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Discrete-time Network-based Control under Try-Once-Discard Protocol and Actuator Constraints

Kun Liu and Emilia Fridman

Abstract—This paper deals with the solution bounds for discrete-time Networked Control Systems (NCSs) via delay-dependent Lyapunov-Krasovskii methods. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. A time-delay approach was introduced recently for continuous-time NCS under a weighted Try-Once-Discard (TOD) protocol in [6], where actuator saturation was not taken into account. In the present paper, we develop the time-delay approach for linear (probably, uncertain) discrete-time NCS under the weighted TOD protocol in the presence of actuator saturation. A novel Lyapunov-based method is presented for finding the domain of attraction. Polytopic uncertainties in the system model can be easily included in our analysis. The efficiency of the time-delay approach is illustrated on the example of a cart-pendulum system.

Keywords: Networked control systems, time-delay approach, scheduling, input saturation, Lyapunov-Krasovskii method.

I. INTRODUCTION

Networked Control Systems are control systems comprised of the system to be controlled and of actuators, sensors, and controllers, the operation of which is coordinated via a communication network [1]. In the present paper, we focus on the stability analysis of discrete-time NCS with communication constraints and actuator constraints. A linear (probably, uncertain) system with distributed sensors is considered.

The time-delay approach has been introduced recently for the stabilization of continuous-time NCS under the Round-Robin (RR) protocol in [7] and under a weighted Try-Once-Discard (TOD) protocol in [6], respectively. The closed-loop system is modeled as a switched system with multiple and ordered time-varying delays under RR scheduling or as a hybrid system with time-varying delays in the dynamics and in the reset equations under the TOD scheduling. Differently from the existing hybrid [4] and discrete-time [3] approaches, the transmission delay is allowed to be larger than the sampling interval. However, actuator saturation has not been taken into account in [6] and [7]. Recently, the stabilization of sampled-data systems under variable samplings and actuator saturation was studied in [10], where scheduling protocols and delays are not included.

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As was pointed out in [5], when we deal with the solution bounds of time-delay systems via Lyapunov-Krasovskii method, the first time-interval of the delay length needs a special analysis. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. This first time-interval does not influence on the stability and the exponential decay rate analysis. The analysis of the first time-interval is important for nonlinear systems e.g., for finding the domain of attraction.

In the present paper, the time-delay approach is extended to the stability analysis of discrete-time NCS under the weighted TOD protocol, actuator saturation, variable sampling intervals and variable delays. Following [5], we present a Lyapunov-based method for finding the domain of attraction under the weighted TOD protocol. The conditions are given in terms of Linear Matrix Inequalities (LMIs). Polytopic uncertainties in the system model can be easily included in the analysis. The efficiency of the presented approach is illustrated by a cart-pendulum system.

Notation: Throughout the paper the superscript ‘ T ’ stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. $\mathbb{Z}$ and $\mathbb{N}$ denote the set of non-negative integers and positive integers, respectively. For any matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$, the notations A_j and x_j denote, respectively, the j^{th} line of matrix A and the j^{th} component of vector x . $\mathbb{Z}$ denotes the set of non-negative integers. Given $\bar{u} = [\bar{u}_1, \dots, \bar{u}_{n_u}]^T$, $0 < \bar{u}_i$, $i = 1, \dots, n_u$, for any $u = [u_1, \dots, u_{n_u}]^T$ we denote by $\text{sat}(u, \bar{u})$ the vector with coordinates $\text{sign}(u_i) \min(|u_i|, \bar{u}_i)$.

II. SYSTEM MODEL

A. Discrete-time NCS model

Consider the system architecture in Fig.1 with discrete-time plant:

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{Z}, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, A and B are (probably, uncertain) system matrices of appropriate dimensions. The initial condition is $x(0) = x_0$. We suppose that the control input is subject to the following amplitude constraints

$$|u_i(t)| \leq \bar{u}_i, \quad 0 < \bar{u}_i, \quad i = 1, \dots, n_u, \quad t \in \mathbb{Z}. \quad (2)$$

