The Klein-Gordon Equation and Pionic Atoms

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Abstract

This bachelor thesis introduces the Klein-Gordon equation and pionic atoms through a historical review. It discusses properties of the equation and its continuity equation based on a comparison with the Schrödinger equation and its continuity equation. The conclusion that the Klein-Gordon equation is a relativistic extension of the Schrödinger equation for spin-0 particles is drawn. This makes it possible to derive a Klein-Gordon equation and Coulomb potential based model for pionic atoms. The transition energies and the charge density for a pionic atom are derived. The model and a Schrödinger equation model are used to draw conclusions on the differences between regular and pionic atoms. Numerical predictions of the models are compared to experimental data and the accuracy of the models is discussed. Properties of the pionic atom are discussed based on the radial charge density.

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Chapter 1

Introduction

The Schrödinger equation, SE, is a powerful and useful equation for describing particles, but it does not obey the rules of special relativity. At relativistic energy the physics behave differently and therefore we need a wave equation for describing relativistic particles. An equation obeying the laws of special relativity is the Klein-Gordon equation, KGE, which describes spin-0 particles with relativistic energy. Such a particle is the pion, the pion. A pion is a short lived subatomic particle that can take the place of an electron in an atom creating a pionic atom [1].

The aim of this report can be specified as follows. Firstly, we intend to introduce the subject of the KGE and pionic atoms. Based on knowledge about the SE, our purpose is to examine the relationship between this non-relativistic equation and the relativistic KGE. Secondly, we plan to investigate the relationship between pionic and regular atoms by using the KGE and the SE to numerically compare theoretical binding energies. Thirdly, we aim to investigate the agreement between the numerical predictions produced by the equations and experimental results for pionic atoms. This will be performed by studying transition energies. Finally, we intend to investigate the radial charge densities of pionic atoms, which are introduced in Section 3.2.

This report is divided into four chapters. The first chapter deals with the KGE from a historical point of view and some background theory of pionic atoms. In the second chapter the properties of the KGE are presented in relation to the SE. In the third chapter energies and charge densities are derived analytically from the KGE. Based on this and the Schrödinger theory the numerical calculations are done. The results are presented and discussed. The fourth chapter contains the summary and conclusions.

1.1 Klein-Gordon Equation in Historical Context

The derivation of the relativistic KGE is based on progress made in physics at the beginning of the 20th century. In 1905 two important papers were published. In the first [2] Einstein presented what is now well known as the theory of special relativity under the two assumptions of a constant light speed, c, and of the so-called principle of relativity. This principle demands the laws of physics to be equivalent in all inertial frames of reference. At the end of the paper he derived a relation between the rest mass of an electron and the kinetic energy of that electron in a moving system. The system moved at constant speed with respect to the electron’s rest system. In the second paper [3] he used this relation to show the equivalence of mass and energy. Based on this theory
a relativistic expression for the total energy of a system can be derived in four-vector notation as

\[ p^\mu p_\mu = m^2 c^2, \quad (1.1) \]

where \( m \) is the rest mass of the particle in question, \( p^\mu = [E, p_x, p_y, p_z] \), \( p_i \) is the momentum in the ith direction and \( E \) is the energy of the particle. This was later used, both in the 1926 paper of Gordon [4] and in the 1927 paper of Klein [5], to derive the equation bearing their names from the Schrödinger formulation of quantum mechanics.

In 1926 Schrödinger produced a series of papers in which he took a new approach to quantum mechanics, describing a quantum mechanical system through use of a wave function \( \psi \) inspired, as he states, by the work of de Broglie proposing progressing phase waves for matter. In the first paper [6] Schrödinger uses the approach of setting the energy \( E \) of an electron in a hydrogen Coulomb potential equal to the Hamiltonian of the system, written in terms of \( \psi \) using the momentum operator. Using spherical coordinates and separating \( \psi \) in radial and angular parts gives Schrödinger the negative energy solution set of the equation as discrete and given by \( E = -\frac{m^2 e^4}{2\hbar^2 n^2} \), where \( \hbar \) is the reduced Planck constant, \( e \) is the elementary charge, \( m \) is the electron mass and \( n = 1, 2, 3, \ldots \). Adjusting the theory to apply to a hydrogen-like atom we simply obtain the energies

\[ E = -\frac{mZ^2 e^4}{2\hbar^2 n^2}, \quad (1.2) \]

where \( Z \) from now on denotes the atomic number of the atom. Nowadays we find the time dependent form of the SE for free particles not taking spin into account, using the energy operator, in basic quantum mechanics textbooks, for example [7], as

\[ \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m}\right) \psi(\mathbf{r}, t) = 0. \quad (1.3) \]

In the fourth paper [8] Schrödinger offers the interpretation of \( \psi^* \) as “eine Art Gewichtsfunktion im Konfigurationenraum des Systems”, translating to a sort of weight function in the configuration space of the system. Furthermore, he states that the system could be viewed as being in every kinematically possible configuration at the same time, the function \( \psi^* \) deciding how “strongly” each configuration contributes to the overall state. In the same paper he tries to make a relativistic generalization of another form of his equation for an electron, which takes general electromagnetic interactions into account, but considers his theory to be incomplete.

As already stated, the derivation of a relativistic generalization of the Schrödinger theory was performed by Klein and Gordon, but there were others working on the same problem, including V. Fock, J. Kudar, T. de Donder, and H. van Dungen as stated in Ref. [1], the works of whom will not be commented on further in this report.


1.2 Pions and Pionic Atoms

Pions are created naturally on Earth when high energy particles, cosmic rays, hit the matter in the atmosphere. Creation of pions can also be achieved in particle accelerators by collision of high energy hadrons, such as protons. A pion can be either positively, $\pi^+$, negatively, $\pi^-$ or neutrally charged, $\pi^0$. The $\pi^+$ and the $\pi^-$ are a particle-antiparticle pair [9].

