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Uniformly Improving Maximum-Likelihood SNR Estimation of Known Signals in Gaussian Channels

Efthymios Stathakis, Joakim Jaldén, Lars K. Rasmussen and Mikael Skoglund

Abstract—The signal-to-noise ratio (SNR) estimation problem is considered for an amplitude modulated known signal in Gaussian noise. The benchmark method is the maximum-likelihood estimator (MLE), whose merits are well-documented in the literature. In this work, an affinely modified version of the MLE (AMMLE) that uniformly outperforms, over all SNR values, the traditional MLE in terms of the mean-square error (MSE) is obtained in closed-form. However, construction of an AMMLE whose MSE is lower, at every SNR, than the unbiased Cramer-Rao bound (UCRB), is shown to be infeasible. In light of this result, the AMMLE construction rule is modified to provision for an a priori known set $\mathcal{S}$, where the SNR lies, and the MSE enhancement target is pursued within $\mathcal{S}$. The latter is realized through proper extension of an existing framework, due to Eldar [13], which settles the design problem by solving a semidefinite program. The analysis is further extended to the general case of vector signal models. Numerical results show that the proposed design demonstrates enhancement of the MSE for all the considered cases.

I. INTRODUCTION

In this paper we consider a known signal $s(n)$, which is amplified by an *unknown* scalar constant A and perturbed by Gaussian noise. For *real-valued* quantities, the noisy observation $x(n)$ is expressed according to

$$x(n) = As(n) + w(n), \quad n = 0, \dots, N-1, \quad (1)$$

where $w(n)$ is the additive white Gaussian noise sequence with unknown variance σ_w^2 and where N is the sequence length. For the model (1), we are interested in recovering the average, over the entire observation period, signal-to-noise ratio (SNR) which is defined as $\alpha \triangleq \frac{A^2}{\sigma_w^2}$.

The problem of SNR estimation is relevant in many practical systems, e.g., optical-fiber systems, satellite and radar communications. The quality-of-service (QoS) delivered by such technology is highly influenced by the estimation accuracy of the SNR, whose (imperfect) knowledge has a non-trivial impact on the systems' performance; for example in a wireless communication system SNR estimation is involved in various parts of the transceiver chain such as modulation [1], coding [2], [3] and detection [4]. The problem of accurately recovering the SNR from a noisy data sequence was addressed quite early in literature [5] and different estimators, tailored to the signal-specific model, have been proposed in later studies, see [6] for an overview. Among the different approaches, employed for

SNR recovery, are the data-aided (DA) maximum-likelihood estimator (MLE) [7], the decision-directed (DD) estimator [8] and the method-of-moments [9].

The MLE is a natural choice for practical systems, because of the established theoretical performance guarantees [10], i.e., convergence in probability to the true parameter (consistency), asymptotic achievability of the unbiased Cramer-Rao bound (UCRB), and being efficient whenever such an estimator exists. However, the target of operating systems close to their theoretical performance limits dictates further improvements on the MLE. The performance of an estimator $\hat{\alpha}$ is typically assessed by the mean square-error (MSE) between the estimator and the actual parameter α . Therefore, an enhanced estimator would be designed to achieve a lower MSE than the MLE, in other words it should *dominate*¹ the MLE.

Efforts, in the aforementioned direction, have focused on the construction of linearly and affinely modified versions of the MLE. The main advantage, in considering such modifications, stems from the fact that theoretical analysis remains tractable. Direct minimization of the MSE expression, for the retrieval of the relevant design parameters, was successfully applied in a particular parametric estimation problem in [11]. However, this approach is by no means universal and will most often result in an unrealizable estimator. The benefits of linearly/affinely modified versions of MLE were highlighted in the pioneering work of [12], [13] where a general framework, addressing parametric *vector* estimation was established. For certain families of MSE functions, the sought modified estimators were found as the solution to a min max optimization problem. This approach is universal in the sense that it always returns a realizable estimator and a detailed account along with other non-linear approaches can be found in [14]. The ideas explored within this framework were also applied in the SNR problem (see [13, Section VII]) but the design was based on the Cramer-Rao bounds of the MLE and its corresponding biased counterpart, rather than the actual MSE expressions. The actual MSE formulas, though, are different than the Cramer-Rao bounds because for the SNR problem an efficient estimator, i.e., an estimator that attains the UCRB, *does not exist*. It also turns out (refer to Section III-D for details) that the framework of [12]–[14] is non-informative for many variations of the SNR estimation problem, because it does not yield unique estimators.

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¹If $\hat{\alpha}_1, \hat{\alpha}_2$ are estimators for the same parameter α , then $\hat{\alpha}_1$ is said to dominate $\hat{\alpha}_2$ if $\text{MSE}(\hat{\alpha}_1) \leq \text{MSE}(\hat{\alpha}_2)$, where $\text{MSE}(\hat{\alpha}_k) \triangleq \mathbf{E}\{(\hat{\alpha}_k - \alpha)^2\}$, $k \in \{1, 2\}$, holds for all possible values of the parameter α with strict inequality for at least one value.

Our work elaborates on the problem of constructing an affinely modified version of the MLE $\hat{\alpha}$, namely the AMMLE $\bar{\alpha}$, such that it outperforms, in terms of achievable MSE, the MLE for the SNR estimation problem. The main contribution is that our analysis is based on the actual MSE expressions for the SNR, which characterize performance more accurately. Since naive minimization of the MSE is not applicable in this case, the approach, proposed in [14], is adopted to set up the problem. Our mathematical analysis, though, is based on different techniques and we are able to establish the desired estimator in *closed-form*, thus obviating the need for solving an optimization problem. In light of the fact that the MLE is not efficient we seek a construction rule for the AMMLE such that it dominates the UCRB. A negative answer motivates the investigation for the existence of an AMMLE that dominates the UCRB within a specified set of values, where the SNR is assumed to lie, but the objective function cannot be handled under the original framework [12]–[14]. Therefore, our investigation is carried out by extending the minmax approach and introducing new techniques to accommodate the efficient numerical solution of the resulting optimization problem. Furthermore the extension to the multiple-input multiple-output (MIMO) model is discussed and analyzed to capture the general and more interesting, from a practical viewpoint, vector case. A notable difference with the treatment of the SNR estimation problem in [12]–[14] is that our approach always returns a unique solution, thus establishing a non-heuristic direction towards the MSE performance enhancement goal. Numerical results indicate an MSE improvement over the traditional MLE and the related UCRB for all the scenarios under consideration.

This paper is organized as follows. In Section II, the existing results are briefly summarized while a thorough analysis of the framework for the construction of the proposed estimator is analyzed in Section III. Generalization of the results to the general vector signal model is given in Section IV. Numerical results are presented in Section V and finally, Section VI summarizes the most important contributions of this work.

Notation: Bold lower case letters denote vectors and bold upper case letters denote matrices. The $n \times n$ identity and the $n \times n$ all-zero matrices are denoted by $\mathbf{I}_n$ and $\mathbf{0}_n$, respectively. The operator $\|\cdot\|_2$ stands for the Euclidean norm and $\{x\}^+$ is equivalent to $\max\{x, 0\}$.

II. PROBLEM STATEMENT

A. Background

For ease of exposition we shall work with an equivalent formulation of (1) which follows after defining the quantities $\mathbf{x} \triangleq [x(0) \dots x(N-1)]^T$, $\mathbf{s} \triangleq [s(0) \dots s(N-1)]^T$ and $\mathbf{w} \triangleq [w(0) \dots w(N-1)]^T$. Based on these definitions, (1) can be recast in the following equivalent vector form

$$\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{w}, \quad (2)$$

which is a more compact representation. Without loss of generality it is assumed that the input signal has unit energy, which is equivalent to imposing the constraint $\frac{1}{N} \sum_{n=0}^{N-1} s^2(n) = 1$.

