Master Thesis

Perturbative and non-perturbative aspects of Chern-Simons

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To my parents and my fiancée, for their love and support.
Abstract

In this master thesis, we focus on both perturbative and non-perturbative methods of evaluating Chern-Simons partition function on $S^3$ with gauge group $SU(2)$. We first introduce the relation between Morse theory and supersymmetry, and show its application in a simple instanton calculation. Since Chern-Simons is a crucial Morse function, we then calculate the first loop correction of it. The result is Ray-Singer torsion of $S^3$ with respect to a flat connection. We also give a full explanation of framing dependence of the theory. The last part serves as a non-perturbative comparison. We introduce affine Lie algebra and show that the modular invariance of its character can be used to calculate the partition function with arguments from simple surgeries of $S^3$. 
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Introduction

The early motivation of this thesis is to understand recent developments in relations between quantization and categorification[20, 21]. Also various naive interests of the author in mysterious relations between number theory and physics turn out to be much profound in modern understanding of non-Abelian extension of class field theory, also called the geometric Langlands program. The electromagnetic duality plays central role in physical construction of the program[22, 27, 18]. A crucial ingredient in the field of study is the Chern-Simons theory which has a long history in physics whilst it became more and more vital whence people started using topological information to study theories of different dimensions. One can find a more detailed motivation and discussion of results in the epilogue, and here we will outline the contents of this thesis.

As one of the most celebrating theories of 20-th century, quantum field theory suffers serious problem of divergences even disrespect the unsatisfactory underlining mathematical formalism and the non-conventional physical reality that physicists do not concern too much about since none of them seems solvable before the divergences being removed. Therefore various solutions appeared since the theory developed. It is interesting that if the extension is “appropriate”, one gets more detail structures, and the problems remained do not worry people any more. However, divergences still exist until string theory and supersymmetry were introduced to the game. The ultraviolet divergence in QFT is mainly caused by using of point like particle, and when interactions are in short distances or even zero-distance infinity appears. By upgrading to higher dimensional particles, the problem resolved. One interesting fact is that within a 2-dimensional world sheet, conformal symmetry translates a singular point to an infinite tube where different string states exist, this is the operator-state correspondence, a short explanation in Appendix G. If one looks at the Feynman diagram, this appropriate extension is just fatten the one-dimensional world line to world sheet. The infrared divergence is gone by adding supersymmetry. A satisfactory QFT is one major problem for theorists.

Another fundamental problem at central concern is the reconciliation of QFT and gravity. It is the main driving force for many theorists. One seemingly profound tool from Standard Model is gauge symmetry. With it one can gain pure geometric information which is metric independent, thus we really have the desired general covariance. CS captures such information, and among other intuitions Witten
considered it as a QFT in [42]. The partition function of such theory is a generating function of knots and it gives knot invariant, i.e. the Jones polynomial. Knots can be physically represented by non-trivial Wilson loops in the theory, such that knot invariant coincides with observables. On the other hand, the projection of a knot in a plane has a lot of interesting properties related to physics. One simple example is the Yang-Baxter equation from statistic physics and the third Reidemeister move. Many applications to vertex model, quantum group, integrable system, etc tightly related to CS[41, 43].

We will focus on topological information that can be extracted from supersymmetry, therefore the first chapter introduces its relations to differential forms. Then we continue to its beautiful relations to Morse theory which makes it possible to calculate global invariants from local analytic functions. We carry out the instanton calculation of supersymmetric non-sigma model to show how one can get the topological data from the theory. Its realtions to CS is when one considers the Morse function as CS. This is an example of non-relativistic three dimensional field theory, the generalization to relativistic version is Donaldson theory of four manifolds which can be interpreted as a topological QFT. Along Floer homology many interesting theories appear, unfortunately we will not talk about them but only focus on CS first.

In the second chapter, we start with basic facts of CS. Then construct a QFT with it. With standard Faddeev-Povov procedure we arrive the gauge fixed action which has BRST symmetry, thus CS gives a topological theory. The first loop correction carried out by the path integral is proportional to the Ray-Singer torsion, which is topological invariant, up to a metric dependent phase term. Using APS theorem one can see that the extra phase term is actually metric independent up to a framing choice. Moreover, there is always a canonical 2-framing that the phase term disappears. This essentially proves the theory is topological from perturbative point of view.