When the negative pion moves in a stopping material, it is finally captured by an atom and held there by an electromagnetic potential. The result is a pionic atom. The pion is, based on experiments, likely to be captured in a highly excited state. From there it descends to lower energy levels emitting high energy (Auger) electrons and X-rays. The X-ray energies can be measured with gamma detectors of high precision and by doing so, the difference between the energy levels can be studied [10]. The mass of a pion is very large in relation to the electron mass and this leads to a lowering of the Bohr energy levels. This is an advantage when studying properties of the nucleus. The large mass of the pion also leads to the atom becoming a hydrogen-like system, which is computationally beneficial compared to a many-body problem [11].

It is of interest to study pionic atoms because we can learn more about the interactions in the atom, the nucleus and the pion itself. It is for example by studying pionic atoms that the pion mass has been determined. With more knowledge about interactions, pions and their properties, we can make use of this in other areas dealing with nuclear systems, such as medicine and technology [10].

The first real evidence for the existence of pionic atoms was received in 1952 at Rochester by Camac, McGuire, Platt and Schulte [12]. This was made possible through the development of synchrocyclotrons, particle accelerators that produce pions. After the existence of pionic atoms had been discovered, the advancements in the field of gamma-ray detectors led to more research on pionic atoms [11]. Data of transition energies in many different pionic atoms were collected at Berkeley [13, 14, 15, 16], Cern [17, 18, 19, 11] and Virginia [20, 21, 22, 23, 24] from 1965 and onwards. These transition energies are summarized by Backenstoss [11] and used in Chapter 3 to evaluate the accuracy of the KGE and the SE predictions.

To investigate pionic atoms we need a model for the system taking the forces acting on the pion into consideration. We need to simplify the model to make it analytically solvable. Assuming that the dominating interaction is the Coulomb potential, we can approximate the nucleus by a point charge of magnitude $Ze$. The pion can be approximated by a point charge of magnitude $-e$. The same approach was used by Schrödinger for a regular hydrogen atom and could be adjusted for a hydrogen-like regular atom into Eq. (1.2). Exchanging the electron mass for the pion mass gives us an SE model for the pionic atom. A KGE model based on the same potential will be derived in Section 3.1.

However, there are many other interactions besides the Coulomb potential. These other interactions lead to shifts in the Coulomb energy levels. From the data referenced above small shifts in relation to the Coulomb energy could be studied. The interactions contributing the most to the shifts have been described and explained by Backenstoss [11]. In orbits close to the nucleus, the most contributing interactions are the vacuum polarization, the finite-size effect and the strong interaction. Vacuum polarization is a correction originating from quantum-electrodynamics and it increases with lower energy levels and increases the magnitude of the binding energy. The finite-size effect comes from...
the approximation of a point nucleus. It lowers the magnitude of the binding energies and only appears in the low levels. For larger nuclei the low levels are allowed to be of higher energies than for smaller nuclei. The strong interaction is only apparent in the low energy levels as well. The strong interaction between the pion and the nucleus has been a great area of research and is an area of study in Backenstoss’ review. The strong interaction is varying. It is affected by, and affects, the other interactions. Thus, all interactions have to be described at the same time to be able to describe the strong interaction. In levels far from the nucleus the most contributing interactions affecting the energy levels, apart from the Coulomb potential, are the vacuum polarization and electron screening. Electron screening is not that apparent in the lower levels. Due to the large mass of the pion, the orbit which it is in will be the orbit closest to the nucleus with the electron orbits outside. Electron screening is only apparent for \( n > \sqrt{\frac{m_e}{m_\pi}} \) translating to \( n > 16 \), with \( m_\pi \) being the mass of the pion and \( m_e \) being the mass of the electron. This condition comes from examining the binding energies.

A common approach to describe the strong interaction between pion and nucleus is to use the combination of a real and an imaginary potential. This combination is called an optical potential. The imaginary part is to describe \( \pi^- \) absorption that can occur by the nucleus. A fundamental potential for the real part that has been used is the Kisslinger potential by Leonard S. Kisslinger. Added to this was the important Ericson-Ericson correction [25]. According to Hüfner, [10] a landmark paper in describing pionic atoms was the paper [26], developed by Ericsson and Ericsson in 1966. With these works as a base, many theoretical developments of potentials have been made. The subject of optical potentials is worth to mention, but we will not include such substantial theory in this report. Instead we focus on the basics of pionic atoms and leave this area to the interested reader. From our literature search for this report, we can conclude that a lot of theoretical work has been done in order to analyze data from experiments, with the goal of describing the different phenomena and interactions in pionic atoms. Examples of such studies are given in In-medium nuclear interactions of low-energy hadrons by E. Friedman and A. Gal, [27] from 2007 or Strong Interaction Physics from Hadronic Atoms by C.J. Batty, E. Friedman and A. Gal, [28] from 1997.
Chapter 2

Background

In this chapter we derive the KGE and study it in the non-relativistic limit, motivating its use as a relativistic extension of the SE. We continue by comparing the free particle solutions and continuity equations of the KGE and the SE, leading us to the definition of the charge density.

2.1 Derivation of the Klein-Gordon Equation

The relativistic expression for energy (1.1) is equivalent to

\[ E^2 = p^2 c^2 + m^2 c^4 \]  

(2.1)

in three-vector notation where \( p = [p_x, p_y, p_z] \) since \( p_\mu = g_{\mu\nu} p^\nu \) where

\[
g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
\]

Proceeding along the line of Gordon [4], we can derive the Klein-Gordon equation from Eq. (1.1) using the Schrödinger approach of replacing the momentum with the momentum operator \( \hat{p}_\mu = i\hbar \left[ \frac{\partial}{\partial (ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \) and applying this to a wave function \( \psi \). This gives us the Klein-Gordon equation as

\[
\hat{p}_\mu \hat{p}_\mu \psi = m^2 c^2 \psi.
\]

(2.2)

Defining the so-called d’Alembertian as \( \Box \psi = \left( \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \psi \) and performing the scalar multiplication in equation (2.2) produces

\[
\left( \Box + \frac{m^2 c^2}{\hbar^2} \right) \psi = 0.
\]

(2.3)

The same result is yielded if we use the common operator replacements for both energy and momentum used when deriving the time-dependent Schrödinger equation, \( \hat{E} = i\hbar \frac{\partial}{\partial t} \) and \( \hat{\mathbf{p}} = -i\hbar \mathbf{\nabla} \), in Eq. (2.1).
2.1.1 Non-relativistic limit