The baseline method that will serve as benchmark for the evaluation of the design, proposed in this paper, is the maximum-likelihood estimator (MLE) $\hat{\alpha}$ which is associated with the SNR α . Invoking the invariance property [9] it can be directly observed that $\hat{\alpha} = \frac{\hat{A}^2}{\hat{\sigma}_w^2}$, where $\hat{A}$, $\hat{\sigma}_w^2$ are the MLEs associated with the signal amplitude A and the noise variance σ_w^2 , respectively. For further notational simplicity we let $\boldsymbol{\theta} = [A \ \sigma_w^2]^T$ be the vector of the unknown parameters. Starting from the pdf $p(\mathbf{x}; \boldsymbol{\theta})$ which is expressed as

$$p(\mathbf{x}; \boldsymbol{\theta}) = \frac{1}{(2\pi\sigma_w^2)^{\frac{N}{2}}} \exp \left\{ -\frac{1}{2\sigma_w^2} \|\mathbf{x} - \mathbf{A}\mathbf{s}\|_2^2 \right\} \quad (3)$$

and invoking the necessary conditions [9] that need to be satisfied by the ML estimates we obtain the following system of equations,

$$\begin{aligned} \frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})}{\partial A} \Big|_{A=\hat{A}} &= 0 & 2(\mathbf{s}^T \mathbf{s})\hat{A} - 2(\mathbf{s}^T \mathbf{x}) &= 0 \\ \frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})}{\partial \sigma_w^2} \Big|_{\sigma_w^2=\hat{\sigma}_w^2} &= 0 & -\frac{N}{2\hat{\sigma}_w^2} + \frac{\|\mathbf{x} - \hat{A}\mathbf{s}\|_2^2}{2(\hat{\sigma}_w^2)^2} &= 0. \end{aligned}$$

Thus $\hat{A}$ and $\hat{\sigma}_w^2$ are recovered from the above system, whose solution yields

$$\hat{A} = \frac{\mathbf{s}^T \mathbf{x}}{\|\mathbf{s}\|_2^2}, \quad \hat{\sigma}_w^2 = \frac{1}{N} \|\boldsymbol{\Pi}_s^\perp \mathbf{x}\|_2^2, \quad (4)$$

where $\boldsymbol{\Pi}_s^\perp = \mathbf{I}_N - \frac{\mathbf{s}\mathbf{s}^T}{\|\mathbf{s}\|_2^2}$ is the projection onto the orthogonal complement of the space spanned by $\mathbf{s}$ implying that $\hat{A}^2$ and $\hat{\sigma}_w^2$ are independent. It holds that $\hat{A} \sim \mathcal{N}(A, \sigma_w^2/N)$ and we can further conclude that the random variable $y \triangleq \frac{1}{\hat{\sigma}_w^2} \|\boldsymbol{\Pi}_s^\perp \mathbf{x}\|_2^2 = \frac{1}{\hat{\sigma}_w^2} \|\boldsymbol{\Pi}_s^\perp \mathbf{w}\|_2^2$ is chi-square distributed with $(N-1)$ degrees of freedom,² i.e., $y \sim \chi_{N-1}^2(0)$. It can then be shown that $\hat{\alpha} = \frac{1}{N-1} Z$ where Z is a random variable with singly non-central F -distribution, i.e., $Z \sim \mathcal{F}_{1, N-1}(N\alpha, 0)$. From existing results [5] in the literature it follows that the MSE of $\hat{\alpha}$ is expressed as

$$\text{MSE}(\hat{\alpha}) = \frac{(2\alpha^2 + 4\alpha)N + (15\alpha^2 + 10\alpha + 3)}{(N-3)(N-5)}. \quad (5)$$

It should be noted here that (5) holds for $N \geq 6$, which is a necessary condition in order to guarantee that the first two moments of the singly non-central distribution of Z (and subsequently of $\hat{\alpha}$) are finite. Furthermore, the UCRB for α is known [9] to be $\text{UCRB}(\alpha) = \frac{1}{N}(2\alpha^2 + 4\alpha)$ and by inspection of (5) we can verify the asymptotic efficiency of $\hat{\alpha}$. Therefore, by the properties of the MLE, $\hat{\alpha}$ will be asymptotically distributed ($\overset{\alpha}{\sim}$) as $\hat{\alpha} \overset{\alpha}{\sim} \mathcal{N}(\alpha, \frac{1}{N}(2\alpha^2 + 4\alpha))$.

III. DOMINATING THE MLE THROUGH AFFINE TRANSFORMATION

In this section we investigate a modified version of the MLE $\hat{\alpha}$ as the candidate that dominates $\hat{\alpha}$, in terms of MSE, over the entire range of admissible SNR values. Moreover, we study the properties of such an estimator and discuss its relation to the UCRB.

²A formal proof of this statement is based on the observation that $\boldsymbol{\Pi}_s^\perp$ is an idempotent matrix, of rank $(N-1)$, thus it has $(N-1)$ unit eigenvalues. If we perform an eigendecomposition on $\boldsymbol{\Pi}_s^\perp$ the result follows by invoking the invariance of Rayleigh distributed vectors, with i.i.d elements, in unitary transformations.

A. Affinely Modified MLE

The estimator under consideration is the AMMLE which is mathematically defined as $\bar{\alpha} \triangleq \rho\hat{\alpha} + c$. The class of AMMLEs was considered in detail in [13], in pursuit of estimators which perform better than the UCRB for problems where the UCRB has a certain form, as a function of θ . In particular, parametric estimation problems, where the inverse Fischer information matrix $\mathbf{J}^{-1}$ had a quadratic form with respect to the sought vector quantity, were considered. The property that renders the AMMLE an appealing choice is the fact that the first two moments of the modified estimators are linear functions of the respective moments of the MLE, hence the evaluation of the associated MSE expression will be feasible for the majority of the cases. For our current estimator $\bar{\alpha}$, it is given by the following proposition.

Proposition 1. *The MSE of the AMMLE $\bar{\alpha}$ is given as*

$$\text{MSE}(\bar{\alpha}) = \frac{A_0(\rho, c)\alpha^2 + B_0(\rho, c)\alpha + C_0(\rho, c)}{(N-3)(N-5)}, \quad (6)$$

where the coefficients are given as

$$\begin{aligned} A_0(\rho, c) &= (N\rho)^2 - 2N(N-5)\rho + (N-3)(N-5) \\ B_0(\rho, c) &= 6N\rho^2 - 2(N-3)(N-5)c + 2(N-5)(Nc-1)\rho \\ C_0(\rho, c) &= 3\rho^2 + 2(N-5)\rho c + (N-3)(N-5)c^2. \end{aligned}$$

Proof. Given in Appendix A. $\square$

The aim is to properly pick ρ and c such³ that $\bar{\alpha}$ dominates $\hat{\alpha}$, i.e., $\text{MSE}(\bar{\alpha}) \leq \text{MSE}(\hat{\alpha}) \forall \alpha \geq 0$ with strict inequality for at least one value of α . Inspired by the minmax approach, established and analyzed in [12]–[14], we define the relevant quantity $\Delta\text{MSE}(\alpha, \xi) \triangleq \text{MSE}(\bar{\alpha}) - \text{MSE}(\hat{\alpha})$, where $\xi \triangleq [\rho \ c]^T$. The quantity $\Delta\text{MSE}(\alpha, \xi)$ can now be written as

$$\Delta\text{MSE}(\alpha, \xi) = \frac{A_1(\rho, c)\alpha^2 + B_1(\rho, c)\alpha + C_1(\rho, c)}{(N-3)(N-5)}, \quad (7)$$

where $A_1(\rho, c) = (N\rho)^2 - 2N(N-5)\rho + N(N-10)$, $B_1(\rho, c) = 6N\rho^2 - 2(N-5)\rho - 2(N-3)(N-5)c + 2N(N-5)\rho c - (4N+10)$ and $C_1(\rho, c) = 3\rho^2 + 2(N-5)\rho c + (N-3)(N-5)c^2 - 3$. If we set $g(\rho, c) \triangleq \max_{\alpha \geq 0} [\Delta\text{MSE}(\alpha, \xi)(N-3)(N-5)]$ then⁴ it is readily seen that $g(1, 0) = 0$, which guarantees that zero is *always* an achievable value for ΔMSE , over all feasible α . Thus if $g(\rho, c) \leq 0$ for some pair $(\rho, c) \neq (1, 0)$ and $\Delta\text{MSE}(\alpha, \xi) < 0$ for at least one value of α then $\bar{\alpha}$ will dominate $\hat{\alpha}$. In order to settle the problem of determining the optimal pair (ρ^*, c^*) which minimizes g and thus yields the sought AMMLE, we solve the following minmax problem

$$\begin{aligned} &\underset{\xi}{\text{minimize}} \quad \max_{\alpha} [\Delta\text{MSE}(\alpha, \xi)(N-3)(N-5)] \\ &\text{subject to} \quad \alpha \geq 0. \end{aligned} \quad (\text{P1})$$

Problem (P1) is convex [15], because the objective is the pointwise supremum of convex functions in ξ over a set.

³The values of ρ and c will probably have some underlying relation to the observation length N . However this relation will be explicit (in closed-form) or implicit depending on the adopted approach.

⁴The reason for multiplying the objective with the term $(N-3)(N-5)$ is just to simplify the exposition by studying this simpler equivalent formulation.

Fig. 1. Plot of the theoretical results of Proposition 2 for various values of N . Theoretical values are verified through comparison with numerical solution of (P1).

