is straightforward (i.e., by multiplying each subvector in \( V \) by \( G_n \)).

The achievability theorems (Theorem 3.1, 3.3 or 3.7) ensure that there exists a sequence of PCs such that
\[ \|T_{X,W} - P_{X,W}\|_{TV} \to 0 \] (3.85)
in probability. Since the total variation is a bounded measure, this implies that
\[ \lim_{n \to \infty} \mathbb{E}_{X,W} \{\|T_{X,W} - P_{X,W}\|_{TV}\} = 0. \] (3.86)
The expectation, which is taken with respect to the induced distribution $Q_{X,W}$, can be written as

$$
\mathbb{E}_{X,W} \{\|T_{X,W} - P_{X,W}\|_{TV}\} = \mathbb{E}_{X,V} \{\|T_{X,V} - P_{X,W}\|_{TV}\} \tag{3.87}
$$

$$
= \sum_{x,v} Q(x,v)\|T_{x,v} - P_{X,W}\|_{TV} \tag{3.88}
$$

$$
= \sum_{v_{\mathcal{F}}} \sum_{x,v_{\mathcal{F}C}} Q(x,v_{\mathcal{F}C},v_{\mathcal{F}})\|T_{x,v} - P_{X,W}\|_{TV}
$$

$$
= \mathbb{E}_{\mathcal{F}} \{\mathbb{E}_{X,V_{\mathcal{F}C}} \{\|T_{X,V_{\mathcal{F}C},V_{\mathcal{F}}} - P_{X,W}\|_{TV}\}\}. \tag{3.89}
$$

In (3.87) and (3.88) we have used the translation between $V$ and $W$ to change the arguments of $Q$ and $T$ so that they depend on $v$. The notation $T_{x,v_{\mathcal{F}C},v_{\mathcal{F}}} = T_{x,v}$ in (3.89) simply makes explicit the fact that $v$ has two parts: $(v_{\mathcal{F}C}$ and $v_{\mathcal{F}})$, where $\mathcal{F}$ identifies the elements in each of the frozen sets. For the outer expectation in (3.89) to go to zero with the block length there must exist a fixed choice of the values of the frozen bits $\tilde{v}_{\mathcal{F}}$ such that

$$
\mathbb{E}_{X,V_{\mathcal{F}C}} \{\|T_{X,V_{\mathcal{F}C},\tilde{v}_{\mathcal{F}}} - P_{X,W}\|_{TV}\} \leq \mathbb{E}_{X,V} \{\|T_{X,V} - P_{X,W}\|_{TV}\}. \tag{3.90}
$$

Finally, since convergence in the first mean implies convergence in probability, we conclude that there exists a sequence of PCs for coordination and a choice of the frozen bits that achieves empirical coordination.
In this chapter, we consider polar codes (PCs) for relay channels with orthogonal receivers. Our main result is a construction for compress-and-forward (CF) relaying. The structure is similar to that of the coordination codes presented in Chapter 3, in particular those for the cascade network. We also show that our construction can be specialized to the case of compress-and-forward relaying based on Slepian-Wolf coding, which achieves the capacity of the channel in some special cases. We conclude the chapter by numerically evaluating the performance of polar codes for compress-and-forward and decode-and-forward (DF) relaying with finite block lengths. Although their error performance is insufficient for practical applications, the simplicity of the constructions allows us to identify the different bottlenecks in the system. As before, all the proofs are provided in the appendix.

4.1 Preliminaries

In this chapter, we consider the relay channel with orthogonal receivers introduced in Section 2.2.1 and depicted in Figure 4.1. The distribution governing this channel has the form

$$P_{Y_{SD},Y_{SR}|X}(y_{sd},y_{sr}|x)P_{Y_{RD}|X_R}(y_{rd}|x_r).$$

(4.1)

In Figure 4.1 we have included the different vectors involved in the communication. We describe them in the following: $M$ contains the information and the frozen symbols (the latter are also available at the destination) transmitted by the source.
\( \hat{M} \) is the corresponding estimate at the destination. \( X \) is the vector of symbols put into the channel by the source, and \( Y_{SR} \) and \( Y_{SD} \) are the channel outputs at the relay and at the destination, respectively. Similarly, \( X_R \) is the vector put into the channel by the relay and \( Y_{RD} \) is the observation at the destination. The vector \( Y_Q \) is a compressed version of \( Y_{SR} \) produced by the relay and \( \hat{Y}_Q \) is the corresponding estimate generated by the destination. We will refer to the marginal pmfs \( P_{Y_{SR}|X}, P_{Y_{SD}|X} \), and \( P_{Y_{RD}|X_R} \) as the source-relay, source-destination, and relay-destination channels. To avoid a cumbersome notation, we will denote them by \( W_{SR}, W_{SD}, \) and \( W_{RD} \), respectively.

We now restate the bounds on the capacity of the relay channel that were introduced in Section 2.2.1, adapted to polar codes.

**Definition 4.1** (Cut-set upper bound).

\[
C \leq \max_{P_X, P_{X_R}} \min \{ I(X; Y_{SD}Y_{SR}), I(X; Y_{SD}) + I(X_R; Y_{RD}) \}. \tag{4.2}
\]

**Definition 4.2** (Symmetric DF rate for relay channels with orthogonal receivers).

\[
R^{s}_{DF} = \min \{ I(X; Y_{SR}), I(X; Y_{SD}) + I(X_R; Y_{RD}) \} \tag{4.3}
\]

with \( X \) and \( X_R \) uniformly distributed over their respective alphabets.

Whenever the source-relay, source-destination, and relay-destination channels are symmetric, the symmetric decode-and-forward rate \( R^{s}_{DF} \) coincides with the decode-and-forward rate \( R_{DF} \) in Definition 2.42. Recall also that decode-and-forward relaying achieves the capacity of the physically degraded relay channel [CG79].
4.1 Preliminaries

Definition 4.3 (Symmetric CF rate for relay channels with orthogonal receivers).

\[ R_{CF}^s = \sup I(X; Y_Q Y_{SD}). \]  \hspace{1cm} (4.4)

The supremum is over all conditional distributions \( P_{Y_Q|Y_{SR}} \) that induce a uniform distribution on \( Y_Q \), where \( |Y_Q| \) is a prime number, and such that \( I(X_R; Y_{RD}) \geq I(Y_Q; Y_{SR}|Y_{SD}). \) \( X \) and \( X_R \) are uniformly distributed over their respective alphabets.

When comparing the inner bounds to the original formulations in Definitions 2.42 and 2.43, we observe the following additional constraints:

- \( X \) and \( X_R \) must follow uniform distributions.
- The admissible conditional probabilities \( P_{Y_Q|Y_{SR}} \) in the characterization of the CF rate must induce a uniform distribution on \( Y_Q \).

They are natural consequences of the special properties of PCs discussed in previous chapters. The methods that can be used to extend the results beyond these constraints were discussed in Section 3.3. We do not consider them explicitly in this chapter because they bring no insight into the problem discussed here.

Although this chapter is mainly concerned with compress-and-forward relaying, in the numerical evaluation section we will consider both decode-and-forward and compress-and-forward relaying using polar codes.

Theorem 4.4 (Symmetric decode-and-forward relaying with polar codes, [ART+10, BTA+12]). Consider a stochastically degraded relay channel with orthogonal receiver components. For any transmission rate \( R < R_{DF}^s \), there exists a sequence of \((n, 2^{nR})\)-polar codes (with \( n = 2^p \), \( p \in \mathbb{N} \)) with block error probability

\[ \Pr(\hat{M} \neq M) \leq O(2^{-n\beta}) \]  \hspace{1cm} (4.5)

for any \( 0 < \beta < \frac{1}{2} \). The complexity of the source, relay, and destination operations is \( O(n \log n) \) for each.

The basic idea behind decode-and-forward relaying is the following. Transmission is divided into blocks. In all but the last one, the source transmits a new message. At the end of each block, the relay decodes the message transmitted by the source and then, forwards some information about it in the next block. The destination decodes each of the messages using the contributions from source and relay in two consecutive blocks. In this way, a total of \( b \) messages are transmitted over \( b + 1 \) blocks.
4.2 Main Results

The main result of this chapter is the following theorem:

**Theorem 4.5** (Symmetric compress-and-forward relaying using polar codes). Consider a relay channel with orthogonal receiver components. For any transmission rate \( R < R_{\text{CF}}^s \), there exists a sequence of \((n,2^nR)\)-polar codes (with \( n = 2^p \), \( p \in \mathbb{N} \)) with block error probability

\[
\Pr(\hat{M} \neq M) \leq O(2^{-n^\beta})
\]

for any \( 0 < \beta < \frac{1}{2} \). The complexity of the source, relay, and destination operations is \( O(n \log n) \) for each.

**Proof.** The proof is provided in Appendix 4.A. \( \blacksquare \)

As in Chapter 3 in the proof of the theorem, we assume that all the nodes have access to a common source of randomness used to generate the frozen symbols. It follows, then, that there must exist a fixed choice of these symbols with the performance described in Theorem 4.5.

**Compress-and-Forward Relaying Based on Slepian-Wolf Coding**

We obtain the following result by specializing Theorem 4.5 to the case where the relay does not perform any lossy source compression (i.e., \( Y_Q = Y_{SR} \)).

**Corollary 4.6** (CF relaying with PCs based on Slepian-Wolf coding). Consider a relay channel with orthogonal receiver components. Let \( R_{\text{CF-SW}}^s \equiv I(X;Y_{SD}Y_{SR}) \) for a uniform distribution on \( X \). For any transmission rate \( R < R_{\text{CF-SW}}^s \), there exists a sequence of \((n,2^nR)\)-polar codes (with \( n = 2^p \), \( p \in \mathbb{N} \)) with block error probability \( \Pr(\hat{M} \neq M) \leq O(2^{-n^\beta}) \) for any \( 0 < \beta < \frac{1}{2} \), as long as \( H(Y_{SR}|Y_{SD}) \leq I(X_R;Y_{RD}) \) for a uniform distribution on \( X_R \). The complexity of the source, relay, and destination operations is \( O(n \log n) \) for each. \( \blacksquare \)

Note that the absence of a polar code for compressing \( Y_{SR} \) into \( Y_Q \) simplifies the construction. Moreover, since \( Y_Q = Y_{SR} \), the constraint on the rate \( R_{RD} \) reduces to the Slepian-Wolf result \[SW73\]:

\[
R_{RD} > I(Y_Q;Y_{SR}) - I(Y_Q;Y_{SD})
\]

\[
= H(Y_Q) - H(Y_Q|Y_{SR}) - H(Y_Q) + H(Y_Q|Y_{SD})
\]

\[
= H(Y_{SR}|Y_{SD}).
\]

\(^2\)In this case, the restrictions on \( Y_Q \) described in Section 4.1 must be satisfied by \( Y_{SR} \).
In some special circumstances, the rate $R^*_\text{CF-SW}$ coincides with the cut-set bound. Hence, in these cases, the strategy achieves the capacity of the relay channel.

**Corollary 4.7.** If all the channels are symmetric and $H(Y_{SR}|Y_{SD}) \leq I(X_R;Y_{RD})$ for a uniform distribution on $X_R$, then compress-and-forward relaying based on Slepian-Wolf coding achieves the cut-set bound, which is given by the term $I(X;Y_{SD}Y_{SR})$.

**Proof.** In this case the two terms in the cut-set bound satisfy:

$$I(X;Y_{SD}Y_{SR}) \leq I(X;Y_{SD}) + H(Y_{SR}|Y_{SD}) \leq I(X;Y_{SD}) + I(X_R;Y_{RD}).$$

(4.10)

Therefore, the rate $R^*_\text{CF-SW} = I(X;Y_{SD}Y_{SR})$ coincides with the cut-set bound. \qed

### 4.3 Numerical Evaluation

In this section, we numerically evaluate the performance of polar codes for decode-and-forward and compress-and-forward relaying. As discussed in Section 2.3, designing polar codes essentially amounts to selecting the frozen set. That is, it is necessary to know the Bhattacharyya parameters of each of the synthetic channels $W_n^{(i)}$, $i \in \{1, \ldots, n\}$. Unfortunately, there is no simple method to calculate Bhattacharyya parameters. However, using density evolution methods [RU08], it is simple to obtain the error probability of each synthetic channel $W_n^{(i)}$. We can use the error probabilities to specify the frozen sets (i.e., by replacing $Z(W_n^{(i)})$ by the probability of error of $W_n^{(i)}$ in the definitions). This type of construction was introduced in [MT09] and can be shown to perform no worse than Arıkan’s original codes. In addition, we note that, as opposed to the method used in the proofs, we choose the frozen set for given rate and block length by selecting the fraction (equal to the rate) of synthetic channels with smallest error probability.

**Decode-and-forward relaying.** We first consider PCs for DF relaying in a physically degraded relay channel (cf. Definition 2.46) with the following characteristics: $W_{SR}(y_{sr}|x)$ and $P_{Y_{SD}|Y_{SR}}(y_{sd}|y_{sr})$ are independent binary symmetric channels (BSC) with crossover probabilities of 0.05 and 0.15, respectively. We study two different realizations of the relay-destination channel: an error-free channel, and an independent BSC with crossover probability 0.1. We will only consider cases where the capacity of the relay channel equals that of the source-relay channel, that is, $C = I(W_{SR}) \approx 0.71$. The relay-destination BSC has $I(W_{RD}) \approx 0.53$. Direct transmission without cooperation is limited by $I(W_{SD}) \approx 0.31$. We show in Figure 4.2 the bit error rate ($\Pr(\hat{M} \neq M)$) for the constructions of polar codes for DF relaying from [ART+10, BTA+12] for different values of
the transmission rates used by the source ($R$, coordinate axis) and the relay ($R_{RD}$, line face), and block lengths ($n = 2^p$, line marker) for the case of an error-free relay-destination channel (with capacity equal to $R_{RD}$). As one would expect, it is possible to lower the BER by increasing $p$ (and hence the complexity and delay) and also by reducing $R$ (i.e., the efficiency). Moreover, increasing $R_{RD}$ also yields a lower BER. This happens because we are reducing the amount of information that has to be decoded from the direct link observation at no cost since the relay-destination channel is error-free.

We show in Figure 4.3 the BER when the relay-destination channel is a BSC. In this case, we observe that increasing $R_{RD}$ does not always improve the performance. The reason for this is that now transmission from relay to destination takes place over a channel that introduces errors. At some point, the effect of these errors becomes the bottleneck of the system since the destination cannot recover the additional information conveyed by the relay that is necessary to decode the direct link observation.
4.3 Numerical Evaluation

Figure 4.3: Performance of DF relaying with PCs: binary symmetric relay-destination channel. (© 2012 IEEE. Reused with permission.)

**Compress-and-forward relaying.** We now consider CF relaying. We model the source-relay and source-destination channels as independent BSCs with crossover probabilities of 0.1 and 0.05, respectively. The relay compresses its observation at a rate of 0.8 bits per sample (i.e., $\frac{|F_Q|}{n} = 0.8$ in (4.22)) using a PC designed using a BSC as the test channel. As before, we will first model the relay-destination channel as an error-free link with limited capacity equal to $R_{RD}$ and then as an independent BSC, in this case with capacity equal to 0.85. Numerical evaluation of the limits in Theorem 4.5 results in $I(X; Y_Q Y_{SD}) \approx 0.81$ and $I(Y_Q; Y_{SR} | Y_{SD}) \approx 0.44$. Without cooperation the scenario is limited by $I(W_{SD}) \approx 0.71$.

We show in Figure 4.4 the BER of our construction for CF relaying for the case of an error-free relay-destination channel. Again, performance improves with larger block lengths $n = 2^p$ and $R_{RD}$, or with lower $R$. However, in this case we observe a saturation effect if only one of the rates is changed. For example, for $R < 0.7$ the BER curves flatten out. This happens because, in this region, the error probability is dominated by the errors of the channel code used to minimize the communication rate over $W_{RD}$ (i.e., the first term in (4.44)). A similar effect is observed if only
When $R_{RD}$ is increased. For example, for $R_{RD} > 0.65$ the reduction in BER is negligible. In this case the method used to minimize the rate over $W_{RD}$ becomes nearly error free and the probability of error is dominated by the weakness of the PC used by the source (i.e., the second term in (4.44)).

Finally, in Figure 4.5 we show the BER when the relay-destination channel is a BSC. As for DF relaying, we observe that increasing $R_{RD}$ may degrade the performance (in this case for $R_{RD} > 0.65$). The reason is similar: a larger value of $R_{RD}$ brings the PC used for transmission from relay to destination close to the capacity of the channel. At some point, the destination cannot decode the relay contribution without errors. Using the wrong estimate $\hat{Y}_Q$ implies error propagation when decoding the message transmitted by the source. This means a larger BER.

As a final remark, we note that for moderate block lengths, the performance of our constructions is quite far away from the asymptotical limits (i.e., $n \rightarrow \infty$). This is common to all designs based on Arıkan’s construction of PCs. In more recent works, the finite block length performance of polar codes has been improved.
4.4 Summary and Concluding Remarks

In this chapter, we have shown that polar codes are suitable for compress-and-forward relaying in channels with orthogonal receivers. Our construction combines elements for source coding and for channel coding. The former adapt the quality of the description provided by the relay. The latter allow for a reduction of the rate required for communication over the relay-destination channel by exploiting the side information available at the destination in the form of the direct-link observation.

We have also numerically evaluated the performance of polar codes for decode-and-forward and compress-and-forward relaying for finite block lengths. Our results show that these constructions are not directly applicable to moderate block lengths but they provide useful insights into the implementation of the two relaying protocols with structured codes. In addition, they allow for the identification of the different performance bottlenecks.
4.A Proof of Theorem 4.5

We assume that all the nodes share a source of common randomness that allows them to generate frozen symbols.

Proof. Fix $0 < \beta < \frac{1}{2}$ and let $\delta_n \triangleq \frac{1}{n} 2^{-n^\beta}$. Fix a valid $P_{YQ|Y_{SR}}$ in (4.4) and let

$$W_Q(y_{sr}|y_q) \triangleq P_{Y_Q|Y_{SR}}(y_q|y_{sr})P_{Y_{SR}}(y_{sr})|Y_Q|$$

where

$$P_{Y_{SR}}(y_{sr}) = \sum_{y_{sd},x} P_{Y_{SD},Y_{SR}|X}(y_{sd},y_{sr}|x) \frac{1}{|\mathcal{X}|}.$$  

Observe that we are implicitly assuming that $X$ and $Y_Q$ are uniformly distributed. Define also

$$W(y_{sd},y_q|x) = \sum_{y_{sr}} P_{Y_Q|Y_{SR}}(y_q|y_{sr})P_{Y_{SD},Y_{SR}|X}(y_{sd},y_{sr}|x)$$

and let $R_S = I(X;Y_QY_{SD}) - \epsilon$ for arbitrary $\epsilon > 0$.

A total of $b$ messages of equal size are transmitted over $b + 1$ blocks, each consisting of $n$ channel uses. In the first block, the source transmits the first message whereas the relay transmits no information. In the second block, the source transmits the second message and the relay forwards some information about its observation from the previous block. After the second block, the destination decodes the first message transmitted by the source using the two contributions. This process is repeated for $b$ blocks: the source transmits a new message in block $i$, the relay compresses its observation and forwards the information in block $i + 1$. At the end of the block $i + 1$, the destination decodes the message transmitted by the source. During block $b + 1$, the source does not transmit any information and the relay forwards the last compressed observation. Thus, the transmission rate is

$$\frac{b}{b + 1} R_S \to R_S$$

as $b \to \infty$.

In the following, we concentrate on an arbitrary pair of consecutive blocks and describe the process at source (first block), and relay and destination (second block).

**Encoding at the source node.** The source node encodes the information and frozen symbols $M$ using a PC that is capacity achieving for the channel $W(y_{sd},y_q|x)$. Consider the following definition of the frozen sets for a sequence of polar codes for channel coding:

$$\mathcal{F}_S \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W^{(i)}) \geq \delta_n \right\}.$$  

(4.16)
By Theorem 2.67 we know that

$$\frac{|F_S|}{n} \geq R_S$$

(4.17)

for sufficiently large $n$.

Assume that the information symbols are chosen i.i.d. randomly from a uniform distribution. Similarly, let the frozen symbols be chosen i.i.d. from a uniform distribution (using the source of common randomness, so that they are also available at the destination node). Then the observation at the relay $Y_{SR}$ has the distribution:

$$P_{Y_{SR}}(y_{sr}) = \sum_x \frac{1}{q^n} \prod_{i=1}^n W_{SR}(y_{sr,i}|x_i)$$

(4.18)

$$= \prod_{i=1}^n \frac{1}{q} \sum_{x_i} W_{SR}(y_{sr,i}|x_i)$$

(4.19)

$$= \prod_{i=1}^n P_{Y_{SR}}(y_{sr,i}).$$

(4.20)

That is, it behaves like a discrete memoryless source.

**Processing at the relay.** The relay uses a PC with design based on $W_Q$ to compress $Y_{SR}$ into $Y_Q$. Consider the following definition of the frozen set for a sequence of polar codes:

$$F_Q = \left\{ i \in \{1, \ldots, n\} : I(W_Q^{(i)}) \leq \delta_n^2 \right\}.$$  

(4.21)

By Theorem 2.74 we know that

$$\frac{|F_Q|}{n} \leq I(W_Q) + \epsilon$$

(4.22)

for sufficiently large $n$. The frozen symbols $U_{F_Q}$ used for compression are generated using the source of common randomness so that they are also available at the destination. The compressed observation can be reconstructed as $Y_Q = UG_n$.

Now, we only need to make $Y_Q$ available at the destination. Transmitting the symbols $U_{F_Q}$ directly over the relay-destination channel requires a communication rate of roughly $I(W_Q)$ [bpcu], which is higher than allowed by Definition 4.3. Thus, we use the same mechanism as for the cascade and broadcast networks in Chapter 3 and exploit the correlation between $Y_Q$ and the direct-link observation at the destination $Y_{SD}$. The exact distribution of this correlation is determined by the channel distribution and the PC used for source compression. Rather than considering the true distribution, we simply model the correlation as if $Y_Q$ were
transmitted through the DMC

\[ W_V(y_{sd}|y_q) = \sum_{y_{sr}} \left( W_Q(y_{sr}|y_q) \frac{1}{P_{Y_{SR}}(y_{sr})} \sum_{x \in \mathcal{X}} \frac{1}{|\mathcal{X}|} P_{Y_{SD},Y_{SR}|X}(y_{sd},y_{sr}|x) \right), \]

(4.23)

and use the SC decoding algorithm to produce an estimate \( \hat{Y}_Q \). For this purpose, the SC decoding algorithm needs to know the values of the frozen symbols \( U_{F_V} \), where the sequence of frozen sets for channel coding is defined as

\[ F_V \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W_V^{(i)}) \geq \delta'_n \right\}. \]

(4.24)

Consider also the auxiliary sequence of sets\(^3\)

\[ F_Q \triangleq \left\{ i \in \{1, \ldots, n\} : Z(W_Q^{(i)}) \geq 1 - \delta'_n \right\}. \]

(4.25)

Note now that, \( W_V \) defines a Markov chain: \( Y_Q - Y_{SR} - Y_{SD} \). That is, \( W_V \) is stochastically degraded (in fact, physically) with respect to \( W_Q \). Thus, by virtue of Lemma 2.70, \( F_Q \subseteq F_V \) for sufficiently large \( n \), so that \( \delta'_n < 1 - \delta'_n \). If we choose \( \delta'_n = \frac{q^{-q^1-q^2}}{q-1} \), then Lemma 2.69 ensures that \( \delta'_n \leq \delta'_n \leq O(2^{-n^\beta}) \) and \( F_Q \subseteq F_Q' \). The concatenation of the two set relations yields \( F_Q \subseteq F_Q' \). Since the values \( U_{F_Q} \) are already known at the destination, only the symbols in \( F_V \cap F_Q^c \) need to be transmitted (see Figure 4.6). This requires a communication rate equal to \( \frac{|F_V| - |F_Q|}{n} [bpcu] \). By (4.22) and Theorem 2.67, we have that

\[ \frac{|F_V| - |F_Q|}{n} = \frac{|F_Q^c| - |F_V^c|}{n} \leq I(W_Q) + \epsilon - I(W_V) + \epsilon. \]

(4.26)

(4.27)

To show that this is equal to \( I(Y_Q;Y_{SR}|Y_{SD}) \), we consider the Markov chain \( Y_Q - Y_{SR} - Y_{SD} \):

\[ I(Y_Q;Y_{SR}) = I(Y_Q;Y_{SD}Y_{SR}) \]

(4.28)

\[ = I(Y_Q;Y_{SD}) + I(Y_Q;Y_{SR}|Y_{SD}). \]

(4.29)

This holds for any distribution on \( Y_Q \) as long as the Markov chain relationship is satisfied. In particular, when \( Y_Q \) is uniformly distributed we have that \( I(Y_Q;Y_{SR}|Y_{SD}) = I(W_Q) - I(W_V) \) (using the same units on both sides).

