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This is the published version of a paper presented at *IEEE International Instrumentation and Measurement Technology Conference (I2MTC), 2013*.

Citation for the original published paper:

Negusse, S., Händel, P., Zetterberg, P. (2013)

On SNR estimation using IEEE-STD-1057 three-parameter sine wave fit.

In:

N.B. When citing this work, cite the original published paper.

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<http://urn.kb.se/resolve?urn=urn:nbn:se:kth:diva-136145>

On SNR estimation using IEEE-STD-1057 three-parameter sine wave fit

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Abstract—In this paper, theoretical properties of a maximum-likelihood (ML) estimator of signal-to-noise ratio (SNR) is discussed. The three-parameter sine fit algorithm is employed on a finite and coherently sampled measurement set corrupted by additive white Gaussian noise. Under the Gaussian noise model, the least squares solution provided by the three-parameter sine fit is also ML estimator. Exact distribution and finite sample properties of the SNR estimate are derived. Moreover, an explicit expression for the mean squared error (MSE) of the estimator is given. Simulation results are shown to verify the underlying theoretical results.

Index Terms—Sine-fit algorithm, maximum-likelihood, coherent sampling, Signal-to-Noise Ratio.

I. INTRODUCTION

The three-parameter and four-parameter sine fit algorithms defined in IEEE standard 1057 [1] and IEEE standard 1241 [2] provide a straight forward technique for extracting parameters from a set of samples which are crucial, for example, in characterizing and testing of devices such as analog-to-digital converters (ADCs).

Study of estimation performance, as well as performance comparison of the three-parameter and four-parameter sine fit based on the underlying noise model is presented in [3]–[5]. In [3] and [4], the asymptotic Cramér-Rao bound (CRB) is used to evaluate the performance of the three and four-parameter sine fit algorithms. The parsimony principle is further used in [4] to present detailed comparison of the mean squared error performance of the three and four-parameter fit showing that the latter method often performs better than the former in terms of a smaller expected sum-squared error.

In this paper, we consider the three-parameter sine fit algorithm for SNR estimation in a Gaussian noise scenario, assuming that the frequency is known. Previous works in [6] and [7] presented detailed analysis of the statistical properties of the estimates of amplitude and the squared amplitude using the three-parameter fit. The Rician distribution of the amplitude estimate obtained in [7], which is derived from the noncentral chi-square distribution of the squared amplitude estimate, led to a simplified derivation of the bias and variance. The work in [8] presents the statistical properties of the amplitude estimation as well as its relation to the input frequency considering optimal and suboptimal values for sampling rate and number of samples. Further analysis comparing performance of the three-parameter and four-parameter sine fit algorithms employed for noise variance estimation under noncoherent sampling is discussed in [9]. In this paper, we derive an

exact expression for distribution of the maximum-likelihood estimate of SNR, based on which detailed analysis of the finite sample properties of the estimator will be presented. Assuming coherent sampling, we use explicit expression of the distribution of the noise power estimate to derive properties of the SNR estimate based on analysis of the ratio distribution. Coherently sampled simulation data is used to verify the given theoretical results on the statistical properties of the estimated SNR. Moreover, by relaxing the assumption, we will present simulation result for a non-coherently sampled data set which shows a good match to the derived results.

II. SNR ESTIMATION USING THREE-PARAMETER SINE FIT

A. Three-parameter sine wave fit

Consider the N -vector $\mathbf{y}$ of *measurements*,

$$\mathbf{y} = (y(0), \dots, y(N-1))^T \quad (1)$$

where T denotes the transpose operator. The vector entry $y(n)$ for a given time instant n is assumed to fulfill the *signal model*

$$y(n) = s(n; \theta) + v(n) \quad (2)$$

for some *true* θ_0 where $s(n; \theta)$ is the deterministic signal model parameterized using the generic parameter vector θ , and $v(n)$ is a zero mean white Gaussian noise of unknown variance σ^2 , which is assumed to model the imperfections and noise not captured by the deterministic part. Further, the deterministic signal model is given by

$$s(n; \theta) = A \cos(\omega n) + B \sin(\omega n) + C. \quad (3)$$

In (3), $\theta = (A, B, C)^T$ defines the parameter vector with the inphase A , quadrature B and the direct current C components. The angular frequency $\omega = 2\pi F/F_s$ is assumed known, with the absolute frequency F [s^{-1}] and the sampling frequency F_s [s^{-1}].

