Essays in Finance Related Problems with PDE Approach

SADNA SAJADINI

Doctoral Thesis
Stockholm, Sweden 2013

TRITA-MAT-13-MA-05
ISSN 1401-2278
ISRN KTH/MAT/DA 13/04-SE
ISBN 978-91-7501-959-8

KTH
Institutionen för Matematik
100 44 Stockholm
SWEDEN

Akademisk avhandling som med tillstånd av Kungliga Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie doktorsexamen i matematik måndagengen den 16 december 2013 kl 13.00 i sal F3, Kungliga Tekniska högskolan, Lindstedtsvägen 26, Stockholm.

Tryck: Universitetsservice US AB

Abstract

This thesis is comprised of three scientific articles, all in PDE problems related to finance. The first two papers address problems concerning bonds and the third one treats a symmetry problem in the mean field game theory.

We investigate the free boundary for the problem of valuation of American type convertible bond with extra call feature, in paper one. In this case the free boundary behaviour is of interest since it shows the optimal conversion strategy. We study the regularity of the free boundary and also the solution to the pricing of this special financial derivative. The touching point of the free boundary and fixed boundary is discussed, as well.

The second paper is devoted to the study of bonds with call feature and conversion. Further, it is shown that the price of a Bermudan type bond tends to that of an American one on condition that the conversion time goes to the whole life-span of the bond.

Our main goal in the third paper is to investigate a symmetry case regarding the stationary problem in mean field game of Lasry-Lions. Starting with a priori unknown ring-shaped region, taking the boundaries as level sets and admitting harmonic solutions, it is implied that the region has spherical symmetry. Moreover, therein we attempt to state the model to an higher dimensional case.

Sammanfattning

Denna avhandling består av tre artiklar inom partiella differntiella elvaktioner relaterade till finans matematik. De första två artiklarna handlar om obligationer, medan den tredje är en så-kallade mean field game problem.

I första artikeln undersöker vi fria randen hos värdefunktionen av en (amerikansk) konvertibel obligation som ger utfärdaren rätt att köpa tillbaka. Här studerar vi regulariteten hos lösningen och fria randen som uppkommer vid prissättningen av denna speciella finansiella instrument. Den punkt där den fria och den fasta rander möts diskuteras.

I den andra artikeln gör vi en utredning av obligationer med köp och konvertering möjligheter. Vidare visas att priset på bermudiansk typ av obligationer konvergerar mot en amerikansk obligation då konverteringstiden går mot hela livslängd hos obligationen.

Vårt främsta mål i den tredje artikeln är att utforska ett symmetriskt fall angående stationära problemet i mean filed game teorin av Lasry-Lions. Här visar vi kvalitativa egenskaper hos lösningar till överdeterminerade problem, som kan tänkas vara högre dimensionella versioner av Lasry-Lions modelen.

Contents

Acknowledgements

There are absolutely no words that can express my deepest appreciation towards my esteemed teacher Professor Henrik Shahgholian, for his constant support, patience and guidance. His footprints can easily be observed all over this thesis. What he did, means a world to me.

I would like also to thank my second advisor, Professor Boualem Djehiche, specially for what I learned during the course I had with him.

There are several people in the department of Mathematics, that I am grateful to for all the moments we have shared. My officemate, Ornella, takes a unique place among them, particularly for all our conversations on many different and exciting topics. I shall surely miss you all.

The help and contribution of my mathematical brothers Andreas, Martin and Reza, during my graduate program, are really appreciated. I am also so grateful and indebted for the fruitful comments I got from Dr. John Andersson and Dr. Erik Lindgren.

Finally I wish to thank my dear family, my parents and my sister for their non-stop and overall help and encouragement, and also M. Reza for supporting me and being there patiently, whenever I needed him. A very special thank goes to my little daughter, Viana, for injecting enthusiasm and energy into my life and filling my heart with everlasting love.

Chapter 1

Introduction

1.1 Background

In this thesis we treat two different problems in mathematical finance, from a pure mathematical point of view. We use methods developed by free boundary experts to tackle these problems. The two different free boundary problems that arise in the thesis are the well-known obstacle (variational inequality) like problem and the Bernoulli type free boundary. We have chosen not to give a general account of free boundaries, and instead we shall discuss the two free boundaries arising in this thesis separately in each subtopic below, in the context of the applications in finance.

1.2 Pricing financial securities

The first problem concerns the so-called convertible bonds of American style, with or without conversion. We shall study various aspects of this problem, along with its Bermudan counterpart (see below for definitions). The second problem relates to the so-called mean field games, in a non-standard way, and in higher dimensions; mean field games are mostly considered in one-dimensional setting.

