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Discrete-time Network-based Control under Schedulings and Actuator Constraints

Kun Liu and Emilia Fridman

Abstract

This paper is concerned with the solution bounds for discrete-time networked control systems via delay-dependent Lyapunov-Krasovskii methods. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. The time-delay approach has been introduced recently for the stabilization of continuous-time networked control systems under the Round-Robin protocol in [11] and under a weighted Try-Once-Discard protocol in [10], respectively. Actuator saturation has not been taken into account in [10] and [11]. In the present paper, for the first time the time-delay approach is extended to the stability analysis of the discrete-time networked control systems under both scheduling protocols and actuators saturation. The communication delays are allowed to be larger than the sampling intervals. A novel Lyapunov-based method is presented for finding the domain of attraction. Polytopic uncertainties in the system model can be easily included in our analysis. The efficiency of the time-delay approach is illustrated on the example of a cart-pendulum system.

Keywords: networked control systems, time-delay approach, scheduling, input saturation, Lyapunov-Krasovskii method.

I. INTRODUCTION

Networked Control Systems (NCSs) are control systems comprised of the system to be controlled and of actuators, sensors, and controllers, the operation of which is coordinated via a communication network [1]. In many such systems, only one node is allowed to use the

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communication channel at once. In the present paper, we focus on the stability analysis of discrete-time NCSs with communication constraints and actuator constraints. The scheduling of sensor information towards the controller is ruled by the Round-Robin (RR) protocol and by a weighted Try-Once-Discard (TOD) protocol, respectively. A linear (probably, uncertain) system with distributed sensors is considered. Three recent approaches for NCSs are based on discrete-time systems [2] [6], impulsive/hybrid systems [8] [13] and time-delay systems [3] [4] [7].

The time-delay approach has been introduced for the stabilization of continuous-time NCSs under the RR protocol in [11] and under a weighted TOD protocol in [10], respectively. The closed-loop system is modeled as a switched system with multiple and ordered time-varying delays under RR scheduling or as a hybrid system with time-varying delays in the dynamics and in the reset equations under the TOD scheduling. Differently from the existing hybrid and discrete-time approaches, the time-delay approach allows treating the case of large communication delays. Actuator saturation has not been taken into account in [10] and [11]. Recently, the stabilization of sampled-data systems under variable samplings and actuator saturation was studied in [15], where scheduling protocols and delays were not included.

As shown in [9], when one deals with the solution bounds of time-delay systems via Lyapunov-Krasovskii method, the first time-interval of the delay length needs a special analysis. Solution bounds are widely used for systems with input saturation caused by actuator saturation or by the quantizers with saturation. This first time-interval does not influence on the stability and the exponential decay rate analysis. The analysis of the first time-interval of the delay length is important for nonlinear systems e.g., for finding the domain of attraction.

In the present paper, the time-delay approach is extended to the stability analysis of discrete-time NCSs with actuator constraints under the RR [11] or under a weighted TOD [10] scheduling. Following [9], we present a direct Lyapunov approach for finding the domain of attraction under the both scheduling protocols. The conditions are given in terms of Linear Matrix Inequalities (LMIs). Polytopic uncertainties in the system model can be easily included in the analysis. The efficiency of the presented approach is illustrated by a cart-pendulum system.

Notation: Throughout the paper the superscript ‘ T ’ stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive definite.

The symmetric elements of the symmetric matrix will be denoted by $*$. For any matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$, the notations A_j and x_j denote, respectively, the j^{th} line of matrix A and the j^{th} component of vector x . $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}^+$ denote the set of non-negative integers, positive integers and non-negative real numbers, respectively. Given $\bar{u} = [\bar{u}_1, \dots, \bar{u}_{n_u}]^T$, $0 < \bar{u}_i$, $i = 1, \dots, n_u$, for any $u = [u_1, \dots, u_{n_u}]^T$ we denote by $\text{sat}(u)$ the vector with coordinates $\text{sign}(u_i) \min(|u_i|, \bar{u}_i)$. *MATI* denotes the Maximum Allowable Transmission Interval.

II. STABILIZATION OF DISCRETE-TIME NCSs WITH ACTUATOR SATURATION UNDER RR SCHEDULING

A. Problem formulation and a switched system model

Consider the system architecture in Fig.1 with plant

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{Z}, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, A and B are (probably, uncertain) system matrices with appropriate dimensions. The initial condition is given by $x(0) = x_0$. We suppose that the control input is subject to the following amplitude constraints

$$|u_i(t)| \leq \bar{u}_i, \quad 0 < \bar{u}_i, \quad i = 1, \dots, n_u, \quad t \in \mathbb{Z}. \quad (2)$$

The NCS has several nodes (distributed sensors, a controller node and an actuator node) which are connected via networks. For the sake of simplicity, we consider two sensor nodes $y^i(t) = C^i x(t)$, $i = 1, 2$ and we denote $C = \begin{bmatrix} C^1 \\ C^2 \end{bmatrix}$, $y(t) = \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \in \mathbb{R}^{n_y}$. The results can be easily extended to any finite number of sensors. We let s_k denote the unbounded and monotonously increasing sequence of sampling instants, i.e.,

$$0 = s_0 < s_1 < \dots < s_k < \dots, \quad k \in \mathbb{Z}, \quad \lim_{k \rightarrow \infty} s_k = \infty, \quad s_{k+1} - s_k \leq \text{MATI}, \quad (3)$$

where $\{s_k\}$ is a subsequence of $\mathbb{Z}$. At each sampling instant s_k , one of the outputs $y^i(t) \in \mathbb{R}^{n_i}$ ($n_1 + n_2 = n_y$) is sampled and transmitted via the network. Firstly we consider the RR scheduling protocol for the choice of the active output node: the outputs are transmitted one after another, i.e., $y^i(t) = C^i x(t)$, $t \in \mathbb{Z}$ is transmitted only at the sampling instant $t = s_{2p+i-1}$, $p \in \mathbb{Z}$, $i = 1, 2$. We denote $C = \begin{bmatrix} C^1 \\ C^2 \end{bmatrix}$, $y(t) = \begin{bmatrix} y^1(t) \\ y^2(t) \end{bmatrix} \in \mathbb{R}^{n_y}$, $t \in \mathbb{Z}$. After each transmission and reception, the values in $y^i(t)$ are updated with the newly received values, while the values of