The NCS has several nodes (distributed sensors, a controller node and an actuator node) which are connected via networks. For the sake of simplicity, we consider two sensor nodes $y^i(t) = C^i x(t)$, $i = 1, 2$ and we denote $C = \begin{bmatrix} C^1 \\ C^2 \end{bmatrix}$, $y(t) = \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \in \mathbb{R}^{n_y}$. The results can be easily extended to any finite number of sensors. We let s_k denote the unbounded and monotonously increasing sequence of sampling instants

$$0 = s_0 < s_1 < \dots < s_k < \dots, \quad k \in \mathbb{Z}, \quad \lim_{k \rightarrow \infty} s_k = \infty, \quad (3)$$

where $\{s_k\}$ is a subsequence of $\mathbb{Z}$. At each sampling instant s_k , one of the outputs $y^i(t) \in \mathbb{R}^{n_i}$ ($n_1 + n_2 = n_y$) is sampled and transmitted over the network.

Fig. 1. NCS with actuator saturation under TOD protocol

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the two networks (between the sensor and the actuator) is subject to a variable total delay $\eta_k = \eta_k^{sc} + \eta_k^{ca}$, where $\eta_k^{sc} \in \mathbb{Z}$ and $\eta_k^{ca} \in \mathbb{Z}$ are the network-induced delays from the sensor to the controller and from the controller to the actuator, respectively. Then $t_k = s_k + \eta_k$ is the updating time instant of the Zero-Order Hold (ZOH), where $\{t_k\}$ is a subsequence of $\mathbb{Z}$.

Following [7] and [8], we allow the delay to be non-small provided that the old sample cannot get to the destination (to the controller or to the actuator) after the most recent one. Assume that the network-induced delay η_k and the time span between the updating and the most recent sampling instants are bounded:

$$\begin{aligned} t_{k+1} - t_k + \eta_k &\leq 1 + \tau_M, \quad i.e. \quad (t_{k+1} - 1) - s_k \leq \tau_M, \\ 0 &\leq \eta_m \leq \eta_k \leq \eta_M, \quad k \in \mathbb{Z}. \end{aligned} \quad (4)$$

Here τ_M , η_m and η_M are known non-negative integers and $\tau_M = MATI + \eta_M$, where $MATI$ denotes the Maximum Allowable Transmission Interval. Then

$$\begin{aligned} (t_{k+1} - 1) - t_k &\leq \tau_M - \eta_m, \\ (t_{k+1} - 1) - t_{k-1} + \eta_{k-1} &\leq 2\tau_M - \eta_m + 1 \triangleq \bar{\tau}_M. \end{aligned} \quad (5)$$

Note that the first updating time t_0 corresponds to the first data received by the actuator, which means that $u(t) = 0$, $t \in [0, t_0 - 1]$. Then for $t \in [0, t_0 - 1]$, (1) is given by

$$x(t+1) = Ax(t), \quad t = 0, 1, \dots, t_0 - 1, \quad t \in \mathbb{Z}. \quad (6)$$

B. TOD protocol and a discrete-time hybrid time-delay model

In [6], a weighted TOD protocol was introduced for the stabilization of continuous-time NCS in terms of time-delay approach. Actuator saturation was not taken into account. In this section, we consider discrete-time NCS with actuator saturation under this weighted TOD protocol. Following [6], consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$e(t) = \text{col}\{e_1(t), e_2(t)\} \triangleq \hat{y}(s_{k-1}) - y(s_k), \quad t \in [t_k, t_{k+1}-1], \quad t \in \mathbb{Z}, k \in \mathbb{Z}, \quad \hat{y}(s_{-1}) \triangleq 0, \quad e(t) \in \mathbb{R}^{n_y}.$$

Let $Q_i > 0$, $i = 1, 2$ be some weighting matrices (they will be found from LMIs in Lemma 2 below). The node that has the largest error, $|\sqrt{Q_i}e_i(t)|^2$, $i = 1, 2$ is granted access to the network. Consider the first case of

$$|\sqrt{Q_1}e_1(t)|^2 \geq |\sqrt{Q_2}e_2(t)|^2, \quad t \in [t_k, t_{k+1}-1], \quad t \in \mathbb{Z}. \quad (7)$$

Due to the control bounds defined in (2), the effective control signal to be applied to the system (1) is given by

$$\begin{aligned} u(t) &= \text{sat}[K^1 y^1(t_k - \eta_k) + K^2 \hat{y}^2(t_{k-1} - \eta_{k-1})], \\ t &\in [t_k, t_{k+1}-1], \quad t \in \mathbb{Z}. \end{aligned}$$