Many problems in finance are formulated in terms of an underlying process S_t, and its behavior. Here S_t can be the price of a stock, index, or any commodity. These processes (due to their uncertain/random nature) are represented by a stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dX_t,$$

where r, σ are given values (e.g. interest rate, and volatility of the underlier), and X_t is the standard Brownian motion. Here we only consider one dimensional processes.

The valuation of securities in finance (options, convertible bonds, etc) is now a well-established area of research in mathematical finance and it has been subject for intense studies in the last few decades. The price function, call it V, of a security, can in general be represented by

$$V(x) = E^x(g(S_\tau))$$

where g is a reward function (the amount to be paid/received), τ is an optimal stopping time (i.e., best time to stop the process) and E^x is expected value with respect to a probability law P^x of S_t (and of course that of X_t), with $S_0 = x$. Although, many times, the models are much more complicated, for clarity of exposition, we shall stick to this simple case. This gives rise to elliptic problems and in order to involve time, one takes more general forms of the above expression.

Our model problems in this thesis consists of cases, where one wants to maximizes (from a sellers perspective) the value

$$V(x) := \sup_\tau E^x(g(S_\tau)).$$

This naturally leads us to a constraint problem, in the sense that

$$V(x) = \sup_{\tau < N} E^x(g(S_\tau)) \geq E^x(g(S_0)) = g(S_0) = g(x),$$

where N is a given value, e.g. the first exit time of a bounded domain.

Having this fixed, one can relate this model to the theory of partial differential equations, through infinitesimal generator of the underlying stochastic process (see [10]). Since we do not treat our problem from a stochastic point of view, this introduction will be the only place that we mention the background in terms of stochastic models of the rewards.

Below, we shall derive the PDE model from a pure financial perspective using Black-Scholes type of approach (see [7] and [11]) with a simple use of Ito's formula (Taylor expansion for random variables) by accepting formally $\sqrt{dt} = dX_t$. The latter is the corner stone of the stochastic calculus theory.

Financial jargon related to Bonds

In this section we shall introduce basics concepts in finance, that has been investigated in the thesis. We shall also present certain mathematical formulas and equations relevant to the thesis.

Bond is a financial contract which is paid in advance and yields a predetermined amount on a known date usually called maturity date. The issuer (the government or big company) guarantees the holder an specified amount will be paid at the maturity date. Therefore it can be considered as a loan to the shareholder and the reason of its issuing, is to raise the capital. Bonds may pay a known cash dividend, commonly named coupon, at specified dates up to and including maturity time. If there is no such coupons, the bond is called a zero-coupon bond. Furthermore obviously, the longer a bond has to live the less it is worth today.

European Convertible Bond (CB) is a bond that could be converted into a predetermined number (say γ) of stocks or assets, only at the maturity.

American style CB is a convertible bond which allows conversion at any time prior (including the maturity time) to expiry.

Callability is the right of the company to buy back the bond for an amount K. It is clear that $V(S,t) \leq K$ and a convertible bond with this feature is worth less than a usual one. Moreover, callability is forced when the value of the underlier riches K/γ. This in turn gives a natural boundary to the mathematical model, in the sense that the governing equations will take place in the region $0 < S < K/\gamma$.

Bermudian Convertible Bond lies in between European and American types of CBs. If a convertible bond may be converted only at certain period during its life time, then we call it Bermudan style.

Pricing CBs Using PDE method

Pricing a CB has further complexity since we are dealing with both debt and equity. Recall the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dX_t,$$

that presents the value of the stock and consider a portfolio consisting of one CB and $-\triangle$ of the stock;

$$\Pi = V - \triangle S$$

One can find the change of the value in portfolio, by taking into account, coupon payments c, dividend D, interest rate r, volatility σ,

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \triangle DSdt - \triangle dS + cdt.$$

By setting $\triangle = \frac{\partial V}{\partial S}$, using the expression for dS, along with fact that the return should be at most that of a bank deposit, one gets rid of the term dX and ends up with

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV + c \leq 0.$$

Since the bond may be exchanged into γ assets, one also has the constraint

$$V \geq \gamma S.$$

The model can now be completed by terminal value, as well as the behavior of V for large values of S.

Qualitative Analysis

In the thesis we shall establish various properties of the value function V for the American and Bermudan convertible bond, as well as the behavior of the free boundary, i.e., boundary of the conversion region. We shall also present a convergence result for the Bermudan type convert to American convert. These will be discussed in Paper I and II.

