Estimation of a Liquidity Premium for Swedish Inflation Linked Bonds

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Master of Science Thesis
Stockholm, Sweden 2014

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Master's Thesis in Mathematical Statistics (30 ECTS credits)
Master Programme in Mathematics (120 credits)
Royal Institute of Technology year 2014
Supervisor at KTH was Henrik Hult
Examiner was Henrik Hult

TRITA-MAT-E 2014:27
ISRN-KTH/MAT/E--14/27-SE

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Abstract

It is well known that the inflation linked breakeven inflation, defined as the difference between a nominal yield and an inflation linked yield, sometimes is used as an approximation of the market's inflation expectation. D'Amico *et al.* (2009, [5]) show that this is a poor approximation for the US market. Based on their work, this thesis shows that the approximation also is poor for the Swedish bond market. This is done by modelling the Swedish bond market using a five-factor latent variable model, where an inflation linked bond specific premium is introduced. Latent variables and parameters are estimated using a Kalman filter and a maximum likelihood estimation. The conclusion is drawn that the modelling was successful and that the model implied outputs gave plausible results.

Keywords: Inflation linked yields, State space model, Kalman filter, Maximum likelihood estimation, Stochastic calculus

Acknowledgements

We would like to thank our supervisor at *KTH Mathematical Statistics*, Henrik Hult, for great feedback and guidance. We would also like to thank Mats Hydén, Chief Analyst at *Nordea Markets*, for introducing us to this interesting subject and for his many thoughtful insights.

Stockholm, May 29, 2014

Magnus Bergroth and Anders Carlsson

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CHAPTER 1

Introduction

This chapter will cover a short background for the main subject of this thesis, with the intention to give the reader an understanding of the topic at hand.

In Section 1.2 the problem formulation will be given. In Section 1.3 the implementation of the used model will be explained. In Section 1.4 an initial study will be performed giving an impression of the later introduced liquidity premium in the inflation linked bond market. Finally in Section 1.5, the outline for the thesis will be given.

1.1 Background

In the article *Tips from TIPS: the informational content of Treasury Inflation-Protected Security prices* (D'Amico, Kim & Wei 2010, [5]) the scientists Stefania D'Amico, Don H. Kim and Min Wei are trying to model the US inflation linked bond market using a five factor model in the no arbitrage framework. They introduce an inflation linked bond specific liquidity premium, defined as the difference between the inflation linked yield and the real yield. State variables and parameters are estimated using a Kalman filter in combination with a maximum likelihood estimation. Further, they show that by including the inflation linked bond specific liquidity premium, their results are considerably improved.

One of many conclusions that can be drawn from their work is that the inflation linked breakeven inflation, defined as the difference between a nominal yield and an inflation linked yield, falsely has been used as a measure of the market implied inflation expectation, see [1]. Furthermore, they present many interesting yield decompositions from which the correlation structure in the bond market can be analysed.

Based on the above mentioned article, see [5], it is therefore interesting to investigate whether a model including an inflation linked liquidity premium can be used to model the Swedish market for inflation linked bonds.

1.2 Problem Formulation

The aim of this thesis is to investigate the possibility of fitting a model, including an inflation linked specific liquidity premium, to the Swedish bond market, as was done for the US bond market in the article *Tips from TIPS: the informational content of Treasury Inflation-Protected Security prices* (D'Amico, Kim & Wei 2010, [5]).

1.3 Practical Implementation

This section provides an overview of how the model, *Model L-II* given in the above mentioned article [5], practically was implemented to the Swedish market.

It was of great importance to break down the very extensive article into its pieces. After this was done, the main steps could be identified and a roadmap for the thesis could be laid out. The following points were set up and performed during the thesis.

1. Identify the separate parts of the market that needed to be modelled (the real part, the nominal part, the inflation linked part etc.), see Chapter 3. Also, the corresponding solutions or systems of ODE:s (later needed to be solved numerically) had to be identified.

2. Identify the needed data, see Chapter 4.

3. Perform an initial study on the Swedish market, see Section 1.4.

4. Derive the solutions or the later numerically solved ODE systems. These derivations are given in the appendix, see Appendix A. The derivations were performed to further understand the implications of the different model assumptions.

5. Derive the state space model, containing the separate models for each part of the market that was modelled, see Section 2.2.

6. Derive the needed components in both the predication and the update phase of the Kalman filter, see section 2.3. The Kalman filter was later used to estimate the state variables.

7. Derive the multivariate log-likelihood function later used to estimate the parameters, see Section 2.4.

8. Implement the Kalman filter and the maximum likelihood estimation into MATLAB, as well as all introduced parameters and the corresponding solutions and systems of ODE:s. In this implementation it was of great importance to find numerical methods which correctly could solve this particular problem, see Section 5.3. Further, since all introduced model parts could not be solved analytically, an ODE solver was needed to be implemented. The MATLAB implementation was very time consuming since there were so many different parts that needed to be implemented and fully working together.

9. When the estimation was performed and all parameters and state variables had been given values, the results could be produced, see Chapter 6, using the assumed model relations, see Chapter 3.

1.4 Initial Study

Before going deeper into the subject, it seems reasonable to perform some initial studies aimed to investigate whether there is any evidence for the presence of a liquidity premium in the Swedish inflation linked bond market.

Regression Analysis

Before the analysis can be done it needs to be stated that the Swedish inflation linked bonds also are referred to as linkers.

The first analysis is done by running a simple regression. Further, some definitions are needed. The linkers breakeven inflation and the true breakeven inflation are defined as

$$y_{t,\tau}^{BEI,L} = y_{t,\tau}^{N} - y_{t,\tau}^{L} \qquad \text{and} \qquad y_{t,\tau}^{BEI} = y_{t,\tau}^{N} - y_{t,\tau}^{R}, \tag{1.1}$$

respectively, where $y_{t,\tau}^{N}$ is the nominal yield, $y_{t,\tau}^{L}$ is the inflation linked yield and $y_{t,\tau}^{R}$ is the real yield.

When analysing the inflation, the market often uses the linkers breakeven inflation as a proxy for the true breakeven inflation. Based on this, which will be further discussed below, it can be interesting to run a simple regression of the 10-year linkers breakeven inflation onto the 3-month, 2-year and 10-year nominal yields, including a constant, given as

$$y_{t,\tau}^{BEI,L} = \alpha + \beta_1 y_{t,0.25}^N + \beta_2 y_{t,2}^N + \beta_3 y_{t,10}^N + e_t. \tag{1.2}$$

The nominal yield at time t with maturity at time $t+\tau$ is defined as follows,

$$y_{t,\tau}^N = y_{t,\tau}^R + I_{t,\tau} + \wp_{t,\tau}^I, \tag{1.3}$$

where $y_{t,\tau}^R$ is the real yield, $I_{t,\tau}$ is the inflation expectation and $\wp_{t,\tau}^I$ is the inflation risk premium.

Therefore, by assuming that the linkers breakeven inflation is an acceptable proxy for the true breakeven inflation, giving that the two expressions in (1.1) are equal, and using the above decomposition of the nominal yield, some interesting results can be produced. By this assumption the left hand side of the regression in (1.2) would simply be a sum of the expected inflation and the inflation risk premium

$$\begin{aligned} y_{t,\tau}^{BEI,L} &= y_{t,\tau}^{BEI} = y_{t,\tau}^N - y_{t,\tau}^R \\ &\Leftrightarrow \\ I_{t,\tau} + \wp_{t,\tau}^I &= \alpha + \beta_1 y_{t,0.25}^N + \beta_2 y_{t,2}^N + \beta_3 y_{t,10}^N + e_t. \end{aligned} \tag{1.4}$$

Then, since both the expected inflation and the inflation risk premium are included in the nominal yields, one would expect a high R^2-value when running the above given regression.

Statistics from running the regression on both daily values and daily changes are given in Table 1.1. Noticeable is that the R^2 equals 0.66 for the regression using daily values and 0.71 for the regression using daily changes. This can be compared to a regression of a nominal yield, or its daily changes, onto other nominal yields, or their changes, which gives R^2:s in the region $0.95 - 0.99$. Therefore, these R^2 values suggest that there are some other factors, not included in the nominal yields, that partly determine the level of the linkers breakeven inflation.

Table 1.1: Regression Analysis

Coefficients				R^2
Constant	*3-month*	*2-year*	*10-year*	
Daily Values				
0.0106	-0.0002	0.0868	0.2072	0.6553
(0.0002)	(0.0120)	(0.0178)	(0.0117)	
Daily Changes				
0.0000	-0.0060	-0.0496	0.5952	0.7111
(0.0002)	(-0.0060)	(-0.0496)	(0.0117)	

Principal Component Analysis

The second analysis is a principal component analysis of the cross section of the nominal yields and the inflation linked yields. The data set used in this analysis comprises nominal yields with maturity 3- and 6-months and 1-, 2-, 4-, 7- and 10-years and inflation linked yields with maturity 5-, 7- and 10-years.

Before considering the results, given in Table 1.2 and Table 1.3, it is worth mentioning the interpretations of the first three principal components of a nominal yield curve:

- PC1 - The level of the yield curve
- PC2 - The slope of the yield curve
- PC3 - The curvature of the yield curve

Further, it is well known that the first three principal components can explain most of the variations in the nominal yield curve, see [6].

Table 1.2 provides the aggregated variations explained when adding each principal component, for two different data sets. Firstly, a set containing the nominal yields and secondly a set containing both the nominal yields and the inflation linked yields. In Table 1.2 it can be seen that the first three principal components, for the first data set, together explain most of the variations in the nominal yields. Then, if looking at the result for the second data set, when the inflation linked yields are added to the nominal yields, it can be seen that a fourth component is needed to explain the same amount of variations as three components did for the first data set. It is therefore possible to draw the conclusion that there is something in the inflation linked yields that is not contained in the nominal yields.

Table 1.2: Variance Portion Explained by Principal Components

Principal Component	Nominal Yields Only	Nominal & Linkers Yields
1^{st}	0.711	0.690
2^{nd}	0.899	0.855
3^{rd}	0.955	0.914
4^{th}	0.978	0.956

Then since the interpretations of the nominal principal components are well known, it could also be interesting to get an impression of the interpretations of the principal components for the second data set. One way of doing this could be to evaluate the correlations between the principal components for the two data sets. Then, based on the similarity of the sets, if two components would have a high correlation it would seems reasonable to give these components the same interpretation, see [5].

Table 1.3 provides the correlations between the principal components of the two data sets. Based on the argumentation in the previous paragraph, it can be noticed that the first and the second components, of the different data sets, are highly correlated and therefore can be given the same interpretations. Further, the third component of the second data set has a fairly high correlation to the third component of the first data set. Still, it is not of the same magnitude as the correlations between the first and the second components, for the respective data sets. Therefore it can not be concluded that the third component of the second data set can be given the same interpretation as the third component of the first data set. Also it can be seen that the fourth component of the second data set has a higher correlation to the third component of the first data set, than the third component of the second data set has. Still, the third component of the second data set explains a larger portion of the variations in the second data set.

Therefore, it can be concluded that the third component of the second data set seems to contain some factor that is not present in any of the first three components in the first data set.

Table 1.3: Correlation of Principal Components

		Nominal & Linkers Yields			
		PC1	PC2	PC3	PC4
Nominal	PC1	0.993	-0.052	-0.093	0.052
Yields	PC2	0.038	0.991	-0.117	0.055
Only	PC3	0.020	0.030	0.629	0.776
	PC4	-0.002	-0.002	-0.015	0.008

A Linker Specific Liquidity Premium

Based on the previous section there is evidence pointing in the direction that there is something except for the nominal yields that determines the levels of the inflation linked yields.

In the earlier mentioned article, see [5], the authors investigated the presence of a liquidity premium in the US inflation linked bond market. During their chosen time period they could identify a large steady growth in both the total outstanding- and transactional volumes in the US inflation linked bond market. Hence they concluded that the liquidity conditions clearly had enhanced during their chosen time period. Consequently the extra premium could be viewed primarily as a liquidity premium, even though some other potential factors also were mentioned. Based on this insight they chose to use a deterministic trend when modelling the liquidity premium in one of their introduced models. This model was later shown to be their best model.

Based on their results it is of great interest to look at the total outstanding- and transactional volumes in the Swedish inflation linked bond market, visualized in Figure 1.1. The conclusion can be drawn that the liquidity conditions did not improve significantly during the period between 2005-2014. Therefore, it cannot be argued that the other factors mainly can be identified as a liquidity premium, as was done for the US market.

Still, in the US article they were trying to fit more than one model. Among those models, one had the same characteristics as the above mentioned model, except for the deterministic trend. Also when using this model a significant liquidity premium could be identify for the entire time period.

To simplify the estimation, the sample period in this thesis was chosen to not include any deterministic trend and therefore the model excluding the deterministic trend in the liquidity premium was chosen. In the US article this model is referred to as *Model L-II*.

Thus, even though liquidity might not be the only reason for having an inflation linked bond specific premium in the Swedish bond market, the earlier introduced notation, the liquidity premium, will be used throughout this thesis.

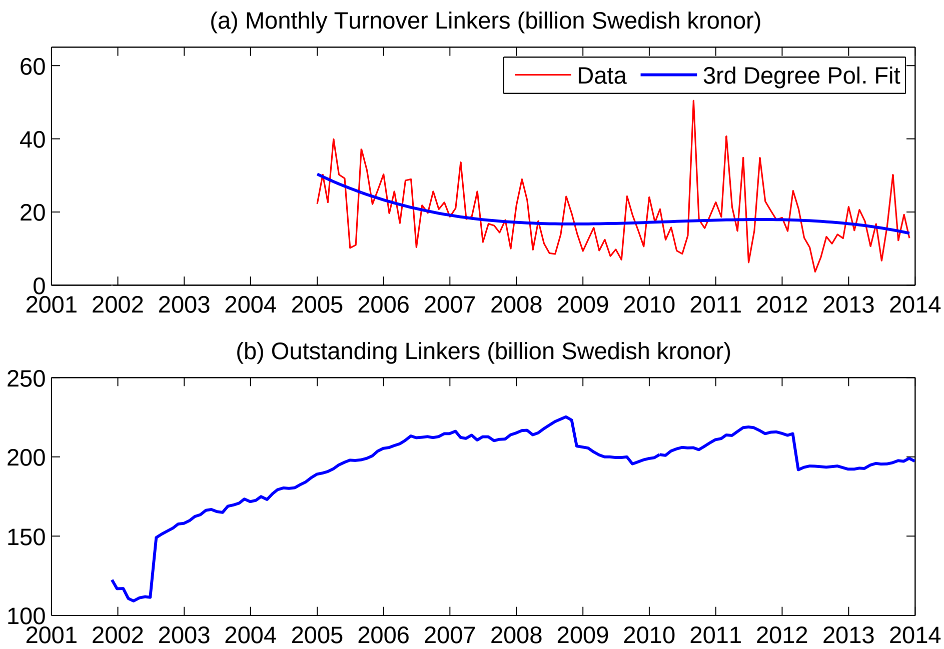

Figure 1.1: Linker Transaction- and Outstanding Volumes

1.5 Outline

The thesis will further be divided into seven separate parts.

The first part is a theoretical background, see Chapter 2, giving the needed theory to easily understand the later introduced models and the upcoming estimation methodology.

The second part contains the used model, see Chapter 3. In this chapter all model assumptions are stated and the corresponding parameters are introduced. Also, solutions or ordinary differential equations to all separate models are given.

The third part, see Chapter 4, gives the reader some insight into the used data and how some of the data samples are produced.

The fourth part renders the used method, see Chapter 5. Here the parameter estimation procedure is presented as well as how one numerically estimates the corresponding state variables.

The fifth part is where all the results are given, see Chapter 6. Among those, parameter values are given as well as graphical decompositions of the introduced yields.

In the sixth part the results are analysed and interpretations are given, see Chapter 7. Further, conclusions are drawn with the main focus to answer the earlier stated problem formulation.

The seventh part is an Appendix, see Appendix A, where all formula derivations are given separately.

CHAPTER 2

Theoretical Background

This chapter is aimed to give the reader sufficient theory to later understand the introduced model, given in Chapter 3, and the method used for estimating the state variables and the parameters in Chapter 5. Hence, the theoretical background should be considered as a tool for understanding these chapters and not as containing an adequate given theory on its own.

First, an introduction to the Swedish government bond market will be given, including a short review on the relation between the yield and the price of a bond.

Secondly, the state space model equations will be derived, followed by an introduction to the Kalman filter and the associated log-likelihood function.

Finally, a part with stochastic calculus theory will be given.

2.1 Government Bonds

This section provides a brief background on how to calculate yields of nominal- and inflation linked government bonds. There is also a summary of the Swedish interest rate market and the drivers of its yields.

2.1.1 Price and Yield

Below a discussion is given about the relationship between price and yield for nominal- and inflation linked government bonds.

Nominal Bonds

A bond can be traded either on price or yield. A fixed coupon nominal bond is issued at par value with fixed yearly coupons. During its term to maturity the bond is traded in the market and thus the bond price can deviate from its par value. Given a market price of a nominal fixed coupon bond, P_t^N, the market implied nominal yield to maturity, y_t^N, can be calculated using (2.1). The yield can then be used when comparing two separate bonds. For simplicity (2.1) is accurate for yield calculations instantly after a coupon payment, otherwise one would have to compensate for accrued interest rate,

$$P_t^N = \frac{C_{t+\Delta t_1}}{(1+y_t^N)^{\Delta t_1}} + \frac{C_{t+\Delta t_2}}{(1+y_t^N)^{\Delta t_2}} + ... + \frac{C_{t+\Delta t_n}}{(1+y_t^N)^{\Delta t_n}}, \tag{2.1}$$

where t is the present time and $\Delta t_1, \Delta t_2, ..., \Delta t_n$ are the times from t to the respective future fixed cash flows $C_{t+\Delta t_1}, C_{t+\Delta t_2}, ..., C_{t+\Delta t_n}$.

Inflation Linked Bonds

Inflation linked bonds, also called linkers, promise to protect the investors purchasing power, whereby the coupon payments are linked to the inflation. Thus the size of the coupons are adjusted in line with the increase or decrease of the inflation. Therefore it repays the bondholders' principal in an amount that maintains the purchasing power of their original investment, see [6].

The main difference between the nominal bond and the inflation linked bond, when calculating the yield, is that an index factor is multiplied to the right hand side of (2.1), thus

$$P_t^L = I_t \cdot \left(\frac{C_{t+\Delta t_1}}{(1+y_t^L)^{\Delta t_1}} + \frac{C_{t+\Delta t_2}}{(1+y_t^L)^{\Delta t_2}} + ... + \frac{C_{t+\Delta t_n}}{(1+y_t^L)^{\Delta t_n}} \right), \tag{2.2}$$

where P_t^L equals the market price of the inflation linked bond. The index factor I_t expresses the change in the Swedish consumer pricing index as the quote between the index level at the present time t (reference index) and the index level at the issuance time 0 (base index), given as

$$I_t = \frac{Ref.index}{Base.index} = \frac{CPI_t}{CPI_0}.$$

Thus the present yield to maturity for an inflation linked bond is given by solving (2.2) for y_t^L. Notice that the yield to maturity in (2.1) and (2.2) need to be solved numerically (unless there is only one payment left).

The yield curve, as a function of time to maturity, is given if the yield is calculated for each outstanding bond within an asset class, for example government bonds, and thereafter interpolate between these yields.

2.1.2 Swedish Interest Rate Market

The main instruments in the Swedish fixed income market are:

- Government Bonds
- Covered Bonds
- Derivatives

Swedish government bonds are issued by the Swedish national debt office that uses the bonds to finance the government's debt. Typical holders of these assets might be pension funds, mutual funds, foreign central banks or hedge funds.

Government bonds are distributed over different debt classes, where nominal bonds and inflation linked bonds are two of them. As of today there are 10 nominal and 6 inflation linked issues traded in the market with different maturities up to 30 years. As a percentage of the total outstanding volumes of Swedish government bonds, the target is to have a diversification of:

- 60 percent in nominal bonds
- 25 percent in inflation linked bonds
- 15 percent in foreign currency bonds

Credit rating is an important aspect that affects the yield level of a bond. A credit-rating is an evaluation of the credit-worthiness of a debtor. Ratings are provided from credit rating agencies such as *Standard & Poor's* and *Moody's* for companies and governments, but not for individual consumers. The evaluation gives a judgement of a debtor's ability to pay back the debt and the likelihood of default. Thus one can say that credit rating is a tool to evaluate the risk of a counterpart. By investing in Swedish government bonds with the highest possible credit rating (AAA), one can consider the investment as "risk-free" and the credit risk is therefore considered to be close to negligible. Instead there are other risk factors that are having larger impact on the yield levels of Swedish government bonds, some of these factors are discussed below.

2.1.3 Drivers of Yields

The Swedish central bank, *Riksbanken*, is the anchor of the Swedish yield curves since it determines the level of the short rates by adjusting *the repo rate*. Drivers of yields vary among different maturities but they still correlate to a large extent. The following part in this section will give a brief background on some of the factors that might drive the yields across the different maturities and asset classes.

Front-end yields are ranging in maturities up to approximately 3 years. Drivers of front-end yields might be domestically rate expectations which reflect the anticipated macro development in the years to come. Further, the international outlooks affect small open economies, such as Sweden, to a very large extent. For instance, the dependency on foreign policy can materializes through the exchange rate channel. Thus decisions that for example are delivered by the European central bank will spill over to the Swedish economy. Liquidity- and bank risk factors also affect the spread between *the repo rate* and the 3-month Stockholm interbank offered rate. As of the financial crisis in 2007-2008 and the default of Lehman Brothers, these kind of spreads have increased considerably and have been a game-changer for the front-end valuation.

Long-end yields are considered to be those with maturities longer than the short-end yields. One of the drivers of long-end yields is the policy rate cycle. It does not have instant effect as for the short-end yields, but it sets the direction for the long-end yields. Other factors that are linked to the long-end might be; supply of bonds, equity performance, general perception of risk and quantitative easing.

Yield curves are often judged by comparing yields of different maturities, for example the yield spread between a 2-year and a 10-year government bond. A steep and upward sloping yield curve generally indicates future economic improvement and vice versa. A driver of the yield curve might be the policy rate cycle. Yet again, Sweden is a small open economy and the shape of the yield curve is strongly correlated with foreign yield curves. These curves are in turn driven by their respective policy cycles, view on risk premiums, quantitative easing and economic forecasts. Thus there are several factors, both domestically and foreign, that drive the Swedish yield curves.

Further, a yield spread is the difference between yield levels of different yield curves, for example government bonds versus covered bonds (mortgage spread), government bonds versus swap rates (asset swaps) and corporate bonds versus government bonds (credit spreads). A yield spread can be used as a measure to identify if a curve is rich or cheap in comparison to another yield curve. Drivers of the yield spreads might be; relative liquidity risk and credit risk between the curves, supply and demand factors, yield pick-up (rich/cheap) and risk appetite.

Conclusions

The bottom-line from the past section; Swedish government bond yields are affected by several factors and among these liquidity is one of the drivers. High liquidity implies smaller liquidity risk that lowers the yields and vice versa. But equally well, there are other factors that also influence the yields to a very large extent. Thus it would be naive to direct all characteristics of the resulting liquidity premium, in this thesis, to precisely liquidity without also taking other factors into concern.

The reader should bear this in mind when approaching the results in this thesis. This is further discussed in the result- and analysis chapters, see Chapter 6 and Chapter 7.

2.2 State Space Model

The model construction and its assumptions are motivated in the model chapter, see Chapter 3, and this section will give a background on how the model is being discretized and transferred to state space form.

A state of a dynamic system refers to the minimum set of variables (state variables) that can describe all dynamics of the system and its outputs at a given time. The mathematical description of a system's dynamics can be expressed through a number of first order differential equations, also referred to as the state equations. These state equations, when discretized, can be written on matrix form and thus yield the matrix state equation, derived in Section 2.2.1. The outputs of the system are given by the vector observation equation. The vector observation equation is a linear combination of the state variables, including an additive error term, that is assumed to equal the observed data, measured with error, derived in Section 2.2.2.

2.2.1 State Equation

The later introduced model consists of three stochastic differential equations that describe the model dynamics through the latent variables q_t, $\mathbf{x}_t = [x_1, x_2, x_3]$ and $\tilde{x}_t$. These differential equations are later defined by (3.11), (3.1) and (3.36), but are here being restated.

The one-dimensional stochastic differential equation of the logarithmic price level process, (3.11), takes the form

$$\begin{aligned} dq_t = d(logQ_t) &= \pi(\mathbf{x}_t)dt + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P} \\ &= \left\{ (3.12),\ \pi(\mathbf{x}_t) = \rho_0^{\pi} + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_t \right\} \\ &= \rho_0^{\pi} dt + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_t dt + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P}, \end{aligned} \tag{2.3}$$

where ρ_0^{π} and $\sigma_q^{\perp}$ are constants, $\boldsymbol{\rho}_1^{\pi}$ and $\boldsymbol{\sigma}_q$ are constant vectors, $\boldsymbol{B}_t^P$ is a three-dimensional vector of Brownian motions and $B_t^{\perp,P}$ is a Brownian motion such that $d\boldsymbol{B}_t^P dB_t^{\perp,P} = \mathbf{0}$.

The three-dimensional stochastic differential equation that drives the real yield, the nominal yield and the expected inflation, (3.1), is given by

$$d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P, \tag{2.4}$$

where $\boldsymbol{\mu}$ is a constant vector and both $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\Sigma}$ are constant matrices.

The one-dimensional stochastic differential equation (3.36) that, together with (2.4), drive the liquidity spread follows a Vasicek process, given as

$$d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma} d\tilde{B}_t^P, \tag{2.5}$$

where $\tilde{\kappa}, \tilde{\mu}$ and $\tilde{\sigma}$ are constants and $\tilde{B}_t^P$ is a Brownian motion such that $d\boldsymbol{B}_t^P d\tilde{B}_t^P = \mathbf{0}$.

The model drivers (2.3), (2.4) and (2.5) are stated in continuous time but the observations are given on discrete time steps, thus these equations are being discretized. The discretization step, $dt = h$, coincide with the time between two subsequent observations. The Euler forward method is used for the discretization; when applied to (2.3), (2.4) and (2.5) it gives that

$$\begin{aligned} \frac{q_t - q_{t-h}}{h} &= \rho_0^{\pi} + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_{t-h} + \boldsymbol{\sigma}_q' \frac{\boldsymbol{B}_t^P - \boldsymbol{B}_{t-h}^P}{h} + \sigma_q^{\perp} \frac{B_t^{\perp,P} - B_{t-h}^{\perp,P}}{h} \\ &= \rho_0^{\pi} + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_{t-h} + \boldsymbol{\sigma}_q' \frac{\boldsymbol{\eta}_t}{h} + \sigma_q^{\perp} \frac{\eta_t^{\perp}}{h} \end{aligned} \tag{2.6}$$

$$\frac{\mathbf{x}_t - \mathbf{x}_{t-h}}{h} = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_{t-h}) + \boldsymbol{\Sigma}\frac{\boldsymbol{B}_t^P - \boldsymbol{B}_{t-h}^P}{h} = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_{t-h}) + \boldsymbol{\Sigma}\frac{\boldsymbol{\eta}_t}{h} \tag{2.7}$$

$$\frac{\tilde{x}_t - \tilde{x}_{t-h}}{h} = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_{t-h}) + \tilde{\sigma}\frac{\tilde{B}_t^P - \tilde{B}_{t-h}^P}{h} = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_{t-h}) + \tilde{\sigma}\frac{\tilde{\eta}_t}{h}, \tag{2.8}$$

where, by the definition of Brownian motion, see Section 2.5.1, $\boldsymbol{\eta}_t \sim N(\mathbf{0}, \mathbf{1} \cdot h)$, $\eta_t^{\perp} \sim N(0, h)$ and $\tilde{\eta}_t \sim N(0, h)$.

Equations (2.6), (2.7) and (2.8) can respectively be solved for q_t, $\mathbf{x}_t$ and $\tilde{x}_t$:

$$q_t = \rho_0^{\pi} h + q_{t-h} + \boldsymbol{\rho}_1^{\pi\prime} h \mathbf{x}_{t-h} + \boldsymbol{\sigma}_q^{\prime} \boldsymbol{\eta}_t + \sigma_q^{\perp} \eta_t^{\perp} \tag{2.9}$$

$$\mathbf{x}_t = \boldsymbol{\mathcal{K}} \boldsymbol{\mu} h + (\mathbf{I} - \boldsymbol{\mathcal{K}} h) \mathbf{x}_{t-h} + \boldsymbol{\Sigma} \boldsymbol{\eta}_t \tag{2.10}$$

$$\tilde{x}_t = \tilde{\kappa} \tilde{\mu} h + (1 - \tilde{\kappa} h) \tilde{x}_{t-h} + \tilde{\sigma} \tilde{\eta}_t \tag{2.11}$$

The discretized model dynamics are given by equation (2.9), (2.10) and (2.11). These equations can be written on matrix form, giving the matrix state equation

$$\mathbf{x}_t^{\star} = \mathbf{G}_h + \boldsymbol{\Gamma}_h \mathbf{x}_{t-h}^{\star} + \boldsymbol{\eta}_t^{\mathbf{x}}, \tag{2.12}$$

where

$$\mathbf{G}_h = \begin{bmatrix} \rho_0^{\pi} h \\ \boldsymbol{\mathcal{K}} \boldsymbol{\mu} h \\ \tilde{\kappa} \tilde{\mu} h \end{bmatrix}, \quad \boldsymbol{\Gamma}_h = \begin{bmatrix} 1 & \boldsymbol{\rho}_1^{\pi\prime} h & 0 \\ \mathbf{0} & \mathbf{I} - \boldsymbol{\mathcal{K}} h & \mathbf{0} \\ 0 & \mathbf{0}' & 1 - \tilde{\kappa} h \end{bmatrix}, \quad \boldsymbol{\eta}_t^{\mathbf{x}} = \begin{bmatrix} \boldsymbol{\sigma}_q^{\prime} \boldsymbol{\eta}_t + \sigma_q^{\perp} \eta_t^{\perp} \\ \boldsymbol{\Sigma} \boldsymbol{\eta}_t \\ \tilde{\sigma} \tilde{\eta}_t \end{bmatrix},$$

in which $\boldsymbol{\eta}_t$, $\eta_t^{\perp}$ and $\tilde{\eta}_t$ are independent of each other and where the state vector $\mathbf{x}_t^{\star}$ is defined by

$$\mathbf{x}_t^{\star} = \begin{bmatrix} q_t \\ \mathbf{x}_t \\ \tilde{x}_t \end{bmatrix}.$$

For easing the notation, the $\star$ will be dropped from $\mathbf{x}_t^{\star}$ and the state vector will be written as $\mathbf{x}_t$ from this point until the end of Section 2.4.

2.2.2 Observation Equation

The set of observations are being specified as;

$$\mathbf{Y}_t^N = \begin{bmatrix} y_{t,3m}^N & y_{t,6m}^N & y_{t,1y}^N & y_{t,2y}^N & y_{t,4y}^N & y_{t,7y}^N & y_{t,10y}^N \end{bmatrix}',$$

which are the nominal yields,

$$\mathbf{Y}_t^L = \begin{bmatrix} y_{t,5y}^L & y_{t,7y}^L & y_{t,10y}^L \end{bmatrix}',$$

which are the inflation linked yields as well as q_t, the logarithmic consumer pricing index price level.