Let us consider the nonrelativistic limit of our derived relativistic equation. Following Greiner [25] in this limit, we have \( E \approx mc^2 \). We take \( \psi(\mathbf{r}, t) = \varphi(\mathbf{r}, t)e^{-\frac{i\hbar mc^2 t}{\hbar}} \) as an Ansatz, where we have separated the time-dependence into two parts. Here \( \varphi(\mathbf{r}, t) \) represents the part that is not relativistic, hence we would like to obtain the equation for \( \varphi \). When taking the non-relativistic limit, we rewrite the KGE (2.3) as

\[
\frac{1}{c^2} \frac{\partial^2 \psi(\mathbf{r}, t)}{\partial t^2} = \left( \nabla^2 - \frac{mc^2}{\hbar^2} \right) \psi(\mathbf{r}, t). \tag{2.4}
\]

To insert our Ansatz we need the second time derivative of \( \psi \), which simplifies with the fact that \( |i\hbar \frac{\partial \varphi}{\partial t}| \ll mc^2 \varphi \). This can be explained by the fact that the rest mass is much larger than all other expressions for energy in the non-relativistic case. For this reason we discard all terms not containing the rest mass outside of the exponential in the approximation of the second time-derivative:

\[
\frac{\partial \psi}{\partial t} = \left( \frac{\partial \varphi}{\partial t} - \frac{i mc^2}{\hbar} \varphi \right) e^{-\frac{i\hbar mc^2 t}{\hbar}} \approx -\frac{i}{\hbar} mc^2 \varphi e^{-\frac{i\hbar mc^2 t}{\hbar}}, \tag{2.5}
\]

\[
\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial t} - \frac{i mc^2}{\hbar} \varphi \right) e^{-\frac{i\hbar mc^2 t}{\hbar}} = \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{imc^2}{\hbar} \frac{\partial \varphi}{\partial t} - \frac{imc^2}{\hbar} \frac{\partial \varphi}{\partial t} - \frac{m^2c^4}{\hbar^2} \varphi \right) e^{-\frac{i\hbar mc^2 t}{\hbar}} \approx -\left( \frac{2imc^2 \varphi}{\hbar} + \frac{m^2c^4}{\hbar^2} \varphi \right) e^{-\frac{i\hbar mc^2 t}{\hbar}}. \tag{2.6}
\]

Inserting the second time-derivative in Eq. (2.4) gives

\[
-\frac{1}{c^2} \left( \frac{i2mc^2}{\hbar} \frac{d \varphi}{dt} + \frac{m^2c^4}{\hbar^2} \varphi \right) e^{-\frac{i\hbar mc^2 t}{\hbar}} = \left( \nabla^2 - \frac{m^2c^2}{\hbar^2} \right) \varphi e^{-\frac{i\hbar mc^2 t}{\hbar}}, \tag{2.7}
\]

which reduces to

\[
-\frac{2m}{\hbar} \frac{\partial \varphi}{\partial t} = \nabla^2 \varphi. \tag{2.8}
\]

Multiplying both sides with \(-\frac{\hbar^2}{2m}\) gives

\[
i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \varphi. \tag{2.9}
\]

This is our well-known SE. Through the correspondence principle we conclude that the KGE is a relativistic extension of the SE.
2.2 Comparison of the Klein-Gordon Equation and the Schrödinger Equation

Having derived the KGE and studied its non-relativistic limit, we now continue by studying it in more detail and compare it to the familiar SE.

2.2.1 The solutions

Following Greiner [25], an Ansatz for the wave function solutions for free particles is given by

\[ \psi(r, t) = Ae^{i \mathbf{p} \cdot \mathbf{r} - Et} \]  

(2.10)

with \( A \) as a normalization constant. The KGE (2.2) can be written as

\[ (\hat{p}^\mu \hat{p}_\mu - m^2 c^2)\psi = 0. \]  

(2.11)

From this and with our Ansatz for the free particle solutions (2.10) we get that

\[ (\hat{p}^\mu \hat{p}_\mu - m^2 c^2) = E^2 c^2 - \mathbf{p}^2 - m^2 c^2 = 0 \]  

(2.12)

has to be fulfilled. This condition leads to the energies

\[ E = \pm \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \]  

(2.13)

and the solutions

\[ \psi_\pm(r, t) = A_\pm e^{i \mathbf{p} \cdot \mathbf{r} \mp |E| t}. \]  

(2.14)

Thus, the KGE gives us two different solutions for every momentum, one with positive energy and one with negative energy. The energy of a free particle is in the SE case given by the well-known expression \( E = \frac{\mathbf{p}^2}{2m} + V(r, t) \), where the first term correspond to the kinetic energy of the particle and the second term comes from the potential energy.

From this we conclude that there is a difference in what the free particle solutions to the equations describe. The SE solution describes just a free particle. For the KGE, the two different solutions are defined to describe two different particles - the particle and the antiparticle. The particle with negative energy is defined to be the antiparticle.
2.2.2 The continuity equation

A continuity equation represents conservation of a quantity through a density within a surface area depending on a current through that area. We start by studying this for both our particle-describing equations.

The Schrödinger continuity equation

To obtain the SE continuity equation given by Schwabl [29] we multiply the SE (1.3) with the complex conjugate $\psi^*$ from the left-hand side, which gives

$$\psi^*(\mathbf{r}, t) \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} \right) \psi(\mathbf{r}, t) = 0 \quad (2.15)$$

and then subtracting the complex conjugate of this expression to this expression

$$\psi^*(\mathbf{r}, t) \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} \right) \psi(\mathbf{r}, t) - \psi(\mathbf{r}, t) \left( -i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} \right) \psi^*(\mathbf{r}, t) = 0 \quad (2.16)$$

gives us the continuity equation after division by $i\hbar$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (2.17)$$

where

$$\rho = |\psi(\mathbf{r}, t)|^2 = \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t), \quad \mathbf{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi(\mathbf{r}, t) - \psi(\mathbf{r}, t) \nabla \psi^*(\mathbf{r}, t)). \quad (2.18)$$

This continuity equation is equivalent to conservation of probability. The flow out from a volume is in proportion to the change of probability in this volume. Here $\rho$ is the probability density and $\mathbf{j}$ is the probability current.