Fig. 1. NCSs with actuator saturation under RR scheduling

$y^j(t)$ for $j \neq i$ remain the same, as no additional information is received. This leads to the constrained data exchange expressed as

$$y_k^i = \begin{cases} y^i(s_k) = C^i x(s_k), & k = 2p + i - 1, \\ y_{k-1}^i, & k \neq 2p + i - 1, \end{cases} \quad p \in \mathbb{Z}.$$

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the two networks (between the sensor and the actuator) is subject to a variable delay $\eta_k = \eta_k^{sc} + \eta_k^{ca}$, where η_k^{sc} and η_k^{ca} are the network-induced delays from the sensor to the controller and from the controller to the actuator, respectively. Then $t_k = s_k + \eta_k$ is the updating instant time of the Zero-Order Hold (ZOH), where $\{t_k\}$ is a subsequence of $\mathbb{Z}$.

As in [11] and [12], we allow the delay to be non-small provided that the old sample cannot get to the destination (to the controller or to the actuator) after the most recent one. Assume that the network-induced delay η_k and the time span between the updating and the most recent sampling instants are bounded:

$$t_{k+1} - t_k + \eta_k \leq 1 + \tau_M, \quad 0 \leq \eta_m \leq \eta_k \leq \eta_M, \quad k \in \mathbb{Z}, \quad (4)$$

where τ_M, η_m and η_M are known non-negative integers. Then

$$\begin{aligned} (t_{k+1} - 1) - s_k &= s_{k+1} - s_k + \eta_{k+1} - 1 \leq MATI + \eta_M - 1 = \tau_M, \\ (t_{k+1} - 1) - s_{k-1} &= s_{k+1} - s_{k-1} + \eta_{k+1} - 1 \leq 2MATI + \eta_M - 1 = 2\tau_M - \eta_M + 1 \triangleq \bar{\tau}_M, \\ t_{k+1} - t_k &\leq \tau_M - \eta_m + 1. \end{aligned} \quad (5)$$

In [11], a time-delay approach was introduced for the stability and L_2 -gain analysis of continuous-time NCSs with RR scheduling. Actuator saturation was not taken into account. In this section, we consider the stability analysis of discrete-time NCSs with actuator saturation under RR scheduling protocol. Due to the control bounds defined in (2), the effective control signal to be applied to the system (1) is given by

$$u(t) = \text{sat}(K^1 y_k^1 + K^2 y_k^2), \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{N}, \quad k \in \mathbb{N},$$

with $u(t) = \text{sat}(K^1 y_0^1)$, $t \in [t_0, t_1)$, $t \in \mathbb{N}$, where $K = [K^1 \ K^2]$, $K^1 \in \mathbb{R}^{n_u \times n_1}$, $K^2 \in \mathbb{R}^{n_u \times (n_y - n_1)}$ such that $A + BKC$ is Schur.

We define the polyhedron

$$\mathcal{L}(K^j, \bar{u}) = \{x \in \mathbb{R}^n : |(K^j C^j)_i x| \leq \frac{1}{2} \bar{u}_i, \quad i = 1, \dots, n_u\}, \quad j = 1, 2.$$

If the control is such that $x(t) \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$, then $|(K^1 C^1)_i x + (K^2 C^2)_i x| \leq \bar{u}_i$. Following [11], the closed-loop system with RR scheduling is modeled as a switched system:

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_{k-1} - \eta_{k-1}) + A_2 x(t_k - \eta_k), \quad t \in [t_k, t_{k+1} - 1], \\ x(t+1) &= Ax(t) + A_1 x(t_{k+1} - \eta_{k+1}) + A_2 x(t_k - \eta_k), \quad t \in [t_{k+1}, t_{k+2} - 1], \end{aligned} \quad (6)$$

where $k = 2p - 1$, $p \in \mathbb{N}$, $A_i = BK^i C^i$, $i = 1, 2$.

For $t \in [t_k, t_{k+1} - 1]$ we can represent $t_k - \eta_k = t - \tau_1(t)$, $t_{k-1} - \eta_{k-1} = t - \tau_2(t)$, where

$$\begin{aligned} \tau_1(t) &= t - t_k + \eta_k < \tau_2(t) = t - t_{k-1} + \eta_{k-1}, \\ \tau_1(t) &\in [\eta_m, \tau_M], \quad \tau_2(t) \in [\eta_m, \bar{\tau}_M], \quad t \in [t_k, t_{k+1} - 1]. \end{aligned}$$

Therefore, (6) for $t \in [t_k, t_{k+1} - 1]$ can be considered as a system with two time-varying interval delays, where $\tau_1(t) < \tau_2(t)$. Similarly, for $t \in [t_{k+1}, t_{k+2} - 1]$ (6) is a system with two time-varying delays, one of which is less than another.

B. Solution bounds

Applying the following discrete-time Lyapunov-Krasovskii Functional (LKF) to system (6) with time-varying delay from the maximum delay interval $[\eta_m, \bar{\tau}_M]$:

$$\begin{aligned} V_{RR}(t) &= x^T(t) P x(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) \\ &\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_0 \eta(s) + \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\ &\quad + (\bar{\tau}_M - \eta_m) \sum_{j=-\bar{\tau}_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_1 \eta(s), \quad \eta(t) = \eta(t+1) - \eta(t), \\ &\quad P > 0, \quad S_i > 0, \quad R_i > 0, \quad i = 0, 1, \quad 0 < \lambda < 1, \end{aligned} \quad (7)$$

and following [11], we find conditions that guarantee

$$V_{RR}(t+1) - \lambda V_{RR}(t) \leq 0, \quad t = t_1, t_1 + 1, \dots \quad (8)$$

Then $V_{RR}(t) \leq \lambda^{t-t_1} V(t_1)$, $t = t_1, t_1 + 1, \dots$. Following the arguments in [9], we define (for simplicity)

$$x(t) = x_0, \quad t \leq 0. \quad (9)$$

In order to derive a bound on $V_{RR}(t_1)$ in terms of x_0 in a simple way, on the controller side we need to wait for all (both) latest transmitted measurements $y_1(s_0)$, $y_2(s_1)$ and then send them together to the actuator side. So the first updating time $t_0 = t_1$ corresponds to the first data received by the actuator, which means that $u(t) = 0$, $t \in [0, t_1 - 1]$. Then for $t \in [0, t_1 - 1]$, (1) is given by

$$x(t+1) = Ax(t), \quad t = 0, 1, \dots, t_1 - 1, \quad t \in \mathbb{Z}. \quad (10)$$