The three-parameter least squares fit is obtained by minimizing the loss function [1]

$$\mathcal{V}(\theta) = \frac{1}{N} \sum_{n=0}^{N-1} (y(n) - s(n; \theta))^2 \quad (4)$$

with respect to θ , that is

$$\hat{\theta} = \arg \min_{\theta} \mathcal{V}(\theta) \quad (5)$$

where $\hat{\theta}$ denotes the least-squares solution. Equation (5) can be written in closed-form employing a vector notation

$$\hat{\theta} = (\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T \mathbf{y} \quad (6)$$

where the matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \begin{pmatrix} \cos \omega \cdot 0 & \sin \omega \cdot 0 & 1 \\ \cos \omega \cdot 1 & \sin \omega \cdot 1 & 1 \\ \vdots & \vdots & \vdots \\ \cos \omega \cdot (N-1) & \sin \omega \cdot (N-1) & 1 \end{pmatrix}. \quad (7)$$

For the given Gaussian noise model in (2), the maximum likelihood (ML) estimator

$$\hat{\theta}_{\text{ML}} = \arg \min_{\theta} \ln p(\mathbf{y}; \theta) \quad (8)$$

where

$$p(\mathbf{y}; \theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp \left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (y(n) - s(n; \theta))^2 \right] \quad (9)$$

is also given by the least squares solution in (6) [10].

B. SNR estimation

Once $\hat{\theta}$ is obtained, the residual error $e(n)$ is given by

$$e(n) = y(n) - s(n; \hat{\theta}). \quad (10)$$

The ML estimator of the noise variance, σ^2 , is given by the sample variance of the residual error $e(n)$ [10]

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{n=0}^{N-1} e^2(n) = \mathcal{V}(\hat{\theta}) \quad (11)$$

where the second equality follows from (6) and (10). An estimate of the SNR is finally obtained by the invariance principle of the method of ML, that is

$$\widehat{\text{SNR}}_{\text{ML}} = \frac{\hat{A}^2 + \hat{B}^2}{2\hat{\sigma}^2} = \frac{\hat{\alpha}^2}{2\hat{\sigma}^2} \quad (12)$$

where $\hat{A}$ and $\hat{B}$ are the first two entries of $\hat{\theta}$, and the estimated squared amplitude $\hat{\alpha}^2 = \hat{A}^2 + \hat{B}^2$ is introduced in the second equality.

Employing Cochran's theorem [11], which states that the sample mean and sample variance of a set of independent Gaussian distributed random variables are independent, it can be concluded that $\hat{\theta}$ determined from the weighted sample mean (6) and the noise variance estimate $\hat{\sigma}^2$ in (11) are independent. Consequently, $\hat{\alpha}^2$ and $\hat{\sigma}^2$ are also independent.

III. ANALYSIS OF ML

In order to perform the analysis, we will assume that the data is coherently sampled, that is the N -vector $\mathbf{y}$ in (1) contains exactly an integer J periods of the sine wave, that is $F/F_s = J/N$. The assumption may at first glance seem restrictive, although it makes it possible to derive exact closed-form properties of the estimator performance.

Recall (1)-(2) and introduce the vector notation $\mathbf{y} = \mathbf{s}(\theta_0) + \mathbf{v}$ for the modeled data. Then, the estimator $\hat{\theta}$ in (6) can be expressed in terms of the true vector θ_0 as

$$\begin{aligned} \hat{\theta} &= (\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T (\mathbf{s}(\theta_0) + \mathbf{v}) \\ &= \theta_0 + (\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T \mathbf{v} \end{aligned} \quad (13)$$

where in the second equality $\mathbf{s}(\theta_0) = \mathbf{D} \theta_0$ was used. With $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ (with $\mathbf{I}$ being the unity matrix of suitable order, when appropriate the order N is explicitly given as $\mathbf{I}_N$), it follows that $\hat{\theta}$ is multivariate Gaussian $\mathcal{N}(\theta_0, \mathbf{\Gamma}_\theta)$. The covariance matrix $\mathbf{\Gamma}_\theta$ is given by

$$\mathbf{\Gamma}_\theta = \frac{\sigma^2}{N} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (14)$$

where the diagonal structure follows from the coherency of data [4].

Under the given assumptions, properties of the squared amplitude estimate, $\hat{\alpha}^2$, are discussed in [6], [7]. It was shown that the normalized estimate $\hat{\alpha}^2 N / (2\sigma^2)$ is non-centrally chi-squared distributed with $k = 2$ degrees of freedom and non-centrality parameter $\lambda = \text{NSNR}$.