The Klein-Gordon continuity equation

To obtain the Klein-Gordon continuity equation we proceed in the same way, following Schwabl [30]. Starting by multiplying the relativistic equation (2.2) from the left with the complex conjugate $\psi^*$ we obtain

$$\psi^* \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 - \frac{m^2 c^2}{\hbar^2} \right) \psi = 0. \quad (2.19)$$

By subtracting the complex conjugate of Eq. (2.19) from Eq. (2.19) we obtain

$$\frac{\partial}{\partial t} \frac{1}{c^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) + \nabla (\psi^* \nabla \psi - \psi \nabla \psi^*) = 0. \quad (2.20)$$

Multiplying Eq. (2.20) by $\frac{\hbar}{2mi}$ gives us the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (2.21)$$

with

$$\rho = \frac{i\hbar}{2mc^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right), \quad \mathbf{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*). \quad (2.22)$$

Thus in the relativistic case we get a continuity equation where the probability current $\mathbf{j}$ is the same as for the non-relativistic case, but there is a difference in the density expression, $\rho$. We now need to interpret this result.
Interpretation of the density for the Klein-Gordon continuity equation

To interpret the density we start our analysis following Landau [31]. Using the Ansatz (2.10) for the wave function of a free particle, with the normalization constant $A$, where the energy for every momentum $p$ has both a positive and a negative value from Eq. (2.13), we can put these solutions, one at a time, into the expression for $\rho$ and obtain the normalized densities

$$\rho^+ = + \frac{|E|}{mc^2} |\psi^+(r)|^2, \quad \rho^- = - \frac{|E|}{mc^2} |\psi^-(r)|^2.$$  \hspace{1cm} (2.23)

By studying this result, we understand that in the relativistic case, $\rho$ cannot be interpreted as a probability density, like when we were dealing with the SE continuity equation. This is due to the fact that $\rho$ is not positive definite as in the SE case. We observe that in the non-relativistic limit, when $E$ approaches $mc^2$, we get the non-relativistic density expression for probability $\rho = |\psi(r)|^2$, which we would have expected. However with the unclarity of how to interpret the different signs of the density, it is clear that we need a new physical interpretation for it.

To produce a continuity equation making physical sense we follow Greiner [25] and multiply Eq. (2.21) by the elementary charge $e$, resulting in

$$\rho = \frac{i\hbar e}{2mc^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right), \quad j = \frac{\hbar e}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*).$$  \hspace{1cm} (2.24)

We now call $\rho$ the charge density and $j$ the charge current density. If we insert the wave function (2.14) into the density expression in Eq. (2.24) we obtain

$$\rho_\pm = \pm \frac{e|E|}{mc^2} \psi^\pm \psi_\pm.$$  \hspace{1cm} (2.25)

Here we interpret the wave functions as $\psi_+$ being the solution for particles with a positive elementary charge and $\psi_-$ being the solution for particles with negative charge, meaning antiparticles. With this approach we interpret the continuity equation as a conservation of charge density. For particles with zero charge we have $\rho = 0$ which leads to $\psi^* = \psi$. The charge-current-density for neutral particles will then also be equal to zero. Hence, for neutral particles there is no conservation of charge density. This holds with our interpretation.

Another way to interpret the continuity equation is to use field theory. This is the subject of quantum field theory and will not be covered in this report [1].
Chapter 3

The Klein-Gordon Equation for Pionic Atoms

In this chapter we start by presenting and motivating a KGE model for pionic atoms. We continue by deriving the energy and charge density produced by this KGE model. The numerical calculations are described, followed by our results. This is followed by a discussion of our results.

3.1 A Klein-Gordon Model for Pionic Atoms

The same assumption of a Coulomb potential as used for deriving the SE model for pionic atoms in Section 1.2 produces a simple KGE model. For Eq. (2.2) to be applicable on pionic atoms we thus need to incorporate the electromagnetic potential energy. In four-vector notation the electromagnetic potential reads $[A_0, A]$, where the approximated Coulomb potential of the nucleus gives $A = 0$ and $A_0 = \frac{Ze}{4\pi\epsilon_0 rc}$, where $\epsilon_0$ is the vacuum permittivity and $r$ is the distance from the nucleus.

Due to approximating the pion charge as point-like, its coupling to the electromagnetic field from the nucleus can be approximated as minimal. This means that only the lowest multipole moment contributes to the interaction. We can, as done in Ref. [1], replace the first term in the momentum operator in Eq. (2.2) by $\hat{p}_0 = i\hbar \frac{\partial}{\partial (ct)} - q A_0$, where $q$ is the charge of $\pi^-$. The resulting Klein-Gordon equation for a pionic atom is given by

$$\hat{p}^\mu \hat{p}_\mu \psi = \left[ \left( i\hbar \frac{\partial}{\partial (ct)} - q A_0 \right)^2 + \hbar^2 \Delta \right] \psi = m^2 c^2 \psi, \quad (3.1)$$

where $m$ is the pionic mass. Taking the non-relativistic limit of Eq. (3.1) in analogy with Subsection 2.1.1 produces

$$i\hbar \frac{\partial}{\partial t} \psi = \left( -\frac{\hbar^2 \nabla^2}{2m} + q A_0 \right) \psi. \quad (3.2)$$

This being the SE for particles in a Coulomb potential, not taking spin into account, suggests that the KGE is the relativistic extension of the SE for spin-0 particles. The accuracy of our model is discussed in Section 3.5 when we compare experimental values to the values produced by this model.
3.2 Analytical Calculations