Therefore we can implement efficient numerical solutions, such as interior-point methods, but (P1) needs to be reformulated in standard convex form, as in [13], in order to facilitate their implementation. Remarkably, this turns out to be unnecessary because we show that the AMMLE is one of the few cases [11] for which the desired estimator can be established in closed-form as described by the following result.

Proposition 2. *Let (ρ^*, c^*) be the optimal pair and p^* be the optimal value for Problem (P1). Then it holds that $(\rho^*, c^*) = (\{\frac{N-10}{N}\}^+, -\{\frac{N-10}{N(N-3)}\}^+)$ and $p^* = \{2(\frac{N-2}{N-3})(\rho^*)^2 - 3\}$, implying that $p_\alpha^* \triangleq \max_{\alpha \geq 0} \Delta\text{MSE}(\alpha, \xi^*) = \frac{p^*}{(N-3)(N-5)}$.*

Proof. Given in Appendix B. $\square$

An interesting aspect of the proof of Proposition 2 is that it is not carried out along the lines of [11] which dictate direct minimization, through derivation of the objective. In fact the proof-line is a modification of the approach proposed in [14], which involves reformulation of the inner maximization problem, in (P1), in standard convex form through its dual. The latter is facilitated via application of the Schur complement on the epigraph form of the dual problem, which results in a cone program [15]. Introduction of slack variables and linearization of the resulting cone constraints eventually yields an equivalent semidefinite program (SDP). The approach, adopted herein, albeit based on the same framework provides an alternative path to the result. Moreover our solution, given in closed-form, rather than in the form of an optimization problem, lends itself to analysis of the estimator's properties. In particular, after obtaining the Lagrangian, in lieu of the SDP construction steps we analyze the optimal solution by exploiting the resulting formulation and using various key-insights we directly establish the optimal point in closed-form. The visual certificate of Proposition 2 is available in Figure 1, where the theoretical values are plotted against the numerical solution of (P1).

Proposition 2 offers a neat solution to the problem at hand that renders the acquisition of the optimal, in the min max sense, AMMLE straightforward; once $\hat{\alpha}$, whose computational complexity is $\mathcal{O}(N)$, has been constructed then $\bar{\alpha}$ is readily obtained after proper scaling and shifting. The optimal such parameters ρ^* and c^* are functions of the observation length alone and can easily be evaluated, in $\mathcal{O}(1)$ time, for any value of N . Thus, the proposed solution to Problem (P1) is not only theoretically tractable but also suggests that essentially no additional computations are needed to enhance the performance of the MLE $\hat{\alpha}$. Further, this observation can be exploited to reduce the complexity in applications where the MLE is updated online. Instead of storing the values of ρ^* and c^* for every N , that would entail high storage needs for large sequences, one can rely on our insights to calculate the optimal pair on-the-fly without being concerned about processing delay or higher computational costs.

It is worth pointing out that the optimal c is not zero, in contrast to [13] where the analogue of (P1) was solved using the UCRB expression for the AMMLE and c^* turned out to be zero for all N (see Section III-D). We conclude that, when the actual MSE expression is optimized, the presence of a constant term can possibly bring some gain into the performance of the modified MLE. It is further inferred from the closed-form solution that, for large values of N the AMMLE converges to the MLE, i.e., $\rho^* \rightarrow 1$ and $c^* \rightarrow 0$. The same holds if we confine the estimator to be simply a linear scaled version $\tilde{\alpha} \triangleq \rho\alpha$ of the MLE, namely the linearly modified MLE (LMMLE) $\tilde{\alpha}$, whose optimal scaling parameter is the same as that for the AMMLE (see Remark 2 in Appendix B). Notably the structural difference between the optimal $\hat{\alpha}$ and $\tilde{\alpha}$, i.e., the term c^* , does not have practical implications in performance and is not reflected on the MSE results, acquired in simulated scenarios pertaining to (P1).

B. The AMMLE vs the UCRB

In the previous section, we established a closed-form solution for the construction of the best, in the min max sense, estimator that dominates the MLE for the estimation problem at hand. If the MLE is efficient then the min max approach of [14] produces a (biased) superefficient estimator; an estimator which can resolve the sought parameter more accurately than what the UCRB indicates as the ultimate achievable MSE for unbiased estimators [16]. This aspect is pertinent to parametric estimation since it establishes a rigorous performance guarantee. Although it was early observed [10] that the MSE performance can be improved by properly exploiting the bias-variance trade-off, inherent in any estimator, it was not clear how, or if, a superefficient estimator could be systematically constructed. For a certain class of problems, this is settled by the framework proposed in [14]. However, the ML estimate $\hat{\alpha}$ is *biased* and only asymptotically achieves the UCRB (not efficient) implying that our design does not guarantee to enhance the MSE further than what is dictated by the UCRB. Thus, it is interesting to investigate the possibility of extrapolating the min max concept, using the expression for the UCRB instead of the $\text{MSE}(\hat{\alpha})$, in order to construct a

superefficient estimator.⁵ To this end, we define the quantity $\Delta\text{BMSE}(\alpha, \xi) \triangleq \text{MSE}(\bar{\alpha}) - \text{UCRB}(\alpha)$ as

$$\Delta\text{BMSE}(\alpha, \xi) = A_2(\rho, c)\alpha^2 + B_2(\rho, c)\alpha + C_2(\rho, c), \quad (8)$$

where $A_2(\rho, c) = \frac{A_0(\rho, c)}{(N-3)(N-5)} - \frac{2}{N}$, $B_2(\rho, c) = \frac{B_0(\rho, c)}{(N-3)(N-5)} - \frac{4}{N}$ and $C_2(\rho, c) = \frac{C_0(\rho, c)}{(N-3)(N-5)}$. In this context, the min max approach, yields the following optimization problem

$$\begin{aligned} & \underset{\xi}{\text{minimize}} \quad \max_{\alpha} \Delta\text{BMSE}(\alpha, \xi) \\ & \text{subject to} \quad \alpha \geq 0. \end{aligned} \quad (P2)$$

Unlike $\Delta\text{MSE}(\alpha, \xi)$, which is zero for $(\rho, c) = (1, 0)$, it is not obvious if there exists a corresponding pair (ρ, c) such that $\Delta\text{BMSE}(\alpha, \xi) = 0 \quad \forall \alpha \geq 0$. It is a striking fact that it is not actually possible to find any pair (ρ, c) such that $\Delta\text{BMSE}(\alpha, \xi) < \infty \quad \forall \alpha \geq 0$, as stated by the following result.

Proposition 3. *Problem (P2) is infeasible, i.e., its optimal value is $p^* = +\infty$.*⁶

Proof. Given in Appendix C. $\square$

This result indicates that there exists no pair (ρ, c) such that the optimal value of (P2) is finite. The implication is that we cannot construct an AMMLE whose MSE is uniformly lower than the UCRB over all $\alpha \geq 0$, in sharp contrast to the cases studied in [14]. That work considers problems for which the MLE is efficient and for which (P1) and (P2) coincide. Therefore, it is not straightforward how to further improve the MSE performance of the sought estimator along the direction of affinely transforming the MLE.

C. Bounding the SNR

In most practical scenarios, the SNR is not likely to cover the entire span of (non-negative) values but instead it will vary within some range. This observation, in combination with the result of Proposition 3, motivates the investigation of a *trust region* $\mathcal{S}_{a,b}(\alpha) \triangleq \{(\alpha - a)(\alpha - b) \leq 0\}$ for the SNR, i.e., a set of values $a \leq \alpha \leq b$ where the SNR is a priori known to lie. Within $\mathcal{S}_{a,b}$ it might be possible to obtain a superefficient design, since constraining the feasible set of SNR values to $[a, b]$ could facilitate uniform improvement of the $\text{MSE}(\hat{\alpha})$ over the UCRB. In absence of closed-form solutions we answer this question experimentally, through numerical simulations. Upon defining the function $f_{a,b}(\rho, c) \triangleq \max_{\alpha \in \mathcal{S}(a,b)} \Delta\text{BMSE}(\alpha, \xi)$ we set up a minimization problem, similar to (P1) except for the feasible set, which is stated below

$$\underset{\xi}{\text{minimize}} \quad f_{a,b}(\rho, c). \quad (P3)$$

In contrast to (P1) where the solution could be established in closed-form, it is difficult to carry out a similar analysis for

⁵The idea of a superefficient estimator should be treated carefully because it might lure the reader to erroneously relate superefficiency with uniformly minimum variance estimation within the class of biased estimators. The existence of such estimators holds true for the class of the unbiased counterparts, however it is known that “asymptotically there exists no uniformly minimum variance parameter estimator” [16].

⁶Technically speaking one should replace min and max with inf and sup such that infinity can be a feasible value for (P2).