Finally, note that if \( |Y_Q| \neq |X_R| \), then it is necessary to change the representation of the symbols from \( |Y_Q|\)-ary to \( |X_R|\)-ary. To transmit the information

\(^3\)\( F_Q' \) is an auxiliary set that is only used to show that \( F_Q \subseteq F_V \). No polar codes are explicitly built using this set.
Figure 4.6: Nested structure of PCs for compress-and-forward relaying. The symbols $u_{F_Q}$ are fixed when performing source compression with PCs at the relay and therefore known by both the relay and the destination. The destination needs to obtain the remaining symbols $u_{F_Q \setminus F_Q^c}$. If the relay communicates only the symbols $u_{F_V \setminus F_Q^c}$ the destination can use its own channel observation to recover the remaining symbols $u_{F_V}$ using the SC algorithm. (© 2012 IEEE. Reused with permission.)

over $W_{RD}$, the relay uses a polar code for channel coding. Consider the following sequence of frozen sets for channel coding

$$ F_{RD} \triangleq \{ i \in \{1, \ldots, n\} : Z(W_{RD}^{(i)}) \geq \delta_n \}. $$

(4.30)

By Theorem 2.67 we know that

$$ \frac{|F_{RD}^c|}{n} \geq I(W_{RD}) - \epsilon $$

(4.31)

for sufficiently large $n$. Putting together the two requirements we obtain

$$ I(Y_Q; Y_{SR}|Y_{SD}) + 2\epsilon \leq I(W_{RD}) - \epsilon $$

(4.32)

(using the same units on both sides).

**Decoding at the destination.** The destination performs three decoding steps. First it decodes the message transmitted by the relay, that is, it obtains $U_{F_V \setminus F_Q^c}$. These symbols together with $U_{F_Q}$ are all the symbols in the frozen set $F_V$. Knowing them allows the destination node to decode the compressed vector $Y_Q = UG_n$ from $Y_{SD}$ using the SC algorithm. Finally, the destination decodes the message $M$ from the estimate $\hat{Y}_Q$ and $Y_{SD}$ using the SC algorithm.

**Analysis of the error probability.** We want to evaluate the probability of the event $\mathcal{E} = \{ \hat{M} \neq M \}$ over the distribution induced by the channel and the different codes in our scheme. To simplify the notation, we denote this distribution as $Q_S(s)$ where $S$ is the set of all the random vectors present in the scenario. To emphasize the dependency of $Q_S(s)$ on the distribution $Q_{Y_{SR}, Y_{Q}}$ induced by the PC used for
source coding at the relay, we define \( S_s = S \setminus \{ Y_{SR}, Y_Q \} \) (i.e., the subset of \( S \) that excludes \( Y_{SR}, Y_Q \)). Using this, we write

\[
Q_S(s) = Q_{Y_{SR}, Y_Q}(y_{sr}, y_q)s \mid Y_{SR}, Y_Q(s_\ell | y_{sr}, y_q).
\]

(4.33)

Consider now the optimal coupling (cf. Lemma 2.4) \( C_{PQ}(y_{sr}, y_q; \tilde{y}_{sr}, \tilde{y}_q) \) between \( Q_{Y_{SR}, Y_Q} \) and the distribution

\[
P_{Y_{SR}, Y_Q}(y_{sr}, y_q) = \prod_{i=1}^{n} P_{Y_Q|Y_{SR}}(y_{q,i}|y_{sr,i})P_{Y_{SR}}(y_{sr,i}).
\]

(4.34)

By the properties of the optimal coupling, the probability of the event

\[
\mathcal{E}_E \triangleq \{(Y_{SR}, Y_Q) \neq (\tilde{Y}_{SR}, \tilde{Y}_Q)\},
\]

(4.35)

evaluated over \( C_{PQ} \), satisfies

\[
\Pr(\mathcal{E}_E) = \| P_{Y_{SR}, Y_Q} - Q_{Y_{SR}, Y_Q} \|_{TV}.
\]

(4.36)

Moreover, for our choice of \( F_Q \), we know from Theorem 2.73 that this probability, evaluated over \( C_{PQ} \), is bounded as

\[
\Pr(\mathcal{E}_E) \leq O(2^{-n^\beta}).
\]

(4.37)

To evaluate \( \Pr(\mathcal{E}) \), we replace the distribution \( Q_{Y_{SR}, Y_Q} \) in (4.33) by the optimal coupling \( C_{PQ} \). Observe that this change does not alter the value of \( \Pr(\mathcal{E}) \) because only the marginal, which is equal to \( Q_{Y_{SR}, Y_Q} \), is relevant here. Consider the events

\[
\mathcal{E}_{Y_Q} \triangleq \{ \tilde{Y}_Q \neq Y_Q \}
\]

(4.38)

and \( \mathcal{E}_{RD} \), which denotes an erroneous relay-destination transmission. Using this, we write

\[
\Pr(\mathcal{E}) = \Pr(\mathcal{E} | \mathcal{E}_{RD}) \Pr(\mathcal{E}_{RD}) + \Pr(\mathcal{E}, \mathcal{E}_{RD}) \leq \Pr(\mathcal{E}_{RD}) + \Pr(\mathcal{E}, \mathcal{E}_{RD}).
\]

(4.39)

(4.40)

By Theorem 2.67 we know that \( \Pr(\mathcal{E}_{RD}) \leq O(2^{-n^\beta}) \) given our choice of \( F_{RD} \). We rewrite the second term in (4.40) as

\[
\Pr(\mathcal{E}, \mathcal{E}_{RD}) = \Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_E) + \Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_{E}^c)
\]

(4.41)

\[
\leq \Pr(\mathcal{E}_E) + \Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_E^c).
\]

(4.42)

The first term was already bounded in (4.37). We bound the second term in (4.42) as

\[
\Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_E^c) = \Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_{E}^c, \mathcal{E}_{Y_Q}^c) + \Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_{E}^c, \mathcal{E}_{Y_Q}^c)
\]

(4.43)

\[
\leq \Pr(\mathcal{E}_{RD}, \mathcal{E}_{E}^c, \mathcal{E}_{Y_Q}^c) + \Pr(\mathcal{E}, \mathcal{E}_{RD}, \mathcal{E}_{E}^c, \mathcal{E}_{Y_Q}^c).
\]

(4.44)
The first term in (4.44) is easily upper bounded by the probability that decoding of $Y_Q$ fails under the design conditions. That is, when the decoder has access to the frozen symbols (i.e., $E_{RD}$) and when the relation between $Y_Q$ and $Y_{SD}$ is indeed (4.23). This corresponds to the conditions of design of a polar code used for channel coding. Thus, using similar steps as in (3.50)-(3.52) and Theorem 2.67, $\Pr(E_Q, E_{cRD}, E_{cE}) \leq O(2^{-n^\beta})$. The second term in (4.44) is bounded similarly. In this case, we are analyzing the probability that decoding of $M$ under the design conditions fails. Again, using Theorem 2.67, $\Pr(E, E_{cRD}, E_{cE}, E_{cY_Q}) \leq O(2^{-n^\beta})$.

Collecting the different terms we obtain the desired bound on the error probability. The claim on the complexity follows from the fact that we have only used the encoding and decoding algorithms for polar codes.
Chapter 5

Coordination for Interference Control

In this chapter, we consider the problem of coordinating communications to control the interference created to an external observer, measured in terms of its type (i.e., empirical distribution). As we will see, communicating and controlling the interference are conflicting goals, and there is a fundamental tradeoff between them. We will study this tradeoff for two different cases.

First, we consider the scenario in which there is only one transmitter-receiver pair whose communication affects a third party, the observer. In this case, the tradeoff involves only the communication rate and the type of the interference. Our main result for this setting is a complete characterization of the set of achievable communication rate-interference type pairs. Then, we look at the scenario in which two user pairs transmit over independent channels but create a joint interference at the observer. The two transmitters are allowed to coordinate their actions by means of a unidirectional communication phase that takes place prior to transmission. This yields a tradeoff between communication, coordination, and interference. Our main result for this case is an achievable communication-interference region. The model is quite general, in the sense that we do not make any assumptions on the way the two transmitters interfere with the observer. As usual, all the proofs are provided in the appendices.

5.1 Preliminaries

Interference is a very complex phenomenon inherent to wireless communication that yields performance tradeoffs between the different elements in the scenario. From an information-theoretic point of view, our understanding of interference and how to deal with it is only partial. For example, the capacity region of the interference channel, which is the canonical model for the study of interference, is still unknown. Part of the complexity of the issue lies on the fact that, contrary to our basic intuition, interference is not always detrimental [Car75]. In fact, very
strong interference is easier to handle than milder interference. Similarly, in large networks, the interference created by channels with some special structures can be exploited to achieve higher communication rates [NG11].

In recent work in [BG11], Bandemer and El Gamal treated the problem of communication with disturbance constraints. Their approach is to measure the interference in terms of the average (undesired) mutual information. That is, they endow disturbance with an informational meaning. In contrast, we do not give interference any such interpretation as we simply measure it in terms its type. Thus, our approach is more related to the study of the empirical distribution [SV97] and the output statistics [HV93], although our metric of interest will be total variation rather than divergence.

5.1.1 Notation

In this chapter, we will explicitly state the length of all (column) vectors using the notation $X^n$ (instead of the usual $X$). The reason for this is that most of our statements concern sequences of vectors indexed by the length $n$.

Convergence of Sequences

Most of our statements will involve the convergence of sequences of distributions. This convergence will be expressed in different ways depending on the nature of the sequences (i.e., random or deterministic) but, ultimately, they all refer to the same notion of convergence based on total variation. In the following, we review these different statements of convergence.

Consider a sequence of random vectors $\{X^n\}$ with $X^n \sim P_{X^n}$ for some sequence of distributions $\{P_{X^n}\}$, and the corresponding sequence of types $\{T_{X^n}\}$. Note that each $T_{X^n}$ is itself random. Consider a deterministic distribution $G_1$. We say that the sequence $\{T_{X^n}\}$ converges in probability in total variation to the distribution $G_1$ if

$$\lim_{n \to \infty} \Pr(\|T_{X^n} - G_1\|_{TV} \geq \epsilon) = 0$$

(5.1)

for all $\epsilon > 0$. As in previous chapters, we use the shorthand notation

$$\|T_{X^n} - G_1\|_{TV} \to 0 \text{ in probability}$$

(5.2)

to denote this type of convergence.

Consider now a sequence of deterministic distributions $\{G_2^{(n)}\}$. We say that the sequence $\{T_{X^n}\}$ converges in probability in total variation to the sequence $\{G_2^{(n)}\}$ if

$$\lim_{n \to \infty} \Pr(\|T_{X^n} - G_2^{(n)}\|_{TV} \geq \epsilon) = 0$$

(5.3)
for all $\epsilon > 0$. We use the shorthand notation

$$\|T_{X^n} - G_{2}^{(n)}\|_{TV} \to 0 \text{ in probability}$$

(5.4)

to denote this type of convergence.

Finally, consider two sequences of deterministic distributions $\{G_3^{(n)}\}$ and $\{G_4^{(n)}\}$. We say that $\{G_3^{(n)}\}$ converges in total variation to the sequence $\{G_4^{(n)}\}$ (or vice versa) if

$$\lim_{n \to \infty} \|G_3^{(n)} - G_4^{(n)}\|_{TV} = 0.$$ (5.5)

We use the shorthand notation

$$\|G_3^{(n)} - G_4^{(n)}\|_{TV} \to 0$$

(5.6)

to denote this type of convergence. Since all three types of convergence involve total variation, we will often omit the qualifier “in total variation”.

We will construct sequences of codes in a similar way as we did in the preceding chapters (e.g., Section 2.2.1). Consequently, we should talk about the sequences of distributions (or types) induced by the sequences of codes. However, as before, we will simply refer to them as induced distributions and use the notation $P_{X^n}$ instead of $\{P_{X^n}\}$.

5.2 Single User

Consider the scenario depicted in Figure 5.1. This corresponds to a channel with one input $X$ and two outputs $Y$ and $Z$. The former is the observation at the intended receiver, while the latter corresponds to an undesired interference created to an external observer. The channel is governed by a pmf $P_{Y,Z|X}$. The transmitter-receiver pair can use the channel for communication as long as the interference $z^n$ has a certain shape, measured in terms of its type $T_{z^n}(z)$. For this purpose, they use a code.

**Definition 5.1 (Code).** An $(n, 2^{nR})$-code for the DMC $P_{Y,Z|X}$ consists of:

- a message set $\mathcal{M} \triangleq \{1, \ldots, [2^{nR}]\}$,
- an encoding function $x^n : \mathcal{M} \to X^n$,
- a decoding function $\hat{m} : Y^n \to \mathcal{M} \cup \{e\}$.

We assume that the message is uniformly distributed over the message set.
Figure 5.1: Scenario for single-user communication with interference constraint.

**Definition 5.2 (Achievability).** We say that the communication rate $R$ is achievable with interference type $G_Z$ if there exists a sequence of $(n, 2^{nR})$-codes such that

$$\lim_{n \to \infty} \Pr(\hat{M} \neq M) = 0,$$

$$\|T_{Z^n} - G_Z\|_{TV} \to 0 \text{ in probability}$$

under the distribution induced by the codes. $\Box$

The communication-interference capacity region $\mathcal{C}$ of the DMC $P_{Y,Z|X}$ is the closure of the set of all rate-interference type tuples $(R, G_Z)$ that are achievable.

### 5.2.1 Main Results

Our main result for the channel model in Figure 5.1 is a complete characterization of the communication-interference capacity region (Theorem 5.5). We start by introducing two important properties.

**Lemma 5.3.** The communication-interference capacity region $\mathcal{C}$ is convex. $\Box$

*Proof.* The proof is provided in Appendix 5.A.1 $\blacksquare$

As for the case of many other multi-terminal problems (e.g., broadcast channel [Cov72]), the communication-interference capacity region only depends on the marginals of $P_{Y,Z|X}$.

**Theorem 5.4.** The communication-interference capacity region $\mathcal{C}$ of the DMC $P_{Y,Z|X}$ only depends on the marginals

$$P_{Y|X} = \sum_z P_{Y,Z|X},$$

$$P_{Z|X} = \sum_y P_{Y,Z|X}.$$
Proof. The proof is provided in Appendix 5.A.2.

This result is instrumental in establishing the following characterization of $C$:

**Theorem 5.5.** The communication-interference capacity region $C$ of the DMC $P_{Y,Z|X}$ is the set of rate-interference type tuples $(R,G_Z)$ such that

$$R \leq \max_{P_X \in \mathcal{P}} I(X;Y)$$

where

$$\mathcal{P} \triangleq \left\{ P_X : \sum_x P_X P_{Z|X} = G_Z \right\}.$$ 

Proof. The proof is divided into two parts, achievability and converse, which are provided in Appendices 5.A.3 and 5.A.4 respectively.

Observe that the result agrees with our basic understanding of communication and coordination (see Section 2.2). In particular, the capacity expression is reminiscent of that for the point-to-point channel (cf. Theorem 2.36) but the maximization is over the restricted set of input distributions $P_X$ that induce the desired interference type $G_Z$.

The main tool to prove the converse result in Theorem 5.5 is Lemma 5.20 (in Appendix 5.A.4), which states that the expectation of the type of the codewords $E\{T_{X^n}\}$ must converge to some sequence $\{P_X^{(n)}\}$ with $P_X^{(n)} \in \mathcal{P}$, that is,

$$\|E\{T_{X^n}\} - P_X^{(n)}\|_{TV} \to 0.$$ 

At first sight, one might be tempted to conjecture that the type itself $T_{X^n}$ converges to some sequence $\{P_X^{(n)}\}$ with $P_X^{(n)} \in \mathcal{P}$, that is,

$$\|T_{X^n} - P_X^{(n)}\|_{TV} \to 0 \text{ in probability.}$$

However, this is not true in general. In fact, the type need not converge at all. We illustrate this with the following example.

**Example 5.6.** Consider the channel $P_{Z|X}$ in Figure 5.2 and let the desired interference type be $G_Z(z_1) = G_Z(z_2) = \frac{1}{2}$. First, consider a codebook that only uses input symbols from $\{x_1, x_3\}$ and achieves the point $(1, G_Z)$ in the communication-interference capacity region. The type of the codewords $T_{X^n}$ of a sequence of codebooks constructed in this manner clearly converges (in the sense of (5.13)) to a distribution that places half mass on the symbols $x_1$ and $x_3$. 


Figure 5.2: Example of a channel \( P_{Z|X} \) for which convergence in probability of the output type \( T_{Z^n} \) does not imply converge in probability of the input type \( T_{X^n} \).

Now, construct a new codebook by keeping half of the codewords from the original codebook and by applying the substitution \( x_1 \mapsto x_2 \) and \( x_3 \mapsto x_4 \) to the other half. This code also achieves the point \((1, G_Z)\) in the communication-interference capacity region. However, the type of the codewords \( T_{X^n} \) of a sequence of codebooks constructed in this manner cannot converge (in the sense of (5.14)) to any distribution: for half of its codewords (i.e., half of the probability) the type will distribute the probability mass between \( x_1 \) and \( x_3 \). For the other half, between \( x_2 \) and \( x_4 \).

The stronger convergence in (5.14) is easily shown to hold if the distribution \( G_Z \) is such that \( \mathcal{P} = \{P_X\} \) for a single distribution \( P_X \) (see Corollary 5.21 in Appendix 5.A.4).

5.3 Multiple Users

Consider the scenario depicted in Figure 5.3. In this case, two transmitters want to communicate with their respective receivers. Although they do not hamper each other’s transmission, they create interference at a third external node, the observer. The channel is governed by a product pmf

\[
P_{Y_1,Y_2,Z|X_1,X_2} = P_{Y_1|X_1} P_{Y_2|X_2} P_{Z|X_1,X_2}.
\]

The marginals \( P_{Y_1|X_1} \) and \( P_{Y_2|X_2} \) model orthogonal communication channels between pairs of encoders and decoders, whereas \( P_{Z|X_1,X_2} \) models the joint disturbance that the two transmissions create to the observer. To control this interference, the two transmitters have access to a unidirectional rate-limited noiseless link from the first to the second encoder. They can use this resource to coordinate their transmissions and shape the type of the interference \( T_{Z^n}(z) \).

Observe that the factorization in (5.15) entails a loss of generality, as opposed to the single-user scenario that we considered in Section 5.2 (cf. Theorem 5.4). There are two good reasons for restricting the attention to orthogonal communication channels (i.e., \( P_{Y_1|X_1} P_{Y_2|X_2} \)): i) there is currently no complete single-letter
characterization of channels of the type $P_{Y_1,Y_2|X_1,X_2}$ (i.e., general interference channels); and ii) if the channels are not orthogonal, the presence of the coordination link allows for transferring information about the message $M_1$ from Encoder 2 to Decoder 1. This is certainly interesting but outside the scope of this work, which studies coordination for controlling the interference.\footnote{Using the same arguments as in the proof of Theorem 5.4, it is possible to prove that, in the general case, the communication-interference capacity region only depends on the marginals $P_{Z|X_1,X_2}$ and $P_{Y_1,Y_2|X_1,X_2}$.}

The scenario in Figure 5.3 can be seen, for example, as a simple model for a cognitive radio environment in which two secondary users can make use of the communication resources licensed to the primary users as long as their interference is within some limits. Before proceeding with the formal analysis of the scenario, we present a couple of examples that illustrate the generality of the model. In particular, they emphasize that this framework for studying the interference does not rely on a specific structure of the channel $P_{Z|X_1,X_2}$ (i.e., the way the two transmitters interfere with the observer) or on highly synchronized transceivers.

**Example 5.7** (Carrier/phase-level synchronization). Let $X_1 = X_2 = \{\pm 1\}$ and $Z = \{0, \pm 2\}$. The conditional distribution

$$P_{Z|X_1,X_2}(z|x_1,x_2) = \mathbf{1}\{z = x_1 + x_2\}$$

models a noiseless scenario in which the interference is created by the coherent combination of the signals sent by the secondary users.

\[\diamond\]
Example 5.8 (Symbol-level synchronization). Let $\mathcal{X}_1 = \mathcal{X}_2 = \{\pm 1, \pm 3\}$ and $\mathcal{Z} = \{0, \pm 2, \pm 4, \pm 6\}$. The conditional distribution

$$
P_{Z|X_1, X_2}(z|x_1 = \pm 3, x_2 = \pm 3) = \frac{1}{7} \text{ for } z \in \mathcal{Z},$$

(5.17)

$$
P_{Z|X_1, X_2}(z|x_1 = \pm 3, x_2 = \pm 1) = \begin{cases} 
1/5 \text{ for } z \in \{0, \pm 2, \pm 4\}, \\
0 \text{ for } z \in \{\pm 6\},
\end{cases}$$

(5.18)

$$
P_{Z|X_1, X_2}(z|x_1 = \pm 1, x_2 = \pm 3) = \begin{cases} 
1/5 \text{ for } z \in \{0, \pm 2, \pm 4\}, \\
0 \text{ for } z \in \{\pm 6\},
\end{cases}$$

(5.19)

$$
P_{Z|X_1, X_2}(z|x_1 = \pm 1, x_2 = \pm 1) = \begin{cases} 
1/3 \text{ for } z \in \{0, \pm 2\}, \\
0 \text{ for } z \in \{\pm 4, \pm 6\},
\end{cases}$$

(5.20)

models a scenario in which the signals of the secondary users combine at symbol level but without carrier/phase coherence. This illustrates that, in the worst-case, the interference created by two symbols with small amplitudes (e.g., $\pm 1$) is not as harmful as the worst-case interference created by two symbols with large amplitudes (e.g., $\pm 3$).

We now introduce the necessary definitions and state our main results for this scenario.

Definition 5.9 (Code). An $(n, 2^nR_1, 2^nR_2, 2^nR_c)$-code for the scenario in Figure 5.3 consists of:

- three sets of messages
  \[ \mathcal{M}_1 \triangleq \{1, \ldots, [2^nR_1]\}, \]
  (5.21)
  \[ \mathcal{M}_2 \triangleq \{1, \ldots, [2^nR_2]\}, \]
  (5.22)
  \[ \mathcal{M}_c \triangleq \{1, \ldots, [2^nR_c]\}, \]
  (5.23)

- two encoding functions
  \[ x^n_1 : \mathcal{M}_1 \to \mathcal{X}_1^n, \]
  (5.24)
  \[ x^n_2 : \mathcal{M}_2 \times \mathcal{M}_c \to \mathcal{X}_2^n, \]
  (5.25)

- a coordination function
  \[ c : \mathcal{M}_1 \to \mathcal{M}_c, \]
  (5.26)

- and two decoding functions
  \[ \hat{m}_1 : \mathcal{Y}_1^n \to \mathcal{M}_1 \cup \{e\}, \]
  (5.27)
  \[ \hat{m}_2 : \mathcal{Y}_2^n \to \mathcal{M}_2 \cup \{e\}. \]
  (5.28)
We assume that the message pair \((M_1, M_2)\) is uniformly distributed over the set \(M_1 \times M_2\).

