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Fast Argument Quantization

John-Olof Nilsson and Peter Händel

A quantized low-resolution argument of a complex number or two-dimensional vector is required in many digital signal processing algorithms. Examples include: amplitude and phase-shift keying (APSK) code demodulation for which it may be used to evaluate the Voronoi diagram; low-level processing for many computer vision methods which exploit histograms of gradient sample arguments, e.g. scale-invariant-feature-transform (SIFT) and histogram-of-oriented-gradients (HOG); and phase tracking/frequency estimation for which it may be used as a low-cost phase approximation. Often, such quantized arguments will have to be computed a large number of times and under real-time constraints. Therefore, efficient solutions to these calculations are of interest.

The argument quantization can be viewed as two-dimensional data binning in equidistant sectors as illustrated in Figure 1. Consequently, we present three efficient solutions to this data binning problem. First, a straight-forward solution explicitly calculating the complex/vector argument is presented. Secondly, we present a solution based on projection and bisection which can be implemented without any division and with only integer data representation. Thirdly, we present a phase rotation based solution suitable for vectorization/parallelization. The solutions we refer to as *binning by argument* (BBA), *binning by projection* (BBP), and *binning by rotation* (BBR). We also present ways of finding the first 3 bits, i.e. the half-plane, quad-

rant, and octant sector, using bit-manipulations. Finally, a table of types and numbers of required operations for the implementation of each solution is presented. Further details and extensions of the presented methods can be found in the report [1].

Fig. 1: Sample x falls in sector 1.

Sector binning

The data binning in sectors, or *sector binning*, entails finding the sector that a complex number $x = x_0 + jx_1$ or two-dimensional vector $x = (x_0, x_1)$ occupies. The problem is illustrated in Figure 1 for 16 equidistant sector bins, corresponding to a 4-bit argument quantization. Formally, the sector binning can be described by the mapping

$$g : x \mapsto n$$

$$\text{s.t. } 2\pi \frac{n}{N} \leq \arg(x) < 2\pi \frac{n+1}{N} \quad (1)$$

where n is the sector bin number, N is the total number of sectors, $\arg(\cdot)$ is the complex/vector argument defined on $[0, 2\pi)$, and $2\pi \frac{n}{N}$ and $2\pi \frac{n+1}{N}$ are the delimiting arguments of the sector n . The sector binning (1) may also be extended to the non-equidistant case with arbitrary delimiting arguments [1].

Binning by argument

The straight-forward way of implementing the sector binning, as described by (1), is by explicitly calculating the argument $\arg(x)$. By simple manipulations, (1) yields

$$n = \lfloor \frac{N}{2\pi} \arg(x) \rfloor \quad (2)$$

where $\lfloor \cdot \rfloor$ denotes the floor-operation. Consequently, by calculating $\arg(x)$, the sector n can easily be found. Now, $\arg(x) = \text{atan2}(x_1, x_0)$, which is $\text{atan}(\frac{x_1}{x_0})$ mapped onto the four quadrants. Unfortunately, the evaluation of $\text{atan}(\cdot)$ can incur a significant computational cost.

Multiple fast approximations of $\text{atan}(\cdot)$ and $\text{atan2}(\cdot, \cdot)$ are available in the literature [2][3] and have previously been presented in this column [4][5]. For the sector binning we suggest using an $\text{atan2}(\cdot, \cdot)$ implementation based on the $\text{atan}(\cdot)$ approximation number 7 from [5],

$$\text{atan}(z) \approx \frac{\pi}{4} z + 0.273 z (1 - |z|).$$

Together with the relation [6]

$$\text{atan}(\frac{x_1}{x_0}) = \frac{\pi}{4} + \text{atan}(z)$$

where

$$z = \frac{|x_1| - |x_0|}{|x_1| + |x_0|},$$

it gives

$$\frac{N}{2\pi} \text{atan}(\frac{x_1}{x_0}) \approx \frac{N}{8} + \frac{N}{8} z + \frac{0.273 N}{2\pi} z (1 - |z|). \quad (3)$$

This approximation lets us exploit the factor $\frac{N}{2\pi}$ to simplify the implementation and has an accuracy to computational cost trade-off suitable for up to some hundred equidistant sector bins (maximum error corresponding to 0.22°).

