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Transmission of Structure-borne Sound in Buildings above Railway Tunnels

by

Osama Hassan

Stockholm 2001

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TRANSMISSION OF STRUCTURE-BORNE SOUND IN BUILDINGS ABOVE RAILWAY TUNNELS

Licentiate of Engineering Thesis
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November 2001

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PREFACE

This licentiate thesis is the result of my research work at the division of Building Technology, Department of Civil Engineering and Building Sciences at the Royal Institute of Technology, Stockholm during the period Oct. 1999- Nov. 2001. In this context, I would like to acknowledge the financial support of the Swedish Transport and Communications Research Board (KFB).

I would like to express my sincere gratitude to my supervisor, Professor Sten Ljunggren, for giving me the opportunity to carry out this research work and for his encouragement and guidance. Thanks also to Dr. Olivier Fégeant for reading and commenting on the manuscript of the second paper.

Stockholm, in November 2001

The thesis comprises an introduction and the following papers:

- Paper I O. A .B. Hassan, “ Transmission of Structure-borne Sound in Building above Railway Tunnels” *Building Acoustics*, 8(4), 269-299.
- Paper II O. A .B. Hassan, “ A Tentative Noise Control Method for Buildings above Railway Tunnels” (Submitted for publication in *Building Acoustics*).

INTRODUCTION

The increase in site prices in the central parts of large cities has made it appropriate to build close to and above railways. With relatively short distance from railways and tunnels, the structure-borne sound is of great importance with regard to the building design, as the costs of noise control measures are relatively high. Therefore, a prediction scheme for sound transmission from building foundation up to structural elements is thought to be of intrinsic interest.

In the first paper, the transmission of structure-borne sound in buildings above railway tunnels is investigated in a specific case: the sound is assumed to emanate from railway traffic and to be transmitted through the rock foundation into the building. Analytical models are presented for the propagation of structure-borne sound upward in buildings. The models are based on the assumption that the vibrational energy propagates from the foundation in form of quasi-longitudinal waves. A finite ground impedance is taken into account as well as wave reflection from the roof. Results are presented for buildings constructed with either load-bearing walls or columns. The building under study has a typical configuration in the sense that the walls/columns are aligned on top of each other.

In the second paper, a design method to reduce the transmission of structure-borne sound in buildings above railway tunnels is studied. The case studied is similar to the one studied in the first paper in the sense that the building is founded on rock. However, in the present case, the load-bearing columns on the first floor are not placed above other load-bearing columns or walls. Analytical models for the propagation of structure-borne sound upward in buildings are proposed. In the same way as in the first paper, the models are based on the assumption that the initial vibrational energy that propagates from the foundation is in form of quasi-longitudinal waves, and a finite ground impedance is taken into account as well as wave reflection from the roof. The propagation of bending and quasi-longitudinal waves in the load-bearing columns is considered. A parametric survey is conducted to investigate the influence of certain important geometrical properties of the first floor on the sound transmission. Finally, the results are compared with previous results for a typical building configuration and it is shown that the dislocation of the columns can significantly reduce the sound transmission upward in the building.

In the second paper, the phenomenon of scattering of flexural waves on the first floor from region of columns is investigated. For a similar case, Leviatan et al.¹ proposed a representation of the scattered response from a compact region by a superposition of responses from point sources of images. Lapin² considered a scattering problem within the context of exact elasticity theory for thick plates. Norris and Vemula^{3,4} discussed the idea of energy flux and scattering cross-section and derived an optical theorem for flexural waves. Miklowitz⁵ expanded the general solution to a line load on a thick plate in the high frequency region. However, none of these authors considered the present problem: the scattering of a propagating wave and an exponential near field by circular column attached to the plate.

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Paper I

TRANSMISSION OF STRUCTURE-BORNE SOUND IN BUILDINGS ABOVE RAILWAY TUNNELS

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ABSTRACT

Transmission of structure-borne sound in buildings above railway tunnels is studied in a specific case: the sound is assumed to emanate from railway traffic and to be transmitted through the rock foundation into the building. Analytical models are presented for the propagation of structure-borne sound upward in buildings. The models are based on the assumption that the vibrational energy propagates from the foundation in form of quasi-longitudinal waves. A finite ground impedance is taken into account as well as wave reflection from the roof. Results are presented for buildings constructed with either load-bearing walls or columns.

1. INTRODUCTION

Increasing site prices in the central parts of large cities has made it interesting to build close to and above railways. With relatively short distance from railways and tunnels, the ground-borne sound is of great importance with regard to the building design, as the costs of noise control measures are relatively high. However, it has been pointed out that the noise level inside the building depends on the building design¹. Therefore, a prediction scheme for sound transmission from building foundation up to structural elements is thought to be of intrinsic interest.

In broader sense, two categories of models for the prediction of structure-borne sound from underground railways can be found in the literature: (1) models for soft ground, (2) models for stiff ground. An underlying assumption for the first category is that the ground acts as a constant force source, that is, the impedance of the ground is low in comparison with the impedance of the building, see e.g. Melke and Ungar et al.^{2,3}. The second category of prediction models is derived under the assumption that the ground has high impedance compared with that of the building; the ground is in this case considered as a constant velocity source⁴. This difference must be taken into account for the transmission upwards in the building.

For the case of buildings on a ground with high impedance, Ljunggren¹ investigated the transmission of structure-borne sound in building above railways and showed that a major part of the sound is transmitted upwards in the building in the form of quasi-longitudinal waves. He presented a theoretical model for the transmission based on "Ketten-Leiter" theory, where the load bearing walls/columns and the floors are modeled as an infinite cascade of longitudinal rods alternating with impedance elements. Ljunggren's model, however, is restricted to the case of sound transmission in the intermediate regions of high-rise buildings. Therefore, it cannot be used to predict the sound transmission in other regions; in particular, it did not consider the influence of wave reflection from the building roof, which may be important in low-rise buildings. A literature search resulted in no work on this subject apart from that of Ljunggren^{1,4}.

The present paper is devoted to the case of buildings on a ground with high impedance. Analytical models are presented for the propagation of structure-borne sound upward in buildings. The models are based on the assumption that the vibrational energy propagates from the foundation in form of quasi-longitudinal waves. The main difference from Ljunggren's work is that a finite ground impedance as well as wave reflection from the roof are taken into account. The load-bearing members (walls and columns) of the building are regarded as wave-guides. The vertical velocities of all the junctions between walls/columns and floors are determined using the appropriate boundary conditions at the foundation, all the junctions and the roof. The bending wave fields of the floors are then determined using a statistical energy analysis (SEA) approach.

2. MAIN ASSUMPTIONS AND LIMITATIONS

A spatially periodic building structure is considered, see Figure 1. The main assumptions and limitation of the study are summed up in the following.

2.1 Dimensions of structural elements

The thicknesses of the floors are assumed to be the same throughout the buildings. In the same way, the thicknesses of the walls and the diameters of the columns are assumed to be the same. All the vertical load-bearing members are assumed to be placed on top of each other; an interesting example where this is not the case will be treated in a subsequent paper. The bearing elements and floors are of uniform thickness and made of an isotropic, homogeneous, and linearly elastic material. In addition, the cross-sectional areas of the bearing columns are circular.

2.2 Frequency range

A typical spectrum of velocity level due to a passing train, measured on bare rock shows that the most important part of the spectrum lies between 25 and 250 Hz, with a broad peak at 63 Hz¹. Consequently, the prediction model of sound transmission will be focused on relatively low frequencies.

2.3 Physical properties of structure

The low frequencies of interest here imply that the floor and the vertical load-bearing elements can be considered as thin compared to the wavelength λ , i.e. the condition $\lambda > 6h$, where h is the respective thickness, is fulfilled. The storey height L_y is defined as in Figure 1. The junction beam is supposed to transfer displacements and angular velocities unchanged. The common joints between walls and slabs are looked upon as rigid joints.

2.4 Impedance of the ground

The buildings are assumed to rest on rock, a common situation in Sweden. As mentioned earlier, the ground is assumed to possess very high impedance. This restriction implies that the excitation of the system occurs via vertical ground velocity component, which couples to longitudinal waves in the load-bearing members of the building.

It is worth noting that for this case, the horizontal component of ground motion, under the present prerequisites, is not considered as it was shown by Ljunggren¹ that more power is transmitted by means of the longitudinal waves than by the bending waves.

3. COMPLETE MODEL FOR BUILDINGS WITH LOAD-BEARING WALLS

The complete model is obtained by considering the foundation and roof as global boundary for the sound propagation through the cascade system. The word complete is here used in order to distinguish this solution from an approximate one described in Section 4. In Figure 1 is shown a schematic view of the structure junctions. Floor N denotes here the roof of the complete building. The solution is derived in two parts. First, the foundation junction is analyzed to relate the velocity of foundation with that of the propagating waves of the wall. Second, the other junctions are analyzed with the help of the reflection coefficient of the roof.

3.1 Wave transmission through the foundation (junction 0)

For the model, the ground is substituted by an equivalent waveguide, carrying longitudinal volume waves in the vertical direction. The horizontal cross-section of the waveguide is the same as that of the building wall immediately above the ground. This model is justified by the well-known Thevenin theorem⁵.

The free velocity v_{free} and the input force impedance of the wave-guide are the same as those of the ground.

The velocity of the equivalent waveguide is expressed as

$$v_g(y) = (v_g^+ e^{-jk_g y} + v_g^- e^{jk_g y}) e^{j\omega t} \quad (1)$$

where k_g is the wave number of the (equivalent) longitudinal wave. For the sake of simplicity, all factors indicating sinusoidal time-dependencies $e^{+j\omega t}$, where ω is the angular frequency, $\omega = 2\pi f$, are omitted in the remainder of the paper.

Obviously, v_g^+ is related to the free velocity of the ground in the y -direction as

$$v_{free} = 2v_g^+ \quad (2)$$

The corresponding force that acts in the waveguide is obtained as

$$F_g(y) = Z_\infty^F (v_g^+ e^{-jk_g y} - v_g^- e^{jk_g y}) \quad (3)$$

where Z_∞^F is the force impedance of a semi-infinite medium which here is taken as (see appendix)

$$Z_\infty^F = \frac{\pi E_g}{j\omega 2 (1 - \mu_g^2) \ln(\frac{L_z}{2h})} \quad (4)$$

Figure 1. Schematic view of a typical building under excitation.

where L_z and h are the length of the wall in the z -direction and the wall thickness, respectively. E and μ here are the modulus of elasticity and the Poisson's ratio, respectively. It can be shown that the ground impedance according to Equation (4) is much higher than the impedance of the wall for the common value of dimensions and material data.

The velocity of the wall is taken as

$$v_\theta(y) = v_\theta^+ e^{-jk_L y} + v_\theta^- e^{jk_L y} \quad (5)$$

where k_L is the wave number of quasi-longitudinal wave in the wall, $k_L = \omega / c_L$ and c_L is the phase velocity of the quasi-longitudinal waves. The corresponding force in the wall that acts at the junction is given by

$$F_\theta(y) = Z (v_\theta^+ e^{-jk_L y} - v_\theta^- e^{jk_L y}) \quad (6)$$

where Z , is the impedance of the wall,

$$Z = \rho h c_L \quad (7)$$

where ρ is the density of the wall material. The continuity of velocity and forces equilibrium at $y=0$ gives

$$v_\theta^+ + v_\theta^- = v_g^+ + v_g^- \quad (8)$$

$$Z(v_\theta^+ - v_\theta^-) = Z_\infty^F (v_g^+ - v_g^-). \quad (9)$$

3.2. Reflection at and transmission into the roof (junction N)

The velocity field in the wall is taken as,

$$v_{N-I}(y) = v_{N-I}^+ e^{-jk_L y} + v_{N-I}^- e^{jk_L y} \quad (10)$$

The incident wave in the bearing wall couples to bending waves of roof. The field of the roof is taken as

$$u_N(x) = u_N^+ e^{-jk_B x} + u_N^- e^{-jk_B x} \quad (11)$$

where u_N^+ , u_N^j refer to the propagating bending wave and the decaying near-field in the roof, respectively; k_b here is the wave number of bending wave. At the present stage, the roof is considered as a semi-finite plate⁶. The influence of finite size of floor plates will be considered later on.