We define the polyhedron

$$\mathcal{L}(K^j, \bar{u}) = \{x \in \mathbb{R}^n : |(K^j C^j)_i x| \leq \frac{1}{2} \bar{u}_i, \quad i = 1, \dots, n_u\},$$

where $j = 1, 2$. If the control is such that $x \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$, then $|(K^1 C^1)_i x + (K^2 C^2)_i x(t)| \leq \bar{u}_i$. Following [6], we obtain thus the impulsive closed-loop model with the following discrete dynamics:

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_k - \eta_k) + B_2 e_2(t), \\ e(t+1) &= e(t), \quad t \in [t_k, t_{k+1}-1], \quad t \in \mathbb{Z}, \end{aligned} \quad (8)$$

where $A_1 = B K C$, $B_i = B K^i$, $K = [K^1 \ K^2]$, $i = 1, 2$. The delayed reset system is given by

$$\begin{aligned} x(t_{k+1}) &= x(t_{k+1}^-), \\ e(t_{k+1}) &= \begin{bmatrix} 0 \\ e_2(t_{k+1}^-) \end{bmatrix} + Cx(t_k - \eta_k) - Cx(t_{k+1} - \eta_{k+1}). \end{aligned} \quad (9)$$

Similarly if

$$|\sqrt{Q_2}e_2(t)|^2 > |\sqrt{Q_1}e_1(t)|^2, \quad t \in [t_k, t_{k+1}-1], \quad t \in \mathbb{Z}, \quad (10)$$

then the closed-loop system has the form

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_k - \eta_k) + B_1 e_1(t), \\ e(t+1) &= e(t), \quad t \in [t_k, t_{k+1}-1], \quad t \in \mathbb{Z} \end{aligned} \quad (11)$$

with the delayed reset system

$$\begin{aligned} x(t_{k+1}) &= x(t_{k+1}^-), \\ e(t_{k+1}) &= \begin{bmatrix} e_1(t_{k+1}^-) \\ 0 \end{bmatrix} + Cx(t_k - \eta_k) - Cx(t_{k+1} - \eta_{k+1}), \end{aligned} \quad (12)$$

where $A_1 = BKC$, $B_i = BK^i$, $i = 1, 2$. The initial condition for (7)-(12) has the form of (6) and $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$.

Denote by $x(t, x_0)$ the state trajectory of (7)-(12) with respect to x and with the initial condition $x_0 \in \mathbb{R}^n$. Then the domain of attraction of the origin of the closed-loop system (7)-(12) is the set $\mathcal{A} = \{x_0 \in \mathbb{R}^n : \lim_{t \rightarrow \infty} x(t, x_0) = 0\}$. Given K^1, K^2 and non-negative integers $0 \leq \eta_m \leq \eta_M < \tau_M$, our objective is to get $\mathcal{X}_\beta \subset \mathcal{A}$ (as large as we can get) on the domain of attraction, for which exponential stability of the closed-loop system (7)-(12) with respect to x is ensured, where

$$\mathcal{X}_\beta = \{x_0 \in \mathbb{R}^n : x_0^T P x_0 \leq \beta^{-1}\}, \quad (13)$$

and where $\beta > 0$ is a scalar, $P > 0$ is an $n \times n$ -matrix.

III. PARTIAL EXPONENTIAL STABILITY OF THE HYBRID DELAYED SYSTEM WITHOUT ACTUATOR SATURATION

Definition 1: Hybrid system (7)-(12) with initial conditions (6) and $e(t_0)$ is said to be partially exponential stability with respect to x if there exist constants $b > 0, 0 < \kappa < 1$ such that the following holds

$$|x(t)|^2 \leq b\kappa^{t-t_0} [|x_0|^2 + |e(t_0)|^2], \quad t \geq t_0$$

for the solutions of the hybrid system initialized with (6) and $e(t_0) \in \mathbb{R}^{n_y}$.