**Definition 5.10 (Achievability).** We say that the tuple \((R_1, R_2, R_c, Q_Z)\) is achievable if there exists a sequence of \((n, 2^{nR_1}, 2^{nR_2}, 2^{nR_c})\)-codes such that

\[
\lim_{n \to \infty} \Pr((\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)) = 0, \quad (5.29)
\]

\[
\|T_{Z^n} - Q_Z\|_{TV} \to 0 \text{ in probability} \quad (5.30)
\]

under the distribution induced by the codes.

The communication-interference capacity region \(C\) is the closure of the set of all tuples \((R_1, R_2, R_c, Q_Z)\) that are achievable.

### 5.3.1 Main Results

As for the scenario in the previous section, the communication-interference capacity region \(C\) is convex.

**Lemma 5.11.** The communication-interference capacity region \(C\) is convex. ■

**Proof.** The proof is identical to that for Lemma 5.3 which is provided in Appendix 5.A.1 ■

Consider the following set:

\[
\mathcal{R} \triangleq \left\{ (R_1, R_2, R_c, Q_Z) \text{ s.t.} \begin{array}{l}
\exists P_U P_{X_1|U} P_{X_2|U} \text{ s.t.} \\
R_1 < I(X_1; Y_1), \\
R_2 < [I(X_2; Y_2) - I(U; X_2)]^+, \\
R_c > I(U; X_1), \\
\sum_{u, x_1, x_2} P_U P_{X_1|U} P_{X_2|U} P_Z|X_1=x_1, x_2 = Q_Z
\end{array} \right\} \quad (5.31)
\]

where \([x]^+ = \max(x, 0)\). Let \(\text{conv}(\mathcal{R})\) denote the convex hull of \(\mathcal{R}\). Our main result for the channel model in Figure 5.3 is the following partial characterization.

**Theorem 5.12.** The communication-interference capacity region \(C\) satisfies

\[
\text{conv}(\mathcal{R}) \subseteq C. \quad (5.32)
\]

**Proof.** The proof is provided in Appendix 5.B ■
Observe the following two facts about $\mathcal{R}$: i) the random variable $U$ plays the role of the coordination message sent from Encoder 1 to Encoder 2. By setting $U = \emptyset$, we obtain $R_c = 0$ and recover the case where the users are not coordinated (i.e., $X_1$ and $X_2$ are independent). As we will shortly see, for most distributions $P_{Z|X_1, X_2}$, our strategy strictly improves upon uncoordinated communication. ii) The coordination message $U$ couples the rates $R_1$ and $R_2$ in two ways. First, their choices of input distributions have to be compatible in the sense that they yield the desired $G_Z$. In addition, the rate for Encoder 2 has a penalty term that reflects that the transmitted signals are correlated. That is, $X_2$ carries information about $X_1$. This is similar to the situation in Gel’fand Pinsker coding, where the transmission is aligned with the channel state and thus carries information about it [GP80]. These considerations are illustrated by the following example.

**Example 5.13.** Consider the scenario in which each of the two encoders can make use of the set of 16 symbols depicted in Figure 5.4 as inputs to the channel. Assume that the observer tolerates only low and mild levels of interference. This means that the two encoders are not allowed to use the black-circle symbols simultaneously. For simplicity, assume that the channels $P_{Y_1|X_1}$ and $P_{Y_2|X_2}$ are noiseless.

Without coordination, one of the two users is restricted to use only the subset of red-diamond symbols. Assume that the restriction is placed on the second user. This yields the rate pair $(R_1, R_2) = (4, 2)$. In contrast, if Encoder 1 uses the coordination link to declare whether it will use a black-circle or a red-diamond symbol, Encoder 2 can opportunistically choose its constellation to boost its communication rate. For example, if Encoder 1 makes use of all 16 symbols with equal frequency, then Encoder 2 is forced to use the red-diamond symbols (i.e., transmit 2 [bpcu]) 75% of the times. However, in the remaining 25%, it can use any of the black-circle symbols (i.e., $\log_2 12$ [bpcu]). This yields

\[
R_2 = \frac{3}{4} 2 + \frac{1}{4} \log_2 12 \\
\approx 2.4 \text{ [bpcu].}
\]

Thus, we have $(R_1, R_2) = (4, 2.4)$. Observe that the constraint placed by the observer does not preclude Encoder 2 from using *any* of the symbols in Figure 5.4 when Encoder 1 sends a red-diamond symbol. However, Decoder 2 needs to know whether the transmitted symbol corresponds to 2 or 4 bits. In this way, Encoder 2 is conveying information about the message of Encoder 1, namely that the current input consists of one of the red-diamond symbols.

A coordination rate equal to $R_c = 0.81$ [bpa] is sufficient to implement this protocol if Encoder 1 uses a lossless source coding algorithm to declare its intentions for a batch of channel uses.

The preceding example only requires symbol-level synchronization. Consider now the following tightly synchronized situation.
Example 5.14. Let $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0, 1\}$ and
\[
P_{Z\mid X_1, X_2} = \mathbb{1}\{Z = X_1 \oplus X_2\}.
\] (5.35)

Assume that the target distribution is
\[
G_Z(Z = 0) = 1,
\] (5.36)
\[
G_Z(Z = 1) = 0.
\] (5.37)

This models an observer that does not tolerate any random interference. If $R_c = 0$ then neither of the transmitters can make use of the channel at a positive rate. In contrast, for $R_c > 0$, the encoders can coordinate their transmissions so that $X_1 \oplus X_2 = 0$. In this simple way, all tuples $(R_1, 0, R_c, G_Z)$ such that
\[
0 \leq R_1 \leq \min\{R_c, \max_{P_{X_1}} I(X_1; Y_1)\}
\] (5.38)
are achievable.

Similar observations can be made for other distributions $P_{Z\mid X_1, X_2}$, including those in Examples 5.7 and 5.8.

Discussion on Optimality

The result in Theorem 5.12 is unsatisfactory in two ways. First, it does not include an outer bound. We tried, without success, to establish a meaningful outer bound that could shed some light on the optimal coding strategy for the scenario in Figure 5.3. As for the single-user scenario, obtaining such a bound would involve two steps: i) using the operational characterization of a good sequence of
(n, 2^{nR_1}, 2^{nR_2}, 2^{nR_c})-codes (i.e, (5.29) and (5.30)) to derive some properties distribution induced by the sequence, and ii) using this distribution along with its properties to obtain a single-letter expression. There are two reasons that make step ii) especially challenging. First, the sequence that needs to be described by the coordination function (i.e., X_1) consists of correlated symbols. This correlation poses and additional difficulty because lower bounding quantities like H(X_1) by single-letter expressions is no longer an easy task. Second, since the coordination rate R_c is limited, the description of X_1 cannot be complete in general. These two problems are also encountered when studying channels with state. In neither case a single-letter expression for the channel capacity is known (see [HG83] and the discussion in [GK11, Section 7.8]).

For the inner bound, we conjecture that it is in fact not tight. The reason is that the code does not exploit the structure present in X_1 when generating the coordination message. That is, although X_1 is a codeword from a fixed codebook, the coordination function treats it as if it were a sequence of i.i.d. symbols drawn from the distribution P_{X_1}.

### 5.4 Summary and Concluding Remarks

In this chapter, we have considered the problem of coordinating communications to control the interference. We have shown that there is a fundamental tradeoff between communication, coordination, and undesired interference. Our study is very general, in the sense that the models that we have used do not rely on a certain structure of the channel or on specific features of the transmitters such as tight synchronization.

By studying the relationship between input and output distributions, we have established a complete characterization of the communication-interference capacity region for the scenario with a single transmitter-receiver pair and one observer, and an achievable region for a multiple user scenario.
5.A Proofs for Single User

5.A.1 Proof of Lemma 5.3

Proof of Lemma 5.3. The proof follows by observing that if we do time sharing between two codes, the resulting communication rate and interference type are convex combinations of the original rates and types of each of the individual codes. The coefficient of the convex combination corresponds to the time-sharing factor. ■

5.A.2 Proof of Theorem 5.4

Proof of Theorem 5.4. For given $\epsilon > 0$, the error probability is

$$\Pr(\mathcal{E}) = \Pr(\{\hat{M} \neq M\} \cup \{\|T_{Z^n} - G_{Z}\|_{TV} \geq \epsilon\}) \quad (5.39)$$

and satisfies

$$\Pr(\mathcal{E}) \geq \max(\Pr(\hat{M} \neq M), \Pr(\|T_{Z^n} - G\|_{TV} \geq \epsilon)) \quad (5.40)$$

$$\Pr(\mathcal{E}) \leq \Pr(\hat{M} \neq M) + \Pr(\|T_{Z^n} - G\|_{TV} \geq \epsilon). \quad (5.41)$$

Note that each of the terms on the right-hand side of (5.40) and (5.41) only depend on the marginals $P_{Y|X}$ and $P_{Z|X}$. Therefore, whether $\Pr(\mathcal{E})$ vanishes or not depends only on these marginal distributions. ■

5.A.3 Proof of Achievability in Theorem 5.5

The achievability result follows from Shannon’s coding theorem. Thus, we only include here a short version of the proof that emphasizes the convergence of the interference type.

Proof of achievability in Theorem 5.5. Fix $\epsilon > 0$, choose $(R, G_Z)$ in the interior of $\mathcal{C}$ and let $P_X$ be the corresponding pmf.

Codebook generation. For every $m \in \mathcal{M}$, generate a codeword $x^n(m)$ according to $\prod_{i=1}^n P_X(x_i)$.

Encoding. To transmit the message $m$, the encoder puts the codeword $x^n(m)$ into the channel.

Decoding. Given the observation $y^n$, the decoder searches for a unique $\hat{m}$ such that $(x^n(\hat{m}), y^n) \in \mathcal{T}^{(n)}(P_X, y)$. If no such $\hat{m}$ is found or if it is not unique, the decoder declares an error.
Analysis of the error probability. We consider the error probability averaged over the ensemble of codebooks. Let $\mathcal{E}$ denote the error event. Due to the symmetry in the generation of the codebooks, we can assume that $M = 1$ without loss of generality, that is,

$$\Pr(\mathcal{E}) = \Pr(\mathcal{E}|M = 1). \quad (5.42)$$

To bound the error probability, consider the following events:

$$\mathcal{E}_M \triangleq \{\hat{M} \neq 1\}, \quad (5.43)$$

$$\mathcal{E}_Z \triangleq \{\|T^*_{\mathcal{Z}^n} - G_Z\|_{TV} \geq \epsilon\}. \quad (5.44)$$

We have that

$$\Pr(\mathcal{E}) \leq \Pr(\mathcal{E}_M) + \Pr(\mathcal{E}_Z). \quad (5.45)$$

The first term in (5.45) corresponds to the probability of decoding error in a point-to-point DMC (cf. Theorem 2.36), which vanishes as $n \to \infty$ for every $R < I(X;Y)$ (see, e.g., [CT06]). To bound the second term in (5.45), consider the event

$$\mathcal{E}_{XZ} \triangleq \{\|T^*_{x^n}P_X z^n - P_X z^n X\|_{TV} \geq \epsilon\} \quad (5.46)$$

and note that, by Lemma 2.8, for every $(x^n, z^n)$ such that

$$\|T^*_{x^n}z^n - P_X z^n X\|_{TV} < \epsilon, \quad (5.47)$$

we have

$$\|T^*_{z^n}z^n - G_Z\|_{TV} < \epsilon. \quad (5.48)$$

Therefore,

$$\Pr(\mathcal{E}_Z) \leq \Pr(\mathcal{E}_{XZ}). \quad (5.49)$$

By the law of large numbers, $\Pr(\mathcal{E}_{XZ}) \to 0$ as $n \to \infty$ for any $\epsilon > 0$. This completes the proof of achievability. 

5.A.4 Proof of Converse in Theorem 5.5

In this section, we prove the converse result in Theorem 5.5. That is, that any sequence of $(n, 2^{nR})$-codes such that

$$\lim_{n \to \infty} \Pr(\hat{M} \neq M) = 0, \quad (5.50)$$

$$\|T^*_{\mathcal{Z}^n} - G_Z\|_{TV} \to 0 \quad \text{in probability}, \quad (5.51)$$
must satisfy \((R, G_Z) \in \mathcal{C}\). The uniform distribution on the messages, together with the code, induces the joint distribution

\[
\frac{1}{|\mathcal{M}|} P_{X^n|M}(x^n|m)P_{Y^n|X^n}(y^n|x^n)P_{Z^n|X^n}(z^n|x^n)P_{M|Y^n}(m|y^n),
\]

with \(P_{Y^n|X^n}(y^n|x^n) = \prod_{i=1}^{n} P_{Y|x}(y_i|x_i)\) and \(P_{Z^n|X^n}(z^n|x^n) = \prod_{i=1}^{n} P_{Z|x}(z_i|x_i)\). Observe that in (5.52), we have restricted our attention to distributions \(P_{Y,Z|X} = P_{Y|X}P_{Z|X}\). By virtue of Theorem 5.4, this entails no loss of generality.

To prove the converse, we need to relate the operational characterization of a sequence of \((n, 2^{nR})\)-codes in (5.50) and (5.51) to the distribution in (5.52). Condition (5.50) will enter the converse in the usual way through Fano’s inequality (Lemma 2.30). Condition (5.51) will enter the converse through Lemma 5.20. To show this result, we first present some auxiliary lemmata. The set of distributions on \(X\) that are mapped onto a given \(G_Z\) plays a fundamental role in the following:

**Definition 5.15 (Pre-image of \(G_Z\)).** For a given \(G_Z\), we define the set \(\mathcal{P}\) as its pre-image under \(P_{Z|X}\):

\[
\mathcal{P} \triangleq \left\{ P_X : \sum_x P_X P_{Z|X} = G_Z \right\}.
\]

\(\diamondsuit\)

**Lemma 5.16.** The interference type \(T_{Z^n}\) induced by a sequence of \((n, 2^{nR})\)-codes can only converge in probability to distributions \(G_Z\) with non-empty pre-image set, that is, \(\mathcal{P} \neq \emptyset\).

\(\square\)

**Proof.** Assume that the type \(T_{Z^n}\) induced by a sequence of \((n, 2^{nR})\)-codes converges to some \(G_Z\) such that \(\mathcal{P} = \emptyset\). First, note that \(\mathcal{P}^c\) is the whole simplex of probability distributions on \(X\). Thus,

\[
\bar{P}_X \triangleq \mathbb{E}\{T_X^n\} \in \mathcal{P}^c.
\]

Now, use this result and Lemma 2.10 to write

\[
\mathbb{E}\{T_{Z^n}\} = \sum_x \mathbb{E}\{T_{X^n,Z^n}(x,z)\}
\]

\[
= \sum_x \mathbb{E}\{T_{X^n}(x)\}P_{Z|X}(z|x)
\]

\[
= \sum_x \bar{P}_X(x)P_{Z|X}(z|x).
\]

At the same time, convergence in probability

\[
\|T_{Z^n} - G_Z\|_{TV} \to 0
\]
implies that
\[ E\{\|T_{Z^n} - G_Z\|_{TV}\} \to 0 \] (5.59)
because the total variation is bounded. In turn, this means that
\[ E\{T_{Z^n}\} \to G_Z \] (5.60)
by a simple application of Jensen’s inequality. The result in (5.57), combined with (5.60) leads to a contradiction because we assumed that \( P \) was an empty set. Therefore, \( P \neq \emptyset \).

Lemma 5.17. For given \( G_Z \), its pre-image set \( P \) is closed and convex.

Proof. That \( P \) is closed follows from the fact that it is defined as the pre-image of the closed set \( G_Z \) under a continuous function. To prove convexity, let \( P_1, P_2 \in P \) and observe that
\[
\sum_x (\lambda P_1 + (1 - \lambda)P_2)P_{Z|X} = \lambda \sum_x P_1 P_{Z|X} + (1 - \lambda) \sum_x P_2 P_{Z|X} \\
= \lambda G_Z + (1 - \lambda)G_Z,
\] (5.61)
for any \( \lambda \in [0, 1] \).

Lemma 5.18. Let \( G_Z \) be given and have pre-image set \( P \) such that \( P \neq \emptyset \) and \( P^c \neq \emptyset \). Consider the sets
\[
\tilde{P}_\epsilon \triangleq \{ \tilde{P}_X : \|\tilde{P}_X - P_X\|_{TV} \geq \epsilon \text{ for all } P_X \in P \},
\] (5.63)
\[
\tilde{G} \triangleq \left\{ \tilde{G}_Z : \sum_x P_{Z|X} \tilde{P}_X = \tilde{G}_Z \text{ for some } \tilde{P}_X \in \tilde{P}_\epsilon \right\},
\] (5.64)
defined for any fixed \( \epsilon > 0 \) such that \( \tilde{P}_\epsilon \neq \emptyset \). Let
\[ d^* = \inf_{G_Z \in \tilde{G}} \|G_Z - \tilde{G}_Z\|_{TV} \] (5.65)
Then, we have that \( d^* > 0 \).

Proof. Assume that \( d^* = 0 \). Note that \( \tilde{P}_\epsilon \) is a compact set and that \( \tilde{G}_Z \) is a continuous function of \( \tilde{P}_X \). Therefore, \( \tilde{G} \) is a compact set, too. Note also that \( \|G_Z - \tilde{G}_Z\|_{TV} \) is a continuous function of \( \tilde{G}_Z \). Thus, by Weierstrass’ extreme value theorem [Rud76, Theorem 4.16], there must exist some \( \tilde{G}_Z \in \tilde{G} \) (and hence some \( \tilde{P}_X \in \tilde{P}_\epsilon \)) such that
\[ \|G_Z - \tilde{G}_Z\|_{TV} = 0. \] (5.66)
That is, \( G_Z = \tilde{G}_Z \). However, this would imply that \( \tilde{P}_X \in P \), contradicting our initial hypothesis. Thus, we must have \( d^* > 0 \).
Lemma 5.19. Let \( \epsilon > 0 \) and consider two arbitrary pmfs \( Q_Z \) and \( \tilde{Q}_Z \) defined on \( Z \) with typical sets \( T^{(n)}_\epsilon(Q_Z) \) and \( T^{(n)}_\epsilon(\tilde{Q}_Z) \), respectively. If the variational distance between the pmfs satisfies \( \| Q_Z - \tilde{Q}_Z \|_{TV} > 2\epsilon \) then the two typical sets are disjoint. That is, \( T^{(n)}_\epsilon(Q_Z) \cap T^{(n)}_\epsilon(\tilde{Q}_Z) = \emptyset \).

Proof. Let \( z^n \in T^{(n)}_\epsilon(Q_Z) \), that is,

\[
\| Q_Z - T_{z^n} \|_{TV} < \epsilon. \tag{5.67}
\]

Then

\[
\| \tilde{Q}_Z - T_{z^n} \|_{TV} = \| \tilde{Q}_Z - Q_Z + Q_Z - T_{z^n} \|_{TV} \geq \| \tilde{Q}_Z - Q_Z \|_{TV} - \| Q_Z - T_{z^n} \|_{TV} > 2\epsilon - \epsilon. \tag{5.69}
\]

That is, \( z^n \notin T^{(n)}_\epsilon(\tilde{Q}_Z) \). Therefore, \( T^{(n)}_\epsilon(Q_Z) \cap T^{(n)}_\epsilon(\tilde{Q}_Z) = \emptyset \). \( \square \)

Lemma 5.20. Let \( G_Z \) be fixed and have pre-image \( \mathcal{P} \). If a sequence of \( (n, 2^n)^R \)-codes induces an interference type \( T_{Z^n} \) such that

\[
\| T_{Z^n} - G_Z \|_{TV} \to 0 \quad \text{in probability}, \tag{5.71}
\]

then the expectation of the type of the codewords \( \mathbb{E}\{T_{X^n}\} \) satisfies

\[
\| \mathbb{E}\{T_{X^n}\} - P^{(n)}_X \|_{TV} \to 0 \tag{5.72}
\]

for some sequence \( \{P^{(n)}_X\} \) with \( P^{(n)}_X \in \mathcal{P} \) for all \( n \).

Proof. First, note that \( \mathcal{P} \neq \emptyset \) by virtue of Lemma 5.16. Moreover, if \( \mathcal{P} = \emptyset \) the proof is
trivial. We prove the lemma for the case \( P \neq \emptyset, P^c \neq \emptyset \) in two steps. i) First, we show that (5.71) implies that \( \lim_{n \to \infty} \Pr(X^n \notin T^{(n)}_{\epsilon}(P)) = 0 \) for any \( \epsilon > 0 \), where

\[
T^{(n)}_{\epsilon}(P) \triangleq \{ x^n : \| T_{x^n} - P_X \|_{TV} < \epsilon \text{ for some } P_X \in P \}.
\]

(5.73)

(The set \( T^{(n)}_{\epsilon}(P) \) is a straightforward generalization of the typical set \( T^{(n)}_{\epsilon}(P) \) in Definition 2.7). ii) Then, we show that this implies (5.72).

i) We prove the first step by contradiction. Assume that (5.71) is satisfied by some sequence of \( (n, 2^n R) \)-codes with distribution \( P_X^n \) for which there exist \( \delta > 0 \) and \( \epsilon_x > 0 \) such that

\[
\delta \leq \limsup_{n \to \infty} \Pr(X^n \notin T^{(n)}_{\epsilon_x}(P)).
\]

(5.74)

Note that the set \( \{ x^n \notin T^{(n)}_{\epsilon_x}(P) \} \) is equivalent to \( \{ x^n : T_{x^n} \in \tilde{P}_{\epsilon_x} \} \), with

\[
\tilde{P}_{\epsilon_x} \triangleq \{ \tilde{P}_X : \| \tilde{P}_X - P_X \|_{TV} \geq \epsilon_x \text{ for all } P_X \in P \}.
\]

(5.75)

Observe that \( \tilde{P}_{\epsilon_x} \neq \emptyset \) for sufficiently small \( \epsilon_x \) because \( \tilde{P}_{\epsilon_x} \subseteq P^c \) and \( P^c \) is a non-empty open set. In addition, for every \( \epsilon'_x \) such that \( 0 < \epsilon'_x < \epsilon_x \) we have \( \tilde{P}_{\epsilon_x} \subseteq \tilde{P}_{\epsilon'_x} \) and this implies that \( \Pr(X^n \notin T^{(n)}_{\epsilon_x}(P)) \leq \Pr(X^n \notin T^{(n)}_{\epsilon'_x}(P)) \). Thus, without loss of generality, we assume that \( \tilde{P}_{\epsilon_x} \neq \emptyset \).