At the roof junction, Figure1, the boundary conditions are obtained as follows.

α) The angular velocity of the roof,

$$\left(\frac{\partial u_N(x)}{\partial x} \right)_{x=0} = 0 \quad (12)$$

β) Continuity of velocity at $y = L_y$, $x=0$ leads to

$$u_N^+ + u_N^j = v_{N-1}^+ e^{-jk_L L_y} + v_{N-1}^- e^{jk_L L_y} \quad (13)$$

γ) Force equilibrium.

Two types of forces are present at the junction: axial forces and shear forces. The axial forces are related to the quasi-longitudinal waves in the walls and are given by

$$F_{N-1} = \frac{-E_w h}{j\omega(1 - \mu_w^2)} \frac{\partial v_{N-1}(y)}{\partial y} \quad (14)$$

where the subscript w denotes the wall. The shear forces are related to the bending waves of the roof and can be taken as

$$F_{nf} = -j\omega m_f \int u_N(x) dx \quad (15)$$

where m_f is the surface density of floor plate, $m_f = \rho h_f$. Obviously, the shear forces on both sides of the excitation area are equal due to symmetry.

The solution of equations (10-15) can be written as

$$u_N^+ = \frac{2\beta}{\beta(1-j)+4} v_{N-1}^+ \quad (16)$$

where β is defined as

$$\beta = \frac{E_w h}{c_L c_B m_f (1 - \mu_w^2)} \quad (17)$$

In addition, the solution yields the reflection coefficient at the roof, which is defined as

$$r_N = \frac{v_{N-l}^-}{v_{N-l}^+} = \frac{\beta(1-j) - 4}{\beta(1-j) + 4} e^{-j2k_L L_y} \quad (18)$$

3.3. Wave transmission through junction (n)

In addition to Equation (5), the velocity expressions at junction (n) are taken as

$$v_n(y) = v_n^+ e^{-jk_L y} + v_n^- e^{jk_L y} \quad (19)$$

$$u_n(x) = u_n^+ e^{-jk_B x} + u_n^- e^{jk_B x} \quad (20)$$

In order to relate the velocity amplitudes, the boundary conditions are applied. The boundary conditions are, of course, in principle the same as those of the roof junction. Accordingly, solving Equations (5), (19) and (20) with regard of the boundary conditions at junction (n) yields

$$u_n^+ = \frac{v_n^+ + v_n^-}{1-j} \quad (21)$$

$$v_n^+ = v_{n-l}^+ e^{-jk_L L_y} - \frac{2(v_{n-l}^+ e^{-jk_L L_y} + v_{n-l}^- e^{jk_L L_y})}{\beta(1-j)} \quad (22)$$

$$v_n^- = v_{n-l}^- e^{+jk_L L_y} + \frac{2(v_{n-l}^+ e^{-jk_L L_y} + v_{n-l}^- e^{jk_L L_y})}{\beta(1-j)} \quad (23)$$

3.4 Solution of the equations of motions for the entire cascade

To solve the motion equations at the entire cascade, the procedure can be explained as follows. The relation between the propagating wave in the upward direction and reflected ones in the wall of each junction can be obtained by starting solving the equations from the roof reflection and down to the foundation junction.

Consequently, the resulting equation, which relates v_o^+ and v_o^- , is then solved simultaneously together with Equation (8) and Equation (9) to obtain v_o^+ . Finally, the calculated v_o^+ and v_o^- are sufficient to determine the respective field variables at the remaining junctions upwards to the roof junction.

3.4 Average velocity of the floor

In the steady state, the dissipated power of floor plate must be equal to power supplied to the system. The incident power of the bending wave at the junctions obeys the general relation

$$P_{in} = \frac{I}{2} \operatorname{Re} \{ \hat{F} \hat{v}^* \} L_z \quad (24)$$

where $\hat{F}$ is shear force (peak value) as shown in Equation (15) and $\hat{v}^*$ is the complex conjugate of the velocity at the junction.

The dissipated power over the entire area S of the floor can be written as, see Cremer et al.⁶,

$$P_f = \omega \eta \int_S m_f \langle u^2 \rangle dx dy \quad (25)$$

provided that the plate is not heavily damped. The term η denotes the total loss factor of the floor. Accordingly, the time and space-average velocity of floor can be written as

$$\langle u^2 \rangle = \frac{2 P_{in}}{\omega m_f S \eta} \quad (26)$$

The latter expression takes into account that excitation of the floor occurs via two bearing walls as indicated in Figure 2.

Figure 2. The floor excitation by two bearing walls

The total loss factor η can be taken as

$$\eta = \frac{C_B \sum_{i=1}^n \alpha_i L_i}{\omega \pi S} + \eta_i \quad (27)$$

where α_i is the absorption coefficient for each absorbing line of length L_i , and η_i is the internal material damping.

For the usual case with floors and walls of equal thickness, where $\alpha = 0.25^7$, it can be shown that

$$\frac{\langle u^2 \rangle}{|v|^2} = \frac{2\pi L_z}{L_z + \eta_i \pi k_b S} \quad (28)$$

where v is the RMS value of the vertical velocity at a junction and $\langle u^2 \rangle$ is the RMS value of the bending wave velocity of the floor connected to the junction.

In many cases in practice, the internal losses are small compared to the boundary losses (hard coupling). In such a case, Equation (28) is reduced to

$$\frac{\langle u^2 \rangle}{|v|^2} = 2\pi \quad (29)$$

The velocity level decrease between the foundation (junction 0) and the floor at the

junction is defined as

$$\Delta L_v = 10 \log \left| \frac{v^2}{v_{free}^2} \right| \quad (30)$$

3.5 Presentation and discussion

In Figures 3-6 is shown the velocity level difference for the four frequencies 31.5, 63, 125, and 250 Hz. Each curve is assigned to a building with the number of storeys given in the legend; the last junction is that with the roof of the building. The material and geometrical properties of the buildings here and in the remainder of the text are presented in tables 1 and 2. The ground is assumed to have the density 2700 kg/m³ and the longitudinal propagation velocity of 6300 m/s.

For the chosen configuration, it can be shown from Equation (18) that the phase angle of the reflection coefficient is close to zero. Furthermore, Equations (22) and (23) show that the phase shift of the propagating quasi-longitudinal waves is small at the junctions. As the ground impedance is high, a resonance should be expected at the frequency where the wavelength λ equals four times the building height. Accordingly, the resonance frequency is determined from $f_r = c' / \lambda$, where c' is the trace velocity, $c' = \omega L_y / \text{Im}\{g\}$, where g is the propagation constant, see Equation (33) in the next section. The

calculated resonance frequency for a 2-storey building is ~ 120 Hz, and for 3-storey building is ~ 80 Hz. Clearly, this agrees with the resonance curves showed in Figures 7 and 8.

Table 1
Material properties of a typical multistorey building

Property	Value
Young's modulus	26×10^9 Pa
Poisson's ratio	0.3
Concrete internal loss factor	0.005
Density	2400 kg/m^3
Phase velocity of quasi-longitudinal waves in walls	3450 m/sec
Phase velocity of quasi-longitudinal waves in columns	3290 m/sec

Table 2
Dimensions of load-bearing elements

Height of the wall, L_y ,	2.7 m
Width of the floor, L_z ,	12 m

Figure 3. Calculated velocity level difference for buildings with bearing walls at 31.5 Hz. The thickness of the floor and the wall is 200 mm, (—) 9-storey building, (— · —) 8-storey building, (.....) 7-storey building, (— · —) 6-storey, (— · — · —) 5-storey building, (— · — · —) 4-storey building, (.....) 3-storey building, (— · —) 2-storey building.

Figure 4. Calculated velocity level difference for buildings with bearing walls at 63 Hz. The thickness of the floor and the wall is 200 mm, (—) 9-storey building, (— · —) 8-storey building, (.....) 7-storey building, (— · — ·) 6-storey, (— · · —) 5-storey building, (— · —) 4-storey building, (.....) 3-storey building, (—) 2-storey building.

Figure 5. Calculated velocity level difference for buildings with bearing walls at 125 Hz. The thickness of the floor and the wall is 200 mm, (—) 9-storey building, (— · —) 8-storey building, (.....) 7-storey building, (— · — ·) 6-storey, (— · · —) 5-storey building, (— · —) 4-storey building, (.....) 3-storey building, (—) 2-storey building.

Figure 6. Calculated velocity level difference for buildings with bearing walls at 250 Hz. The thickness of the floor and the wall is 200 mm, (—) 9-storey building, (— · —) 8-storey building, (.....) 7-storey building, (— · — · —) 6-storey, (— · — · —) 5-storey building, (— · — · —) 4-storey building, (.....) 3-storey building, (—) 2-storey building.

Figure 7. Calculated velocity level difference for a 2-storey building with load-bearing walls. The thickness of the floor and the wall is 200 mm, (—) junction 1, (-----) junction 2 (the roof).

Figure 8. Calculated velocity level difference for a 3-storey building with load-bearing walls. Thickness of the floor and the wall is 200 mm, (-----) junction 1, (——) junction 2, (.....) junction 3 (the roof).

4. APPROXIMATE MODEL FOR HIGH-RISE BUILDINGS WITH LOAD-BEARING WALLS

The approximate model is obtained by assuming an infinite building height and neglecting any potential effect from sound reflection from the building roof. For this case, an arbitrary junction located far away from the roof is considered, Figure 9.

4.1 Determination of the attenuation per storey

The boundary conditions at the junction, see Figure 9 are similar to those of junction (n), described in section 3.3. The additional relations needed for the solution of the problem are obtained by introducing a propagation constant g , see e.g. Cremer et al⁶,

Figure 9. The field variables at an arbitrary junction located in the intermediate region of a high-rise building.

$$v_{+(n)} e^{-jkL_v} = v_{+(n-1)} e^{-g} \quad (31)$$

$$v_{-(n)} e^{jkL_v} = v_{-(n-1)} e^{-g} . \quad (32)$$

The above equations together with equations (12)-(15) can be solved with the result

$$\cosh(g) = \frac{e^{-j2kL} (\beta - j\beta - 2) + \beta(1-j) + 2}{2\beta e^{-jkL} (1-j)} \quad (33)$$

The attenuation per storey can be calculated from the propagation constant g as

$$\Delta L_v = 20 \lg \left| e^{-g} e^{jk_L L_v} \right| = -8.7 \operatorname{Re}\{g\} \text{ dB} \quad (34)$$

The result is shown in Figure 10. It can be seen that the attenuation is fairly small. The influence of the material loss factor of the walls has been investigated by means of a complex wave number, k_L . It was found that this loss factor had a negligible effect on the result for appropriate values of the loss factor.

Figure 10. Calculated attenuation per storey for a building with load-bearing walls. The thickness of the floor is 200 mm and the thickness of the wall is 150 mm (---), 200 mm (—) and 250 mm (·····).