The observation data are collected in a observation vector $\mathbf{y}_t$ defined as

$$\mathbf{y}_t = \begin{bmatrix} q_t & \mathbf{Y}_t^N & \mathbf{Y}_t^L \end{bmatrix}'. \tag{2.13}$$

Further, it is assumed that all nominal and inflation linked yields are being observed with error. The observation equation therefore takes the form

$$\mathbf{y}_t = \mathbf{A}_t + \mathbf{B}_t \mathbf{x}_t + \mathbf{e}_t, \tag{2.14}$$

where

$$\mathbf{A}_t = \begin{bmatrix} 0 \\ \mathbf{A}^N \\ \tilde{\mathbf{a}} + \mathbf{A}^L \end{bmatrix}, \quad \mathbf{B}_t = \begin{bmatrix} 1 & \mathbf{0}' & 0 \\ \mathbf{0} & \mathbf{B}^{N\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{L\prime} & \tilde{\mathbf{b}} \end{bmatrix}, \quad \boldsymbol{e}_t = \begin{bmatrix} 0 \\ \mathbf{e}_t^N \\ \mathbf{e}_t^L \end{bmatrix},$$

in which

$$\mathbf{A}^N = \left[\begin{array}{ccccccc} a^N_{3m} & a^N_{6m} & a^N_{1y} & a^N_{2y} & a^N_{4y} & a^N_{7y} & a^N_{10y} \end{array}\right]'$$
$$\mathbf{B}^N = \left[\begin{array}{ccccccc} \mathbf{b}^N_{3m} & \mathbf{b}^N_{6m} & \mathbf{b}^N_{1y} & \mathbf{b}^N_{2y} & \mathbf{b}^N_{4y} & \mathbf{b}^N_{7y} & \mathbf{b}^N_{10y} \end{array}\right]',$$

are the nominal yield loadings on $\mathbf{x}_t$ given by equation (3.10),

$$\mathbf{A}^L = \left[\begin{array}{ccc} a^L_{5y} & a^L_{7y} & a^L_{10y} \end{array}\right]' \quad \text{and} \quad \mathbf{B}^L = \left[\begin{array}{ccc} \mathbf{b}^L_{5y} & \mathbf{b}^L_{7y} & \mathbf{b}^L_{10y} \end{array}\right]',$$

are the inflation linked yield loadings on $\mathbf{x}_t$, given by equation (3.41), and where

$$\tilde{\mathbf{a}} = \left[\begin{array}{ccc} \tilde{a}^N_{5y} & \tilde{a}^N_{7y} & \tilde{a}^N_{10y} \end{array}\right]' \quad \text{and} \quad \tilde{\mathbf{b}} = \left[\begin{array}{ccc} \tilde{b}^N_{5y} & \tilde{b}^N_{7y} & \tilde{b}^N_{10y} \end{array}\right]',$$

collect the inflation linked yield loadings on the independent liquidity factor $\tilde{x}_t$, also given by equation (3.41). Further, it is assumed a structure of identical and independently distributed measurement errors such that

$$\begin{aligned} e^N_{t,\tau_N} &\sim N(0, \delta^2_{N,\tau_N}) \qquad \text{for} \quad \tau_N = 3m,\ 6m, 1y\ , 2y,\ 4y,\ 7y,\ 10y \\ e^L_{t,\tau_L} &\sim N(0, \delta^2_{L,\tau_L}) \qquad \text{for} \quad \tau_L = 5y,\ 7y,\ 10y. \end{aligned}$$

The state space model is now fully defined by equations (2.12) and (2.14). Based on the state space model it is straightforward to implement the Kalman filter for estimating the state variables, see Section 2.3, and estimate the model parameters using maximum likelihood estimation, see Section 2.4. What can be noticed is that the matrices in the observation equation, $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{e}_t$ are time dependent, this is of technical concerns when implementing the Kalman filter. Since the consumer pricing index data is available once a month, meanwhile the nominal and inflation linked yields are provided on a weekly basis, one must let the dimensions of observation equation matrices vary with time. This implies that the Kalman filter will update the state vector on weekly frequency, in accordance with the state equation, but that the consumer pricing index data only will add in to the estimation every fourth iteration.

2.3 Kalman Filter

The Kalman filter is a recursive algorithm that estimates the states of a dynamical system from a set of incomplete noisy observations. The basic recursive Kalman filter used in this thesis is based on the least squares method, fitting a linear model. The Kalman filter algorithm can naturally be divided into two parts; the prediction phase and the update phase. These two phases will be described in Section 2.3.1 and Section 2.3.2, respectively.

2.3.1 Prediction

The prediction phase of the Kalman filter is making a prediction, $\hat{\mathbf{x}}_{t,t-h}$, of the state vector based on the previous state estimate, $\hat{\mathbf{x}}_{t-h,t-h}$. Where $\hat{\mathbf{x}}_{t,t-h}$ is commonly known as the *a prior* state estimate and $\hat{\mathbf{x}}_{t-h,t-h}$ is commonly known as the *a posteriori* state estimate, see Section 2.3.2. Even though the *a prior* state estimate is an estimate of the current state vector, it is not based on the current observation data.

The evolution from a previous state to the current state is described by the state equation (2.12). Given an initial guess of the state factor, $\hat{\mathbf{x}}_0$, the *a priori* state estimate at time $t = h$, is given by

$$\begin{aligned} \hat{\mathbf{x}}_{h,0} :&= E\big[\mathbf{x}_h | \Im_0\big] \\ &= \Big\{(2.12),\ \mathbf{x}_h = \mathbf{G}_h + \mathbf{\Gamma}_h \mathbf{x}_0 + \boldsymbol{\eta}^x_0\Big\} \\ &= E\big[\mathbf{G}_h + \mathbf{\Gamma}_h \mathbf{x}_0 + \boldsymbol{\eta}^x_0 | \Im_0\big] \\ &= \mathbf{G}_h + \mathbf{\Gamma}_h \hat{\mathbf{x}}_0. \end{aligned}$$

More generally the *a priori* state estimate, $\hat{\mathbf{x}}_{t,t-h}$, at time step t is given by

$$\begin{aligned}
\hat{\mathbf{x}}_{t,t-h} :&= E\big[\mathbf{x}_t|\Im_{t-h}\big] \\
&= \Big\{(2.12),\ \mathbf{x}_t = \mathbf{G}_h + \mathbf{\Gamma}_h\mathbf{x}_{t-h} + \boldsymbol{\eta}_t^{\mathbf{x}}\Big\} \\
&= E\big[\mathbf{G}_h + \mathbf{\Gamma}_h\mathbf{x}_{t-h} + \boldsymbol{\eta}_t^{\mathbf{x}}|\Im_{t-h}\big] \\
&= \mathbf{G}_h + \mathbf{\Gamma}_h\hat{\mathbf{x}}_{t-h,t-h},
\end{aligned} \tag{2.15}$$

where $\Im_{t-h}$ denotes all given information prior to this time step. The corresponding variance-covariance matrix, $\mathbf{Q}_{t,t-h}$, of the *a priori* state estimate is given by

$$\begin{aligned}
\mathbf{Q}_{t,t-h} :&= Var(\mathbf{x}_t|\Im_{t-h}) \\
&= E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big] \\
&= \Big\{(2.12),\ \mathbf{x}_t = \mathbf{G}_h + \mathbf{\Gamma}_h\mathbf{x}_{t-h} + \boldsymbol{\eta}_t^{\mathbf{x}};\quad (2.15),\ \hat{\mathbf{x}}_{t,t-h} = \mathbf{G}_h + \mathbf{\Gamma}_h\hat{\mathbf{x}}_{t-h,t-h}\Big\} \\
&= E\big[(\mathbf{G}_h + \mathbf{\Gamma}_h\mathbf{x}_{t-h} + \boldsymbol{\eta}_t^{\mathbf{x}} - \mathbf{G}_h - \mathbf{\Gamma}_h\hat{\mathbf{x}}_{t-h,t-h}) \\
&\qquad \cdot (\mathbf{G}_h + \mathbf{\Gamma}_h\mathbf{x}_{t-h} + \boldsymbol{\eta}_t^{\mathbf{x}} - \mathbf{G}_h - \mathbf{\Gamma}_h\hat{\mathbf{x}}_{t-h,t-h})'|\Im_{t-h}\big] \\
&= E\big[\big(\mathbf{\Gamma}_h(\mathbf{x}_{t-h} - \hat{\mathbf{x}}_{t-h,t-h}) + \boldsymbol{\eta}_t^{\mathbf{x}}\big)\big(\mathbf{\Gamma}_h(\mathbf{x}_{t-h} - \hat{\mathbf{x}}_{t-h,t-h}) + \boldsymbol{\eta}_t^{\mathbf{x}}\big)'|\Im_{t-h}\big] \\
&= \mathbf{\Gamma}_h E\big[(\mathbf{x}_{t-h} - \hat{\mathbf{x}}_{t-h,t-h})(\mathbf{x}_{t-h} - \hat{\mathbf{x}}_{t-h,t-h})'|\Im_{t-h}\big]\mathbf{\Gamma}_h' + E\big[\boldsymbol{\eta}_t^{\mathbf{x}}\boldsymbol{\eta}_t^{\mathbf{x}\prime}\big] \\
&= \mathbf{\Gamma}_h\mathbf{Q}_{t-h,t-h}\mathbf{\Gamma}_h{}' + \mathbf{\Omega}_{t-h}^{x},
\end{aligned} \tag{2.16}$$

where

$$\mathbf{\Omega}_{t-h}^{x} = E\big[\boldsymbol{\eta}_t^{\mathbf{x}}\boldsymbol{\eta}_t^{\mathbf{x}\prime}\big]$$

and

$$\mathbf{Q}_{t-h,t-h} = E\big[(\mathbf{x}_{t-h} - \hat{\mathbf{x}}_{t-h,t-h})(\mathbf{x}_{t-h} - \hat{\mathbf{x}}_{t-h,t-h})'|\Im_{t-h}\big]$$

is identified as the variance-covariance matrix of the *a posteriori* state estimate, see Section 2.3.2.

2.3.2 Update

In the update phase the current *a priori* state estimate, $\hat{\mathbf{x}}_{t,t-h}$, is combined with the current observation data to improve the state estimate. This refined state estimate, $\hat{\mathbf{x}}_{t,t}$, is referred to as the *a posteriori* estimate of the state vector with variance-covariance matrix $\mathbf{Q}_{t,t}$. This variance-covariance matrix is part of the expression for the next *a priori* variance-covariance matrix (2.16).

Then the problem boils down to finding the optimal *a posteriori* estimate of the state. Having the particular set-up with both $\mathbf{x}_t$ and $\mathbf{y}_t$ being linear and Gaussian, it will be shown that the best *a posteriori* estimate of the state vector is given by a regression function of $\mathbf{x}_t$ on $\mathbf{y}_t$. In this case the regression function is given as the conditional expectation of $\mathbf{x}_t$ given $\mathbf{y}_t = \mathbf{y}_t^{obs}$ and all the information available up until time $t-h$

$$\hat{\mathbf{x}}_{t,t} = E\big[\mathbf{x}_t|\mathbf{y}_t = \mathbf{y}_t^{obs}, \Im_{t-h}\big],$$

with variance-covariance matrix

$$\mathbf{Q}_{t,t} = Var\big[\mathbf{x}_t|\mathbf{y}_t = \mathbf{y}_t^{obs}, \Im_{t-h}\big].$$

This result requires motivation from probability theory. The following theorems, definitions and assumptions are based on the theory from the book *An Intermediate Course in Probability* (Gut 2009, [7]).

Optimality for Jointly Gaussian Variables

Let x and y be jointly distributed random variables and let

$$h(y^\circ) = E\big[x|y = y^\circ\big],$$

where the function h is defined as the regression function x on y.

A predictor for x based on y is a function $d(y)$. Predictors are used to predict and the prediction error is given by the random variable

$$x - d(y).$$

Furthermore, the expected quadratic prediction error is defined as

$$E\big[x - d(y)\big]^2,$$

in which; if d_1 and d_2 are predictors, d_1 is better than d_2 if $E\big[x - d_1(y)\big]^2 < E\big[x - d_2(y)\big]^2$.

Theorem 2.1 *Suppose that* $E\big[x^2\big] < \infty$*. Then* $h(y^\circ) = E\big[x|y = y^\circ\big]$ *is the best predictor of* x *based on* y*.*

Sometimes it is hard to determine regression functions explicitly and in such cases one might be satisfied with the best linear predictor. This means that one wishes to minimize $E\big[x-(a+by)\big]^2$ as a function of a and b, which leads to the well-known method of least squares. The solution of this problem is given by the following theorem:

Theorem 2.2 *Suppose that* $E\big[x^2\big] < \infty$ *and* $E\big[y^2\big] < \infty$*. Set* $\mu_x = E\big[x\big]$, $\mu_y = E\big[y\big]$, $\sigma_x^2 = Var\big[x\big]$, $\sigma_y^2 = Var\big[y\big]$, $\sigma_{xy} = Cov\big[x, y\big]$ *and* $\rho = \sigma_{xy}/\sigma_x\sigma_y$*. The best linear predictor of* x *based on* y *is*

$$L(y) = \alpha + \beta y,$$

where

$$\alpha = \mu_x - \frac{\sigma_{yx}}{\sigma_y^2}\mu_y = \mu_x - \rho\frac{\sigma_x}{\sigma_y}\mu_y \quad \text{and} \quad \beta = \frac{\sigma_{yx}}{\sigma_y^2} = \rho\frac{\sigma_x}{\sigma_y}.$$

Thus the best linear predictor, given by Theorem 2.2, becomes

$$L(y) = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y). \tag{2.17}$$

Definition: Regression Line

The line $x^\circ = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y^\circ - \mu_y)$ *is called the regression line* x *on* y*. The slope,* $\rho\frac{\sigma_x}{\sigma_y}$*, of the line is called the regression coefficient.*

The expected quadratic prediction error of the best linear predictor of x based on y is given by the following theorem:

Theorem 2.3 $E\big[x - L(y)\big]^2 = \sigma_x^2(1 - \rho^2)$.

Next consider the case of x and y being jointly normal such that $(x, y)' \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$, where $E[x] = \mu_x$, $E[y] = \mu_y$, $Var[x] = \sigma_x^2$, $Var[y] = \sigma_y^2$, $Corr[x, y] = \rho$, where $|\rho| < 1$. Then the conditional density function for x given y is

$$
\begin{aligned}
f_{x|y=y^\circ}(x^\circ) &= \frac{f_{x,y}(x^\circ, y^\circ)}{f_y(y^\circ)} \\
&= \frac{\frac{1}{2\pi\sigma_y\sigma_x\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}\left(\left(\frac{y^\circ-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(y^\circ-\mu_y)(x^\circ-\mu_x)}{\sigma_y\sigma_x} + \left(\frac{x^\circ-\mu_x}{\sigma_x}\right)^2\right)\right\}}{\frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{ -\frac{1}{2}\left(\frac{y^\circ-\mu_y}{\sigma_y}\right)^2\right\}} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}\left(\left(\frac{y^\circ-\mu_y}{\sigma_y}\right)^2\rho^2 - 2\rho\frac{(y^\circ-\mu_y)(x^\circ-\mu_x)}{\sigma_y\sigma_x} + \left(\frac{x^\circ-\mu_x}{\sigma_x}\right)^2\right)\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2\sigma_x^2(1-\rho^2)}\left(x^\circ - \mu_x - \rho\frac{\sigma_x}{\sigma_y}(y^\circ - \mu_y)\right)^2\right\}. \qquad (2.18)
\end{aligned}
$$

This density can be recognized as the density of a normal distribution with mean $\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y^\circ - \mu_y)$ and variance $\sigma_x^2(1-\rho^2)$, thus it follows that

$$E[x|y = y^\circ] = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y^\circ - \mu_y), \qquad (2.19)$$

$$Var[x|y = y^\circ] = \sigma_x^2(1-\rho^2). \qquad (2.20)$$

Now the explicitly given regression function, i.e the best predictor of x based on y as given by Theorem 2.1, in (2.19) is linear and coincides with the regression line as given by the definition on the last page. Further, the conditional variance in (2.20) coincides with the quadratic prediction error of the best linear predictor, Theorem 2.3.

Thus, in particular, if (x, y) have a joint Gaussian distribution, it turns out that the best linear predictor is, in fact, also the best predictor.

The Kalman filter uses the least squares method for the *a posteriori* estimate of the state. The least squares solution is provided as the best linear predictor of x based on y, given by Theorem 2.2. Further, if the innovations of the observation equation (2.14) and the state equation (2.12) are Gaussian white noise, the *a posteriori* estimate coincides with the regression function of x based on y and therefore it is the best estimate possible, as is motivated above.

The derivation of optimality was given for the bi-variate case, but the results hold for multi-variate normal random variables of higher dimensions as well. In the case of having two multi-variate normal distributed variables $\mathbf{x}$ and $\mathbf{y}$, their joint distribution is given by

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu_x} \\ \boldsymbol{\mu_y} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{xx}} & \boldsymbol{\Sigma}_{\mathbf{xy}} \\ \boldsymbol{\Sigma}_{\mathbf{yx}} & \boldsymbol{\Sigma}_{\mathbf{yy}} \end{bmatrix}\right), \qquad (2.21)$$

where $\boldsymbol{\mu_x} = E[\mathbf{x}]$, $\boldsymbol{\Sigma}_{\mathbf{xx}} = Var[\mathbf{x}]$, $\boldsymbol{\mu_y} = E[\mathbf{y}]$, $\boldsymbol{\Sigma}_{\mathbf{yy}} = Var[\mathbf{y}]$, $\boldsymbol{\Sigma}_{\mathbf{xy}} = Cov[\mathbf{x}, \mathbf{y}]$ and $\boldsymbol{\Sigma}_{\mathbf{yx}} = Cov[\mathbf{y}, \mathbf{x}]$.

The distribution of $\mathbf{x}$ conditioned on $\mathbf{y} = \mathbf{y}^\circ$ is also multivariate normal

$$(\mathbf{x}|\mathbf{y} = \mathbf{y}^\circ) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\mu} = E[\mathbf{x}|\mathbf{y} = \mathbf{y}^\circ] = \boldsymbol{\mu_x} + \boldsymbol{\Sigma_{xy}}\boldsymbol{\Sigma_{yy}}^{-1}(\mathbf{y}^\circ - \boldsymbol{\mu_y}) \qquad (2.22)$$

and

$$\boldsymbol{\Sigma} = Var[\mathbf{x}|\mathbf{y} = \mathbf{y}^\circ] = \boldsymbol{\Sigma_{xx}} - \boldsymbol{\Sigma_{xy}}\boldsymbol{\Sigma_{yy}}^{-1}\boldsymbol{\Sigma_{yx}}. \qquad (2.23)$$

Thus to perform the update step of the Kalman filter one need to calculate the joint distribution of $\mathbf{x}_t$ and $\mathbf{y}_t$, conditional on all information available up until time $t-h$.

Given any *a priori* estimate of $\mathbf{x}_t$ one can compute a forecast for the observable variables based on all information available up until time $t-h$

$$\begin{aligned}\hat{\mathbf{y}}_{t,t-h} &= E\big[\mathbf{y}_t|\Im_{t-h}\big]\\ &= \Big\{(2.14),\ \mathbf{y}_t = \mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t\Big\}\\ &= E\big[\mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t|\Im_{t-h}\big]\\ &= \mathbf{A}_t + \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h}.\end{aligned} \tag{2.24}$$

The variance-covariance matrix of the observable variables based on all information available up until time $t-h$, is given by

$$\begin{aligned}\mathbf{V}_{t,t-h} &= Var\big[\mathbf{y}_t|\Im_{t-h}\big]\\ &= E\big[(\mathbf{y}_t - \hat{\mathbf{y}}_{t,t-h})(\mathbf{y}_t - \hat{\mathbf{y}}_{t,t-h})'|\Im_{t-h}\big]\\ &= \Big\{(2.14),\ \mathbf{y}_t = \mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t;\quad (2.24),\ \hat{\mathbf{y}}_{t,t-h} = \mathbf{A}_t + \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h}\Big\}\\ &= E\big[(\mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t - \mathbf{A}_t - \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h})(\mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t - \mathbf{A}_t - \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\\ &= E\big[(\mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t - \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h})(\mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t - \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\\ &= \mathbf{B}_t E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\mathbf{B}_t' + E\big[\mathbf{e}_t\mathbf{e}_t'\big]\\ &= \mathbf{B}\mathbf{Q}_{t,t-h}\mathbf{B}' + \mathbf{\Omega}_t^e\end{aligned} \tag{2.25}$$

where

$$\mathbf{\Omega}_t^e = E\big[\mathbf{e}_t\mathbf{e}_t'\big]$$

and

$$\mathbf{Q}_{t,t-h} = E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]$$

is identified from (2.16).

Next, the variance-covariance matrix between the state variables, $\mathbf{x}_t$, and the observable variables, $\mathbf{y}_t$, based on all information available up until time $t-h$, takes the form

$$\begin{aligned}\mathbf{V}^{\mathbf{xy}}_{t,t-h} &= Cov\big[\mathbf{x}_t, \mathbf{y}_t|\Im_{t-h}\big]\\ &= E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{y}_t - \hat{\mathbf{y}}_{t,t-h})'|\Im_{t-h}\big]\\ &= \Big\{(2.14),\ \mathbf{y}_t = \mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t;\quad (2.24),\ \hat{\mathbf{y}}_{t,t-h} = \mathbf{A}_t + \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h}\Big\}\\ &= E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{A}_t + \mathbf{B}_t\mathbf{x}_t + \mathbf{e}_t - \mathbf{A}_t - \mathbf{B}_t\hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\\ &= E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\mathbf{B}_t'\\ &= \mathbf{Q}_{t,t-h}\mathbf{B}'.\end{aligned} \tag{2.26}$$

Conversely the variance-covariance matrix between the observable variables, $\mathbf{y}_t$, and the state variables, $\mathbf{x}_t$, based on all information available up until time $t-h$, takes the form

$$\begin{aligned}\mathbf{V}^{\mathbf{yx}}_{t,t-h} &= Cov\big[\mathbf{y}_t, \mathbf{x}_t|\Im_{t-h}\big]\\ &= E\big[(\mathbf{y}_t - \hat{\mathbf{y}}_{t,t-h})(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\\ &= \mathbf{B}E\big[(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})(\mathbf{x}_t - \hat{\mathbf{x}}_{t,t-h})'|\Im_{t-h}\big]\\ &= \mathbf{B}\mathbf{Q}_{t,t-h}.\end{aligned} \tag{2.27}$$

The joint distribution for $\mathbf{x}_t$ and $\mathbf{y}_t$ at time t conditional on all information available up until time $t-h$ therefore is given by

$$\left[\begin{array}{c}\mathbf{x}_t\\ \mathbf{y}_t\end{array}\right]|\Im_{t-h} \sim N\left(\left[\begin{array}{c}\hat{\mathbf{x}}_{t,t-h}\\ \hat{\mathbf{y}}_{t,t-h}\end{array}\right],\left[\begin{array}{cc}\mathbf{Q}_{t,t-h} & \mathbf{V}^{\mathbf{xy}}_{t,t-h}\\ \mathbf{V}^{\mathbf{yx}}_{t,t-h} & \mathbf{V}_{t,t-h}\end{array}\right]\right). \tag{2.28}$$

Then, the distribution of $\mathbf{x}_t$ conditioned on $\mathbf{y}_t = \mathbf{y}_t^{obs}$ and on all information available up until time $t-h$ is also normally distributed

$$\left(\mathbf{x}_t|\mathbf{y}_t = \mathbf{y}_t^{obs}, \Im_{t-h}\right) \sim N\left(\hat{\mathbf{x}}_{t,t}, \mathbf{Q}_{t,t}\right),$$

where according to (2.22) and (2.23)

$$\hat{\mathbf{x}}_{t,t} = E\left[\mathbf{x}_t|\mathbf{y}_t = \mathbf{y}_t^{obs}, \Im_{t-h}\right] = \hat{\mathbf{x}}_{t,t-h} + \mathbf{V}^{\mathbf{xy}}_{t,t-h}\mathbf{V}^{-1}_{t,t-h}(\mathbf{y}_t^{obs} - \hat{\mathbf{y}}_{t,t-h}) \tag{2.29}$$

and

$$\mathbf{Q}_{t,t} = Var\left[\mathbf{x}_t|\mathbf{y}_t = \mathbf{y}_t^{obs}, \Im_{t-h}\right] = \mathbf{Q}_{t,t-h} - \mathbf{V}^{\mathbf{xy}}_{t,t-h}\mathbf{V}^{-1}_{t,t-h}\mathbf{V}^{\mathbf{yx}}_{t,t-h}. \tag{2.30}$$

The variance-covariance matrix of the updated *a posteriori* state vector, $\mathbf{Q}_{t,t}$, will be smaller than the variance-covariance matrix of the *a priori* estimate of the state vector, $\mathbf{Q}_{t,t-h}$, due to the new information added through the observation $\mathbf{y}_t^{obs}$.

To sum up, in this section one iteration of the Kalman filter was derived, whereby the *a priori* estimate of the state vector is given by (2.15) and the *a posteriori* estimate of the state vector is given by (2.29).

Noticeable is that, when performing an iteration of the Kalman filter as well as estimating the state variables at time t, only the last *a posteriori* estimate, $\hat{\mathbf{x}}_{t-h,t-h}$, of the state variables and its variance-covariance matrix, $\mathbf{Q}_{t-h,t-h}$, are needed. As mentioned in Section 2.2.2, the dimension of the matrices in the observation equation (2.14) can vary. Thus the Kalman filter can run, estimating the state variables, only based on the *a posteriori* estimate of the state variables and its variance-covariance matrix. Although, for that case, without any new information, $\mathbf{y}_t^{obs}$, added to the system, the uncertainty of the state estimates will grow.

2.4 Log-likelihood Function

This section successively builds up the log-likelihood function used for the parameter estimation. Beginning from the basic one-dimensional definition of the likelihood function, naturally followed by a description of the maximum likelihood method. Thereafter this is expanded to the multivariate case and the Kalman filter specific logarithmic likelihood function is presented.

2.4.1 Likelihood function

The likelihood function for a set of parameters $\theta_1, \theta_2, ..., \theta_n$, given the outcome y^{obs} of a stochastic variable y for a continuous distribution, is equal to the probability density function of y evaluated in y^{obs}, given the parameter set $\theta_1, \theta_2, ..., \theta_n$,

$$\mathcal{L}\left(\theta_1, \theta_2, ..., \theta_n|y^{obs}\right) = f_y\left(y^{obs}|\theta_1, \theta_2, ..., \theta_n\right). \tag{2.31}$$

Further for a sample of size T/h with outcomes $y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}$ of independent and identically distributed stochastic variables $y_h, y_{2h}, ..., y_T$; the likelihood function in $\theta_1, \theta_2, ..., \theta_n$ given the outcomes $y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}$ equals the joint probability density function evaluated in $y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}$ given the parameters $\theta_1, \theta_2, ..., \theta_n$

$$\mathcal{L}\left(\theta_1, \theta_2, ..., \theta_n|y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}\right) = f_{y_h, y_{2h}, ..., y_T}\left(y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}|\theta_1, \theta_2, ..., \theta_n\right). \tag{2.32}$$

If the sample comes from independent and identically distributed stochastic variables, the joint probability density function can be written as products of the individual probability density functions

$$\begin{aligned}
&f_{y_h,y_{2h},...,y_T}\big(y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}|\theta_1, \theta_2, ..., \theta_n\big) = \\
&= f_{y_h}\big(y_h^{obs}|\theta_1, \theta_2, ..., \theta_n\big) \cdot f_{y_{2h}}\big(y_{2h}^{obs}|\theta_1, \theta_2, ..., \theta_n\big) \cdot ... \cdot f_{y_T}\big(y_T^{obs}|\theta_1, \theta_2, ..., \theta_n\big) \\
&= \prod_{t=h}^{T} f_{y_t}\big(y_t^{obs}|\theta_1, \theta_2, ..., \theta_n\big). \qquad (2.33)
\end{aligned}$$

Thus, inserting (2.33) into (2.32) gives that

$$\mathcal{L}\big(\theta_1, \theta_2, ..., \theta_n|y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}\big) = \prod_{t=h}^{T} f_{y_t}\big(y_t^{obs}|\theta_1, \theta_2, ..., \theta_n\big). \qquad (2.34)$$

2.4.2 Maximum Likelihood Estimation

Maximum-likelihood estimation is an approach to estimate parameters in a probability distribution given a sample of the distribution in fact. If the sample $y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}$ is assumed to be independent and identically distributed, the maximum likelihood estimation is performed by maximizing the likelihood function (2.34) with respect to the unknown parameters $\theta_1, \theta_2, ..., \theta_n$

$$max_{\theta_1,\theta_2,...,\theta_n}\mathcal{L}\big(\theta_1, \theta_2, ..., \theta_n|y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}\big) = \prod_{t=h}^{T} f_{y_t}\big(y_t^{obs}|\theta_1^*, \theta_2^*, ..., \theta_n^*\big),$$

where $\theta_1^*, \theta_2^*, ..., \theta_n^*$ is the optimal parameter-set. For practical reasons it is often more convenient to take the logarithm of the likelihood function, (2.34), before maximizing. The log-likelihood function is given as

$$\begin{aligned}
&log\Big(\mathcal{L}\big(\theta_1, \theta_2, ..., \theta_n|y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}\big)\Big) = \\
&= log\Big(f_{y_h}\big(y_h^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big) + log\Big(f_{y_{2h}}\big(y_{2h}^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big) + ... + log\Big(f_{y_T}\big(y_T^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big) \\
&= \sum_{t=h}^{T} log\Big(f_{y_t}\big(y_t^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big). \qquad (2.35)
\end{aligned}$$

Since the logarithmic function is a monotonically increasing function, defined on the open interval $(0, \infty)$ and the probability density functions $f_{y_t}(y_t^{obs}|\theta_1, \theta_2, ..., \theta_n)$ only take positive values, the following conclusion can be drawn; the optimal parameter-set that maximizes the likelihood function (2.32) will coincide with the parameter-set maximizing the logarithmic likelihood function (2.35).

The expression of the logarithmic log likelihood function in (2.35) is a sum of T/h terms containing the logarithm of the individual probability density functions. For the analytic approach, having an expression of sums eases the work when calculating derivatives. Further, the density function often consists of an exponential expression that gets simplified when taking the logarithm.