(P3). Therefore the conversion of (P3) into standard convex-form becomes indispensable for the application of numerical methods [17]. The latter is realized after the following result,

Proposition 4. *Problem (P3) is equivalent to the following problem*

$$\begin{aligned} & \underset{\boldsymbol{\eta}}{\text{minimize}} \quad t \\ & \text{subject to} \quad \begin{bmatrix} -1 & f_1(\rho, c) & f_2(\rho, c) \\ f_1(\rho, c) & [\boldsymbol{\Omega}]_{11} & [\boldsymbol{\Omega}]_{12} \\ f_2(\rho, c) & [\boldsymbol{\Omega}]_{21} & [\boldsymbol{\Omega}]_{22} \end{bmatrix} \preceq \mathbf{0} \quad (\text{P4}) \\ & \quad \rho^2 \leq s, c^2 \leq q, \lambda \geq 0, \end{aligned}$$

where $\boldsymbol{\eta} = [\rho \ s \ c \ q \ \lambda \ t]^T$, $f_1(\rho, c) = \frac{N\rho}{\sqrt{(N-3)(N-5)}}$, $f_2(\rho, c) = \frac{\rho + (N-5)c}{\sqrt{(N-3)(N-5)}}$ and the remaining elements $[\boldsymbol{\Omega}]_{ij}$, of the block-matrix $\boldsymbol{\Omega}$, are given as follows

$$\begin{aligned} [\boldsymbol{\Omega}]_{11} &= \frac{-2N(N-5)\rho}{(N-3)(N-5)} + \frac{N-2}{N} - \lambda \\ [\boldsymbol{\Omega}]_{21} &= [\boldsymbol{\Omega}]_{12} = \frac{1}{2} \left[\frac{4Ns - 2(N-5)\rho}{(N-3)(N-5)} - \frac{2Nc + 4}{N} + \lambda(a+b) \right] \\ [\boldsymbol{\Omega}]_{22} &= \frac{2[s + (N-5)q]}{(N-3)(N-5)} - \lambda ab - t. \end{aligned}$$

Proof. Given in Appendix D. $\square$

The proof-line for this result is an extension of the approach proposed in [13]. After the formulation of the cone program a different strategy needs to be adopted, in order to construct the equivalent SDP. The AMMLE MSE expression does not fall into the family of functions considered in [13], hence the framework established therein, albeit covering the general vector parameter case, does not include (6) as a special case. A specific property that renders [13] inapplicable for (6) is that $C_0(\rho, c)$ contains a cross-product term between ρ and c . Thus, new stages are introduced and different mathematical techniques are used to facilitate the conversion of the original problem (P3) into standard-form.

Problem (P4) is an SDP, in the vector variable $\boldsymbol{\eta}$, and can be efficiently solved numerically. Since the SDP is part of the estimator design phase and not of the estimation process, it needs to be solved only once, for a given N , yielding a value for $\boldsymbol{\xi}$ which is independent of α , thus ensuring that our estimator is *realizable*. Since it cannot be guaranteed that the optimal pair (ρ^*, c^*) will satisfy $f_{a,b}(\rho^*, c^*) \leq 0$ we experimentally establish ranges where $\text{MSE}(\bar{\alpha})$ is lower than $\text{UCRB}(\alpha)$, by solving (P4) for different values of a, b and N . Thus we will investigate numerically only for which values of α our proposed estimator will possess the desired attribute $\Delta \text{BMSE}(\alpha, \boldsymbol{\xi}^*) \leq 0 \forall \alpha \in \mathcal{S}_{a,b}$, with strict inequality for some α .

Remark 1: As pointed out by one of the reviewers, one can design the AMMLE against the MSE of the unmodified MLE over a trust region by solving (P1) over some $\mathcal{S}_{a,b} \subset \mathbb{R}_+$. We note that a closed-form solution is not available in that case, yet one can resolve the sought design by constructing an SDP in the exact same way, as demonstrated in Proposition 4.

D. Comparison with previous work

In this section we analyze how the SNR estimation problem is addressed under the umbrella of [12]–[14]. In that framework the AMMLE parameters are also obtained by solving a min max problem, but the optimization problem is based on the UCRB rather than the actual MSE expressions. In particular, for the SNR estimation problem with trust region $\mathcal{S}_{a,b}$ the AMMLE parameters are obtained by solving (see [13, Eq. (6)] or [14, Eq. (98)])

$$\begin{aligned} f^* &= \min_{M,u} \max_{\alpha \in \mathcal{S}_{a,b}} (M\alpha + u)^2 + [(1+M)^2 - 1]J^{-1}(\alpha) \\ &= \min_{M,u} \max_{\alpha \in \mathcal{S}_{a,b}} \underbrace{(M\alpha + u)^2 + M(M+2)\frac{(4\alpha + 2\alpha^2)}{N}}_{f(M,u,\alpha)}, \quad (9) \end{aligned}$$

where the notation $J^{-1}(\alpha) \equiv \text{UCRB}(\alpha)$, $M \equiv \rho - 1$ and $u \equiv c$ is borrowed from [14] to maintain consistency with the definitions therein. While (9) provides a useful heuristic⁷ for some SNR trust regions, we argue that it is inapplicable whenever the trust region contains the zero SNR.

To see this, note first that $f^* \leq 0$ for any $\mathcal{S}_{a,b}$ as $M = 0$ and $u = 0$ yields $f(M, u, \alpha) = f(0, 0, \alpha) = 0$ for any $\alpha \in \mathcal{S}_{a,b}$. It also holds that $f(M, u, 0) = u^2 \geq 0$ for any u which implies that $f^* \geq 0$ and thus $f^* = 0$ whenever $0 \in \mathcal{S}_{a,b}$. It follows that the unmodified MLE obtained by choosing $M = 0$ and $u = 0$, or equivalently $\rho = 1$ and $c = 0$, is optimal with respect to (9) whenever $0 \in \mathcal{S}_{a,b}$. In other words, the method of [12]–[14] is not able to identify an AMMLE that dominates the MLE for this particular case. We hasten to add that *any* solution where $M \in [\frac{-2(2b+4)}{Nb+(2b+4)}, 0]$ and $u = 0$ can in fact be shown to be optimal with respect to (9) when $0 = a < b$, and a numerical solver can thus potentially give a solution that differs from $(M, u) = (0, 0)$. Thus, we can judiciously argue that the framework in [12]–[14] is non-informative in this case, even though [14] originally proposed (9) as a solution for the unconstrained case considered in Section III-A where $a = 0$ and $b = \infty$. On the contrary, (P1) guarantees a unique solution that dominates the MLE also in this case, rendering the AMMLE (LMMLE) parameter selection process informative.

IV. EXTENSION TO THE MIMO CASE

Our analysis has addressed, so far, the design of estimators applicable to scalar models. However this is a special case of the more generic MIMO signal model, where both the input and the output are multidimensional vectors. In this section we extend the design, tailored to (1), to the vector case and generalize the associated mathematical results. For the MIMO model we consider a known signal vector $\mathbf{s}(n) \in \mathbb{R}^{t \times 1}$ that is amplified by an *unknown* matrix $\mathbf{A} \in \mathbb{R}^{r \times t}$ and perturbed by the noise vector $\mathbf{w}(n) \in \mathbb{R}^{r \times 1}$, where $\mathbf{w}(n) \sim \mathcal{N}(\mathbf{0}_r, \sigma_w^2 \mathbf{I}_r)$ and σ_w^2 is unknown. For real-valued quantities, the noisy observation $\mathbf{x}(n)$ is

$$\mathbf{x}(n) = \mathbf{A}\mathbf{s}(n) + \mathbf{w}(n), \quad n = 0, \dots, N-1, \quad (10)$$

⁷The reason that (9) is a heuristic is the lack of an efficient estimator for the SNR estimation problem, as stated earlier in the section, implying that the UCRB is not equivalent to the MSE and that the MLE is not unbiased.