Similarly, properties of the noise power estimate $\hat{\sigma}^2$ can be derived from its probability density function. Consider a vector representation $\mathbf{e}$ of the residual error in (10), one has

$$\mathbf{e} = \mathbf{y} - \mathbf{s}(\hat{\theta}) = \mathbf{y} - \mathbf{D} \hat{\theta}. \quad (15)$$

Then using (2) and (13), (15) reduces to

$$\mathbf{e} = \underbrace{(\mathbf{I} - \mathbf{D}(\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T)}_{\mathbf{\Pi}^\perp} \mathbf{v} \quad (16)$$

where the matrix $\mathbf{\Pi}^\perp$ is symmetric and idempotent (that is, $\mathbf{\Pi}^\perp \mathbf{\Pi}^\perp = \mathbf{\Pi}^\perp$). The vector of residuals $\mathbf{e}$ is therefore multivariate Gaussian $\mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_e)$ with covariance matrix $\mathbf{\Gamma}_e = \mathbb{E}[\mathbf{e}\mathbf{e}^T]$, obtained using the symmetry and idempotency property of the matrix $\mathbf{\Pi}^\perp$, given by

$$\mathbf{\Gamma}_e = \mathbb{E}[\mathbf{\Pi}^\perp \mathbf{v} \mathbf{v}^T (\mathbf{\Pi}^\perp)^T] = \sigma^2 \mathbf{\Pi}^\perp. \quad (17)$$

The sum of the square of the residuals in (11), which is also an ML estimate of the noise variance, can be expressed in a quadratic form as

$$\hat{\sigma}^2 = \frac{\mathbf{e}^T \mathbf{e}}{N} = \frac{\mathbf{v}^T \mathbf{\Pi}^\perp \mathbf{v}}{N}. \quad (18)$$

Then, from [12], the distribution of the quadratic form given by

$$\hat{\sigma}^2 \frac{N}{\sigma^2} = \frac{\mathbf{v}^T \mathbf{\Pi}^\perp \mathbf{v}}{\sigma^2} \quad (19)$$

is the chi-square distribution with m degrees of freedom where m is the rank of the symmetric idempotent matrix $\mathbf{\Pi}^\perp$. The rank of $\mathbf{\Pi}^\perp$ is equal to $m = \text{tr}(\mathbf{\Pi}^\perp)$ and is evaluated as

$$m = \text{tr}(\mathbf{I}) - \text{tr}(\mathbf{D}(\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T) = N - 3 \quad (20)$$

where $\text{tr}(\cdot)$ denotes the trace operator and the second equality follows from $\text{tr}(\mathbf{D}(\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T) = \text{tr}(\mathbf{I}_3) = 3$, noting that

$$\text{tr}(\mathbf{D}(\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T) = \text{tr}((\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T \mathbf{D}) \quad (21)$$

since the trace-operator is invariant under cyclic permutations.

Therefore, let S be a chi-square distributed stochastic variable with $s = N - 3$ degrees of freedom, and let A be a non-centrally chi-squared distributed with $a = 2$ degrees of freedom and non-centrality parameter $\lambda = N\alpha^2/2\sigma^2 = N\text{SNR}$. Then, from (12), the estimator $\widehat{\text{SNR}}_{\text{ML}}$ reads

$$\widehat{\text{SNR}}_{\text{ML}} = \frac{2}{N-3} \frac{\left(\frac{\hat{\sigma}^2 N}{2\sigma^2}\right)/2}{\left(\frac{N\hat{\sigma}^2}{\sigma^2}\right)/(N-3)} = \frac{2}{N-3} \frac{A/a}{S/s}. \quad (22)$$

If $\mathcal{F} = (A/2)/(S/(N-3))$, then $\mathcal{F}$ obeys a noncentral $F'_{2,(N-3)}(N\text{SNR})$ -distribution since A and S are statistically independent [12].

The expectation (for $N > 5$) of $\widehat{\text{SNR}}_{\text{ML}}$ can be found in the literature [12], that is

$$\mathbb{E}[\widehat{\text{SNR}}_{\text{ML}}] = \frac{N}{N-5} \text{SNR} + \frac{2}{N-5} \quad (23)$$

and the variance (for $N > 7$) reads

$$\text{Var}[\widehat{\text{SNR}}_{\text{ML}}] = 2 \frac{(2 + N\text{SNR})^2 + (2 + 2N\text{SNR})(N-5)}{(N-5)^2(N-7)}. \quad (24)$$