Now, we define the following finite cover \( Q_{\epsilon_c} \) of \( \tilde{P}_{\epsilon_x} \): given \( \epsilon_c \) such that \( 0 < \epsilon_c < \epsilon_x \), the set \( Q_{\epsilon_c} \) is a finite set of distributions on \( X \) such that for every \( \tilde{P}_X \in \tilde{P}_{\epsilon_x} \) there exists some \( P_X \in Q_{\epsilon_c} \) with

\[
\| P_X - \tilde{P}_X \|_{TV} < \epsilon_c.
\]

(5.76)

Such cover exists because the set \( \tilde{P}_{\epsilon_x} \) is compact. In fact, there exist more than one set with these properties. For convenience, we choose one (any) such set with the smallest possible cardinality. Thus, any pmf in \( \tilde{P}_{\epsilon_x} \) can be approximated by an element in the finite set \( Q_{\epsilon_c} \) with an error, in terms of the variational distance, not exceeding \( \epsilon_c \). Fix an arbitrary ordering of the elements in \( Q_{\epsilon_c} \)

\[
Q_{\epsilon_c} = \{ Q_{X,1}, Q_{X,2}, \ldots, Q_{X,|Q_{\epsilon_c}|} \},
\]

(5.77)

and let

\[
\tilde{Q}_i \triangleq \{ P_X \in \tilde{P}_{\epsilon_x} : \| Q_{X,i} - P_X \|_{TV} < \epsilon_c \}
\]

(5.78)

for \( i \in \{ 1, \ldots, |Q_{\epsilon_c}| \} \). To avoid the possibility that \( \tilde{P}_X \in \tilde{Q}_i \) and \( \tilde{P}_X \in \tilde{Q}_j \) for \( i \neq j \), we define the following disjoint sets

\[
Q_1 \triangleq \tilde{Q}_1,
\]

(5.79)

\[
Q_i \triangleq \tilde{Q}_i \setminus \bigcup_{j=1}^{i-1} \tilde{Q}_j
\]

(5.80)
for \( i \in \{2, \ldots, |Q_{e_i}|\} \). Observe that \( \cup_i Q_i = \hat{P}_{e_x} \). Thus, for each \( x^n \notin T^{(n)}_{e_x}(\mathcal{P}) \) its type \( T_{e_x} \) satisfies \( T_{e_x} \in Q_i \) for one, and only one, \( i \in \{1, \ldots, |Q_{e_i}|\} \). Using this covering in disjoints sets, we write

\[
\sum_{x^n \notin T^{(n)}_{e_x}(\mathcal{P})} P_{X^n}(x^n) = \sum_{i=1}^{|Q_{e_i}|} \sum_{x^n \in T_{e_x} \cap Q_i} P_{X^n}(x^n). \tag{5.81}
\]

Now, for arbitrary \( \epsilon > 0 \), write

\[
\sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n}(z^n) = \sum_{x^n} P_{X^n}(x^n) \sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n|X^n}(z^n|x^n) \geq \sum_{x^n \notin T^{(n)}_{e_x}(\mathcal{P})} P_{X^n}(x^n) \sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n|X^n}(z^n|x^n) = \sum_{x^n : T_{e_x} \in Q_1} P_{X^n}(x^n) \sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n|X^n}(z^n|x^n) + \sum_{x^n : T_{e_x} \in Q_2} P_{X^n}(x^n) \sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n|X^n}(z^n|x^n) + \ldots \tag{5.82}
\]

Consider the \( i^{th} \) term in (5.84). First, note that each of the sequences \( x^n \) in the sum belongs to the typical set \( T^{(n)}_{e_x}(Q_{X,i}) \). Now, define \( Q_{Z,i} = \sum_{x} P_{Z|X} Q_{X,i} \) and consider the set \( T^{(n)}_{e_x}(Q_{Z,i}) \) of sequences \( z^n \) that are typical according to \( Q_{Z,i} \).

From Lemma 5.18 we know that, given \( e_x \), there exists a fixed \( d^* > 0 \) such that \( \|GZ - Q_{Z,i}\|_{TV} \geq d^* \) for all \( Q_{Z,i} \) \( i \in \{1, \ldots, |Q_{e_i}|\} \). Thus, for any \( \epsilon \) such that \( 0 < \epsilon < \frac{d^*}{2} \), applying Lemma 5.19 we see that \( T^{(n)}_{e_x}(GZ) \cap T^{(n)}_{e_x}(Q_{Z,i}) = \emptyset \). Using this, we write

\[
\sum_{x^n : T_{e_x} \in Q_1} P_{X^n}(x^n) \sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n|X^n}(z^n|x^n) \geq \sum_{x^n : T_{e_x} \in Q_1} P_{X^n}(x^n) \sum_{z^n \in T^{(n)}_{e_x}(Q_{Z,i})} P_{Z^n|X^n}(z^n|x^n). \tag{5.83}
\]

Moreover, by Corollary 2.13 on the conditional typicality lemma, we see that

\[
\sum_{z^n \in T^{(n)}_{e_x}(Q_{Z,i})} P_{Z^n|X^n}(z^n|x^n) \geq 1 - \delta_{e_x,\epsilon}(n) \tag{5.84}
\]

for every \( x^n \) such that \( T_{e_x} \in Q_i \). The term \( \delta_{e_x,\epsilon}(n) \) goes to 0 with \( n \) and is fixed given the cover \( Q_{e_x} \). Thus,

\[
\sum_{x^n : T_{e_x} \in Q_1} P_{X^n}(x^n) \sum_{z^n \notin T^{(n)}_{e_x}(GZ)} P_{Z^n|X^n}(z^n|x^n) \geq (1 - \delta_{e_x,\epsilon}(n)) \sum_{x^n : T_{e_x} \in Q_1} P_{X^n}(x^n). \tag{5.85}
\]
Using this, we rewrite (5.84) as
\[
\sum_{z^n \notin T_{\epsilon}^n(G_Z)} P_{Z^n}(z^n) \geq \sum_{i=1}^{\|Q_{e_{\epsilon}}\|} \sum_{x^n : T_{x^n} \in Q_i} P_{X^n}(x^n)(1 - \delta_{e_{\epsilon},\epsilon}(n)) \geq (1 - \delta_{e_{\epsilon},\epsilon}(n)) \sum_{x^n \notin T_{\epsilon}^n(\mathcal{P})} P_{X^n}(x^n). \tag{5.89}
\]

Therefore, for any \(0 < \epsilon < \frac{d^*}{2}\) we have
\[
\limsup_{n \to \infty} \sum_{z^n \notin T_{\epsilon}^n(G_Z)} P_{Z^n}(z^n) \geq \limsup_{n \to \infty} (1 - \delta_{e_{\epsilon},\epsilon}(n)) \sum_{x^n \notin T_{\epsilon}^n(\mathcal{P})} P_{X^n}(x^n) \geq \delta \geq 0. \tag{5.90}
\]

This contradicts our initial hypothesis that \(P_{X^n}\) induces a type \(T_{Z^n}\) that satisfies (5.71). Thus, we must have \(\lim_{n \to \infty} \Pr(X^n \notin T_{\epsilon}^n(\mathcal{P})) = 0\) for any \(\epsilon > 0\).

ii) Now, we show that this implies (5.72). To this end, we write
\[
\|E\{T_{X^n}\} - P_{X^n}\|_{TV} = \|E\{T_{X^n} | X^n \in T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \in T_{\epsilon}^n(\mathcal{P}))
\]
\[
+ E\{T_{X^n} | X^n \notin T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \notin T_{\epsilon}^n(\mathcal{P})) - P_{X^n}\|_{TV}
\]
\[
\leq \|E\{T_{X^n} | X^n \in T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \in T_{\epsilon}^n(\mathcal{P})) - P_{X^n}\|_{TV}
\]
\[
+ \|E\{T_{X^n} | X^n \notin T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \notin T_{\epsilon}^n(\mathcal{P}))\|_{TV}\tag{5.91}
\]

for arbitrary \(\epsilon > 0\). Note that, for any two sequences \(x^n \) and \(\tilde{x}^n \) in \(T_{\epsilon}^n(\mathcal{P})\), the convex combination of their types \(T_{x^n}\) and \(T_{\tilde{x}^n}\) satisfies
\[
\|\lambda T_{x^n} + (1 - \lambda) T_{\tilde{x}^n} - P_X\|_{TV} < \epsilon \tag{5.92}
\]
for some \(P_X \in \mathcal{P}\) and any \(\lambda \in [0, 1]\). Thus, since
\[
E\{T_{X^n} | X^n \in T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \in T_{\epsilon}^n(\mathcal{P})) \tag{5.93}
\]
is a convex combination of types of sequences in \(T_{\epsilon}^n(\mathcal{P})\), we have that
\[
\|E\{T_{X^n} | X^n \in T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \in T_{\epsilon}^n(\mathcal{P})) - P_{X^n}\|_{TV} < \epsilon \tag{5.94}
\]
for some \(P_{X^n} \in \mathcal{P}\). Regarding the second term in (5.95), we see that
\[
\|E\{T_{X^n} | X^n \notin T_{\epsilon}^n(\mathcal{P})\} \Pr(X^n \notin T_{\epsilon}^n(\mathcal{P}))\|_{TV}
\]
\[
= \Pr(X^n \notin T_{\epsilon}^n(\mathcal{P})) \|E\{T_{X^n} | X^n \notin T_{\epsilon}^n(\mathcal{P})\}\|_{TV}\tag{5.95}
\]
\[
\leq \Pr(X^n \notin T_{\epsilon}^n(\mathcal{P})) \tag{5.96}
\]
\[
< \epsilon, \tag{5.97}
\]

(iii) The final step follows by applying the same argument to the conditional distributions.

Thus, we have shown that (5.72) holds for arbitrary \(\epsilon > 0\).
where the inequality is satisfied for sufficiently large $n$. Combining the two bounds, we see that
\[
\|E\{T_X^n\} - P_X^{(n)}\|_{TV} < 2\epsilon. \tag{5.103}
\]
Finally, we complete the proof by letting $\epsilon \to 0$. ■

This lemma has the following simple corollary.

**Corollary 5.21.** Let $G_Z$ be fixed and have pre-image consisting of a single distribution (i.e., $P = \{P_X\}$). If a sequence of $(n, 2^{nR})$-codes induces an interference type $T_{Z^n}$ such that
\[
\|T_{Z^n} - G_Z\|_{TV} \to 0 \text{ in probability}, \tag{5.104}
\]
then the input type must converge in probability to $P_X$, that is,
\[
\|T_{X^n} - P_X\|_{TV} \to 0 \text{ in probability.} \tag{5.105}
\]
□

*Proof.* The proof follows directly from the first half of the proof of Lemma 5.20 ■

We are now ready to prove converse result in Theorem 5.5.

**Proof of the converse result in Theorem 5.5.** Consider a sequence of $(n, 2^{nR})$-codes that achieve the rate-interference type pair $(R, G_Z)$. First, using the standard arguments based on Fano’s inequality (Lemma 2.30), we see that a vanishing error probability (i.e., (5.50)) implies that
\[
\begin{align*}
nR &= H(M) \\
&\leq I(M; Y^n) + n\epsilon_n \tag{5.106} \\
&= I(X^n; Y^n) + n\epsilon_n \tag{5.107} \\
&\leq \sum_{q=1}^{n} I(X_q; Y_q) + n\epsilon_n \tag{5.108} \\
&= n\sum_{q=1}^{n} \frac{1}{n} I(X_q; Y_q | Q = q) + n\epsilon_n \tag{5.109} \\
&= nI(X_Q; Y_Q | Q) + n\epsilon_n \tag{5.110} \\
&\leq nI(QX_Q; Y_Q) + n\epsilon_n \tag{5.111} \\
&= nI(X_Q; Y_Q) + n\epsilon_n \tag{5.112} \\
\end{align*}
\]
where $Q$ is a random variable uniformly distributed on $\{1, \ldots, n\}$ and independent of $(X^n, Y^n, Z^n)$, and $\epsilon_n \to 0$ as $n \to \infty$. The equality in (5.113) is justified by the
fact that the DMC establishes the Markov chain $Q - X_Q - Y_Q$. Diving by $n$, we obtain

$$R \leq I(X_Q;Y_Q) + \epsilon_n.$$  

(5.114)

This mutual information is evaluated for $P_{X_Q,Y_Q}$, which can be written as

$$P_{X_Q,Y_Q}(x,y) = P_{X_Q}(x)P_{Y|X}(y|x)$$  

(5.115)

$$= \mathbb{E}\{T_{X^n}(x)\}P_{Y|X}(y|x).$$  

(5.116)

The first equality comes from the Markov chain $Q - X_Q - Y_Q$. The second equality is due to Lemma 2.9.

Now, condition (5.51) on the type of the interference for a sequence of $(n, 2^{nR})$-codes that achieves the pair $(R,G_Z)$, combined with Lemma 5.20, implies that the expectation of the type of the input to the channel $\mathbb{E}\{T_{X^n}\}$ must converge to a sequence $\{P^{(n)}_X\}$ with $P^{(n)}_X \in \mathcal{P}$ for all $n$. That is,

$$\mathbb{E}\{T_{X^n}(x)\}P_{Y|X}(y|x) \rightarrow P^{(n)}_X(x)P_{Y|X}(y|x)$$  

(5.117)

or, equivalently,

$$P_{X_Q,Y_Q}(x,y) \rightarrow P^{(n)}_X(x)P_{Y|X}(y|x).$$  

(5.118)

Since the mutual information is a continuous function of the input distribution, this convergence implies that any sequence of $(n, 2^{nR})$-codes must satisfy

$$R \leq \sup_n I(X;Y)|_{P^{(n)}_X} + \epsilon_n$$  

(5.119)

$$\leq \max_{P_X \in \mathcal{P}} I(X;Y) + \epsilon_n.$$  

(5.120)

Finally, letting $n \rightarrow \infty$ we establish that $(R,G_Z) \in \mathcal{C}$. ■

5.B Proof for Multiple Users

Proof of Theorem 5.12. Fix arbitrary $\epsilon > 0$ and let $\delta(\epsilon) > 0$ be some function such that $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Choose a tuple $(R_1, R_2, R_c, Q_Z) \in \mathcal{R}$ and let $\tilde{R}_2 > R_2$. Let $P_{U}P_{X_1|U}P_{X_2|U}$ be the corresponding distribution.

Codebook generation.

- For every $m_c \in \mathcal{M}_c$, generate a sequence $u^n(m_c)$ according to $\prod_{i=1}^{n} P_U(u_i)$.
- For every $m_1 \in \mathcal{M}_1$, generate a codeword $x^n_1(m_1)$ according to $\prod_{i=1}^{n} P_{X_1}(x_{1i})$.
- For every $m_2 \in \mathcal{M}_2$ and every $l \in \{1, \ldots, \lfloor 2^{n(\tilde{R}_2-R_2)} \rfloor \}$, generate a codeword $x^n_2(l,m_2)$ according to $\prod_{i=1}^{n} P_{X_2}(x_{2i})$. 

Encoding.

1. To transmit the message \( m_1 \), Encoder 1 puts the codeword \( x_1^n(m_1) \) into the channel.

2. To generate the coordination message given \( x_1^n(m_1) \), Encoder 1 searches for an index \( m_c \) such that \((u^n(m_c), x_1^n(m_1)) \in T_e^{(n)}(P_{U,X_1})\). If more than one such \( m_c \) exist, it chooses one at random among the candidates. If none exist, then it chooses \( m_c = 1 \). Finally, it conveys the index \( m_c \) to Encoder 2.

3. To transmit the message \( m_2 \), Encoder 2 searches for an index \( l \) such that \((u^n(m_c), x_2^n(l, m_2)) \in T_e^{(n)}(P_{U,X_2})\). If more than one such \( l \) exist, it chooses one at random among the candidates. If none exist, then it chooses \( l = 1 \). Finally, it puts the codeword \( x_2^n(l, m_c) \) into the channel.

Decoding.

- Given the observation \( y_1^n \), Decoder 1 searches for a unique index \( \hat{m}_1 \) such that \((x_1^n(\hat{m}_1), y_1^n) \in T_e^{(n)}(P_{X_1,Y_1})\). If no such \( \hat{m}_1 \) is found or if it is not unique, the decoder declares an error.

- Given the observation \( y_2^n \), Decoder 2 searches for a unique index \( \hat{m}_2 \) such that \((x_2^n(\hat{l}, \hat{m}_2), y_2^n) \in T_e^{(n)}(P_{X_2,Y_2})\) for some \( \hat{l} \in \{1, \ldots, \lceil 2^n(R_2 - R_1) \rceil \} \). If no such \( \hat{m}_2 \) is found or if it is not unique, the decoder declares an error.

Analysis of the error probability. We consider the error probability averaged over the ensemble of codebooks. Let \( \mathcal{E} \) denote the error event. Due to the symmetry in the generation of the codebooks, we can assume that \( M_1 = M_2 = 1 \) without loss of generality. That is,

\[
\Pr(\mathcal{E}) = \Pr(\mathcal{E}|(M_1, M_2) = (1, 1)).
\]

To bound the error probability, consider the following events:

\[
\mathcal{E}_Z \triangleq \{\|T^n - Q_Z\|_{TV} \geq \epsilon\},
\]

\[
\mathcal{E}_i \triangleq \{\hat{M}_i \neq 1\}
\]

for \( i = \{1, 2\} \). The error probability satisfies

\[
\Pr(\mathcal{E}) \leq \Pr(\mathcal{E}_Z|(M_1, M_2) = (1, 1)) + \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2).
\]

We bound each of the three terms individually.
To bound \( \Pr(\mathcal{E}_Z) \), first consider the event

\[
\mathcal{E}_{UXYZ} \triangleq \{\|T_{U^n,X_1^n(1),X_2^n(l,1),Z^n} - P_{Z|X_1,X_2,P_{X_1|U,P_{X_2|U}}P_U}\|_{TV} \geq \epsilon\}
\]

(5.125)

(5.124)
and note that, by Lemma 2.8, for every \((u^n, x^n_1, x^n_2, z^n)\) such that
\[
\|T_{u^n, x^n_1, x^n_2, z^n} - P_{Z|X_1, X_2} P_{X_1|U} P_{X_2|U}\|_{TV} < \epsilon,
\]
(5.126)
we have
\[
\|T_{z^n} - Q_{Z}\|_{TV} < \epsilon.
\]
(5.127)
Therefore,
\[
\Pr(\mathcal{E}_Z|(M_1, M_2) = (1, 1)) \leq \Pr(\mathcal{E}_{UXYZ}).
\]
(5.128)
Now, let \(\epsilon' = \frac{\epsilon}{4}\) and
\[
\mathcal{E}_Z \triangleq \{(U^n(m_c), X^n_1(1)) \notin \mathcal{T}^{(n)}_{\epsilon}(P_{U,Y_1}) \text{ for all } m_c \in \mathcal{M}_c\},
\]
(5.129)
\[
\mathcal{E}_{Z2} \triangleq \{(U^n(M_c), X^n_2(l, 1)) \notin \mathcal{T}^{(n)}_{\epsilon}(P_{U,Y_2}) \text{ for all } l \in \{1, \ldots, [2^n(\bar{R}_2-R_2)]\}\},
\]
(5.130)
\[
\mathcal{E}_{Z3} \triangleq \{(U^n(M_c), X^n_1(1), X^n_2(L, 1)) \notin \mathcal{T}^{(n)}_{\epsilon}(P_{U,Y_1,Y_2})\},
\]
(5.131)
\[
\mathcal{E}_{Z4} \triangleq \{(U^n(M_c), X^n_1(1), X^n_2(L, 1), Z^n) \notin \mathcal{T}^{(n)}_{\epsilon}(P_{U,Y_1,Y_2,Z})\}.
\]
(5.132)
We have that
\[
\Pr(\mathcal{E}_{UXYZ}) \leq \Pr(\mathcal{E}_{Z1}) + \Pr(\mathcal{E}_{Z2}) + \Pr(\mathcal{E}_{Z3} \cap (\mathcal{E}^c_{Z1} \cap \mathcal{E}^c_{Z2}) \cap \mathcal{E}^c_{Z3}).
\]
(5.133)
By the covering lemma, \(\Pr(\mathcal{E}_{Z1}) \to 0\) as \(n \to \infty\) if \(R_c > I(U; X_1) - \delta(\epsilon')\). For the second term in (5.133), note that the distribution of \((U^n(M_c), X^n_2(L, 1))\) is the same for all values of \(M_c\) and \(l\); they are independent. Thus, again by the covering lemma, \(\Pr(\mathcal{E}_{Z2}) \to 0\) as \(n \to \infty\) if \(R_2 - R_2 > I(U; X_2) - \delta(\epsilon')\).

Regarding the third term in (5.133), we observe the following. Given \(\mathcal{E}^c_{Z1}\), we have that \((U^n(M_c), X^n_1(1)) \in \mathcal{T}^{(n)}_{\epsilon}(P_{U,Y_1})\). Similarly, given \(\mathcal{E}^c_{Z2}\), we have that \((U^n(M_c), X^n_2(L, 1)) \in \mathcal{T}^{(n)}_{\epsilon}(P_{U,Y_2})\). Observe that \(X_1 - U - X_2\) and that the distribution of \(X^n_2\) is permutation invariant with respect to \(u^n\) (cf. Lemma 2.17). Thus, by Lemma 2.17, \(\Pr(\mathcal{E}_{Z3} \cap (\mathcal{E}^c_{Z1} \cap \mathcal{E}^c_{Z2})) \to 0\) as \(n \to \infty\).

Finally, for the last term in (5.133), we have that \(Z^n\) is generated by passing a \(\epsilon\)-typical pair \((X^n_1, X^n_2)\) through the channel \(P_{Z|X_1, X_2}\). Thus, by the law of large numbers, \(\Pr(\mathcal{E}_{Z4} \cap \mathcal{E}^c_{Z3}) \to 0\) as \(n \to 0\).

We now turn our attention to the term \(\Pr(\mathcal{E}_1)\) in (5.124). Consider the following events
\[
\mathcal{E}_{11} \triangleq \{(X^n_1(1), Y^n_1) \notin \mathcal{T}^{(n)}_{\epsilon}(P_{X_1,Y_1})\},
\]
(5.134)
\[
\mathcal{E}_{12} \triangleq \{(X^n_1(\hat{m}_1), Y^n_1) \in \mathcal{T}^{(n)}_{\epsilon}(P_{X_1,Y_1}) \text{ for some } \hat{m}_1 \neq 1\}.
\]
(5.135)
We have that
\[
\Pr(\mathcal{E}_1) \leq \Pr(\mathcal{E}_{11}) + \Pr(\mathcal{E}_{12}),
\]
(5.136)
where \( \Pr(\mathcal{E}_{11}) \to 0 \) as \( n \to 0 \) by the law of large numbers, and \( \Pr(\mathcal{E}_{12}) \to 0 \) as \( n \to 0 \) if

\[
R_1 < I(X_1; Y_1) - \delta(\epsilon)
\]

(5.137)

by the packing lemma.

Similarly, to bound \( \Pr(\mathcal{E}_2) \) in (5.124), consider the following events

\[
\mathcal{E}_{21} \triangleq \{(X^n_2(L, 1), Y^n_2) \notin T^{(n)}_\epsilon(P_{X_2, Y_2})\},
\]

(5.138)

\[
\mathcal{E}_{22} \triangleq \{(X^n_2(\hat{l}, \hat{m}_2), Y^n_2) \in T^{(n)}_\epsilon(P_{X_2, Y_2}) \text{ for some } \hat{m}_2 \neq 1
\]

and some \( \hat{l} \in \{1, \ldots, \lceil 2^n(\tilde{R}_2 - R_2) \rceil \} \} \}.
\]

(5.139)

We have that

\[
\Pr(\mathcal{E}_2) \leq \Pr(\mathcal{E}_{21}) + \Pr(\mathcal{E}_{22}),
\]

(5.140)

where \( \Pr(\mathcal{E}_{21}) \to 0 \) as \( n \to 0 \) by the law of large numbers. For \( \mathcal{E}_{22} \), note that

\( X^n_2(\hat{l}, \hat{m}_2) \) is independent of \( Y^n_2 \) for all \( \hat{m}_2 \neq 1 \), regardless of \( \hat{l} \). Thus, by the packing lemma, \( \Pr(\mathcal{E}_{22}) \to 0 \) as \( n \to 0 \) if

\[
\tilde{R}_2 < I(X_2; Y_2) - \delta(\epsilon).
\]

(5.141)

Combining all terms and letting \( \epsilon \to 0 \), we obtain

\[
R_c > I(U; X_1),
\]

(5.142)

\[
R_1 < I(X_1; Y_1),
\]

(5.143)

\[
R_2 < \tilde{R}_2 - I(U; X_2) = I(X_2; Y_2) - I(U; X_2),
\]

(5.144)

as desired. The remaining tuples in the convex hull are achieved by time sharing. □
In this chapter, we turn our attention to a different type of problem: compressive sensing. We study the fundamental relationship between two basic quantities: the measurement rate and the mean square estimation error (MSE). More specifically, we consider the estimation of a deterministic but unknown vector with a fixed level of sparsity and derive sufficient conditions on the measurement rate to achieve a certain mean square error. The structure of the chapter is the following.

First, we motivate the problem discussed in this chapter and present a formulation that defines two sub-problems of interest: partial support recovery and estimation using partial support set information (Section 6.1). For the first sub-problem, we establish a condition on the number of measurements that are asymptotically sufficient for reliable recovery of parts of the support containing a certain fraction of the total power (Section 6.2.1). Our results show that it is possible to reduce the measurement rate at the expense of recovering smaller parts of the support set. For the second sub-problem, we characterize the mean square estimation error performance averaged over the ensemble of Gaussian measurement matrices when the sparse estimator relies only on partial knowledge about the support set (Section 6.2.2).