4.2 Comparison of the two solutions

In order to compare the approximate solution with the complete one, a high-rise building with 9 storeys is studied. The results are shown in Figures 11 to 14.

Figure 11. Velocity level difference between the junctions and the foundation of a typical 9-storey building with bearing walls. (—) complete model; (-----) approximate model. The frequency is 31.5 Hz and the thickness of the floor and wall is 200 mm.

It is seen that the agreement between the complete and the approximate solution is not too bad at 250 Hz. At this frequency, the behavior of the load-bearing walls is more or less that of a damped, semi-infinite wave-guide. However, at lower frequencies, the modal behavior becomes more prominent, which makes the approximate solution inappropriate.

Figure 12. Velocity level difference between the foundation and storeys of a typical 9-storey building with bearing-walls. (—) complete model; (-----) approximate model. The frequency is 63 Hz and the thickness of the floor h_f and wall h is 200 mm.

Figure 13. Velocity level difference between the foundation and storeys of a typical 9-storey building with bearing-walls. (—) complete model; (-----) approximate model. The frequency is 125 Hz and the thickness of the floor h_f and wall h is 200 mm.

Figure 14. Velocity level difference between the foundation and storeys of a typical 9-storey building with bearing-walls. (—) complete model; (-----) approximate model. The frequency is 250 Hz and the thickness of the floor h_f and wall h is 200 mm.

5. SOUND TRANSMISSION IN BUILDINGS WITH LOAD-BEARING COLUMNS

5.1 Complete model for buildings with circular bearing columns

In principle, the solution procedure follows the procedure of Section 3. The sole difference lies in the boundary conditions.

5.1.1 Wave coupling at the foundation (junction 0)

The relations of the respective field variables at the foundation junction, e.g. junction 0, remain the same as before (Section 3.1). Solely the ground and wall impedances must be changed as the circular column replaces the bearing wall. In the present case, the first-order expression given by Boussinesq⁸ will be used

$$Z_{\infty}^F = 4G_g r_0 / j\omega(1 - \mu_g) \quad (35)$$

where G_g is the shear modulus, $G_g = E_g / 2(1 + \mu_g)$ and r_0 is the radius of the column.

5.1.2 Reflection at and transmission into the roof

The solution can be obtained from the following boundary conditions:

- (α) The displacement of the floor must be axially symmetric.
- (β) At the excitation area, the angular displacement must vanish.
- (γ) At the excitation area, the sum of shear forces must be equal to the exciting force.
- (δ) The solution for the bending waves in the roof must satisfy the so-called Sommerfeld “radiation condition” i.e., the displacement at large distances from the excitation point must have the form of a wave propagating away from the area of excitation.
- (ε) The excitation area of floor and column is plane, and the column moves the plate as a stiff body. The analysis is analogous to the procedure of bearing-walls (Section 3.1)

The notations assigned for the problem of bearing walls along with Figure 1 and Figure 9 will be used also here, but with a cylindrical coordinate system for the floors.

The velocities of the column obey the same relations for the bearing wall, Equation (10), whereas for the roof the velocity is taken as

$$u_N(r) = u_N^+ H_0^{(2)}(k_B r) + u_N^j H_0^{(2)}(-jk_B r) \quad (36)$$

where $H_0^{(2)}$ is the Hankel function of second kind and zero order and where r represents the distance from the center of the column. Analogous to the problem of bearing walls, the boundary conditions are as follows.

α) Continuity of velocity at $y = L$, $r = r_0$ leads to

$$u_N^+ H_0^{(2)}(k_B r_0) + u_N^j H_0^{(2)}(-jk_B r_0) = v_{N-I}^+ e^{-jk_L L_y} + v_{N-I}^- e^{jk_L L_y} \quad (37)$$

where r_0 is the radius of column.

β) The angular velocity of the roof,

$$\left(\frac{\partial u_N(r)}{\partial r} \right)_{r=r_0} = 0 \quad (38)$$

γ) Force equilibrium.

As mentioned in the case of load-bearing walls, the excitation by longitudinal waves in the building generates two types of forces at the junction: axial forces and shear forces. The axial forces, Figure 1, obey

$$F_n = \frac{-ES}{j\omega} \frac{\partial v_{N-1}(y)}{\partial y} \quad (39)$$

where S is the cross-sectional area of the column. The total shear force acting on the floor, F_{nf} , is expressed as

$$F_{nf} = 2\pi r_0 Q_r \quad (40)$$

Q_r being the shear force per unit length. Q_r is in the book by Cremer et al.⁶ given by

$$Q_r = B \frac{\partial \Delta \eta}{\partial r} \quad (41)$$

where η is the displacement of plate under cylindrical excitation and Δ is the Laplacian operator, which in cylindrical coordinates can be written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \phi} \quad (42)$$

The later expression is further modified with view of Equation (41) to,

$$Q_r = \frac{B}{j\omega} \frac{\partial}{\partial r} (\nabla^2 u_N^+ H_0^{(2)}(k_B r) + \nabla^2 u_N^- H_0^{(2)}(-jk_B r)) \quad (43)$$

where B is the flexural stiffness of the floor, $B = EI_f / (1 - \mu_f^2)$, where I_f is the moment of inertia of the plate.

If the latter expression is substituted into Equation (41) and use is made of the relation $\nabla^2 H_0^{(2)}(k_B r) = -k_B^2 H_0^{(2)}(k_B r)$ and $\nabla^2 H_0^{(2)}(-jk_B r) = k_B^2 H_0^{(2)}(-jk_B r)$, the general relation for the shear forces is obtained,

$$F_{nf} = 4\pi r_0 \left(\frac{Bk_B^3}{j\omega} \right) u_n^+ H_1^{(2)}(k_B r_0) \quad (44)$$

where $H_1^{(2)}$ is the Hankel function of the second kind and first order.

Combining the equations of the boundary conditions and isolating the term

$$H = \frac{j\omega r_0 E_c}{c_L Bk_B^3} \quad (45)$$

yields

$$r_N = \frac{v_{N-l}^-}{v_{N-l}^+} = \frac{1 - 4\Xi(k_B r_0)}{1 + 4\Xi(k_B r_0)} e^{-j2k_L L_v} \quad (46)$$

and

$$u_n^+ = \frac{2v_{N-l}^+ e^{-jk_L L_v}}{(H_0^{(2)}(k_B r_0) + \varphi(-jk_B r_0)H_1^{(2)}(k_B r_0)/j)(1 + 4\Xi(k_B r_0))} \quad (47)$$

where $\varphi(-jk_B r_0)$ reads

$$\varphi(-jk_B r_0) = \frac{-jK_0(k_B r_0)}{K_1(k_B r_0)} \quad (48)$$

K_0, K_1 are here the modified Bessel functions of zero and first order, respectively. The function $\Xi(k_B r)$ reads

$$\Xi(k_B r) = \frac{H_1^{(2)}(k_B r_0)}{(H_0^{(2)}(k_B r_0) + \varphi(-jk_B r_0)H_1^{(2)}(k_B r_0)/j)H} \quad (49)$$

5.1.3 Wave transmission through junction (n)

The boundary conditions here are in principle the same as those of the roof junction. The velocity expressions for the columns obey Equation (5) and Equation (19), Figure 1, whereas for the floor the velocity is taken in the same form of Equation (36). Accordingly, the solution becomes

$$u_n^+ = \frac{v_n^+ + v_n^-}{(H_0^{(2)}(kr_0) + \varphi(-jkr_0)H_1^{(2)}(kr_0)/j)} \quad (50)$$

and consequently,

$$v_n^+ = v_{n-l}^+ e^{-jk_L L_v} - [2(v_{n-l}^+ e^{-jk_L L_v} + v_{n-l}^- e^{jk_L L_v})\Xi(k_B r_0)] \quad (51)$$

$$v_n^- = v_{n-1}^- e^{jk_L L_v} + \left[2(v_{n-1}^+ e^{-jk_L L_v} + v_{n-1}^- e^{jk_L L_v}) \Xi(k_B r_0) \right] \quad (52)$$

5.1.4 Solution of the equations of motions for the entire cascade

In principle, the procedure used to derive the motion equations for the bearing walls case (Section 3.3) can be used to obtain the solution for the bearing-columns.

5.1.5 Determination of the time and space-average velocity of floor

The incident power input of the bending wave at junctions is taken as

$$P_m = \frac{I}{2} \operatorname{Re} \{ \hat{Q} \hat{v}^* \} 2\pi r_0 \quad (53)$$

where $\hat{Q}$ is shear stress (peak value) as depicted in Equation (43) and $\hat{v}$ is the complex conjugate of the velocity at the respective junction.

For the present case, the time and space-average velocity of floor is obtained as

$$\langle u^2 \rangle = \frac{P_f}{\omega m_f S \eta_f} \quad (54)$$

It can be shown that

$$\frac{\langle u^2 \rangle}{|v|^2} = \frac{4\pi r_0}{k_B S_0 \eta_0 |\epsilon(k_B r_0)|^2}, \quad (55)$$

where $\epsilon(k_B r_0)$ reads

$$\epsilon(k_B r_0) = 1 + \left(\frac{H_1^{(2)}(k_B r_0) H_0(-jk_B r_0)}{j H_0^{(2)}(k_B r_0) H_1(-jk_B r_0)} \right). \quad (56)$$

v is here the RMS value of the vertical velocity at a junction and $\langle u^2 \rangle$ is the RMS value of the bending wave velocity of the floor connected to the junction.

5.1.6 Presentation and discussion

In Figures 15-18 is shown the velocity level difference (level at a junction–level of free ground, see Equation (30)), for the four frequencies 31.5, 63, 125, and 250 Hz. Each curve is assigned to a building with the number of storeys given in the legend; the last junction is that with the roof of the building. The material properties of the buildings are presented in table 1.

It is seen that the building in this case acts mainly as a damped wave-guide, the modal character is much less pronounced than in the case of load-bearing walls. However, the reflection from the roof is always evident.

Figure 15. Calculated velocity level difference for buildings with columns at 31.5 Hz. The thickness of the floor is 200 mm and the column diameter is 400 mm, (—) 9-storey building, (— —) 8-storey building, (.....) 7-storey building, (— · —) 6-storey, (— · —) 5-storey building, (— — —) 4-storey building, (.....) 3-storey building, (——) 2-storey building.

Figure 16. Calculated velocity level difference for buildings with columns at 63 Hz. The thickness of the floor is 200 mm and the column diameter is 400 mm, (—) 9-storey building, (— —) 8-storey building, (.....) 7-storey building, (— · —) 6-storey, (— · —) 5-storey building, (— — —) 4-storey building, (.....) 3-storey building, (——) 2-storey building.

Figure 17. Calculated velocity level difference for buildings with columns at 125 Hz . The thickness of the floor is 200 mm and the column diameter is 400 mm, (—) 9-storey building, (— —) 8-storey building, (.....) 7-storey building, (— ·) 6-storey, (— ·) 5-storey building, (— — —) 4-storey building, (.....) 3-storey building, (—) 2-storey building.

Figure 18. Calculated velocity level difference for buildings with columns at 250 Hz . The thickness of the floor is 200 mm and the column diameter is 400 mm, (—) 9-storey building, (— —) 8-storey building, (.....) 7-storey building, (— ·) 6-storey, (— ·) 5-storey building, (— — —) 4-storey building, (.....) 3-storey building, (—) 2-storey building.