We extend here the result of [6] to the discrete-time case. Consider the Lyapunov-Krasovskii Functional (LKF) of the form:

$$\begin{aligned} V_e(t) &= V_{TOD}(t) + \min_{i=1,2} \{e_i^T(t) Q_i e_i(t)\}, \\ V_{TOD}(t) &= \tilde{V}(t) + (\tau_M - \eta_m) \sum_{s=t_k - \eta_k}^{t-1} \lambda^{t-s-1} \eta^T(s) Q \eta(s), \\ \tilde{V}(t) &= x^T(t) P x(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) \\ &\quad + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\ &\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_0 \eta(s) \\ &\quad + (\tau_M - \eta_m) \sum_{j=-\tau_M}^{-\eta_m} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_1 \eta(s), \\ P &> 0, S_i > 0, R_i > 0, Q > 0, 0 < \lambda < 1, \\ i &= 0, 1, j = 1, 2, t \in [t_k, t_{k+1} - 1], t \in \mathbb{Z}, k \in \mathbb{Z}, \end{aligned} \quad (14)$$

where we follow the arguments in [5] and define (for simplicity)

$$x(t) = x_0, \quad t \leq 0. \quad (15)$$

The term $(\tau_M - \eta_m) \sum_{s=t_k - \eta_k}^{t-1} \lambda^{t-s-1} \eta^T(s) Q \eta(s)$ is inserted to cope with the delays in the reset conditions. It does not grow at the jumps $t = t_k$, since

$$\begin{aligned} &(\tau_M - \eta_m) \left[\sum_{s=t_k - \eta_k}^{t_k - 1} \lambda^{t_k - s - 1} \eta^T(s) Q \eta(s) \right. \\ &\quad \left. - \sum_{s=t_{k-1} - \eta_{k-1}}^{t_k - 1} \lambda^{t_k - s - 1} \eta^T(s) Q \eta(s) \right] \\ &\leq -(\tau_M - \eta_m) \lambda^{\tau_M} \sum_{s=t_{k-1} - \eta_{k-1}}^{t_k - \eta_k - 1} \eta^T(s) Q \eta(s) \\ &\leq -\lambda^{\tau_M} |\sqrt{Q} [x(t_{k-1} - \eta_{k-1}) - x(t_k - \eta_k)]|^2, \end{aligned} \quad (16)$$

where we applied Cauchy-Schwartz inequality (see e.g., [2]). The following lemma gives sufficient conditions for the exponential stability of (7)-(12) with respect to x :

Lemma 1: If there exist positive constant $\lambda < 1$ and $V_e(t)$ of (14) such that along (7)-(9)

$$\begin{aligned} V_e(t+1) - \lambda V_e(t) - \frac{1}{\tau_M - \eta_m} |\sqrt{Q_2} e_2(t)|^2 &\leq 0, \\ t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}, \end{aligned} \quad (17)$$

and along (10)-(12)

$$\begin{aligned} V_e(t+1) - \lambda V_e(t) - \frac{1}{\tau_M - \eta_m} |\sqrt{Q_1} e_1(t)|^2 &\leq 0, \\ t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}, \end{aligned} \quad (18)$$

hold with

$$Q \lambda^{\tau_M} \geq C^i T Q_i C^i, \quad i = 1, 2. \quad (19)$$

Then $V_e(t)$ does not grow at the jumps along (7)-(9), where

$$V_e(t_{k+1}) - V_e(t_{k+1}^-) \leq -|\sqrt{Q_i} e_i(t_k)|^2, \quad i = 2, \quad (20)$$

and along (10)-(12), where (20) holds with $i = 1$. Moreover, the following bound is valid for the solution of (7)-(12) with the initial condition (6) and $e(t_0) \in \mathbb{R}^{n_y}$:

$$\begin{aligned} V_{TOD}(t) &\leq \lambda^{t-t_0} V_e(t_0), \quad t \geq t_0, \quad t \in \mathbb{Z}, \\ V_e(t_0) &= V_{TOD}(t_0) + \min_{i=1,2} \{|\sqrt{Q_i} e_i(t_0)|^2\}, \end{aligned} \quad (21)$$

implying the exponential stability of (7)-(12) with respect to x .