Definition of the matrices $\mathbf{X} \triangleq [\mathbf{x}(0) \dots \mathbf{x}(N-1)]$, $\mathbf{S} \triangleq [\mathbf{s}(0) \dots \mathbf{s}(N-1)]$ and $\mathbf{W} \triangleq [\mathbf{w}(0) \dots \mathbf{w}(N-1)]$ facilitates a more compact description of (10), which is

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{W}, \quad (11)$$

where the matrix $\mathbf{S}$ is assumed to obey the orthogonality shape constraint $\frac{1}{N}\mathbf{S}\mathbf{S}^T = \mathbf{I}_t$. Setting $\mathbf{x} \triangleq \text{vec}(\mathbf{X})$, $\boldsymbol{\tau} \triangleq \text{vec}(\mathbf{A})$, $\mathbf{w} \triangleq \text{vec}(\mathbf{W})$ and $\mathbf{H} \triangleq \mathbf{S}^T \otimes \mathbf{I}_r$ yields, under trivial manipulations, the vectorized counterpart of (11), expressed as

$$\mathbf{x} = \mathbf{H}\boldsymbol{\tau} + \mathbf{w}, \quad (12)$$

For the model (12), the signal-to-noise ratio is $\alpha_m = \frac{\|\mathbf{H}\boldsymbol{\tau}\|_2^2}{\mathbf{E}\|\mathbf{w}\|_2^2} = \frac{N\|\boldsymbol{\tau}\|_2^2}{Nr\sigma_w^2} = \frac{\|\boldsymbol{\tau}\|_2^2}{r\sigma_w^2}$. Let $\boldsymbol{\mu} \triangleq [\boldsymbol{\tau}^T \sigma_w^2]^T$ be the collection of the unknown parameters and $\hat{\boldsymbol{\tau}}, \hat{\sigma}_w^2$ be their associated ML estimates. Standard analysis on the joint pdf $p(\mathbf{x}; \boldsymbol{\mu})$ yields

$$\hat{\boldsymbol{\tau}} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{x}, \quad \hat{\sigma}_w^2 = \frac{1}{Nr} \|\Pi_{\mathbf{H}}^\perp \mathbf{w}\|_2^2,$$

where $\Pi_{\mathbf{H}}^\perp = \mathbf{I}_{Nr} - \mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T$ is again the projection onto the orthogonal complement of the space spanned by the columns of $\mathbf{H}$. The sought estimate is $\hat{\alpha}_m = \frac{\|\hat{\boldsymbol{\tau}}\|_2^2}{r\hat{\sigma}_w^2}$, where $\|\hat{\boldsymbol{\tau}}\|_2^2 \sim \frac{\sigma_w^2}{N} \chi_{rt}^2(\lambda)$, $\lambda = Nr\alpha_m$ is the non-centrality parameter, and $\sigma_w^2 \sim \frac{\sigma_w^2}{Nr} \chi_{(N-t)r}^2(0)$. It follows that $\hat{\alpha}_m = \frac{t}{N-t} Z$, where $Z \sim \mathcal{F}_{tr, (N-t)r}(Nr\alpha_m, 0)$, generalizing the scalar case. The first two moments of $\hat{\alpha}_m$ are given as follows

$$\begin{aligned} \mathbf{E}(\hat{\alpha}_m) &= \frac{tr + Nr\alpha_m}{(N-t)r - 2}, \\ \mathbf{E}(\hat{\alpha}_m^2) &= \frac{(Nr)^2 \alpha_m^2 + (2Ntr^2 + 4Nr)\alpha_m + (2tr + (tr)^2)}{[(N-t)r - 2][(N-t)r - 4]}, \end{aligned}$$

where $N > \frac{4}{r} + t$ should hold for the second moment to be finite. Tedious algebraic steps yield the following expression for the MSE of $\hat{\alpha}_m$.

$$\text{MSE}(\hat{\alpha}_m) = \frac{A_m(t, r)\alpha_m^2 + B_m(t, r)\alpha_m + C_m(t, r)}{[(N-t)r - 2][(N-t)r - 4]}, \quad (13)$$

where $A_m(t, r) = 2Nr + t^2r^2 + 6tr + 8$, $B_m(t, r) = 4Nr + 2t^2r^2 + 8tr$ and $C_m(t, r) = 2tr + t^2r^2$. Proper benchmarking of candidate estimators necessitates the acquisition of the respective UCRB that, using [9, Eqs. (3.30), (3.31)], can be found to be $\text{UCRB}(\alpha_m) = \frac{1}{Nr}(2\alpha_m^2 + 4\alpha_m)$. Notice that this expression is a scaled, by $\frac{1}{r}$, version of the UCRB for the degenerate case, when $r = 1$, and irrespective of the value of t . This is due to the shape constraint, imposed on the training matrix $\mathbf{S}$, which results in uncorrelated estimates and therefore a diagonal Fischer information matrix.

Capitalizing on the results for the scalar case, we can set up the min max framework in order to obtain the AMMLE $\hat{\alpha}_m \triangleq \rho_m \hat{\alpha}_m + c_m$ that uniformly outperforms the MLE in terms of MSE. In that case one should define the equivalent, for the MIMO case, objective $g_m(\rho_m, c_m) \triangleq \max_{\alpha_m \geq 0} [\text{MSE}(\hat{\alpha}_m) - \text{MSE}(\hat{\alpha}_m)]$ and recover the design parameters ρ_m, c_m that minimize this function. The optimal such pair (ρ_m^*, c_m^*) is determined by the following proposition.

Fig. 2. MSE performance comparison between the modified estimators (AMMLE, LMMLE) and the traditional MLE.

Proposition 5. Let (ρ_m^*, c_m^*) be the set of parameters that minimize $g_m(\rho_m, c_m)$. Then it holds that $(\rho^*, c^*) = (\{\frac{Nr-2(tr+4)}{Nr}\}^+, -\frac{tr\rho^*}{[(N-t)r-2]})$.

Proof. Given in Appendix E. $\square$

It is clear that since a superefficient AMMLE design is infeasible for the scalar case (see Proposition 3) then it will also be infeasible for the general MIMO model. Nonetheless, it is again possible to uniformly improve the MSE performance of the AMMLE over the UCRB locally, by carrying out the design over a trust region. In this case, the parameters are retrieved by setting up a semidefinite program, following the rationale of Proposition 4.

V. NUMERICAL RESULTS

In this section we investigate numerically some scenarios through computer simulations in order to illustrate the performance of the AMMLE designs, developed in the previous sections. In all our setups the value of A is kept fixed and is equal to one while the noise variance σ_w^2 is varying. For the scenarios where a trust region is assumed we present, for each estimator, two curves; one corresponds to the theoretical MSE, i.e., the MSE obtained by plugging the pair (ρ^*, c^*) in (6) whereas the other presents the MSE based on the truncated estimator SNR values, i.e., whenever the estimate falls outside the given trust region then it is mapped to its closest boundary value a or b . It is clear that truncation can only reduce the MSE.

Figure 2 illustrates the performance of our proposed estimators $\hat{\alpha}, \tilde{\alpha}$ against $\hat{\alpha}$. It is obvious that our design uniformly outperforms the MLE for all SNR values, and this performance improvement is more salient for small N . Thus, our numerical evaluation verifies that both $\hat{\alpha}$ and $\tilde{\alpha}$ dominate $\hat{\alpha}$. Observe that the MSE curves of the AMMLE and the LMMLE coincide, implying that the introduction of a constant is not particularly helpful in the unconstrained case. The set $\mathcal{S}_{0,+\infty}$, of feasible

Fig. 3. Theoretical and simulated MSE performance comparison between the modified estimators (AMMLE, LMMLE) and the UCRB for a certain trust region and fixed sequence length.

SNR values enforces the “conservative” choice $c^* = -\frac{\rho^*}{N-3}$. This parameterization results in a small, and decreasing with N , value for c , which cannot further influence the AMMLE’s performance in practice.

A different situation is depicted in Figure 3 where the estimators’ MSE performance is compared against the UCRB, for certain trust regions. As stated in Section III-C, the optimal pair (ρ^*, c^*) does not guarantee that $f_{a,b}(\rho^*, c^*) \leq 0$ holds, thus we check the estimators’ performance visually. According to the theoretical predictions the AMMLE $\bar{\alpha}$ is expected to achieve an MSE that is lower than the UCRB, for the designated range $\mathcal{S}_1 = \{-3\text{dB} \leq \alpha \leq 7\text{dB}\}$ (dB denotes logarithmic scale) whereas the LMMLE is expected to fail this goal. However, we can observe that the simulated estimators demonstrate better, than the predicted, performance due to the truncation operation. It is seen that the AMMLE $\bar{a}$ performs better than the LMMLE $\bar{a}$ for the largest fraction of $\mathcal{S}_1$ and, in particular, for the range $\{-1\text{dB} \leq \alpha \leq 7\text{dB}\}$ while $\text{MSE}(\bar{\alpha}) < \text{MSE}(\bar{a})$ holds for the rest part. Therefore, the existence of the constant term in the AMMLE is actually influential in this case. Overall, and albeit the negative result of Proposition 3, we can exploit the idea of constraining the set of admissible SNR values within reasonable bounds such that we can further improve the estimation accuracy of the sought parameter.