The approximation (3) gives the scaled argument of x in the first quadrant. Now, we want to extend it to all four quadrants. This can be done with a series of bit manipulations and an addition. Let the symbols $\oplus$ and $\wedge$ denote the logical XOR and AND operations, respectively. Let the single bit s indicate the even/odd status of the quadrant of x , i.e.

$$s = \text{sb}(x_0) \oplus \text{sb}(x_1)$$

where $\text{sb}(\cdot)$ denotes the sign-bit (1 if the number is negative and 0 if it is positive). Let r be an integer number with the first bit equal to $\text{sb}(x_0)$, the second bit equal to $\text{sb}(x_1) \wedge s$, and the remaining bits zero (i.e. $r = 0, 1, 2$). Further, define the sign-bit-copy function

$$\text{sbc}(\alpha, \beta) = \begin{cases} \beta & \text{if } \alpha = 0, \\ -\beta & \text{if } \alpha = 1, \end{cases}$$

where α is a single bit and β is a positive number. Then (3) can be mapped onto the four quadrants by

$$\frac{N}{2\pi} \text{atan2}(x_1, x_0) = \frac{N}{2} r + \text{sbc}(s, \frac{N}{2\pi} \text{atan}(|\frac{x_1}{x_0}|)).$$

Apart from bit-manipulations, this requires 1 division, 2-3 multiplications, and 6 additions; and no conditional branches are required. In case N is a power of 2, the multiplications with $\frac{N}{2}$ and $\frac{N}{8}$ can be implemented with left shifts. In case N is not a power of 2, $\frac{N}{2}r$ can preferably be implemented with a memory access by r in the 3 element look-up table $[0, \frac{N}{2}, N]$.

With this approximation of $\frac{N}{2\pi} \text{atan2}(x_1, x_0)$, the relation (2) gives the sector bin n . We refer to this as binning by argument or BBA. While simple and straight-forward, this solution to the sector binning unfortunately has a number of undesirable properties. First of all, it requires a division operation, which is slow on many general purpose processing platforms and requires a large number of gates if implemented in hardware. Secondly, it requires

non-integer data types, which obviously complicates the implementation and adds numerical approximation issues. Thirdly, it exploits the function approximation (3), which gives systematic errors in the binning, i.e. the *de facto* delimiting arguments will be slightly off from those in (1). (See [5] for a detailed error plot.) These properties hold for any implementation in which the argument is explicitly calculated. Therefore, for many platforms and applications, alternatives are desirable.

Binning by projection

Implementing the sector binning by computing the argument might seem a little backward. A highly accurate argument is calculated while all we need to know is between which two lines in Figure 1 that x falls. Obviously, the argument is not needed to find that out.

If p is a vector, such that $\arg(p) = 2\pi \frac{n}{N}$ and q is a vector rotated counter clock-wise by $\pi/2$ relative p . Then, as shown for two samples x and x' in Figure 2, if the argument difference between a sample and p lies between π and $-\pi$, the sign (bit) of the projection of the sample on q can be used to determine whether its argument is smaller or larger than the argument $2\pi \frac{n}{N}$ of p .

Fig. 2: Projection of x and x' on q .

The sign of the projection is the same as the sign of the scalar product between x and q . By defining vectors q_n such that

$$\arg(q_n) \stackrel{2\pi}{\equiv} 2\pi \frac{n}{N} + \frac{1}{2}\pi \quad (4)$$

where $\stackrel{2\pi}{\equiv}$ denotes equality modulus 2π , the sector binning (1) can equiv-

alently be written

$$\begin{aligned} g : x &\mapsto n \\ \text{s.t. } x \cdot q_n &\geq 0 > x \cdot q_{n+1}, \\ -\pi &< 2\pi \frac{n}{N} - \arg(x) \leq \pi, \\ -\pi &< 2\pi \frac{n+1}{N} - \arg(x) \leq \pi \end{aligned} \quad (5)$$

where $\cdot$ denote the scalar product. This projection approach partially resembles/generalizes the ideas presented in [7][8].