Denote by $x(t, x_0)$ the state trajectory of (6), (10) with the initial condition $x_0 \in \mathbb{R}^n$. Then the domain of attraction of the closed-loop system (6), (10) is the set $\mathcal{A} = \{x_0 \in \mathbb{R}^n : \lim_{t \rightarrow \infty} x(t, x_0) = 0\}$. Given K^1, K^2 and positive integers $0 \leq \eta_m \leq \eta_M < \tau_M$, our objective is to get $\mathcal{X}_\beta \subset \mathcal{A}$ (as large as we can get) on the domain of attraction, for which exponential stability of the closed-loop system is ensured, where

$$\mathcal{X}_\beta = \{x_0 \in \mathbb{R}^n : x_0^T P x_0 \leq \beta^{-1}\}, \quad (11)$$

and where $\beta > 0$ is a scalar, $P > 0$ is an $n \times n$ -matrix.

By extending the direct Lyapunov approach suggested in [9] for time-delay systems to the switched systems given by (6), (10), we obtain

Lemma 1: Consider LKF $V_{RR}(t)$ given by (7) and denote $V_0(t) = x^T(t) P x(t)$. Under (9) let there exist $0 < \lambda < 1$ and $\mu > 1$ such that the following inequalities

$$V_0(t+1) - \mu V_0(t) \leq 0, \quad t = 0, 1, \dots, t_1 - 1, \quad (12a)$$

$$V_{RR}(t+1) - \lambda V_{RR}(t) - (\mu - 1)V_0(t) \leq 0, \quad t = 0, 1, \dots, t_1 - 1. \quad (12b)$$

hold along (10). Then the solutions of (6), (10) at time t_1 satisfy:

$$V_{RR}(t_1) \leq x_0^T [\lambda^{\eta_m} \Xi_{RR} + (\mu^{\tau_M+1} - 1)P] x_0, \quad (13)$$

where

$$\Xi_{RR} = P + \eta_m S_0 + \lambda^{\eta_m} (\bar{\tau}_M - \eta_m) S_1. \quad (14)$$

Proof: From (12a), $V_0(t) \leq \mu^t V_0(0)$ for $t = 0, 1, \dots, t_1$. Under the constant initial condition (9) and $V_{RR}(t)$ of (7), we have for $t = 0$

$$V_{RR}(0) = x_0^T P x_0 + \sum_{s=-\eta_m}^{-1} \lambda^{-s-1} x^T(s) S_0 x(s) + \sum_{s=-\bar{\tau}_M}^{-\eta_m-1} \lambda^{-s-1} x_0^T S_1 x_0.$$

Hence, $V_{RR}(0) \leq x_0^T \Xi_{RR} x_0$. Noting that $\eta_m < t_1 \leq \tau_M + 1$, (12b) implies

$$\begin{aligned} V_{RR}(t_1) &\leq \lambda^{t_1} V_{RR}(0) + (\mu^{t_1} - 1) x_0^T P x_0 \leq x_0^T \left[\lambda^{t_1} \Xi_{RR} + (\mu^{t_1} - 1) P \right] x_0 \\ &\leq x_0^T \left[\lambda^{\eta_m} \Xi_{RR} + (\mu^{\tau_M+1} - 1) P \right] x_0. \quad \square \end{aligned}$$

Lemma 1 implies the following result:

Theorem 1: Given positive scalars $\lambda < 1, \beta, \mu > 1, \sigma$, positive integers $0 \leq \eta_m \leq \eta_M < \tau_M$ and K^1, K^2 , let there exist scalars $n \times n$ matrices $P > 0, S_\vartheta > 0, R_\vartheta > 0 (\vartheta = 0, 1), G_1^i, G_2^i, G_3^i$ ($i = 1, 2$) such that $\eta_m S_0 + \lambda^{\eta_m} (\bar{\tau}_M - \eta_m) S_1 \leq \sigma P$ and the following matrix inequalities are feasible:

$$\Omega_i = \begin{bmatrix} R_1 & G_1^i & G_2^i \\ * & R_1 & G_3^i \\ * & * & R_1 \end{bmatrix} \geq 0, \quad (15)$$

$$\begin{bmatrix} -\mu P & A^T P \\ * & -P \end{bmatrix} < 0, \quad (16)$$

$$\begin{bmatrix} P \rho_{RR}^{-1} & (K^i C^i)_j^T \\ * & \frac{1}{4} \beta \bar{u}_j^2 \end{bmatrix} \geq 0, \quad j = 1, \dots, n_u, \quad (17)$$

$$F_0^T P F_0 + \Sigma + F_{01}^T H F_{01} - F_{12}^T R_0 \lambda^{\eta_m} F_{12} - F^T \Omega_i \lambda^{\bar{\tau}_M} F < 0, \quad (18)$$

$$\bar{F}_0^T P \bar{F}_0 + \bar{\Sigma} + \bar{F}_{01}^T H \bar{F}_{01} - F_{12}^T R_0 \lambda^{\eta_m} F_{12} - F^T \Omega_1 \lambda^{\bar{\tau}_M} F < 0, \quad i = 1, 2, \quad (19)$$

where

$$\begin{aligned}
 F_0 &= [A \ 0 \ A_{3-i} \ A_i \ 0], \ F_j = [\Gamma_1 \ \Gamma_2 \ \Gamma_3 \ \Gamma_4 \ \Gamma_5], \ \Gamma_j = I, \ \Gamma_l = 0, \ l = 1, \dots, 5, \ l \neq j, \\
 F_{kj} &= F_k - F_j, \ k, j = 0, 1, \dots, 5, \ F = [F_{23}^T \ F_{34}^T \ F_{45}^T]^T, \\
 \bar{F}_0 &= [A \ 0 \ 0 \ 0 \ 0], \ \bar{F}_{01} = \bar{F}_0 - F_1, \\
 \Sigma &= \text{diag}\{S_0 - \lambda P, -(S_0 - S_1)\lambda^{\eta_m}, 0, 0, -S_1\lambda^{\bar{\tau}_M}\}, \\
 \bar{\Sigma} &= \text{diag}\{S_0 - \lambda P + (1-\mu)P, -(S_0 - S_1)\lambda^{\eta_m}, 0, 0, -S_1\lambda^{\bar{\tau}_M}\}, \\
 \rho_{RR} &= \lambda^{\eta_m}(1 + \sigma) + \mu^{\tau_M+1} - 1, \\
 H &= \eta_m^2 R_0 + (\bar{\tau}_M - \eta_m)^2 R_1, \ i = 1, 2.
 \end{aligned} \tag{20}$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, the closed-loop system (6), (10) is exponentially stable.