We use these two results to derive a characterization of the tradeoff between the measurement rate and the mean square estimation error (Section 6.2.3). This tradeoff is asymptotically achievable by adopting a natural two-step approach: first support recovery, then estimation of the detected active components. Our results show that it is often possible to significantly reduce the measurement rate at a

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1Parts of the material presented in this chapter are based on/a transcript of our work previously published in \[BZS^{+}13a\] (© 2013 IEEE) or submitted for publication in \[BZS^{+}13b\] to the IEEE Transactions on Signal Processing. Conditioned on the acceptance for publication, the copyright of \[BZS^{+}13b\] has been transferred to the IEEE too. Material is reused with permission.
minimal cost in terms of MSE. Indeed, we derive conditions on the signal of interest that guarantee that partial support recovery yields the aforementioned reductions (Section 6.2.4).

We conclude our study by analyzing the advantages of partial support set recovery methods in the scenario where the vector of interest is drawn from a random distribution (Section 6.3). We observe that neglecting the elements in the support set that have little power relaxes the requirements in terms of measurement rate and thus reduces the number of measurement outages. At the same time, the contribution of these elements to the estimation error is small. In fact, thanks to the reduction of measurement outages, the MSE performance averaged over the distribution of the signals is significantly improved when compared to methods that rely on complete support recovery. Equivalently, by using partial support recovery it is possible to reduce the measurement signal-to-noise ratio while achieving the same performance in terms of MSE or outage probability.

For clarity of exposition, we present the proofs of all the results in the appendices.

### 6.1 Preliminaries

#### 6.1.1 System Model and Motivation

Recall the model for sparse signals introduced in Section 2.4. The $k$-sparse vector $x \in \mathbb{R}^n$ is defined component-wise as

$$x_i = \begin{cases} w_j & \text{if } i = s_j, \\ 0 & \text{if } i \notin S \end{cases} \quad (6.1)$$

for $i = \{1, \ldots, n\}$, where $S$ is the support set, $[s_1, \ldots, s_k]^T$ is an arbitrary ordering of the elements in $S$, and $w \in \mathbb{R}^k$ is a deterministic but unknown vector with $k$ non-zero entries sorted by decreasing magnitude. The vector $x$ is observed through the random measurement matrix $\Phi \in \mathbb{R}^{m \times n}$ in the presence of random noise $Z$:

$$Y = \Phi x + Z \quad (6.2)$$

The entries of the measurement matrix are independent and identically distributed (i.i.d.) according a Gaussian distribution $\mathcal{N}(0, P_\Phi)$. Similarly, we have i.i.d. $Z_i \sim \mathcal{N}(0, P_Z)$. We define the measurement signal-to-noise ratio as $\text{SNR} \triangleq P_\Phi / P_Z$.

We are interested in estimating $x$. In particular, we construct an estimator $\hat{X}$ that first attempts to detect the support set and then estimates the values of the corresponding entries. We study sufficient conditions on the number of measurements that ensure that the MSE

$$\text{mse}(x) \triangleq \mathbb{E} \left\{ \|x - \hat{X}\|^2 \right\} \quad (6.3)$$

is arbitrarily close to a target value.
In our analysis, we allow both the signal dimension $n$ and the number of measurements $m$ to go to infinity while keeping $k$, $S$, and $\mathbf{w}$ fixed. That is, formally, we consider sequences of vectors \( \{\mathbf{x} \in \mathbb{R}^n\} \), all of them with the same support set $S$ and non-zero entries $\mathbf{w}$, and sequences of measurements $\{\mathbf{y} \in \mathbb{R}^{mn}\}$. We index the sequences by $n$ and thus refer to the number of measurements by $m_n$.

Our formulation divides the problem into two parts: first, support recovery, and then, signal estimation. We will formulate both sub-problems separately in Section 6.1.2 but first, we motivate the necessity for partial support recovery.

Recall that Theorem 2.78 specifies that, for complete detection of the support set of $\mathbf{x}$, it is necessary and sufficient to measure at rate (cf. Definition 2.77) $r > \frac{1}{c(\mathbf{w})}$, where

\[
c(\mathbf{w}) \triangleq \min_{i \in \{1, \ldots, k\}} \frac{1}{2i} \log_2 \left( 1 + \frac{P_{\Phi}}{P_{\mathcal{Z}}} \sum_{j=k+1-i}^{k} w_j^2 \right).
\]

(6.4)

The following example illustrates some of the challenges implied by this result:

**Example 6.1.** Let $\frac{P_{\Phi}}{P_{\mathcal{Z}}} = 10$ and $\mathbf{w} \in \mathbb{R}^2$.

- Case 1: $w_1^2 = w_2^2 = 0.5$ yields $\frac{1}{c(\mathbf{w})} = 1.16$.
- Case 2: $w_1^2 = 0.95$ and $w_2^2 = 0.05$ yield $\frac{1}{c(\mathbf{w})} = 3.42$.
- Case 3: $w_1^2 = 0.99$ and $w_2^2 = 0.01$ yield $\frac{1}{c(\mathbf{w})} = 14.54$.

In all three cases, the sparse signal has $\|\mathbf{w}\|^2 = 1$. A simple inspection of (6.4) reveals that the uneven distribution of the power between the two components varies significantly the requirements for complete support recovery. The requirement for Case 2 is three times larger than that for Case 1. For Case 3, the increase is almost 13-fold.

In some cases, all components in the support set are equally important, independently of their magnitudes. These situations are challenging twice over: all entries in the support set must be detected and nothing else (i.e., no false alarm events). However, in many practical situations this may not be the case. For example, here we are only interested as in the support set as a means for estimating the sparse vector $\mathbf{x}$. Some of the non-zero components of $\mathbf{x}$ may have a negligible impact on the estimation error while, at the same time, they may push the requirements on the measurement matrix beyond what is acceptable. From Example 6.1 we know that this is indeed the trend. Thus, it is more reasonable to aim at partial but reliable detection of the support set.
Analogies with the Multiple Access Channel

As we discussed in Section 2.4, the problem of detection of the support set of a sparse signal and the problem of communication over multiple access channels (with some additional constraints such as unique Gaussian codebook, etc.) are fundamentally related. Therefore, the consequences of Theorem 2.78 discussed in Example 6.1 (i.e., that reliable detection of the complete support set is extremely challenging in the presence of small non-zero entries) can also be given a multiple access channel interpretation. This corresponds to the problem of decoding the messages from users with very small channel gains. In this scenario, one possibility is to decode only the strongest users, treating the rest as noise. This is a sensible strategy in a block fading channel where the transmitters do not have channel state information. In this type of channels, it is usually impossible to transmit reliably (in the sense of Section 2.2.1) at any positive communication rate.

Drawing on this analogy, we develop methods for recovering the parts of the support set that contain a certain fraction of the power. Our goal is to find conditions for partial support recovery that are less stringent than those in Theorem 2.78.

6.1.2 Problem Formulation

As we said, we are interested in finding sufficient conditions on the signal dimension $n$ and the measurement rate $r$ that ensure that it is possible to estimate a $k$-sparse vector $x$ with a given MSE. Our approach is to divide the problem into two parts: first, partial support recovery, and then, signal estimation. In the following, we formulate each of these two sub-problems separately.

Partial Support Set Recovery

To study the problem of partial support recovery, we introduce the following definitions that extend the framework for perfect support recovery presented in Section 2.4.2. Our central element is the $\gamma$-support set $\mathcal{S}_\gamma$ of $x$.

Definition 6.2 ($\gamma$-support set). For given $\gamma \in (0, 1]$, a $\gamma$-support set $\mathcal{S}_\gamma$ of $x$ is any subset of $\{1, \ldots, n\}$ such that

\[
\sum_{i \in \mathcal{S}_\gamma} x_i^2 \geq \gamma \|x\|^2
\]  

and that has the smallest possible size $\ell = |\mathcal{S}_\gamma|$. 

Observe that the definition of $\mathcal{S}_\gamma$ precludes $x_i = 0$ for any $i \in \mathcal{S}_\gamma$. Note also that there might exist more than one $\gamma$-support set but they all have the same size $\ell$, which depends on both $\gamma$ and $x$ (or equivalently $w$, cf. (6.1)). In addition, we
define
\[ S_\gamma \triangleq \{ T \in \mathcal{P}(\{1, \ldots, n\}) : T \text{ is a } \gamma\text{-support set of } x \} \tag{6.6} \]
\[ S_\gamma \triangleq \bigcup_{T \in S_\gamma} T, \tag{6.7} \]
where \( \mathcal{P}(\{1, \ldots, n\}) \) is the power set of \( \{1, \ldots, n\} \) (i.e., the set of all possible subsets of \( \{1, \ldots, n\} \)). Unlike \( S_\gamma \), the sets \( S_\gamma \) and \( S_\gamma \) are always unique given a fixed \( \gamma \). For the case \( \gamma = 1 \), we have \( S_1 = \mathcal{S}_\gamma = S \) and \( S_\gamma = \{S\} \).

To estimate a \( \gamma \)-support set of a sparse signal, we use a support recovery map \( d_\gamma \), which maps every pair of vector of observations and measurement matrix \( (y, \Phi) \) to a subset of \( \{1, \ldots, n\} \):
\[ d_\gamma : \mathbb{R}^{mn} \times \mathbb{R}^{mn \times n} \to \mathcal{P}(\{1, \ldots, n\}). \tag{6.8} \]
Note that we want the support recovery maps to be specified independently of \( x \); otherwise, the problem has a trivial solution. As opposed to most previous works, we do not endow the support recovery map with explicit knowledge on the sizes of \( S_\gamma \) or \( S \), (i.e., \( \ell \) or \( k \), respectively). We only assume that the support recovery map knows an upper bound \( k_{\text{max}} \) on \( k \) that is independent of \( n \) and \( mn \).

For given vector \( x \) and support recovery map \( d_\gamma \), we define the error probability as
\[ P_e(x, d_\gamma) \triangleq \Pr(d_\gamma(Y, \Phi) \notin S_\gamma). \tag{6.9} \]
Here the probability is taken over the distributions of the noise and the measurement matrix. As for the case of complete support recovery, it is interesting to characterize the regime of \((mn, n)\) for which partial support recovery with lower error probability \( P_e(x, d_\gamma) \) is possible. Our first goal is to find sufficient conditions on the measurement rate (Definition 2.77)
\[ r \triangleq \limsup_{n \to \infty} \frac{mn}{\log_2 n} \tag{6.10} \]
that ensure the existence of (sequences of) support recovery maps for which the error probability can be made arbitrarily small. We defer this to Section 6.2.1.

**Signal Estimation Using Partial Support Set Knowledge**

In the second part of our study, we determine the MSE performance of an estimator that uses a \( \gamma \)-support set. That is, we are interested in the MSE performance that is achievable by an estimator \( \hat{X}(S_\gamma) \),
\[ \text{mse}(x, S_\gamma) \triangleq \mathbb{E}\left\{ \|x - \hat{X}(S_\gamma)\|^2 \right\} \tag{6.11} \]
that knows \( S_\gamma \in S_\gamma \). We consider this in Section 6.2.2.
In [XLNH12], the authors studied linear minimum mean square error estimators with partially correct support set, a problem which is related to the sub-problem described here. However, our interest extends beyond characterizing (6.11) as we draw an explicit connection between the MSE and the measurement rate (Section 6.2.3).

6.2 Main Results

In this section, we present the main contributions of this chapter. First, we discuss partial support set recovery, followed by signal estimation with partial support set knowledge. We then combine our results on these two topics to establish an achievable measurement rate-MSE tradeoff for the performance of sparse signal estimation using partial support recovery methods. Finally, we study the type of signals for which partial support recovery methods yield a reduction in measurement rate requirements.

6.2.1 Partial Support Set Recovery

For any sequence \( \{x \in \mathbb{R}^n\} \) of \( k \)-sparse vectors with non-zero entries \( w \in \mathbb{R}^k \) (sorted by decreasing magnitude), and any \( \gamma \in (0, 1] \), we define

\[
c_i(w, \gamma) \triangleq \frac{1}{2^i} \log_2 \left( \frac{P\Phi \sum_{j=\ell-i+1}^{k} w_j^2 + P_Z}{(1 - \gamma) \|w\|^2 P\Phi + P_Z} \right),
\]

for \( i \in \{1, \ldots, \ell\} \), where \( \ell \) is the size of the \( \gamma \)-support set (cf. Definition 6.2), and

\[
r^*(w, \gamma) \triangleq \max_{i \in \{1, \ldots, \ell\}} \frac{1}{c_i(w, \gamma)}.
\]

Furthermore, assume that \( k \leq k_{\max} \) for some fixed, known \( k_{\max} \).

**Theorem 6.3.** There exists a fixed sequence of support recovery maps \( \{d^{(n)}_{\gamma}\} \) with

\[
\lim_{n \to \infty} P_e(x, d^{(n)}_{\gamma}) = 0 \tag{6.14}
\]

for any \( \gamma \in (0, 1] \) and any \( \{x \in \mathbb{R}^n\} \) as long as the measurement rate satisfies \( r > r^*(w, \gamma) \). In particular, there exists one such sequence \( \{d^{(n)}_{\gamma}\} \) with

\[
P_e(x, d^{(n)}_{\gamma}) \leq o(1/m_n). \tag{6.15}
\]
Proof. The proof is provided in Appendix 6.A.1.

The main implication of Theorem 6.3 is that, to reliably detect a γ-support set, it suffices that the number of measurements \( m \) grows with the signal dimension \( n \) so that \( r > r^*(w, \gamma) \).

Remark 6.4. By setting \( \gamma = 1 \) (i.e., detecting all \( k \) components in the support set), we recover the direct result in Theorem 2.78. Note that Theorem 6.3 implies that this is possible even without knowledge of the support set size \( k \), as long as it is fixed.

Remark 6.5. As in Theorem 2.78, the statement of Theorem 6.3 only depends on the sparse vector \( x \) through the non-zero components \( w \). The ordering of the components of \( w \) yields a bound \( r^*(w, 1) \) that is easier to evaluate than the original expression in [JKR11].

To illustrate the advantages of partial support recovery, we revisit Example 6.1.

Example 6.6. Let \( \frac{P_k}{P_z} = 10 \) and \( w \in \mathbb{R}^2 \).

- For case 2 in Example 6.1 (i.e., \( w_1^2 = 0.95 \) and \( w_2^2 = 0.05 \)), the choice \( \gamma = 0.95 \) yields \( r^*(w, \gamma) = 0.71 \). In this case, partial support recovery (only the position of the entry \( w_1 \) is detected) requires roughly one fifth of the measurements of complete support recovery.

- For case 3 in Example 6.1 (i.e., \( w_1^2 = 0.99 \) and \( w_2^2 = 0.01 \)), the choice \( \gamma = 0.99 \) yields \( r^*(w, \gamma) = 0.60 \). In this case, partial support recovery (only the position of the entry \( w_1 \) is detected) requires roughly 4% of the number of measurements of complete support recovery.

This shows that large reductions of measurement rate are possible. We emphasize that the savings depend on the choice of \( \gamma \), as shown in the following example.

Example 6.7. Let \( w \in \mathbb{R}^2 \) with \( w_1^2 = 0.7, w_2^2 = 0.3 \). For both \( \gamma_1 = 0.4 \) and \( \gamma_2 = 0.6 \), the \( \gamma_i \)-support set is just the position of \( w_1 \) in \( x \). However, \( r^*(w, \gamma_1) > r^*(w, \gamma_2) \) for all SNR.

In fact, it is easy to show (see Lemma 6.17 in Appendix 6.F) that, for a given \( \ell \), \( r^*(w, \gamma) \) is minimized by choosing \( \gamma = \gamma_{\ell, \text{opt}} \), where

\[
\gamma_{\ell, \text{opt}} = \frac{\sum_{j=1}^{\ell} w_j^2}{\|w\|^2}.
\]

It is necessary to emphasize that, although partial support recovery is often possible at lower measurements rates than complete support recovery, this is not always the case. We illustrate this with a final example.
Example 6.8. Consider case 1 in Example 6.1 (i.e., $w_1^2 = w_2^2 = 0.5$). For any measurement SNR, complete support recovery (i.e., $\gamma = 1$) yields a measurement rate requirement that is lower than that for partial support recovery. That is, $r^*(w, \gamma) > r^*(w, 1)$ for any $\gamma \in (0, 1)$.

These last two examples show that the choice of $\gamma$ should be influenced by our prior knowledge of $w$, if any. We will discuss in Section 6.2.4 under which conditions partial support recovery is possible at lower measurement rate than complete support recovery.

One small problem with the formulation of Theorem 6.3 is that there is not always a unique $\gamma$-support set (see Section 6.1.2). To avoid the ambiguity regarding the output of the support recovery map, we extend the result to the detection of $\mathcal{S}_\gamma$, which is unique.

Corollary 6.9. Under the conditions of Theorem 6.3, there exists a sequence of partial support recovery maps $\{d_n^{(n)}\}$ such that

$$\Pr\left(d_n^{(n)}(Y, \Phi^{(n)}) \neq \mathcal{S}_\gamma\right) \leq o(1/m_n).$$

Proof. The proof is provided in Appendix 6.A.5.

6.2.2 Mean Square Estimation Error

Consider now the measurement model given in (6.2) for fixed dimensions $(m, n)$. Let $x$ be a $k$-sparse vector with support set $\mathcal{S}$.

Theorem 6.10. Given some $\mathcal{T} \subseteq \mathcal{S}$, it is possible to estimate any $x$ with MSE

$$\text{mse}(x, \mathcal{T}) = \|x_{\mathcal{T}^c}\|^2 + \frac{\xi l}{m - l - 1},$$

where $l \triangleq |\mathcal{T}|$, $\xi \triangleq \|P_Z P_X + \|x_{\mathcal{T}^c}\|^2$, and $x_{\mathcal{T}^c}$ is the subvector of $x$ that contains the non-zero entries of $x$ not included in $\mathcal{T}$ (i.e., in $\mathcal{S} \setminus \mathcal{T}$).

Proof. The proof is provided in Appendix 6.B.

The first part of the MSE expression in (6.18) corresponds to the residual entries in $x$; that is, those whose index was not part of $\mathcal{T}$. This term has fixed magnitude, independent of $m_n$ and $n$. The second part of the MSE corresponds to the estimation noise and consists of two contributions (cf. $\xi$): the measurement noise $Z$
and the residual entries. This term depends on the matrix dimensions and decays roughly as $1/m$ with the number of measurements.

The behavior in (6.18) is not unexpected, since given (a fraction of) the support set, the estimation of $x$ amounts to finding the solution of a noisy overdetermined system of equations. In this sense, the MSE expression in (6.18) has an oracular interpretation (cf. Lemma 2.79), similar to those in [BKT09, CT07, and BHE10], for the case where the oracle provides only partial information about the support set.

Remark 6.11. For the case $T = S_\gamma$ (i.e., if $T$ is a $\gamma$-support set), we have

$$\text{mse}(x, S_\gamma) = \|x_{S_\gamma^c}\|^2 + \frac{\xi \ell}{m - \ell - 1}. \quad (6.19)$$

Similarly, if $T = S_\gamma$, then

$$\text{mse}(x, S_\gamma) = \|x_{S_\gamma^c}\|^2 + \frac{\xi \ell}{m - \ell - 1}, \quad (6.20)$$

where $\ell \triangleq |S_\gamma|$. ♦

Note that, in general, (6.19) depends on $x$ and the choice of $S_\gamma$ but (6.20) depends only on $x$ and $\gamma$. Before drawing a connection between Corollary 6.9 and Theorem 6.10 we introduce a general expression for the MSE behavior:

$$\text{mse}^*(w, \gamma) \triangleq \|x_{S_\gamma^c}\|^2 + O(1/m_n). \quad (6.21)$$

6.2.3 Measurement Rate-MSE Tradeoff

Let $\{x \in \mathbb{R}^n\}$ be a sequence of $k$-sparse vectors with common support $S$ and non-zero entries $w \in \mathbb{R}^k$. Consider the measurement model in (6.2) and assume that $k \leq k_{\text{max}}$ for some fixed, known $k_{\text{max}}$. The concatenation of the results in Sections 6.2.1 and 6.2.2 yields the following characterization of the tradeoff between the measurement rate and the MSE.

**Theorem 6.12.** For any $\gamma \in (0, 1]$, there exists a fixed sequence of estimators $\{\hat{x}\}$ with mean square error

$$\text{mse}(x) = \text{mse}^*(w, \gamma) \quad (6.22)$$

for any $\{x\}$ as long as the measurement rate $r$ satisfies

$$r > r^*(w, \gamma). \quad (6.23)$$

The sequence $\{\hat{x}\}$ depends on the dimensions $(m_n, n)$ but is otherwise independent of $\{x\}$. 
Proof. The proof is provided in Appendix 6.C.

Remark 6.13. The pairs \((r, \text{mse})\) satisfying the conditions in Theorem 6.12 are achievable using a two step approach that consists of (partial) support set recovery followed by the estimation of the detected active components.

We note that, even though the tradeoff refers to the performance that is achievable by first recovering \(\hat{S}_\gamma\), the MSE bound in (6.22) is not equal to the expression in (6.20); the former is worse than the latter. The difference comes mainly from the fact that (6.20) is obtained under the assumption that a correct \(S_\gamma\) is available. In contrast, (6.22) is derived using an estimate \(\hat{S}_\gamma\), which is correct with high probability but not always. The support recovery map has no means of telling whether a specific \(\hat{S}_\gamma\) is correct or not and, thus, it cannot avoid making bad estimates. In some works, this issue is neglected and it is decided that \(\hat{x} = 0\) whenever \(\hat{S}_\gamma\) is incorrect. This simplifies the analysis and yields a more optimistic expression for the MSE. We believe that it is necessary to bound properly the MSE in this cases too. Nonetheless, asymptotically, the difference between (6.22) and (6.20) is negligible, as the gap vanishes with \(m\).

In Figure 6.1, we show typical examples of pairs \((r, \text{mse})\). This corresponds to a realization of \(w\) with \(k = 10\) and i.i.d. \(w_j \sim N(0, 1/\sqrt{k})\). The MSE is normalized by \(\|w\|^2\) so that the values range from 0 to 1. The solid line represents the boundary of the region of pairs \((r, \text{mse})\) in Theorem 6.12. All pairs above this curve are asymptotically achievable by selecting \(\gamma\) appropriately. The staircase-like behavior is due to the discrete nature of the support set: once a certain measurement rate is achieved it is possible to recover a larger fraction of the support set and thus abruptly reduce the MSE. The outer corner points of the region correspond to pairs with

\[
\begin{align*}
    r & = r^*(w, \gamma_i^{opt}), \\
    \text{mse} & = \text{mse}^*(w, \gamma_i^{opt})
\end{align*}
\]

(6.24) (6.25)

for some \(i \in \{1, \ldots, k\}\) with \(\gamma_i^{opt}\) as defined in (6.16). In practice one usually has no knowledge about the structure of \(w\) and thus \(\gamma\) needs to be chosen arbitrarily. To illustrate the performance in this case, we have included the \((r^*, \text{mse}^*)\) pair for several arbitrary choices of \(\gamma\). This figure shows that it is often possible to drastically reduce the measurement rate at a very small loss in terms of MSE. For example, a reduction of the measurement rate from \(r \approx 38\) (corresponding to \(\gamma = 1\), complete support recovery [JKR11]) to \(r \approx 7\) only incurs in a relative MSE of 0.0028 if \(\gamma\) is chosen carefully. Even a blind choice of \(\gamma = 0.99\) yields a reduction to \(r \approx 12\) for the same increase in relative MSE.
6.2 Main Results

Figure 6.1: Measurement rate vs. normalized MSE: all pairs \((r, \text{mse})\) above the solid line are asymptotically achievable. The markers identify the \((r^\ast, \text{mse}^\ast)\) pairs achievable using several different values of \(\gamma\). (© 2013 IEEE. Reused with permission.)

6.2.4 Region of Interest

We observed in Example 6.8 that partial support recovery does not always yield a measurement rate requirement that is lower than that for complete support recovery. In some situations, this is the case even if the parameter \(\gamma\) for partial support recovery is optimally chosen, as we also saw in the example. In this section, we find sufficient conditions that guarantee that partial support recovery is advantageous in terms of required measurement rate.