3.2.1 Energies and radial wave functions

To investigate the energies and wave functions we want to solve the time-independent form of Eq. (3.1). Following Ohlsson [1] we separate the wave function into one time-dependent and one space-dependent function \( \psi = T(t) \Psi(r, \theta, \phi) \), choose spherical coordinates due to the central nature of the Coulomb potential and make the common Ansatz for time-dependence \( T(t) = e^{-i \epsilon t} \). Here \( \epsilon \) is the energy level of the pion, which as in Eq. (2.13) is negative for the antiparticle \( \pi^- \) and positive for the particle \( \pi^+ \). Taking the time derivative in Eq. (3.1) results in

\[
-\hbar^2 c^2 \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Psi = \left( (\epsilon - qA_0) - m^2 c^4 \right) \Psi.
\]

(3.3)

The further analysis in this subsection is performed following Greiner [25]. Some steps of his calculations are left out and comments are added where we felt the need of further explanation. Separating \( \Psi \) into one radial and one angular function \( \Psi = u(r) Y(\theta, \phi) \) using a separation constant \( \lambda \), we get the angular equation

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} + \lambda Y = 0.
\]

(3.4)

This is solved by the spherical harmonics giving \( \frac{1}{\hbar^2 c^2} = l(l + 1) \) where \( l = 0, 1, 2, \ldots \). The separation also gives a radial equation

\[
\hbar^2 c^2 \left[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + (\epsilon - qA_0) - m^2 c^4 - \frac{l(l+1)}{r^2} \right] u = 0.
\]

(3.5)

Using the Ansatz \( u(r) = \frac{R(r)}{r} \) in equation (3.5), the previously given \( A_0 \) and \( q = -e \), then rearranging, gives us

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{(Z\alpha)^2}{\hbar^2 c^2} + \frac{2Z\alpha}{\hbar c r} - \frac{mc^4 - \epsilon^2}{\hbar^2 c^2} \right] R(r) = 0,
\]

(3.6)

where \( \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \) is the fine-structure constant. With the binding energy defined as

\[
E_{\text{binding}} = |\epsilon| - mc^2
\]

(3.7)

and pions being bound for negative binding energies, we need to search for energy solutions in the interval \(-mc^2 < \epsilon < mc^2\). The substitutions

\[
\beta = 2 \left( \frac{mc^4 - \epsilon^2}{\hbar c} \right)^{1/2}, \quad \mu = \sqrt{\frac{l(l+1)}{2} - (Z\alpha)^2}, \quad \lambda = \frac{2Z\alpha}{\hbar c \beta}, \quad \rho = \beta r,
\]

(3.8)

with the interval \( 0 < \rho < \infty \) because of the search interval for \( \epsilon \), transforms the equation into

\[
\left( \frac{d^2}{d\rho^2} - \frac{\mu^2 - \frac{1}{4}}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4} \right) R(\rho) = 0.
\]

(3.9)
Studying equation (3.9) as \( \rho \to \infty \) and \( \rho \to 0 \) gives means to determine a suitable Ansatz for \( R(\rho) \). In the first case the second and third terms can be neglected, producing an equation with the solutions \( R(\rho \to \infty) = ae^{-\mu/2} + be^{\mu/2} \) where \( b \) can be set to zero on grounds of the normalization requirement of the solution. In the second case the third and fourth terms can be neglected and the Ansatz \( R(\rho \to 0) = \alpha \rho^\lambda \) gives \( \nu = \frac{1}{2} \pm \mu \). For \( l > 0, \mu \) takes on positive values since \( \alpha \approx \frac{1}{137} \) and \( Z \) is less than 137. This means that we have to choose \( \nu = \frac{1}{2} + \mu \) when \( l > 0 \) in order to avoid a divergence at the origin for \( R \). This is also sufficient for avoiding a divergence at the origin for \( u = \frac{R}{r} \). For \( l = 0, \mu \) can be either positive or imaginary (when \( Z > 68 \)) and the motivation for the decision of \( \nu \) is more complicated. However \( \nu = \frac{1}{2} + \mu \) is the necessary choice in this case as well, giving us allowed solutions for \( l = 0 \) and \( Z < 69 \) with this method. In this case there is a singularity for \( u \) at the origin, but since it can be integrated the wave function can be normalized. The analysis of limits above leads to the Ansatz \( R(\rho) = N'\rho^{\frac{1}{2}+\nu}e^{-\frac{\rho}{2}}f(\rho) \). Inserting this into Eq. (3.9) produces

\[
\frac{d^2 f}{d\rho^2} + \left( \frac{d}{\rho} - 1 \right) \frac{df(\rho)}{d\rho} - \frac{b}{\rho} = 0 \tag{3.10}
\]

if \( 2\mu + 1 = d \) and \( \mu + \frac{1}{2} - \lambda = b \). Attempting to solve this with a power series expansion \( f(\rho) = \sum_{n=0}^{\infty} a_n \rho^{n'} \) gives us

\[
\sum_{n'=2}^{\infty} n'(n'-1)a_{n'}\rho^{n'-2} + \sum_{n'=1}^{\infty} n'a_{n'}\rho^{n'-2} - \sum_{n'=1}^{\infty} n'a_{n'}\rho^{n'-1} - b \sum_{n'=0}^{\infty} a_{n'}\rho^{n'-1} = 0. \tag{3.11}
\]

For the coefficients of terms of equal power to be equal we conclude that

\[
a_j = \frac{a_{j-1}(b+j-1)}{j(d+j-1)} \tag{3.12}
\]

has to be true, making it possible to write

\[
f(\rho) = a_0 \left( 1 + \frac{b}{d} + \frac{b}{d+1} \right) \rho^{\frac{1}{2}} + \frac{b}{d+2} \rho^{\frac{3}{2}} + \cdots. \tag{3.13}
\]

For \( \rho \to 0 \) this \( f(\rho) \) is a constant and thus normalizable. For \( \rho \to \infty, \) \( f(\rho) \) diverges since the series is infinite, making it necessary to require a cut-off value for \( n' \), say \( n' \), after which all terms are zero. This would make it possible to meet the normalization condition due to the decreasing exponential term of \( R(\rho) \) dominating over the largest term in a finite power series as \( \rho \to \infty \). We find the cut-off value by requiring \( b \) to be a negative integer such that \( b + n' = 0 \) for the value \( n' = n' \). Using this and the definitions of \( b \) and \( \lambda \) gives us \( \mu + \frac{1}{2} + n' = \frac{Z\alpha}{2\sqrt{m\gamma^2}} \) and the energies can be found as

\[
\epsilon_{n',l} = \pm mc^2 \left( 1 + \frac{(Z\alpha)^2}{\left( n' + \frac{1}{2} + \left( l + \frac{1}{2} \right)^2 - (Z\alpha)^2 \right)^{\frac{1}{2}}} \right)^{-\frac{1}{2}}. \tag{3.14}
\]