Proof: Suppose that $x(t) \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$. Following [11], we take advantage of the ordered delays and use convex analysis of [14]. We show that (16) and (15), (19) guarantee, respectively, (12a) and (12b) for $t = 0, \dots, t_1 - 1$; (15) and (18) guarantee (8) for $t = t_1, t_1 + 1, \dots$

Noting that $\eta_m S_0 + \lambda^{\eta_m}(\bar{\tau}_M - \eta_m)S_1 \leq \sigma P$, we arrive at

$$\begin{aligned}
 x^T(t)Px(t) &\leq V_{RR}(t) \leq \lambda^{t-t_1}V_{RR}(t_1) \\
 &\leq x_0^T[\lambda^{\eta_m}\Xi_{RR} + (\mu^{\tau_M+1} - 1)P]x_0 \\
 &\leq x_0^T[\lambda^{\eta_m}(P + \sigma P) + (\mu^{\tau_M+1} - 1)P]x_0 \\
 &= \rho_{RR}x_0^T Px_0, \quad t = t_1, t_1 + 1, \dots
 \end{aligned}$$

So for all $x(t) : x^T(t)Px(t) \leq \rho_{RR}\beta^{-1} \Rightarrow x^T(t)(K^j C^j)_i^T (K^j C^j)_i x(t) \leq \frac{1}{4}\bar{u}_i^2$, if

$$\begin{aligned}
 &x^T(t)(K^j C^j)_i^T (K^j C^j)_i x(t) \\
 &\leq \frac{1}{4}\beta\rho_{RR}^{-1}x^T(t)Px(t)\bar{u}_i^2, \quad i = 1, \dots, n_u, \quad j = 1, 2.
 \end{aligned}$$

The latter inequality is guaranteed if $\frac{1}{4}\beta\rho_{RR}^{-1}P\bar{u}_i^2 - (K^j C^j)_i^T (K^j C^j)_i \geq 0$, and, thus, by Schur complements if (17) is feasible. Hence the trajectories of the system (6), (10) converge to the origin exponentially, provided that $x_0 \in \mathcal{X}_\beta$. $\square$

III. DISCRETE-TIME NCSS WITH ACTUATOR SATURATION UNDER A WEIGHTED TOD PROTOCOL

A. Problem formulation and a hybrid time-delay model

In [10], a weighted TOD protocol was analyzed for the stabilization of continuous-time NCSS in the framework of time-delay approach. Actuator saturation was not taken into account. In

this section, we consider discrete-time NCSs with actuator saturation under this weighted TOD protocol. Denote by $\hat{y}(s_k) = \begin{bmatrix} \hat{y}_1(s_k) \\ \hat{y}_2(s_k) \end{bmatrix} \in \mathbb{R}^{n_y}$ the output information submitted to the scheduling protocol, i.e. the “constrained” version of the system output $y(s_k)$. At each sampling instant s_k , one of the system nodes $i \in \{1, 2\}$ is active, that is only one of $\hat{y}_i(s_k)$ values is updated with the recent output $y_i(s_k)$. The choice of the active output node is ruled by a weighted TOD protocol. Following [10], consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$\begin{aligned} e(t) &= \text{col}\{e_1(t), e_2(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k), \quad t \in [t_k, t_{k+1} - 1], \\ t &\in \mathbb{Z}, k \in \mathbb{Z}, \quad \hat{y}(s_{-1}) \triangleq 0, \quad e(t) \in \mathbb{R}^{n_y}. \end{aligned}$$

Let $Q_i > 0, i = 1, 2$ be some weighting matrices (they will be found from LMIs in Lemma 3 below). The node that has the largest error, $|\sqrt{Q_i}e_i(t)|^2, i = 1, 2$ is granted access to the network. Consider the first case of

$$|\sqrt{Q_1}e_1(t)|^2 \geq |\sqrt{Q_2}e_2(t)|^2, \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}. \quad (21)$$

Due to the control bounds defined in (2), the effective control signal to be applied to the system (1) is given by

$$u(t) = \text{sat}[K^1 y^1(t_k - \eta_k) + K^2 \hat{y}^2(t_{k-1} - \eta_{k-1})], \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}.$$

If the control is such that $x(t) \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$, then $|(K^1 C^1)_i x + (K^2 C^2)_i x(t)| \leq \bar{u}_i$.

We obtain thus the impulsive closed-loop model with the following discrete dynamics:

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_k - \eta_k) + B_2 e_2(t), \\ e(t+1) &= e(t), \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}, \end{aligned} \quad (22)$$

where $A_1 = BKC$, $B_i = BK^i$, $K = [K^1 \ K^2]$, $i = 1, 2$. The delayed reset system is given by

$$\begin{aligned} x(t_{k+1}) &= x(t_{k+1}^-), \\ e(t_{k+1}) &= \begin{bmatrix} 0 \\ e_2(t_{k+1}^-) \end{bmatrix} + Cx(t_k - \eta_k) - Cx(t_{k+1} - \eta_{k+1}). \end{aligned} \quad (23)$$

Similarly if

$$|\sqrt{Q_2}e_2(t)|^2 > |\sqrt{Q_1}e_1(t)|^2, \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}, \quad (24)$$

then the closed-loop system has the form

$$\begin{aligned} x(t+1) &= Ax(t) + A_1 x(t_k - \eta_k) + B_1 e_1(t), \\ e(t+1) &= e(t), \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z} \end{aligned} \quad (25)$$

with the delayed reset system

$$\begin{aligned} x(t_{k+1}) &= x(t_{k+1}^-), \\ e(t_{k+1}) &= \begin{bmatrix} e_1(t_{k+1}^-) \\ 0 \end{bmatrix} + Cx(t_k - \eta_k) - Cx(t_{k+1} - \eta_{k+1}), \end{aligned} \quad (26)$$

where $A_1 = BKC$, $B_i = BK^i$, $i = 1, 2$. The initial condition for (21)-(26) has the form of $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$ and

$$x(t+1) = Ax(t), \quad t = 0, 1, \dots, t_0 - 1, \quad t \in \mathbb{Z}. \quad (27)$$