In the last chapter, we use non-perturbative method to derive the exact result of CS partition function on $S^3$ and compare its asymptotic behavior to results from previous chapter. This is the first example of AdS$_3$/CFT$_2$ correspondence. To arrive the comparison, we briefly review the root lattice of simple Lie algebra and extension to Kac-Moody algebra. Then derive the Kac-Weyl character formula which can be used to construct the partition function. One can explicitly compute the partition function by canonical quantization of CS on product manifolds, geometric quantization or holomorphic quantization. Any of the cases shows the partition function has modular invariance. However we choose an easier path. The arriving of exact results is from simple arguments on surgeries of 3-manifolds which can be considered as symmetry of modular forms. Finally the exact result gives same classical behavior from perturbation point of view.

In the Epilogue we give a full motivation and discussions of the results for a future reference. Hopefully this thesis can serve as a general introduction to different aspects of CS and provide a overall guidance to proceed to modern theory of gauge theory and low-dimensional topology.
Chapter 1

Supersymmetry and Morse theory

1.1 Supersymmetry

In this section we will present a brief introduction to supersymmetry by using language of differential forms which might be easier to grasp than the more physical intuition of finding a non-trivial combination of spacetime Poincaré group with internal group. However, before getting in abstract mathematics it is always good to have some physical pictures in mind.

The Standard Model, among other theories, is one of the most successes of mankind. It gives us exceedingly precise theoretical predictions within proper scale limit of spacetime. However, most theories about natural have had their glory in the past and the Standard Model will join them in the near future. The main problem is that the way we understanding nature, namely the reductionist approach, reproduces the scale limit once we refine the definition of elementary.

The very first elementary particle electron is assumed as point like in the classical electromagnetism. The theory is only valid at distance larger than $2.8 \times 10^{-13}$ cm, the Coulomb self-energy ruins everything beyond the limit. The same divergence also exists in quantum world whilst the bizarreness of this level makes it cancelled out. The key point is that when the vacuum fluctuation creates an electron-positron pair, there are chances the original electron is annihilated by the created positron and the created electron left be real. Such contribution of the electron self-energy cancels the leading term in the Coulomb self-energy and it is proportional to the "bare" mass.

The bizarreness, by doubling the degrees of freedom gives us a new elementary world. Meanwhile it introduces a new symmetry, the chiral symmetry. Electrons gain masses by exchanging between left and right hand fields, and if the symmetry is not breaking they remain massless. Unfortunately, the Standard Model suffers
Chapter 1. Supersymmetry and Morse theory

the same problem as classical model, the Higgs boson of it gives a divergent self-energy correction and the scale limit is about $10^{(-17)}$ cm. Therefore supersymmetry is introduced to double the degrees of freedom again and to cancel the divergence of self-energy at scales beyond.

There are also many other beautiful theories to get beyond the Standard Model. Since experiment scale has not reach TeV level yet, we cannot be so obsessed by one’s beauty. However, we will focus on supersymmetry now as it’s natural existence in the string theory and profound connections with geometry and topology.

1.2 Differential forms and supersymmetry

Supersymmetry is a symmetry between boson and fermion. Bosons are relatively easier to imagine than fermions as anticommutativity is indeed unfamiliar to us, just looking at $ab = -ba$. But how about the world of anticommutate objects? They may find us the peculiar one. This suggests a duality between these two different perspectives. Can we find some rigorous mathematics to realize such duality?

Let $M$ be a $n$-dimensional Riemannian manifold with metric $\gamma$, denote as $(M, \gamma)$. At each point $p$ on $M$ we can form tangent vectors $a^k$ and their dual $a^\dagger_k$. Noticing that $a^\dagger_k$ can be regarded as operators on the exterior algebra at $p$, $a^\dagger_k \leftrightarrow dx^k \wedge$. With the Hodge star $\ast$ which maps $\Omega^q(M) \rightarrow \Omega^{n-q}(M)$, it follows that $a^k$ are operators with operation being interior multiplication, $a^k \leftrightarrow (-1)^{nq+n+1} \ast dx^k \ast$. What we have done is nothing but separating the operations of exterior derivative and its dual, see later Eq.(1.2). Most importantly, they obey anticommutation relations

$$\{a^i, a^j\} = 0, \quad \{a^i, a^\dagger_j\} = 0, \quad \{a^i, a^\dagger_j\} = \gamma^{ij}. \tag{1.1}$$