Figure 4 shows the simulated MSE performance of our proposed (truncated) AMMLE, designed against both the UCRB and the unmodified MLE MSE (see Remark 1), and the one designed as in [13]. Performance evaluation of our design, when implemented against the UCRB (Problem (P3)), is carried out under two different bounding sets, $\mathcal{S}_2 = \{-6\text{dB} \leq \alpha \leq 6\text{dB}\}$ and $\mathcal{S}_3 = \{-6\text{dB} \leq \alpha \leq 3\text{dB}\}$. Pertaining to the design against the UCRB one observes that when the SNR is known to lie within $\mathcal{S}_2$ then both approaches, AMMLE and AMMLE [13], demonstrate lower,

Fig. 4. Simulated MSE performance comparison among various AMMLE designs for different trust regions designated by LB and UB (correspond to a , respectively b , in (P4)). The estimators indicated by AMMLE, AMMLE [13] and AMMLE (P1) are obtained from (P3), (9) and $\min_{\rho,c} g_{a,b}(\rho, c)$.

Fig. 5. Simulated MSE performance comparison between our estimators and those constructed using the framework of [12], [13], for fixed SNR and different values of N .

than the UCRB, achievable MSE. Albeit our design is better in the largest part of the region, AMMLE [13] performs better for $\alpha \in [3\text{dB}, 6\text{dB}]$. This actually conforms with the intuition following our previous comment (see footnote 5), i.e., that no uniformly minimum variance estimator exists asymptotically, within the class of superefficient estimators. It is further observed that AMMLE (P1), constructed by minimizing $g_{a,b}(\rho, c) \triangleq \max_{\alpha \in \mathcal{S}_{a,b}} \Delta \text{MSE}(\alpha, \xi)$ has different behavior since it is optimized against the unmodified MLE MSE. Hence, it can perform better or worse than the other candidates in different segments of the trust region.

Another interesting observation relates to the trend of the achievable MSE curves. Intuitively, whenever the AMMLE is

designed for the set $\mathcal{S}_3$, it is not expected to provide better resolution than that dictated by the UCRB over the interval $[-6\text{dB}, 6\text{dB}]$. Extending the trust region to $\mathcal{S}_2 \supset \mathcal{S}_3$ results in a more conservative design because it addresses the worst-case scenario. Hence, it follows that the maximum gap between the AMMLE MSE and the UCRB is larger under $\mathcal{S}_3$ than $\mathcal{S}_2$ since, in the latter case, the design has to account for a wider set of possibilities. However, due to the provision for the worst possible case only, our approach by no means dictates the behaviour of the estimator *within the designated region*. Therefore, it is not necessary that the AMMLE will exhibit uniformly better performance under $\mathcal{S}_3$ and this is clearly seen in Figure 4, though it is still superior for the largest fraction of $\mathcal{S}_3$. We can infer the existence of an inherent trade-off between worst-case MSE behavior and best achievable performance (in terms of improving resolution with respect to the UCRB) which, however, does not exclusively dictate the intra-region performance.

Finally, Figure 5 illustrates the MSE performance of the various designs, for a fixed SNR value, with respect to the number of samples. It is seen that all estimators can achieve higher accuracy than the UCRB and all curves follow a similar trend, with increasing sequence length. Furthermore, our proposed estimators demonstrate better performance than their counterparts constructed under the framework of [12]–[14], for this SNR value and in line with intuition all the curves converge to a single one as the number of samples grows. Note, however, that these results are not expected to hold for all SNR values, given that the estimators are optimized in the min max sense and not for every particular SNR value.

VI. CONCLUSIONS

In this work we studied the MSE performance enhancement of the MLE, for the SNR estimation problem, which was realized through transformations of the original estimator. An affinely modified version of the MLE, the AMMLE, was established in closed-form and it was analytically shown that it uniformly outperforms the MLE, over the entire span of SNR values. However, further analysis revealed that no AMMLE can achieve uniformly better estimation accuracy than what the UCRB predicts. This barrier was bypassed by confining our search for feasible designs within designated trust regions, where the SNR is known to lie. The latter design was realized by solving a convex optimization problem. Generalization to the vector case was also briefly outlined and discussed.

In all cases of interest, the proposed estimator is realizable and the simulation results indicate significant performance improvement, at little additional computational cost. Albeit the MIMO case is not studied experimentally, we expect that further aspects will arise concerning the impact of the system dimensions on the MSE. Overall, the AMMLE appears to be a strong candidate to implement SNR estimation in modern systems. The positive outlook of the invoked performance-complexity trade-off supports further investigation of new construction rules, even non-linear such as iteratively updating the MLE based on some criterion, that could further contribute to MSE performance improvement.

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APPENDIX

A. Proof of Proposition 1

Let again $\tilde{\alpha} = \rho\hat{\alpha}$ be the linearly modified (LMMLE) version of the MLE $\hat{\alpha}$. We then have

$$\begin{aligned} \text{MSE}(\tilde{\alpha}) &= \mathbf{E}\{(\rho\hat{\alpha} + c - \alpha)^2\} \\ &= \mathbf{E}\{(\rho\hat{\alpha} - \alpha)^2\} + 2c\mathbf{E}\{(\rho\hat{\alpha} - \alpha)\} + c^2 \quad (\text{A1}) \\ &= \text{MSE}(\tilde{\alpha}) + 2c\mathbf{E}\{(\rho\hat{\alpha} - \alpha)\} + c^2. \end{aligned}$$

In order to provide a complete and self-contained view on how the various quantities, involved in (A1), are evaluated we need to state the mean and the variance of an $\mathcal{F}$ -distributed random variable $Z \sim \mathcal{F}_{\nu_1, \nu_2}(\lambda, 0)$ [18].

$$\begin{aligned} \mathbf{E}[Z] &= \frac{\nu_2(\nu_1 + \lambda)}{\nu_1(\nu_2 - 2)}, \quad \nu_2 > 2 \\ \text{Var}[Z] &= 2 \frac{(\nu_1 + \lambda)^2 + (\nu_1 + 2\lambda)(\nu_2 - 2)}{(\nu_2 - 2)^2(\nu_2 - 4)} \left(\frac{\nu_2}{\nu_1}\right)^2, \quad \nu_2 > 4. \end{aligned}$$

Note that if the conditions on ν_2 are not satisfied then these quantities cannot be defined. Recalling from Section II-A that $\nu_1 = 1, \nu_2 = N - 1$ and $\lambda = N\alpha$, it follows after algebraic manipulations that the moments of $\hat{\alpha} = \frac{1}{N-1}Z$ are given as $\mathbf{E}[\hat{\alpha}] = \frac{(1+N\alpha)}{(N-3)}$ and $\mathbf{E}[\hat{\alpha}^2] = \frac{N^2\alpha^2 + 6N\alpha + 3}{(N-3)(N-5)}$, which are finite for $N \geq 6$. Using these two fundamental results we can show after straightforward but tedious calculations that

$$\text{MSE}(\tilde{\alpha}) = \frac{A'_0(\rho)\alpha^2 + B'_0(\rho)\alpha + C'_0(\rho)}{(N-3)(N-5)}, \quad (\text{A2})$$

where the coefficient are quadratic functions of ρ , given as

$$A'_0(\rho) = (N\rho)^2 - 2N(N-5)\rho + (N-3)(N-5) \quad (\text{A3})$$

$$B'_0(\rho) = 6N\rho^2 - 2(N-5)\rho \quad (\text{A4})$$

$$C'_0(\rho) = 3\rho^2. \quad (\text{A5})$$

Finally, we can evaluate the second term in (A1) as

$$2c\mathbf{E}\{(\rho\hat{\alpha} - \alpha)\} = 2c\rho \frac{1 + N\alpha}{N-3} - 2\alpha c. \quad (\text{A6})$$

Putting together (A2)–(A6) and carrying out simple algebraic steps yields the result of Proposition 1 and completes the proof.