The mean-square error (MSE) is then computed as the sum of the variance and the squared bias [10]

$$\text{MSE}[\widehat{\text{SNR}}_{\text{ML}}] = \text{Var}[\widehat{\text{SNR}}_{\text{ML}}] + \left(\mathbb{E}[\widehat{\text{SNR}}_{\text{ML}}] - \text{SNR}\right)^2 \quad (25)$$

where $\mathbb{E}[\widehat{\text{SNR}}_{\text{ML}}] - \text{SNR}$ is the bias and using (23) and (24) on (25), the MSE of the ML-estimator becomes

$$\text{MSE}[\widehat{\text{SNR}}_{\text{ML}}] = \frac{\text{SNR}^2(2N+35) + \text{SNR}(4N+28)}{(N-5)(N-7)} + \frac{(8N-40)}{(N-5)^2(N-7)}. \quad (26)$$

The noncentral $F'_{2,(N-3)}(N\text{SNR})$ -distribution has known asymptotic properties, that is, it approaches the normal distribution as $N \rightarrow \infty$. Further, established theories on ML-estimator reveal that it is asymptotically efficient, as well as consistent and that the distribution of the parameters of interest converges to Gaussian distribution as the number of samples N increases [10].

IV. NUMERICAL RESULTS

Computer simulations were run employing the three-parameter sine fit algorithm for SNR estimation in order to verify the theoretical results.

Fig. 1. Histogram of simulated and normalized $\widehat{\text{SNR}}$ and analytical probability function of $F'_{2,N-3}(N\text{SNR})$ distribution (solid line) for $N = 12$ and $\text{SNR} = 100$ in linear scale.

A. Coherent data

In order to investigate the characteristics of the SNR estimators, 10^5 simulation runs were performed on a coherent set of sine wave signal for various number of samples N and corresponding signal period integer J such that the normalized angular frequency is $\omega = 2\pi J/N$. Fig. 1 shows that for $N = 12$ samples and $\text{SNR} = 20\text{dB}$, the histogram of normalized SNR matches the predicted theoretical probability density function which is non-central $F'_{k_1,k_2}(N\text{SNR})$ distribution where $k_1 = 2$, $k_2 = 9$ and $N\text{SNR} = 1200$. In a similar manner, in order to assess the asymptotic distribution, Fig. 2 shows the histogram of normalized SNR for a larger number of samples, i.e. $N = 1092$, such that the non-centrality parameter becomes, $N\text{SNR}$, very large. As described in [12], for large $N\text{SNR}$, $F'_{2,N-3}(N\text{SNR})$ approaches a non-central chi-square distribution with $N-3$ degrees of freedom and a non-centrality parameter $N\text{SNR}$. Further, for large $N\text{SNR}$, it is known that the non-central chi square distribution approaches the normal distribution, which is also shown to be true from the simulation result in Fig. 2.

In Fig. 3, the theoretical expressions of the squared bias, variance and MSE for $N = 12$ are plotted showing good match with simulation values.

B. Non-coherent data

By relaxing the restriction on the coherency of the data set, we further, investigate the distribution for a case when the normalized frequency is randomly drawn from the uniform distribution $[2\pi/N, \pi - 2\pi/N]$. Fig. 4 shows the histogram of the normalized SNR in close match with the pdf of $F'_{2,N-3}(N\text{SNR})$ for $N = 24$ and $\text{SNR} = 20\text{dB}$. Therefore, it can be stated that the derived theoretical result for coherent sample data also applies, with good approximation, to non-coherent samples.

Fig. 2. Histogram of simulated and normalized $\widehat{\text{SNR}}$ and analytical probability function of $F'_{2,N-3}(\text{NSNR})$ distribution (solid line) for $N = 1092$ and $\text{SNR} = 100$ in linear scale.

Fig. 4. Histogram of simulated and normalized $\widehat{\text{SNR}}$ and analytical probability function of $F'_{2,N-3}(\text{NSNR})$ distribution (solid line) for $N = 24$ and $\text{SNR} = 100$ in linear scale.

Fig. 3. Theoretical and simulated plots of variance, squared bias and MSE of SNR_{ML} for $N = 12$ as a function of SNR .

V. CONCLUSION

This paper deals with analysis of SNR estimator employing the three-parameter fit of a sine wave in additive white Gaussian noise under coherent sampling, which is ML. Exact distribution of the noise variance estimate is derived, which in turn is used to derive the distribution of the SNR estimate. Moreover, the distribution of the ML estimator was used to derive explicit expression for the MSE. Simulation results are shown to have excellent match with the theoretical results. By relaxing the coherent sampling assumption, it was shown that the theoretical results also hold for non-coherently sampled data set. Further study of the MSE performance compared to the Cramér–Rao Bound (CRB) for an unbiased estimator

is presented in [13]. With the aim of improving the MSE, alternative estimators are derived which result in superior performance compared to the ML estimator.

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