The Multivariate Normal Case

Assume that the observations $y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}$ are outcomes from the $N(\mu, \sigma)$-distributed stochastic variables $y_h, y_{2h}, ..., y_T$. Then the unknown parameters μ (expected value) and σ (standard deviation) are estimated by inserting $y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}$ into (2.35) and maximize with respect to μ and σ

$$\begin{aligned}
&max_{\mu,\sigma} log\Big(\mathcal{L}\big(\mu, \sigma | y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}\big)\Big) = \\
&= log\Big(f_{y_h}\big(y_h^{obs}|\mu^*, \sigma^*\big)\Big) + log\Big(f_{y_{2h}}\big(y_{2h}^{obs}|\mu^*, \sigma^*\big)\Big) + ... + log\Big(f_{y_T}\big(y_T^{obs}|\mu^*, \sigma^*\big)\Big) \\
&= \sum_{t=h}^{T} log\Big(f_{y_t}\big(y_t^{obs}|\mu^*, \sigma^*\big)\Big),
\end{aligned} \tag{2.36}$$

where μ^* and σ^* are the optimal parameters and the individual normal probability density functions are given by the one-dimensional Gaussian probability density function

$$f_{y_t}\big(y_t^{obs}|\mu, \sigma\big) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_t^{obs}-\mu)^2}{2\sigma^2}} \qquad \text{for} \quad t = h, 2h, ..., T. \tag{2.37}$$

Inserting (2.37) into (2.36) gives

$$\begin{aligned}
&max_{\mu,\sigma} log\Big(\mathcal{L}\big(\mu, \sigma | y_h^{obs}, y_{2h}^{obs}, ..., y_T^{obs}\big)\Big) = \\
&= log\left(\frac{1}{\sigma^*\sqrt{2\pi}} e^{-\frac{(y_h^{obs}-\mu^*)^2}{2\sigma^{*2}}}\right) + log\left(\frac{1}{\sigma^*\sqrt{2\pi}} e^{-\frac{(y_{2h}^{obs}-\mu^*)^2}{2\sigma^{*2}}}\right) + ... + log\left(\frac{1}{\sigma^*\sqrt{2\pi}} e^{-\frac{(y_T^{obs}-\mu^*)^2}{2\sigma^{*2}}}\right) \\
&= \frac{T}{h} log\left(\frac{1}{\sigma^*\sqrt{2\pi}}\right) - \sum_{t=h}^{T} \frac{(y_t^{obs}-\mu^*)^2}{2\sigma^{*2}} \\
&= -\frac{T}{2h} log(2\pi) - \frac{T}{2h} log(\sigma^{*2}) - \frac{1}{2\sigma^{*2}} \sum_{t=h}^{T} (y_t^{obs}-\mu^*)^2.
\end{aligned}$$

In the case of having a multivariate sample of independent and identically distributed outcome-vectors the described method can easily be expanded, where the multivariate log-likelihood function is given as

$$\begin{aligned}
&log\Big(\mathcal{L}\big(\theta_1, \theta_2, ..., \theta_n | \mathbf{y}_h^{obs}, \mathbf{y}_{2h}^{obs}, ..., \mathbf{y}_T^{obs}\big)\Big) = \\
&= log\Big(f_{\mathbf{y}_h}\big(\mathbf{y}_h^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big) + log\Big(f_{\mathbf{y}_{2h}}\big(\mathbf{y}_{2h}^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big) + ... + log\Big(f_{\mathbf{y}_T}\big(\mathbf{y}_T^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big) \\
&= \sum_{t=h}^{T} log\Big(f_{\mathbf{y}_t}\big(\mathbf{y}_t^{obs}|\theta_1, \theta_2, ..., \theta_n\big)\Big),
\end{aligned} \tag{2.38}$$

where $\mathbf{y}_t^{obs} = [y_{t,1}^{obs}, y_{t,2}^{obs}, ..., y_{t,k}^{obs}]'$ is an outcome-vector of the k-dimensional stochastic vector $\mathbf{y}_t = [y_{t,1}, y_{t,2}, ..., y_{t,k}]'$. In the multivariate normal case, the probability density functions are given as

$$f_{\mathbf{y}_t}\big(\mathbf{y}_t^{obs}|\theta_1, \theta_2, ..., \theta_n\big) = \frac{1}{\sqrt{(2\pi)^k |\mathbf{\Sigma}|}} e^{-\frac{1}{2}(\mathbf{y}_t^{obs}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{y}_t^{obs}-\boldsymbol{\mu})} \qquad \text{for} \quad t = h, 2h, ..., T, \tag{2.39}$$

where $\boldsymbol{\mu}$ is the k-dimensional mean vector of $\mathbf{y}_t$ and $\mathbf{\Sigma}$ is the $k \times k$ variance-covariance matrix of $\mathbf{y}_t$. Inserting (2.39) into (2.38) gives

$$
\begin{aligned}
&log\Big(\mathcal{L}(\theta_1,\theta_2,...,\theta_n|\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs})\Big) = \\
&= log\left(\frac{1}{\sqrt{(2\pi)^k|\mathbf{\Sigma}|}}e^{-\frac{1}{2}(\mathbf{y}_h^{obs}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{y}_h^{obs}-\boldsymbol{\mu})}\right) \\
&\quad + log\left(\frac{1}{\sqrt{(2\pi)^k|\mathbf{\Sigma}|}}e^{-\frac{1}{2}(\mathbf{y}_{2h}^{obs}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{y}_{2h}^{obs}-\boldsymbol{\mu})}\right) + ... + log\left(\frac{1}{\sqrt{(2\pi)^k|\mathbf{\Sigma}|}}e^{-\frac{1}{2}(\mathbf{y}_T^{obs}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{y}_T^{obs}-\boldsymbol{\mu})}\right) \\
&= \frac{T}{h}log\left(\frac{1}{\sqrt{(2\pi)^k|\mathbf{\Sigma}|}}\right) + \sum_{t=h}^{T} log\Big(e^{-\frac{1}{2}(\mathbf{y}_t^{obs}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{y}_t^{obs}-\boldsymbol{\mu})}\Big) \\
&= -\frac{Tk}{2h}log(2\pi) - \frac{T}{2h}log(|\mathbf{\Sigma}|) - \frac{1}{2}\sum_{t=h}^{T}(\mathbf{y}_t^{obs}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{y}_t^{obs}-\boldsymbol{\mu}). \qquad (2.40)
\end{aligned}
$$

2.4.3 Log-likelihood Function and The Kalman filter

The implementation of the maximum likelihood estimation, when having estimates from a Kalman filter, deviates from the above described procedure in the sense that the observation vectors are neither independently nor identically distributed. The one-dimensional likelihood function, (2.32), expanded to the multivariate case, as a function of the parameters $\theta_1, \theta_2, ..., \theta_n$ given the multivariate observation vectors $\mathbf{y}_h^{obs}, \mathbf{y}_{2h}^{obs}, ..., \mathbf{y}_T^{obs}$, is given as

$$
\mathcal{L}(\theta_1,\theta_2,...,\theta_n|\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs}) = f_{\mathbf{y}_h,\mathbf{y}_{2h},...,\mathbf{y}_T}(\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs}|\theta_1,\theta_2,...,\theta_n). \qquad (2.41)
$$

Taking the logarithm of (2.41) yields

$$
log\Big(\mathcal{L}(\theta_1,\theta_2,...,\theta_n|\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs})\Big) = log\Big(f_{\mathbf{y}_h,\mathbf{y}_{2h},...,\mathbf{y}_T}(\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs}|\theta_1,\theta_2,...,\theta_n)\Big). \qquad (2.42)
$$

In the Kalman filter the observation vectors $\mathbf{y}_t^{obs}$ are dependent on their predecessors. Therefore it is not straight forward to rewrite expression (2.42) as a sum of the individual logarithmic density functions, as in the independent case, (2.38). Instead one can factorize the joint conditional probability density function by repeated use of the multiplication rule for conditional probabilities

$$
\begin{aligned}
&f_{\mathbf{y}_h,\mathbf{y}_{2h},...,\mathbf{y}_T}(\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs}|\theta_1,\theta_2,...,\theta_n) = \\
&= f_{\mathbf{y}_T}(\mathbf{y}_T^{obs}|\mathbf{y}_{T-h}^{obs},\mathbf{y}_{T-2h}^{obs},...,\mathbf{y}_h^{obs},\theta_1,\theta_2,...,\theta_n) \\
&\quad \cdot f_{\mathbf{y_{T-h}}}(\mathbf{y}_{T-h}^{obs}|\mathbf{y}_{T-2h}^{obs},\mathbf{y}_{T-3h}^{obs},...,\mathbf{y}_h^{obs},\theta_1,\theta_2,...,\theta_n) \cdot ... \cdot f_{\mathbf{y}_h}(\mathbf{y}_h^{obs}|\theta_1,\theta_2,...,\theta_n) \\
&= \prod_{t=2h}^{T} f_{\mathbf{y}_t}(\mathbf{y}_t^{obs}|\mathbf{y}_{t-h}^{obs},\mathbf{y}_{t-2h}^{obs},...,\mathbf{y}_h^{obs},\theta_1,\theta_2,...,\theta_n) \cdot f_{\mathbf{y}_h}(\mathbf{y}_h^{obs}|\theta_1,\theta_2,...,\theta_n). \qquad (2.43)
\end{aligned}
$$

When inserting notation, in line with Section 2.3, where $\Im_t$ represents all information available from time h up until time t, (2.43) becomes

$$
\begin{aligned}
&f_{\mathbf{y}_h,\mathbf{y}_{2h},...,\mathbf{y}_T}(\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs}|\theta_1,\theta_2,...,\theta_n) = \\
&= f_{\mathbf{y}_T}(\mathbf{y}_T^{obs}|\Im_{T-h},\theta_1,\theta_2,...,\theta_n) \cdot f_{\mathbf{y_{T-h}}}(\mathbf{y}_{T-h}^{obs}|\Im_{T-2h},\theta_1,\theta_2,...,\theta_n) \cdot ... \cdot f_{\mathbf{y}_h}(\mathbf{y}_h^{obs}|\Im_0,\theta_1,\theta_2,...,\theta_n) \\
&= \prod_{t=h}^{T} f_{\mathbf{y}_t}(\mathbf{y}_t^{obs}|\Im_{t-h},\theta_1,\theta_2,...,\theta_n). \qquad (2.44)
\end{aligned}
$$

Now inserting (2.44) into (2.42) gives the multivariate log-likelihood function for conditionally dependent observations

$$\begin{aligned} &log\Big(\mathcal{L}(\theta_1,\theta_2,...,\theta_n|\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs})\Big) = \\ &= log\Big(f_{\mathbf{y}_T}(\mathbf{y}_T^{obs}|\Im_{T-h},\theta_1,\theta_2,...,\theta_n)\Big) \\ &\quad + log\Big(f_{\mathbf{y_{T-h}}}(\mathbf{y}_{T-h}^{obs}|\Im_{T-2h},\theta_1,\theta_2,...,\theta_n)\Big) + ... + log\Big(f_{\mathbf{y}_h}(\mathbf{y}_h^{obs}|\theta_1,\theta_2,...,\theta_n)\Big) \\ &= \sum_{t=h}^{T} log\Big(f_{\mathbf{y}_t}(\mathbf{y}_t^{obs}|\Im_{t-h},\theta_1,\theta_2,...,\theta_n)\Big). \end{aligned} \tag{2.45}$$

Let $\mathbf{y}_t^{obs}$ be multivariate observations of the conditionally dependent stochastic Gaussian sequence $\mathbf{y}_t$. Then each of the conditional probability density functions in (2.45) are Gaussian with conditional mean vectors

$$\boldsymbol{\mu}_{t,t-h} = E\big[\mathbf{y}_t|\Im_{t-h}\big]$$

and conditional variance-covariance matrices

$$\boldsymbol{\Sigma}_{t,t-h} = Cov\big[\mathbf{y}_t|\Im_{t-h}\big].$$

Then (2.45) becomes

$$\begin{aligned} &log\Big(\mathcal{L}(\theta_1,\theta_2,...,\theta_n|\mathbf{y}_h^{obs},\mathbf{y}_{2h}^{obs},...,\mathbf{y}_T^{obs})\Big) = \\ &= \sum_{t=h}^{T} log\left(\frac{1}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma}_{t,t-h}|}} e^{-\frac{1}{2}(\mathbf{y}_t^{obs}-\boldsymbol{\mu}_{t,t-h})'\boldsymbol{\Sigma}_{t,t-h}^{-1}(\mathbf{y}_t^{obs}-\boldsymbol{\mu}_{t,t-h})}\right). \end{aligned} \tag{2.46}$$

For the Kalman filter, when conditioning on all previous information, the means of the observable variables, $\mathbf{y}_{t,t-h}$, are given by (2.24) and the corresponding variance-covariance matrices are given by (2.25). Therefore substituting $\boldsymbol{\Sigma}_{t,t-h}$ with $\mathbf{V}_{t,t-h}$ and $\boldsymbol{\mu}_{t,t-h}$ with $\hat{\mathbf{y}}_t$ in (2.46) gives the multivariate log-likelihood function for the Kalman filter

$$\begin{aligned} &log\Big(\mathcal{L}(\theta_1,\theta_2,...,\theta_n|\mathbf{y}_h,\mathbf{y}_{2h},...,\mathbf{y}_T)\Big) = \\ &= \sum_{t=h}^{T} log\left(\frac{1}{\sqrt{(2\pi)^k|\mathbf{V}_{t,t-h}|}} e^{-\frac{1}{2}(\mathbf{y}_t^{obs}-\hat{\mathbf{y}}_t)'\mathbf{V}_{t,t-h}^{-1}(\mathbf{y}_t^{obs}-\hat{\mathbf{y}}_t)}\right) \\ &= -\frac{Tk}{2h}log(2\pi) - \frac{1}{2}\sum_{t=h}^{T} log(|\mathbf{V}_{t,t-h}|) - \frac{1}{2}\sum_{t=h}^{T}(\mathbf{y}_t^{obs}-\hat{\mathbf{y}}_t)'\mathbf{V}_{t,t-h}^{-1}(\mathbf{y}_t^{obs}-\hat{\mathbf{y}}_t). \end{aligned} \tag{2.47}$$

2.5 Stochastic Calculus

Since the later introduced models in this thesis are based on stochastic differential equations it is of great importance to give some basic theory within the area. This section therefore covers the most important stochastic calculus theory, giving an even more inexperienced stochastic calculus reader a possibility to understand the later introduced models in Chapter 3 and their proofs given in the Appendix. The theory in this section are based on the theory from the book *Arbitrage Theory in Continuous Time* (Björk 2009, [2]) and the compendium *Stochastic Calculus - An Introduction with Applications* (Djehiche 2000, [4]).

2.5.1 Stochastic Differential Equations (SDE)

A stochastic differential equation is a differential equation where one or more of the terms is a stochastic process such as

$$dx_t = \mu(t, x_t)dt + \sigma(t, x_t)dB_t, \qquad (2.48)$$

where x_t is a stochastic process, $\mu(t, x_t)$ and $\sigma(t, x_t)$ are deterministic functions and B_t is a Brownian motion. A Brownian motion is defined by the following properties:

Definition: Brownian motion

A stochastic process B is called a Brownian motion if the following conditions hold:

1. $B(0) = 0$.
2. *The process $B(t)$ has independent increments, i.e. if $r < s \leq t < u$ then $B(u) - B(t)$ and $B(s) - B(r)$ are independent stochastic variables.*
3. *For $s < t$ the stochastic variable $B(t) - B(s)$ has the Gaussian distribution $N[0, t-s]$.*
4. *$B(t)$ has continuous trajectories.*

Expression (2.48) implies that the infinitesimal change $dx_t = x_{t+dt} - x_t$ is caused by a change dt of time at a rate $\mu(t, x_t)$ and a change $dB_t = B_{t+dt} - B_t$ at rate $\sigma(t, x_t)$, where $\mu(t, x_t)$ is called the drift and $\sigma(t, x_t)$ is called the diffusion.

Vasicek Process

One of the most commonly used stochastic differential equations for modelling short interest rates of lending and borrowing is *the Vasicek Interest Rate Model.* In this model the interest rate, r_t, is the solution to the following linear stochastic differential equation

$$dr_t = c(\mu - r_t)dt + \sigma dB_t, \quad r_0 = constant, \qquad (2.49)$$

where c, μ and σ are constants, B_t is a Brownian motion and r_0 is the time 0 short rate.

As can be seen in the above given equation the Vasicek drift is given by $c(\mu - r_t)$, where μ can be interpreted as the long run equilibrium rate and c reflects the speed of adjustments between each time interval. The c parameter is positively defined which easily can be seen if looking at very high/low yields when the rate is needed to go down/up to get back to the long run equilibrium rate level, μ. This is called *mean reversion*; the model will over time ensure that the rate level gets back to its long run equilibrium. The Vasicek diffusion is given by σ and determines the volatility of the interest rate.

The solution to (2.49) is given by

$$r_t = r_0 e^{-ct} + \mu(1 - e^{-ct}) - \sigma e^{-ct} \int_0^t e^{cs} dB_s. \qquad (2.50)$$

Therefore it can be seen that the process r_t, in this set up, is a Gaussian process with mean function

$$r_0 e^{-ct} + \mu(1 - e^{-ct}) \qquad (2.51)$$

and variance

$$\frac{\sigma^2}{2c}(1 - e^{-2ct}). \qquad (2.52)$$

Multivariate Gaussian Process

The above given Vasicek process is a one factor model which is used when only one variable is under consideration. In a more sophisticated model, when modelling more variables at the same time such as the nominal rate, the real rate and the inflation expectation, there will be correlations between the variables which need to be included in the model. In this situation the use of multivariate models are essential.

A model containing three latent variables, $\mathbf{x}_t = [x_{1t},\ x_{2t},\ x_{3t}]^T$, can be stated as

$$d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t, \tag{2.53}$$

where $\boldsymbol{B}_t$ is a vector of Brownian motions, $\boldsymbol{\mu}$ is a 3×1 constant vector and both $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\Sigma}$ are 3×3 constant matrices.

The drift term in this model is given by $\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)$, where $\boldsymbol{\mu}$ can be interpreted as a vector containing the long run equilibrium levels and $\boldsymbol{\mathcal{K}}$ reflects the speed of adjustments between each time interval. The big difference between this model compared to the Vasicek model is that the factors are allowed to be correlated to each other. This is realized by the structure of $\boldsymbol{\Sigma}$, which can be interpreted as the variance-covariance matrix of $\mathbf{x}_t$.

Still, this model has the same *mean reversion* property as the Vasicek model, meaning that it will over time ensure that the rate levels get back to the long run equilibriums.

2.5.2 Stochastic Differentials

The main property of the stochastic differential equations that diversifies them from most other differential equations, is that they are not differentiable. This is due to the fact that a path of a Brownian motion is nowhere differentiable. In light of this, the famous *Itô's formula* can be used to calculate stochastic differentials.

Theorem 2.4 *Itô's Formula*
Let the n-dimensional vector process $\boldsymbol{x} = (x_1, x_2, ..., x_n)'$ *have the dynamics*

$$d\boldsymbol{x}(t) = \boldsymbol{\mu}(t)dt + \boldsymbol{\sigma}(t)d\boldsymbol{B}(t),$$

where $\boldsymbol{\mu}$ *is an n-dimensional drift vector,* $\boldsymbol{B}$ *is a d-dimensional vector containing d number of independent Brownian motions and* $\boldsymbol{\sigma}$ *an* $n \times d$*-dimensional diffusion matrix given by*

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_d \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nd} \end{bmatrix}. \tag{2.54}$$

Then, let the process Z be defined by

$$Z(t) = f(t, \boldsymbol{x}(t)), \tag{2.55}$$

where $f : R_+ \times R^n \to R$ *is a* $C^{1,2}$ *mapping. Then, the process* $f(t, \boldsymbol{x}(t))$ *has a stochastic differential given by*

$$df(t, \boldsymbol{x}(t)) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dx_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}dx_i dx_j, \tag{2.56}$$

with the formal multiplication table

$$\begin{cases} (dt)^2 = 0, & \\ dt \cdot dB_i = 0, & \\ (dB_i)^2 = dt, & i = 1, ..., d, \\ dB_i \cdot dB_j = 0, & i = j. \end{cases}$$

2.5.3 Risk Neutral Valuation

The upcoming models will exclusively be defined using the no arbitrage framework and therefore it is of great importance to give some theory about the risk neutral valuation set up. As mentioned in the beginning of this chapter, the reader is recommended to find more extensive theory in relevant literature, see [2] and [4].

Theorem 2.5 *The First Fundamental Theorem of Asset Pricing*
A discrete market, on a discrete probability space $(\Omega, \mathcal{F}, \mathcal{P})$*, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure,* P*.*

In light of Theorem 2.5, a commonly used notation for the risk neutral measure is the Q-measure. Without any further introduction to measure theory, this convention will be used for the rest of this thesis when referring to the risk neutral measure.

Proposition: Risk Neutral Bond Price

The bond price at time t *with maturity* $T = t + \tau$ *is given by the formula* $p(t,T) = F(t, r(t); T)$ *where*

$$F(t, r(t); T) = E^Q_{t,r}\left[e^{-\int_t^T r(s)ds}\right], \tag{2.57}$$

where Q *denotes the risk neutral measure. This together with the subscript* t, r *denote that the expectation shall be taken given the following dynamics for the short rate*

$$\begin{aligned} dr(s) &= (\mu - \lambda\sigma)ds + \sigma dB^Q(s) \\ r(t) &= r, \end{aligned}$$

Notable in this proposition is that the dynamics are given under the risk neutral measure, which always gives an arbitrage free price.

2.5.4 Change of Measure

When modelling economic features, such as interest rates, one often uses stochastic differential equations evaluated under the P-measure. Still, to assure that the pricing does not imply any arbitrage, the pricing of the corresponding securities are needed to be done under the risk neutral measure Q, see Theorem 2.5. The Girsanov theorem then is of great use, giving a framework for how to change measures and hence be able to price the securities correctly.

Theorem 2.6 *The Girsanov Theorem*
Let $\boldsymbol{B}^P_t$ *be a d-dimensional standard P-Brownian motion on* $(\Omega, \mathcal{F}, \mathcal{P}, \underline{\mathcal{F}})$ *and let* $\boldsymbol{\varphi_t}$ *be any d-dimensional adapted column vector process. Choose a fixed* T *and define the process* L *on* $[0, T]$ *by*

$$dL_t = \boldsymbol{\varphi}'_t L_t d\boldsymbol{B}^P_t \tag{2.58}$$

$$L_0 = 1, \tag{2.59}$$

i.e.

$$L_t = exp\left\{\int_0^T \boldsymbol{\varphi}'_s d\boldsymbol{B}^P_s - \frac{1}{2}\int_0^T \|\boldsymbol{\varphi}_s\|^2 ds\right\} \tag{2.60}$$

Assume that

$$E^P[L_T] = 1, \tag{2.61}$$

and define the new probability measure Q on $\mathcal{F}_T$ by

$$L_T = \frac{dQ}{dP}, \quad on \ \ \mathcal{F}_T. \tag{2.62}$$

Then

$$d\boldsymbol{B}_t^P = \boldsymbol{\varphi_t} dt + d\boldsymbol{B}_t^Q, \tag{2.63}$$

where $\boldsymbol{B}^Q$ is a Q-Brownian motion.

Definition: Market Price of Risk

In Theorem 2.6 the variable $\boldsymbol{\varphi}_t$ denotes the Girsanov kernel for which the following relation holds

$$\boldsymbol{\varphi}_t = -\boldsymbol{\lambda}_t = -\big(\lambda_0 + \boldsymbol{\lambda}_1' \boldsymbol{r}_t\big), \tag{2.64}$$

where $\boldsymbol{\lambda}_t$ denotes the market price of risk for $\boldsymbol{r}_t$. The market price of risk can be interpreted as a risk premium per unit risk that the market implies for holding a specific position.

2.5.5 Affine Term Structure

Definition: Affine Term Structure

If the term structure $\{p(t,T); 0 \le t \le T, T > 0\}$ has the form

$$p(t,T) = F(t, r(t); T), \tag{2.65}$$

where F has the form

$$F(t, r(t); T) = e^{A(t,T)+B(t,T)r(t)} \tag{2.66}$$

and where A and B are deterministic functions, then the model is said to possess an **affine term structure**.

Proposition: Affine Term Structure

If the stochastic process $r(t)$ is assumed to have the dynamics

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dB^Q(t). \tag{2.67}$$

Furthermore, it can be assumed that the model possess an affine term structure as in the above given definition, i.e. that

$$p(t,T) = F(t, r(t); T) = e^{A(t,T)+B(t,T)r(t)}. \tag{2.68}$$

Then, assume that $\mu(t, r(t))$ and $\sigma(t, r(t))$ are of the form

$$\begin{cases} \mu(t, r(t)) = \alpha(t)r(t) + \beta(t) \\ \sigma(t, r) = \sqrt{\gamma(t)r(t) + \delta(t).} \end{cases} \tag{2.69}$$

Then A and B satisfy the following system of ordinary differential equations

$$\begin{cases} B_t(t,T) - \alpha(t)B(t,T) - \dfrac{1}{2}\gamma(t)B^2(t,T) = -1 \\ \qquad\qquad\qquad\qquad\qquad\qquad B(T,T) = 0 \end{cases} \tag{2.70}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) + \dfrac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{2.71}$$

2.5.6 Example: Bond Pricing (Vasicek dynamics)

Let the stochastic process $r(t)$ follow the Vasicek dynamics

$$dr(t) = c(\mu - r(t))dt + \sigma dB^P(t), \quad r(0) = constant, \tag{2.72}$$

where $dB^P(t)$ implies that $r(t)$ is defined under the P-measure. Then, maybe it is of interest to price a bond using the risk neutral bond price formula given by (2.57). Firstly it needs to be assumed that the bond price possesses an affine term structure as given by (2.68)

$$p(t,T) = e^{A(t,T)+B(t,T)r(t)}. \tag{2.73}$$

Then, since the short rate dynamics in (2.72) are given under the P-measure, a measure transformation is needed. By *the Girsanov Theorem*, Theorem 2.6, and by the definition of the market price of risk, it follows that

$$dB_t^P = \varphi_t dt + dB_t^Q = -\big(\lambda_0 + \lambda_1 r(t)\big)dt + dB_t^Q, \tag{2.74}$$

where $\lambda = \lambda_0 + \lambda_1 r(t)$ is the market price of risk, for $r(t)$. Then, if this transformation is inserted into (2.72) the risk free Vasicek dynamics are given as

$$dr(t) = c\big(\mu - r(t)\big)dt + \sigma\Big(- \big(\lambda_0 + \lambda_1 r(t)\big)dt + dB_t^Q\Big) \tag{2.75}$$

$$= c\bigg(\Big(\mu - \frac{\sigma\lambda_0}{c}\Big) - \Big(1 + \frac{\sigma\lambda_1}{c}\Big)r(t)\bigg)dt + \sigma dB_t^Q. \tag{2.76}$$

Then, by identification in (2.69) it can be seen that,

$$\begin{cases} \alpha^v(t) = -c\Big(1 + \frac{\sigma\lambda_1}{c}\Big) \\ \beta^v(t) = c\Big(\mu - \frac{\sigma\lambda_0}{c}\Big) \\ \gamma^v(t) = 0 \\ \delta^v(t) = \sigma^2. \end{cases} \tag{2.77}$$

Then, by (2.70) and (2.71) it follows that $A(t,T)$ and $B(t,T)$ in (2.73) solve

$$\begin{cases} B_t(t,T) - \alpha^v(t)B(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{2.78}$$

$$\begin{cases} A_t(t,T) = \beta^v(t)B(t,T) + \frac{1}{2}\delta^v(t)B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{2.79}$$

Equation (2.78) is, when having a fixed T, a simple linear ordinary differential equation in t. Then, following standard procedure, the solution to B_t is given as

$$B(t,T) = \frac{1}{\alpha^v}\Big\{1 - e^{-\alpha^v(T-t)}\Big\}. \tag{2.80}$$

The corresponding solution to $A(t,T)$ is given by integrating (2.79) and inserting the solution to $B(t,T)$

$$A(t,T) = \frac{\big\{B(t,T) - T + t\big\}\big(- \alpha^v\beta^v - \frac{1}{2}\delta^v\big)}{(\alpha^v)^2} - \frac{\delta^v B^2(t,T)}{4(\alpha^v)}. \tag{2.81}$$

These solutions can now be inserted into (2.73) giving arbitrage free bond prices for all points in time t given a fixed future maturity T.

CHAPTER 3

Model

This chapter will cover the different stochastic models used for modelling the nominal-, real- and inflation linked yields. Full proofs of the models are given in the Appendix. Once again it needs to be stated that the model are completely based on *Model L-II* given in the article *Tips from TIPS: the informational content of Treasury Inflation-Protected Security prices*, see [5].

3.1 Nominal Yields and Nominal Bond Prices

The assumption is made that the real yield, the nominal yield and the expected inflation are driven by a vector of three latent variables, $\mathbf{x}_t = [x_{1t},\ x_{2t},\ x_{3t}]^T$, which is assumed to follow a multivariate Gaussian process, for more theory see Section 2.5.1,

$$d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P, \tag{3.1}$$

where $\boldsymbol{B}_t$ is a 3×1 vector of P Brownian motions, $\boldsymbol{\mu}$ is a 3×1 constant vector and both $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\Sigma}$ are 3×3 constant matrices.

Then, the nominal short rate and the nominal price of risk are assumed to be affine functions of the three latent variables

$$r^N(\mathbf{x}_t) = \rho_0^N + \boldsymbol{\rho}_1^{N\prime}\mathbf{x}_t \tag{3.2}$$

$$\boldsymbol{\lambda}^N(\mathbf{x}_t) = \boldsymbol{\lambda}_0^N + \boldsymbol{\Lambda}^N\mathbf{x}_t, \tag{3.3}$$

where ρ_0^N is a constant, $\boldsymbol{\rho}_1^N$ and $\boldsymbol{\lambda}_0^N$ are 3×1 constant vectors and $\boldsymbol{\Lambda}^N$ is a 3×3 constant matrix.

Furthermore, the nominal stochastic discount factor, sometimes referred to as the nominal pricing kernel, can be written as

$$\frac{dM_t^N}{M_t^N} = -r^N(\mathbf{x}_t)dt - \boldsymbol{\lambda}^N(\mathbf{x}_t)' d\boldsymbol{B}_t^P. \tag{3.4}$$

Now, by applying the no arbitrage framework, the nominal bond prices can be calculated.