1.3 A symmetry problem related to mean field games

Lasry-Lions Model

The mean field game theory, which was introduced by Lasry-Lions [5] on 2007, is about the decision making of a large number of players such that every individual has a negligible impact on the result. It has vast applications investigated by many people recently but we consider the economical and financial parts.

A Bid-Ask spread is the difference in price between the best ask (lowest amount that a seller is willing to sell a certain good) and the best bid (the highest price that a buyer is willing to buy the same good). In practice, many economical and financial problems are concerned with a number of rational players. As it is stated in Section 3, in [5], a simple case of a mean field game model introduces for the price formation and dynamical equilibrium. We consider a class of rational players consisting of two groups of buyers and vendors of an specified good. We denote the number of potential buyers (sellers) by a non-negative function, $f_B(x,t)$ ($f_V(x,t)$), where t represents the time and x the price of the good at time t. Furthermore suppose that $2a$ is the bid-ask spread, we obtain the following mean-field equation

$$\begin{cases} \frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = -\frac{\sigma^2}{2}\frac{\partial f}{\partial x}(p(t),t)\{\delta_+(x) - \delta_-(x)\} & \text{on}\, \mathbb{R}\times(0,\infty) \\ f(x,t) > 0 \quad \text{if} \quad x < p(t), t \geq 0; \qquad f(x,t) < 0 \quad \text{if} \quad x > p(t), t \geq 0, \\ f\lfloor_{t=0} = f_0 \quad \text{on} \quad \mathbb{R}, \quad p(0) = p_0, \end{cases}$$

where $\delta_\pm(x) = \delta(x - p(t) \pm a)$ and

$$f(x,t) = \begin{cases} f_B(x,t), & \text{if} \quad x \leq p(t), \\ -f_V(x,t), & \text{if} \quad x > p(t), \end{cases}$$

and the initial condition f_0 is given by

$$f_B\lfloor_{t=0} = f_B^0, \quad f_V\lfloor_{t=0} = f_V^0.$$

Here $\sigma > 0$ measures the randomness and λ states the number of transaction at time t;

$$\lambda = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t) = \frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t).$$

If we put the extra assumption that f is a smooth function on $\mathbb{R}$ with fast decay at infinity, then there exists a unique smooth solution (f,p) to the above mentioned problem. We refer to the articles [1], [2], [3], [4].

In the thesis, we shall discuss, a toy model of this problem in higher dimension and in a stationary case; when time tends to infinity one obtains an elliptic problem. The problem that we study here, is actually an overdetermined problem, in qualitative PDE, related to symmetry. In other words with some extra assumptions one can reduce a higher dimensional version of this model to a pure symmetry problem in elliptic PDE. Then using classical methods of moving planes, one can prove symmetry results for the solution. Similar results from a pure mathematical point of view and in a different settings have been discussed earlier in [6], [8], [9].

Chapter 2

Summary of results

In this chapter, we aim to provide some details of results we have dealt with in this thesis, which consist of three papers in financial problems from a pure mathematical standpoint. These are articles with the following titles

- Analysis of the Free Boundary for Convertible Bonds.
- Analysis for Bonds with conversion/call/Bermudan feature.
- A symmetry problem with applications to finance.

2.1 Paper I

In the first paper, we investigate the behavior of the free boundary arising from a callable (American style) convertible bond. The qualitative analysis of solution of such a bond is of main interest. Mathematical interpretation of this financial problem (bond pricing) gives the following double constraint problem.

$$\begin{cases} \mathcal{L}V = c, & D_T \cap \{\gamma x < V < K\}, \\ \mathcal{L}V \geq c, & D_T, \\ V(K/\gamma, t) = K, & 0 \leq t \leq T, \\ V(x,T) = K, & 0 \leq x \leq K/\gamma, \end{cases} \tag{2.1}$$

where $D_T = (0, \frac{K}{\gamma}) \times (0, T)$ and

$$\mathcal{L} = -\partial_t - \frac{1}{2}\sigma^2 x^2 \partial_{xx} - (r-q)x\partial_x + r.$$

All the constants above have financial meaning, but we refrain ourselves from defining them here. The PDE methods we apply in our arguments are developed in [12], [13] and [14]. We list the main result in Paper I, here below.