5.2 APPROXIMATE MODEL FOR HIGH-RISE BUILDING WITH LOAD-BEARING COLUMNS

5.2.1 Determination of the attenuation per storey

Here, the boundary conditions of junction (n), section 5.1.3, are valid for the arbitrary junction, Figure 9. Equation (31) and Equation (32) give the additional equations necessary to acquire the solution. After some manipulation, it is possible to obtain the solution

$$\cosh(g) = \cos(k_L L_y) - \frac{2 \sin(k_L L_y)}{H(j\varphi(k_B r_0) + \varphi(-jk_B r_0))} \quad (57)$$

where $\varphi(k_B r_0)$ reads

$$\varphi(k_B r_0) = \frac{H_0^{(2)}(k_B r_0)}{H_1^{(2)}(k_B r_0)}. \quad (58)$$

This expression, however, is not valid for very small excitation areas, since the shear forces in the floor plate becomes so big that it results in the appearance of shear deformation, thereby invalidating the classical theory of plate. Of course, such excitation is rarely present in practice.

The attenuation upward in the buildings can be calculated by the same formula as for the case with walls, namely Equation (34). The attenuation ΔL_v is plotted in Figure 19. It is seen that the attenuation per storey is comparatively high and that it increases fast with increasing ratio floor thickness /column diameter. In addition, it is also seen that the result obtained by Ljunggren¹ agrees very closely with the present result.

Figure 19. Calculated attenuation per storey for a building with load-bearing columns. Floor thickness/column diameter is 0.5 (—), 0.8(-----) and 1.0(.....).The line (— · —) indicates the value obtained by Ljunggren for floor thickness/column thickness equal to 0.5.

5.2.2 Comparison of the two solution

As in the case with load-bearing walls, a high-rise building, e.g. a 9-storey building is used for the comparison of the approximate solution with the complete one. The results are shown in Figures 20-23.

It can be seen that the disagreement between the two solutions is in general fairly small and diminishes as the frequency increases. However, the approximate model cannot model the influence of the reflection, which is important at low frequencies.

Figure 20. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model. The frequency is 31.5 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

Figure 21. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model. The frequency is 63 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

Figure 22. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model. The frequency is 125 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

Figure 23. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model. The frequency is 250 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

6. PARAMETRIC STUDY

The parametric study is carried out for a 9-storey building using the complete model. Comparison is made between two bearing elements: walls and columns. The results are plotted as the level difference, at 63 Hz and 250 Hz as these frequencies are thought of as representing the most important range of frequencies in case of a passing train. The level difference is defined here as the velocity level of the floor minus the level of the free ground,

$$\Delta L_v = 10 \log \left| \frac{\langle u^2 \rangle}{v_{free}^2} \right| \quad (59)$$

6.1 Structure description

The buildings under study have the basic configuration defined in table 1. The effect of geometrical parameters is studied as depicted in tables 2, 3 and 4. The values for the absorption coefficients are taken from Craik⁷.

Table 3
Geometric data of building with load bearing walls

Wall thickness h , m	Floor thickness h_f , m	Absorption coefficient at the intermediate junctions	Absorption coefficient of the roof
0.15	0.20	0.31	0.32
0.20	0.20	0.25	0.30
0.25	0.20	0.20	0.25
0.2	0.15	0.16	0.21
0.2	0.3	0.32	0.33

Table 4
Geometric data of building with load bearing columns

Column diameter D , m	Floor thickness h_f , m
0.4	0.20
0.25	0.20
0.2	0.20
0.2	0.15
0.2	0.3

6.2 Building with load-bearing walls

The results are shown in Figure 24 and Figure 25. It is observed that the velocity levels depend mainly on the frequency: increasing the frequency implies a decrease in the

velocity levels. Concerning the ratio h_f/h , it is seen that high ratios lead to increase the sound attenuation.

Irrespective of the frequency, the levels in the storeys close to foundation are the highest for $h_f=0.15$ m and $h=0.2$ m, whereas for $h_f=0.3$ m and $h=0.2$ m, the levels are the lowest for storeys close to the roof.

Figure 24. Calculated level difference at 63 Hz; (—) $h_f/h=1$; (.....) $h_f/h=1.33$; (- - -) $h_f/h=0.80$; (- · - ·) $h_f/h=0.75$; (□) $h_f/h=1.5$. The data is according to table 3.

Figure 25. Calculated level difference at 250 Hz; (—) $h_f/h=1$; (.....) $h_f/h=1.33$; (- - -) $h_f/h=0.80$; (- · - ·) $h_f/h=0.75$; (□) $h_f/h=1.5$. The data is according to table 3.

6.3 Buildings with load-bearing columns

The results are shown in Figures 26 and 27. For the chosen configurations, the loss factor and the area of the floor are taken as $\eta_0 = 0.02$ and $S_0 = 10 \text{ m}^2$, respectively. It is seen that the sound attenuation increases with increasing floor thickness.

The greater attenuation obtained by columns as compared with bearing walls is, of course, due to the fact that the ratio of floor and column impedance is much higher than the floor and wall impedance.

Figure 26. Calculated Level difference at 63 Hz between velocities of the floors supported on columns and the building foundation. (—) $h/D=1$; (.....) $h/D=0.75$; (---) $h/D=0.50$; (- - -) $h/D=0.8$; (□) $h/D=1.5$. The data is according to table 4.

Figure 27. Calculated Level difference at 250 Hz between velocities of the floors supported on columns and the building foundation. (—) $h/D=1$; (.....) $h/D=0.75$; (---) $h/D=0.50$; (- - -) $h/D=0.8$; (□) $h/D=1.5$. The data is according to table 4.

6.4 The influence of ground impedance

A simple way to describe the influence of ground impedance is to change the modulus of elasticity E_g in Equation (4). A typical value for the Swedish rock is 10^{11} Pa. Under the present conditions, it can be shown that a reduction in the ground impedance from values less than 10^{10} Pa, will decrease the velocity levels; on the other hand, an increase of E_g from 10^{11} Pa, does not influence the levels in the building.

7. COMPARISON THEORY-EXPERIMENT

Measurements have previously been carried out in an office building. The building is a 9-storey building close to a railway tunnel in central Stockholm. The floors and the load-bearing walls are made from reinforced concrete with a thickness of 200 mm and 180 mm, respectively. The velocity level difference between foundation velocity and the second floor was measured.

A comparison between predicted and measured level difference between the second floor and the building foundation is shown in Figure 28. The calculation of velocity levels was carried out using the complete and the approximate models, Equation (59).

The agreement is convincing for low frequencies. However, the discrepancy between the predicted and measured values becomes high for frequencies above 250 Hz. The reason for this is thought to be due to the influence of higher-order modes in the floors. It has been shown, see, e.g. Ljunggren¹⁰, that such modes will build up reactive fields at discontinuities, and that these reactive fields can have a substantial influence on the power flow in the case of high-impedance sources.

Figure 28. Velocity level difference between velocities of the second floor and the building foundation; (—) measured; (■) predicted (complete model); (▲) predicted (approximate model).

8. CONCLUDING REMARKS

The solution put forward in this paper is based on the hypothesis that the dominant vibrational energy transmitted from the rock foundation to the bearing elements (walls or columns) occurs via quasi-longitudinal waves due to the high impedance of the ground. The complete solution takes into account the wave reflection from the roof, while the approximate solution is suggested for high-rise buildings without consideration of any influence from the wave reflection from the roof.

From the comparison of the results obtained by means of the approximate and complete models, it is found that the agreement between the two models is in general fairly good for buildings with columns, but not for bearing walls at low frequencies. Additionally, for both bearing walls and columns, a discrepancy between the two models is always observed at the roof.

Calculated results shows that the sound level decreases comparatively slowly in the building. This is especially the case in buildings with load-bearing walls; the calculated decrease is usually more rapid for a structure with load-bearing columns.

The results also show that the sound attenuation increases with increasing floor thickness. Moreover, the calculated attenuation for a building with bearing columns shows a decrease as the diameter of the column increases.

As shown above, the complete model approximately describes a damped beam, which is clamped at one end and free at the other. For such a configuration the loss factor can be approximated as the ratio between the sum of the power injected in the lateral sides and the total mechanical energy in one cycle of vibration. It can be shown that the loss factor computed in this manner, agrees convincingly with the loss factor calculated by half-bandwidth of the spectrum of longitudinal waves.

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APPENDIX

Estimation of ground impedance Z_{∞}^F

In order to estimate the ground impedance, a rigid and massless plate is assumed to rest on and excite a semi-infinite medium, see Figure A1. The width of the plate is $2b$ and the length L_z . As comparatively low frequencies are of interest here, the width of the plate can be assumed to be much smaller than the wavelengths of the propagating waves in the medium.

The line force is approximated by a number of point forces. Each point force acts on the semi-infinite medium by means of a circular, rigid and massless indenter, the radius of which is taken b . The vertical velocity v' on the surface of the semi-infinite medium due to each point source can be written as ⁹

Figure A1. A vertical point force excites a semi-infinite plate.

$$v' = F' M_{LR}^F \frac{2b}{\pi r}, \quad (\text{A.1})$$

$$M_{LR}^F = \frac{j\omega(1 - \mu_g^2)}{2E_g b}. \quad (\text{A.2})$$

where M_{LR}^F is the point mobility and where r is the distance from the point of observation to the indenter center. The circular indenters are now assumed to be placed in a row on the surface of the medium with each indenter touching the neighboring ones. The velocity on the surface in the middle of the row, v_i , is taken as superposition of the displacements of all the point sources,

$$v_i = 2 \sum_{n=1}^N F' M_{LR}^F \frac{2b}{\pi n 2b} \approx 2 \int_{2b}^{L_z/2} F' M_{LR}^F \frac{dr}{\pi r} = \frac{F'(1 - \mu_g^2)}{b\pi E_g} \ln\left(\frac{L_z}{4b}\right), \quad (\text{A.3})$$

where N is one half of the number of indenters,

$$N = \frac{1}{2} \left(\frac{L_z}{2b} - 1 \right). \quad (\text{A.4})$$

The ground impedance per unit length is determined from the expression,

$$Z_{\infty}^F = \frac{F'}{2b v_i} . \quad (\text{A.5})$$

After rearrangement, the latter expression can finally be rewritten as

$$Z_{\infty}^F = \frac{\pi E_g}{j\omega 2 (1 - \mu_g^2) \ln(\frac{L_z}{4b})} . \quad (\text{A.6})$$

Paper II

A TENTATIVE NOISE CONTROL METHOD FOR BUILDINGS ABOVE RAILWAY TUNNELS

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ABSTRACT

In this paper, a design method to reduce the transmission of structure-borne sound in buildings to be built above railway tunnels is studied. The case studied is similar to the one studied previously by the author in the sense that the building is founded on rock. However, in the present case, the load-bearing columns on the first floor are not placed above other load-bearing columns or walls. Analytical models for the propagation of structure-borne sound upward in buildings are derived. The models are based on the assumption that the initial vibrational energy that propagates from the foundation is in form of quasi-longitudinal waves. The propagation of bending and quasi-longitudinal waves in the load-bearing columns is considered. A finite ground impedance is taken into account as well as wave reflection from the roof. A parametric survey is conducted to investigate the influence of certain important geometrical properties of the first floor on the sound transmission. Finally, the results are compared with previous results for a "normal" building and it is shown that the dislocation of the columns can considerably reduce the sound transmission upward in the building.

1. INTRODUCTION

In recent years, concern about noise and vibration in residential buildings near underground railways has led to an increased awareness of the problem by residents, local authorities and building designers.

With relatively short distance from the railway tracks and tunnels stations, the structure-borne noise is of great importance. As a matter of fact, the costs for reducing, to a sufficient degree, the train noise is often substantially higher for structure-borne sound than for air-borne sound.