Intuitively, whenever one of the entries is significantly smaller than the others, partial support recovery has a lower measurement rate requirement (cf. (6.12) and (6.13)). In fact, for any \(w\), if the minimum in the definition of \(r^\ast(w, 1)\) corresponds to \(i = 1\), then discarding the smallest component always lowers the requirement in terms of measurement rate. The following theorem generalizes this observation by establishing a sufficient condition in terms of the values of entries of \(w\). Consider
the following definitions:

\[ c_{i}^{ex} \triangleq c_{i}(w, 1) \]  \hspace{1cm} (6.26)

\[ = \frac{1}{2i} \log_{2} \left( 1 + \text{SNR} \sum_{j=k-i+1}^{k} w_{j}^{2} \right) , \]  \hspace{1cm} (6.27)

\[ r^{ex} \triangleq r^{\star}(w, 1) \]  \hspace{1cm} (6.28)

for \( i \in \{1, \ldots, k\} \), and

\[ c_{i,par,\ell} \triangleq c_{i}(w, \gamma_{\ell}^{opt}) \]  \hspace{1cm} (6.29)

\[ = \frac{1}{2i} \log_{2} \left( 1 + \frac{\text{SNR} \sum_{j=\ell-i+1}^{\ell} w_{j}^{2}}{1 + \text{SNR} \sum_{j=\ell+1}^{k} w_{j}^{2}} \right) , \]  \hspace{1cm} (6.30)

\[ r_{par,\ell} \triangleq r^{\star}(w, \gamma_{\ell}^{opt}) \]  \hspace{1cm} (6.31)

for \( i \in \{1, \ldots, \ell\} \) with \( \ell < k \) and \( \gamma_{\ell}^{opt} \) as in (6.16). Equations (6.26)-(6.28) characterize the requirement for complete support recovery, whereas (6.29)-(6.31) characterize the recovery of the support corresponding to the largest \( \ell \) entries when \( \gamma \) is optimally chosen (see Lemma 6.17 in Appendix 6.F).

\textbf{Theorem 6.14.} If \( c_{i}^{ex} \leq c_{t}^{ex} \) for all \( i \in \{1, \ldots, k-\ell\} \) and \( t \in \{k-\ell+1, \ldots, k\} \) then \( r^{ex} \geq r_{par,\ell} \).

\textbf{Proof.} The proof is provided in Appendix 6.D.1 \hspace{1cm} \blacksquare

In fact, partial support recovery strictly reduces the required measurement rate whenever the inequality in the condition is strict.

\textbf{Corollary 6.15.} If \( c_{i}^{ex} < c_{t}^{ex} \) for all \( i \in \{1, \ldots, k-\ell\} \) and \( t \in \{k-\ell+1, \ldots, k\} \) then \( r^{ex} > r_{par,\ell} \).

\textbf{Proof.} The proof is provided in Appendix 6.D.2 \hspace{1cm} \blacksquare

\subsection*{6.3 Random Signals}

Our derivation of the measurement rate-mean square error tradeoff in the previous section relied on support recovery with a vanishing error probability. This was possible thanks to the deterministic nature of \( x \) (or more precisely of \( w \), see [JKR11]), by increasing \((m_n, n)\) at a measurement rate above \( r^{\star}(w, \gamma) \). Consider now the following measurement model (cf. 6.2)

\[ Y = \Phi X + Z, \]  \hspace{1cm} (6.32)
where $X \in \mathbb{R}^n$ is a $k$-sparse random vector (with fixed $k \in \mathbb{N}$) drawn according to some probability distribution. The measurement matrix $\Phi \in \mathbb{R}^{mn \times n}$ and the noise $Z \in \mathbb{R}^{mn}$ follow Gaussian distributions, as before. Let also $W \in \mathbb{R}^k$ be the random vector that contains the non-zero entries of $X$, sorted in decreasing order of magnitude.

Given (6.32), for any fixed measurement rate $r$ and most distributions of interest for $X$, there is a non-vanishing probability of error in support detection [JKR11]. The reason is that it is always possible that some realizations of $X$ contain small entries that push the measurement requirements beyond $r$. We refer to this type of errors as measurement outages in analogy with the multiple access fading channel. These outages can be of two types: (i) failing to detect some of the entries of the support set and (ii) producing false estimates. In this section, we study the advantages of partial but reliable support set estimation for the case of random $X$ in terms of the measurement outage probability and the average mean square error.

### 6.3.1 Measurement Outage Probability

For any fixed $\gamma \in (0, 1]$, we define the measurement outage probability as

$$P_o \triangleq \Pr (r \leq r^*(W, \gamma)). \quad (6.33)$$

Intuitively, $P_o$ is the probability that the resolution of the tandem of measurement matrix and support recovery map is too low for the current realization of the sparse vector $X$. Indeed for random $W$, $P_o$ is an upper bound on the probability that the output of a good sequence of support recovery maps does not yield a correct $\gamma$-support set (see [JKR11, Theorem 5]) or, more generally, all $\gamma$-support sets, as in Corollary 6.9.

Let $W$ have independent Gaussian distributed entries with zero mean and unit variance (i.e., $W \sim \mathcal{N}(0, I_k)$) and let the support set be randomly distributed over all size-$k$ subsets of $\{1, \ldots, n\}$. In Figure 6.2, we illustrate the measurement outage probability $P_o$ as a function of the SNR for different values of $\gamma$ and $k = 10$. We observe that, for fixed SNR, partial support recovery with a properly chosen $\gamma$ yields significantly lower measurement outage probabilities. Conversely, given a target measurement outage probability $P_o$ it is possible to significantly reduce the SNR if the components in $W$ with less power are of minor concern.

Observe that the optimal choice of $\gamma$ depends on the measurement SNR. Moreover, note that the measurement outage probability curves flatten off at large SNR for all $\gamma < 1$. This is a consequence of the fact that $\gamma$ is set independently of the realization of $W$. Indeed, for any realization $w$ we have that

$$\lim_{\text{SNR} \to \infty} c_i(w, \gamma) = \frac{1}{2i} \log_2 \left( \frac{\sum_{j=\ell-i+1}^{\ell} w_j^2}{(1 - \gamma) \|w\|_2^2} \right). \quad (6.34)$$

Intuitively, by letting SNR $\to \infty$ we are increasing the power contribution of $x$. This includes also the power of the smaller entries in $x$ that are treated as noise.
Figure 6.2: Measurement outage probability $P_o$ as a function of the measurement SNR for partial support recovery of 90%, 95%, 99%, and 99.9% of the signal power, and for complete support recovery of a random Gaussian $X$. (© 2013 IEEE. Reused with permission.)

when performing partial support recovery (i.e., the denominator in (6.12)). Thus, for fixed $\gamma \in (0, 1)$ it is not possible to arbitrarily reduce the required measurement rate simply by letting SNR $\to \infty$. In addition, we observe the following effect. Consider $c_i(w, \gamma)$ for $i = 1$; although we always have that

$$\sum_{j=\ell}^k w_j^2 > (1 - \gamma) \|w\|^2,$$  \hspace{1cm} (6.35)

there exist realizations of $W$ that yield an arbitrarily small gap between both sides in (6.35), given that $\gamma$ is chosen independently. That is, if $\ell$ is the size of the $\gamma$-support sets, the parameter $\gamma$ is too close (from above) to the fraction of power encompassed by the largest $\ell - 1$ components. This effect, illustrated in Figure 6.3, renders support set detection extremely difficult in the presence of noise disturbances. In terms of required measurement rate, $c_1(w, \gamma)$ vanishes and thus
The support recovery map treats the fraction \((1 - \gamma)\) of power, which increases with the SNR, as noise. (© 2013 IEEE. Reused with permission.)

\[ r^\ast(w, \gamma) \text{ becomes arbitrarily large. However, this only becomes noticeable in terms of } P_o \text{ at high SNR. For low and moderate values of the SNR, it is the presence of small entries in } w \text{ that is the dominating source of measurement outages.} \]

### 6.3.2 Mean Square Error

We have seen that partial support recovery increases the reliability of detection by neglecting some of the entries in \(X\). In this section, we evaluate the implications that this has in terms of the estimation MSE when \(X\) is random. For any \(\gamma \in (0, 1]\), consider the expectation of the MSE

\[
\text{mse}_\gamma \triangleq \mathbb{E}_X \{ \text{mse}(X) \} \tag{6.36}
\]

taken over the distribution of \(X\). The results in the previous sections are a valid characterization for those realizations \(x\) whose vector \(w\) falls in \(R \triangleq \{ r > r^\ast(w, \gamma) \}\). Establishing an MSE description for those \(x\) with \(w\) in \(R^c \triangleq \{ r \leq r^\ast(w, \gamma) \}\) would require a complete description of the support recovery map, even when it errs. Such description is beyond the scope of this work. Instead, we establish a bound on the MSE and study its behavior.

As discussed before, a (partial) support recovery map has no means of detecting whether the estimate of the support set is correct or not. Thus, in the event of a support recovery error, the squared error can be arbitrarily large for a given realization of \(x\). Nevertheless, the following theorem establishes that the MSE averaged over the distribution of \(X\) is bounded. Again, we make the additional assumption that the size of the support set is bounded by some arbitrary but fixed and known \(k_{\text{max}}\).
Theorem 6.16. Let $X$ be a $k$-sparse random vector with a distribution that has bounded support. For any measurement rate $r$ there exists a sequence of estimates of $X$ with average MSE that satisfies

$$\limsup_{m_n \to \infty} \text{mse} \gamma \leq \text{mse} \gamma$$

(6.37)

where

$$\text{mse} \gamma \triangleq \int \|x_{S \gamma}\|^2 f(x) dx + \int \|x\|^2 f(x) dx.$$  (6.38)

Proof. The proof is provided in Appendix 6.6. ■

The first integral in (6.38) corresponds to the mean square error incurred in the estimation of those vectors whose $\gamma$-support sets are correctly detected. The second integral in (6.38) corresponds to the estimation error otherwise and amounts to the whole contribution of $x$. That is, even though the estimator cannot know that $S_{\gamma}$ was not correctly detected, the resulting MSE contribution does not exceed $\|x\|^2$ on average.

In Figure 6.4, we show the behavior of $\text{mse} \gamma$ for the setting introduced in Section 6.3.1 (i.e., entries $W \sim N(0, I_k)$ for $k = 10$) for different values of the measurement SNR. We observe the following remarkable facts. First, if $\text{mse} \gamma$ is the metric of interest, insisting on complete support recovery (i.e., $\gamma = 1$) is detrimental; the same performance can be attained at a much lower SNR through partial support recovery by selecting the parameter $\gamma$ properly. Second, observe again that, for $\gamma < 1$, the curves flatten off at high SNR. Two factors contribute to this: the residual error corresponding to the terms that are discarded (i.e., the first integral in (6.38)) and the measurement outage events (i.e., the second integral in (6.38)), which are insensitive to the growth in SNR, as discussed before.

6.4 Summary and Concluding Remarks

In this chapter, we have studied the fundamental tradeoff between the measurement rate and the mean square estimation error in compressive sensing. First, we have derived guarantees for asymptotically reliable recovery of a fraction of the support set. Then, we have derived an expression for the mean square error that is also achievable in the asymptotic regime. The combination of these two results allowed us to determine an achievable measurement rate-MSE region.

Our results show that, in general, methods based on partial support recovery can significantly reduce the measurement rate whenever there are some signal components that are distinctly smaller than the rest. Furthermore, the degradation in
Figure 6.4: Mean square error estimate bound $\text{MSE}_\gamma$ as a function of the measurement SNR for estimation using partial support recovery of 90%, 95%, 99%, and 99.9% of the signal power, and for complete support recovery of a random Gaussian $\mathbf{X}$. (© 2013 IEEE. Reused with permission.)

terms of relative MSE is negligible in these cases. When applied to the estimation of random signals, partial support recovery methods can drastically reduce the occurrence of measurement outages at a minimal cost in terms of MSE. Conversely, large savings in SNR are possible by using partial support recovery-based estimation.
6.A Proofs for Partial Support Recovery

6.A.1 Proof of Theorem 6.3

The proof of Theorem 6.3 uses the random coding method of the proof of [JKR11, Theorem 1]. However, we encounter the following differences:

1. The size of the $\gamma$-support set depends on the value of $\gamma$. Thus, the support recovery map needs to estimate this parameter too. Our approach is to attempt to detect the $\gamma$-support set in an iterative fashion, increasing the size of the candidate set in each iteration. That is, first we assume that the $\gamma$-support set consists of a single element and try to estimate it. If no single-element set passes the test, we increase the size of the candidate set to two and start a new iteration. The process continues until an estimate is found. As we shall show, the estimate is correct with very high probability.

2. Complete support recovery (i.e., [JKR11, Theorem 1]) uses a detection method that only depends on the statistics of the noise and of the entries in the measurement matrix, in particular their second moments. In contrast, partial support recovery needs some knowledge about the power of the signal to be estimated.

3. The support set of a signal is by definition unique. In contrast the $\gamma$-support set need not be unique, thus there might be some ambiguity at the output of the support recovery map. We address this in Corollary 6.9.

Proof of Theorem 6.3. Let $\gamma \in (0, 1]$ and let $\epsilon_n, \epsilon_{1,n}$ be two sequences of positive numbers such that $\epsilon_n \to 0$ and $\epsilon_{1,n} \to 0$ as $n \to \infty$. These two sequences cannot be arbitrary; we will characterize them in the following. A summary of their properties is included at the end of this section.

Support recovery map. Consider the following variation of the support recovery map described in [JKR11]. Given the vector of measurements $Y$:

1. Form an estimate of $\|w\|$ as

$$
\hat{R} = \sqrt{\frac{1}{m_n} \|Y\|^2 - P_Z}{P_\Phi}.
$$

(6.39)

Note that since $x$ is $k$-sparse we have that $\|x\| = \|w\|$ and thus $\hat{R}$ is also an estimate of $\|x\|$.

2. Repeat for $l = 1, \ldots, k_{max}$, in increasing order:
a) Let $B_l(\hat{R} + \epsilon_{1,n})$ be the $l$-dimensional ball of radius $\hat{R} + \epsilon_{1,n}$ with respect to the Frobenius norm:

$$B_l(\hat{R} + \epsilon_{1,n}) \triangleq \{ b \in \mathbb{R}^l : \| b \| \leq \hat{R} + \epsilon_{1,n} \}. \quad (6.40)$$

For the given $\epsilon_{1,n}$, consider the (non-unique) sets of points in $B_l(\hat{R} + \epsilon_{1,n})$ such that $l$-dimensional balls of radius $\epsilon_{1,n}$ centered on these points cover the whole ball $B_l(\hat{R} + \epsilon_{1,n})$. Let $Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n})$ be one such set that has the smallest number of points. That is, for every $b \in B_l(\hat{R} + \epsilon_{1,n})$, there exists at least one $\hat{w} \in Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n})$ such that

$$\| b - \hat{w} \|^2 \leq \epsilon_{1,n} \tag{6.41}$$

and $|Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n})|$ is minimal. Observe that $B_l$ is compact and, thus, a minimal set $Q_l$ exists $\text{[Rud76]}$.

b) Find a set $T \subseteq \{1, \ldots, n\}$ of size $l$ such that

$$\frac{1}{m_n} \left\| Y - \sum_{i=1}^l \hat{w}_t \Phi_i^{(n)} \right\|^2 \leq (1 - \gamma) \hat{R}^2 P + P_Z + \epsilon_n \tag{6.42}$$

for some $\hat{W} = [\hat{w}_1, \ldots, \hat{w}_l]^T \in Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n})$, where $[t_1, \ldots, t_l]$ is the natural ordering of the elements in $T$ and $\Phi_i^{(n)}$ is the column of $\Phi^{(n)}$ in position $t_i$. The process stops when the first set $T$ that satisfies (6.42) is found. This set is the desired estimate (i.e., $\hat{S}_\gamma = T$). If no such set of size $l$ is found, increase $l$ and start again.

For each $l$, $Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n})$ is a set of length-$l$ vectors that are estimates of the first $l$ components of $w$ (i.e., $w_1^T$) in arbitrary order. In Appendix 6.C we have included some basic properties of the sets $Q_l$ that are used in this proof. In particular, note that the size of the sets may grow as $\epsilon_{1,n} \to 0$. As we will see, $\epsilon_{1,n}$ has to decay slowly enough to ensure that this growth is asymptotically negligible (in terms of error probability).

Error analysis. Consider the event

$$\mathcal{E} \triangleq \{ d_\gamma^{(n)}(Y, \Phi^{(n)}) \notin S_\gamma \}. \tag{6.43}$$

With this notation, $\Pr(\mathcal{E})$ corresponds to the average error probability in (6.9). We are interested in showing that this probability can be made arbitrarily small. Recall that $S$ and $S_\gamma$ are the support set and the set of all $\gamma$-support sets of $x$, respectively. Let $I_{\epsilon_{1,n}}$ denote the interval $(-\epsilon_{1,n}, \epsilon_{1,n})$ and consider the following events:

$$\mathcal{E}_W \triangleq \left\{ \hat{R}^2 - \| w \|^2 \in I_{\epsilon_{1,n}} \right\}, \tag{6.44}$$

$$\mathcal{E}_T \triangleq \left\{ \exists \hat{W} \in Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ such that } (6.42) \text{ holds} \right\}. \tag{6.45}$$
defined for any \( \mathcal{T} \subseteq \{1, \ldots, n\} \), and

\[
\mathcal{E}_{aux} \triangleq \mathcal{E}_W \cap \left\{ \frac{\|\Phi^{(n)}\|^2}{nm_n} - P_\Phi \in \mathcal{I}_{\epsilon_1, n} \right\} \cap \left\{ \bigcap_{i \in S} \left\{ \frac{Z^T \Phi_i^{(n)}}{m_n} \in \mathcal{I}_{\epsilon_1, n} \right\} \right\}
\]

\[
\cap \left\{ \frac{\|\mathbf{Z}\|^2}{m_n} - P_Z \in \mathcal{I}_{\epsilon_1, n} \right\} \cap \left\{ \bigcap_{i \in S} \bigcap_{j \in S, j \neq i} \left\{ \frac{1}{m_n} \Phi_i^{(n)} \Phi_j^{(n)} \in \mathcal{I}_{\epsilon_1, n} \right\} \right\}. \tag{6.46}
\]

Using these events, we obtain

\[
\Pr(\mathcal{E}) \leq \Pr \left( \bigcup_{l=1}^{\ell} \left( \bigcup_{\mathcal{T} : |\mathcal{T}| = l, \mathcal{T} \notin S_\gamma} \mathcal{E}_\mathcal{T} \right) \cup \left( \bigcup_{\mathcal{T} \in S_\gamma} \mathcal{E}_\mathcal{T} \right)^c \right) \leq \Pr(\mathcal{E}_{aux}^c) + \Pr \left( \bigcup_{l=1}^{\ell} \left( \bigcup_{\mathcal{T} : |\mathcal{T}| = l, \mathcal{T} \notin S_\gamma} \mathcal{E}_\mathcal{T} \right) \cup \left( \bigcup_{\mathcal{T} \in S_\gamma} \mathcal{E}_\mathcal{T} \right)^c \cap \mathcal{E}_{aux} \right). \tag{6.47}
\]

The first term in (6.48) is an upper bound on the probability that the realization of the measurement matrix or the noise significantly deviate from their expected behavior. In Appendix 6.A.2, we show that, by restricting \( \epsilon_{1, n} \) to decay slow enough, we can ensure that

\[
\Pr(\mathcal{E}_{aux}^c) \leq o(1/m_n). \tag{6.49}
\]

For conciseness, let \( P_1 \) denote the second term in (6.48). Using basic set operations, we see that

\[
P_1 \leq \Pr \left( \bigcap_{\mathcal{T} \in S_\gamma} (\mathcal{E}_\mathcal{T}^c \cap \mathcal{E}_{aux}) \right) + \Pr \left( \bigcup_{l=1}^{\ell} \left( \bigcup_{\mathcal{T} : |\mathcal{T}| = l, \mathcal{T} \notin S_\gamma} \mathcal{E}_\mathcal{T} \right) \cap \mathcal{E}_{aux} \right) \leq \sum_{\mathcal{T} \in S_\gamma} \Pr(\mathcal{E}_\mathcal{T}^c \cap \mathcal{E}_{aux}) + \sum_{l=1}^{\ell} \sum_{\mathcal{T} : |\mathcal{T}| = l, \mathcal{T} \notin S_\gamma} \Pr(\mathcal{E}_\mathcal{T} \cap \mathcal{E}_{aux}). \tag{6.50}
\]

Again for conciseness, let \( P_2 \) and \( P_3 \) denote the first and second terms in (6.51), respectively. Each of the summands in \( P_2 \) is the probability that a \( \gamma \)-support set
ing each of the terms in the double summation defining \( \mathcal{P} \mathcal{R} \) on the deterministic quantity \( \|w\| \).

Note that in event \( E \), we show that, for sufficiently large \( n \), we have

\[
\frac{1}{mn} \left\| Y - \sum_{i=1}^{l} \hat{W}_i \Phi^{(n)}_{t_i} \right\|^2 \leq (1 - \gamma)(\|w\|^2 - \epsilon_{1,n}) P_\Phi + P_Z + \epsilon_n
\]

(6.54)

where the right-hand side is now deterministic. We have that

\[
\Pr(\mathcal{E}_\mathcal{T} | \mathcal{E}_{aux}) = \Pr \left( \exists \hat{W} \in \mathcal{Q}_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. } (6.32) \text{ holds} \big| \mathcal{E}_{aux} \right)
\]

\[
\geq \Pr \left( \exists \hat{W} \in \mathcal{Q}_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. } (6.54) \text{ holds} \big| \mathcal{E}_{aux} \right)
\]

(6.55)

To establish (6.56) we have used that \( \|w\|^2 - \epsilon_{1,n} < \hat{R}^2 \) by the condition \( \mathcal{E}_{aux} \). In Appendix 6.A.3, we show that

\[
\Pr \left( \exists \hat{W} \in \mathcal{Q}_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. } (6.58) \text{ holds} \big| \mathcal{E}_{aux} \right) = 1
\]

(6.57)

if \( \epsilon_n > \delta^{(n)}(\epsilon_{1,n}) \), where \( \delta^{(n)}(\epsilon_{1,n}) \) is a positive function of \( \epsilon_{1,n} \) given in Appendix 6.A.3. Moreover, \( \delta^{(n)}(\epsilon_{1,n}) \to 0 \) as \( \epsilon_{1,n} \to 0 \). Consequently, \( \Pr(\mathcal{E}_\mathcal{T} \cap \mathcal{E}_{aux}) = 0 \) for \( T \in S_\gamma \) and, thus, \( P_2 = 0 \) if \( \epsilon_n > \delta^{(n)}(\epsilon_{1,n}) \).

We encounter a similar problem with the random threshold when upper bounding each of the terms in the double summation defining \( P_3 \). Consider the inequality

\[
\frac{1}{mn} \left\| Y - \sum_{i=1}^{l} \hat{W}_i \Phi^{(n)}_{t_i} \right\|^2 \leq (1 - \gamma)(\|w\|^2 + \epsilon_{1,n}) P_\Phi + P_Z + \epsilon_n.
\]

(6.58)

We have that

\[
\Pr(\mathcal{E}_\mathcal{T} \cap \mathcal{E}_{aux}) \leq \Pr(\mathcal{E}_\mathcal{T} | \mathcal{E}_{aux})
\]

\[
\leq \Pr \left( \exists \hat{W} \in \mathcal{Q}_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. } (6.58) \text{ holds} \big| \mathcal{E}_{aux} \right).
\]

(6.59)

To establish (6.60) we have used that \( \hat{R}^2 < \|w\|^2 + \epsilon_{1,n} \) by the condition \( \mathcal{E}_{aux} \). In Appendix 6.A.4, we show that, for sufficiently large \( n \),

\[
\Pr(\mathcal{E}_\mathcal{T} \cap \mathcal{E}_{aux}) \leq q_l(\epsilon_{1,n})q \left( \frac{P_\Phi \sum_{j=d+1}^{k} w_j^2 + P_Z - \delta_1(\epsilon_{1,n})}{(1 - \gamma)(\|w\|^2 + \epsilon_{1,n}) P_\Phi + P_Z + \epsilon_n} \right)
\]

(6.60)
where \( l = |T| \), \( d \) is the number of correct estimates in the set \( T \) (i.e., \( d = |T \cap S| \)), \( q_l(\epsilon_1, n) \triangleq |Q_l(\|w\| + \delta_2(\epsilon_1, n), \epsilon_1, n)|, l \in \{1, \ldots, \ell\} \), and \( \delta_1(\epsilon_1, n) \) and \( \delta_2(\epsilon_1, n) \) are positive functions of \( \epsilon_1, n \) that tend to 0 as \( \epsilon_1, n \to 0 \).