Proof: Consider first the case of (7), where

$$V_e(t) = V_{TOD}(t) + |\sqrt{Q_2} e_2(t_k)|^2, \quad t \in [t_k, t_{k+1} - 1].$$

Since $\sum_{s=t_k+1}^t \lambda^{t-s} \leq \tau_M - \eta_m$, $t \in (t_k, t_{k+1} - 1]$, by the comparison principle, (17) implies

$$V_e(t) \leq \lambda^{t-t_k} V_e(t_k) + |\sqrt{Q_2} e_2(t_k)|^2, \quad t \in [t_k, t_{k+1} - 1]. \quad (22)$$

Therefore,

$$V_{TOD}(t) \leq \lambda^{t-t_k} V_e(t_k), \quad t \in [t_k, t_{k+1} - 1]. \quad (23)$$

If

$$|\sqrt{Q_1} e_1(t_{k+1})|^2 - |\sqrt{Q_2} e_2(t_{k+1})|^2 \geq 0 \quad (24)$$

then

$$\begin{aligned} V_e(t_{k+1}) &= \tilde{V}(t_{k+1}) + |\sqrt{Q_2} e_2(t_{k+1})|^2 \\ &\quad + (\tau_M - \eta_m) \sum_{s=t_k - \eta_k}^{t_{k+1} - 1} \lambda^{t_{k+1} - s - 1} \eta^T(s) Q \eta(s) \end{aligned}$$

and due to (9)

$$|\sqrt{Q_1} e_1(t_{k+1})|^2 = |\sqrt{Q_1} C_1 [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2.$$

Denote $\Theta \triangleq V_e(t_{k+1}) - V_e(t_{k+1}^-) + |\sqrt{Q_2} e_2(t_k)|^2$. Taking into account (16), we obtain

$$\Theta \leq |\sqrt{Q_2} e_2(t_{k+1})|^2 - \lambda^{\tau_M} |\sqrt{Q} [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2.$$

Due to (24), we have further

$$\begin{aligned} \Theta &\leq |\sqrt{Q_1} e_1(t_{k+1})|^2 - \lambda^{\tau_M} |\sqrt{Q} [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2 \\ &\leq -[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]^T [Q \lambda^{\tau_M} - C_1^T Q_1 C_1] \\ &\quad \times [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})] \leq 0, \end{aligned}$$

where the latter inequality follows from (19) with $i = 1$.

If $|\sqrt{Q_2} e_2(t_{k+1})|^2 > |\sqrt{Q_1} e_1(t_{k+1})|^2$ then

$$\begin{aligned} V_e(t_{k+1}) &= \tilde{V}(t_{k+1}) + |\sqrt{Q_1} e_1(t_{k+1})|^2 \\ &\quad + (\tau_M - \eta_m) \sum_{s=t_k - \eta_k}^{t_{k+1} - 1} \lambda^{t_{k+1} - s - 1} \eta^T(s) Q \eta(s) \end{aligned}$$

and similarly

$$\begin{aligned} \Theta &\leq (|\sqrt{Q_1}e_1(t_{k+1})|^2 - |\sqrt{Q_2}e_2(t_k)|^2) + |\sqrt{Q_2}e_2(t_k)|^2 \\ &\quad - \lambda^{\tau_M} |\sqrt{Q_1}[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2 \\ &\leq -[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]^T [Q\lambda^{\tau_M} - C_1^T Q_1 C_1] \\ &\quad \times [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})] \leq 0. \end{aligned}$$

Therefore, (20) with $i = 2$ is valid and, together with (22) for $t = t_{k+1}^-$, implies

$$V_e(t_{k+1}) \leq \lambda^{t_{k+1}-t_k} V_e(t_k). \quad (25)$$

Similarly, if $|\sqrt{Q_2}e_2(t_k)|^2 > |\sqrt{Q_1}e_1(t_k)|^2$, then (18) and (19) with $i = 2$ imply (23), (20) with $i = 1$ and (25). Hence, (25) holds for all $k \in \mathbb{Z}$ and

$$\begin{aligned} V_e(t_{k+1}) &\leq \lambda^{t_{k+1}-t_{k-1}} V_e(t_{k-1}) \\ &\leq \lambda^{t_{k+1}-t_0} V_e(t_0). \end{aligned}$$

The latter inequality with $k+1$ replaced by k and (23) imply (21) that completes the proof. $\square$

Based on the above lemma, by using the standard arguments for the delay-dependent analysis [9], we derive LMI conditions for the exponential stability with respect to x :