There are two conventional types of measures against the structure-borne sound: track insulation with so-called ballast mats and insulation of the buildings with a kind of elastic pad, usually a foundation mat.

The insulation of an existing track can be problematic and expensive for several reasons. First, the planning situation does not place the builder of a new building in an advantageous position when negotiating with the railway authorities. Second, the cost of putting a mat under the ballast is not insignificant. Third, from the viewpoint of the railway authorities, there is a doubt about the potential effect of these mats on the track corrugation in the long run. Fourth, limited heights of some tunnels will make it practically impossible to install ballast mats. The use of vibration isolators in building, on the other hand, also leads to several practical problems and high costs connected with the necessity to avoid sound bridges. Thus, new noise control methods for buildings above railway tunnels are certainly desirable, and one possible solution is presented here.

The case studied is similar to the one studied previously¹ in that the building is founded on rock. However, in the present case, the load-bearing columns on the first floor are not placed above other load-bearing columns or walls as shown in Figure 1. This figure illustrates a modern station building in Stockholm.

From the acoustical point of view it is evident that such configuration must have a substantial influence on the transmission of the quasi-longitudinal waves upward in the building.

In present paper, analytical models for the propagation of structure-borne sound upward in buildings of the present type are presented. The models are based on the assumption that the vibrational energy propagates from the foundation in form of quasi-longitudinal waves. In the columns, not only quasi-longitudinal but also bending waves are considered. The motion of all the junctions between columns and floors are determined using the appropriate boundary conditions at the foundation, all the floor junctions and the roof. The bending wave fields of the floors are then determined using a statistical energy analysis (SEA) approach.

Figure 1. Cross-section of a part of the railway station building at Stockholm Södra.

2. COMPLETE MODEL FOR A BUILDING WITH LOAD-BEARING COLUMNS ON A RAILWAY TUNNEL

The complete model is obtained by considering the foundation and roof as global boundary for the sound propagation through the cascade system. The solution is derived in three steps. First, the foundation and wall junctions are analyzed to relate the velocity of foundation with that of the waves of the wall and the first floor. Second, the scattered field around a column is determined, and the total displacement of the floor is related to the displacements of the column. Third, the influence of the other junctions including the roof junction is determined. In Figure 2 is shown a detailed schematic view of the structure junctions for the building in Figure 1. Floor N denotes here the roof of the complete building.

2.1 Wave transmission through the foundation and wall junction

2.1.1 Foundation junction

In the same way as in the previous paper¹, the ground is substituted by an equivalent waveguide, carrying longitudinal volume waves in the vertical direction. The horizontal cross-section of the waveguide is the same as that of the building wall immediately above the ground. The free velocity v_{free} and the input force impedance of the wave-guide are the same as those of the ground. The vertical particle velocity of the equivalent waveguide is expressed as

$$v_g(y) = (v_g^+ e^{-jk_g y} + v_g^- e^{jk_g y}) e^{j\omega t} \quad (1)$$

where k_g is the wave number of the (equivalent) longitudinal wave. For the sake of simplicity, all factors containing the sinusoidal time-dependence $\exp(+j\omega t)$, where ω is the angular frequency, $\omega = 2\pi f$, are omitted in the remainder of the paper. The velocity amplitude v_g^+ is related to the free velocity of the ground in the y -direction as

$$v_{free} = 2v_g^+ \quad (2)$$

The corresponding force that acts in the waveguide is obtained as

$$F_g(y) = Z_\infty^F (v_g^+ e^{-jk_g y} - v_g^- e^{jk_g y}) \quad (3)$$

where Z_∞^F is the force impedance of a semi-infinite medium. In the present case, the approximate expression given in reference (1) will be used,

$$Z_\infty^F = \frac{\pi E_g}{j\omega 2(1 - \nu_g^2) \ln(\frac{L_z}{2h})} \quad (4)$$

where h is the wall thickness, L_z is the wall length in the z -direction and ν is the Poisson's ratio; the subscript g denotes the ground. The velocity field in the bearing wall is taken as

$$v_\theta(y) = v_\theta^+ e^{-jk_L y} + v_\theta^- e^{jk_L y} \quad (5)$$

2.1.2 Wall junction

The junction is composed of a floor and bearing-wall, Figure (2). The velocity field in the floor at the junction is taken as

$$u_f(x) = u_f^+ e^{-jk_{B1}x} + u_f^- e^{jk_{B1}x} \quad (6)$$

where k_{B1} denotes the wave number of the bending wave of the first floor.

At junction 1, Figure 2, the boundary conditions require that

$$1. \quad u_f(0) = v_\theta(L_w) \quad (7)$$

$$2. \quad \frac{\partial u_f(0)}{\partial x} = 0 \quad (8)$$

$$3. \quad F_w(L) - 2F_{f1}(0) = 0 \quad (9)$$

where F_w , F_{f1} read

$$F_w = \frac{-E_w h}{j\omega(I - \nu_w^2)} \left(\frac{\partial v_\theta}{\partial y} \right) \quad (10)$$

$$F_{f1} = \frac{B_{f1}}{j\omega} \left(\frac{\partial^3 u_f}{\partial x^3} \right) \quad (11)$$

Here, $B_{f1} = E_f I_{f1} / (I - \nu_{f1}^2)$, I_{f1} is the moment of inertia of the first floor, $I_{f1} = h_{f1}^3 / 12$, h_{f1} is the thickness of the first floor, and E the Young's modulus. The subscripts f and w refer to the floor and wall, respectively.

Note that the solution above neglects the wave reflection from the column as the influence of this reflection is thought to be very small.

The above boundary conditions as well as the continuity of velocity and forces at junction 0 gives the solution for the unknown amplitudes of the bending waves of the first floor,

$$\mathbf{P}_{0,1}(\mathbf{X}_{0,1})^{-1} = \mathbf{Q}_{0,1} \quad (12)$$

where $\mathbf{Q}_{0,1} = \begin{bmatrix} v_g^- & v_\theta^+ & v_\theta^- & u_f^+ & u_f^- \end{bmatrix}^T$, $\mathbf{P}_{0,1} = v_g^+ \begin{bmatrix} I & Z_\infty^F & 0 & 0 & 0 \end{bmatrix}^T$ and where

Figure 2. Schematic view of the building with columns

$$\mathbf{X}_{0,1} = \begin{pmatrix} -I & I & I & 0 & 0 \\ Z_\infty^F & Z_w & -Z_w & 0 & 0 \\ 0 & e^{-jk_L L_w} & e^{jk_L L_w} & -I & -I \\ 0 & 0 & 0 & -j & -I \\ 0 & \beta_1 e^{-jk_L L_w} & -\beta_1 e^{jk_L L_w} & -2j & 2 \end{pmatrix} \quad (13)$$

β is here defined as

$$\beta_1 = \frac{j\omega h E_w}{c_L B_{f1} (1 - v_w^2) k_{B1}^3} \quad (14)$$

L_w as the height of the wall as defined in Figure 2, and Z_w is the impedance of the wall, $Z_w = \rho h c_L$.

It is important to notice that the solution neglects the effect of wave reflection from the edges of the first floor. In many cases in practice, however, a reverberant field will be excited in the first floor. The effect of this can easily be incorporated in the analysis as shown in Section 5.

2.2. Wave transmission through junction (2)

2.2.1 Representation of the scattered field

A circular column in $r \leq r_0$, where r_0 is the radius of the column, is taken as perfectly bonded to the floor along the boundary $r = r_0$, Figure 3. The scattered flexural waves can be represented as²

$$U_s = \sum_{m=0}^{\infty} H_m^{(2)}(kr) (A_m \cos m\theta + A'_m \sin m\theta) + \sum_{n=0}^{\infty} K_n(kr) (B_n \cos n\theta + B'_n \sin n\theta) \quad (15)$$

where $H_m^{(2)}(.)$ is the Hankel function of the second kind and order m and $K_m(.)$ is the modified Bessel function of the second kind. The choice of wave functions is dictated by the condition that the scattered field must fulfill the Sommerfeld radiation condition.

2.2.2 Total displacement

The total displacement field in the first floor is composed of the incident waves, U_i and the scattered one, U_s ,

$$U(r, \theta) = U_i(r, \theta) + U_s(r, \theta) \quad (16)$$

The incident wave is taken in the positive x -direction as shown in Figure (3) so that $A'_m = B'_m = 0$ in Equation (15). Thus, the scattered field can be represented as

$$U_s(r, \theta) = \sum_{m=0}^{\infty} [A_m H_m^{(2)}(k_{B1} r) + B_m K_m(k_{B1} r)] \cos m\theta, \quad r \geq r_0 \quad (17)$$

The incident plane field in polar coordinates, Figure 3, can be written as

$$U_i(r, \theta) = u_i^+ e^{-jk_{B1}(r \cos \theta + \Delta L)} + u_i^- e^{-k_{B1}(r \cos \theta + \Delta L)} \quad (18)$$

Figure 3. Scattering of a plane wave by cylindrical column

By making use of the generation relations⁵, $e^{0.5z(t-t^{-1})} = \sum_{m=-\infty}^{\infty} t^m J_m(z)$ and

$e^{0.5z(t+t^{-1})} = \sum_{m=-\infty}^{\infty} t^m I_m(z)$, a Bessel-function representation of the incident field is obtained,

$$U_i(r, \theta) = \sum_{m=0}^{\infty} \left[u_i^+ e^{-jk_{B1}\Delta L} (-j)^m J_m(k_{B1}r) + u_i^- e^{-k_{B1}\Delta L} (-1)^n I_m(k_{B1}r) \right] \xi_m \cos m\theta \quad (19)$$

where $\xi_0 = 1, \xi_n = 2, n \geq 1$ and where $I(\cdot)$ is the modified Bessel function of the first kind.

2.2.3 Boundary conditions

In order to relate the total displacement field in the plate with the field variables in the column-plate intersection, the boundary conditions must be applied. The velocity field of bending and longitudinal waves in the column can be written as

$$u_z(y, \theta) = \left(u_z^+ e^{-jk_{B1}y} + u_z^- e^{-k_{B1}y} + u_z^+ e^{jk_{B1}y} + u_z^- e^{k_{B1}y} \right) \cos \theta \quad (20)$$

$$v_z(y) = \left(v_z^+ e^{-jk_{L1}y} + v_z^- e^{jk_{L1}y} \right) \quad (21)$$

where the superscript c refers to the column.

For the junction, only the $m=0$ and $m=1$ modes are of interest here as only these modes contribute to the transmission upwards in the building. For the mode $m=0$, the boundary conditions are

$$1. \quad U(r_0, 0) = v_2(0) \quad (22)$$

$$2. \quad \frac{\partial U(r_0, 0)}{\partial r} = 0 \quad (23)$$

$$3. \quad F_r(r_0, 0) - F_y(0) = 0 \quad (24)$$

where F_r is the shear force in the plate³

$$F_r = 2\pi r_0 \frac{B_f}{j\omega} \frac{\partial}{\partial r} \Delta U \quad (25)$$

and where F_y is the axial force in column,

$$F_y = \frac{-E_c S_c}{j\omega} \frac{\partial v}{\partial y} \quad (26)$$

S_c is here the cross-sectional area of the column.