We emphasize that (6.61) is only valid for sufficiently large \( n \). This is a consequence of the fact that, without knowledge on the structure of \( w \), we cannot choose a fixed threshold \( \epsilon_n \) that discriminates between correct and incorrect estimates of \( \gamma \)-support sets for all possible \( w \) and \( \gamma \). Therefore, the rest of the proof assumes implicitly that \( n \) is sufficiently large. This is of minor concern here, since we are precisely interested in this asymptotic regime.

The exponential bound in (6.61) only depends on \( T \) through \( d \). Thus, we can replace the second summation in the definition of \( P_3 \) that runs over sets with a fixed cardinality by a summation that runs over the parameter \( d \). For given \( k \) and \( n \) there exist

\[
d! \binom{k}{d} \binom{n-k}{l-d} \binom{1}{l-d} \Pr(E_T \cap E_{aux}).
\]

(6.63)

Note that by Lemma 6.18 in Appendix 6.G we have that \( q_l(\epsilon_1, n) \leq q_\ell(\epsilon_1, n) \) for all \( l \in \{1, \ldots, \ell\} \). For the sake of compactness we define

\[
c(\epsilon_1, n) \triangleq (\ell!)^2 \binom{k}{\lfloor \ell/2 \rfloor} q_\ell(\epsilon_1, n) \tag{6.64}
\]

that groups and upper bounds all terms that are independent of \( m_n \) and \( n \). We use this to write

\[
P_3 \leq c(\epsilon_1, n) \sum_{l=1}^{\ell} \sum_{d=0}^{l} \binom{n}{l} \binom{1}{l-d} \Pr(E_T \cap E_{aux}).
\]

(6.65)

Now, using the change of variables \( i = l - d \), and the bound

\[
\binom{n}{i} \leq n^i = 2^i \log_2 n
\]

(6.66)

we obtain

\[
P_3 \leq c(\epsilon_1, n) \sum_{l=1}^{\ell} \sum_{i=0}^{l} 2^i \log_2 n \binom{n}{i} \binom{1}{l-i} \Pr(E_T \cap E_{aux}).
\]

(6.67)
It is easy to see that, in order to enforce $P_3 \to 0$ as $m_n$ increases, it suffices to consider the terms in the sum corresponding to $l = \ell$ only. In addition, note that $c(\epsilon_{1,n})$ has to grow slowly enough so that $P_3$ can be made arbitrarily small. From Lemma 6.18-3 in Appendix 6.G, we know that $c(\epsilon_{1,n})$ grows with $\epsilon_{1,n}$, as

$$(1/\epsilon_{1,n})^\ell,$$  \hspace{1cm} (6.68)

at most. Observe that if $(1/\epsilon_{1,n})$ grows at most polynomially with $m_n$, then

$$\lim_{n \to \infty} \left(1/\epsilon_{1,n}\right)^\ell 2^{-m_n/\log_2 n} = 0$$  \hspace{1cm} (6.69)

for every $a > 0$. The support recovery map does not know the exact value of $\ell$ when choosing $\epsilon_{1,n}$ but it can use the conservative upper bound given by $k_{max}$.

Using this result, together with the fact that $\epsilon_n \to 0$ and $\epsilon_{1,n} \to 0$ as $n \to \infty$, we see that

$$\liminf_{n \to \infty} m_n \frac{\log_2 n}{n} > \frac{1}{2i} \log_2 \left( \frac{P_{\Phi} \sum_{j=\ell-i+1}^k w_j^2 + P_Z}{(1 - \gamma) \|w\|^2 P_{\Phi} + P_Z} \right)^{-1}$$  \hspace{1cm} (6.70)

for all $i \in \{1, \ldots, \ell\}$ is a sufficient condition for recovery of a $\gamma$-support set with $P_3 \to 0$ exponentially as $m_n \to \infty$. Collecting the bounds for $\Pr(\mathcal{E}_{aux}^c)$, $P_2$, and $P_3$, we conclude that $\Pr(\mathcal{E}) \leq o(1/m_n)$. This completes the proof. ■

To summarize the properties of $\epsilon_{1,n}$ and $\epsilon_n$: the sequence $\epsilon_{1,n} > 0$ is chosen to decay to 0 slow enough with $n$ so that: i) $\Pr(\mathcal{E}_{aux}^c) \leq o(1/m_n)$ (see Appendix 6.A.2) and ii) $(\epsilon_{1,n})^{-k_{max}}$ grows at most polynomially with $m_n/\log_2 n$. The sequence $\epsilon_n > 0$ also decays to 0 and must satisfy $\epsilon_n > \delta(n)(\epsilon_{1,n})$ for each $n$, where $\delta(n)(\epsilon_{1,n})$ is given in Appendix 6.A.3.

6.A.2 Proof of (6.49)

To show that $\Pr(\mathcal{E}_{aux}^c) \to 0$ at the desired decay rate with $m_n$, we first write

$$\mathcal{E}_{aux}^c \triangleq \left\{ \hat{R}^2 - \|w\|^2 \notin \mathcal{I}_{\epsilon_{1,n}} \right\} \cup \left\{ \|\Phi \|^2/nm_n - P_{\Phi} \notin \mathcal{I}_{\epsilon_{1,n}} \right\} \cup \left\{ \|Z\|^2/m_n - P_Z \notin \mathcal{I}_{\epsilon_{1,n}} \right\}$$

$$\cup \left\{ \bigcup_{i \in S} \left\{ \frac{Z^T \Phi_i^{(n)}}{m_n} \notin \mathcal{I}_{\epsilon_{1,n}} \right\} \right\} \cup \left\{ \bigcup_{i \in S} \bigcup_{j \notin S, j \neq i} \left\{ \frac{1}{m_n} \Phi_i^{(n)} \Phi_j^{(n)} \notin \mathcal{I}_{\epsilon_{1,n}} \right\} \right\}. $$  \hspace{1cm} (6.71)
First, we upper bound $\Pr(\mathcal{E}_{aux}^c)$ by the sum of the probability of all the events. Observe that the number of events is fixed (i.e., it does not depend on $m$ or $n$). Each of them can be bounded using either Corollary 2.19 or Corollary 2.20. Moreover, the existence of a choice of $\epsilon_{1,n}$ such that

$$\Pr(\mathcal{E}_{aux}^c) \leq o(1/mn)$$

follows by applying either Corollary 2.21 or 2.22 to the largest upper bound for the probability of the events in $\mathcal{E}_{aux}$.

### 6.A.3 Proof of (6.57)

Consider the following sets:

$$\mathcal{D} \triangleq \mathcal{T} \cap \mathcal{S},$$

$$\mathcal{F} \triangleq \mathcal{T} \cap \mathcal{S}^c,$$

$$\mathcal{U} \triangleq \mathcal{S} \setminus \mathcal{D} = \mathcal{S} \setminus \mathcal{T}.$$  

(6.73)  
(6.74)  
(6.75)

That is, $\mathcal{D}$ contains the elements in $\mathcal{T}$ that are correct guesses (i.e., they belong to the support set $\mathcal{S}$), $\mathcal{F}$ contains the rest of elements of $\mathcal{T}$ (i.e., wrong guesses), and $\mathcal{U}$ contains the elements in the support set that are not in $\mathcal{T}$ (i.e., undetected). Observe that, for $\mathcal{T} \in \mathcal{S}_\gamma$, we have $\mathcal{F} = \emptyset$.

Given a set $\mathcal{T}$ and a vector $\hat{w} \in \mathcal{Q}_l$, we construct $\hat{x}$ component-wise as follows

$$\hat{x}_i = \begin{cases} 
\hat{w}_j & \text{if } i = t_j, \\
0 & \text{otherwise},
\end{cases}$$

(6.76)

where $t_1 < t_2 < \ldots$ is the natural ordering of the elements in $\mathcal{T}$. Let

$$C_1 \triangleq \frac{1}{mn} \left\| Y - \sum_{i=1}^l \hat{W}_i \Phi^{(n)}_{i_i} \right\|^2$$

(6.77)

$$= \frac{1}{mn} \left\| \sum_{j \in \mathcal{T}} (x_j - \hat{x}_j) \Phi_j + \sum_{j \in \mathcal{U}} x_j \Phi_j + Z \right\|^2$$

(6.78)

where $\hat{W}$ and $\hat{X}$ are related as in (6.70). Given $\mathcal{E}_{aux}$, we know that $\|w\|^2 < \hat{R}^2 + \epsilon_{1,n}$. Thus, by construction, there will be at least one $\hat{w} \in \mathcal{Q}_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n})$ such that

$$\sum_{j \in \mathcal{T}} (x_j - \hat{x}_j)^2 \leq \epsilon_{1,n},$$

(6.79)
where \( \hat{x} \) is constructed from \( \hat{w} \) and \( T \) as in (6.76). For this \( \hat{x} \), if we expand the squared norm in \( C_1 \) and use the condition \( E_{aux} \), we see that

\[
C_1 \in \left( P_\Phi \sum_{j \in U} x_j^2 + P_Z - \delta_3(\epsilon_{1,n}), P_\Phi \sum_{j \in U} x_j^2 + P_Z + \delta_3(\epsilon_{1,n}) \right)
\]

(6.80)

where the function \( \delta_3(\cdot) \) is

\[
\delta_3(t) \triangleq t \left( 1 + P_\Phi + 4k_{max}^2 \left( \hat{R}^2 + 2t + \sqrt{t} + (1 + \sqrt{t})\sqrt{\hat{R}^2 + t} \right) \right).
\]

(6.81)

Observe that \( \delta_3(\epsilon_{1,n}) \) is a positive function of \( \epsilon_{1,n} > 0 \) such that \( \delta_3(\epsilon_{1,n}) \to 0 \) as \( \epsilon_{1,n} \to 0 \). Moreover, it only depends on parameters that are known by the support recovery map and, thus, can be constructed.

Now, let \( \delta^{(n)}(\epsilon_{1,n}) \triangleq \delta_3(\epsilon_{1,n}) + (1 - \gamma)P_\Phi \epsilon_{1,n} \). Observe that if \( \epsilon_n > \delta^{(n)}(\epsilon_{1,n}) \), then

\[
C_1 \leq P_\Phi \sum_{j \in U} x_j^2 + P_Z + \delta^{(n)}(\epsilon_{1,n})
\]

(6.82)

\[
\leq (1 - \gamma)(\|w\|^2 - \epsilon_{1,n})P_\Phi + P_Z + \epsilon_n
\]

(6.83)

always, because

\[
P_\Phi \sum_{j \in U} x_j^2 < (1 - \gamma)\|w\|^2 P_\Phi
\]

(6.84)

if \( T \in S_\gamma \). The condition \( \epsilon_n > \delta^{(n)}(\epsilon_{1,n}) \) expresses the fact that the sensitivity threshold of the support recovery map has to be large enough to allow for the small deviations in the power of the measurement matrices, noise, etc.

6.A.4 Proof of (6.61)

The proof of (6.61) follows closely the steps in [JKR11, Theorem 1]. We are interested in bounding

\[
\Pr \left( \exists \hat{W} \in Q_l(\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. } (6.58) \text{ holds} \mid E_{aux} \right)
\]

(6.85)

for \( T \notin S_\gamma \) with \( |T| \leq \ell \). Consider the sets \( D, F, \) and \( U \), defined in (6.73), (6.74), (6.75), respectively. Consider also the event

\[
E_{\text{cond}} \triangleq \left( \bigcap_{i \in S} \{ \Phi_i = \phi_i \} \right) \cap \{ Z = z \} \cap E_{aux}
\]

(6.86)

\[2\text{We have used very loose upper bounds to obtain a compact expression for } \delta_3(\cdot) \text{ (e.g., } 1 \leq k_{max}). \text{ Our only goal was to show that such a function can be written explicitly and that it only depends on known parameters. Much tighter but equally valid expressions are possible.} \]
and note that we can write (6.85) as
\[
\int P_4 \underbrace{\Pr \left( \exists \hat{\mathbf{W}} \in Q_l (\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. (6.58) holds} \right)}_{E_{\text{cond}}} \times f(\phi_1, \ldots, \phi_k, z | E_{\text{aux}}) d\phi_1 \ldots d\phi_k dz, \tag{6.87}
\]
where \( f(.) \) is the relevant probability density function. We concentrate on \( P_4 \) and distinguish two cases: i) \( D = T \) and ii) \( D \neq T \), which we analyze in the following. Recall that, for given \( T \), we construct \( \hat{\mathbf{x}} \) from \( \hat{\mathbf{w}} \) as in (6.76) and similarly for \( \hat{\mathbf{W}} \) and \( \hat{\mathbf{X}} \).

**Case 1: \( D = T \)**

In this case, all elements are correct estimates (i.e., \( T \subseteq S \)) but they do not encompass enough power:
\[
\sum_{i \in T} x_i^2 < \gamma \| \mathbf{w} \|^2. \tag{6.88}
\]

First, let
\[
C_3 \triangleq \frac{1}{m n} \left\| \sum_{j \in D} (x_j - \hat{x}_j) \phi_j + \sum_{j \in \mathcal{U}} x_j \phi_j + z \right\|^2. \tag{6.89}
\]
We have that
\[
\Pr(E_T|E_{\text{cond}}) = \Pr \left( \exists \hat{\mathbf{W}} \in Q_l (\hat{R} + \epsilon_{1,n}, \epsilon_{1,n}) \text{ s.t. } C_3 \leq (1 - \gamma) \hat{R}^2 P_\Phi + P_Z + \epsilon_n \left| E_{\text{cond}} \right. \right)
\leq \Pr \left( \exists \hat{\mathbf{W}} \in Q_l (\| \mathbf{w} \| + \delta_2(\epsilon_{1,n}, \epsilon_{1,n})) \text{ s.t. } C_3 \leq (1 - \gamma)(\| \mathbf{w} \|^2 + \epsilon_{1,n}) P_\Phi + P_Z + \epsilon_n \left| E_{\text{cond}} \right. \right)
\leq \sum_{\hat{\mathbf{w}} \in Q_l (\| \mathbf{w} \| + \delta_2(\epsilon_{1,n}, \epsilon_{1,n}))} \Pr \left( C_4 \leq (1 - \gamma)(\| \mathbf{w} \|^2 + \epsilon_{1,n}) P_\Phi + P_Z + \epsilon_n \left| E_{\text{cond}} \right. \right) \tag{6.92}
\]
where \( \delta_2(\epsilon_{1,n}) \triangleq \epsilon_{1,n} + \sqrt{\| \mathbf{w} \|^2 + \epsilon_{1,n} - \| \mathbf{w} \|} \) and
\[
C_4 \triangleq \frac{1}{m n} \left\| \sum_{j \in D} (x_j - \hat{x}_j) \phi_j + \sum_{j \in \mathcal{U}} x_j \phi_j + z \right\|^2. \tag{6.93}
\]
Thus, all terms inside the probability expression for $P_4$ are deterministic quantities. Moreover, if we express the norm in terms of all cross products and use the properties of the event $E_{aux}$, we see that

$$C_4 \in \left( P\sum_{j \in U} x_j^2 + P_Z - \delta_4(\epsilon_1,n), P\sum_{j \in U} x_j^2 + P_Z + \delta_4(\epsilon_1,n) \right)$$

(6.94)

where $\delta_4(\epsilon_1,n)$ is a positive function of $\epsilon_1,n$ such that $\delta_4(\epsilon_1,n) \to 0$ as $\epsilon_1,n \to 0$.

Now, write

$$P\sum_{j \in U} x_j^2 + P_Z - \delta_4(\epsilon_1,n) \leq C_4$$

(6.95)

and note that the second inequality will not be satisfied for sufficiently large $n$ because $\epsilon_1,n \to 0$ and $\epsilon_n \to 0$ and

$$P\sum_{j \in U} x_j^2 > (1 - \gamma) \|w\|^2 P\Phi + P_Z + \epsilon_n$$

(6.96)

Thus, for sufficiently large $n$, we have that $\Pr(E_T|E_{cond}) = 0$ for Case 1.

Case 2: $D \neq T$

In this case some of the elements in $T$ are incorrect estimates. Let

$$C_5 \triangleq \frac{1}{mn} \left\| \sum_{j \in D} (x_j - \hat{x}_j) \phi_j + \sum_{j \in U} x_j \phi_j - \sum_{j \in F} \hat{x}_j \Phi_j + z \right\|^2.$$  

(6.98)

We have that

$$\Pr(E_T|E_{cond}) = \Pr \left( \exists \hat{W} \in Q_l(\hat{R} + \epsilon_1,n, \epsilon_1,n) \text{ s.t. } C_5 \leq (1 - \gamma) \hat{R}^2 P\Phi + P_Z + \epsilon_n \bigg| E_{cond} \right)$$

(6.99)

$$\leq \Pr \left( \exists \hat{W} \in Q_l(\|w\| + \delta_2(\epsilon_1,n), \epsilon_1,n) \text{ s.t. } C_5 \leq (1 - \gamma)(\|w\|^2 + \epsilon_1,n) P\Phi + P_Z + \epsilon_n \bigg| E_{cond} \right)$$

(6.100)

$$\leq \sum_{\hat{w} \in Q_l(\|w\| + \delta_2(\epsilon_1,n), \epsilon_1,n)} \Pr \left( C_6 \leq (1 - \gamma)(\|w\|^2 + \epsilon_1,n) P\Phi + P_Z + \epsilon_n \bigg| E_{cond} \right)$$

(6.101)

where

$$C_6 \triangleq \frac{1}{mn} \left\| \sum_{j \in D} (x_j - \hat{x}_j) \phi_j + \sum_{j \in U} x_j \phi_j - \sum_{j \in F} \hat{x}_j \Phi_j + z \right\|^2.$$  

(6.102)
We now use Lemma 2.23 with
\[ u = \sum_{j \in D} (x_j - \hat{x}_j) \phi_j + \sum_{j \in U} x_j \phi_j + z, \]  
(6.103)
\[ V = - \sum_{j \in F} \hat{x}_j \Phi_j. \]  
(6.104)

For this choice of \( u \), we have
\[ \frac{1}{m_n} \|u\|^2 \in (\alpha - \beta, \alpha + \beta) \]  
(6.105)
with
\[ \alpha = \left( \sum_{j \in D} (x_j - \hat{x}_j)^2 + \sum_{j \in U} x_j^2 \right) P_\Phi + P_Z, \]  
(6.106)
\[ \beta = \delta_1(\epsilon_{1,n}), \]  
(6.107)
\[ \lambda = (1 - \gamma)(\|w\|^2 + \epsilon_{1,n}) P_\Phi + P_Z + \epsilon_n, \]  
(6.108)

where \( \delta_1(\epsilon_{1,n}) \) is a positive function of \( \epsilon_{1,n} \) such that \( \delta_1(\epsilon_{1,n}) \rightarrow 0 \) as \( \epsilon_{1,n} \rightarrow 0 \). Observe also that, for sufficiently large \( n \),
\[ \lambda < \sum_{j \in U} x_j^2 P_\Phi + P_Z - \delta_1(\epsilon_{1,n}) \]  
(6.109)
\[ \leq \alpha - \beta, \]  
(6.110)

as required by Lemma 2.23. We further see that
\[ \alpha \geq P_\Phi \sum_{j \in U} x_j^2 + P_Z \]  
(6.111)
\[ \geq P_\Phi \sum_{j=|D|+1}^{k} w_j^2 + P_Z. \]  
(6.112)

To obtain (6.112) we have used the fact that the sum of the magnitudes of any set of \( k - |D| \) non-zero entries in \( x \) must be no less than the sum of the \( k - |D| \)
smallest-magnitude entries. These correspond to the last $k - |D|$ entries in $w$. Thus,
\[
P_4 \leq \sum_{\hat{w} \in Q_l(\|w\| + \delta_2(\epsilon_{1,n}), \epsilon_{1,n})} 2^{\frac{-m\mu}{2}} \log \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \quad (6.113)
\]
\[
\leq \sum_{\hat{w} \in Q_l(\|w\| + \delta_2(\epsilon_{1,n}), \epsilon_{1,n})} 2^{\frac{-m\mu}{2}} \log \left( \frac{P_{\Phi} \sum_{|D|+1}^k w_j^2 + P_Z - \delta_1(\epsilon_{1,n})}{(1-\gamma)(\|w\|^2 + \epsilon_{1,n})P_{\Phi} + P_Z + \epsilon_n} \right) \quad (6.114)
\]
\[
\leq |Q_l(\|w\| + \delta_2(\epsilon_{1,n}), \epsilon_{1,n})| \quad (6.115)
\]
Finally, note that this bound does not depend on any of the integration variables in (6.87). Combining the results for Cases 1 and 2, (6.61) follows immediately.

### 6.A.5 Proof of Corollary 6.9

**Proof of Corollary 6.9**

Consider a variation of the support recovery method used in the proof of Theorem 6.3 that instead of stopping after finding the first set $\hat{S}_\gamma$ that satisfies (6.42) continues performing the test until all possible sets with the same size have been checked. Let $\{\hat{S}_{\gamma,i}\}$ denote the collection, indexed by $i$, of all the sets that pass the test. The output of the support recovery method is then
\[
\hat{S}_\gamma = \bigcup_i \hat{S}_{\gamma,i}. \quad (6.116)
\]

Let $\tilde{E} = \{\hat{S}_\gamma \neq S_\gamma\}$. Following as in the proof of Theorem 6.3 we have that
\[
\Pr(\tilde{E}) \leq \Pr \left( \bigcup_{l=1}^\ell \left( \bigcup_{T : |T| = l} \mathcal{E}_T \right) \cup \left( \bigcup_{T \in S_\gamma} \mathcal{E}_T^c \right) \right). \quad (6.117)
\]
The bound in (6.51) was purposely derived so that it applies here too. Therefore, the proof readily follows from that of Theorem 6.3. ■

### 6.B Proof of Theorem 6.10

We now study the performance in terms of the MSE of an estimator $\hat{X}$ for $x$ that uses a set $T \subseteq S$. To make explicit the dependency of $\hat{X}$ on $T$ we write $\hat{X}(T)$. Recall that in our model both the noise and the measurement matrix are random. We restrict our attention to the set $\Omega$ consisting of all $\phi$ with full column rank and all $z$. This restriction entails no loss of generality because the set of column-rank deficient $\phi$ has Lebesgue measure zero.
Proof of Theorem 6.10. Let \( T \subseteq S \) and let \( T^c \) identify the rest of the support set (i.e., \( T^c = S \setminus T \)). Given \( T \), with elements \( t_1 < t_2 < \ldots \), we construct the estimate \( \hat{X}(T) \) of \( x \) component-wise as

\[
\hat{X}_i = \begin{cases} 
\hat{X}_{T,j} & \text{for } i = t_j, \\
0 & \text{for } i \notin T 
\end{cases}
\]

(6.118)

for \( i \in \{1, \ldots, n\} \) and some estimator \( \hat{X}_T \) that will be specified later. The MSE of \( \hat{X}(T) \) is defined as

\[
\text{mse}(x, T) \triangleq E_{Z, \Phi} \{ \| x - \hat{X}(T) \|^2 \} 
\]

(6.119)

with the implicit assumption that the expectation, over noise and measurement matrices, is restricted to \( \Omega \). We have that

\[
\text{mse}(x, T) = E_{Z, \Phi} \left\{ \text{mse}(x_T) + \| x_{T^c} \|^2 \right\}
\]

(6.120)

Let \( \text{mse}(x_T) \) denote the first term in (6.120). We see that

\[
\text{mse}(x_T) = E_{\Phi_T} \left\{ E_{Z, \Phi_{T^c} | \Phi_T} \{ \| x_T - \hat{X}_T \|^2 \} \right\}
\]

(6.121)

\[
= E_{\Phi_T} \left\{ E_{Z, \Phi_{T^c}} \{ \| x_T - \hat{X}_T \|^2 \} \right\}.
\]

(6.122)

To obtain (6.122) we have used the fact that the columns of the measurement matrix are independently generated. For the inner expectation expression, we see that conditioned on \( \Phi_T \) the vector of measurements is

\[
Y = \phi_T x_T + \sum_{i \in T^c} x_i \Phi_i + Z.
\]

(6.123)

That is, both the noise and the residual terms (i.e., those in \( \Phi_{T^c} \)) are independent Gaussian random processes. Regarding the residual terms, we see that

\[
E \left\{ \sum_{i \in T^c} \sum_{j \in T^c} x_i x_j \Phi_i \Phi_j^T \right\} = P_\Phi \| x_{T^c} \|^2 I_m
\]

(6.124)

and thus, the covariance matrix of the residual terms plus noise is \( \hat{\xi} I_m \) with \( \hat{\xi} \triangleq P_Z + P_\Phi \| x_{T^c} \|^2 \). Conditioned on \( \Phi_T \), the estimation of \( x_T \) corresponds to a linear estimation problem in white Gaussian noise. For the class of unbiased estimators the MSE satisfies:

\[
\text{mse}(x_T | \Phi_T = \phi_T) \geq \hat{\xi} \text{tr} \left\{ (\phi_T^T \phi_T)^{-1} \right\}
\]

(6.125)
with equality for $\hat{X}_\mathcal{T} = \phi_T^\dagger Y$, where

$$\phi_T^\dagger \triangleq (\phi_T^T \phi_T)^{-1} \phi_T^T.$$  (6.126)

Thus, for this choice of estimate,

$$\text{mse}(\hat{x}_T) = \xi E_{\Phi, \gamma} \left\{ \text{tr} \left\{ (\Phi_T^T \Phi_T)^{-1} \right\} \right\}$$

$$= \frac{\xi l}{m - l - 1}.$$  (6.127)

(6.128)

where $\xi \triangleq \tilde{\xi}/P_\Phi$ and $l \triangleq |\mathcal{T}|$. To obtain the exact result for the expectation in (6.128) we have used Lemma 2.24.