B. Partial exponential stability of the hybrid delayed system without actuator saturation

Definition 1: Hybrid system (21)-(26) with initial conditions (27) and $e(t_0)$ is said to be partially exponential stability with respect to x if there exist constants $b > 0, 0 < \kappa < 1$ such that the following holds

$$|x(t)|^2 \leq b\kappa^{t-t_0} [|x_0|^2 + |e(t_0)|^2], \quad t \geq t_0$$

for the solutions of the hybrid system initialized with (27) and $e(t_0) \in \mathbb{R}^{n_y}$.

We extend here the result of [10] to the discrete-time case. Consider the LKF of the form:

$$\begin{aligned} V_e(t) &= V_{TOD}(t) + \min_{i=1,2} \{e_i^T(t) Q_i e_i(t)\}, \\ V_{TOD}(t) &= \tilde{V}(t) + (\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t-1} \lambda^{t-s-1} \eta^T(s) Q \eta(s), \\ \tilde{V}(t) &= x^T(t) P x(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) + \sum_{s=t-\tau_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\ &\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_0 \eta(s) + (\tau_M - \eta_m) \sum_{j=-\tau_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_1 \eta(s), \\ P &> 0, S_i > 0, R_i > 0, Q > 0, Q_j > 0, 0 < \lambda < 1, i = 0, 1, j = 1, 2, t \in [t_k, t_{k+1} - 1], t \in \mathbb{Z}, k \in \mathbb{Z}. \end{aligned} \quad (28)$$

The term $(\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t-1} \lambda^{t-s-1} \eta^T(s) Q \eta(s)$ is inserted to cope with the delays in the reset conditions. It does not grow in the jumps (when $t = t_k$), since

$$\begin{aligned} &(\tau_M - \eta_m) \left[\sum_{s=t_k-\eta_k}^{t_k-1} \lambda^{t_k-s-1} \eta^T(s) Q \eta(s) - \sum_{s=t_{k-1}-\eta_{k-1}}^{t_k^- - 1} \lambda^{t_k-s-1} \eta^T(s) Q \eta(s) \right] \\ &\leq -(\tau_M - \eta_m) \lambda^{\tau_M} \sum_{s=t_{k-1}-\eta_{k-1}}^{t_k-\eta_k-1} \eta^T(s) Q \eta(s) \\ &\leq -\lambda^{\tau_M} |\sqrt{Q} [x(t_{k-1} - \eta_{k-1}) - x(t_k - \eta_k)]|^2, \end{aligned} \quad (29)$$

where we applied Cauchy-Schwartz inequality (see e.g., [5]). The following lemma gives sufficient conditions for the exponential stability of (21)-(26) with respect to x :

Lemma 2: If there exist positive constant $\lambda < 1$ and $V_e(t)$ of (28) such that along (21)-(23)

$$V_e(t+1) - \lambda V_e(t) - \frac{1}{\tau_M - \eta_m} |\sqrt{Q_2} e_2(t)|^2 \leq 0, \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z}, \quad (30)$$

and along (24)-(26)

$$V_e(t+1) - \lambda V_e(t) - \frac{1}{\tau_M - \eta_m} |\sqrt{Q_1} e_1(t)|^2 \leq 0, \quad t \in [t_k, t_{k+1} - 1], \quad t \in \mathbb{Z} \quad (31)$$

with

$$Q \lambda^{\tau_M} \geq C^{iT} Q_i C^i, \quad i = 1, 2. \quad (32)$$

Then $V_e(t)$ does not grow in the jumps along (21)-(23), where

$$V_e(t_{k+1}) - V_e(t_{k+1}^-) \leq -|\sqrt{Q_i} e_i(t_k)|^2, \quad i = 2, \quad (33)$$

and along (24)-(26), where (33) holds with $i = 1$. Moreover, the following bound is valid for the solution of (21)-(26) with the initial condition (27) and $e(t_0) \in \mathbb{R}^{n_y}$:

$$\begin{aligned} V_{TOD}(t) &\leq \lambda^{t-t_0} V_e(t_0), \quad t \geq t_0, \quad t \in \mathbb{Z}, \\ V_e(t_0) &= V_{TOD}(t_0) + \min_{i=1,2} \{|\sqrt{Q_i} e_i(t_0)|^2\}, \end{aligned} \quad (34)$$

implying the exponential stability of (21)-(26) with respect to x .

Proof: Consider first the case of (21), where

$$V_e(t) = V_{TOD}(t) + |\sqrt{Q_2} e_2(t_k)|^2, \quad t \in [t_k, t_{k+1} - 1].$$

Since $\sum_{s=t_k+1}^t \lambda^{t-s} \leq \tau_M - \eta_m$, $t \in (t_k, t_{k+1} - 1]$, by the comparison principle, (30) implies

$$V_e(t) \leq \lambda^{t-t_k} V_e(t_k) + |\sqrt{Q_2} e_2(t_k)|^2, \quad t \in [t_k, t_{k+1} - 1]. \quad (35)$$

Therefore,

$$V_{TOD}(t) \leq \lambda^{t-t_k} V_e(t_k), \quad t \in [t_k, t_{k+1} - 1]. \quad (36)$$

If

$$|\sqrt{Q_1} e_1(t_{k+1})|^2 - |\sqrt{Q_2} e_2(t_{k+1})|^2 \geq 0 \quad (37)$$

then

$$V_e(t_{k+1}) = \tilde{V}(t_{k+1}) + |\sqrt{Q_2} e_2(t_{k+1})|^2 + (\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t_{k+1}-1} \lambda^{t_{k+1}-s-1} \eta^T(s) Q \eta(s)$$

and due to (23)

$$|\sqrt{Q_1} e_1(t_{k+1})|^2 = |\sqrt{Q_1} C_1 [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2.$$

Denote $\Theta \triangleq V_e(t_{k+1}) - V_e(t_{k+1}^-) + |\sqrt{Q_2} e_2(t_k)|^2$. Taking into account (29), we obtain

$$\Theta \leq |\sqrt{Q_2} e_2(t_{k+1})|^2 - \lambda^{\tau_M} |\sqrt{Q} [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2.$$

Due to (37), we have further

$$\begin{aligned}\Theta &\leq |\sqrt{Q_1}e_1(t_{k+1})|^2 - \lambda^{\tau_M} |\sqrt{Q}[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2 \\ &\leq -[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]^T [Q\lambda^{\tau_M} - C_1^T Q_1 C_1] [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})] \leq 0,\end{aligned}$$

where the latter inequality follows from (32) with $i = 1$.