The solution can be written as a matrix

$$\mathbf{X}_2^0 \mathbf{Q}_2^0 = \mathbf{P}_2^0 \quad (27)$$

where $\mathbf{Q}_2^0 = [A_0 \ B_0 \ v_2^+ \ v_2^-]^T$, $\mathbf{P}_2^0 = [-\Psi_1(u_1^+, u_1^j) \ \Psi_2(u_1^+, u_1^j) \ -2\Psi_3(u_1^+, u_1^j)]^T$ and where

$$\mathbf{X}_2^0 = \begin{pmatrix} H_0^{(2)}(k_{B1}r_0) & K_0(k_{B1}r_0) & -1 & -1 \\ -H_1^{(2)}(k_{B1}r_0) & -K_1(k_{B1}r_0) & 0 & 0 \\ 2H_1^{(2)}(k_{B1}r_0) & -2K_1(k_{B1}r_0) & -C_1 & C_1 \end{pmatrix} \quad (28)$$

where

$$C_1 = \frac{j\omega r_0 E_c}{c_L B_{f1} (k_{B1})^3} \quad (29)$$

The functions in the vector $\mathbf{P}_2^0$ read

$$\begin{aligned} \Psi_1(u_1^+, u_1^j) &= u_1^+ \exp(-jk_{B1}\Delta L) J_0(k_{B1}r_0) + u_1^j \exp(-k_{B1}\Delta L) I_0(k_{B1}r_0) \\ \Psi_2(u_1^+, u_1^j) &= u_1^+ \exp(-jk_{B1}\Delta L) J_1(k_{B1}r_0) - u_1^j \exp(-k_{B1}\Delta L) I_1(k_{B1}r_0) \\ \Psi_3(u_1^+, u_1^j) &= u_1^+ \exp(-jk_{B1}\Delta L) J_1(k_{B1}r_0) + u_1^j \exp(-k_{B1}\Delta L) I_1(k_{B1}r_0) \end{aligned} \quad (30)$$

The superscript in the vector notation denotes the mode number, m .

For the mode $m=1$, there are four boundary conditions requiring that

$$1,2. u_2(0, \theta) = U(r, \theta) = 0 \quad (31)$$

$$3. \frac{\partial U(r_0, \theta)}{\partial r} = \frac{\partial u_2(0, \theta)}{\partial y} \quad (32)$$

$$4. (2\pi r_0)M'_r(r_0, \theta) - M_c(0, \theta) = 0 \quad (33)$$

where the expressions for M'_r and M_c in cylindrical coordinates read⁴

$$M'_r(r, \theta) = -\frac{B_{f1}}{j\omega} \left[\frac{\partial^2 U}{\partial r^2} + \nu_{f1} \left(\frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \right) \right], \quad M_c(y, \theta) = \frac{-B_c}{j\omega} \frac{\partial^2 u}{\partial y^2} \quad (34), (35)$$

After applying the boundary conditions the junction matrix is obtained,

$$\mathbf{X}_2^1 \mathbf{Q}_2^1 = \mathbf{P}_2^1 \quad (36)$$

where $\mathbf{Q}_2^1 = \begin{bmatrix} A_1 & B_1 & u_2^+ & u_2^j & u_2^- & u_2^{-j} \end{bmatrix}^T$, and where

$$\mathbf{X}_2^1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ \alpha_1 \Phi_1(k_{B1}r_0) & -\alpha_1 \Lambda_1(k_{B1}r_0) & j & 1 & -j & -1 \\ H_1^{(2)}(k_{B1}r_0) & K_1(k_{B1}r_0) & 0 & 0 & 0 & 0 \\ \Theta_1(k_{B1}r_0) & \Theta_2(k_{B1}r_0) & 1 & -1 & 1 & -1 \end{pmatrix} \quad (37)$$

The matrix elements read

$$\left. \begin{aligned} \Phi_1(k_{B1}r_0) &= H_0^{(2)}(k_{B1}r_0) - H_2^{(2)}(k_{B1}r_0) \\ \Lambda_1(k_{B1}r_0) &= K_0(k_{B1}r_0) + K_2(k_{B1}r_0) \\ \Theta_1(k_{B1}r_0) &= \gamma_1 \Phi_2(k_{B1}r_0) + 2\gamma_2 \Phi_1(k_{B1}r_0) - 4\gamma_3 H_1^{(2)}(k_{B1}r_0) \\ \Theta_2(k_{B1}r_0) &= -\gamma_1 \Lambda(k_{B1}r_0) - 2\gamma_2 \Lambda_1(k_{B1}r_0) - 4\gamma_3 K_1^{(2)}(k_{B1}r_0) \end{aligned} \right\} \quad (38)$$

where

$$\begin{aligned} \Phi_2(k_{B1}r_0) &= -3H_1^{(2)}(k_{B1}r_0) + H_3^{(2)}(k_{B1}r_0) \\ \Lambda_2(k_{B1}r_0) &= -3K_1(k_{B1}r_0) - K_3(k_{B1}r_0) \end{aligned} \quad (39)$$

The parameters in the matrix, Equation (37) are defined as $\alpha_1 = k_{B1} / k_B^c$,

$\gamma_1 = (k_{B1})^2 \pi r_0 B_{f1} / B_c (k_B^c)^2$, $\gamma_2 = \nu_{f1} k_{B1} \pi B_{f1} / B_c (k_B^c)^2$, and $\gamma_3 = \nu_{f1} \pi B_{f1} / r_0 B_c (k_B^c)^2$.

The vector $\mathbf{P}_2^1$ in Equation (36) is $\mathbf{P}_2^1 = \begin{bmatrix} 0 & \Psi_4(u_1^+, u_1^j) & \Psi_5(u_1^+, u_1^j) & -\Psi_6(u_1^+, u_1^j) \end{bmatrix}^T$ and its elements read

$$\begin{aligned}
\Psi_4(u_i^+, u_i^-) &= ju_i^+ \exp(-jk_{B1}\Delta L)\Omega_1(k_{B1}r_0) + u_i^- \exp(-k_{B1}\Delta L)\Xi_1(k_{B1}r_0) \\
\Psi_5(u_i^+, u_i^-) &= ju_i^+ \exp(-jk_{B1}\Delta L)J_1(k_{B1}r_0) + u_i^- \exp(-k_{B1}\Delta L)I_1(k_{B1}r_0) \\
\Psi_6(u_i^+, u_i^-) &= -u_i^+ \exp(-jk_{B1}\Delta L)\Theta_3(k_{B1}r_0) - u_i^- \exp(-k_{B1}\Delta L)\Theta_4(k_{B1}r_0)
\end{aligned} \tag{40}$$

where

$$\left. \begin{aligned}
\Omega_1(k_{B1}r_0) &= J_0(k_{B1}r_0) - J_2(k_{B1}r_0) \\
\Xi_1(k_{B1}r_0) &= I_0(k_{B1}r_0) + I_2(k_{B1}r_0) \\
\Theta_3(k_{B1}r_0) &= -j\gamma_1\Omega_2(k_{B1}r_0) - 2j\gamma_2\Omega_1(k_{B1}r_0) + 4\gamma_3J_1(k_{B1}r_0) \\
\Theta_4(k_{B1}r_0) &= -\gamma_1\Xi_2(k_{B1}r_0) - 2\gamma_2\Xi_1(k_{B1}r_0) + 4\gamma_3I_1(k_{B1}r_0) \\
\Omega_2(k_{B1}r_0) &= -3J_1(k_{B1}r_0) + J_3(k_{B1}r_0) \\
\Xi_2(k_{B1}r_0) &= 3I_1(k_{B1}r_0) + I_3(k_{B1}r_0)
\end{aligned} \right\} \tag{41}$$

2.3. Wave transmission through junction (n)

In addition to Equation (20) and Equation (21), the other velocity fields in column, Figure 2, are taken as

$$u_n(y, \theta) = \left(u_n^+ e^{-jk_B y} + u_n^- e^{-k_B y} + u_n^+ e^{jk_B y} + u_n^- e^{k_B y} \right) \cos \theta \tag{42}$$

$$v_n(y) = \left(v_n^+ e^{-jk_L y} + v_n^- e^{jk_L y} \right) \tag{43}$$

The velocity field of the floor has two modes: the first one ($m=0$) is the plate reaction due to the translational motion by the axial forces for which the velocity is taken as

$$u_{nf}^0(r) = u_{nf}^{0+} H_0^{(2)}(k_B r) + u_{nf}^{0-} H_0^{(2)}(-jk_B r) \tag{44}$$

The second one is the reaction due to rotation ($m=1$) and the velocity can be written as

$$u_{nf}^1(r, \theta) = \left(u_{nf}^{1+} H_1^{(2)}(k_B r) + u_{nf}^{1-} H_1^{(2)}(-jk_B r) \right) \cos \theta \tag{45}$$

r represents here the distance from the center of the column.

For the $m=0$ mode, the boundary conditions require that

$$1, 2. v_{n-1}(L) = u_{nf}^0(r_0) = v_n(0) \tag{46}$$

$$3. \frac{\partial u_{nf}(r)}{\partial r} = 0 \tag{47}$$

$$4. F_{n-1}(L) - (2\pi r) Q_r(r) - F_n(0) = 0 \tag{48}$$

where the axial force, F_{n-1} is defined in Equation (26). The shear force per unit length Q_r is

$$Q_r = \frac{B}{j\omega} \frac{\partial}{\partial r} \Delta u \quad (49)$$

Accordingly the junction matrix can be obtained as

$$\mathbf{X}_n^0 \mathbf{Q}_n^0 = 0 \quad (50)$$

where $\mathbf{Q}_n^0 = [v_{n-l}^+ \ v_{n-l}^- \ u_{nf}^{0+} \ u_{nf}^{0j} v_n^+ \ v_n^-]^T$ and where

$$\mathbf{X}_n^0 = \begin{pmatrix} e^{-jk_L^c L} & e^{jk_L^c L} & -H_0^{(2)}(k_B r_0) & -H_0^{(2)}(-jk_B r_0) & 0 & 0 \\ 0 & 0 & H_0^{(2)}(k_B r_0) & H_0^{(2)}(-jk_B r_0) & -1 & -1 \\ C_n e^{-jk_L^c L} & -C_n e^{jk_L^c L} & -2H_1^{(2)}(k_B r_0) & -2jH_1^{(2)}(-jk_B r_0) & -C_n & C_n \\ 0 & 0 & -H_1^{(2)}(k_B r_0) & jH_1^{(2)}(-jk_B r_0) & 0 & 0 \end{pmatrix} \quad (51)$$

$$\text{where} \quad C_n = \frac{jk_L^c (r_0 E_c)}{B_f (k_B)^3} \quad (52)$$

For the $m=I$ mode, the boundary conditions become

$$1-3. \ u_{n-l}(L, \theta) = u_n(0, \theta) = u_{nf}^I(r_0, \theta) = 0 \quad (53)$$

$$4,5. \ \frac{\partial u_{n-l}(L, \theta)}{\partial y} = \frac{\partial u_n(0, \theta)}{\partial y} = \frac{\partial u_{nf}^I(r_0, \theta)}{\partial r} \quad (54)$$

$$6. \ M_c(L, \theta) - (2\pi r_0) M_r'(r_0, \theta) - M_c(0, \theta) = 0 \quad (55)$$

where M_c obeys Equation (35) and where M_r' is defined as

$$M_r'(r, \theta) = -\frac{B_f}{j\omega} \left[\frac{\partial^2 u}{\partial r^2} + v_f \left(\frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \right] \quad (56)$$

where u in the latter expression is given by Equation (45).