This completes the asymptotic characterization of the MSE averaged over the ensemble of measurement matrices:

$$\text{mse}(x, \gamma) = \|x_{\mathcal{T}c}\|^2 + \frac{\xi l}{m - l - 1}.$$  (6.129)


Proof of Theorem 6.12. Consider an estimator that, given the vector of measurements $Y$, first obtains an estimate $\hat{S}_\gamma$ of the union of all support sets $S_\gamma$ and then uses it to produce an estimate $\hat{X}$ of $x$. We have that:

$$\text{mse}(x, \gamma) = \int_{\Omega} \|x - \hat{x}\|^2 f(\phi, z) d\phi dz$$  (6.130)

where $f(\phi, z)$ is the joint probability density of the noise and the measurement matrix, and $\Omega$ is the set of all $\phi$ with full column rank and all $z$. We split the integral in (6.130) into

$$I_1 \triangleq \int_{E^c} \|x - \hat{x}\|^2 f(\phi, z) d\phi dz,$$

$$I_2 \triangleq \int_{E} \|x - \hat{x}\|^2 f(\phi, z) d\phi dz$$  (6.131)

where $E \triangleq \{(z, \phi) \in \Omega : \hat{S}_\gamma \neq S_\gamma\}$, and $E^c$ is the complement of $E$ with respect to $\Omega$.

The term $I_1$ corresponds to the MSE for those realizations of the noise $z$ and the measurement matrix $\phi$ for which the support recovery map correctly estimates all $\gamma$-support sets, that is, with $\hat{S}_\gamma = S_\gamma$. In this case, the estimate $\hat{x}$ is precisely...
\( \hat{x}(\mathcal{S}_\gamma) \) and thus, we can write

\[
I_1 = \int_{\mathcal{E}} \| x - \hat{x}(\mathcal{S}_\gamma) \|^2 f(\phi, z) d\phi dz
\]

\[
\leq \int_{\Omega} \| x - \hat{x}(\mathcal{S}_\gamma) \|^2 f(\phi, z) d\phi dz
\]

\[
= \text{mse}^*(w, \gamma),
\]

(6.135)

where we have used the result in Theorem 6.10 to establish (6.135). Similarly, the term \( I_2 \) corresponds to the MSE for those \( z \) and \( \phi \) such that \( \hat{S}_\gamma \neq \mathcal{S}_\gamma \). Note that for a given \( \hat{S}_\gamma \), the estimate \( \hat{x} \) sets \( \hat{x}_i = 0 \) for all \( i \notin \hat{S}_\gamma \). Using this we write

\[
I_2 = \int_{\mathcal{E}} \left( \| x_{\mathcal{S}_\gamma^c} \|^2 + \| x_{\hat{S}_\gamma} - \hat{x}_{\hat{S}_\gamma} \|^2 \right) f(\phi, z) d\phi dz
\]

\[
= I_3 + I_4
\]

(6.137)

where

\[
I_3 \triangleq \int_{\mathcal{E}} \| x_{\mathcal{S}_\gamma^c} \|^2 f(\phi, z) d\phi dz,
\]

\[
\leq \| x \|^2 \int_{\mathcal{E}} f(\phi, z) d\phi dz
\]

(6.139)

\[
\leq o(1/m_n).
\]

(6.140)

The bound in (6.140) is due to Corollary 6.9 and relies on the assumption that \( r > r^*(w, \text{mse}) \). Regarding \( I_4 \), using the expressions for the estimate \( \hat{x}_{\hat{S}_\gamma} = \phi_{\hat{S}_\gamma}^\dagger y \), where \( \phi_{\hat{S}_\gamma}^\dagger \) is defined analogous to (6.126), and for the measurement vector \( y = \phi x + z \), we write

\[
I_4 \triangleq \int_{\mathcal{E}} \| x_{\hat{S}_\gamma} - \hat{x}_{\hat{S}_\gamma} \|^2 f(\phi, z) d\phi dz
\]

\[
= \int_{\mathcal{E}} \| x_{\hat{S}_\gamma} - \phi_{\hat{S}_\gamma}^\dagger \phi x - \phi_{\hat{S}_\gamma}^\dagger \phi x \|^2 f(\phi, z) d\phi dz.
\]

(6.142)

Further, using \( \phi x = \phi_{\hat{S}_\gamma} x_{\hat{S}_\gamma} + \phi_{\mathcal{S}_\gamma^c} x_{\mathcal{S}_\gamma^c} \) and the fact that \( \phi_{\hat{S}_\gamma}^\dagger \phi_{\hat{S}_\gamma} = I \) for full
column-rank matrices, we obtain

\[ I_4 = \int_{\mathcal{E}} \| \Phi_{y_0}^\dagger \Phi_{y_0}^c \Phi_{y_0}^{c\dagger} + \Phi_{y_0}^\dagger z \|^2 f(\phi, z) \, d\phi \, dz \]  
(6.143)

\[ \leq \int_{\Omega} \| \Phi_{y_0}^\dagger \Phi_{y_0}^c \Phi_{y_0}^{c\dagger} + \Phi_{y_0}^\dagger z \|^2 f(\phi, z) \, d\phi \, dz \]  
(6.144)

\[ = \int_{\Omega} \left( \| \Phi_{y_0}^\dagger \Phi_{y_0}^c \Phi_{y_0}^{c\dagger} \|^2 + \| \Phi_{y_0}^\dagger z \|^2 \right. 
\left. + 2 \text{tr} \left\{ (\Phi_{y_0}^\dagger \Phi_{y_0}^c \Phi_{y_0}^{c\dagger})^T \Phi_{y_0}^\dagger z \right\} \right) f(\phi, z) \, d\phi \, dz \]  
(6.145)

\[ = I_5 + I_6 + 2I_7, \]  
(6.146)

where

\[ I_5 \overset{\Delta}{=} \mathbb{E}_Z, \Phi \left\{ \| \Phi_{y_0}^\dagger \Phi_{y_0}^c \Phi_{y_0}^{c\dagger} \|^2 \right\}, \]  
(6.147)

\[ I_6 \overset{\Delta}{=} \mathbb{E}_Z, \Phi \left\{ \| \Phi_{y_0}^\dagger Z \|^2 \right\}, \]  
(6.148)

\[ I_7 \overset{\Delta}{=} \mathbb{E}_Z, \Phi \left\{ \text{tr} \left\{ (\Phi_{y_0}^\dagger \Phi_{y_0}^c \Phi_{y_0}^{c\dagger})^T \Phi_{y_0}^\dagger Z \right\} \right\}, \]  
(6.149)

with the implicit assumption that the expectation is taken over \( \Omega \). Noting that \( I_5 \) is independent of \( Z \) and using the submultiplicativity property of the Frobenius norm we see that

\[ I_5 \leq \mathbb{E}_\Phi \left\{ \| \Phi_{y_0}^\dagger \Phi_{y_0}^c \|^2 \right\} \]  
(6.150)

\[ \leq \| \Phi \|^2 \mathbb{E}_\Phi \left\{ \text{tr} \left\{ \Phi_{y_0}^\dagger \mathbb{E}_Z \left\{ \Phi_{y_0}^c \Phi_{y_0}^{c\dagger} \right\} (\Phi_{y_0}^c)^T \right\} \right\} \]  
(6.151)

\[ = l \| \Phi \|^2 P_\Phi \mathbb{E}_{\Phi_{y_0}} \left\{ \text{tr} \left\{ (\Phi_{y_0}^c)^{-1} \right\} \right\}, \]  
(6.152)

where \( l = |\mathcal{S}_{y_0}^c| \). In obtaining (6.151) we have used the bound \( \| \Phi_{y_0}^c \| \leq \| \Phi \| \) for all subsets, and the fact that the submatrices \( \Phi_{y_0}^c \) and \( \Phi_{y_0}^c \) are independently generated in the Gaussian ensemble. Similarly, for \( I_6 \) we can write

\[ I_6 = \mathbb{E}_\Phi \left\{ \text{tr} \left\{ \Phi_{y_0}^\dagger \mathbb{E}_Z \left\{ ZZ^T \right\} (\Phi_{y_0}^c)^T \right\} \right\} \]  
(6.153)

\[ = P_Z \mathbb{E}_{\Phi_{y_0}} \left\{ \text{tr} \left\{ (\Phi_{y_0}^c)^{-1} \right\} \right\}. \]  
(6.154)

As in Section 6.B, using Lemma 2.24 we see that

\[ \mathbb{E}_{\Phi_{y_0}} \left\{ \text{tr} \left\{ (\Phi_{y_0}^c)^{-1} \right\} \right\} = \frac{l}{P_\Phi (m_n - l - 1)}, \]  
(6.155)
Since the cardinality of the support set is upper bounded we have that \( l \leq k_{\max} \) and thus we obtain \( I_5, I_6 \leq O(1/m_n) \). Finally, using the independence of the zero-mean noise, we readily see that \( I_7 = 0 \). Grouping all terms, we conclude that the MSE performance averaged over the ensemble of estimates is \( \text{mse}(x, \gamma) = \text{mse}^*(w, \gamma) \). ■

6.D Proofs for the Region of Interest

6.D.1 Proof of Theorem 6.14

Proof of Theorem 6.14 Note that by assumption we have that

\[
\frac{1}{r_{\text{ex}}} = \min_{i \in \{1, \ldots, k\}} \{c_{i}^{\text{ex}}\} 
= \min_{i \in \{1, \ldots, k-\ell\}} \{c_{i}^{\text{ex}}\} 
\leq c_{k-\ell}^{\text{ex}} \tag{6.157}
\leq c_{t}^{\text{ex}} \tag{6.158}
\]

for any \( t \in \{k - \ell + 1, \ldots, k\} \). Note also that \( c_{k-\ell}^{\text{ex}} \leq c_{t}^{\text{ex}} \) implies

\[
\left(1 + \text{SNR} \sum_{j=k-\ell+1}^{k} w_{j}^2\right)^{\frac{t}{t+\ell-k}} \leq \left(1 + \text{SNR} \sum_{j=k-\ell+1}^{k} w_{j}^2\right)^{\frac{k-\ell}{t+\ell-k}}. \tag{6.160}
\]

We use this to write

\[
c_{k-\ell}^{\text{ex}} \leq c_{t}^{\text{ex}} \tag{6.161}
= \frac{1}{2t} \log_{2} \left( \left(1 + \text{SNR} \sum_{j=k-\ell+1}^{k} w_{j}^2\right)^{\frac{k-\ell}{t+\ell-k}} \right) \tag{6.162}
\leq \frac{1}{2t} \log_{2} \left( \frac{1 + \text{SNR} \sum_{j=k-\ell+1}^{k} w_{j}^2}{1 + \text{SNR} \sum_{j=\ell+1}^{k} w_{j}^2} \right) \tag{6.163}
\leq \frac{1}{2(t+\ell-k)} \log_{2} \left( \frac{1 + \text{SNR} \sum_{j=k-\ell+1}^{\ell} w_{j}^2}{1 + \text{SNR} \sum_{j=\ell+1}^{k} w_{j}^2} \right) \tag{6.164}
= c_{i}^{\text{par},\ell}. \tag{6.165}
\]
To obtain (6.163) we have used inequality (6.160). In (6.165) we have used the change of variables $i = t + \ell - k$. Thus, for $i \in \{1, \ldots, \ell\}$, we have established that

$$c_{k-\ell}^{ex} \leq c_i^{par,\ell}.$$  \hfill (6.166)

So, in particular,

$$\frac{1}{r^{ex}} \leq c_{k-\ell}^{ex} \leq \min_i c_i^{par,\ell} \leq \frac{1}{r^{par,\ell}}.$$  \hfill (6.168)

That is,

$$r^{ex} \geq r^{par,\ell},$$  \hfill (6.170)

as we wanted to prove. \hfill \blacksquare

### 6.D.2 Proof of Corollary 6.15

**Proof of Corollary 6.15**. The proof of the corollary follows by noting that with the stronger assumption, the inequality in (6.160) is strict. This in turn implies that (6.163) is also a strict inequality. This establishes the claim. \hfill \blacksquare

### 6.E Proof of Theorem 6.16

Before proving Theorem 6.16, note that the result of Theorem 6.3 and Corollary 6.9 is easily extended to the case of deterministic $w$ and random support set $\mathcal{S}$. In particular, this holds if the support set is chosen uniformly at random from the set of all possible size-$k$ subsets of $\{1, \ldots, n\}$. To prove this, observe that, by the symmetry of the generation of the measurement matrices and the support recovery maps, we have that:

$$\mathbb{E}_{\Phi^{(n)}},\mathcal{S}\left\{P_e(X, \Phi^{(n)}, d^{(n)}_\gamma)\right\} = \mathbb{E}_{\Phi^{(n)}},\mathcal{S}\left\{P_e(x, \Phi^{(n)}, d^{(n)}_\gamma)\right\}.$$  \hfill (6.171)

That is, $\Pr(\mathcal{E}) = \Pr(\mathcal{E}|\mathcal{S})$, where the right-hand side corresponds to the probability of the event in (6.43), which was analyzed in Section 6.A.1.

**Proof of Theorem 6.16**. Consider any fixed $\gamma \in (0, 1]$ and $r > 0$ and let $\Omega$ be the set of all $\phi$ with full column rank and all $z$. Consider also

$$\mathcal{R} \triangleq \{x : x \leq r^*(w, \gamma)\},$$  \hfill (6.172)

$$A_1 = \{(x, \phi, z) : x \in \mathcal{R}^c, (\phi, z) \in \Omega, \hat{\mathcal{S}}_\gamma = \mathcal{S}_\gamma\},$$  \hfill (6.173)

$$A_2 = \{(x, \phi, z) : x \in \mathcal{R}^c, (\phi, z) \in \Omega, \hat{\mathcal{S}}_\gamma \neq \mathcal{S}_\gamma\}.$$  \hfill (6.174)
where $R^c$ is the complement of $R$. The MSE averaged over the distribution of $X$, which is defined on $X$, is

$$\text{mse}_\gamma \triangleq \int_X \int_\Omega \|x - \hat{x}\|^2 f(x, \phi, z)dx d\phi dz$$  \hspace{1cm} (6.175)

$$= T_1 + T_2 + T_3$$  \hspace{1cm} (6.176)

where $f(x, \phi, z)$ is the joint probability density of the signal, the measurement matrix, and the noise, and

$$T_1 \triangleq \int_{A_1} \|x - \hat{x}\|^2 f(x, \phi, z)dx d\phi dz,$$  \hspace{1cm} (6.177)

$$T_2 \triangleq \int_{A_2} \|x - \hat{x}\|^2 f(x, \phi, z)dx d\phi dz,$$  \hspace{1cm} (6.178)

$$T_3 \triangleq \int_{R^c} \int_\Omega \|x - \hat{x}\|^2 f(x, \phi, z)dx d\phi dz.$$  \hspace{1cm} (6.179)

For $T_1$, following similar steps to (6.133)-(6.135) and noting that the support of $X$ is bounded, we obtain

$$T_1 \leq \int_{R^c} \|x\|^2 \Pr(\hat{S}_\gamma \neq S_\gamma | x) f(x)dx$$  \hspace{1cm} (6.180)

$$= \int_{R^c} \|x_{S_\gamma}\|^2 f(x)dx + O(1/n).$$  \hspace{1cm} (6.181)

Similarly to (6.136)-(6.139), for $T_2$ we obtain

$$T_2 \leq \int_{R^c} \|x\|^2 \Pr(\hat{S}_\gamma \neq S_\gamma | x) f(x)dx$$

$$+ \int_{A_2} \|x_{S_\gamma} - \hat{x}_{S_\gamma}\|^2 f(x, \phi, z)dx d\phi dz. \hspace{1cm} (6.182)$$

Using the reverse Fatou lemma [Wil91] and applying the proof of Corollary 6.9 to the integrand (recall that the support of $X$ is bounded) we see that

$$\limsup_{n \to \infty} \int_{R^c} \|x\|^2 \Pr(\hat{S}_\gamma \neq S_\gamma | x) f(x)dx = 0.$$  \hspace{1cm} (6.183)

Similarly,

$$T_3 \leq \int_{R^c} \|x\|^2 f(x)dx$$

$$+ \int_{R^c} \int_\Omega \|x_{\hat{S}_\gamma} - \hat{x}_{\hat{S}_\gamma}\|^2 f(x, \phi, z)dx d\phi dz. \hspace{1cm} (6.184)$$
Finally, proceeding as for the term $I_4$ in the proof of Theorem 6.12, we bound the sum of $T_4$ (in (6.182)) and $T_5$ (in (6.184))

$$T_4 + T_5 \leq \int_X \int_{\Omega} \|x_{\hat{S}_\gamma} - \hat{x}_{\hat{S}_\gamma}\|^2 f(x, \phi, z) dx d\phi dz$$

$$= O(1/m_n).$$

Thus, collecting the terms and taking the limit (recall that $m_n$ grows with $n$) we establish the desired result

$$\limsup_{n \to \infty} \text{mse}_{\gamma} \leq \int_{\mathcal{R}^c} \|x_{\hat{S}_\gamma}\|^2 f(x) dx + \int_{\mathcal{R}} \|x\|^2 f(x) dx.$$  

6.F Optimal Parameter for Partial Support Recovery

Lemma 6.17. For $i \in \{1, \ldots, k\}$ let

$$\gamma_i \triangleq \sum_{j=1}^{i} \frac{w_j^2}{\|w\|^2}.$$  

Given the size of the $\gamma$-support set $\ell$, the value of $\gamma$ that results in the lowest measurement rate requirement is $\gamma_\ell$.

Proof. Given $\ell$, $\gamma$ must lie in the interval $[\gamma_\ell, \gamma_{\ell+1})$. Note that $c_i(w, \gamma)$ is strictly decreasing in $\gamma$ for all $i \in \{1, \ldots, \ell\}$. Thus, $\gamma = \gamma_\ell$ results in the lowest $r^*(w, \gamma)$.

6.G Properties of the Covering Sets

Lemma 6.18. Consider the sets $Q_l(r, \zeta)$ introduced in Section 6.A.1. These sets satisfy the following properties:

1. $Q_l(r, \zeta)$ is non-decreasing in $r$ for fixed $l$ and $\zeta$.
2. $Q_l(r, \zeta)$ is non-decreasing in $l$ for fixed $r$ and $\zeta$.
3. $|Q_l(r, \zeta)|$ behaves as $O(\zeta^{-l})$ with $\zeta$ for fixed $l$ and $r$.

Proof. Property 1 was proved in [JKR11]. The proof of Property 2 is straightforward. The proof of Property 3 follows from basic results on coverings [SC99].
Conclusion

7.1 Summary

In this thesis, we have studied the problems of coordination and compression in communication networks. For both problems, we have analyzed theoretical aspects and discussed practical considerations. We summarize our results in the following.

• In Chapter 3 we constructed polar codes for coordination for a variety of network topologies. These codes combined elements of source coding with elements of channel coding. The former were used for producing the coordinated actions whereas the latter were used to minimize the rate of communication. We expect that these two blocks together with the basic constructions presented in this thesis play an important role in designs for coordination in general networks. Unfortunately, it is unlikely that the combination of these two blocks alone will solve the general problem.

• In Chapter 4 we constructed polar codes for compress-and-forward relaying. Our code design was based on the same principles as those for the codes used for coordination. Although their performance for finite blocks is not good enough for practical purposes, polar codes have a simple structure that allows for easy identification of the critical aspects of code designs.

• In Chapter 5 we introduced a model to study the fundamental tradeoffs between coordination, communication, and interference in communication networks. The model was general, in the sense that it did not rely on a specific type of channel or on a particular transceiver architecture. We obtained a complete characterization of a simple scenario and a partial one for a multi-user setting. Our results show that the efficiency of multi-user communications can be improved by means of simple coordination mechanisms.
• Finally, in Chapter 6 we studied partial support recovery methods for compressive sensing. Building on a multiple-access channel analogy, we derived sufficient conditions for partial but reliable detection of the support set in terms of the measurement rate. We used this to establish a characterization of the tradeoff between measurement rate and mean square estimation error. We also demonstrated the potentials of partial support recovery methods to avoid measurement outages.

7.2 Future Work

For each of the answers found in this thesis a new set of questions has emerged. We summarize some of the most relevant ones in the following:

• The polar codes presented in this thesis are another step forward to obtain constructions that achieve the information-theoretic limits in communication networks. Unfortunately, their finite block performance is far from satisfactory. Recently, there have been some important advances in polar coding in the non-asymptotic regime [TV11]. It would be interesting to apply these new constructions to the scenarios studied in Chapters 3 and 4.

• The study in Chapter 5 can be extended in many different directions. First, it would be desirable to have a versatile tool for establishing outer bounds for capacity regions. That is, a communication-interference extension of the cut-set bound. Perhaps more importantly, it would be very interesting to extend the results to the continuous case. This is certainly not straightforward, given that we used a characterization of the interference that relied on its discrete nature. It is also possible to consider more general topologies than the ones in Chapter 5. In view of the difficulties in simpler scenarios, it seems unlikely that we will be able to characterize them completely. However, an analysis in terms of types could reveal interesting insights into the tradeoffs in the network.

• Finally, further research is also possible on partial support recovery methods for compressive sensing presented in Chapter 6. First of all, we have only been able to establish the sufficiency of some conditions for partial support recovery. It would be very interesting to determine whether these are necessary conditions as well and, in case they are not, to strengthen the result. In addition, our results on the measurement rate-mean square error tradeoff can be generalized by considering ensembles of sparse vectors with some prior distribution. Note that, to obtain such generalization, a careful redefinition of the problem might be necessary.
Bibliography