Here the minus sign corresponds to bound \( \pi^- \) and the plus sign to bound \( \pi^+ \). Both types of particles have the energy \( |\epsilon_{n',l}| \). If the principal quantum number is chosen as
\( n = n' + l + 1 \), \( l \) is limited by \((n - 1)\) since \( n' \) is 0 at its lowest, whereas \( n \) can take integer values between 1 and infinity since \( l \) is 0 at its lowest as well. Rewriting Eq. (3.14) with the definition of the principal quantum number gives us the final form of the energies as

\[
\epsilon_{n,l} = \pm mc^2 \left( 1 + \frac{(Z\alpha)^2}{\left\{ n - l - \frac{1}{2} + \left[ (l + \frac{1}{2})^2 - (Z\alpha)^2 \right]^{1/2} \right\}^2} \right)^{-\frac{1}{2}}. \tag{3.15}
\]

We can also write the radial wave functions corresponding to these \( n \) and \( l \) values as

\[
u(r) = \frac{R(r)}{r} = N \left[ (\beta r)^{\frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2}} e^{-\frac{\beta r}{2}} f(\beta r) \right]. \tag{3.16}
\]

Here \( N \) is the normalization constant and \( f(\rho) \) is as in Eq. (3.13), with the difference that the sum of the terms in the series is taken from \( n' = 0 \) to \( n' = n - l - 1 \). This can be done because of the cut-off condition \( b + n' = 0 \) and \( n = n' + l + 1 \), since all terms with larger values of \( n' \) will be zero.

Through a series expansion in terms of \( Z\alpha \) around zero the binding energy can be expressed as

\[
E_{\text{binding}} = -mc^2 \left[ \frac{(Z\alpha)^2}{2n^2} + \frac{(Z\alpha)^4}{2n^4} \left( \frac{n}{l + \frac{1}{2}} - \frac{3}{4} \right) + \ldots \right], \tag{3.17}
\]

where the first term corresponds to the SE energy and the others constitute the relativistic KGE correction to this energy.

### 3.2.2 Radial charge densities for bound pions

The charge densities (2.25) are only valid for free particles. To be able to study the charge densities for a pionic atom described by a Coulomb potential we need to obtain them for an electromagnetic field. This is done following Greiner [25]. The electromagnetic potential is described by \( A_\mu = [A_0, A] = g^{\mu\nu} A_\nu \) and we have \( A_\mu = [A_0, -A] = g_{\mu\nu} A^\nu \).

Replacing the four momentum operators in Eq. (2.2) by the operators describing an electromagnetic field, \( \hat{p}^\mu = i\hbar \left[ \frac{\partial}{\partial (ct)} - qA_0, \frac{\partial}{\partial x} - qA_1, \frac{\partial}{\partial y} - qA_2, \frac{\partial}{\partial z} - qA_3 \right] \), gives us

\[
\left[ g^{\mu\nu} \left( i\hbar \frac{\partial}{\partial x^\nu} - qA_\nu \right) \left( i\hbar \frac{\partial}{\partial x^\mu} - qA_\mu \right) \right] \psi = m^2 c^2 \psi \tag{3.18}
\]
We now derive the continuity equation by multiplying Eq. (3.18) from the left with \( \psi^\ast \) and then subtracting the complex conjugate, yielding

\[
0 = \psi^\ast \left[ g^{\mu\nu} \left( i\hbar \frac{\partial}{\partial x^\nu} - qA_\nu \right) \left( i\hbar \frac{\partial}{\partial x^\mu} - qA_\mu \right) \right] \psi \\
- \psi \left[ g^{\mu\nu} \left( -i\hbar \frac{\partial}{\partial x^\nu} - qA_\nu \right) \left( -i\hbar \frac{\partial}{\partial x^\mu} - qA_\mu \right) \right] \psi^\ast \\
= g^{\mu\nu} \left[ -\hbar^2 \psi^\ast \frac{\partial}{\partial x^\nu} \psi - \psi^\ast i\hbar \frac{\partial}{\partial x^\mu} qA_\mu \psi - \psi i\hbar A_\nu \frac{\partial}{\partial x^\nu} \psi \right] \\
+ g^{\mu\nu} \left[ \hbar^2 \psi \frac{\partial}{\partial x^\nu} \psi^\ast - \psi \frac{\partial}{\partial x^\mu} qA_\mu \psi^\ast - \psi \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \psi \right] \\
= g^{\mu\nu} \left[ \frac{\partial}{\partial x^\nu} \left( -\hbar^2 \psi^\ast \frac{\partial}{\partial x^\mu} \psi + \hbar^2 \psi \frac{\partial}{\partial x^\nu} \psi^\ast \right) - 2 \frac{\partial}{\partial x^\mu} \psi i\hbar A_\nu \psi^\ast \right]
\]

Multiplying this by \(-\frac{ie^2}{2m} \) gives us the continuity equation for a particle bound in an electromagnetic field

\[
\frac{\partial \rho'}{\partial t} + \nabla \mathbf{j}' = 0
\]

with

\[
\rho' = \frac{i\hbar e}{2mc^2} \left( \psi^\ast \frac{\partial}{\partial t} \psi - \psi \frac{\partial}{\partial t} \psi^\ast \right) - \frac{eq}{mc} A_0 \psi,
\]

\[
\mathbf{j}' = -\frac{i\hbar e}{2m} (\psi^\ast \nabla \psi - \psi \nabla \psi^\ast) - \frac{eq}{m} \mathbf{A} \psi \psi^\ast.
\]

Applying this to our pionic atom described with the Coulomb potential, we have \( A_0 = \frac{Ze^2}{4\pi\epsilon_0rc} \) and \( \mathbf{A} = 0 \). When studying the charge densities for our model we look at a stationary state. Since the angular part consists of the \( Y \)-functions, which are the same as for the SE case, we want to study the \( u(r) \) part from Eq. (3.16). To normalize the charge densities we are allowed to normalize the radial part separately due to \( Y \) being normalized to unity. The radial part of Eq. (3.20) takes the form

\[
\rho(r) = Ne \left[ \frac{\epsilon - \frac{qZe}{4\pi\epsilon_0rc}}{mc^2} \right] u(r) \psi^\ast(r).
\]

Here \( N \) is the normalization constant obtained by \( \int \rho(r)dr = -e \) for the negatively charged pion. The \( \pi^- \) particle is an antiparticle. The quantity \( \epsilon \) is the energy given by Eq. (3.15) and is therefore negative in our case.