Lemma 2: Given scalar $0 < \lambda < 1$, non-negative integers $0 \leq \eta_m \leq \eta_M < \tau_M$, and K^1, K^2 . If there exist $n \times n$ matrices $P > 0$, $Q > 0$, $S_i > 0$, $R_i > 0$ ($i = 0, 1$), S_{12} , matrices $Q_1 > 0$, $Q_2 > 0$ such that (19) and

$$\Omega = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \geq 0, \quad (26)$$

$$F_0^T P F_0 + \Sigma + F_{01}^T W F_{01} - F_{12}^T R_0 \lambda^{\eta_m} F_{12} - F^T \Omega \lambda^{\tau_M} F < 0, \quad (27)$$

are feasible, where

$$\begin{aligned} F_0 &= [A \ 0 \ A_1 \ 0 \ B_{3-i}], \ F_j = [\Gamma_1 \ \Gamma_2 \ \Gamma_3 \ \Gamma_4 \ \Gamma_5], \\ \Gamma_j &= I, \Gamma_l = 0, l = 1, \dots, 5, \ l \neq j, \\ F_{kj} &= F_k - F_j, \ k, j = 0, 1, \dots, 5, \\ F_{01} &= F_0 - F_1, \ F = [F_{23}^T \ F_{34}^T]^T, \\ \Sigma &= \text{diag}\{S_0 - \lambda P, -(S_0 - S_1)\lambda^{\eta_m}, 0, -S_1\lambda^{\tau_M}, \varphi\}, \\ \varphi &= -[\frac{1}{\tau_M - \eta_m} - (1 - \lambda)]Q_{3-i}, \\ W &= \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m)Q, \ i = 1, 2. \end{aligned} \quad (28)$$

Then solutions of the hybrid system (7)-(12) satisfy the bound (21) and are exponentially stable with respect to x .

IV. PARTIAL EXPONENTIAL STABILITY OF THE HYBRID DELAYED SYSTEM WITH ACTUATOR SATURATION

Under the constant initial condition (15) and $V_{TOD}(t)$ of (14), we have for $t = 0$

$$\begin{aligned} V_{TOD}(0) &= x_0^T P x_0 + \sum_{s=-\eta_m}^{-1} \lambda^{-s-1} x_0^T S_0 x_0 \\ &\quad + \sum_{s=-\tau_M}^{-\eta_m-1} \lambda^{-s-1} x_0^T S_1 x_0. \end{aligned}$$

Thus, $V_{TOD}(0) \leq x_0^T \Xi_{TOD} x_0$, where

$$\Xi_{TOD} = P + \eta_m S_0 + \lambda^{\eta_m} (\tau_M - \eta_m) S_1.$$

In order to derive the bound on $V_{TOD}(t_0)$ in terms of x_0 , by the arguments in [5] we consider $V_0(t) = x^T(t) P x(t)$, $P > 0$ and have for $t = 0, 1, \dots, t_0 - 1$:

$$V_0(t+1) - \mu V_0(t) \leq 0, \ \mu > 1, \quad (29a)$$

$$V_{TOD}(t+1) - \lambda V_{TOD}(t) - (\mu - 1)V_0(t) \leq 0. \quad (29b)$$

Then from (29a), $V_0(t) \leq \mu^t V_0(0)$ for $t = 0, 1, \dots, t_0$. Noting that $\eta_m \leq t_0 = \eta_0 \leq \eta_M$, we have from (29b)

$$\begin{aligned} V_{TOD}(t_0) &\leq \lambda^{t_0} V_{TOD}(0) + (\mu^{t_0} - 1)x_0^T P x_0 \\ &\leq x_0^T [\lambda^{t_0} \Xi_{TOD} + (\mu^{t_0} - 1)P] x_0 \\ &\leq x_0^T [\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_M} - 1)P] x_0. \end{aligned} \quad (30)$$

Theorem 1: Given positive scalars $\lambda < 1, \beta, \mu > 1, \sigma, \bar{\sigma}$, non-negative integers $0 \leq \eta_m \leq \eta_M < \tau_M$, and K^1, K^2 . Suppose there exist $n \times n$ matrices $P > 0$, $Q > 0$, $S_i > 0$, $R_i > 0$ ($i = 0, 1$), S_{12} , matrices $Q_1 > 0$, $Q_2 > 0$ such that (19), (26), (27) and

$$\begin{bmatrix} -\mu P & A^T P \\ * & -P \end{bmatrix} < 0, \quad (31)$$

$$\eta_m S_0 + \lambda^{\eta_m} (\tau_M - \eta_m) S_1 \leq \sigma P, \quad (32)$$