If N is a power of 2 ($N = 2^M$), then (5) is easily solved by bisection. Let n be an integer variable initialized to 0 and let a leading superscript $i(\cdot)$ indicate the i th bit of the variable. Then the loop

$$\begin{aligned} \text{loop } i &= M - 1 : 1 \\ i_n &= 1 \\ d &= x \cdot q_n \\ i_n &= i_n \oplus \text{sb}(d) \end{aligned} \quad (6)$$

gives the solution to the sector binning (5). This is what we refer to as binning by projection or BBP. Note that here $=$ has been used to denote assignment and that the loop goes from higher to lower values of i . The projections (inner products) in the loop is illustrated in Figure 3 where q_n of the ℓ th loop is denoted with q_{n_ℓ} . Each projection on q_{n_ℓ} successively halves the search space (with middle $\arg(q_{n_\ell}) - \pi/2$) of sectors in which x can be found. The BBP solution can also be extended to the case when N is not a power of 2 and the sector bins are not equidistant [1].

Fig. 3: Projections of x on q_{n_ℓ} .

The beauty of the projection and bisection is that we may choose

q_n arbitrarily, as long as (4) is satisfied. Therefore, q_n may be chosen such that the implementation of the scalar product $x \cdot q_n$ is simplified. This means that one or both of the multiplications in $x \cdot q_n$ can be avoided. Also, the solution only requires additions, multiplications, and bit-manipulations. Therefore, no non-integer data types are required. Obviously BBP does not suffer from the undesirable properties of BBA. However, instead it has a computational cost which grows as the logarithm of N ($M-1$ loop iterations) and the bisection requires data dependent memory accesses in a small q_n look-up table. The latter can be undesirable since the memory accesses may be slow when compared to arithmetic operations and they cannot easily be vectorized.

Binning by rotation

In case data dependent memory accesses are undesirable, the sector binning can be solved in a recursive manner without any memory accesses by manipulating x itself. Treating x as a complex number, means its argument can be manipulated by multiplying it with another complex number. This will rotate x with the argument of the complex number. Rotating x by π and the sign of x_1 will tell us in which half plane x is in. Rotating the product with $\pi/2$ towards zero argument will then result in the sign of x_1 giving us the quadrant of x . Continuing rotating with $\pi/4$ gives the octant and so forth. Along these lines and with the rotating complex numbers c_i , with real parts a_i and imaginary parts b_i , chosen such that

$$\arg(c_i) = \pi 2^{-i+(M-1)}; \quad (7)$$

i.e. $\arg(c_{M-1}) = \pi$, $\arg(c_{M-2}) = \pi/2$, $\arg(c_{M-3}) = \pi/4$, and so forth; the sector bin number n (initialized to 0) of x can recursively be calculated with

$$\begin{aligned} \text{loop } i &= M-1 : 1 \\ i_n &= 1 \\ b_i &= -\text{sign}(x_1) b_i \quad (8) \\ x &= x \times c_i \\ i_n &= i_n \oplus \text{sb}(x_1) \end{aligned}$$

where $\times$ denotes the complex multiplication. We refer to this as binning by rotation or BBR. The switching of the sign of b_i ensures that the rotation of x is always towards zero argument. Note that $-\text{sign}(x_1) b_i$ can be implemented with $\text{sbc}(\neg \text{sb}(x_1), b_i)$, where $\neg$ denotes the NOT operation. Consequently, it only requires bit-manipulations. The manipulation of x by the loop is illustrated in Figure 4 where c_{i_ℓ} is used to denote the rotating complex number of the ℓ th loop.

Fig. 4: Rotations of x by c_i .

Each rotation halves the number of potential sectors and therefore, the loop in essence describes another form of bisection. The number of required arithmetic operations for evaluating (8) is larger when compared to (6). However, the data dependent memory accesses have been avoided. Unfortunately, this solution cannot be extended to non-equidistant and non-power-of-2 number of sector bins. Similar to the choices of q_n in BBP, the primitive rotations c_i can be chosen to simplify the implementation of (8), letting us avoid multiplication operations in $x \times c_i$. Also, only bit-manipulations, additions, and multiplications are required; and therefore, integer data types can be used throughout.