### 3.3 Numerical Calculations

To fulfil the aims of this report we now use some of the previously derived equations to calculate numerical values for regular and pionic atoms using MATLAB. The second aim is to investigate the relation between regular and pionic atoms. This is performed by using the expression for SE binding energies (1.2) and the expression for KGE binding energies (3.7) with (3.15) to calculate binding energies for different values of the parameters \( n \), \( l \) and \( Z \). The third aim is to investigate the agreement between the predictions of the
KGE model and experimental values. This is done by calculating transition energies using the KGE energy (3.15) and the SE energy (1.2) for pionic atoms. We compare these to experimental values for the transitions from Berkeley [13, 14, 15, 16], Cern [17, 18, 19, 11] and Virginia [20, 21, 22, 23, 24]. The experimental values are for the transitions $2p - 1s$, $3d - 2p$, $4f - 3d$ and $5g - 4f$ in atoms of different $Z$. The fourth aim is to investigate the radial charge densities. This is done using Eq. (3.22) to numerically calculate the densities for different values of $r$ and the other parameters $n$, $l$ and $Z$. All of the above calculations are done using the constants given in table 3.1.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Value</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$7.2973525698 \cdot 10^{-3}$</td>
<td>[32]</td>
</tr>
<tr>
<td>$\hbar$</td>
<td>$1.054571726 \cdot 10^{-34}$ Js</td>
<td>[32]</td>
</tr>
<tr>
<td>$c$</td>
<td>$299792458$ m/s</td>
<td>[32]</td>
</tr>
<tr>
<td>$e$</td>
<td>$1.602176565 \cdot 10^{-19}$ C</td>
<td>[32]</td>
</tr>
<tr>
<td>Electron mass</td>
<td>$510.998928$ keV</td>
<td>[32]</td>
</tr>
<tr>
<td>Pion mass</td>
<td>$139.57018$ keV</td>
<td>[33]</td>
</tr>
</tbody>
</table>

### 3.4 Results

#### 3.4.1 Energies

We start by giving an overview of the differences in the binding energies for a pionic atom and a regular atom in Figures 3.1 and 3.2. We have set $l = 0$ in the figures because of this energy level being more relevant in atoms of lower $Z$ than larger $l$'s. In Figure 3.1 the absolute value of the binding energies is plotted against increasing $Z$ and we have set $n = 1$ constant to show the energy dependence on $Z$. We have chosen $n = 1$ because this gives the largest difference for the pionic atom binding energy when using the KGE and the SE. Since, for $l = 0$, our theory is not applicable for $Z > 68$, we have plotted $Z$ in the interval $0 < Z < 69$. The choice of plotting with a logarithmic axis is due to the large difference in binding energy magnitude between regular and pionic atoms. We have taken the absolute value because of the binding energies being negative. In Figure 3.2 we have plotted the energies against different $n$ values with a constant $Z$ to show the energy dependence on $n$. We wanted to have $Z$ as large as possible because this gives the largest differences for the pionic atom binding energy when using the KGE and the SE. Thus we have set $Z = 68$. Changing $l$ and keeping the other parameters constant does not contribute to as large differences in the energy levels as changing $Z$ and $n$. Therefore we refrain from presenting this in a figure. However, the absolute value of the binding energy decreases with increasing $l$. 

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Figure 3.1: The logarithm of the absolute values of the binding energies as a function of the atomic number $Z$ and constant $n = 1, l = 0$.

Figure 3.2: The logarithm of the absolute values of the binding energies as a function of the principal quantum number $n$ and constant $Z = 68, l = 0$. 

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An overview of the KGE and the SE values for transition energies compared to experimental results is given in Figure 3.3. Both the calculated and the experimental values of all transitions are plotted against atomic number $Z$. A closer look at the agreement of the KGE and the SE models with data is given in Figures 3.4 for $5g - 4f$, 3.5 for $4f - 3d$, 3.6 for $3d - 2p$ and 3.7 for $2p - 1s$. The percentual deviation of the experimental data and the SE model values from the KGE model values are plotted as a function of $Z$. We have chosen not to plot the uncertainties in the experimental data. These are small in comparison to the difference between the data and the model predictions.

**Figure 3.3**: From left to right: transition energies for $2p - 1s$, $3d - 2p$, $4f - 3d$ and $5g - 4f$ transitions. The energies come from experimental data and from theoretical calculations with the KGE and the SE models.
Figure 3.4: Transition $5g - 4f$. Percentual deviation of experimental values and the SE model values from the KGE model values.

Figure 3.5: Transition $4f - 3d$. Percentual deviation of experimental values and the SE model values from the KGE model values.
Figure 3.6: Transition $3d - 2p$. Percentual deviation of experimental values and the SE model values from the KGE model values.

Figure 3.7: Transition $2p - 1s$. Percentual deviation of experimental values and the SE model values from the KGE model values.
3.4.2 Charge densities

The normalized radial charge densities are plotted in Figures 3.8 and 3.9 as functions of \( r \), keeping two parameters constant and changing the third. Here \( l \) is set to 1 on grounds of the charge density diverging as \( r \) approaches 0 for \( l = 0 \). This makes it difficult to normalize the charge density using MATLAB. When constant, \( Z \) is arbitrarily set to 39. Now \( n \) is set to 2 when varying \( Z \), because of this being the lowest \( n \) allowing \( l = 1 \). We want \( n \) to be fairly low, due to such an energy level being more relevant in atoms of lower \( Z \) than higher ones. To get a clear view of the general behaviour of the \( Z \) dependence, we have chosen the interval \( 10 < Z < 69 \) with step 3 which produces clearly readable curves in the figure. When keeping \( Z \) and \( n \) constant and varying \( l \), the change in the radial charge density is not as dramatical. An increase of \( l \) leads to the minimum of the radial charge density moving slightly to the right and the curve shape changing slightly.