B. Proof of Proposition 2

In the original problem formulation, stated in Section III-A, we seek to minimize the supremum of $\Delta\text{MSE}(\alpha, \xi)$ over all $\alpha \geq 0$, and this was formulated as the following optimization problem

$$\underset{\xi}{\text{minimize}} \quad g(\rho, c), \quad (\text{BP1})$$

which is convex in ξ . The goal, onwards, is to render (BP1) amenable to analysis, by manipulating the inner problem which is written as

$$\begin{aligned} & \underset{\alpha}{\text{maximize}} \quad \Delta \text{MSE}(\alpha, \xi)(N-3)(N-5) \\ & \text{subject to} \quad \alpha \geq 0. \end{aligned} \quad (\text{BP2})$$

The Lagrangian for (BP2) is

$$L(\alpha, \lambda) = A_1(\rho, c)\alpha^2 + [B_1(\rho, c) + \lambda]\alpha + C_1(\rho, c). \quad (\text{B1})$$

Instead of formulating and analyzing the dual problem, it is possible to directly characterize the optimal solution by means of the Lagrangian. The optimal value p^* of our problem (BP1) is now written, in terms of $L(\alpha, \lambda)$, as

$$\begin{aligned} p^* &= \min_{\rho, c} \min_{\lambda \geq 0} \max_{\alpha} L(\alpha, \lambda) \\ &= \min_{\rho, c} \min_{\lambda \geq 0} -\frac{1}{4} \frac{[B_1(\rho, c) + \lambda]^2}{A_1(\rho, c)} + C_1(\rho, c), \end{aligned} \quad (\text{B2})$$

where the second equality is trivially calculated from the solution of the equation $\frac{\partial L(\lambda, \alpha)}{\partial \alpha} \Big|_{\alpha^*} = 0$ and back-substitution of the corresponding value for α^* . A necessary condition, such that the optimal value p^* is finite, is $A_1(\rho, c) \leq 0$ which renders the first term in the second equality a convex quadratic function of λ . At this point two key observations should be stated:

- (O1): The polynomial function $A_1(\rho, c)$ is quadratic in ρ and independent of c . Therefore, its roots are found as $\rho_{A1} = 1, \rho_{A2} = \frac{N-10}{N}$ and the condition $A_1(\rho, c) \leq 0$ is equivalent to $\frac{N-10}{N} \leq \rho \leq 1$.
- (O2): The quadratic function $C_1(\rho, c)$ is jointly strictly convex in the variables (ρ, c) , which is asserted from the fact that its Hessian is positive definite, i.e., $\nabla_{\xi}^2 C_1(\xi) \succ \mathbf{0}_2$ for $N \geq 6$. Therefore, joint minimization of $C_1(\rho, c)$ over ξ is equivalent to successive minimization over the vector elements ρ and c .

From (O2) it follows that, for any given ρ the unconstrained minimizer of $C_1(\rho, c)$, over c , is retrieved by solving $\frac{\partial C_1}{\partial c} \Big|_{c_\rho} = 0$, which yields $c_\rho = -\frac{\rho}{N-3}$. If we plug this value of c in $B_1(\rho, c)$ then we get

$$\begin{aligned} B_1(\rho, c_\rho) &= 6N\rho^2 - 2(N-5)\rho + 2(N-5)\rho \\ &\quad - 2\frac{N(N-5)}{N-3}\rho^2 - (4N+10) \\ &= 2N\left(\frac{2N-4}{N-3}\right)\rho^2 - (4N+10) \\ &\stackrel{(a)}{\leq} 2N\left(\frac{2N-4}{N-3}\right) - (4N+10) \\ &= \frac{4N^2-8N}{N-3} - \frac{4N^2-2N-30}{N-3} \\ &= -6\left(\frac{N-5}{N-3}\right) \stackrel{(b)}{<} 0, \end{aligned} \quad (\text{B3})$$

where (a) follows from the fact that $\rho^2 \leq 1$, due to the bounds stated in (O1); and (b) is due to the condition $N \geq 6$. The insightful interpretation of (B3) is that if we select the constant c to be c_ρ then the supremum of $B_1(\rho, c_\rho)$, over ρ , is always negative. Therefore $\lambda_\rho = -B_1(\rho, c_\rho)$ will always be a feasible

point, forcing the quadratic term to zero, the latter being also its infimum over λ . The point $(\alpha^*, \rho, c_\rho, \lambda_\rho)$ guarantees the attainment of the optimal values (suprema and infima) of the various terms in (B2), in their respective variables α, λ and c . The remaining optimization problem is a function of ρ only and it is written as

$$\begin{aligned} & \underset{\rho}{\text{minimize}} \quad C_1(\rho, c_\rho) \\ & \text{subject to} \quad A_1(\rho, c_\rho) \leq 0, \end{aligned} \quad (\text{BP3})$$

which is trivially shown, using (O1), to be equivalent to

$$\begin{aligned} & \underset{\rho}{\text{minimize}} \quad 2\left(\frac{N-2}{N-3}\right)\rho^2 - 3 \\ & \text{subject to} \quad \frac{N-10}{N} \leq \rho \leq 1. \end{aligned} \quad (\text{BP4})$$

The optimal point that solves (BP4) is $\rho^* = \left\{\frac{N-10}{N}\right\}^+$ and if we plug this into c_ρ and the objective of (BP4) we recover the closed-form expressions for c^* and p^* , suggested by Proposition 2.

Remark 2: If we consider the simpler case of the LMMLE $\tilde{\alpha}$, where $c = 0$ by definition, then the proof can be drastically shortened. In that case, the key element would be the fact that the roots of $B_1(\rho, 0)$ are $\rho_{B1} = 1, \rho_{B2} = -\frac{1}{3}\left(\frac{2N+5}{N}\right)$ and for $N \geq 6$ it holds that $[\rho_{A2}, \rho_{A1}] \subseteq [\rho_{B2}, \rho_{B1}]$ which is equivalent to

$$A_1(\rho) \leq 0 \implies B_1(\rho) \leq 0. \quad (\text{B4})$$

The condition (B4) implies (B3) and the rest of the proof-line holds unchanged and yields the exact same closed-form expression for ρ^* .

C. Proof of Proposition 3

In order to prove the infeasibility of Problem (P2) we show that $p^* = +\infty$ always holds. The inner optimization problem in (P2) involves the maximization of a quadratic function (in α); thus its optimal value will either be finite if the quadratic is concave or infinite if it is convex. We can actually show that, for *all* possible values of the pair (ρ, c) , this quadratic is convex with respect to α . This is carried out by considering the second-order conditions for convexity, i.e., our function will be convex iff $\frac{\partial^2 \Delta \text{BMSE}}{\partial \alpha^2} \geq 0 \forall \alpha \geq 0$.

We have $\frac{\partial^2 \Delta \text{BMSE}}{\partial \alpha^2} = 2A_2(\rho, c)$, with $A_2(\rho, c)$ being quadratic in ρ and independent of c . The discriminant Δ of $A_2(\rho)$ is $\Delta = \frac{1}{K}[4N^4(N-5)^2 - 4N^3(N-2)(N-3)(N-5)] = -\frac{1}{K}24N^3(N-5)$, where $K = N^2(N-3)^2(N-5)^2$. Since $N \geq 6$, by assumption, then it always holds that $\Delta < 0$ which in turn means that $A_2(\rho)$ has no roots in $\mathbb{R}$. Moreover $A_2(\rho)$ will always be positive, since the sign of a quadratic function, when its discriminant is negative, is identical to the sign of the coefficient of its second-order term, which is positive in our case for $N \geq 6$. Thus, $\Delta \text{BMSE}(\alpha, \xi)$ will always be convex in α and the proof is complete.

D. Proof of Proposition 4

We begin directly with the problem of interest, which corresponds to the inner optimization problem in (P3), written as

$$\begin{aligned} & \underset{\alpha}{\text{maximize}} \quad \Delta \text{BMSE}(\alpha, \xi) \\ & \text{subject to} \quad (\alpha - a)(\alpha - b) \leq 0. \end{aligned} \quad (\text{DP1})$$

The Lagrangian for (DP1) is simply

$$L(\alpha, \lambda) = [A_2(\rho, c) - \lambda]\alpha^2 + [B_2(\rho, c) + \lambda(a + b)]\alpha + C_2(\rho, c) - \lambda ab. \quad (\text{D1})$$

Let $\bar{A}_2(\rho, c) \triangleq A_2(\rho, c) - \lambda$, $\bar{B}_2(\rho, c) \triangleq B_2(\rho, c) + \lambda(a + b)$ and $\bar{C}_2(\rho, c) \triangleq C_2(\rho, c) - \lambda ab$. Then, the dual problem is given, in epigraph form [15], as

$$\begin{aligned} & \underset{\xi, t, \lambda}{\text{minimize}} \quad t \\ & \text{subject to} \quad \begin{bmatrix} \bar{A}_2(\rho, c) & \frac{1}{2}\bar{B}_2(\rho, c) \\ \frac{1}{2}\bar{B}_2(\rho, c) & \bar{C}_2(\rho, c) - t \end{bmatrix} \preceq 0 \\ & \quad \lambda \geq 0, \end{aligned} \quad (\text{DP2})$$

where we have implicitly used the Schur complement to reformulate the resulting inequality as a cone inequality, which is in fact a quadratic matrix inequality (QMI). For simplicity the matrix in the inequality constraint in (DP2) is referred to as $\mathbf{M}$. Irrespectively of the convexity of (BP3), the problem is still not in standard form. The challenge is to express the QMI as a linear matrix inequality (LMI) in order to convert (DP2) into an SDP and facilitate its efficient numerical solution by existing software [17]. The desired representation is obtained after the following two-stage process.