This can be done by considering the risk neutral bond price formula, see the risk neutral bond price proposition in Section 2.5.3, for the bond price at time 0 of a bond with maturity at time T

$$p^N(0, T) = E^Q\left[exp\left\{ - \int_0^T r(s)ds \right\}\right], \tag{3.5}$$

where Q denotes the risk neutral measure, see Section 2.5.3. If then τ is given as the time to maturity from the current time t, an equivalent expression for the bond price at time t with maturity at time $t + \tau$ is given by

$$p^N(t, \tau) = E^Q\left[exp\left\{ - \int_t^{t+\tau} r(s)ds \right\}\right]. \tag{3.6}$$

Then, if it is assumed that the nominal bond price possesses an affine term structure, the bond price can be written as

$$p(t,\tau) = F(t,\tau,\mathbf{x}_t) = e^{A_\tau^N + \boldsymbol{B}_\tau^{N\prime}\mathbf{x}_t}, \tag{3.7}$$

where A_τ^N and $\boldsymbol{B}_\tau^N$ are deterministic functions of τ, taking the maturity time T as a parameter. Then, by the calculations given in the Appendix, Section A.3, the following system of ordinary differential equations is obtained

$$\frac{dA_\tau^N}{d\tau} = -\rho_0^N + \boldsymbol{B}_\tau^{N\prime}(\boldsymbol{\mathcal{K}}\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0^N) + \frac{1}{2}\boldsymbol{B}_\tau^{N\prime}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^N \tag{3.8}$$

$$\frac{d\boldsymbol{B}_\tau^N}{d\tau} = -\boldsymbol{\rho}_1^N - (\boldsymbol{\mathcal{K}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}^N)'\boldsymbol{B}_\tau^N, \tag{3.9}$$

with initial conditions $A_0^N = 0$ and $\boldsymbol{B}_0^N = \mathbf{0}$. These differential equations can be solved using numerical methods. Then, if a_τ^N and $\boldsymbol{b}_\tau^N$ are defined as

$$a_\tau^N = \frac{-A_\tau^N}{\tau}$$

$$\boldsymbol{b}_\tau^N = \frac{-\boldsymbol{B}_\tau^N}{\tau},$$

the nominal yield $y_{t,\tau}^N$ takes an affine form and can be written as

$$y_{t,\tau}^N = a_\tau^N + \boldsymbol{b}_\tau^{N\prime}\mathbf{x}_t. \tag{3.10}$$

3.2 Inflation Expectations

Since the main difference between the real yield and nominal yield is the inflation, it is of great importance to model the market implied inflation. The price level process is assumed to be of the following form

$$dq_t = d(logQ_t) = \pi(\mathbf{x}_t)dt + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^\perp dB_t^{\perp,P}, \tag{3.11}$$

where $\pi(\mathbf{x}_t)$ is the instantaneous expected inflation, $\boldsymbol{\sigma}_q'$ is a 3×1 constant vector and $\sigma_q^\perp$ is a constant. The instantaneous expected inflation is assumed to be an affine function in the latent variables

$$\pi(\mathbf{x}_t) = \rho_0^\pi + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_t. \tag{3.12}$$

The other part of the price level process, $\boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^\perp dB_t^{\perp,P}$, reflects the unexpected inflation. The structure allows the price level to both depend on shocks that affect the nominal interest rate and the expected inflation, $d\boldsymbol{B}_t^P$, and on so called orthogonal shocks, $dB_t^{\perp,P}$, where $d\boldsymbol{B}_t^P dB_t^{\perp,P} = 0$. The orthogonal shock term is included to capture short-run inflation variations that may not be spanned by the yield curve movements, see [5].

The inflation expectation at time t until time $t+\tau$ is defined as

$$I_{t,\tau} = \frac{1}{\tau}E_t\left[log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right]. \tag{3.13}$$

It is also assumed that the inflation expectation takes an affine form in the latent variables

$$I_{t,\tau} = a_\tau^I + \boldsymbol{b}_\tau^{I\prime}\mathbf{x}_t, \tag{3.14}$$

where a_τ^I is a constant and $\boldsymbol{b}_\tau^I$ is a 3×1 constant vector.

Then, by the calculations given in the Appendix, Section A.4, the following expressions are obtained

$$a^I_\tau = \rho^\pi_0 + \frac{1}{\tau}\boldsymbol{\rho}^{\pi\prime}_1 \int_0^\tau \left(\boldsymbol{I} - e^{-\boldsymbol{\mathcal{K}}u}\right)\boldsymbol{\mu} du \tag{3.15}$$

$$\boldsymbol{b}^{I\prime}_\tau = \frac{1}{\tau}\boldsymbol{\rho}^{\pi\prime}_1 \int_0^\tau e^{-\boldsymbol{\mathcal{K}}u} du. \tag{3.16}$$

3.3 Real Yields and Real Bond Prices

Since the nominal short rate and the expected inflation both are defined, it is now possible to model the real yield and the real bond price.

The real and the nominal pricing kernels are set up such that they must fulfil the following no-arbitrage relation

$$M^R_t = M^N_t Q_t, \tag{3.17}$$

where M^N_t and Q_t are defined by (3.4) and (3.11) respectively. Then if one apply Itô's formula to (3.17) the following relation is obtained

$$\frac{dM^R_t}{M^R_t} = \frac{dM^N_t}{M^N_t} + \frac{dQ_t}{Q_t} + \frac{dM^N_t}{M^N_t} \cdot \frac{dQ_t}{Q_t}. \tag{3.18}$$

Through the calculations given in the Appendix, Section A.2, (3.18) can be written as

$$\frac{dM^R}{M^R} = -r^R(\mathbf{x}_t)dt - \boldsymbol{\lambda}^R(\mathbf{x}_t)' d\boldsymbol{B}^P_t + \sigma^\perp_t dB^{\perp,P}_t,$$

where the real short rate and the real price of risk are given by

$$r^R(\mathbf{x}_t) = \rho^R_0 + \boldsymbol{\rho}^{R\prime}_1 \mathbf{x}_t \tag{3.19}$$

$$\boldsymbol{\lambda}^R(\mathbf{x}_t) = \boldsymbol{\lambda}^R_0 + \boldsymbol{\Lambda}^R \mathbf{x}_t, \tag{3.20}$$

where the parameter relations are given as

$$\rho^R_0 = \rho^N_0 - \rho^\pi_0 - \frac{1}{2}(\boldsymbol{\sigma}'_q\boldsymbol{\sigma}_q + {\sigma^\perp_q}^2) + \boldsymbol{\lambda}^{N\prime}_0 \boldsymbol{\sigma}_q \tag{3.21}$$

$$\boldsymbol{\rho}^R_1 = \boldsymbol{\rho}^N_1 - \boldsymbol{\rho}^\pi_1 + \boldsymbol{\Lambda}^{N\prime}\boldsymbol{\sigma}_q \tag{3.22}$$

$$\boldsymbol{\lambda}^R_0 = \boldsymbol{\lambda}^N_0 - \boldsymbol{\sigma}_q \tag{3.23}$$

$$\boldsymbol{\Lambda}^R = \boldsymbol{\Lambda}^N. \tag{3.24}$$

With the real short rate defined, it is now also possible to derive the no arbitrage prices for the real bonds, in the same manner as for the nominal bond prices.

It is assumed that the real bond price possesses an affine term structure

$$p^R(t,\tau) = F(t,\tau,\mathbf{x}_t) = e^{A^R_\tau + \boldsymbol{B}^{R\prime}_\tau \mathbf{x}_t}, \tag{3.25}$$

where A^R_τ and $\boldsymbol{B}^R_\tau$ are deterministic functions of τ, taking the maturity time T as a parameter. Then, by the calculations given in the Appendix, Section A.3, the following differential equations are obtained

$$\frac{dA^R_\tau}{d\tau} = -\rho^R_0 + \boldsymbol{B}^{R\prime}_\tau(\boldsymbol{\mathcal{K}}\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}^R_0) + \frac{1}{2}\boldsymbol{B}^{R\prime}_\tau \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}^R_\tau \tag{3.26}$$

$$\frac{d\boldsymbol{B}^R_\tau}{d\tau} = -\boldsymbol{\rho}^R_1 - (\boldsymbol{\mathcal{K}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}^R)'\boldsymbol{B}^R_\tau, \tag{3.27}$$

with initial conditions $A_0^R = 0$ and $\boldsymbol{B}_0^R = \mathbf{0}$. These differential equations can be solved using numerical methods. Then, if a_τ^R and $\boldsymbol{b}_\tau^R$ are defined as

$$a_\tau^R = \frac{-A_\tau^R}{\tau}$$
$$\boldsymbol{b}_\tau^R = \frac{-\boldsymbol{B}_\tau^R}{\tau},$$

the real yield $y_{t,\tau}^R$ takes an affine form and can be written as

$$y_{t,\tau}^R = a_\tau^R + \boldsymbol{b}_\tau^{R\prime}\mathbf{x}_t. \tag{3.28}$$

3.4 Breakeven Inflations and Inflation Risk Premiums

The true breakeven inflation is defined as the difference between the nominal yield and the real yield

$$y_{t,\tau}^{BEI} = y_{t,\tau}^N - y_{t,\tau}^R. \tag{3.29}$$

The nominal and the real yields are defined by (3.10) and (3.28) respectively. Therefore it follows that (3.29) can be rewritten as

$$y_{t,\tau}^{BEI} = a_\tau^N - a_\tau^R + (\boldsymbol{b}_\tau^N - \boldsymbol{b}_\tau^R)'\mathbf{x}_t. \tag{3.30}$$

Furthermore, an expression for the inflation risk premium can be derived. The inflation risk premium is defined as the difference between the breakeven inflation and the expected inflation

$$\wp_{t,\tau}^I = y_{t,\tau}^{BEI} - I_{t,\tau} = a_\tau^N - a_\tau^R - a_\tau^I + (\boldsymbol{b}_\tau^N - \boldsymbol{b}_\tau^R - \boldsymbol{b}_\tau^I)'\mathbf{x}_t, \tag{3.31}$$

where the breakeven inflation and the expected inflation are given by (3.30) and (3.14) respectively.

3.5 Inflation Linked Yields

The inflation linked yield can be modelled by letting the nominal yield deviate from the market implied true real yield. This is implemented by introducing a inflation linked bond specific liquidity premium, $L_{t,\tau}^L$, later referred to as the liquidity premium, at time t, defined as

$$L_{t,\tau}^L = y_{t,\tau}^L - y_{t,\tau}^R, \tag{3.32}$$

giving that the inflation linked yield is defined by

$$y_{t,\tau}^L = L_{t,\tau}^L + y_{t,\tau}^R. \tag{3.33}$$

The liquidity premium can partly be viewed as an extra compensation investors demand for holding an instrument with less liquid market conditions. It should though, once again, be pointed out that it also will comprise other factors that potentially could drive a wedge between the real and inflation linked yields, see Section 2.1.3.

Moreover, the liquidity premium will in general be positive partly based on the less liquid market conditions for inflation linked bonds, giving lower prices and higher yields.

Mathematically the liquidity premium is defined by adding a positive liquidity spread, l_t^L, to the true real instantaneous short rate, used by investors while discounting the inflation linked bond's cash flow. Hence (3.32) can be rewritten as

$$L_{t,\tau}^L = -\frac{1}{\tau} log E_t^Q \Big[exp\Big(-\int_t^{t+\tau} (r_s^R + l_s^L) ds \Big) \Big] - y_t^R, \tag{3.34}$$

where the expectation is taken under the risk-neutral measure Q. This way of using an extra spread factor while discounting cash flows is analogous to the corporate bond pricing literature, where default-able bond cash flows are discounted with an extra spread factor, see [5].

The liquidity spread is modelled as a function depending on four latent variables. Firstly, one term represented by the latent variables in $\mathbf{x}_t$, letting the liquidity spread being dependent on the state of the economy and secondly, one term that is independent to the state of the economy, $\tilde{x}_t$. Therefore, the liquidity spread is defined by

$$l_t^L = \boldsymbol{\gamma}'\mathbf{x}_t + \tilde{\gamma}\tilde{x}_t, \tag{3.35}$$

where $\boldsymbol{\gamma}$ is a 3×1 constant vector and $\tilde{\gamma}$ is a constant. It is further assumed that $\tilde{x}_t$ follows a Vasicek process which is defined by

$$d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma}d\tilde{B}_t^P, \tag{3.36}$$

where $\tilde{\kappa}$, $\tilde{\mu}$ and $\tilde{\sigma}$ are constants and $\tilde{B}_t^P$ is a P Brownian motion. Since $\tilde{x}_t$ is said to be independent of $\mathbf{x}_t$ it must hold that $d\tilde{B}_t^P d\boldsymbol{B}_t^P = \mathbf{0}$. It is also assumed that the independent liquidity factor possesses a market price of risk defined as an affine function in $\tilde{x}_t$

$$\tilde{\lambda}_t = \tilde{\lambda}_0 + \tilde{\lambda}_1\tilde{x}_t, \tag{3.37}$$

where $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ are constants.

Then, by the calculations given in the Appendix, see Section A.5, the following expression for the liquidity premium can be obtained

$$L_{t,\tau}^L = [\tilde{a}_\tau^L + (a_\tau^L - a_\tau^R)] + \big[(\boldsymbol{b}_\tau^L - \boldsymbol{b}_\tau^R)' \;\; \tilde{b}_\tau^L\big]\begin{bmatrix}\mathbf{x}_t \\ \tilde{x}_t\end{bmatrix}, \tag{3.38}$$

where $\tilde{a}_\tau^L$, $\tilde{b}_\tau^L$ and a_τ^L are constants and $\boldsymbol{b}_\tau^L$ is a 3×1 constant vector given by

$$\begin{aligned}
\tilde{a}_\tau^L &= \tilde{\gamma}\Big[\big(\tilde{\mu}^* - \frac{1}{2}\Big(\frac{\tilde{\sigma}}{\tilde{\kappa}^*}\Big)^2\tilde{\gamma}\big)\big(1 - \frac{\tilde{b}_\tau^L}{\tilde{\gamma}}\big) + \frac{\tilde{\sigma}^2}{4\tilde{\gamma}\tilde{\kappa}^*}\tau(\tilde{b}_\tau^L)^2\Big] \\
\tilde{b}_\tau^L &= \frac{\tilde{\gamma}}{\tilde{\kappa}^*\tau}\Big(1 - e^{-\tilde{\kappa}^*\tau}\Big) \\
\\
a_\tau^L &= -\frac{A_\tau^L}{\tau} \\
\boldsymbol{b}_\tau^L &= -\frac{\boldsymbol{B}_\tau^L}{\tau}
\end{aligned}$$

where A_τ^L and $\boldsymbol{B}_\tau^L$ solve the following system of differential equations

$$\frac{dA_\tau^L}{d\tau} = -\rho_0^R + \boldsymbol{B}_\tau^{L\prime}(\boldsymbol{\mathcal{K}}\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0^R) + \frac{1}{2}\boldsymbol{B}_\tau^{L\prime}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L \tag{3.39}$$

$$\frac{d\boldsymbol{B}_\tau^L}{d\tau} = -(\boldsymbol{\rho}_1^R + \boldsymbol{\gamma}) - (\boldsymbol{\mathcal{K}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}^R)'\boldsymbol{B}_\tau^L, \tag{3.40}$$

with initial conditions $A_0^L = 0$ and $\boldsymbol{B}_0 = \mathbf{0}$. These differential equations can be solved using numerical methods.

From the above formulas it also follows that the inflation linked yield is given as

$$y_{t,\tau}^L = [\tilde{a}_\tau^L + a_\tau^L] + \big[\boldsymbol{b}_\tau^{L\prime} \;\; \tilde{b}_\tau^L\big]\begin{bmatrix}\mathbf{x}_t \\ \tilde{x}_t\end{bmatrix}. \tag{3.41}$$

3.6 Decompositions

Based on the model assumptions made in the previous sections the nominal yield can be decomposed as

$$y^N_{t,\tau} = y^R_{t,\tau} + I_{t,\tau} + \wp^I_{t,\tau}. \tag{3.42}$$

This decomposition is given if one solves for $y^N_{t,\tau}$ in (3.29) and then uses the fact that

$$y^{BEI}_{t,\tau} = \wp^I_{t,\tau} + I_{t,\tau}, \tag{3.43}$$

given by (3.31).

The linkers breakeven inflation is defined as the difference between the nominal yield and the inflation linked yield

$$y^{BEI,L}_{t,\tau} = y^N_{t,\tau} - y^L_{t,\tau}. \tag{3.44}$$

Then, if the above decomposition of the nominal yield, together with the definition of the inflation linked yield, (3.33),

$$y^L_{t,\tau} = L^L_{t,\tau} + y^R_{t,\tau} \tag{3.45}$$

are inserted into (3.44) it follows that the linkers breakeven inflation can be decomposed as

$$y^{BEI,L}_{t,\tau} = I_{t,\tau} + \wp^I_{t,\tau} - L^L_{t,\tau}. \tag{3.46}$$

Hence, if (3.46) is subtracted from (3.43) the difference between true breakeven inflation and the linkers breakeven inflation can be identified as the liquidity premium

$$y^{BEI}_{t,\tau} - y^{BEI,L}_{t,\tau} = \wp^I_{t,\tau} + I_{t,\tau} - (I_{t,\tau} + \wp^I_{t,\tau} - L^L_{t,\tau}) = L^L_{t,\tau}. \tag{3.47}$$

CHAPTER 4

Data

This chapter will give a description of the data used for fitting the model. A thorough review of the model is provided in Chapter 3.

4.1 Time Series

The time series used are having different time steps between the observations, but the time step remain uniform within each series. Therefore, to make the data samples appropriate for being implemented in the Kalman filter, see Section 2.3, some transformations of the original time series have been applied. Below, such transformations and further descriptions are provided for each data sample used in this thesis. The discretization step, h, of the state space model, see Section 2.2, is set to be weekly, thus $h = 1/52$.

Noticeable is that none of the used time series inherit any structural breaks, i.e. no discontinuities, which possibly could have made the data inappropriate to be used in the estimation.

4.1.1 Nominal- and Inflation Linked Yields

The following data was used for fitting the model. Firstly, Swedish nominal government bond yields with maturities 3-month, 6-month, 1- ,2- ,4- ,7- and 10-year. These yields were all extracted from the nominal zero coupon yield curve. Secondly, Swedish inflation linked government bond yields with maturities 5-, 7- and 10-year. These yields were all extracted from the inflation linked zero coupon yield curve. Thirdly, Swedish consumer pricing index data.

The nominal and inflation linked yield data ranges from the 5th of January 2005 until the 4th of February 2014. The yield levels were, as earlier mentioned, extracted form the zero-coupon yield curves, whereas the bootstrap method was used to calculate the term structures. These calculations were performed by *Nordea Analytics*, a software tool used within *Nordea Markets* for analysis of financial time series. The time series were generated with daily intensity.

Since the discretization step, h, was set to be weekly and the time series were given on daily frequency, transformations were applied. For sampling the data of daily observations to a data sample of weekly observations, the following approach was used

$$y_t^{Weekly} = \frac{y_{t-\frac{2}{365}}^{Daily} + y_{t-\frac{1}{365}}^{Daily} + y_t^{Daily} + y_{t+\frac{1}{365}}^{Daily} + y_{t+\frac{2}{365}}^{Daily}}{5} \qquad \text{for} \quad t = h,\ 2h, ...,\ T,$$

with t, the date stamp, synchronized to always occur on a Wednesday.

This formula is complete for weeks containing five business days. For weeks with less than five business days, averaging still were applied, but over the daily observations available within each week. For each of these weeks, the date stamp was set to be on a Wednesday, unconditionally of what weekdays that actually comprised to the averaging.

The yield data is visualized in Figure 4.1, where the nominal yield levels are given in panel (a) and the inflation linked yield levels are given in the panel (b). Panel (c) visualizes the actual breakeven inflations, given as the differences between the nominal and the inflation linked yields.

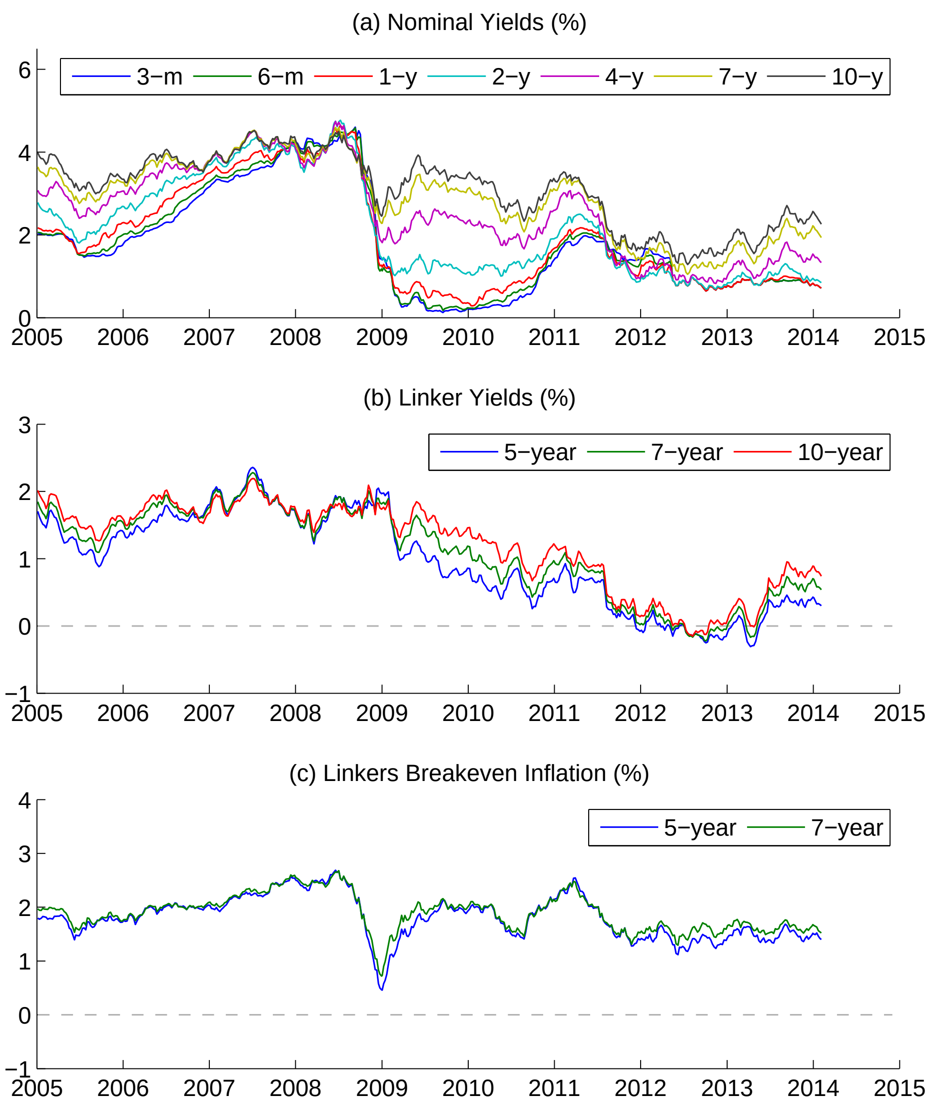

Figure 4.1: Actual - Nominal Yields, Linker Yields and Linkers Breakeven Inflations

4.1.2 Consumer Pricing Index

Besides using nominal- and inflation linked yield data for fitting the model, monthly Swedish consumer pricing index data also was used. This time series ranges from the 2nd of January 2005 until the 1st of January 2014 and was collected from *Statistics Sweden*.

The time series inherits seasonality. Thus, since the model does not accommodate seasonality, the consumer pricing index data was seasonally adjusted before being used, see [5]. The transformation of the original time series, was performed by *Reuters EcoWin*, an analysis and charting software.

To make the time series appropriate for being implemented in the Kalman filter, the date stamps for the respective consumer pricing index observations were set to coincide with the closest available measurements of the weekly sampled yield data. This final consumer pricing index data is given in Figure 4.2.

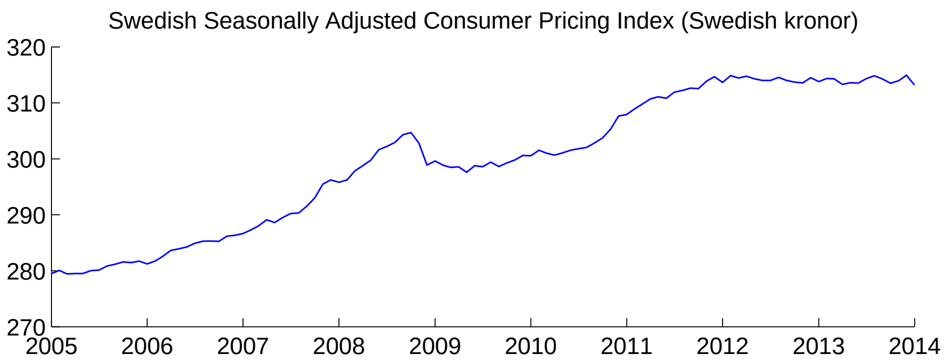

Figure 4.2: Swedish Seasonally Adjusted Consumer Pricing Index

CHAPTER 5

Method

This chapter will provide a thorough description of the used approach for the parameter estimation and the model fitting. The assumptions regarding normalization of parameters and estimation methodology coincide with those used in *Tips from TIPS: the informational content of Treasury Inflation-Protected Security prices*, see [5].

5.1 Normalization

Before fitting the model and performing the estimation of the unknown parameters, the following normalization was employed

$$\boldsymbol{\mu} = \mathbf{0}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 0.01 & 0 & 0 \\ \Sigma_{21} & 0.01 & 0 \\ \Sigma_{31} & \Sigma_{32} & 0.01 \end{bmatrix}, \quad \boldsymbol{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_{11} & 0 & 0 \\ 0 & \mathcal{K}_{22} & 0 \\ 0 & 0 & \mathcal{K}_{33} \end{bmatrix}, \quad \tilde{\sigma} = 0.01.$$

These restrictions were necessary for achieving identification so as to allow a maximal flexible correlation structure between the factors, that was critical in fitting the rich behaviour in the risk premiums that can be observed in the data. The remaining 48 parameters were remained unrestricted.

5.2 Parameter Estimation

To be able to estimate the parameters the model had to be discretized and reformulated on state space form. This was needed to make the continuous model appropriate for being implemented in the Kalman filter.

The general derivation of the state space model is provided in Section 2.2. The specific derivation of the discretized model dynamics are given in Section 2.2.1. The matrix state equation, (2.12), is given by

$$\mathbf{x}_t^\star = \mathbf{G}_h + \boldsymbol{\Gamma}_h \mathbf{x}_{t-h}^\star + \boldsymbol{\eta}_t^{\mathbf{x}}, \tag{5.1}$$

where

$$\mathbf{G}_h = \begin{bmatrix} \rho_0^\pi h \\ \boldsymbol{\mathcal{K}}\boldsymbol{\mu} h \\ \tilde{\kappa}\tilde{\mu} h \end{bmatrix}, \quad \boldsymbol{\Gamma}_h = \begin{bmatrix} 1 & \boldsymbol{\rho}_1^{\pi\prime} h & 0 \\ \mathbf{0} & \mathbf{I} - \boldsymbol{\mathcal{K}} h & \mathbf{0} \\ 0 & \mathbf{0}' & 1 - \tilde{\kappa} h \end{bmatrix}, \quad \boldsymbol{\eta}_t^{\mathbf{x}} = \begin{bmatrix} \boldsymbol{\sigma}_q' \boldsymbol{\eta_t} + \sigma_q^\perp \eta_t^\perp \\ \boldsymbol{\Sigma} \boldsymbol{\eta}_t \\ \tilde{\sigma} \tilde{\eta}_t \end{bmatrix},$$

in which $\boldsymbol{\eta_t}$, $\eta_t^\perp$ and $\tilde{\eta}_t$ are independent of each other and the state vector $\mathbf{x}_t^\star$ is defined by

$$\mathbf{x}_t^\star = \begin{bmatrix} q_t \\ \mathbf{x}_t \\ \tilde{x}_t \end{bmatrix}.$$

Furthermore, the observation equation for the state space model is derived in Section 2.2.2, where the matrix observation equation, (2.14), is given by

$$\mathbf{y}_t = \mathbf{A}_t + \mathbf{B}_t \mathbf{x}_t + \mathbf{e}_t, \tag{5.2}$$

i.e. the observation equation is a linear combination of the state variables, including an additive error term, that is assumed to equal the observed data, measured with error. $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{e}_t$ in (5.2) are given by

$$\mathbf{A}_t = \begin{bmatrix} 0 \\ \mathbf{A}^N \\ \tilde{\mathbf{a}} + \mathbf{A}^L \end{bmatrix}, \quad \mathbf{B}_t = \begin{bmatrix} 1 & \mathbf{0}' & 0 \\ \mathbf{0} & \mathbf{B}^{N\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{L\prime} & \tilde{\mathbf{b}} \end{bmatrix}, \quad \boldsymbol{e}_t = \begin{bmatrix} 0 \\ \mathbf{e}_t^N \\ \mathbf{e}_t^L \end{bmatrix},$$

where a structure of identical and independently distributed measurement errors is assumed such that

$$e^N_{t,\tau_N} \sim N(0, \delta^2_{N,\tau_N}) \qquad \text{for} \quad \tau_N = 3m,\ 6m, 1y\ , 2y,\ 4y,\ 7y,\ 10y \tag{5.3}$$

$$e^L_{t,\tau_L} \sim N(0, \delta^2_{L,\tau_L}) \qquad \text{for} \quad \tau_L = 5y,\ 7y,\ 10y. \tag{5.4}$$

Then, since the errors are assumed to be independent, the linkers breakeven inflation measurement error is defined as

$$e^{BEI,L}_{t,\tau} \sim N(0, \delta^2_{N,\tau} + \delta^2_{L,\tau}) \qquad \text{for} \quad \tau = 7y,\ 10y. \tag{5.5}$$

Having the model given on state space form, consisting of the state equation (5.1) and the observation equation (5.2), the Kalman filter and maximum likelihood estimation were implemented for fitting the model.

Since the innovations of both the state equation and the observation equation are assumed to be normal, the Kalman filter is proved to give the best possible estimates of the state variables, $\mathbf{x}_t^\star$. This is motivated in Section 2.3, where the recursive algorithm of the Kalman filter is being derived.

Thereafter, since the Kalman filter allows to construct the logarithmic likelihood function associated with the state space model, the logarithmic likelihood function could be produced and maximized with respect to the parameter set.

Details of the numerical optimization procedure of the logarithmic likelihood function are given in Section 5.3. The derivations of the logarithmic likelihood function for the Kalman filter and general background on maximum likelihood estimation can be found in Section 2.4.

Finally, noticeable is that the matrices in the observation equation, $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{e}_t$ are time dependent, which is of technical concerns while implementing the Kalman filter. This allows for the case of missing observations, which is needed since the observation data for the nominal- and linker yields are weekly sampled but the consumer pricing index data is only provided every fourth week, see Chapter 4. Thus, the missing price level data can be interpreted as missing observations and hence the dimensions of $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{e}_t$ in (5.2) varies in time.

5.3 Numerical Optimization

5.3.1 Parameters

As later described in Section 5.3.3, unrestricted numerical optimization was used. Thus, there was high demand for having a good initial guess of the parameter values and the choice of starting parameters thereby was essential. Particular in the sense that the logarithmic likelihood function had a 48-dimensional surface, implying that the numerical methods used for the maximization easily converged into local maximum points that gave unrealistic results.

Parameter values were chosen to equal the optimal parameter set, for *Model L-II*, see [5].

5.3.2 State Estimates

When initializing the Kalman filter a qualified guess of the state variables and their variance-covariance matrix was made.

The initial value of the state vector, $\mathbf{x}_0^\star$, was set to be a vector of zeros. This relied on the assumption that the initial parameters, chosen above, would provide a $\mathbf{G}_h$-matrix not to far away from the true *a priori* estimate of the state vector, see (5.2).

As a starting guess of the state variance-covariance matrix, $\mathbf{Q}_{0,0}$, a matrix with moderate sized entries was chosen such that the first couple of iterations would give larger weights on the actual observations and less weights on the predictions.

5.3.3 Numerical Methods

Several different algorithms were tested for maximizing the logarithmic likelihood function with respect to the unknown parameters.