If $|\sqrt{Q_2}e_2(t_{k+1})|^2 > |\sqrt{Q_1}e_1(t_{k+1})|^2$ then

$$V_e(t_{k+1}) = \tilde{V}(t_{k+1}) + |\sqrt{Q_1}e_1(t_{k+1})|^2 + (\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t_{k+1}-1} \lambda^{t_{k+1}-s-1} \eta^T(s) Q \eta(s)$$

and similarly

$$\begin{aligned}\Theta &\leq (|\sqrt{Q_1}e_1(t_{k+1})|^2 - |\sqrt{Q_2}e_2(t_k)|^2) + |\sqrt{Q_2}e_2(t_k)|^2 - \lambda^{\tau_M} |\sqrt{Q}[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]|^2 \\ &\leq -[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})]^T [Q\lambda^{\tau_M} - C_1^T Q_1 C_1] [x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})] \leq 0.\end{aligned}$$

Therefore, (33) with $i = 2$ is valid and, together with (35) for $t = t_{k+1}^-$, implies

$$V_e(t_{k+1}) \leq \lambda^{t_{k+1}-t_k} V_e(t_k). \quad (38)$$

Similarly, if $|\sqrt{Q_2}e_2(t_k)|^2 > |\sqrt{Q_1}e_1(t_k)|^2$, then (31) and (32) with $i = 2$ imply (36), (33) with $i = 1$ and (38). Hence, (38) holds for all $k \in \mathbb{Z}$ and

$$\begin{aligned}V_e(t_{k+1}) &\leq \lambda^{t_{k+1}-t_{k-1}} V_e(t_{k-1}) \\ &\leq \lambda^{t_{k+1}-t_0} V_e(t_0).\end{aligned}$$

The latter inequality with $k + 1$ replaced by k and (36) imply (34) that completes the proof. $\square$

Based on the above lemma, by using the standard arguments for the delay-dependent analysis [14], we derive conditions for the exponential stability of (21)-(26) with respect to x :

Lemma 3: Given scalar $0 < \lambda < 1$, positive integers $0 \leq \eta_m \leq \eta_M < \tau_M$, and K^1, K^2 , if there exist $n \times n$ matrices $P > 0$, $Q > 0$, $S_j > 0$, $R_j > 0$ ($j = 0, 1$), S_{12} , matrices $Q_1 > 0$, $Q_2 > 0$ such that (32) and

$$\hat{\Omega} = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \geq 0, \quad (39)$$

$$\hat{F}_0^T P \hat{F}_0 + \hat{\Sigma} + \hat{F}_{01}^T W \hat{F}_{01} - F_{12}^T R_0 \lambda^{\eta_m} F_{12} - \hat{F}^T \hat{\Omega} \lambda^{\tau_M} \hat{F} < 0, \quad (40)$$

are feasible, where

$$\begin{aligned}\hat{F}_0 &= [A \ 0 \ A_1 \ 0 \ B_{3-i}], \quad \hat{F}_{01} = \hat{F}_0 - F_1, \quad \hat{F} = [F_{23}^T \ F_{34}^T]^T, \\ \hat{\Sigma} &= \text{diag}\{S_0 - \lambda P, -(S_0 - S_1)\lambda^{\eta_m}, 0, -S_1\lambda^{\tau_M}, \varphi\}, \\ \varphi &= -[\frac{1}{\tau_M - \eta_m} - (1 - \lambda)]Q_{3-i}, \\ W &= \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m)Q, \quad i = 1, 2.\end{aligned} \quad (41)$$

Then solutions of the hybrid system (21)-(26) satisfy the bound (34) and are exponentially stable with respect to x .

C. Partial exponential stability of the hybrid delayed system with actuator saturation

Our objective is to derive the bound on $V_{TOD}(t_0)$ in terms of x_0 . By using arguments similar to Lemma 1, we arrive at

Lemma 4: Consider LKF $V_{TOD}(t)$ given by (28) and denote $V_0(t) = x^T(t)Px(t)$. Under (9) let there exist $0 < \lambda < 1$ and $\mu > 1$ such that (12a) and

$$V_{TOD}(t+1) - \lambda V_{TOD}(t) - (\mu - 1)V_0(t) \leq 0, \quad t = 0, 1, \dots, t_0 - 1, \quad (42)$$

hold along (27). Then we have

$$V_{TOD}(t_0) \leq x_0^T \left[\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_m} - 1)P \right] x_0, \quad (43)$$

where

$$\Xi_{TOD} = P + \eta_m S_0 + \lambda^{\eta_m} (\tau_M - \eta_m) S_1. \quad (44)$$

Proof: From (12a), $V_0(t) \leq \mu^t V_0(0)$ for $t = 0, 1, \dots, t_0$. Under the constant initial condition (9) and $V_{TOD}(t)$ of (28), we have for $t = 0$

$$V_{TOD}(0) = x_0^T P x_0 + \sum_{s=-\eta_m}^{-1} \lambda^{-s-1} x_0^T S_0 x_0 + \sum_{s=-\tau_M}^{-\eta_m-1} \lambda^{-s-1} x_0^T S_1 x_0.$$

Thus $V_{TOD}(0) \leq x_0^T \Xi_{TOD} x_0$. Noting that $\eta_m \leq t_0 = \eta_0 \leq \eta_M$, the inequality (42) implies

$$\begin{aligned} V_{TOD}(t_0) &\leq \lambda^{t_0} V_{TOD}(0) + (\mu^{t_0} - 1) x_0^T P x_0 \leq x_0^T \left[\lambda^{t_0} \Xi_{TOD} + (\mu^{t_0} - 1)P \right] x_0 \\ &\leq x_0^T \left[\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_m} - 1)P \right] x_0. \quad \square \end{aligned} \quad (45)$$