T1: Lifting of the cone inequality constraint $\mathbf{M} \preceq \mathbf{0}$ in a higher dimension such that the variables ρ and c decouple.

T2: Introducing slack variables to replace ρ^2, c^2 wherever they appear in the new description and prove tightness of the constraints, involving the slack variables, at the optimum.

The main difficulty in this process lies in the first stage (T1), because to the extent of the authors' knowledge there exists no standardized or structured way to carry out this lifting. In general, we wish to determine coefficients a, b, d, f such that the following representation is an equivalent description to $\mathbf{M} \preceq \mathbf{0}$ in (DP2).

$$\mathbf{P} \triangleq \begin{bmatrix} -1 & \alpha\rho + bc & d\rho + fc \\ \alpha\rho + bc & [\mathbf{W}]_{11} & [\mathbf{W}]_{12} \\ d\rho + fc & [\mathbf{W}]_{21} & [\mathbf{W}]_{22} \end{bmatrix} \preceq \mathbf{0}.$$

At this point, we define the vector $\mathbf{z} = [\alpha\rho + bc \ d\rho + fc]^T$. We know that since $-1 < 0$ then $\mathbf{P} \preceq \mathbf{0}$ if and only if $\mathbf{S} \preceq \mathbf{0}$, where $\mathbf{S} = \mathbf{W} + \mathbf{z}\mathbf{z}^T$ denotes precisely the Schur complement of (-1) in $\mathbf{P}$. Obviously $\mathbf{W}, \mathbf{z}$ should be properly chosen such that $\mathbf{S} = \mathbf{M}$, or equivalently $\mathbf{M} = \mathbf{W} + \mathbf{z}\mathbf{z}^T$, in order to obtain a representation describing the set $\mathbf{M} \preceq \mathbf{0}$. In view of this the coefficients a, b, d, f are selected to map certain

elements of $\mathbf{M}$. In particular, letting $\Pi \triangleq (N-3)(N-5)$ and $(a, b, d, f) = \frac{1}{\sqrt{\Pi}}(N, 0, 1, N-5)$ yields

$$\begin{bmatrix} -1 & \frac{N\rho}{\sqrt{\Pi}} & \frac{\rho+(N-5)c}{\sqrt{\Pi}} \\ \frac{N\rho}{\sqrt{\Pi}} & [\mathbf{W}]_{11} & [\mathbf{W}]_{12} \\ \frac{\rho+(N-5)c}{\sqrt{\Pi}} & [\mathbf{W}]_{21} & [\mathbf{W}]_{22} \end{bmatrix} \preceq \mathbf{0} \quad (\text{S1})$$

and the quantity $\mathbf{z}\mathbf{z}^T$ is calculated as

$$\mathbf{z}\mathbf{z}^T = \frac{1}{\Pi} \begin{bmatrix} N^2\rho^2 & N\rho^2 + N(N-5)\rho c \\ N\rho^2 + N(N-5)\rho c & [\rho + (N-5)c]^2 \end{bmatrix}.$$

Through direct computations the elements of $\mathbf{W}$ are obtained as

$$\begin{aligned} [\mathbf{W}]_{11} &= \frac{-2N(N-5)\rho}{(N-3)(N-5)} + \frac{N-2}{N} - \lambda \\ [\mathbf{W}]_{21} = [\mathbf{W}]_{12} &= \frac{1}{2} \left[\frac{4N\rho^2 - 2(N-5)\rho}{(N-3)(N-5)} - \frac{2Nc + 4}{N} + \lambda(a + b) \right] \\ [\mathbf{W}]_{22} &= \frac{2[\rho^2 + (N-5)c^2]}{(N-3)(N-5)} - \lambda ab - t. \end{aligned}$$

Thus (S1) is an equivalent description for $\mathbf{M} \preceq \mathbf{0}$. It should be noted here that the selection of the coefficients is by no means unique; any selection would serve the target of stage (T1) as long as the bilinear products ρc vanish in the resulting description. The presence of quadratic terms in (S1) is tackled at stage (T2) where the slack variables s and q are introduced such that $\rho^2 \leq s$ and $c^2 \leq q$. The constraint $\mathbf{P} \preceq \mathbf{0}$ is now relaxed as shown below and the representation (S1) becomes

$$\begin{aligned} \mathbf{Q} &\triangleq \begin{bmatrix} -1 & \frac{N\rho}{\sqrt{\Pi}} & \frac{\rho+(N-5)c}{\sqrt{\Pi}} \\ \frac{N\rho}{\sqrt{\Pi}} & [\mathbf{\Omega}]_{11} & [\mathbf{\Omega}]_{12} \\ \frac{\rho+(N-5)c}{\sqrt{\Pi}} & [\mathbf{\Omega}]_{21} & [\mathbf{\Omega}]_{22} \end{bmatrix} \preceq \mathbf{0} \\ &\quad \rho^2 \leq s, c^2 \leq q, \end{aligned} \quad (\text{S2})$$

where $[\mathbf{\Omega}]_{ij}$ are the elements of the inner block-matrix $\mathbf{\Omega}$ and they are given as

$$\begin{aligned} [\mathbf{\Omega}]_{11} &= [\mathbf{W}]_{11} \\ [\mathbf{\Omega}]_{21} = [\mathbf{\Omega}]_{12} &= \frac{1}{2} \left[\frac{4Ns - 2(N-5)\rho}{(N-3)(N-5)} - \frac{2Nc + 4}{N} + \lambda(a + b) \right] \\ [\mathbf{\Omega}]_{22} &= \frac{2[s + (N-5)q]}{(N-3)(N-5)} - \lambda ab - t. \end{aligned}$$

Problem (DP2) can now be rewritten, using the description (S2), as

$$\begin{aligned} & \underset{t}{\text{minimize}} \quad t \\ & \text{subject to} \quad \begin{bmatrix} -1 & f_1(\rho, c) & f_2(\rho, c) \\ f_1(\rho, c) & [\mathbf{\Omega}]_{11} & [\mathbf{\Omega}]_{12} \\ f_2(\rho, c) & [\mathbf{\Omega}]_{21} & [\mathbf{\Omega}]_{22} \end{bmatrix} \preceq \mathbf{0} \\ & \quad \rho^2 \leq s, c^2 \leq q, \lambda \geq 0, \end{aligned} \quad (\text{DP3})$$

where we defined, for brevity, the functions $f_1(\rho, c) \triangleq \frac{N\rho}{\sqrt{(N-3)(N-5)}}$ and $f_2(\rho, c) \triangleq \frac{\rho+(N-5)c}{\sqrt{(N-3)(N-5)}}$. The following proposition establishes the equivalence of the optimization problem at hand and (DP2).

Proposition 6. Let $\boldsymbol{\eta} = [t \ \rho \ s \ c \ q \ \lambda]^T$ be the optimal point for (DP3). Then $\rho^2 = s$ and $c^2 = q$, i.e., the quadratic constraints

are tight at the optimum, implying that (DP2) and (DP3) are equivalent.

Proof. Assume that $c^2 < q$. Then we can construct a point $\eta^* = [t^* \rho s c q^* \lambda]^T$, with $q^* = q - \epsilon$, $t^* = t - \frac{2}{(N-3)}\epsilon$ and ϵ sufficiently small such that the inequality still holds. Then it is easy to see that $\mathbf{Q}(\eta^*) = \mathbf{Q}(\eta) \preceq \mathbf{0}$, i.e., η^* is feasible and results in a lower objective value, which is a contradiction.

In the same spirit we show tightness of $\rho^2 \leq s$. If the constraint is not tight then we can construct a point $\eta^* = [t^* \rho s^* c q \lambda^*]^T$, with $s^* = s - \epsilon$, $\lambda^* = \lambda + \frac{1}{(a+b)} \frac{4N\epsilon}{(N-3)(N-5)}$, $t^* = t - \left(1 + \frac{2Nab}{a+b}\right) \frac{2\epsilon}{(N-3)(N-5)}$ and ϵ sufficiently small such that the inequality still holds. Again $\mathbf{Q}(\eta^*) = \mathbf{Q}(\eta) \preceq \mathbf{0}$ is feasible, resulting in a contradiction. $\square$

E. Proof of Proposition 5

The line-of-proof is identical to that of Proposition 2. Therefore, we briefly outline it here and state explicitly only the key elements. Retrieval of the optimal pair (ρ_m^*, c_m^*) is based on expressing the optimal value of the respective optimization problem in terms of the Lagrangian, as in (B2). One should again exploit the fact that the coefficient of α_m^2 , in the expression for ΔMSE_m , is a polynomial of ρ_m only, whose roots are $\frac{Nr-2(tr+4)}{Nr}$ and 1. One can then repeat the step (B3) and establish the optimization problem, as in (BP4), leading to the sought result.

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