With a large amount of parameters and a rather nested logarithmic likelihood function - the case of reaching a local maximum point was of concern.

According to the book *Introductory Econometrics for Finance*, see [3], one can start by using the *simplex algorithm*, that is a derivative-free algorithm. Since this algorithm does not use derivative methods, it is generally slower in reaching an optimum and it cannot produce standard error estimates. Still, the *simplex algorithm* is useful in the sense that it is more robust to local curvature. Thereafter, it is suggested that the finalization of the optimization can be performed by using the *BHHH algorithm*. This is a derivative method that besides optimizing the logarithmic likelihood function also produces standard error estimates of the parameters.

The above mentioned approach was applied with 300 iterations of the *simplex algorithm* and 30 iterations of the *BHHH algorithm*.

CHAPTER 6

Results

In this Chapter all results will be given without any conclusions or further discussions. In Section 6.1, parameter estimates, model fitting statistics and decomposition of variances will be presented. Furthermore, in Section 6.2, graphs visualizing the model output will be given.

6.1 Tables

6.1.1 Parameter Estimates

Below the optimal parameters are given as well as the corresponding standard deviations. For further interpretation of the parameters, see Chapter 3, and for details regarding the estimation procedure, see Chapter 5.

Table 6.1: Parameter Estimates: State Variables Dynamics

$$d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\mathbf{B}_t^P$$

Parameter	Estimate	Std. Dev.
$\mathcal{K}_{11}$	0.93596	(0.17593)
$\mathcal{K}_{22}$	0.04346	(0.17057)
$\mathcal{K}_{33}$	1.37640	(0.17462)
Σ_{21}	-0.00365	(0.00294)
Σ_{31}	-0.06333	(0.02466)
Σ_{32}	-0.00682	(0.00456)

Table 6.2: Parameter Estimates: Nominal Pricing Kernel

$dM_t^N/M_t^N = -r^N(\mathbf{x}_t)dt - \boldsymbol{\lambda}^N(\mathbf{x}_t)'d\boldsymbol{B}_t^P$

$r^N(\mathbf{x}_t) = \rho_0^N + \boldsymbol{\rho}_1^{N\prime}\mathbf{x}_t, \quad \boldsymbol{\lambda}^N(\mathbf{x}_t) = \boldsymbol{\lambda}_0^N + \boldsymbol{\Lambda}^N\mathbf{x}_t$

Parameter	Estimate	Std. Dev.
ρ_0^N	0.04423	(0.00750)
$\rho_{1,1}^N$	3.98740	(1.57974)
$\rho_{1,2}^N$	0.54568	(0.25100)
$\rho_{1,3}^N$	0.61299	(0.03045)
$\lambda_{0,1}^N$	0.47174	(0.20603)
$\lambda_{0,2}^N$	-0.22702	(0.13116)
$\lambda_{0,3}^N$	-0.29033	(0.10798)
$[\boldsymbol{\Sigma}\Lambda^N]_{11}$	-0.87312	(0.50910)
$[\boldsymbol{\Sigma}\Lambda^N]_{21}$	2.27240	(1.41611)
$[\boldsymbol{\Sigma}\Lambda^N]_{31}$	7.33100	(4.24987)
$[\boldsymbol{\Sigma}\Lambda^N]_{12}$	0.03062	(0.17851)
$[\boldsymbol{\Sigma}\Lambda^N]_{22}$	-0.24799	(0.20410)
$[\boldsymbol{\Sigma}\Lambda^N]_{32}$	-0.96758	(0.98965)
$[\boldsymbol{\Sigma}\Lambda^N]_{13}$	-0.12579	(0.11905)
$[\boldsymbol{\Sigma}\Lambda^N]_{23}$	0.55925	(0.40787)
$[\boldsymbol{\Sigma}\Lambda^N]_{33}$	1.05940	(0.57896)

Table 6.3: Parameter Estimates: Log Price Level

$d(logQ_t) = \pi(\mathbf{x}_t)dt + \boldsymbol{\sigma}_q^T d\boldsymbol{B}_t^P + \sigma_q^\perp dB_t^{\perp,P}$

$\pi(\mathbf{x}_t) = \rho_0^\pi + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_t$

Parameter	Estimate	Std. Dev.
ρ_0^π	0.02459	(0.00378)
$\rho_{1,1}^\pi$	0.04265	(0.70041)
$\rho_{1,2}^\pi$	0.24336	(0.16966)
$\rho_{1,3}^\pi$	-0.01620	(0.14803)
$\sigma_{q,1}$	-0.00115	(0.00044)
$\sigma_{q,2}$	0.00058	(0.00058)
$\sigma_{q,3}$	0.00030	(0.00036)
$\sigma_q^\perp$	0.00717	(0.00036)

Table 6.4: Parameter Estimates: Linker Liquidity Premium

$l_t^L = \boldsymbol{\gamma}'\mathbf{x}_t + \tilde{\gamma}\tilde{x}_t, d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma}d\tilde{B}_t^P$

$\tilde{\lambda}_t = \tilde{\lambda}_0 + \tilde{\lambda}_1\tilde{x}_t$

Parameter	Estimate	Std. Dev.
$\tilde{\gamma}$	0.60298	(0.02272)
$\tilde{\kappa}$	0.23182	(0.27443)
$\tilde{\mu}$	0.01290	(0.01944)
$\tilde{\lambda}_0$	0.30012	(0.22556)
$\tilde{\lambda}_1$	-1.39300	(27.92300)
γ_1	-0.97104	(0.79605)
γ_2	-0.04916	(0.22130)
γ_3	-0.10528	(0.15371)

Table 6.5: Parameter Estimates: Measurement Errors - Nominal Yields

Parameter	Estimate	Std. Dev.
$100 \cdot \delta_{N,3m}$	0.10223	(0.00507)
$100 \cdot \delta_{N,6m}$	-0.02231	(0.00430)
$100 \cdot \delta_{N,1y}$	0.05380	(0.00100)
$100 \cdot \delta_{N,2y}$	0.07122	(0.00197)
$100 \cdot \delta_{N,4y}$	0.03015	(0.00135)
$100 \cdot \delta_{N,7y}$	-0.09724	(0.00357)
$100 \cdot \delta_{N,10y}$	0.04999	(0.00205)

Table 6.6: Parameter Estimates: Measurement Errors - Linker Yields

Parameter	Estimate	Std. Dev.
$100 \cdot \delta_{L,5y}$	0.06657	(0.00228)
$100 \cdot \delta_{L,7y}$	-0.04024	(0.00572)
$100 \cdot \delta_{L,10y}$	-0.06544	(0.00215)

6.1.2 Model Fit Statistics

Table 6.7 presents the model diagnostic statistics for the model implied inflation linked yields when comparing to the observations, i.e. the market data. Table 6.8 presents the corresponding statistics for the model implied linkers breakeven inflations.

Table 6.7: Fitting Test: Linker Yields

Maturity	Corr	RMSE	R^2
5-year	0.99734	0.05422	0.99468
7-year	0.99961	0.01905	0.99921
10-year	0.99671	0.06145	0.99343

Table 6.8: Fitting Test: Linkers Breakeven Inflation

Maturity	Corr	RMSE	R^2
7-year	0.99283	0.05090	0.98572
10-year	0.96808	0.09721	0.93718

6.1.3 Decomposition of Variances

Tables 6.9, 6.10 and 6.11 present the model implied decompositions of the variances for the nominal yields, the inflation linked yields and the linkers breakeven inflations, respectively. Due to the autocorrelation in the time series, the decompositions are produced for the first order differences, denoted by Δ, of each time series. The following formulas are being used

$$var\big(\Delta y^N_{t,\tau}\big) = cov\big(\Delta y^N_{t,\tau}, \Delta y^R_{t,\tau}\big) + cov\big(\Delta y^N_{t,\tau}, \Delta I_{t,\tau}\big) + cov\big(\Delta y^N_{t,\tau}, \Delta \wp^I_{t,\tau}\big),$$
$$var\big(\Delta y^L_{t,\tau}\big) = cov\big(\Delta y^L_{t,\tau}, \Delta y^R_{t,\tau}\big) + cov\big(\Delta y^L_{t,\tau}, \Delta L^L_{t,\tau}\big)$$

and

$$var\big(\Delta y^{BEI,L}_{t,\tau}\big) = cov\big(\Delta y^{BEI,L}_{t,\tau}, \Delta I_{t,\tau}\big) + cov\big(\Delta y^{BEI,L}_{t,\tau}, \Delta \wp^I_{t,\tau}\big) + cov\big(\Delta y^{BEI,L}_{t,\tau}, \Delta - L^L_{t,\tau}\big).$$

Table 6.9: Decomposition of Variance: Nominal Yields

Maturity	Real Yield	Inflation Exp.	Inf. Risk. Premium
3-month	0.96691	0.10647	-0.07338
6-month	0.86299	0.16170	-0.02470
1-year	0.72940	0.19485	0.07574
2-year	0.67517	0.18576	0.13907
4-year	0.66270	0.19870	0.13861
7-year	0.66763	0.21499	0.11738
10-year	0.67553	0.22058	0.10389

Table 6.10: Decomposition of Variance: Linker Yields

Maturity	Real Yield	Liq. Premium
5-year	0.67475	0.32525
7-year	0.79789	0.20211
10-year	0.92043	0.07957

Table 6.11: Decomposition of Variance: Linkers Breakeven Inflation

Maturity	Inflation Exp.	Inf. Risk. Premium	Liq. Premium
5-year	0.18921	0.14387	0.66691
7-year	0.23931	0.15100	0.60970
10-year	0.29678	0.15742	0.54580

6.2 Graphs

6.2.1 Estimates and Model Fitting

In this section the model implied yields are given in combination with a comparison to the observed data. Also, the measurement errors are presented as well as the corresponding quantile-quantile plots. This is done for the model implied inflation linked yields and the model implied breakeven inflations. In the inflation linked yield section, the model implied liquidity premium is plotted for different maturities. Furthermore, the model implied inflation expectations are plotted against the *Prospera* inflation expectation surveys and finally the model implied inflation risk premium is presented for different maturities.

Inflation Linked Yields

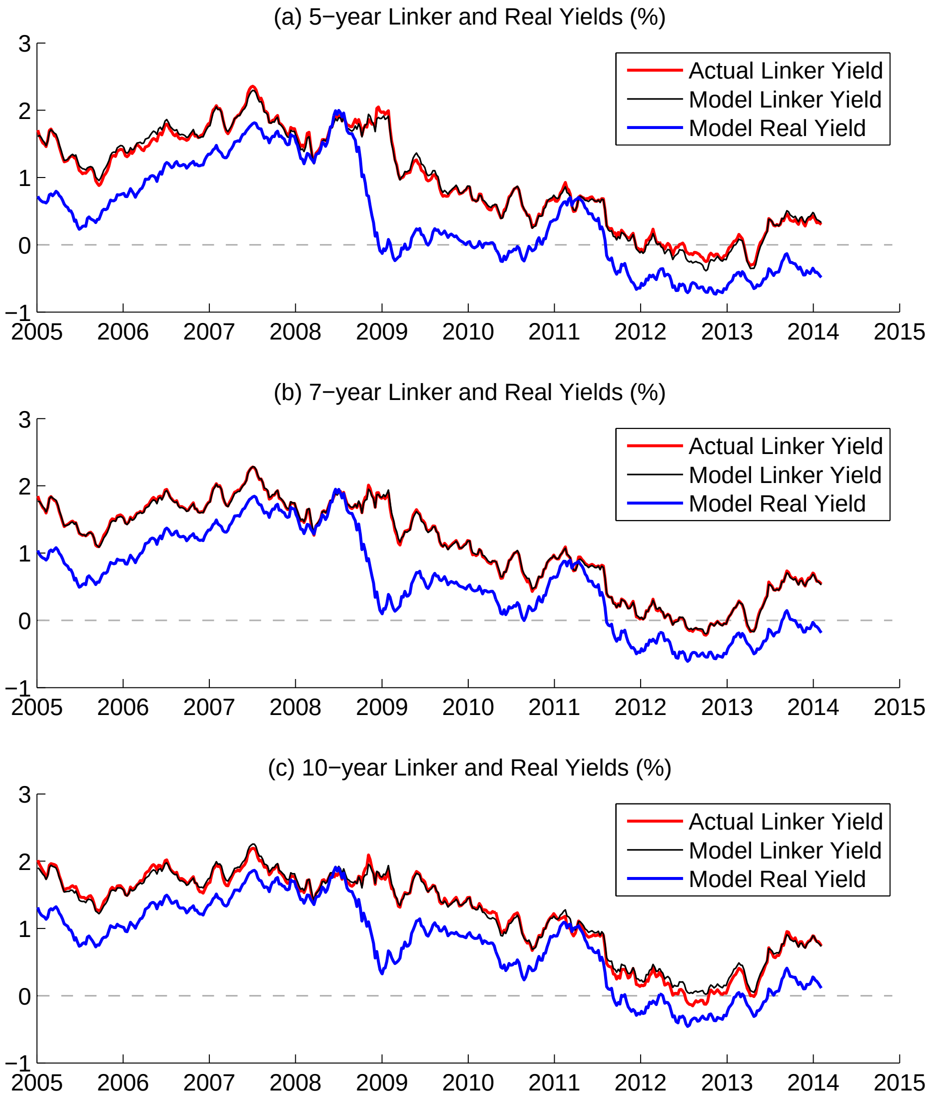

Figure 6.1: Actual- and Model Linker Yields and Real Yields

Figure 6.1 visualizes the model implied inflation linked yields, the black lines, as well as the model implied real yields, the blue lines. The red lines represent the observed market implied inflation linked yields.

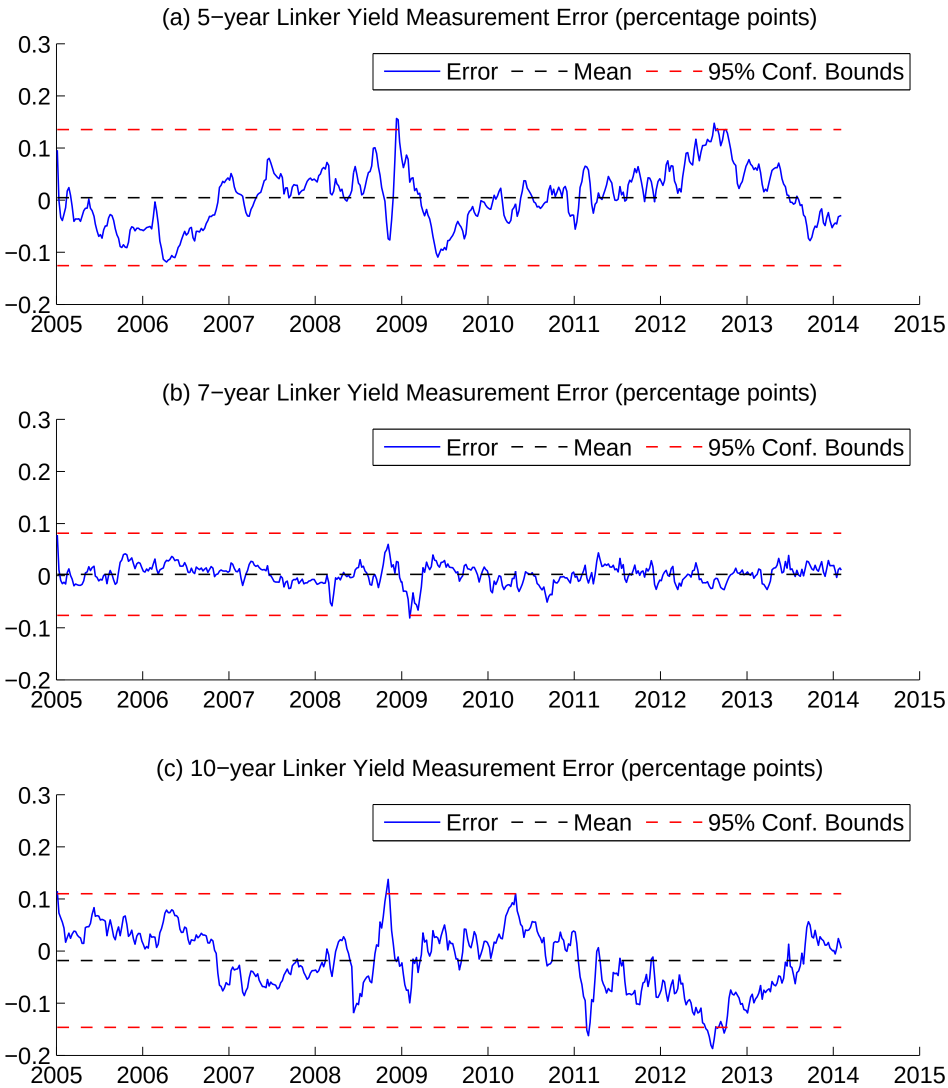

Figure 6.2: Linker Measurement Errors

Figure 6.2 visualizes the measurement error for the inflation linked yields, the blue lines, defined as the differences between the observed market implied inflation linked yields and the corresponding model implied yields. The red dashed lines represent the model implied 95% confidence bounds. Further, it is worth noting that the measurement errors also can be seen in Figure 6.1, represented by the differences between the red lines and the black lines.

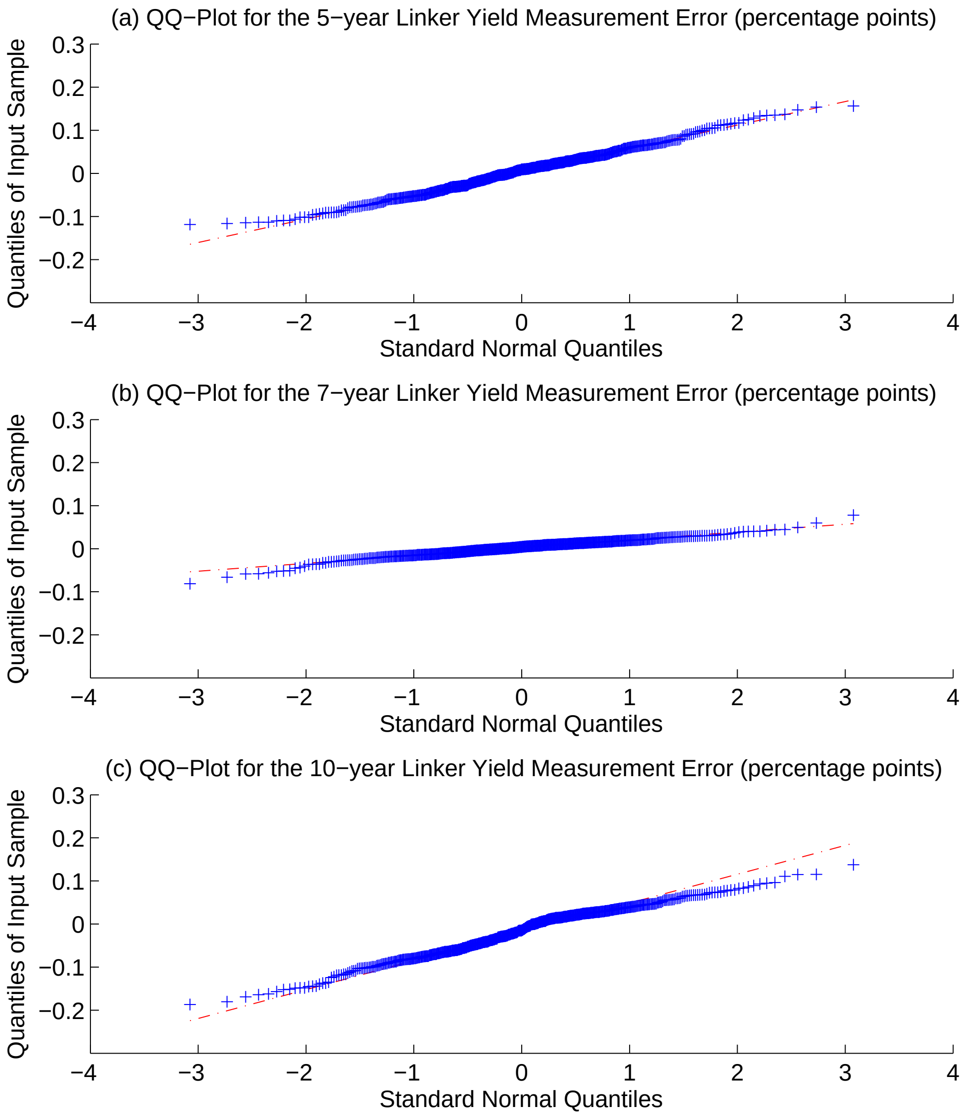

Figure 6.3: Quantile-Quantile Plot of Linker Measurement Errors

Figure 6.3 visualizes the quantile-quantile plots of the inflation linked measurement errors, given in Figure 6.2, as the blue crosses. A standard normal distribution is used as reference distribution.

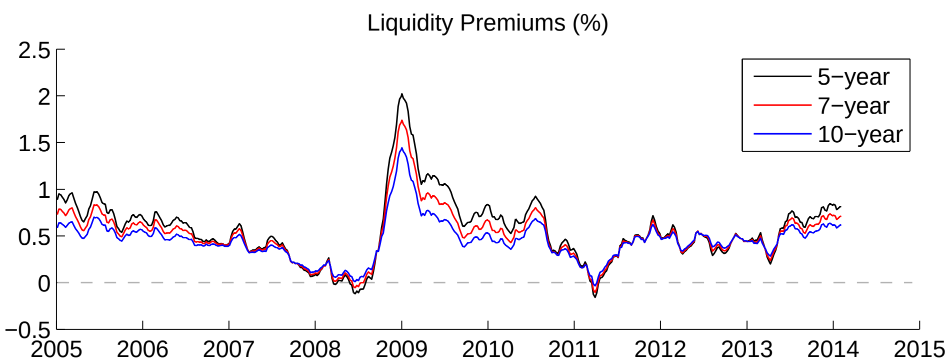

Figure 6.4: Linker Liquidity Premiums

Figure 6.4 visualizes the model implied liquidity premium for the maturities 5-, 7- and 10-years of the corresponding inflation linked bonds.

Breakeven Inflations

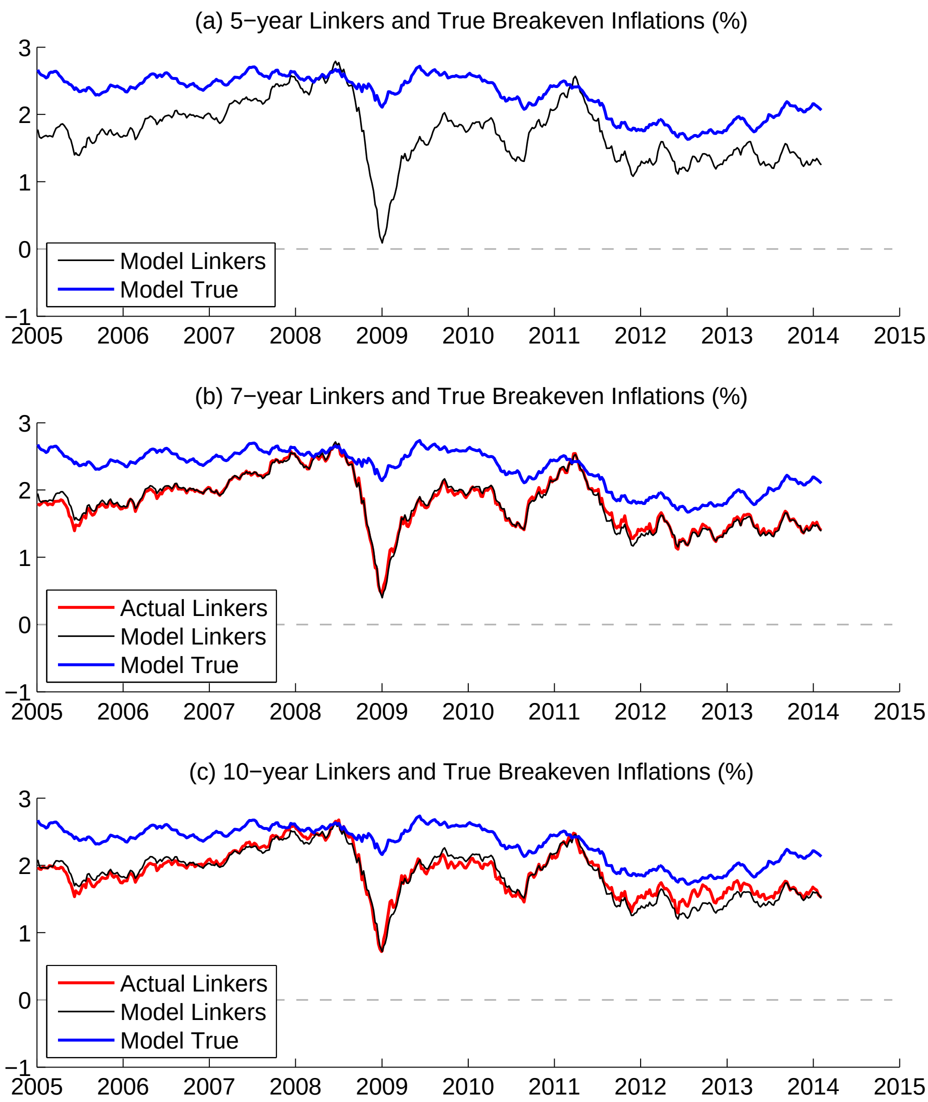

Figure 6.5: Actual- and Model Linkers Breakevens and Model True Breakevens

Figure 6.5 visualizes the model implied linkers breakeven inflations and the model implied true breakeven inflations as the black lines and the blue lines, respectively. The red lines represent the observed market implied linkers breakeven inflations.

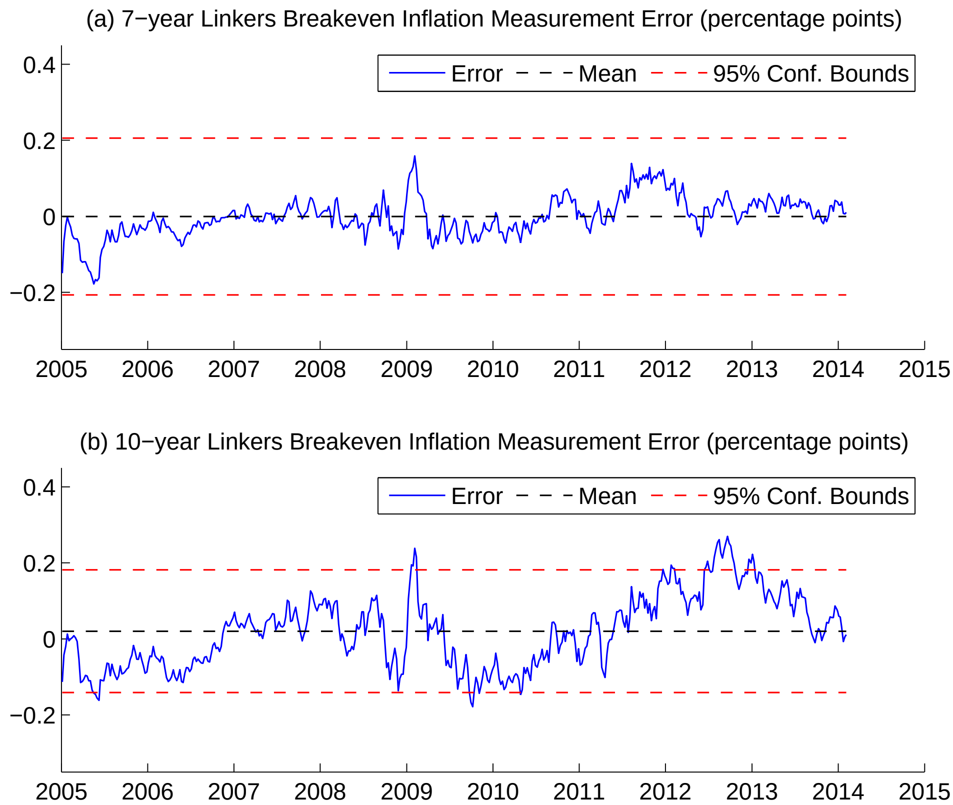

Figure 6.6: Breakeven Inflation Measurement Errors

Figure 6.6 visualizes the measurement error for the inflation linked yields, the blue lines, defined as the differences between the observed market implied inflation linked yields and the corresponding model implied yields. The red dashed lines represent the model implied 95% confidence bounds. Further, it is worth noting that the measurement errors also can be seen in Figure 6.5, represented by the differences between the red lines and the black lines.

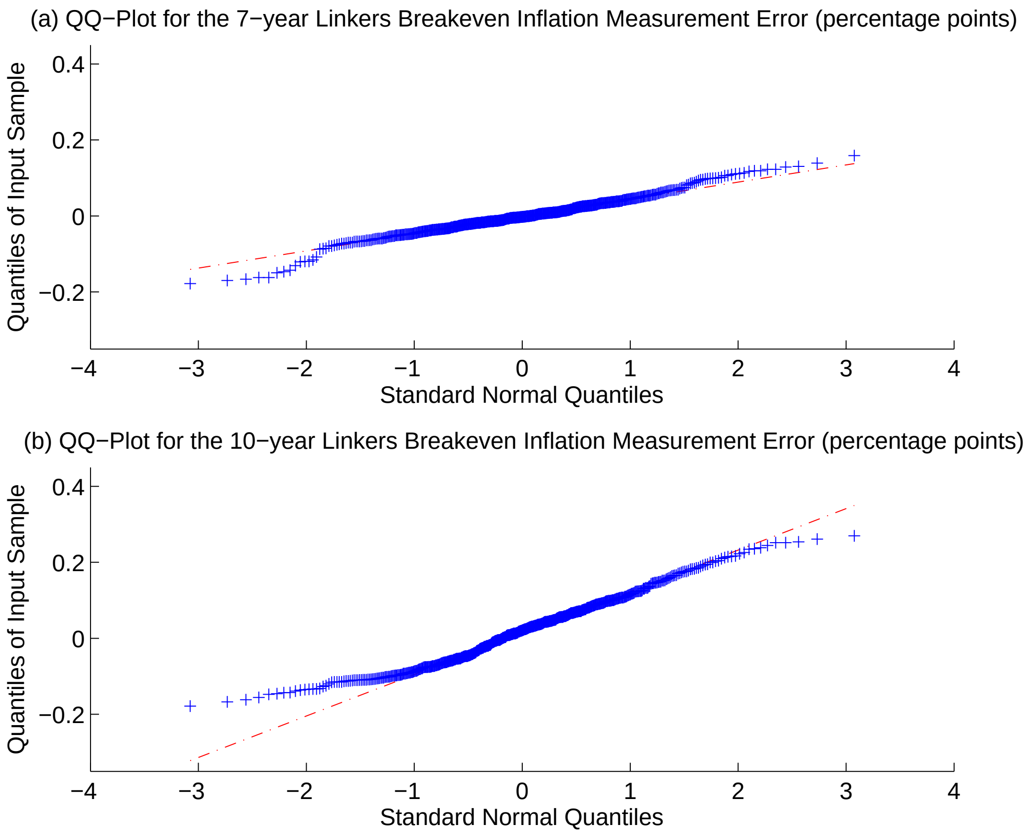

Figure 6.7: Quantile-Quantile Plot of the Breakeven Inflation Measurement Errors

Figure 6.7 visualizes the quantile-quantile plots of the linkers breakeven inflation measurement errors, given in Figure 6.6, as the blue crosses. A standard normal distribution is used as reference distribution.