The combined boundary conditions above can be summarized in a matrix form,

$$\mathbf{X}_n^1 \mathbf{Q}_n^1 = 0 \quad (57)$$

where $\mathbf{Q}_n^1 = [u_{n-l}^+ \ u_{n-l}^j \ u_{n-l}^- \ u_{n-l}^{-j} \ u_{nf}^{I+} \ u_{nf}^{Ij} \ u_n^+ \ u_n^j \ u_n^- \ u_n^{-j}]^T$ and where

$$\mathbf{X}_n^1 = \begin{pmatrix} e^{-jk_B^c L} & e^{-k_B^c L} & e^{jk_B^c L} & e^{k_B^c L} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & H_l^{(2)}(k_B r_0) & H_l^{(2)}(-jk_B r_0) & 0 & 0 & 0 & 0 \\ -e^{-jk_B^c L} & e^{-k_B^c L} & -e^{jk_B^c L} & e^{k_B^c L} & -2\sigma\varphi(k_B r_0) & -2\sigma\varphi(-jk_B r_0) & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \kappa\Phi_l(k_B r_0) & -j\kappa\Phi_l(-jk_B r_0) & 2j & 2 & -2j & -2 \\ -2je^{-jk_B^c L} & -2e^{-k_B^c L} & 2je^{jk_B^c L} & 2je^{k_B^c L} & -\kappa\Phi_l(k_B r_0) & j\kappa\Phi_l(-jk_B r_0) & 0 & 0 & 0 & 0 \end{pmatrix} \quad (58)$$

where κ is defined as $\kappa = k_B / k_B^c$. The functions $\varphi(k_B r_0)$ and $\varphi(-jk_B r_0)$ read

$$\varphi(k_B r_0) = \left[\left(-k_B^2 + \frac{1}{r_0^2} - \frac{2\nu_f}{r_0^2} \right) H_l^{(2)}(k_B r_0) + \left(\frac{-k_B}{r_0} + \frac{\nu_f k_B}{r_0} \right) H_0^{(2)}(k_B r_0) \right] \quad (59)$$

$$\varphi(-jk_B r_0) = \left[\left(k_B^2 + \frac{1}{r_0^2} - \frac{2\nu_f}{r_0^2} \right) H_l^{(2)}(-jk_B r_0) + \left(\frac{jk_B}{r_0} - \frac{j\nu_f k_B}{r_0} \right) H_0^{(2)}(-jk_B r_0) \right] \quad (60)$$

The parameter σ in Equation (57) reads

$$\sigma = \frac{B_f \pi r_0}{B_c (k_B^c)^2} \quad (61)$$

2.4. Reflection at and transmission into the roof (junction N)

The boundary conditions for junction (n) are in principle valid for the junction at the roof; the only difference is that the wave field of the upper column must be deleted.

Accordingly, the roof junction matrix for $m=0$ can be written as

$$\mathbf{X}_N^0 \mathbf{Q}_N^0 = 0 \quad (62)$$

where $\mathbf{Q}_N^0 = [v_{N-l}^+ \ v_{N-l}^- \ u_{Nf}^{0+} \ u_{Nf}^{0j}]^T$ and where

$$\mathbf{X}_N^0 = \begin{pmatrix} e^{-jk_L^c L} & e^{jk_L^c L} & -H_0^{(2)}(k_B r_0) & -H_0^{(2)}(-jk_B r_0) \\ 0 & 0 & -H_l^{(2)}(k_B r_0) & jH_l^{(2)}(-jk_B r_0) \\ C_n e^{-jk_L^c L} & -C_n e^{jk_L^c L} & -2H_l^{(2)}(k_B r_0) & -2jH_l^{(2)}(-jk_B r_0) \end{pmatrix} \quad (63)$$

Likewise, for $m=l$, it follows that

$$\mathbf{X}_N^l \mathbf{Q}_N^l = 0 \quad (64)$$

where $\mathbf{Q}_N^1 = [u_{N-l}^+ u_{N-l}^- u_{N-l}^+ u_{N-l}^- u_{Nf}^+ u_{Nf}^-]^T$ and where

$$\mathbf{X}_N^1 = \begin{pmatrix} e^{-jk_B^c L} & e^{-k_B^c L} & e^{jk_B^c L} & e^{k_B^c L} & 0 & 0 \\ -2je^{-jk_B^c L} & -2e^{-k_B^c L} & 2je^{jk_B^c L} & 2e^{k_B^c L} & -\kappa\Phi_l(k_B r_0) & j\kappa\Phi_l(-jk_B r_0) \\ 0 & 0 & 0 & 0 & H_l^{(2)}(k_B r_0) & H_l^{(2)}(-jk_B r_0) \\ -e^{-jk_B^c L} & e^{-k_B^c L} & -e^{jk_B^c L} & e^{k_B^c L} & -2\sigma\varphi(k_B r_0) & -2\sigma\varphi(-jk_B r_0) \end{pmatrix} \quad (65)$$

In addition, the reflection coefficient of the quasi-longitudinal waves at the roof is defined as¹

$$r_N = \frac{l - 4\zeta(k_B r_0)}{l + 4\zeta(k_B r_0)} e^{-j2k_B^c L} \quad (66)$$

where

$$\zeta(k_B r_0) = \frac{H_l^{(2)}(k_B r_0)}{(H_0^{(2)}(k_B r_0) + \varphi(-jk_B r_0)H_l^{(2)}(k_B r_0)/j)C_n} \quad (67)$$

2.5 Solution of the equations of motions for the entire cascade

To determine the unknown amplitudes of the entire cascade, the matrices of the junctions are combined. The solution is then obtained as

$$\mathbf{Q}_s^m = (\mathbf{X}_s^m)^{-1} \mathbf{P}_s^m \quad (68)$$

where $\mathbf{Q}_s^m$ is the system matrix of the entire cascade and m is the mode number. For the mode $m=0$, it follows that $\mathbf{Q}_s^0 = [A_0 \ B_0 \ v_2^+ v_2^- \dots u_{Nf}^{0+} u_{Nf}^{0-}]^T$, and $\mathbf{X}_s^0$ is the matrix of the entire cascade, defined as

$$\mathbf{X}_s^0 = \begin{pmatrix} \mathbf{x}_2^0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \mathbf{x}_N^0 \end{pmatrix} \quad (69)$$

and $\mathbf{P}_s^0$ reads

$$\mathbf{P}_s^0 = [p_2^0 \ 0 \ 0 \dots 0]^T \quad (70)$$

For the mode, $m=1$, the same procedure can be applied.

2.6 Determination of power into and average velocity of the floor

For the given configuration, a case of simultaneous excitation of force and moment is occurring in which the two modes $m=0$ and $m=1$ must be considered. Accordingly, the

total power is transferred through a combination of the lateral shear Q_r and tangential moment M'_r ,

$$P_{tot} = P_l + P_2 = \frac{I}{2} \operatorname{Re} \left\{ \int_0^{2\pi} u_{nf}^0 Q_r^* r_0 d\theta + \int_0^{2\pi} w M_r'^* r_0 d\theta \right\} \quad (71)$$

where w is the angular velocity. The first integral in Equation (71) can be evaluated as

$$P_l = \frac{I}{2} \operatorname{Re} \left\{ \int_0^{2\pi} u_{nf}^0(r) Q_r^* r_0 d\theta \right\} = \frac{I}{2} \operatorname{Re} \{ u_{nf}^0 Q_r^* \} 2\pi r_0 \quad (72)$$

The latter expression can be written as

$$P_l = \frac{I}{2} \left| u_{nf}^0 H_0^{(2)}(k_B r_0) + u_{nf}^{0j} H_0^{(2)}(-jk_B r_0) \right|^2 \operatorname{Re} \{ Z_f \} \quad (73)$$

where Z_f reads

$$Z_f = \frac{4\pi B_f k_B^3 r_0}{j\omega} \left[\frac{H_0^{(2)}(k_B r_0)}{H_1^{(2)}(k_B r_0)} - \frac{K_0(k_B r_0)}{K_1(k_B r_0)} \right]^{-1} \quad (74)$$

This expression for the force impedance agrees with that previously given by Ljunggren⁶.

For the second integral in Equation (71), the angular velocity obeys Equation (54). The integral can thus be written as

$$P_2 = \frac{I}{2} \operatorname{Re} \left\{ \int_0^{2\pi} \frac{\partial u_{nf}^I(r_0, \theta)}{\partial r_0} (M_r'^*) r_0 d\theta \right\}, \quad (75)$$

where the moment M_r' can be obtained from Equation (56). By applying the same

principles as above and making use of the relation $(H_{l,2}^{(2)}(x))^* = H_{l,2}^{(1)}(x^*)$, the integral

can be written as

$$P_2 = \frac{I}{2} \operatorname{Re} \left\{ \int_0^{2\pi} \frac{-k_B r_0 \cos \theta}{2} u_{nf}^{I+} \Phi_I(k_B r_0) \left(\frac{-B_f \cos \theta}{j\omega} (u_{nf}^{I+})^* \left(-k_B^2 + \frac{I}{r_0^2} - \frac{2v_f}{r_0^2} \right) H_I^{(I)}(k_B r_0) \right) d\theta \right\} +$$

$$\frac{I}{2} \operatorname{Re} \left\{ \int_0^{2\pi} \frac{-k_B r_0 \cos \theta}{2} u_{nf}^{I+} \Phi_I(k_B r_0) \left(\frac{-B_f \cos \theta}{j\omega} (u_{nf}^{I+})^* \left(-\frac{k_B}{r_0} + \frac{v_f k_B}{r_0} \right) H_0^{(I)}(k_B r_0) \right) d\theta \right\} \quad (76)$$

The integration yields

$$P_2 = \frac{-k_B \pi B_f r_0}{4} \operatorname{Re} \left\{ \frac{I}{j\omega} u_{nf}^{I+} \Phi_I(k_B r_0) (u_{nf}^{I+})^* \left(\left(-k_B^2 + \frac{I}{r_0^2} - \frac{2v_f}{r_0^2} \right) H_I^{(I)}(k_B r_0) + \left(-\frac{k_B}{r_0} + \frac{v_f k_B}{r_0} \right) H_0^{(I)}(k_B r_0) \right) \right\} \quad (77)$$

For the present case, the time and space-average velocity of floor is obtained as³

$$\langle u^2 \rangle = \frac{P_{tot}}{\omega m_f S_0 \eta_0} \quad (78)$$

The velocity level difference is defined here as space-average velocity of floor minus the RMS value of velocity of the free ground,

$$\Delta L_v = 10 \log \left| \frac{\langle u^2 \rangle}{\tilde{v}_{free}^2} \right| \quad (79)$$

where the velocity of free ground is given by Equation (1)

2.7 Presentation of results

The velocity level difference for the four frequencies 31.5, 63, 125, and 250 Hz is shown Figures 4 to 7. Each curve is assigned to a building with the number of storeys given in the legend; the last floor is the roof of the building. The material and geometrical properties of the buildings are presented in tables 1 and 2. For the chosen configurations, the loss factor and the area of the floor for each column are taken as $\eta_0 = 0.02$ and

$S_0 = 10 \text{ m}^2$. It should be noted that this reference area is fairly small compared to those occurring in practice; the value of S_0 is chosen merely for convenience.

As can be seen from figures 4-7, the building acts in this case mainly as a damped waveguide; the modal character is little pronounced.

In addition, the effect of sound reflection from the roof is always evident. This behavior of sound attenuation, for this given case, resembles to a great extent the corresponding behavior for the typical case, that is, a building with all the columns on top of each other.