$$Q\lambda^{\tau_M} \leq \bar{\sigma} P, \quad (33)$$

$$\begin{bmatrix} P\lambda^{\eta_M} \rho_{TOD}^{-1} & (K^i C^i)^T_j \\ * & \frac{1}{4}\beta \bar{u}_j^2 \end{bmatrix} \geq 0, \ j = 1, \dots, n_u, \quad (34)$$

$$\tilde{F}_0^T P \tilde{F}_0 + \tilde{\Sigma} + \tilde{F}_{01}^T W \tilde{F}_{01} - \tilde{F}_{12}^T R_0 \lambda^{\eta_m} \tilde{F}_{12} - \tilde{F}^T \Omega \lambda^{\tau_M} \tilde{F} < 0, \quad (35)$$

hold, where the notations are given by (28) and where

$$\begin{aligned} \tilde{F}_0 &= [A \ 0 \ 0 \ 0], \\ \tilde{F}_j &= [\Gamma_1 \ \Gamma_2 \ \Gamma_3 \ \Gamma_4], \Gamma_j = I, \Gamma_l = 0, l = 1, \dots, 4, l \neq j, \\ \tilde{F}_{kj} &= \tilde{F}_k - \tilde{F}_j, \ k, j = 0, 1, \dots, 4, \ \tilde{F} = [\tilde{F}_{23}^T \ \tilde{F}_{34}^T]^T, \\ \tilde{\Sigma} &= \text{diag}\{S_0 - \lambda P + (1 - \mu)P, -(S_0 - S_1)\lambda^{\eta_m}, 0, -S_1\lambda^{\tau_M}\}, \\ \rho_{TOD} &= \lambda^{\eta_m} (1 + \sigma) + (\mu^{\eta_M} - 1) + \bar{\sigma}, \ i = 1, 2. \end{aligned}$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, the closed-loop system (7)-(12) is exponentially stable with respect to x .

Proof: Suppose that $x(t) \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$. As was shown in Lemma 2, LMIs (19), (26) and (27) lead to (21) for $t = t_0, t_0 + 1, \dots$. By applying convex analysis of [9], we show that for $t = 0, 1, \dots, t_0 - 1$, LMIs (31) and (35) with (26) can guarantee (29a) and (29b), respectively.

Next, noting that (19) and (33), we have

$$x_0^T C^i T Q_i C^i x_0 \leq \lambda^{\tau_M} x_0^T Q x_0 \leq \bar{\sigma} x_0^T P x_0, \quad (36)$$

which implies that $|\sqrt{Q_i}e_i(t_0)|^2 = |-\sqrt{Q_i}C^i x_0|^2 < \bar{\sigma} x_0^T P x_0$, $i = 1, 2$.

Therefore, taking into (21), (30), (32), (36) and $t_0 = \eta_0 \leq \eta_M$, we obtain

$$\begin{aligned} x^T(t) P x(t) &\leq V_{TOD}(t) \leq \lambda^{t-t_0} V_e(t_0) \\ &= \lambda^{t-t_0} \{V_{TOD}(t_0) + \min_{i=1,2} \{e_i^T(t_0) Q_i e_i(t_0)\}\} \\ &\leq \lambda^{t-t_0} x_0^T [\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_M} - 1)P + \bar{\sigma} P] x_0 \\ &\leq \lambda^{t-t_0} x_0^T [\lambda^{\eta_m} (1 + \sigma) + (\mu^{\eta_M} - 1) + \bar{\sigma}] P x_0 \\ &= \lambda^{t-t_0} \rho_{TOD} x_0^T P x_0 \leq \lambda^{-\eta_M} \rho_{TOD} x_0^T P x_0, \ t \in \mathbb{Z}. \end{aligned}$$

So for all $x(t) : x^T(t) P x(t) \leq \lambda^{-\eta_M} \rho_{TOD} \beta^{-1} \Rightarrow x^T(t) (K^j C^j)_i^T (K^j C^j)_i x(t) \leq \frac{1}{4} \bar{u}_i^2$, if

$$\begin{aligned} x^T(t) (K^j C^j)_i^T (K^j C^j)_i x(t) &\leq \frac{1}{4} \beta \lambda^{\eta_M} \rho_{Q_P}^{-1} x^T(t) P x(t) \bar{u}_i^2, \ i = 1, \dots, n_u, \ j = 1, 2. \end{aligned}$$