Inflation Expectations

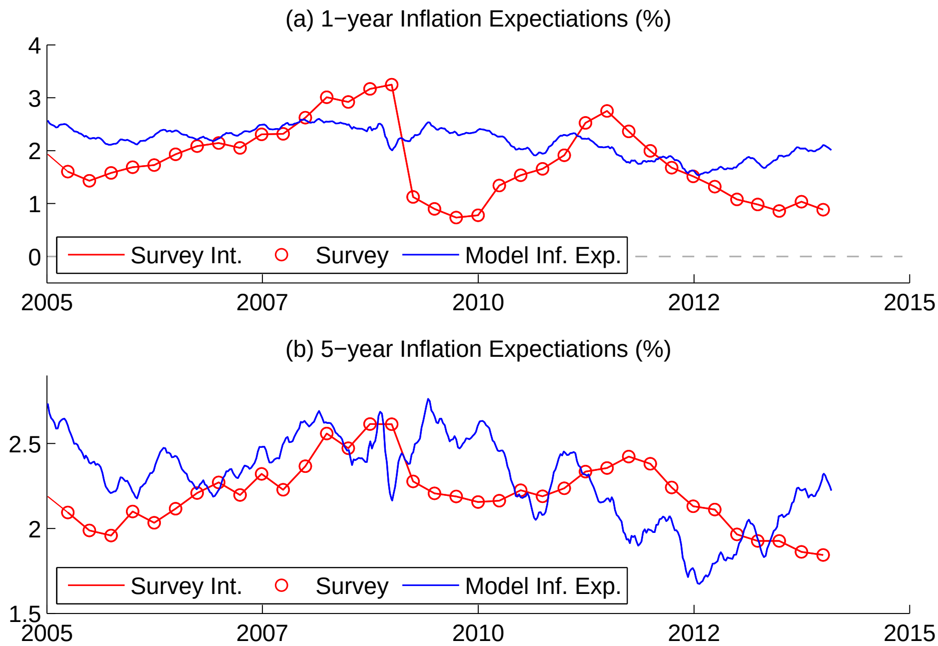

Figure 6.8: Survey Inflation Expectations and Model Inflation Expectations

Figure 6.8 visualizes the model implied inflation expectations, the blue lines, and the *Prospera* inflation expectation surveys, the red dots. The red lines represent linear interpolations for the surveys. The inflation expectations should be interpreted as the expected yearly inflations k-years from each point in time. So, for example, the 5-year inflation expectation curve should be interpreted as the expected inflation 5 years from current point in time.

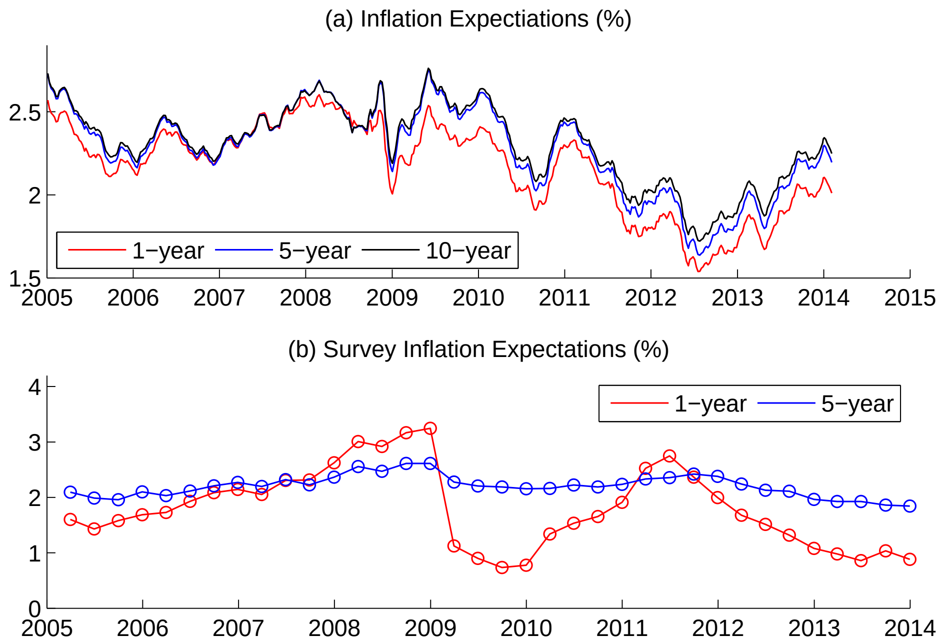

Figure 6.9: Model Inflation Expectations and Survey Inflation Expectations

Figure 6.9 visualizes, in panel (a), the 1-, 5- and 10-year model implied inflation expectations. In panel (b) the *Prospera* 1- and 5-year inflation expectation surveys are given.

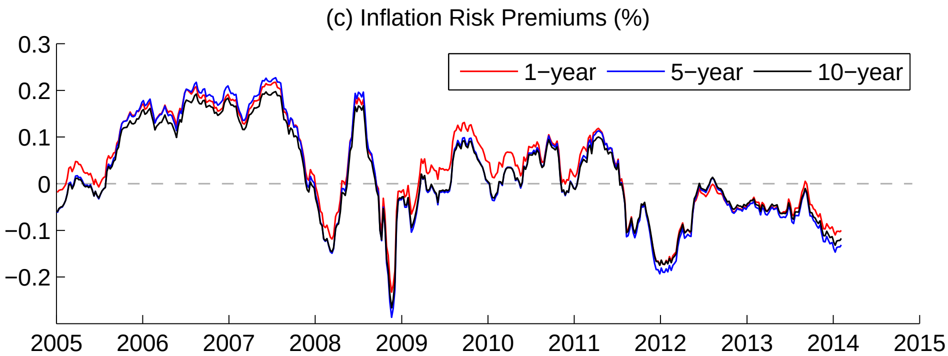

Figure 6.10: Inflation Risk Premiums

Figure 6.10 visualizes the model implied inflation risk premium for three different horizons.

6.2.2 Decompositions

In this section the model implied decompositions of the nominal yield, the inflation linked yield and the linkers breakeven inflation are given. These decompositions are given in the model chapter, see Chapter 3.

Nominal Yields

In the model chapter the nominal yield, see (3.42), is decomposed as

$$y^N_{t,\tau} = y^R_{t,\tau} + I_{t,\tau} + \wp^I_{t,\tau}. \tag{6.1}$$

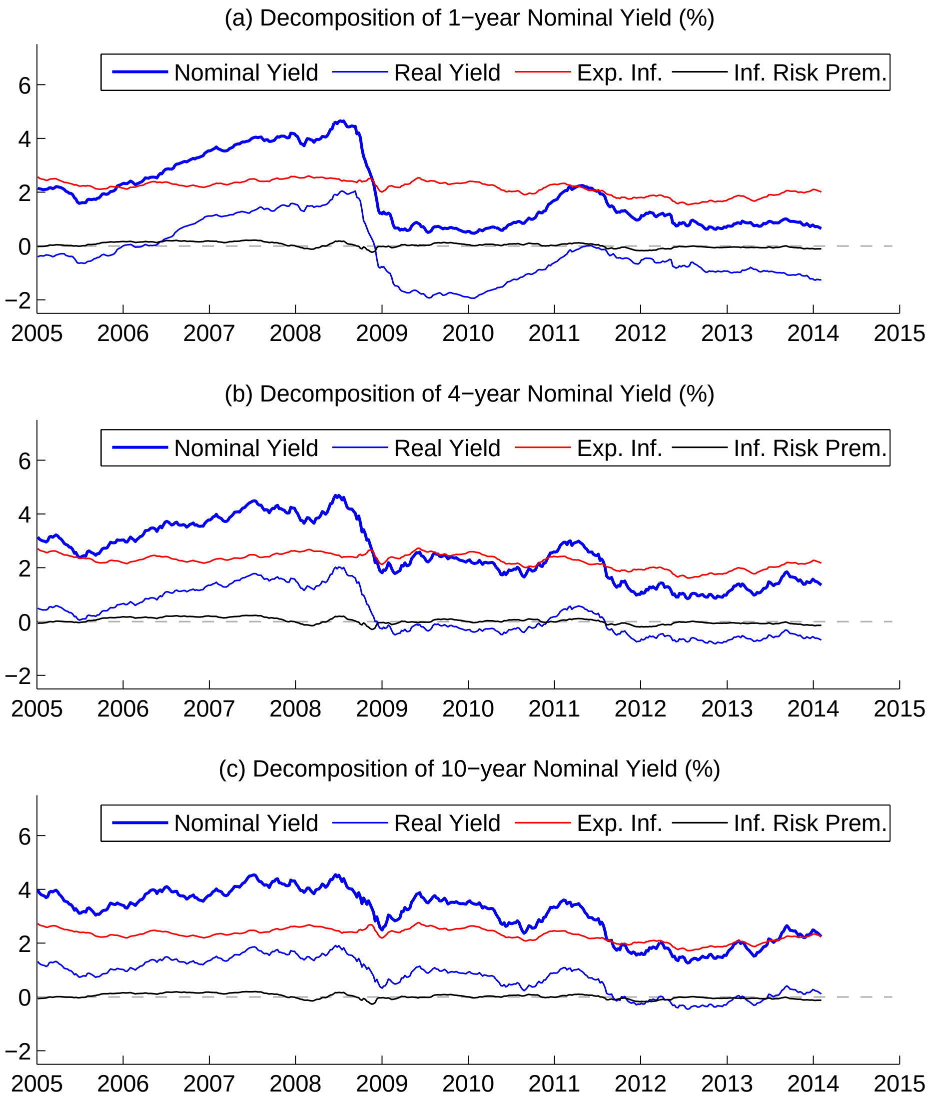

Figure 6.11: Decomposition of Nominal Yields

Figure 6.11 visualizes the decompositions given by (6.1). The bold blue lines represent the model implied nominal yields, the thin blue lines represent the model implied real yields, the red lines represent the model implied inflation expectations and the black lines represent the model implied inflation risk premiums.

Inflation Linked Yields

In the model chapter the inflation linked yield, see (3.33), is decomposed as

$$y^L_{t,\tau} = L^L_{t,\tau} + y^R_{t,\tau}. \tag{6.2}$$

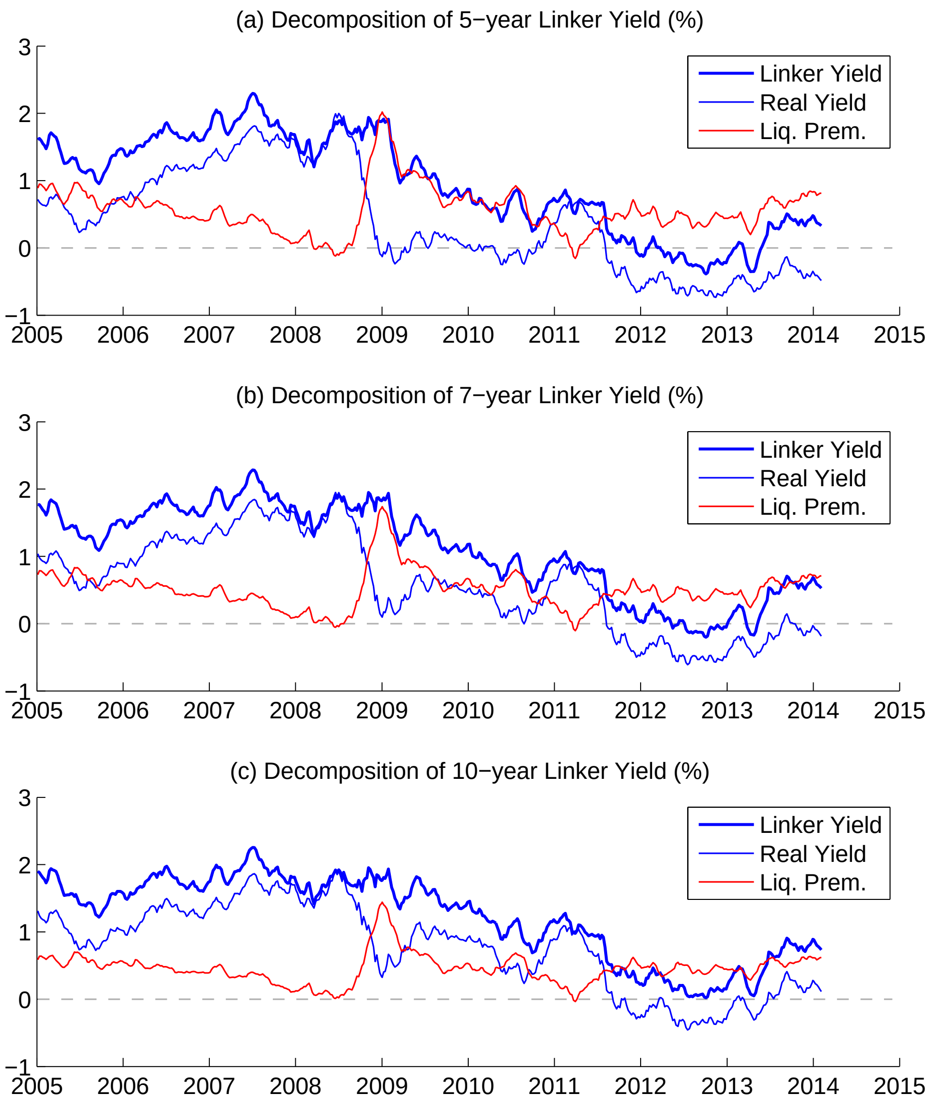

Figure 6.12: Decomposition of Linker Yields

Figure 6.12 visualizes the decompositions given by (6.2). The bold blue lines represent the model implied inflation linked yields, the thin blue lines represent the model implied real yields and the red lines represent the model implied liquidity premiums.

Breakeven Inflations

In the model chapter the linkers breakeven inflation, see (3.46), is decomposed as

$$y_{t,\tau}^{BEI,L} = I_{t,\tau} + \wp_{t,\tau}^{I} - L_{t,\tau}^{L}. \tag{6.3}$$

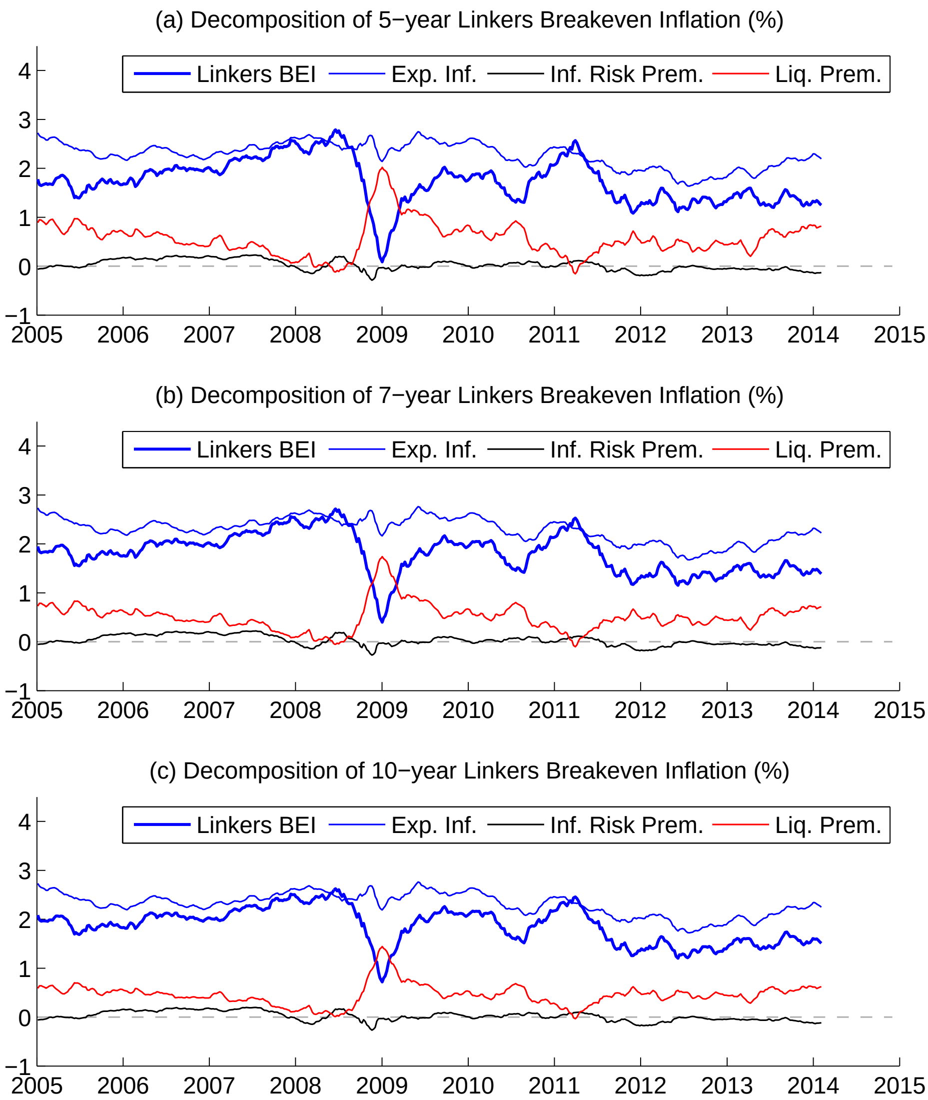

Figure 6.13: Decomposition of Linkers Breakeven Inflations

Figure 6.13 visualizes the decompositions given by (6.3). The bold blue lines represent the model implied linkers breakeven inflations, the thin blue lines represent the model implied inflation expectations, the black lines represent the model implied inflation risk premiums and the red lines represent the model implied liquidity premiums.

CHAPTER 7

Analysis

This chapter will be comprised of three different sections, were conclusions continuously will be drawn.

Firstly, a section where the model will be evaluated with respect to the observed market data, see Section 7.1.

Secondly, a section where discussions will be held regarding some of the model implied relationships in the Swedish bond market, see Section 7.2.

Finally, a short section containing the final conclusion regarding the problem formulation will be given, see Section 7.3.

7.1 Model Evaluation

7.1.1 Inflation Linked Yields

The model implied inflation linked yields can be evaluated against the corresponding market data by considering Figure 6.1. It can be seen that the model implied inflation linked yields well model the market data for each of the three maturities and that the best fit is given for the 7-year maturity.

Table 6.7 states some diagnostic statistics for the comparison between the model implied and market implied inflation linked yields. These statistics show that the model successfully manages to fit the model output to the observed market data and that the best fit is given for the 7-year maturity.

The corresponding measurement errors, given in percentage points, are visualized in Figure 6.2. The error stays fairly small for the entire time period and is close to centred around zero for all maturities. Furthermore, visually it looks like the errors can be seen as Gaussian white noises. This corresponds to the assumption made in the method chapter, see equation (5.4). To further evaluate this assumption Figure 6.3 visualizes the corresponding quantile-quantile plots. Since all quantile-quantile plots are approximately linear this strengthens the Gaussian assumption even more.

Also, in Figure 6.1 the model implied real yields are visualized. It can be concluded that the model implies real yields that do not coincide with the market implied inflation linked yields. The spread between a model implied real yield and a model implied inflation linked yield can be identified as the liquidity premium, see equation (3.32). The liquidity premiums are visualized in Figure 6.4. It can be noted that the liquidity premiums stay in the interval between 0 and 1 percent during most of the time period with one big exception, during the financial crisis. An analysis of what might drive the liquidity premium is given in Section 7.2.3.

7.1.2 Breakeven Inflations

The model implied linkers breakeven inflations can be evaluated against the corresponding market data by considering Figure 6.5. It can be seen that the model implied linkers breakeven inflations well model the corresponding market data. The best fit is once again given for the 7-year maturity.

Table 6.8 states some diagnostic statistics for the comparison between the model implied and the market implied breakeven inflations. These statistics show that the model successfully manages to fit the model output to the observed market data and that the best fit is given for the 7-year maturity. If these statistics are compared to the corresponding once for the inflation linked yields, the conclusion can be drawn that the model gives a better fit for the inflation linked yields. This result follows naturally since when modelling the breakeven inflation, there will be some errors from both fitting the nominal and the inflation linked yields.

The measurement errors for the linkers breakeven inflations, given in percentage points, are visualized in Figure 6.6. The error stays fairly small for the entire time period and is close to centred around zero for both maturities. Once again the errors look like Gaussian white noises, which is assumed in (5.5). To further evaluate this assumption Figure 6.7 visualizes the corresponding quantile-quantile plots. The 7-year plot is approximately linear, meaning that the corresponding measurement error can be approximated by a normal distribution. This conclusion can not be drawn regarding the 10-year plot, that has sort of an s-shape. This indicates that the measurement error has lighter tails than the normal distribution, see [8]. Hence, this puts some doubt on the assumption of independence between the nominal and the inflation linked measurement errors, used in (5.5).

Also, in Figure 6.5 the model implied true breakeven inflations are visualized. It can be concluded that the model implies true breakeven inflations that do not coincide with the market implied linkers breakeven inflations. This follows from the earlier drawn conclusion that the model implied real yields do not coincides with the market implied inflation linked yields.

The spread between a model implied true breakeven inflation and a model implied linkers breakeven inflation can once again be identified as the model implied liquidity premium, see equation (3.32). The liquidity premiums are visualized in Figure 6.4.

7.1.3 Inflation Expectations

The model implied inflation expectations can, in some sense, be evaluated against the *Prospera* inflation expectation surveys, see Figure 6.8. One can notice that the model implied inflation expectations are of the same magnitudes as the inflation expectation surveys but that the model does not reproduce the local variations in the surveys. This outcome was expected, since the survey data was not included in the model fitting procedure.

It is worth noting that, if taking the entire time period under consideration, the magnitudes of the model implied inflation expectations lie around 2%. This is a plausible result since it coincides with the Swedish central bank's, *Riksbankens*, inflation target at 2%, see [10].

In Figure 6.9 panel (a), one can see that the model implied inflation expectation for the different horizons seem to be highly correlated. Noticeable is also that the inflation expectations with shorter horizons constantly are lower then the once with longer horizons. The last mentioned pattern can also be recognized in the inflation expectation surveys during most of the considered time period.

7.2 Discussion

In the previous section the model fitting was evaluated. This section further focus on the model implied market relations and their plausibility. Firstly, this is done by considering the decompositions of the nominal yield, the inflation linked yield and the linkers breakeven inflation. Secondly, further investigations into what might driver the liquidity premium are performed.

7.2.1 Decomposition Analysis

All discussed variables in this part are referring to model implied outputs, hence this is not stated before each variable.

Nominal Yield

In Figure 6.11 the decompositions of the nominal yield are visualized as the sum of the real yield, the expected inflation and the inflation risk premium. Based on these decompositions, Table 6.9 gives the corresponding decompositions of the nominal yields' variances. One can notice that the real yield explains most of the variance for all maturities, but that its influence is lower for longer maturities. Further, the inflation expectation accounts for the second largest portion of the variance and its influence is increasing for longer maturities. It can also be noticed that the inflation risk premium's portion is fairly constant for the longer maturities and has a negative sign for the two shortest maturities. The above given portions can also, to some extent, be recognized by visually comparing the different maturities in Figure 6.11.

Inflation Linked Yield

In Figure 6.12 the decompositions of the inflation linked yield are visualized as the sum of the real yield and the liquidity premium. Based on these decompositions, Table 6.10 gives the decompositions of the inflation linked yields' variances. As with the nominal yield, one can notice that the real yield explains most of the variance for all maturities and that its influence this time increases for longer maturities. The liquidity premium's portion thereby decreases for longer maturities, from a portion of $\sim 33\%$ to a portion of $\sim 8\%$. The above given portions can also, to some extent, be recognized by visually comparing the different maturities in Figure 6.12.

Linkers Breakeven Inflation

In Figure 6.13 the decompositions of the linkers breakeven inflation are visualized as the sum of the expected inflation, the inflation risk premium and the negative liquidity premium. Based on these decompositions, Table 6.11 gives the decompositions of linkers breakeven inflations' variances. One can notice that the liquidity premium explains most of the variance for all maturities, but that its influence slightly decreases for longer maturities. Further, the inflation expectation accounts for the second largest portion of the variance and its influence increases for longer maturities. It can also be worth noting that the inflation expectation accounts for fairly the same portions as it did for the nominal yields. Furthermore, the inflation risk premium explains a nearly constant portion of $\sim 15\%$, for the different maturities. The above given portions can also, to some extent, be recognized by visually comparing the different maturities in Figure 6.13.

7.2.2 Inflation Expectation Analysis

In Figure 6.8 panel (b), the model implied 5-year inflation expectation is plotted versus the *Prospera* 5-year inflation expectation survey. In this graph one might identify some lagged correlation between the time series, where the survey seems to be lagged with $6 - 12$ months. This argument is contradictive in the backwash of the financial crisis, between 2009-2011. Such events are though very rare and it is well known that this particular period generally is very hard to analyse. Thus, this contradictive pattern can be disregarded as an anomaly from the hypothesis of the lagged correlation.

The trend in the model implied inflation expectation seems to revert from its peaks and valleys before the inflation expectation survey.

In that case, since the model inflation expectation has reached a minima between 2012-2013 and currently is trending upwards, one would therefore expect the inflation expectation survey to revert in a near future and start trending upwards.

7.2.3 Drivers of the Liquidity Premium

In light of the previous sections, where the consequences of having introduced a liquidity premium was shown, this section is trying to answer the question of what might drive the liquidity premium.

Can the inflation be identified as the key driver?

Firstly, one driver might be the inflation. As mentioned in Section 2.1.1, inflation linked yield levels are highly affected by inflation, this can be seen in equation (2.2). Then, since the inflation linked yields are partly determined by their respective liquidity premiums, it seems reasonable to believe that the inflation also will affect the liquidity premiums. Also, since the inflation has a great impact on the economy in general, it will affect the risk premiums in the financial market as well.

In Figure 7.1 the model implied 10-year liquidity premium is plotted versus the reversed year-on-year spot inflation.

One can notice that when the inflation is high and rising, the liquidity premium is low and decreasing. This might can be explained by reallocations among investors to inflation linked bonds, to protect their investments from the rising inflation. Therefore, the liquidity increases in inflation linked bond market and consequently the liquidity premium is decreasing.

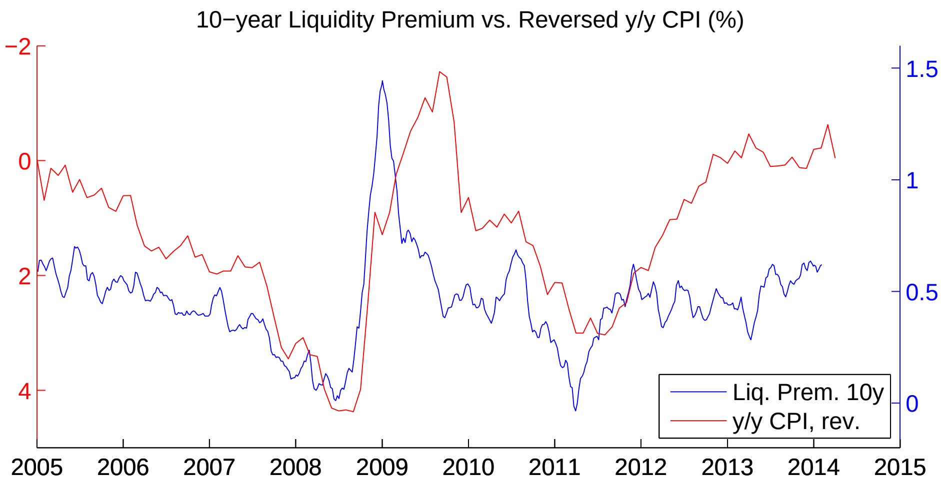

Figure 7.1: 10-year Liquidity Premium vs. y/y Swedish Consumer Pricing Index (reversed)

Can the inflation linked debt share be identified as the key driver?

Secondly, one factor that might can be a driver of the liquidity premium is the supply of inflation linked bonds. As mentioned in Section 2.1.2, the guideline is to have 25 percent of the total national government debt financed with inflation linked bonds. When this share is to low, there is a need for the Swedish national debt office, *Riksgälden*, to issue more inflation linked bonds and conversely, when the share is to high there is a need for buybacks of the excess stock of inflation linked bonds or a decreased issuance. Hence, there seems to be a clear connection between the debt share and the liquidity conditions in the inflation linked bond market. Before investigating whether this relation is realised in the liquidity premium, the following two paragraphs will cover the mechanisms connecting the inflation, the consumer pricing index and the debt share.

As mentioned in Section 2.1.1, the inflation linked yield levels are affected by the consumer pricing index level. When the consumer pricing index is decreasing the inflation linked debt share decreases as well, this can be seen in equation (2.2). Also, there is a tendency for shortage in the government finances when the inflation is low, i.e. when having a downward trending consumer pricing index. Larger shortage leads to an increased requirement for borrowing and this shortage is primarily adjusted by issuing nominal bonds. This implies that the debt share of inflation linked bonds decreases even further.

Conversely, if there is a high economic activity, the inflation will be high and the consumer pricing index will be upward trending. Thus, the inflation linked bond prices will increase with the consumer pricing index and the government earnings will increase due to the high economic activity. In this case the total national debt will decrease, due to buybacks or less issuance of nominal bonds, which implies that the share of inflation linked bonds will grow even further.

Based on the discussion in the previous two paragraphs, it is of interest to add the inflation linked debt share to Figure 7.1. This is done in Figure 7.2 where the 10-year liquidity premium is plotted versus both the reversed year-on-year spot inflation and the reversed inflation linked debt share, given in percent.

In Figure 7.2 the previous stated events can be identified, where a rising consumer pricing index corresponds to an increasing inflation linked debt share. It can also be seen that these events are followed by a decreasing liquidity premium. This well corresponds to the earlier stated hypothesis, that these events probably would enhance the liquidity conditions in the inflation linked bond market and therefore give a lower liquidity premium. Thus the inflation linked debt share might be a driver of the liquidity premium.

The converse situation with a downward trending consumer pricing index and a decreasing inflation linked debt share, can be seen to imply a higher liquidity premium. Once again it is therefore noted that the inflation linked debt share might be a driver of the liquidity premium.

For completeness, it needs to be stated that the inflation linked debt share cannot explain the rapid movements in the liquidity premium during the financial crisis. Still, this period can be considered as an extreme outcome and hence partly be disregard from the analysis.