Table 1
Material properties of a typical multistory building

Property	Value
Young's modulus for the concrete	26×10^9 Pa
Young's modulus for the steel	21×10^{10} Pa
Poisson's ratio	0.3
Concrete internal loss factor	0.005
Concrete density	2400 kg/m^3
Steel Density	7800 kg/m^3
Phase velocity of quasi-longitudinal waves in concrete walls	3450 m/sec
Phase velocity of quasi-longitudinal waves in concrete columns	3290 m/sec
Phase velocity of quasi-longitudinal waves in steel columns	5100 m/sec

Table 2
Dimensions of load-bearing membranes

Heights of the load-bearing wall, L_{wv} , and the columns, L	2.9 m
Horizontal distance between wall and row of columns, ΔL	1 m
Thickness of the first floor, h_{f1}	0.4 m
Diameter of the columns, $2r_0$	0.4 m
Thickness of the floors, h_f	0.2 m

Figure 4. Calculated velocity level difference for buildings with columns at 31.5 Hz, (—) 9-storey building, (— —) 8-storey building, (·····) 7-storey building, (- · -) 6-storey, (— —) 5-storey building, (— —) 4-storey building, (·····) 3-storey building.

Figure 5. Calculated velocity level difference for buildings with columns at 63 Hz, (—) 9-storey building, (— —) 8-storey building, (·····) 7-storey building, (- · -) 6-storey, (— —) 5-storey building, (— —) 4-storey building, (·····) 3-storey building.

Figure 6. Calculated velocity level difference for buildings with columns at 125 Hz, (—) 9-storey building, (- -) 8-storey building, (.....) 7-storey building, (- · -) 6-storey, (—) 5-storey building; (—) 4-storey building, (.....) 3-storey building.

Figure 7. Calculated velocity level difference for buildings with columns at 250 Hz, (—) 9-storey building, (- -) 8-storey building, (.....) 7-storey building, (- · -) 6-storey, (—) 5-storey building; (—) 4-storey building, (.....) 3-storey building.

3. APPROXIMATE MODELS

Approximate models can be obtained by neglecting the generation of bending waves in the columns. As can be shown, this simplification is valid in many practical cases. Moreover, in order to simplify the solution, the near field in the first floor is neglected too. The system will then be treated in the same way as treated before in reference (1) i.e., junction 2 becomes the foundation for the cascade above it.

3.1 Boundary conditions at junction 2

Two boundary conditions are needed in this case:

$$1. v_2(0) = u_1(\Delta L) = u_1^+ e^{-jk_B \Delta L} \quad (80)$$

$$2. Z_{f1} u_1 = Z_c v_2 \quad (81)$$

where Z_{f1} is input impedance of a point-excited plate, which obeys⁶

$$Z_{f1} = \frac{4\pi B_{f1} k_{B1}^3 r_0}{j\omega} \left[\frac{H_0^{(2)}(k_{B1} r_0)}{H_1^{(2)}(k_{B1} r_0)} - \frac{K_0(k_{B1} r_0)}{K_1(k_{B1} r_0)} \right]^{-1} \quad (82)$$

and Z_c is the impedance of the infinite column,

$$Z_c = \rho \pi r_0^2 c_L \quad (83)$$

3.2 Approximate model considering propagation constant

By using the procedure outlined in reference (1), a propagation constant g can be derived. Consequently, the level of junction 2 can be used together with g to determine the levels of all the floors of the building.

3.3 Approximate model considering the roof effect

The vertical velocities in the column at junction 2 is determined using the above boundary conditions, Equations (80) and (81) while the remaining boundary conditions at junctions n to N remain the same as before.

3.4 Comparison of the solutions

In order to compare results from the approximate models with those from the complete model, a high-rise building composed of 9-storeys is studied. The results are shown in Figures 8-11. The material and geometrical properties of the building appears in Tables 1 and 2. As can be seen, the agreement between the two types of models is good; however, the approximate model considering the propagation constant cannot model the influence of the reflection. Moreover, the approximate models neglect the influence of bending waves in the column on the total power injected onto floors.

Figure 8. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model considering the propagation constant; (-+ -+ -+) approximate model considering the roof effect. The frequency is 31.5 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

Figure 9. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model considering the propagation constant; (-+ -+ -+) approximate model considering the roof effect. The frequency is 63 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

Figure 10. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model considering the propagation constant; (-+ -+ -+) approximate model considering the roof effect. The frequency is 125 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

Figure 11. Velocity level difference between the junction and the foundation of a typical 9-storey building with columns. (—) complete model; (-----) approximate model considering the propagation constant; (-+ -+ -+) approximate model considering the roof effect. The frequency is 250 Hz. The thickness of the floor and the diameter of column are 200 mm and 400 mm, respectively.

4. PARAMETRIC STUDY

A limited parametric study has been carried out for a 9-storey building with the help of the complete model. The building is similar to that of Figure 1 and is composed of steel columns (ϕ 200× 15 mm) in the inner row and concrete slabs. The thickness of the tunnel wall is 0.35 m and of the first floor 0.4 m. The thickness of the remaining floors is 0.2 m. Herein, the normalized loss factor, $\eta_0 = 0.02$, and the area of the floor $S_0 = 10 \text{ m}^2$ are considered. The objective is to study the influence of the thickness for the first floor, h_{f1} , and the horizontal distance between the tunnel wall and the columns, ΔL . The material properties are given in table 1. Again, the results are presented as velocity level difference, as defined in Equation (79).

4.1 Influence of the distance ΔL

The results are shown in Figures 12 to 15. As can be seen, the sound attenuation increases with increasing the distance. The difference between the different cases is seen constant and it is about 1 dB/m under the low frequency range. Moreover, at high frequencies, the decrease in the velocity level diminishes slowly.

Figure 12. Calculated Level difference at 31.5 Hz between velocities of the floors supported on columns and the building foundation. (—) $\Delta L=5$; (— —) $\Delta L=3$; (.....) $\Delta L=2$; (— - -) $\Delta L=1$.

Figure 13. Calculated Level difference at 63 Hz between velocities of the floors supported on columns and the building foundation. (—) $\Delta L=5$; (— —) $\Delta L=3$; (.....) $\Delta L=2$; (— · —) $\Delta L=1$.

Figure 14. Calculated Level difference at 125 Hz between velocities of the floors supported on columns and the building foundation. (—) $\Delta L=5$; (— —) $\Delta L=3$; (.....) $\Delta L=2$; (— · —) $\Delta L=1$.

Figure 15. Calculated Level difference at 250 Hz between velocities of the floors supported on columns and the building foundation. (——) $\Delta L=5$; (— —) $\Delta L=3$; (.....) $\Delta L=2$; (— - -) $\Delta L=1$.

4.2 Influence of the floor thickness h_f

The influence of the floor thickness is investigated for three thicknesses: 0.4 m, 0.5 m and 0.6 m, and the distance $\Delta L=3$ m. The results are shown in Figures 16 to 19.

As can be seen, the increase of the thickness has little effect on the velocity level difference in the low frequency range, whereas it has considerable effect on higher frequencies. This results indicates that a resonance is occurring around 250 Hz leading to increase in transmission when the thickness becomes smaller.

Figure 16. Calculated Level difference at 31.5 Hz between velocities of the floors supported on columns and the building foundation. (—) $h_{f1}=0.4$ m; (---) $h_{f1}=0.5$ m; (.....) $h_{f1}=0.6$ m.

Figure 17. Calculated Level difference at 63 Hz between velocities of the floors supported on columns and the building foundation. (—) $h_{f1}=0.4$ m; (---) $h_{f1}=0.5$ m; (.....) $h_{f1}=0.6$ m.

Figure 18. Calculated Level difference at 125 Hz between velocities of the floors supported on columns and the building foundation. (——) $h_{f1}=0.4 \text{ m}$; (— —) $h_{f1}=0.5 \text{ m}$; (.....) $h_{f1}=0.6 \text{ m}$.

Figure 19. Calculated Level difference at 250 Hz between velocities of the floors supported on columns and the building foundation. (——) $h_{f1}=0.4 \text{ m}$; (— —) $h_{f1}=0.5 \text{ m}$; (.....) $h_{f1}=0.6 \text{ m}$.

5. COMPARISON THEORY-EXPERIMENT

Some measurements of the noise situation due to passing trains have previously been performed at an office building close to the railway station illustrated in Figure 1. The foundation is of homogenous concrete, cast in situ. The thickness of the train tunnel wall is 0.45m and of the slab above the tunnel 0.50 m. The superstructure consists of pre-cast slabs on load-bearing steel columns, which are circular ($\varnothing$ 508×12.5 mm) in the inner row and the box-shaped (VKR 400×200 ×12.5 mm) in the outer row. The columns of the outer row are not connected to the train tunnel. The center to center distance of the column is 9.0 m (in each row) and the distance; ΔL is 1.5 m. The slabs consist of hollow concrete elements (HDF 120/27) with no topping. The surface weight of the slabs is 370 kg/m². There are no false ceilings. The outer wall and the wall facing the light-court are made in a lightweight construction as well as all interior partitions. In order to compare the theory with the measurement data, a correction has to be made on the predicted data due to the reverberant field generated in the first slab. That the field is reverberant is a fairly safe assumption since there are 15 eigenfrequencies in the lowest frequency band of interest, i.e. the octave band around 31.5 Hz. In addition, the field of bending vibrations on the slab 2 is certainly reverberant since this slab is slender compared with the first slab and of about the same size. The loss factor of the first floor has been estimated to (mainly coupling losses):

Frequency, Hz	31.5	63	125	250
η_0	9.8%	7.8%	5.8%	4.2%

A comparison between predicted and measured level differences between the second floor and the building foundation is shown in Figure 20. The calculation of velocity levels was carried out using the complete model, the approximate model considering the roof effect and the approximate model considering propagation factor, Equation (79). In addition, the results are compared with the typical building defined above with the help of the complete model in the previous paper¹; the thickness of all floors is 0.2 m.

The agreement is convincing for low frequencies. However, the discrepancy between the predicted and measured values becomes high for at 125 Hz frequency. The reason for this could either be measurement error or some other circumstance that has not been considered in the calculations. In particular, it is thought that the modeling of the junction between the columns and the upper floors is too crude as the existence of, e.g., the load-bearing beams between the columns has been disregarded. On the other hand, the agreement for the average values over four frequencies is good: the measured noise reduction is 9 dB and the calculated one 7 dB.

Figure 20. Level difference between transverse velocities of the second floor and the building foundation; (—) measured; (— —) predicted (complete model); (—●—) predicted (approx. model considering the roof effect); (—▲—) predicted (approx. model considering the propagation constant); (.....) predicted (typical building).

6. CONCLUSION

A design method to reduce the transmission of structure-borne sound in buildings with columns above railway tunnels has been investigated. The case studied is similar to the one studied previously by the author in that the building is founded on rock. However, in the present case, the load-bearing columns on the first floor are not placed above other load-bearing columns or wall. Analytical models for transmission of sound upward in buildings of the present type has been derived. The models are based on the assumption that the vibrational energy propagates from the foundation in form of quasi-longitudinal waves. In the columns, not only quasi-longitudinal but also bending waves are taken into account.

A complete model is presented including wave scattering in the first floor and bending waves in columns. In addition, approximate models based on the solution procedure used in a previous paper are presented. The approximate models neglect the potential effect from the bending waves in columns.

From the comparison of the results obtained by means of the approximate and complete models, it is found that the discrepancy between two types of models is generally small.

A parametric study has been carried out on a building with steel columns and it is found that the horizontal distance that separate the columns from the tunnel wall in the first floor has some influence on the noise level upwards in the building. In addition, the thickness of the first floor is found to have considerable effect on the sound attenuation at higher frequencies.

Predicted results have been compared with measured ones. The agreement has been found reasonable. Thus, it can be concluded that the idea of placing the columns aside the load-bearing wall has a potential of reducing the noise level by approximately 5-10 dB.

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