The latter inequality is guaranteed if $\frac{1}{4}\beta\lambda^{\eta_M}\rho_{TOD}^{-1}P\bar{u}_i^2 - (K^j C^j)^T (K^j C^j)_i \geq 0$, and, thus, by Schur complements if (34) is feasible. Hence, the solutions of hybrid system (7)-(12) with respect to x converge to the origin exponentially, provided that $x_0 \in \mathcal{X}_\beta$. $\square$

Remark 1: Note that

$$x_0^T P x_0 \leq \lambda_{\max}(P) |x_0|^2 < \beta^{-1}, \quad (37)$$

where $\lambda_{\max}(P)$ denotes the largest eigenvalue of P . Hence the following initial region $|x_0|^2 < \beta^{-1}/\lambda_{\max}(P)$ is inside of $\mathcal{X}_\beta$. In order to maximize the initial ball, we can add the condition

$$P - \gamma I < 0, \quad (38)$$

to Theorem 1, where $\gamma > 0$ is minimized.

V. EXAMPLE: DISCRETE-TIME CART-PENDULUM

Consider the following linearized model of the inverted pendulum on a cart [7]:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{a}{M} \\ 0 \\ \frac{-a}{Ml} \end{bmatrix} u(t)$$

with $M = 3.9249\text{Kg}$, $m = 0.2047\text{Kg}$, $l = 0.2302\text{m}$, $g = 9.81\text{N/Kg}$, $a = 25.3\text{N/V}$ and $\bar{u} = 50$. In the model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. Such a model is discretized with a sampling time $T_s = 0.001\text{s}$:

$$\begin{bmatrix} x(t+1) \\ \Delta x(t+1) \\ \theta(t+1) \\ \Delta \theta(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0 & 1 & -0.0005 & 0 \\ 0 & 0 & 1.00 & 0.001 \\ 0 & 0 & 0.0448 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \Delta x(t) \\ \theta(t) \\ \Delta \theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0064 \\ 0 \\ -0.0280 \end{bmatrix} u(t), \quad t \in \mathbb{Z}$$

with $\bar{u} = 50$. The pendulum can be stabilized by a state feedback $u(t) = K \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T$ with the gain $K = [K^1 \quad K^2]$

$$K^1 = \begin{bmatrix} 5.825 & 5.883 \end{bmatrix}, K^2 = \begin{bmatrix} 24.941 & 5.140 \end{bmatrix}, \quad (39)$$

which leads to the closed-loop system eigenvalues $\{0.8997, 0.9980 + 0.0020i, 0.9980 - 0.0020i, 0.9980\}$. Suppose the variables $\theta, \Delta\theta$ and $x, \Delta x$ are not accessible simultaneously. We consider measurements $y^i(t) = C^i x(t)$, $t \in \mathbb{Z}$, where

$$C^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Choose $\lambda = 1$, $\beta = 1$, $\sigma = 1.0 \times 10^{-2}$ and $\eta_m = 1$, $\eta_M = 2$, $\tau_M = 3$. Applying Theorem 1 with $\mu = 1.02$, $\bar{\sigma} = 1.1$ and Remark 1, the closed-loop system (7)-(12) is exponentially stable with respect to x starting from the initial ball $|x_0| < 0.3342$ by weighted TOD protocol.

Then for different μ , applying Theorem 1 with $\lambda = 1$ and Remark 1, we give the corresponding largest ball of admissible initial conditions in Table 1, from which we can see that $|x_0|$ increases with smaller μ .

TABLE I
LARGEST BALL OF ADMISSIBLE INITIAL CONDITIONS FOR DIFFERENT μ

μ	1.02	1.2	3
$ x_0 $ (Theorem 1,TOD)	0.3342	0.3069	0.1541

VI. CONCLUSIONS

In this paper, a time-delay approach was developed to the stability analysis of discrete-time NCS under a weighted TOD protocol in the presence of actuator saturation. A Lyapunov-based method was presented for finding the domain of attraction under the weighted TOD protocol. The conditions are given in terms of LMIs. Polytopic uncertainties in the system model can be easily included in the analysis. Numerical example illustrates the efficiency of our method.

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