Finding half-plane, quadrant, and octant

The first three loops in (6) correspond to finding the half-plane, quadrant, and octant sector which x occupies. These first three loops can be simplified. The first loop/bit (the half plane) is simply given by

the sign bit of x_1

$$M-1n = \text{sb}(x_1). \quad (9)$$

The second loop/bit (the quadrant) is given by

$$M-2n = M-1n \oplus \text{sb}(x_0). \quad (10)$$

Finally, the third loop/bit (octant) is given by

$$M-3n = M-2n \oplus \text{sb}(|x_0| - |x_1|). \quad (11)$$

For subsequent loops, no straightforward simplifications compared to (6) can be found. Using (9)-(11), instead of the first three bisection loops, will significantly reduce the cost of the bisection. Also, for applications in which binning in only 4 or 8 sectors is desired ($M = 2$ and $M = 3$), this will directly give the sector bin number without any multiplications.

Expression (9) and (10) can also be used to replace the first two loops of (8). However, for (8), x needs to be rotated accordingly which can be achieved with the initial reassignment

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} |x_1| \\ -\text{sign}(x_1) x_0 \end{bmatrix}.$$

Computational cost

Apart from the $\text{atan}(\cdot)$ approximation and the non-integer variable and constant representations in BBA and the necessary numerical approximations in the relations (4) and (7) when selecting q_n and c_i for BBP and BBR, respectively, all three solutions, BBA, BBP, and BBR, have the same logical result. Therefore, their performance is primarily determined by their computational requirements and costs.

The exact numbers and types of required operations for the different solutions are dependent on many implementation choices. However, in Table 1, a rough picture of the approximative numbers and types of operations required for BBA, BBP, and BBR is given. The operations have been divided into: additions and subtractions (add/sub), multiplications (mul), divisions (div), absolute values (abs), memory accesses (mov), bit-shifts (shift), bit-assignments (bit-assign), bit-wise XOR/AND/NOT

(bit-op), and other miscellaneous operations/functions (other). Further, it has been assumed that the number of sector bins N is 2^M (a power of 2); the half-plane, quadrant, and octant sector calculations are used; and for both BBP and BBR, one and two multiplications in $x = x \cdot q_i$ and $x = x \times c_i$, respectively, are replaced with bit-shifting and signcopy. Finally, the loops are assumed unrolled, and therefore, no operations for iteration variable manipulation are counted.

Discussion and conclusions

We have presented three different solutions to sector binning of a complex number or two-dimensional vector. This binning can be viewed as a quantization of the related argument. Table 1 shows the strengths and weaknesses of each solution. BBA requires division and non-integer data types, but the computational cost is independent of the number of sector bins. BBP requires the least number of operations for a small number of sectors, but the computational cost scales with the logarithm of N and data dependent memory accesses are required. BBR avoids the data dependent memory accesses at the

cost of a somewhat higher number of required operations. Both BBP and BBR can be implemented with only integer data types.

Which solution to favor is a matter of the intended platform, the number of sector bins/quantization level, and the application. For embedded platforms, a moderate number of bins, and sequential processing typical for communication applications, we would generally favor BBP. However, for processing of large data chunks (e.g. images), opting for vectorization, and for platforms with readily available multipliers, we would favor BBR. Finally, for platforms with a fast division operation and in the case of a large non-power-of-2 number of sector bins or floating-point data, we would favor BBA.

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	BBA	BBP	BBR
add/sub	6	$(M - 3) + 1$	$2(M - 2)$
mul	2	$(M - 3)$	$2(M - 2)$
div	1	0	0
abs	3	2	0
mov	0	$3(M - 3)$	0
shift	2	$(M - 3)$	$2(M - 2)$
bit-assign	3	$2(M - 3) + 3$	$2(M - 2) + 2$
bit-op	2	$(M - 3) + 2$	$(M - 2) + 2$
other	floor, sbc($\cdot$)	M signcopy	sbc($\cdot$), $2M$ signcopy

Table 1: Operations required to perform the sector binning.