We refrain from presenting a figure showing this.

\[
\begin{align*}
0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 & \quad 2.5 \\
\times 10^{-12} & \\
0 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 0 \\
-0.2 & \quad -0.4 & \quad -0.6 & \quad -0.8 & \quad -1 & \quad -1.2 \\
-1.2 & \quad -1 & \quad -0.8 & \quad -0.6 & \quad -0.4 & \quad -0.2 & \quad 0 \\
0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 & \quad 2.5 \\
\times 10^{-6} &
\end{align*}
\]

\textbf{Figure 3.8:} Normalized radial charge densities for \( n = 2, l = 1 \) and \( Z \) from 11 to 68 with step 3.
3.5 Discussion

3.5.1 Energies

From both Figures 3.1 and 3.2, we observe that the difference in the binding energies of a pionic atom and a regular atom is several orders of magnitude. Considering the difference in mass between a pion and an electron this is highly expected.

From Figure 3.1 we note that the absolute value binding energy increases with Z both for a regular- and a pionic atom. This is also expected since more protons are present with higher atomic number, increasing the electromagnetic interaction. Furthermore, we notice that the energy values of the two models for pionic atoms deviate increasingly from each other for higher Z. By considering the series expansion expression (3.7) for the KGE binding energies, we realise that the relativistic terms cause a larger difference in the energies of the different models for higher Z. From the series expansion we also see that this is the consequence for lower n, which is evident in Figure 3.2.

From Figure 3.3 we see that both our models fit rather well with the experimental data, particularly for the two transitions in the middle. However, for the highest transition our SE model seems to be in slightly worse agreement with the experimental data than our KGE model. For the lowest transition, both models seem to be in worse agreement with the data than in the in the middle transitions. Our SE model is in slightly better agreement in this case than our KGE model. To be able to investigate the behaviour of the models further, we need to analyse Figures 3.4, 3.5, 3.6 and 3.7.

Focusing on the behaviour of our SE model for the different transitions, we conclude that its agreement progressively improves for lower transitions. Furthermore, the agreement improves for lower values of Z within one transition. With that said, the agreement
in the three outer transitions is worse than the agreement of the KGE model. In the inner transition the agreement is only slightly better than for our KGE model. This implies that the KGE produces a better model for pionic atoms than the SE does.

Turning to our KGE model, we see that its agreement with the experimental data is better for higher transitions than for lower ones. The agreement is noticeably worse for the lowest transition. Furthermore, the agreement of our KGE model decreases with increasing $Z$ within each transition. Our KGE model only takes the Coulomb potential into consideration. The data thus implies that this is more accurate in the higher transitions than in the lowest one and for lower $Z$ than for higher $Z$. The effects could be attributed to the interactions mentioned in Section 1.2, including the finite size effect, vacuum polarization and strong interaction. For the lowest transition and for large $Z$, our KGE model is thus in need of improvement.

3.5.2 Charge densities

From Figure 3.8 we observe that the minimum of the radial charge density is further away from the nucleus and the density curve is increasingly smeared out for lower values of $Z$. This corresponds to the magnitude of the binding energy increasing for larger nuclei and the pion being closer to the nucleus than for smaller nuclei.

From figure 3.9 we note that the dependence of the position of the minimum of the radial charge density on $n$ is opposite of the dependence on $Z$. The charge density is more smeared out with increasing $n$ in this case. This corresponds to the magnitude of the binding energy increasing for lower $n$-levels and the pion being closer to the nucleus.

The position of the minimum is around $10^{-12}$ m from the nucleus. This is closer to the nucleus than for electrons of the lowest quantum number. The electron screening of the nucleus mentioned in Section 1.2 is thus not likely to have a large effect on the pionic energy levels in the figure.

The fact that the radial charge density does not change as drastically when varying $l$ as when varying $n$ or $Z$ can be explained by looking at the expression for the radial charge density (3.22). This shows that the density is dependent on the energy level (3.15) of the pion. This energy changes less for a change in $l$. 

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Chapter 4

Summary and Conclusions

We have introduced the subject of the Klein-Gordon equation, KGE, through a historical review. Properties of this equation have been studied through a comparison with the Schrödinger equation, SE. This led to the conclusions that the KGE constitutes a relativistic extension of the SE for spin-0 particles and that the density of the KGE continuity equation cannot be interpreted as a probability density, as in the SE case. Instead it must be interpreted as a charge density, making the continuity equation a preservation equation of charge. For every momentum there are a negative and a positive energy solution to the KGE. These can be defined to represent particles and antiparticles. The subject of pions and pionic atoms has been introduced through a review of some research on the subject.

Based on the KGE, a model of pionic atoms has been derived. We have used a point nucleus Coulomb potential and viewed the pions as point charges, $-e$. The energy levels and radial charge density of the KGE model have been derived analytically. These energy levels have been used to examine the relationship between regular and pionic atoms. This has led to the conclusion that pions have higher magnitude of binding energy than electrons.

We have compared the transition energies for our SE and our KGE model to experimental data. From this we could conclude that the KGE model is a better model for pionic atoms than the SE model. We could also conclude that the agreement was better for lower $Z$ and higher transitions for the KGE model. An improved model is needed for higher $Z$ and for the inner transition, possibly due to the finite size effect, vacuum polarization and strong interaction being of larger influence here.

The radial charge densities for pionic atoms have been studied. This has led to the conclusion that the radial charge density minimum is closer to the nucleus for high $Z$ or low $n$. Furthermore the pion is closer to the nucleus than the electrons if $n$ is low enough or $Z$ is high enough. We can also conclude that the radial charge density diverges for $l = 0$ as $r \to 0$.

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Bibliography