Lemmas 3 and 4 imply the following result:

Theorem 2: Given positive scalars $\lambda < 1, \beta, \mu > 1, \sigma, \bar{\sigma}, 0 \leq \eta_m \leq \eta_M < \tau_M$ and K^1, K^2 , if there exist $n \times n$ matrices $P > 0, Q > 0, S_l > 0, R_l > 0 (l = 0, 1), S_{12}$, matrices $Q_1 > 0, Q_2 > 0$ such that (16), (32), (39), (40) and

$$\eta_m S_0 + \lambda^{\eta_m} (\tau_M - \eta_m) S_1 \leq \sigma P, \quad (46)$$

$$Q \lambda^{\tau_M} \leq \bar{\sigma} P, \quad (47)$$

$$\begin{bmatrix} P \rho_{TOD}^{-1} & (K^i C^i)_j^T \\ * & \frac{1}{4} \beta \bar{u}_j^2 \end{bmatrix} \geq 0, \quad j = 1, \dots, n_u, \quad (48)$$

$$\tilde{F}_0^T P \tilde{F}_0 + \tilde{\Sigma} + \tilde{F}_{01}^T W \tilde{F}_{01} - \tilde{F}_{12}^T R_0 \lambda^{\eta_m} \tilde{F}_{12} - \tilde{F}^T \hat{\Omega} \lambda^{\tau_M} \tilde{F} < 0, \quad (49)$$

are feasible, where the notations are given by (41) and where

$$\begin{aligned} \tilde{F}_0 &= [A \ 0 \ 0 \ 0], \\ \tilde{F}_j &= [\Gamma_1 \ \Gamma_2 \ \Gamma_3 \ \Gamma_4], \Gamma_j = I, \Gamma_l = 0, l = 1, \dots, 4, l \neq j, \\ \tilde{F}_{kj} &= \tilde{F}_k - \tilde{F}_j, \ k, j = 0, 1, \dots, 4, \ \tilde{F} = [\tilde{F}_{23}^T \ \tilde{F}_{34}^T]^T, \\ \tilde{\Sigma} &= \text{diag}\{S_0 - \lambda P + (1 - \mu)P, -(S_0 - S_1)\lambda^{\eta_m}, 0, -S_1\lambda^{\tau_M}\}, \\ \rho_{TOD} &= \lambda^{\eta_m}(1 + \sigma) + (\mu^{\eta_M} - 1) + \bar{\sigma}, \ i = 1, 2. \end{aligned} \quad (50)$$

Then, for all initial conditions x_0 belonging to $\mathcal{X}_\beta$, the closed-loop system (21)-(26) is exponentially stable with respect to x .

Proof: Suppose that $x(t) \in \mathcal{L}(K^1, \bar{u}) \cap \mathcal{L}(K^2, \bar{u})$. As shown in Lemma 3, (32), (39) and (40) lead to (34) for $t = t_0, t_0 + 1, \dots$. Following the proof of Theorem 1, for $t = 0, 1, \dots, t_0 - 1$, (16) and (49) with (39) guarantee (12a) and (42), respectively.

Next noting that (32) and (47), we have

$$x_0^T C^i Q_i C^i x_0 \leq \lambda^{\tau_M} x_0^T Q x_0 \leq \bar{\sigma} x_0^T P x_0, \quad (51)$$

which implies that $|\sqrt{Q_i} e_i(t_0)|^2 = |-\sqrt{Q_i} C^i x_0|^2 < \bar{\sigma} x_0^T P x_0$, $i = 1, 2$.

Therefore, taking into (34), (43), (46) and (51), we obtain

$$\begin{aligned} x^T(t) P x(t) &\leq V_{TOD}(t) \leq \lambda^{t-t_0} V_e(t_0) \\ &= \lambda^{t-t_0} \{V_{TOD}(t_0) + \min_{i=1,2} \{e_i^T(t_0) Q_i e_i(t_0)\}\} \\ &\leq x_0^T [\lambda^{\eta_m} \Xi_{TOD} + (\mu^{\eta_M} - 1)P + \bar{\sigma}P] x_0 \\ &\leq x_0^T [\lambda^{\eta_m}(1 + \sigma) + (\mu^{\eta_M} - 1) + \bar{\sigma}] P x_0 \\ &= \rho_{TOD} x_0^T P x_0, \quad t = t_0, t_0 + 1, \dots \end{aligned}$$

So for all $x(t) : x^T(t) P x(t) \leq \rho_{TOD} \beta^{-1} \Rightarrow x^T(t) (K^j C^j)_i^T (K^j C^j)_i x(t) \leq \frac{1}{4} \bar{u}_i^2$, if

$$\begin{aligned} &x^T(t) (K^j C^j)_i^T (K^j C^j)_i x(t) \\ &\leq \frac{1}{4} \beta \rho_{TOD}^{-1} x^T(t) P x(t) \bar{u}_i^2, \quad i = 1, \dots, n_u, \quad j = 1, 2. \end{aligned}$$

The latter inequality is guaranteed if $\frac{1}{4} \beta \rho_{TOD}^{-1} P \bar{u}_i^2 - (K^j C^j)_i^T (K^j C^j)_i \geq 0$, and, thus, by Schur complements if (48) is feasible. Hence, solutions of the hybrid system (21)-(26) initialized with (27) and $e(t_0) \in \mathbb{R}^{n_y}$ converge to the origin exponentially with respect to x , provided that $x_0 \in \mathcal{X}_\beta$. $\square$

Remark 1: Note that

$$x_0^T P x_0 \leq \lambda_{\max}(P) |x_0|^2 < \beta^{-1}, \quad (52)$$

where $\lambda_{\max}(P)$ denotes the largest eigenvalue of P . Hence the following initial region $|x_0|^2 < \beta^{-1}/\lambda_{\max}(P)$ is inside of $\mathcal{X}_\beta$. In order to maximize the initial ball, we can add the condition

$$P - \gamma I < 0, \quad (53)$$

to Theorems 1 and 2, where $\gamma > 0$ is minimized.