Proposition 2.1. *There exists a solution* $V \in W^{2,p}_{x,loc} \cap W^{1,p}_{t,loc}$*,* $1 < p < \infty$ *to Problem (2.1). The solution is unique in the class of all solutions with*

$$\lim_{x\to 0} x^2 V_{xx}(x,t) = \lim_{x\to 0} x V_x(x,t) = 0,$$

and satisfies

$$0 \le V_t, \qquad 0 \le V_x \le \gamma.$$

Consequently, in this class we also have that the exercise region (if non-empty) is an epi-graph in both x *and* t *directions.*

Theorem 2.2. *The solution* V *to Problem (2.1) is uniformly* $C^{1,1}_x \cap C^{0,1}_t$ *in* $\overline{D}_T \setminus \{x = 0\}$.

Theorem 2.3. *The free boundary* $\Gamma = \partial\{V > \gamma x\}$ *in Problem (2.1), is uniformly parabolically tangential to the fixed boundary at* $X^* = (K/\gamma, t^*)$. *In other words there is a modulus of continuity* σ $(\sigma(0^+) = 0)$ *and an* r_0 *such that*

$$\Gamma \cap Q_{r_0}(X^*) \subset \left\{(x,t) : t - t^* \le -\frac{|x - K/\gamma|^2}{\sigma(|x - K/\gamma|)}\right\}.$$

2.2 Paper II

In the second paper, we again discuss bonds but our main focus is on callability and different conversion strategies.
As Paper I, we take the Black-Scholes operator for evaluation a bond,

$$\mathcal{L} = -\partial_t - \frac{1}{2}\sigma^2 x^2 \partial_{xx} - (r-q)x\partial_x + r.$$

If we consider both conversion and call features for the bond the problem of pricing bond can be formulated as the following double obstacle problem

$$\max(\min(\mathcal{L}V - c, V - \gamma S), V - K) = 0, \quad V(S,T) = \max(L, \gamma S),$$

in the region $0 < S < K/\gamma$, and $t < T$.

In case that we just have conversion and no call feature one may consider $K = \infty$ and hence the equation reduces to a one obstacle problem

$$\min(\mathcal{L}V - c, V - \gamma S) = 0, \qquad V(S,T) = \max(L, \gamma S), \quad S > 0. \tag{2.2}$$

When there is no early conversion possibility then again we obtain a one obstacle problem this time from above

$$\max(\mathcal{L}V - c, V - K) = 0, \qquad V(S,T) = \max(L, \gamma S).$$

Then we state how to find the price of a Bermudan type convertible. One has the following Theorem

Proposition 2.4. *The function V, that represents the value function for the Bermudan convertible bond, satisfies the following:*

i) upper-semi continuous from the right hand side at the singletons, and right-end points of the intervals of I,

ii) continuous from the left hand side,

iii) can be constructed as the infimum of all supersolution V to the equation (2.2) *where γS is replaced by $\gamma S \chi_I$.*

Theorem 2.5. *The general Bermudan type converts converges to the American-type converts as the length of the largest non-exercise time interval tends to zero. More exactly*

$$V^n \leq V, \qquad \lim_{n\to\infty} V^n(S,t) = V(S,t),$$

provided $dist(I^n_{j+1}, I^n_j) \to 0$. Here V^n represents the price for Bermudan with dates I^n and V is the price for American convertible.

2.3 Paper III

In the third paper, we study a special case of spherical symmetry problem with an additional extra datum that the boundary is considered to be a level set, where the solution is harmonic. Furthermore we survey the stationary case of mean field game of Lasry-Lions in $\mathbb{R}^n$ $(n \geq 2)$ and try to interpret our symmetry case and relate it to the mean field game model. The main theorem in Paper II, is as follows

Theorem 2.6. *Let $D_+ \subset\subset D_-$ be two bounded C^1 domains in $\mathbb{R}^n$ $(n \geq 2)$, and consider the capacitor potential u in $D_- \setminus D_+$*

$$\begin{cases} \Delta u = 0 & \text{in} \quad D_- \setminus \overline{D_+}, \\ u = \pm 1 & \text{on} \quad \partial D_\pm, \end{cases}$$

with the extra property that, for all $x \in \{u = 0\}$, and some $r_\pm > 0$

$$\operatorname{dist}(x, \partial D_\pm) = r_\pm.$$

Then $D_\pm$ are concentric balls, of appropriate radius.

In one dimension, the stationary case of mean field game theory is formulated as follows

$$-\frac{\sigma^2}{2}\frac{d^2 f}{dx^2} = -\frac{\sigma^2}{2}\frac{df}{dx}(p)(\delta_{p-a} - \delta_{p+a}) \quad \text{in} \, (0, A),$$

where $\frac{df}{dx}(0) = \frac{df}{dx}(A) = 0$ and f is positive on $(0, p)$ and negative on (p, A). Also $N_1 = \int_0^p f$ and $N_2 = \int_p^A -f$ are respectively the total amount of buyers and sellers and p shows the price. In this case the solution is determined explicitly in the paper of Lasry-Lions, [5].