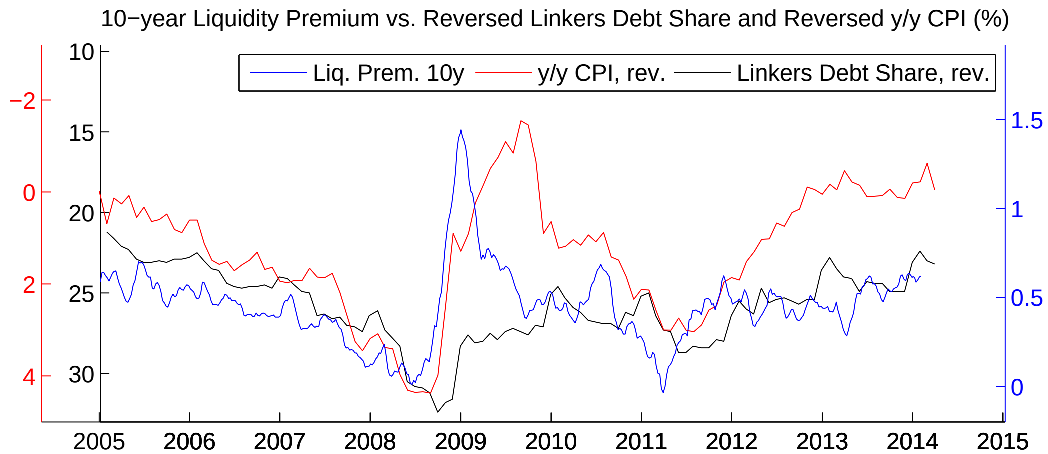

Figure 7.2: 10-year Liquidity Premium vs. y/y Swedish Consumer Pricing Index (reversed) and Inflation Linked Debt Share (reversed)

Further Discussions

It can furthermore be interesting to examine the behavior of the liquidity premium during the 2008 financial crisis. At the height of the crisis the global financial system experienced urgent demand for cash from various sources, including counterparties, short-term creditors and especially existing borrowers. Thus, during this period the demand for reallocation into liquid assets, such as nominal government bonds, increased. Moreover, during a financial crisis of this kind the inflation can be seen to drop rapidly and the need for inflation protection will consequently be very low. Then, which can be seen in Figure 7.2, the combination of the decreased demand for inflation linked bonds and the high demand for reallocation into nominal bonds, caused the liquidity premium to increase considerably during this period. This increase in the liquidity premium could also be identified for inflation linked bonds with other maturities, see Figure 6.4. Hence a driver of the liquidity premium during times of severe financial stress might be the risk averse actions of investors reallocating their investments to more liquid assets.

To sum up, if the hypothesizes presented in this and the previous sections are reflecting what actually is the case, then the following factors might be drivers of the liquidity premium:

- the inflation
- the inflation linked debt share
- the risk awareness among investors (during severe financial stress)

7.3 Final Conclusion

The aim of this thesis was to investigate the possibility of fitting a model, including an inflation linked specific liquidity premium, to the Swedish bond market.

It was found that the estimation methodology and the model fitting were successful. Furthermore, the model implied results seemed to be intuitive and realistic. Thus it can be concluded that the used model was appropriate for modelling the Swedish bond market.

APPENDIX A

Model Derivations

A.1 Latent Variable Models

$$d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P \qquad \mathbf{x}_t = [x_{1t},\ x_{2t},\ x_{3t}]^T \tag{A.1}$$
$$dq_t = d(logQ_t) = \pi(\mathbf{x}_t)dt + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P} \qquad \mathbf{x}_t = [x_{1t},\ x_{2t},\ x_{3t}]^T \tag{A.2}$$
$$d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma} d\tilde{B}_t^P, \tag{A.3}$$

where $\pi(\mathbf{x}_t)$ is the instantaneous expected inflation and is an affine function in the state variables

$$\pi(\mathbf{x}_t) = \rho_0^{\pi} + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_t. \tag{A.4}$$

A.2 Inflation and the Real Pricing Kernel

The real and the nominal pricing kernels must fulfil the following no-arbitrage relation

$$M_t^R = M_t^N Q_t, \tag{A.5}$$

where

$$\frac{dM_t^N}{M_t^N} = -r^N(\mathbf{x}_t)dt - \boldsymbol{\lambda}^N(\mathbf{x}_t)' d\boldsymbol{B}_t^P \tag{A.6}$$
$$dq_t = d(logQ_t) = \pi(\mathbf{x}_t)dt + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P}, \tag{A.7}$$

are the nominal pricing kernel and the price level respectively. Also, in (A.6) it is assumed that

$$r^N(\mathbf{x}_t) = \rho_0^N + \boldsymbol{\rho}_1^{N\prime}\mathbf{x}_t \tag{A.8}$$
$$\boldsymbol{\lambda}^N(\mathbf{x}_t) = \boldsymbol{\lambda}_0^N + \boldsymbol{\Lambda}^N\mathbf{x}_t, \tag{A.9}$$

are the nominal short rate and the nominal prices of risk, respectively. Then, Itô's formula can be applied to (A.5)

$$\begin{aligned} dM_t^R &= Q_t dM_t^N + M_t^N dQ_t + dM^N dQ_t \\ &= \left\{ divide\ on\ both\ sides\ by\ M_t^R = M_t^N Q_t,\ (A.5) \right\} \\ &\Leftrightarrow \\ \frac{dM_t^R}{M_t^R} &= \frac{dM_t^N}{M_t^N} + \frac{dQ_t}{Q_t} + \frac{dM_t^N}{M_t^N}\frac{dQ_t}{Q_t}. \end{aligned} \tag{A.10}$$

One can see that the real pricing kernel is driven by three factors. Firstly, the change in the nominal pricing kernel, secondly, the change in the price level and thirdly, the cross term of the previous two changes. Left is now only to calculate the term $\frac{dQ_t}{Q_t}$, since $\frac{dM_t^N}{M_t^N}$ is given by (A.6). From (A.2) it follows that

$$logQ_t = logQ_0 + \int_0^t \pi(\mathbf{x}_s)ds + \int_0^t \boldsymbol{\sigma}_q' d\boldsymbol{B}_s^P + \int_0^t \sigma_q^{\perp} dB_s^{\perp,P}. \tag{A.11}$$

Hence it can be used that

$$Q_t = e^{F(t,\boldsymbol{B}_t)}, \tag{A.12}$$

where

$$F(t,\boldsymbol{B}_t) = logQ_0 + \int_0^t \pi(\mathbf{x}_s)ds + \int_0^t \boldsymbol{\sigma}_q' d\boldsymbol{B}_s^P + \int_0^t \sigma_q^{\perp} dB_s^{\perp,P}. \tag{A.13}$$

Then, Itô's formula can be applied to (A.12)

$$\begin{aligned}
dQ_t &= \frac{\partial F}{\partial t}Q_t dt + \frac{\partial F}{\partial \boldsymbol{B}^P}Q_t d\boldsymbol{B}_t^P + \frac{\partial F}{\partial B_t^{\perp,P}}Q_t dB_t^{\perp,P} + \frac{1}{2}\frac{\partial^2 F}{\partial^2 \boldsymbol{B}_t^P}Q_t\big((d\boldsymbol{B}^P)'d\boldsymbol{B}^P\big) + \frac{1}{2}\frac{\partial^2 F}{\partial^2 B_t^{\perp,P}}Q_t(dB_t^{\perp})^2 \\
&= \Big\{(dB_t^{\perp,P})^2 = (d\boldsymbol{B}^P)'d\boldsymbol{B}^P = dt\Big\} \\
&= \pi(\mathbf{x}_t)Q_t dt + \boldsymbol{\sigma}_q' Q_t d\boldsymbol{B}_t^P + \sigma_q^{\perp} Q_t dB_t^{\perp,P} + \frac{1}{2}\boldsymbol{\sigma}_q'\boldsymbol{\sigma}_q Q_t dt + \frac{1}{2}{\sigma_q^{\perp}}^2 Q_t dt \\
&= \Big[\pi(\mathbf{x}_t) + \frac{1}{2}\boldsymbol{\sigma}_q'\boldsymbol{\sigma}_q + \frac{1}{2}{\sigma_q^{\perp}}^2\Big]Q_t dt + \boldsymbol{\sigma}_q' Q_t d\boldsymbol{B}_t^P + \sigma_q^{\perp} Q_t dB_t^{\perp,P},
\end{aligned}$$

where it has been used that $d\boldsymbol{B}_t^P dB_t^{\perp,P} = 0$. Then, if this expression together with (A.6) are inserted into (A.10) and using the fact that $d\boldsymbol{B}_t^P dB_t^{\perp,P} = dB_t^{\perp,P}dt = d\boldsymbol{B}_t^P dt = 0$, it follows that

$$\begin{aligned}
\frac{dM^R}{M^R} &= -r^N(\mathbf{x}_t)dt - \boldsymbol{\lambda}^N(\mathbf{x}_t)'d\boldsymbol{B}_t^P + \Big[\pi(\mathbf{x}_t) + \frac{1}{2}\boldsymbol{\sigma}_q'\boldsymbol{\sigma}_q + \frac{1}{2}{\sigma_q^{\perp}}^2\Big]dt \\
&\quad + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P} - \boldsymbol{\lambda}^N(\mathbf{x}_t)'\boldsymbol{\sigma}_q' dt \\
&= \Big[-r^N(\mathbf{x}_t) + \pi(\mathbf{x}_t) + \frac{1}{2}\boldsymbol{\sigma}_q'\boldsymbol{\sigma}_q + \frac{1}{2}{\sigma_q^{\perp}}^2 - \boldsymbol{\lambda}^N(\mathbf{x}_t)'\boldsymbol{\sigma}_q\Big]dt \\
&\quad + \Big[-\boldsymbol{\lambda}^N(\mathbf{x}_t)' + \boldsymbol{\sigma}_q'\Big]d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P} \\
&= \Big\{insert\ r^N(\mathbf{x}_t),\ \ \pi(\mathbf{x}_t)\ \ and\ \ \boldsymbol{\lambda}^N(\mathbf{x}_t); (A.8),\ \ (A.4)\ \ and\ \ (A.9)\Big\} \\
&= \Big[-\rho_0^N - {\boldsymbol{\rho}_1^N}'\mathbf{x}_t + \rho_0^{\pi} + {\boldsymbol{\rho}_1^{\pi}}'\mathbf{x}_t + \frac{1}{2}\boldsymbol{\sigma}_q'\boldsymbol{\sigma}_q + \frac{1}{2}{\sigma_q^{\perp}}^2 - (\boldsymbol{\lambda}_0^N + \boldsymbol{\Lambda}^N\mathbf{x}_t)'\boldsymbol{\sigma}_q\Big]dt \\
&\quad + \Big[-{\boldsymbol{\lambda}_0^N}' - {\boldsymbol{\Lambda}^N}'\mathbf{x}_t + \boldsymbol{\sigma}_q'\Big]d\boldsymbol{B}_t^P + \sigma_q^{\perp} dB_t^{\perp,P} \\
&= -r^R(\mathbf{x}_t)dt - \boldsymbol{\lambda}^R(\mathbf{x}_t)'d\boldsymbol{B}_t^P + {\sigma_t^{\perp}}' dB_t^{\perp,P},
\end{aligned}$$

where

$$r^R(\mathbf{x}_t) = \rho_0^R + {\boldsymbol{\rho}_1^R}'\mathbf{x}_t \tag{A.14}$$

$$\boldsymbol{\lambda}^R(\mathbf{x}_t) = \boldsymbol{\lambda}_0^R + \boldsymbol{\Lambda}^R\mathbf{x}_t, \tag{A.15}$$

where

$$\rho_0^R = \rho_0^N - \rho_0^{\pi} - \frac{1}{2}(\boldsymbol{\sigma}_q'\boldsymbol{\sigma}_q + {\sigma_q^{\perp}}^2) + {\boldsymbol{\lambda}_0^N}'\boldsymbol{\sigma}_q \tag{A.16}$$

$$\boldsymbol{\rho}_1^R = \boldsymbol{\rho}_1^N - \boldsymbol{\rho}_1^{\pi} + {\boldsymbol{\Lambda}^N}'\boldsymbol{\sigma}_q \tag{A.17}$$

$$\boldsymbol{\lambda}_0^R = \boldsymbol{\lambda}_0^N - \boldsymbol{\sigma}_q \tag{A.18}$$

$$\boldsymbol{\Lambda}^R = \boldsymbol{\Lambda}^N. \tag{A.19}$$

A.3 Nominal and Real Bond Prices and Yields

As in standard literature, see [2], the bond price at time 0 of a bond with maturity at time T is given as

$$p(0,T) = E^Q\Big[exp\Big\{-\int_0^T r^i(s)ds\Big\}\Big]. \tag{A.20}$$

If then τ is defined as the time to maturity from the current time t, an equivalent expression for the bond price at time t with maturity at time $t+\tau$ is given by

$$p^i(t,\tau) = E^Q\Big[exp\Big\{-\int_t^{t+\tau} r^i(s)ds\Big\}\Big], \qquad i = N, R, \tag{A.21}$$

where i states if the price is for a nominal or a real bond. As in the above calculations it is assumed that the nominal short rate and the real short rate are given as

$$r^N(x_t) = \rho_0^N + \boldsymbol{\rho}_1^{N\prime}\mathbf{x}_t \tag{A.22}$$

$$r^R(x_t) = \rho_0^R + \boldsymbol{\rho}_1^{R\prime}\mathbf{x}_t, \tag{A.23}$$

where $\mathbf{x}_t$ is given by (A.1). Bond prices are almost always assumed to posses affine term structures which states that the bond prices can be expressed as

$$p^i(t,\tau) = F(t,\tau,\mathbf{x}_t) = e^{A_\tau^i + \boldsymbol{B}_\tau^{i\prime}\mathbf{x}_t}, \qquad i = N, R. \tag{A.24}$$

where A_τ^i and $\boldsymbol{B}_\tau^i$ are deterministic functions of τ, taking the maturity time T as a parameter.

Itô's formula applied to (A.24) then gives

$$\begin{aligned}
dp^i(t,\tau) = dF(t,\tau,\mathbf{x}_t) &= \frac{\partial F}{\partial \tau}d\tau + \frac{\partial F}{\partial \mathbf{x}_t}d\mathbf{x}_t + \frac{1}{2}\frac{\partial^2 F}{\partial \mathbf{x}_t^2}(d\mathbf{x}_t' d\mathbf{x}_t)\\
&= \Big\{notation; \quad \partial A_\tau^i = \frac{\partial A_\tau^i}{\partial \tau}, \quad \partial \boldsymbol{B}_\tau^i = \frac{\partial \boldsymbol{B}_\tau^i}{\partial \tau}\Big\}\\
&= (\partial A_\tau^i + \partial \boldsymbol{B}_\tau^{i\prime}\mathbf{x}_t)Fd\tau + \boldsymbol{B}_\tau^{i\prime}Fd\mathbf{x}_t + \frac{1}{2}\boldsymbol{B}_\tau^{i\prime}\boldsymbol{B}_\tau^i F(d\mathbf{x}_t' d\mathbf{x}_t)\\
&= \Big\{\tau = T - t \Leftrightarrow d\tau = -dt\Big\}\\
&= -(\partial A_\tau^i + \partial \boldsymbol{B}_\tau^{i\prime}\mathbf{x}_t)Fdt + \boldsymbol{B}_\tau^{i\prime}Fd\mathbf{x}_t + \frac{1}{2}\boldsymbol{B}_\tau^{i\prime}\boldsymbol{B}_\tau^i F(d\mathbf{x}_t' d\mathbf{x}_t)\\
&= \Big\{(A.1), \quad d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P\Big\}\\
&= -(\partial A_\tau^i + \partial \boldsymbol{B}_\tau^{i\prime}\mathbf{x}_t)Fdt + \boldsymbol{B}_\tau^{i\prime}F(\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P) + \frac{1}{2}\boldsymbol{B}_\tau^{i\prime}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^i Fdt\\
&= \big[-(\partial A_\tau^i + \partial \boldsymbol{B}_\tau^{i\prime}\mathbf{x}_t) + \boldsymbol{B}_\tau^{i\prime}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{i\prime}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^i\big]Fdt + \boldsymbol{B}_\tau^{i\prime}\boldsymbol{\Sigma} Fd\boldsymbol{B}_t^P\\
&= \Big\{Girsanov\ transformation\ (Theorem\ 2.6); \quad d\boldsymbol{B}_\tau^P = \boldsymbol{\varphi}_t dt + d\boldsymbol{B}_\tau^Q\Big\}\\
&= \Big\{-\boldsymbol{\varphi}_t^i = market\ price\ of\ risk = \boldsymbol{\lambda}^i(\mathbf{x}_t) = \boldsymbol{\lambda}_0^i + \boldsymbol{\Lambda}^i\mathbf{x}_t \qquad i = N, R\Big\}\\
&= \big[-(\partial A_\tau^i + \partial \boldsymbol{B}_\tau^{i\prime}\mathbf{x}_t) + \boldsymbol{B}_\tau^{i\prime}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{i\prime}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^i - \boldsymbol{B}_\tau^{i\prime}\boldsymbol{\Sigma}(\boldsymbol{\lambda}_0^i + \boldsymbol{\Lambda}^i\mathbf{x}_t)\big]Fdt\\
&\quad + \boldsymbol{B}_\tau^{i\prime}\boldsymbol{\Sigma} Fd\boldsymbol{B}_t^Q.
\end{aligned}$$

Since the above expression is given under the risk neutral measure Q, the drift must equal the local rate of return

$$r^i(\mathbf{x}_t) = -(\partial A^i_\tau + \partial \boldsymbol{B}^{i\,\prime}_\tau \mathbf{x}_t) + \boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{B}^i_\tau - \boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\Sigma}(\boldsymbol{\lambda}^i_0 + \boldsymbol{\Lambda}^i \mathbf{x}_t). \tag{A.25}$$

Then, if (A.25) is inserted into (A.22) and (A.23) the following relation is given

$$\big[-\partial A^i_\tau + \boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\mathcal{K}}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}^i_\tau - \boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\Sigma}\boldsymbol{\lambda}^i_0 - \rho^i_0\big] + \big[-\partial \boldsymbol{B}^{i\,\prime}_\tau - \boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\mathcal{K}} - \boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\Sigma}\boldsymbol{\Lambda}^i - \boldsymbol{\rho}^{i\,\prime}_1\big]\mathbf{x}_t = 0.$$

This equation must hold for all t, τ and $\mathbf{x}_t$. If one consider the equation for fixed t and τ it still must hold for all $\mathbf{x}_t$. Hence, the constant term in front of $\mathbf{x}_t$ must equal zero. Therefore the other constant term also must equal zero. The following system of ordinary differential equations is then generated

$$\frac{dA^i_\tau}{d\tau} = -\rho^i_0 + \boldsymbol{B}^{i\,\prime}_\tau(\boldsymbol{\mathcal{K}}\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}^i_0) + \frac{1}{2}\boldsymbol{B}^{i\,\prime}_\tau \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}^i_\tau \tag{A.26}$$

$$\frac{d\boldsymbol{B}^i_\tau}{d\tau} = -\boldsymbol{\rho}^i_1 - (\boldsymbol{\mathcal{K}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}^i)'\boldsymbol{B}^i_\tau, \tag{A.27}$$

with initial conditions $A^i_0 = 0$ and $\boldsymbol{B}^i_0 = \mathbf{0}$. The initial conditions are given by the fact that the bond price must equal the face value at maturity and hence the exponent in (A.24) must equal 0 for $\tau = 0$.

Equation (A.26) and (A.27) can then be solved using numerical methods. Finally, the bond prices are given by inserting these solutions into (A.24).

Also, it follows that the nominal and real yields are given as

$$y^i_{t,\tau} = a^i_\tau + \mathbf{b}^{i\,\prime}_\tau \mathbf{x}_t, \qquad i = N, R. \tag{A.28}$$

Where it has been used that

$$a^i_\tau = -\frac{A^i_\tau}{\tau} \tag{A.29}$$

$$\mathbf{b}^i_\tau = -\frac{\mathbf{B}^i_\tau}{\tau}. \tag{A.30}$$

A.4 Inflation Expectations

The inflation expectation is assumed to take an affine form

$$I_{t,\tau} = \frac{1}{\tau} E_t\Big[log\Big(\frac{Q_{t+\tau}}{Q_t}\Big)\Big] = a_\tau^I + \boldsymbol{b}_\tau^{I\prime}\mathbf{x}_t, \tag{A.31}$$

where Q_t is given by (A.2)

$$d(logQ_t) = \pi(\mathbf{x}_t)dt + \boldsymbol{\sigma}_q' d\boldsymbol{B}_t^P + \sigma_q^\perp dB_t^{\perp,P}. \tag{A.32}$$

The first step in order to identify the constant, a_τ^I, and the constant vector, $\mathbf{b}_\tau^I$, in (A.31) is to integrate the expression in (A.32) and insert into (A.31)

$$\begin{aligned}
\tau I_{t,\tau} &= E_t\Big[log\Big(\frac{Q_{t,\tau}}{Q_t}\Big)\Big] \\
&= \Big\{(A.2), \quad Q_t = exp\Big\{logQ_0 + \int_0^T \pi(\mathbf{x}_s)ds + \int_0^T \boldsymbol{\sigma}_q' d\boldsymbol{B}_s^P + \int_0^T \sigma_q^\perp dB_s^{\perp,P}\Big\}\Big\} \\
&= E\Big[log\Big(exp\Big\{\int_0^{t+\tau} \pi(\mathbf{x}_s)ds + \int_0^{t+\tau} \boldsymbol{\sigma}_q' d\boldsymbol{B}_s^P + \int_0^{t+\tau} \sigma_q^\perp dB_s^{\perp,P}\Big\} \\
&\qquad \cdot exp\Big\{-\Big(\int_0^t \pi(\mathbf{x}_s)ds + \int_0^t \boldsymbol{\sigma}_q' d\boldsymbol{B}_s^P + \int_0^t \sigma_q^\perp dB_s^{\perp,P}\Big)\Big\}\Big)\Big] \\
&= E\Big[\int_t^{t+\tau} \pi(\mathbf{x}_s)ds + \int_t^{t+\tau} \boldsymbol{\sigma}_q' d\boldsymbol{B}_s^P + \int_t^{t+\tau} \sigma_q^\perp dB_s^{\perp,P}\Big] \\
&= \Big\{linearity; \quad E[It\hat{o}\, integral] = 0\Big\} \\
&= E\Big[\int_t^{t+\tau} \pi(\mathbf{x}_s)ds\Big] = \Big\{fubini\Big\} = \int_t^{t+\tau} E\big[\pi(\mathbf{x}_s)\big]ds \\
&= \Big\{(A.4), \quad \pi(\mathbf{x}_t) = \rho_0^\pi + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_t\Big\} \\
&= \int_t^{t+\tau} E\big[\rho_0^\pi + \boldsymbol{\rho}_1^{\pi\prime}\mathbf{x}_s\big]ds = \Big\{linearity\Big\} \\
&= \rho_0^\pi\tau + \boldsymbol{\rho}_1^{\pi\prime}\int_t^{t+\tau} E\big[\mathbf{x}_s\big]ds.
\end{aligned}$$

The above calculations yield the following relation

$$\tau I_{t,\tau} = \rho_0^\pi\tau + \boldsymbol{\rho}_1^{\pi\prime}\int_t^{t+\tau} E\big[\mathbf{x}_s\big]ds. \tag{A.33}$$

Left is now to calculate the integral in (A.33), or more specifically, calculate $E[\mathbf{x}_s]$. This can be done by applying Itô's formula to the expression $e^{\boldsymbol{\mathcal{K}}s}\mathbf{x}_s$, $s \in (t, t+\tau)$,

$$\begin{aligned}
d\big(e^{\boldsymbol{\mathcal{K}}s}\mathbf{x}_s\big) &= \boldsymbol{\mathcal{K}}e^{\boldsymbol{\mathcal{K}}s}\mathbf{x}_s ds + e^{\boldsymbol{\mathcal{K}}s}d\mathbf{x}_s + 0\cdot d\mathbf{x}_s' d\mathbf{x}_s \\
&= \Big\{(A.1), \quad d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma}d\boldsymbol{B}_t^P\Big\} \\
&= \boldsymbol{\mathcal{K}}e^{\boldsymbol{\mathcal{K}}s}\mathbf{x}_s ds + e^{\boldsymbol{\mathcal{K}}s}(\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_s)ds + \boldsymbol{\Sigma}d\boldsymbol{B}_s^P) \\
&= e^{\boldsymbol{\mathcal{K}}s}\boldsymbol{\mathcal{K}}\boldsymbol{\mu}ds + e^{\boldsymbol{\mathcal{K}}s}\boldsymbol{\Sigma}d\boldsymbol{B}_s^P.
\end{aligned}$$

The expression at the last page can then be integrated with respect to s

$$
\begin{aligned}
e^{\boldsymbol{\mathcal{K}}s}\mathbf{x}_s - e^{\boldsymbol{\mathcal{K}}t}\mathbf{x}_t &= \int_t^s e^{\boldsymbol{\mathcal{K}}u}\boldsymbol{\mathcal{K}}\boldsymbol{\mu}du + \int_t^s e^{\boldsymbol{\mathcal{K}}u}\boldsymbol{\Sigma}d\boldsymbol{B}_u^P \\
&= \Big\{ multiply\ by\ e^{-\boldsymbol{\mathcal{K}}s}\ on\ both\ sides \Big\} \\
&\Leftrightarrow \\
\mathbf{x}_s &= e^{\boldsymbol{\mathcal{K}}(t-s)}\mathbf{x}_t + \int_t^s e^{\boldsymbol{\mathcal{K}}(u-s)}\boldsymbol{\mathcal{K}}\boldsymbol{\mu}du + \int_t^s e^{\boldsymbol{\mathcal{K}}(u-s)}\boldsymbol{\Sigma}d\boldsymbol{B}_u^P \\
&= \Big\{ take\ expectations; \quad t\ is\ determinsistic \Big\} \\
&\Leftrightarrow \\
E\big[\mathbf{x}_s\big] &= e^{\boldsymbol{\mathcal{K}}(t-s)}\mathbf{x}_t + E\Big[\int_t^s e^{\boldsymbol{\mathcal{K}}(u-s)}\boldsymbol{\mathcal{K}}\boldsymbol{\mu}du\Big] + E\Big[\int_t^s e^{\boldsymbol{\mathcal{K}}(u-s)}\boldsymbol{\Sigma}d\boldsymbol{B}_u^P\Big] \\
&= \Big\{ funbini; \quad E[It\hat{o}\ integral] = 0 \Big\} \\
&\Leftrightarrow \\
E\big[\mathbf{x}_s\big] &= e^{\boldsymbol{\mathcal{K}}(t-s)}\mathbf{x}_t + \int_t^s E\big[e^{\boldsymbol{\mathcal{K}}(u-s)}\big]\boldsymbol{\mathcal{K}}\boldsymbol{\mu}du \\
&= e^{\boldsymbol{\mathcal{K}}(t-s)}\mathbf{x}_t + \Big[\frac{1}{\boldsymbol{\mathcal{K}}}e^{\boldsymbol{\mathcal{K}}(u-s)}\Big]_t^s\boldsymbol{\mathcal{K}}\boldsymbol{\mu} \\
&= e^{\boldsymbol{\mathcal{K}}(t-s)}\mathbf{x}_t + \Big(\boldsymbol{I} - e^{-\boldsymbol{\mathcal{K}}(s-t)}\Big)\boldsymbol{\mu}.
\end{aligned}
$$

By inserting the last expression into (A.33) it is given that

$$
\begin{aligned}
\tau I_{t,\tau} &= \rho_0^\pi\tau + \boldsymbol{\rho}_1^{\pi\prime}\int_t^{t+\tau}\Big(e^{\boldsymbol{\mathcal{K}}(t-s)}\mathbf{x}_t + \Big(\boldsymbol{I} - e^{-\boldsymbol{\mathcal{K}}(s-t)}\Big)\boldsymbol{\mu}\Big)ds \\
&= \rho_0^\pi\tau + \boldsymbol{\rho}_1^{\pi\prime}\int_t^{t+\tau}\Big(\boldsymbol{I} - e^{-\boldsymbol{\mathcal{K}}(s-t)}\Big)\boldsymbol{\mu}ds + \boldsymbol{\rho}_1^{\pi\prime}\int_t^{t+\tau}e^{\boldsymbol{\mathcal{K}}(t-s)}ds\mathbf{x}_t \\
&= \Big\{ substitution,\ \ u = s - t \ \Big\} \\
&= \rho_0^\pi\tau + \boldsymbol{\rho}_1^{\pi\prime}\int_0^{\tau}\Big(\boldsymbol{I} - e^{-\boldsymbol{\mathcal{K}}u}\Big)\boldsymbol{\mu}du + \boldsymbol{\rho}_1^{\pi\prime}\int_0^{\tau}e^{-\boldsymbol{\mathcal{K}}u}du\mathbf{x}_t.
\end{aligned}
$$

Therefore, the following relation is given

$$
I_{t,\tau} = a_\tau^I + \boldsymbol{b}_\tau^{I\prime}\mathbf{x}_t, \tag{A.34}
$$

where

$$
a_\tau^I = \rho_0^\pi + \frac{1}{\tau}\boldsymbol{\rho}_1^{\pi\prime}\int_0^{\tau}\Big(\boldsymbol{I} - e^{-\boldsymbol{\mathcal{K}}u}\Big)\boldsymbol{\mu}du \tag{A.35}
$$

$$
\boldsymbol{b}_\tau^{I\prime} = \frac{1}{\tau}\boldsymbol{\rho}_1^{\pi\prime}\int_0^{\tau}e^{-\boldsymbol{\mathcal{K}}u}du. \tag{A.36}
$$

A.5 Linker Yields

The inflation linked yield is defined as

$$y^L_{t,\tau} = L^L_{t,\tau} + y^R_{t,\tau}, \tag{A.37}$$

where $L^L_{t,\tau}$ is the liquidity premium and $y^R_{t,\tau}$ is the market implied real yield.

Mathematically the liquidity premium is defined by introducing a positive liquidity spread, l^L_t, to the true real instantaneous short rate, used by investors when discounting an inflation linked bond's cash-flow

$$L^L_{t,\tau} = -\frac{1}{\tau} log E^Q_t \Big[exp\Big\{ - \int_t^{t+\tau} (r^R_s + l^L_s) ds \Big\} \Big] - y^R_t, \tag{A.38}$$

where the expectation is taken under the risk-neutral measure Q and r^R_s is the real short rate.

The liquidity spread is modelled as a function depending on four latent variables. Firstly, one term represented by the latent variables in $\mathbf{x}_t$, modelled by (A.1), letting the liquidity spread being dependent on the state of the economy and secondly, one term independent of the state of the economy, $\tilde{x}_t$. Therefore, the liquidity spread is defined by

$$l^L_t = \boldsymbol{\gamma}' \mathbf{x}_t + \tilde{\gamma} \tilde{x}_t, \tag{A.39}$$

where $\boldsymbol{\gamma}$ is a constant vector and $\tilde{\gamma}$ is a constant. It is further assumed that $\tilde{x}_t$ follows a Vasicek process, which is given as

$$d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t) dt + \tilde{\sigma} d\tilde{B}^P_t, \tag{A.40}$$

where $\tilde{\kappa}$, $\tilde{\mu}$ and $\tilde{\sigma}$ are constants and $\tilde{B}^P_t$ is a P Brownian motion. Since $\tilde{x}_t$ is said to be independent of $\mathbf{x}_t$ it must hold that $d\tilde{B}^P_t d\boldsymbol{B}^P_t = \mathbf{0}$. It is also assumed that the independent liquidity factor possesses a market price of risk defined as an affine function in $\tilde{x}_t$

$$\tilde{\lambda}_t = \tilde{\lambda}_0 + \tilde{\lambda}_1 \tilde{x}_t, \tag{A.41}$$

where $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ are constants.