IV. EXAMPLE: DISCRETE-TIME CART-PENDULUM

Consider the following linearized model of the inverted pendulum on a cart [11]:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\dot{x}}(t) \\ \dot{\theta}(t) \\ \dot{\ddot{\theta}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{a}{M} \\ 0 \\ \frac{-a}{Ml} \end{bmatrix} u(t), \quad t \in \mathbb{R}^+ \quad (54)$$

with $M = 3.9249\text{Kg}$, $m = 0.2047\text{Kg}$, $l = 0.2302\text{m}$, $g = 9.81\text{N/Kg}$, $a = 25.3\text{N/V}$ and $\bar{u} = 50$.

In the model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. Such a model is discretized with a sampling time $T_s = 0.001\text{s}$:

$$\begin{bmatrix} x(t+1) \\ \Delta x(t+1) \\ \theta(t+1) \\ \Delta\theta(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0 & 1 & -0.0005 & 0 \\ 0 & 0 & 1.00 & 0.001 \\ 0 & 0 & 0.0448 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \Delta x(t) \\ \theta(t) \\ \Delta\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0064 \\ 0 \\ -0.0280 \end{bmatrix} u(t), \quad t \in \mathbb{Z} \quad (55)$$

with $\bar{u} = 50$. The pendulum can be stabilized by a state feedback $u(t) = K \begin{bmatrix} x & \Delta x & \theta & \Delta\theta \end{bmatrix}^T$ with the gain $K = [K^1 \quad K^2]$

$$K^1 = \begin{bmatrix} 5.825 & 5.883 \end{bmatrix}, \quad K^2 = \begin{bmatrix} 24.941 & 5.140 \end{bmatrix}, \quad (56)$$

which leads to the closed-loop system eigenvalues $\{0.8997, 0.9980+0.0020i, 0.9980-0.0020i, 0.9980\}$.

Suppose the variables $\theta, \Delta\theta$ and $x, \Delta x$ are not accessible simultaneously. We consider measurements $y^i(t) = C^i x(t)$, $t \in \mathbb{Z}$, where

$$C^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (57)$$

Choose $\lambda = 1$, $\beta = 1$, $\sigma = 1.0 \times 10^{-2}$ and $\eta_m = 1$, $\eta_M = 2$, $\tau_M = 3$. Applying Theorem 2 with $\mu = 1.02$, $\bar{\sigma} = 1.1$ and Remark 1, the closed-loop system (21)-(26) is exponentially stable with

respect to x starting from the initial ball $|x_0| < 0.3342$ by the presented TOD protocol. Applying Theorem 1 with $\mu = 1.02$ and Remark 1, the closed-loop system (6), (10) is asymptotically stable and the largest ball of admissible initial conditions is $|x_0| < 0.4964$.

Then for different μ , by Theorems 1, 2 with $\lambda = 1$ and Remark 1, we give the corresponding largest ball of admissible initial conditions (see Table 1).

TABLE I
LARGEST BALL OF ADMISSIBLE INITIAL CONDITIONS FOR DIFFERENT μ

μ	1.02	1.2	1.5	2	3
$ x_0 (\text{Theorem 1, RR})$	0.4964	0.3557	0.2304	0.1297	0.1339
$ x_0 (\text{Theorem 2, TOD})$	0.3342	0.3069	0.2674	0.2168	0.1541

V. CONCLUSIONS

In this paper, a time-delay approach was developed for the stability analysis of discrete-time NCSs in the presence of actuator saturation under the RR or under a weighted TOD scheduling. A Lyapunov-based method was presented for finding the domain of attraction under the both scheduling protocols. The conditions are given in terms of LMIs. Polytopic uncertainties in the system model can be easily included in the analysis. Numerical example illustrates the efficiency of our method.

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REFERENCES

- [1] P. Antsaklis and J. Baillieul. Special issue on technology of networked control systems. *Proceedings of the IEEE*, 95(1):5–8, 2007.
- [2] M. Donkers, W. Heemels, N. van de Wouw, and L. Hetel. Stability analysis of networked control systems using a switched linear systems approach. *IEEE Transactions on Automatic control*, 56(9):2101–2115, 2011.

- [3] E. Fridman. A refined input delay approach to sampled-data control. *Automatica*, 46(2):421–427, 2010.
- [4] E. Fridman, A. Seuret, and J.P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40(8):1441–1446, 2004.
- [5] E. Fridman and U. Shaked. Stability and guaranteed cost control of uncertain discrete delay systems. *International Journal of Control*, 78(4):235–246, 2005.
- [6] H. Fujioka. Stability analysis of systems with aperiodic sample-and-hold devices. *Automatica*, 45(3):771–775, 2009.
- [7] H. Gao and T. Chen. Network-based H_∞ output tracking control. *IEEE Transactions on Automatic control*, 53(3):655–667, 2008.
- [8] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, and D. Nesic. Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. *IEEE Transactions on Automatic control*, 55(8):1781–1796, 2010.
- [9] K. Liu and E. Fridman. Delay-dependent methods and the first delay interval. *Systems & Control Letters*, 2013, to appear.
- [10] K. Liu, E. Fridman, and L. Hetel. Network-based control via a novel analysis of hybrid systems with time-varying delays. In *Proceedings of the 51th IEEE Conference on Decision and Control*, Hawaii, USA, December 2012.
- [11] K. Liu, E. Fridman, and L. Hetel. Stability and L_2 -gain analysis of networked control systems under Round-Robin scheduling: a time-delay approach. *Systems & Control Letters*, 61(5):666–675, 2012.
- [12] P. Naghshtabrizi, J.P. Hespanha, and A.R. Teel. Stability of delay impulsive systems with application to networked control systems. *Transactions of the Institute of Measurement and Control*, 32(5):511–528, 2010.
- [13] D. Nesic and D. Liberzon. A unified framework for design and analysis of networked and quantized control systems. *IEEE Transactions on Automatic control*, 54(4):732–747, 2009.
- [14] P.G. Park, J.W. Ko, and C. Jeong. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47(1):235–238, 2011.
- [15] A. Seuret and Jr. J. M. Gomes da Silva. Taking into account period variations and actuator saturation in sampled-data systems. *Systems & Control Letters*, 61(12):1286–1293, 2012.