We tried to establish a toy model of this stationary case in higher dimensions, in Paper II.
Consider a Lipschitz function u, which solves

$$\Delta u = \lambda_-(x) dS\lfloor_{S_{-a}} - \lambda_+(x) dS\lfloor_{S_{+a}}, \quad \text{in } \Omega_- \setminus \overline{\Omega}_+,$$

$$\nabla u \cdot \nu = 0 \quad \text{on } \partial(\Omega_- \setminus \overline{\Omega}_+).$$

Here $dS\lfloor_{S_{\pm a}}$ is the surface measure on $S_{\pm a}$, and

$$S_{\pm a} = \{x : \; dist(x, \Gamma_0) = a, \; \pm u(x) > 0\},$$

with $\Gamma_0 = \{u = 0\}$.

Furthermore, suppose the extra conditions

$$N_1 = \int_{\Omega_- \setminus \Omega_+} \max(u, 0), \qquad N_2 = \int_{\Omega_- \setminus \Omega_+} \max(-u, 0),$$

hold, where $N_1 > 0$ is the number of buyers, and $N_2 > 0$ is the number of sellers. The continuous positive functions $\lambda_\pm$ represent the flux of buyers and sellers from one to another side. In one spatial dimension (one product case) this obviously can be taken to be constant and $\lambda_+ = \lambda_-$. In higher dimensions, a simple integration by parts gives

$$\int_{S_{+a}} \lambda_+ dS = \int_{S_{-a}} \lambda_- dS.$$

Moreover we found the solutions in special case of spherical domains, for $n = 2, 3$ during two examples.

References

[1] L. Caffarelli, P. Markowich, J. Pietschmann, Jan-F. On a price formation free boundary model by Lasry and Lions. C. R. Math. Acad. Sci. Paris 349 (2011), no. 11-12, 621-624.

[2] L. Chayes, M. Gualndani, M. Gonzalez, I. C. Kim, Global existence and uniqueness of solutions to a model of price formation. SIAM J. Math. Anal. 41 (2009), no. 5, 2107-2135.

[3] M. Gonzalez, M. Gualdani, Asymptotics for a free-boundary model in price formation, Nonlinear Analysis 74 (2011), pp. 3269-3294.

[4] M. Gonzalez, M. Gualdani, H. Shahgholian, A discrete Bernoulli free boundary problem, To appear in Proceedings of the Conference in honor of O.A. Ladyzhenskaya.

[5] J.M. Lasry, P.L. Lions, Mean field games. Japanese Journal of Mathematics, 2(1) (2007) 229-260.

[6] R. Magnanini, S. Sakaguchi. Nonlinear diffusion with a bounded stationary level surface, Ann. Inst. Henri Poincaré Analyse Nonlinéaire 27 (2010), 937-952.

[7] R.C. Merton, Theory of rational option pricing. Bell J. Econom. and Management Sci. 4 (1973), 141-183.

[8] J. Serrin, A symmetry problem in potential theory. Arch. Rational Mech. Anal. 43 (1971), 304-318.

[9] H. Shahgholian, Diversifications of Serrin's and related symmetry problems.(Submitted)

[10] B. Oksendal, Stochastic differential equations. An introduction with applications. Universitext. Springer-Verlag, Berlin, 1985. xiii+205 pp.

[11] P. Wilmott, S. Howison, J. Dewynne, The mathematics of financial derivatives. A student introduction. Cambridge University Press, Cambridge, 1995. xiv+317 pp.

[12] D. E. Apushkinskaya, N. N. Uraltseva, and H. Shahgholian, Lipschitz Property Of The Free Boundary In The Parabolic Obstacle Problem, St. Petersburg Math. J. Vol. 15 (2004), No. 3, Pages 375-391.

[13] D. E. Apushkinskaya, N. N. Uraltseva, H. Shahgholian, On global solutions of a parabolic problem with an obstacle, (Russian) Algebra i Analiz 14 (2002), no. 1, 3–25; translation in St. Petersburg Math. J. 14 (2003), no. 1, 1-17.

[14] L. Caffarelli, A. Petrosyan, H. Shahgholian, Regularity of a free boundary in parabolic potential theory, J. Amer. Math. Soc. 17 (2004), no. 4, 827-869.