Then, if (A.38) is inserted into (A.37), the following is given

$$\begin{aligned}
y^L_{t,\tau} &= -\frac{1}{\tau} log E^Q_t \Big[exp\Big\{ - \int_t^{t+\tau} (r^R_s + l^L_s) ds \Big\} \Big] \\
&= \Big\{ (A.14),\ r^R(\mathbf{x}_t) = \rho^R_0 + \boldsymbol{\rho}^{R\prime}_1 \mathbf{x}_t; \quad (A.39),\ l^L_t = \boldsymbol{\gamma}' \mathbf{x}_t + \tilde{\gamma} \tilde{x}_t \Big\} \\
&= -\frac{1}{\tau} log E^Q_t \Big[exp\Big\{ - \int_t^{t+\tau} \big(\tilde{\gamma} \tilde{x}_s + \rho^R_0 + (\boldsymbol{\rho}^{R\prime}_1 + \boldsymbol{\gamma}') \mathbf{x}_s \big) ds \Big\} \Big] \\
&= -\frac{1}{\tau} log E^Q_t \Big[exp\Big\{ - \int_t^{t+\tau} \tilde{\gamma} \tilde{x}_s ds \Big\} \cdot exp\Big\{ - \int_t^{t+\tau} \big(\rho^R_0 + (\boldsymbol{\rho}^{R\prime}_1 + \boldsymbol{\gamma}') \mathbf{x}_s \big) ds \Big\} \Big] \\
&= \Big\{ \tilde{x}_t \ and \ \mathbf{x}_t \ are \ independent \Big\} \\
&= -\frac{1}{\tau} log E^Q_t \Big[exp\Big\{ - \int_t^{t+\tau} \tilde{\gamma} \tilde{x}_s ds \Big\} \Big] - \frac{1}{\tau} log E^Q_t \Big[exp\Big\{ - \int_t^{t+\tau} \big(\rho^R_0 + (\boldsymbol{\rho}^{R\prime}_1 + \boldsymbol{\gamma}') \mathbf{x}_s \big) ds \Big\} \Big].
\end{aligned}$$

The expression at the last page can be solved by looking at the two terms separately. The first term,

$$-\frac{1}{\tau}logE_t^Q\Big[exp\Big\{-\int_t^{t+\tau}\tilde{\gamma}\tilde{x}_s ds\Big\}\Big], \tag{A.42}$$

can be solved by letting $\tilde{\gamma}\tilde{x}_t = \tilde{y}_t$. Hence, the following expression is under consideration

$$\tilde{k}^L(t,T) = E_t^Q\Big[exp\Big\{-\int_t^{t+\tau}\tilde{y}_s ds\Big\}\Big], \tag{A.43}$$

where $\tilde{k}^L(t,T)$ can be interpreted as the bond price at time t of a bond with maturity $T = t + \tau$ when $\tilde{y}_t$ represents the market rate, or the local rate of return. It can now be assumed that $\tilde{k}^L(t,\tau)$ possesses an affine term structure

$$\tilde{k}^L(t,\tau) = \tilde{F}(t,\tau,\tilde{x}_t) = e^{\tilde{A}_\tau^L + \tilde{B}_\tau^L\tilde{x}_t} \tag{A.44}$$

where $\tilde{A}_\tau^L$ and $\tilde{B}_\tau^L$ are deterministic functions of τ, taking the maturity time T as a parameter. If (A.44) is inserted into (A.42), the following is given

$$\begin{aligned}-\frac{1}{\tau}logE_t^Q\Big[exp\Big\{-\int_t^{t+\tau}\tilde{\gamma}\tilde{x}_s ds\Big\}\Big] &= -\frac{1}{\tau}log\big(e^{\tilde{A}_\tau^L + \tilde{B}_\tau^L\tilde{x}_t}\big)\\ &= -\frac{\tilde{A}_\tau^L}{\tau} - \frac{\tilde{B}_\tau^L}{\tau}\tilde{x}_t\\ &= \tilde{a}_\tau^L + \tilde{b}_\tau^L\tilde{x}_t,\end{aligned} \tag{A.45}$$

where the these substitutions have been used

$$\tilde{a}_\tau^L = -\frac{\tilde{A}_\tau^L}{\tau} \tag{A.46}$$

$$\tilde{b}_\tau^L = -\frac{\tilde{B}_\tau^L}{\tau}. \tag{A.47}$$

Now, a solution to (A.42) is yielded if one can find $\tilde{A}_\tau^L\tau$ and $\tilde{B}_\tau^L\tau$ such that (A.44) holds for all τ, t and $\tilde{x}_t$. Therefore, Itô's formula is applied to (A.44) which gives that

$$
\begin{aligned}
d\tilde{k}^L(t,\tau) = d\tilde{F}(t,\tau,\tilde{x}_t) &= \frac{\partial \tilde{F}}{\partial \tau} d\tau + \frac{\partial \tilde{F}}{\partial \tilde{x}_t} d\tilde{x}_t + \frac{1}{2}\frac{\partial^2 \tilde{F}}{\partial \tilde{x}_t^2}(d\tilde{x}_t)^2 \\
&= \left\{ notation; \quad \partial \tilde{A}^L_\tau = \frac{\partial \tilde{A}^L_\tau}{\partial \tau}, \quad \partial \tilde{B}^L_\tau = \frac{\partial \tilde{B}^L_\tau}{\partial \tau} \right\} \\
&= (\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t)\tilde{F} d\tau + \tilde{B}^L_\tau \tilde{F} d\tilde{x}_t + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{F}(d\tilde{x}_t)^2 \\
&= \left\{ \tau = T - t \Leftrightarrow d\tau = -dt \right\} \\
&= -(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t)\tilde{F} dt + \tilde{B}^L_\tau \tilde{F} d\tilde{x}_t + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{F}(d\tilde{x}_t)^2 \\
&= \left\{ (A.40), \quad d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma} d\tilde{B}^P_t \right\} \\
&= -(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t)\tilde{F} dt + \tilde{B}^L_\tau \tilde{F}(\tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma} d\tilde{B}^P_t) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 \tilde{F} dt \\
&= \big[-(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t) + \tilde{B}^L_\tau \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 \big] \tilde{F} dt + \tilde{B}^L_\tau \tilde{\sigma} \tilde{F} d\tilde{B}^P_t \\
&= \left\{ Girsanov\ transformation\ (Theorem\ 2.6); \quad d\tilde{B}_t = \tilde{\varphi}_t dt + d\tilde{B}^Q_t \right\} \\
&= \Big[-(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t) + \tilde{B}^L_\tau \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 + \tilde{B}^L_\tau \tilde{\sigma} \tilde{\varphi}_t \Big] \tilde{F} dt \\
&\quad + \tilde{B}^L_\tau \tilde{\sigma} \tilde{F} d\tilde{B}^Q_t \\
&= \left\{ -\tilde{\varphi}_t = market\ price\ of\ risk = \tilde{\lambda}(\tilde{x}_t) = \tilde{\lambda}_t = \tilde{\lambda}_0 + \tilde{\lambda}_1 \tilde{x}_t \right\} \\
&= \Big[-(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t) + \tilde{B}^L_\tau \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 - \tilde{B}^L_\tau \tilde{\sigma}(\tilde{\lambda}_0 + \tilde{\lambda}_1 \tilde{x}_t) \Big] \tilde{F} dt \\
&\quad + \tilde{B}^L_\tau \tilde{\sigma} \tilde{F} d\tilde{B}^Q_t \\
&= \Big[-(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t) + \tilde{B}^L_\tau (\tilde{\kappa}\tilde{\mu} - \tilde{\sigma}\tilde{\lambda}_0) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 - \tilde{B}^L_\tau (\tilde{\kappa} + \tilde{\sigma}\tilde{\lambda}_1)\tilde{x}_t \Big] \tilde{F} dt \\
&\quad + \tilde{B}^L_\tau \tilde{\sigma} \tilde{F} d\tilde{B}^Q_t
\end{aligned}
$$

Since the above expression is given under the risk neutral measure Q, the drift must equal the local rate of return

$$\tilde{y}_t(\tilde{x}_t) = \tilde{\gamma}\tilde{x}_t = -(\partial \tilde{A}^L_\tau + \partial \tilde{B}^L_\tau \tilde{x}_t) + \tilde{B}^L_\tau (\tilde{\kappa}\tilde{\mu} - \tilde{\sigma}\tilde{\lambda}_0) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 - \tilde{B}^L_\tau (\tilde{\kappa} + \tilde{\sigma}\tilde{\lambda}_1)\tilde{x}_t. \tag{A.48}$$

Then if one collect terms, (A.48) is equivalent to

$$
\begin{aligned}
&\Big[-\partial \tilde{A}^L_\tau + \tilde{B}^L_\tau (\tilde{\kappa}\tilde{\mu} - \tilde{\sigma}\tilde{\lambda}_0) + \frac{1}{2}(\tilde{B}^L_\tau)^2 \tilde{\sigma}^2 \Big] \\
&+ \Big[-\partial \tilde{B}^L_\tau - \tilde{B}^L_\tau (\tilde{\kappa} + \tilde{\sigma}\tilde{\lambda}_1) - \tilde{\gamma} \Big] \tilde{x}_t = 0.
\end{aligned} \tag{A.49}
$$

This equation must hold for all t, τ and $\tilde{x}_t$. If one consider the equation for fixed t and τ, it still must hold for all $\tilde{x}_t$. Hence, the constant term in front of $\tilde{x}_t$ must equal zero. Therefore the other constant term also must equal zero. The following system of ordinary differential equation is then generated

$$\frac{d\tilde{A}^L_\tau}{d\tau} = \tilde{\mu}^* \tilde{\kappa}^* \tilde{B}^L_\tau + \frac{1}{2}\tilde{\sigma}^2 (\tilde{B}^L_\tau)^2 \tag{A.50}$$

$$\frac{d\tilde{B}^L_\tau}{d\tau} + \tilde{\kappa}^* \tilde{B}^L_\tau = -\tilde{\gamma}, \tag{A.51}$$

with initial conditions $\tilde{A}_0^L = 0$ and $\tilde{B}_0^L = 0$. The initial conditions are given by the fact that the bond price must equal the face value at maturity and hence the exponent in (A.44) must equal 0 for $\tau = 0$. The following substitutions are being used in (A.50) and (A.51)

$$\tilde{\kappa}^* = \tilde{\kappa} + \tilde{\sigma}\tilde{\lambda}_1 \tag{A.52}$$

$$\tilde{\mu}^* = \frac{1}{\tilde{\kappa}^*}(\tilde{\kappa}\tilde{\mu} - \tilde{\sigma}\tilde{\lambda}_0) \tag{A.53}$$

This system of ordinary differential equations is of the kind that can be solved analytically, hence this will be done below. Since $\tilde{A}_\tau^L$ clearly is a function of $\tilde{B}_\tau^L$, it follows naturally to first solve for $\tilde{B}_\tau^L$. (A.51) can be identified as a first order differential equation and can therefore be solved using the integrating factor method. Multiplying both sides of (A.51) with $e^{\tilde{\kappa}^*\tau}$, $\tau = T - t$, gives that

$$\begin{aligned}
e^{\tilde{\kappa}^*\tau}\frac{d\tilde{B}_\tau^L}{d\tau} + e^{\tilde{\kappa}^*\tau}\tilde{\mu}^*\tilde{\kappa}^*\tilde{B}_\tau^L &= -e^{\tilde{\kappa}^*\tau}\tilde{\gamma} \\
&= \left\{ e^{\tilde{\kappa}^*\tau}\frac{d\tilde{B}_\tau^L}{d\tau} + e^{\tilde{\kappa}^*\tau}\tilde{\mu}^*\tilde{\kappa}^*\tilde{B}_\tau^L = \frac{d}{d\tau}\left(e^{\tilde{\kappa}^*\tau}\tilde{B}_\tau^L\right) \right\} \\
&\Leftrightarrow \\
\frac{d}{d\tau}\left(e^{\tilde{\kappa}^*\tau}\tilde{B}_\tau^L\right) &= -\tilde{\gamma}e^{\tilde{\kappa}^*\tau} \\
&= \left\{ \textit{integrating with respect to } \tau \right\} \\
&\Leftrightarrow \\
e^{\tilde{\kappa}^*\tau}\tilde{B}_\tau^L &= -\frac{\tilde{\gamma}}{\tilde{\kappa}^*}e^{\tilde{\kappa}^*\tau} + C \\
\tilde{B}_\tau^L &= -\frac{\tilde{\gamma}}{\tilde{\kappa}^*} + Ce^{-\tilde{\kappa}^*\tau}
\end{aligned} \tag{A.54}$$

Therefore, if the initial condition, $\tilde{B}_0^L = 0$, is used one get that

$$\begin{aligned}
\tilde{B}_0^L = 0 &= -\frac{\tilde{\gamma}}{\tilde{\kappa}^*} + C \\
&\Leftrightarrow \\
C &= \frac{\tilde{\gamma}}{\tilde{\kappa}^*}.
\end{aligned} \tag{A.55}$$

Then if (A.55) is inserted into (A.54) the solution to $\tilde{B}_\tau^L$ is given as

$$\tilde{B}_\tau^L = -\frac{\tilde{\gamma}}{\tilde{\kappa}^*}\left(1 - e^{-\tilde{\kappa}^*\tau}\right). \tag{A.56}$$

Then, if it is used that $\tilde{b}_\tau^L = -\frac{\tilde{B}_\tau^L}{\tau}$, as in (A.47), (A.56) corresponds to

$$\tilde{b}_\tau^L = \frac{\tilde{\gamma}}{\tilde{\kappa}^*\tau}\left(1 - e^{-\tilde{\kappa}^*\tau}\right). \tag{A.57}$$

Now, (A.50) can be solved by firstly integrating with respect to $\tau \in (0, T-t)$,

$$
\begin{aligned}
\tilde{A}^L_\tau - \underbrace{\tilde{A}^L_0}_{=0} &= \tilde{A}^L_\tau = \tilde{\mu}^*\tilde{\kappa}^* \int_0^\tau \tilde{B}^L_s ds + \frac{1}{2}\tilde{\sigma}^2 \int_0^\tau (\tilde{B}^L_s)^2 ds \\
&= \Big\{ (A.56), \quad \tilde{B}^L_\tau = -\frac{\tilde{\gamma}}{\tilde{\kappa}^*}\Big(1 - e^{-\tilde{\kappa}^*\tau}\Big) \Big\} \\
&= \tilde{\mu}^*\tilde{\kappa}^* \int_0^\tau -\frac{\tilde{\gamma}}{\tilde{\kappa}^*}\Big(1 - e^{-\tilde{\kappa}^* s}\Big) ds + \frac{1}{2}\tilde{\sigma}^2 \int_0^\tau \Big(\frac{\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \Big(1 - e^{-\tilde{\kappa}^* s}\Big)^2 ds \\
&= -\tilde{\mu}^*\tilde{\gamma}^* \Big(\int_0^\tau ds - \int_0^\tau e^{-\tilde{\kappa}^* s} ds \Big) \\
&\quad + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \Big(\int_0^\tau ds - \int_0^\tau 2e^{-\tilde{\kappa}^* s} ds \int_0^\tau e^{-2\tilde{\kappa}^* s} ds \Big) \\
&= -\tilde{\mu}^*\tilde{\gamma}^* \big[s\big]_0^\tau + \tilde{\mu}^*\tilde{\gamma}^* \Big[-\frac{1}{\tilde{\kappa}^*} e^{-\tilde{\kappa}^* s} \Big]_0^\tau \\
&\quad + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \big[s\big]_0^\tau - \Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \Big[-\frac{1}{\tilde{\kappa}^*} e^{-\tilde{\kappa}^* s} \Big]_0^\tau + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \Big[-\frac{1}{2\tilde{\kappa}^*} e^{-2\tilde{\kappa}^* s} \Big]_0^\tau \\
&= -\tilde{\mu}^*\tilde{\gamma}^*\tau + \underbrace{\frac{\tilde{\mu}^*\tilde{\gamma}^*}{\tilde{\kappa}^*}\Big(1 - e^{-\tilde{\kappa}^*\tau}\Big)}_{=-\tilde{\mu}^*\tilde{B}^L_\tau} \\
&\quad + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \tau - \Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \frac{1}{\tilde{\kappa}^*}\Big(1 - e^{-\tilde{\kappa}^*\tau}\Big) - \frac{1}{4}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \frac{1}{\tilde{\kappa}^*}\Big(1 - e^{-2\tilde{\kappa}^*\tau}\Big) \\
&= -\tilde{\mu}^*\tilde{\gamma}^*\tau - \tilde{\mu}^*\tilde{B}^L_\tau + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \tau \\
&\quad \underbrace{-\frac{1}{4}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \frac{1}{\tilde{\kappa}^*}\Big(1 - 2e^{-\tilde{\kappa}^*\tau} + e^{-2\tilde{\kappa}^*\tau}\Big)}_{=-\frac{\tilde{\sigma}^2}{4\tilde{\kappa}^*}(\tilde{B}^L_\tau)^2} - \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \frac{1}{\tilde{\kappa}^*} + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \frac{1}{\tilde{\kappa}^*} e^{-\tilde{\kappa}^*\tau} \\
&= -\tilde{\mu}^*\tilde{\gamma}^*\tau - \tilde{\mu}^*\tilde{B}^L_\tau + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \tau - \frac{\tilde{\sigma}^2}{4\tilde{\kappa}^*}(\tilde{B}^L_\tau)^2 \underbrace{-\frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \frac{1}{\tilde{\kappa}^*}\Big(1 - e^{-\tilde{\kappa}^*\tau}\Big)}_{=\frac{1}{2}(\frac{\tilde{\sigma}}{\tilde{\kappa}^*})^2\tilde{\gamma}\tilde{B}^L_\tau} \\
&= -\tilde{\mu}^*\tilde{\gamma}^*\tau - \tilde{\mu}^*\tilde{B}^L_\tau + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \tau - \frac{\tilde{\sigma}^2}{4\tilde{\kappa}^*}(\tilde{B}^L_\tau)^2 + \frac{1}{2}(\frac{\tilde{\sigma}}{\tilde{\kappa}^*})^2\tilde{\gamma}\tilde{B}^L_\tau \\
&= \Big\{ (A.46), \quad \tilde{A}^L_\tau = -\tau\tilde{a}^L_\tau; \quad (A.47), \quad \tilde{B}^L_\tau = -\tau\tilde{B}^L_\tau \Big\} \\
&\Leftrightarrow \\
-\tau\tilde{a}^L_\tau &= -\tilde{\mu}^*\tilde{\gamma}^*\tau + \tilde{\mu}^*\tau\tilde{b}^L_\tau + \frac{1}{2}\Big(\frac{\tilde{\sigma}\tilde{\gamma}}{\tilde{\kappa}^*}\Big)^2 \tau - \frac{\tilde{\sigma}^2}{4\tilde{\kappa}^*}\tau^2(\tilde{b}^L_\tau)^2 - \frac{1}{2}(\frac{\tilde{\sigma}}{\tilde{\kappa}^*})^2\tilde{\gamma}\tau\tilde{b}^L_\tau \\
&\Leftrightarrow \\
\tilde{a}^L_\tau &= \tilde{\gamma}\Big[\tilde{\mu}^* - \frac{\tilde{\mu}^*\tilde{b}^L_\tau}{\tilde{\gamma}} - \frac{1}{2}\Big(\frac{\tilde{\sigma}}{\tilde{\kappa}^*}\Big)^2\tilde{\gamma} + \frac{\tilde{\sigma}^2}{4\tilde{\gamma}\tilde{\kappa}^*}\tau(\tilde{b}^L_\tau)^2 + \frac{1}{2}(\frac{\tilde{\sigma}}{\tilde{\kappa}^*})^2\tilde{b}^L_\tau\Big] \\
&\Leftrightarrow \\
\tilde{a}^L_\tau &= \tilde{\gamma}\Big[\big(\tilde{\mu}^* - \frac{1}{2}\Big(\frac{\tilde{\sigma}}{\tilde{\kappa}^*}\Big)^2\tilde{\gamma}\big)\big(1 - \frac{\tilde{b}^L_\tau}{\tilde{\gamma}}\big) + \frac{\tilde{\sigma}^2}{4\tilde{\gamma}\tilde{\kappa}^*}\tau(\tilde{b}^L_\tau)^2\Big]. \qquad (A.58)
\end{aligned}
$$

With $\tilde{a}^L_\tau$ defined as in (A.58) and $\tilde{b}^L_\tau$ defined as in (A.57) an analytical solution to (A.43) are then given by inserting $\tilde{a}^L_\tau$ and $\tilde{b}^L_\tau$ into (A.45).

Left is then to calculate the solution to the second term which is given as

$$-\frac{1}{\tau}logE_t^Q\Big[exp\Big\{-\int_t^{t+\tau}(\rho_0^R+(\boldsymbol{\rho}_1^{R'}+\boldsymbol{\gamma}')\mathbf{x}_s)ds\Big\}\Big]. \tag{A.59}$$

Then, to be able to solve this expression it is of interest to use the substitution

$$y_t=\rho_0^R+(\boldsymbol{\rho}_1^{R'}+\boldsymbol{\gamma}')\mathbf{x}_t. \tag{A.60}$$

Then, to be able to solve (A.59) it helps to first consider

$$k^L(t,\tau)=E_t^Q\Big[exp\Big\{-\int_t^{t+\tau}y_s ds\Big\}\Big], \tag{A.61}$$

where $k^L(t,\tau)$ can be interpreted as the bond price at time t of a bond with maturity $t+\tau$ when y_t represents the market rate, or the local rate of return, and is an affine function of the latent variable $\mathbf{x}_t$

$$y_t=\rho_0^R+\mathbf{c}'\mathbf{x}_t, \tag{A.62}$$

where it has been used that $\mathbf{c}=(\boldsymbol{\rho}_1^R+\boldsymbol{\gamma})$. It can now be assumed that $k^L(t,\tau)$ possesses an affine term structure

$$k^L(t,\tau)=F(t,\tau,\mathbf{x}_t)=e^{A_\tau^L+\boldsymbol{B}_\tau^{L'}\mathbf{x}_t}. \tag{A.63}$$

where A_τ^L and $\boldsymbol{B}_\tau^L$ are deterministic functions of τ, taking the maturity time T as a parameter. If (A.63) is inserted into (A.59), the following is given

$$\begin{aligned}-\frac{1}{\tau}logE_t^Q\Big[exp\Big\{-\int_t^{t+\tau}(\rho_0^R+(\boldsymbol{\rho}_1^{R'}+\boldsymbol{\gamma}')\mathbf{x}_s)ds\Big\}\Big]&=-\frac{1}{\tau}log\big(e^{A_\tau^L+\boldsymbol{B}_\tau^{L'}\mathbf{x}_t}\big)\\&=-\frac{A_\tau^L}{\tau}-\frac{\boldsymbol{B}_\tau^{L'}}{\tau}\mathbf{x}_t\\&=a_\tau^L+\boldsymbol{b}_\tau^{L'}\mathbf{x}_t,\end{aligned} \tag{A.64}$$

where the these substitutions have been used

$$a_\tau^L=-\frac{A_\tau^L}{\tau} \tag{A.65}$$

$$\boldsymbol{b}_\tau^L=-\frac{\boldsymbol{B}_\tau^L}{\tau}. \tag{A.66}$$

Now, a solution to (A.59) is yielded if one can find A_τ^L and $\boldsymbol{B}_\tau^L$ such that (A.63) holds for all τ,t

and $\mathbf{x}_t$. Therefore, It$\hat{o}$'s formula is applied to (A.63) which gives that

$$
\begin{aligned}
dk^L(t,\tau) = dF(t,\tau,\mathbf{x}_t) &= \frac{\partial F}{\partial \tau}d\tau + \frac{\partial F}{\partial \mathbf{x}_t}d\mathbf{x}_t + \frac{1}{2}\frac{\partial^2 F}{\partial \mathbf{x}_t^2}(d\mathbf{x}_t' d\mathbf{x}_t) \\
&= \left\{ notation; \quad \partial A_\tau^L = \frac{\partial A_\tau^L}{\partial \tau}, \quad \partial \boldsymbol{B}_\tau^L = \frac{\partial \boldsymbol{B}_\tau^L}{\partial \tau} \right\} \\
&= (\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) F d\tau + \boldsymbol{B}_\tau^{L'} F d\mathbf{x}_t + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{B}_\tau^L F(d\mathbf{x}_t' d\mathbf{x}_t) \\
&= \left\{ \tau = T - t \Leftrightarrow d\tau = -dt \right\} \\
&= -(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) F dt + \boldsymbol{B}_\tau^{L'} F d\mathbf{x}_t + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{B}_\tau^L F(d\mathbf{x}_t' d\mathbf{x}_t) \\
&= \left\{ (A.1), \quad d\mathbf{x}_t = \boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P \right\} \\
&= -(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) F dt + \boldsymbol{B}_\tau^{L'} F(\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t)dt + \boldsymbol{\Sigma} d\boldsymbol{B}_t^P) + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L F dt \\
&= \big[-(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) + \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L \big] F dt + \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma} F d\boldsymbol{B}_t^P \\
&= \left\{ Girsanov\ transformation\ (Theorem\ 2.6); \quad d\boldsymbol{B}_\tau^P = \boldsymbol{\varphi}_t^R dt + d\boldsymbol{B}_\tau^Q \right\} \\
&= \Big[-(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) + \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L + \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\varphi}_t^R \Big] F dt \\
&\quad + \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma} F d\boldsymbol{B}_t^Q \\
&= \Big\{ -\boldsymbol{\varphi}_t^R = market\ price\ of\ risk = \boldsymbol{\lambda}^R(\mathbf{x}_t) = \boldsymbol{\lambda}_0^R + \boldsymbol{\Lambda}^R \mathbf{x}_t \} \\
&= \Big[-(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) + \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L \\
&\qquad - \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}(\boldsymbol{\lambda}_0^R + \boldsymbol{\Lambda}^R \mathbf{x}_t) \Big] F dt + \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma} F d\boldsymbol{B}_t^Q
\end{aligned}
$$

Since the above expression is given under the risk neutral measure Q, the drift must equal the local rate of return

$$
y_t(\mathbf{x}_t) = -(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) + \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L - \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}(\boldsymbol{\lambda}_0^R + \boldsymbol{\Lambda}^R \mathbf{x}_t). \tag{A.67}
$$

Then, if this expression is inserted into (A.62) the following relation is given

$$
-(\partial A_\tau^L + \partial \boldsymbol{B}_\tau^{L'} \mathbf{x}_t) + \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}}(\boldsymbol{\mu} - \mathbf{x}_t) + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L - \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}(\boldsymbol{\lambda}_0^R + \boldsymbol{\Lambda}^R \mathbf{x}_t) = \rho_0^R + \mathbf{c}'\mathbf{x}_t. \tag{A.68}
$$

Then if one collect terms, (A.68) is equivalent to

$$
\begin{aligned}
&\Big[-\partial A_\tau^L + \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L - \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\lambda}_0^R - \rho_0^R \Big] \\
&+ \Big[-\partial \boldsymbol{B}_\tau^{L'} - \boldsymbol{B}_\tau^{L'}\boldsymbol{\mathcal{K}} - \boldsymbol{B}_\tau^{L'}\boldsymbol{\Sigma}\boldsymbol{\Lambda}^R - \mathbf{c}' \Big] \mathbf{x}_t = 0.
\end{aligned} \tag{A.69}
$$

This equation must hold for all t, τ and $\mathbf{x}_t$. If one consider the equation for fixed t and τ it still must hold for all $\mathbf{x}_t$. Hence, the constant term in front of $\mathbf{x}_t$ must equal zero. Therefore the other constant term also must equal zero. The following system of ordinary differential equations is then generated

$$\frac{dA_\tau^L}{d\tau} = -\rho_0^R + \boldsymbol{B}_\tau^{L\prime}(\boldsymbol{\mathcal{K}}\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0^R) + \frac{1}{2}\boldsymbol{B}_\tau^{L\prime}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{B}_\tau^L \tag{A.70}$$

$$\frac{d\boldsymbol{B}_\tau^L}{d\tau} = -(\boldsymbol{\rho}_1^R + \boldsymbol{\gamma}) - (\boldsymbol{\mathcal{K}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}^R)'\boldsymbol{B}_\tau^L, \tag{A.71}$$

with initial conditions $A_0^L = 0$ and $\boldsymbol{B}_0^L = \mathbf{0}$. The initial conditions are given by the fact that the bond price must equal the face value at maturity and hence the exponent in (A.63) must equal 0 for $\tau = 0$.

Then, a solution to the second term is given if (A.70) and (A.71) are solved using numerical methods. Then by inserting this solution together with the analytical solution to the first term into (A.38) the following expression for the liquidity premium is given

$$L_{t,\tau}^L = \tilde{a}_\tau^L + \tilde{b}_\tau^L \tilde{x}_t + a_\tau^L + \boldsymbol{b}_\tau^{L\prime}\mathbf{x}_t - y_t^R. \tag{A.72}$$

Then if the solution to y_t^R, (A.29), is inserted to (A.72) the liquidity premium can be expressed as

$$L_{t,\tau}^L = [\tilde{a}_\tau^L + (a_\tau^L - a_\tau^R)] + \left[(\boldsymbol{b}_\tau^L - \boldsymbol{b}_\tau^R)' \ \ \tilde{b}_\tau^L\right]\begin{bmatrix}\mathbf{x}_t \\ \tilde{x}_t\end{bmatrix}, \tag{A.73}$$

where

$$\tilde{a}_\tau^L = \tilde{\gamma}\Big[\big(\tilde{\mu}^* - \frac{1}{2}\Big(\frac{\tilde{\sigma}}{\tilde{\kappa}^*}\Big)^2\tilde{\gamma}\big)\big(1 - \frac{\tilde{b}_\tau^L}{\tilde{\gamma}}\big) + \frac{\tilde{\sigma}^2}{4\tilde{\gamma}\tilde{\kappa}^*}\tau(\tilde{b}_\tau^L)^2\Big]$$

$$\tilde{b}_\tau^L = \frac{\tilde{\gamma}}{\tilde{\kappa}^*\tau}\Big(1 - e^{-\tilde{\kappa}^*\tau}\Big)$$

$$a_\tau^L = -\frac{A_\tau^L}{\tau}$$

$$\boldsymbol{b}_\tau^L = -\frac{\boldsymbol{B}_\tau^L}{\tau}$$

$$a_\tau^R = -\frac{A_\tau^R}{\tau}$$

$$\mathbf{b}_\tau^R = -\frac{\mathbf{B}_\tau^R}{\tau}.$$

Furthermore, the inflation linked yield is given if (A.38) and (A.29), for $i = R$, are inserted into (A.37)

$$y_{t,\tau}^L = [\tilde{a}_\tau^L + a_\tau^L] + \left[\boldsymbol{b}_\tau^{L\prime} \ \ \tilde{b}_\tau^L\right]\begin{bmatrix}\mathbf{x}_t \\ \tilde{x}_t\end{bmatrix}. \tag{A.74}$$

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TRITA-MAT-E 2014:27
ISRN-KTH/MAT/E—14/27-SE