These are desired fermionic relations we want. To involve bosons, recall the bosonic momentum operators $p_k \leftrightarrow -i\partial_k$. With these identifications, we can rewrite the exterior derivative $d$ and the co-differential $d^\dagger$ as

$$d = ia^\dagger_k p_k, \quad d^\dagger = -ia^k p_k. \tag{1.2}$$

Therefore we have commutation relations

$$[d, x^k] = a^\dagger_k, \quad [d^\dagger, x^k] = -a^k, \tag{1.3}$$

and anticommutation relations

$$\{d, a^\dagger_j\} = 0, \quad \{d, a^j\} = i\gamma^{jk} p_k, \quad \{d^\dagger, a^\dagger_j\} = -i\gamma^{jk} p_k, \quad \{d^\dagger, a^j\} = 0. \tag{1.4}$$

From these relations, we can regard $d$ and $d^\dagger$ as generators of exchanges between fermionic operators $a^i, a^\dagger_i$ and bosonic operators $p_i$. They form a $\mathbb{Z}_2$ graded algebra.
We can now define supercharges \( Q_i \), generators of supersymmetry, as linear combinations of \( d \) and \( d^\dagger \) to be
\[
Q_1 = d + d^\dagger, \quad Q_2 = i(d - d^\dagger).
\] (1.5)

The fact that \( d^2 = d^\dagger 2 = 0 \) implies \( \{Q_1, Q_2\} = 0 \) and \( Q_1^2 = Q_2^2 = \{d, d^\dagger\} \), which is the Laplace-Beltrami operator \( \Delta \). If we can consider \( \Delta \) as the Hamiltonian operator \( H \), then we have the simplest supersymmetric quantum mechanics. The Lorentz invariance indicates that the momentum operator \( P \) is zero in our case, that is to say we are dealing with \( P = 0 \) subspace \( \mathcal{H}_0 \) of the total Hilbert space \( \mathcal{H} \).

Accordingly, we see that supersymmetry is of beauty with language of differential forms. Such elegant connection between physics and mathematics is introduced by Witten in early 80s[38]. The motivation of Witten might be the deep connection of supersymmetry and topology. Let us keep looking at some physical reasoning again before going any further.

The first question about supersymmetry is that we cannot see it in nature, at present scale limit. We have to know how to determine its presence in one theory. Like other symmetries, supersymmetry must commute with the Hamiltonian operator
\[
[Q_i, H] = 0, \quad i = 1, \ldots, N,
\] (1.6)

where \( N \) is the number of supercharges. However, there are huge differences between spontaneous breaking of supersymmetry than other internal symmetries. Since \( H = Q_1^2 \), any supersymmetric state \( |0\rangle \), which is a state annihilated by the \( Q_i \), must has zero-energy and it is the true vacuum state. Therefore only if there is no such a state, supersymmetry is spontaneously broken and the ground state will have positive energy. For ordinary internal symmetries, the criteria for spontaneously broken of symmetry is independent of the energy of ground state, e.g. the Mexican hat potential.

Moreover, if supersymmetry is not broken, we can define an operator to distinguish bosonic and fermionic states. Recall supersymmetry is a \( \mathbb{Z}_2 \) graded algebra, we can decompose the Hilbert space \( \mathcal{H}_0 \) into a direct sum two subspaces, \( \mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathcal{H}_0^- \), where \( \mathcal{H}_0^+ \) for bosonic states and \( \mathcal{H}_0^- \) for fermionic states. Then we can define \( (-1)^F \) as \( (-1)^F |b\rangle = |b\rangle \) for \( |b\rangle \in \mathcal{H}_0^+ \), and \( (-1)^F |f\rangle = -|f\rangle \) for \( |f\rangle \in \mathcal{H}_0^- \). Consequently, we have a basic condition must be satisfied by each \( Q_i \)
\[
(-1)^F Q_i + Q_i (-1)^F = 0.
\] (1.7)

A crucial observation by Witten[37] recognizes the trace of \( (-1)^F \), the Witten index, as a topological invariant. Let us assume there is only one supercharge \( Q \). Since supersymmetry is not broken, any state of non-zero energy has a super partner
state connected by $Q$. If state $|b\rangle$ has energy $E$, we can define state $|f\rangle = 1/\sqrt{E}|b\rangle$. Therefore $|b\rangle$ and $|f\rangle$ transform into each other by $Q$ as

$$Q|b\rangle = \sqrt{E}|f\rangle, \quad Q|f\rangle = \sqrt{E}|b\rangle,$$

(1.8)

and $Q^2 = H$ is satisfied. For states of zero-energy, $Q|b\rangle = 0$ and $Q|f\rangle = 0$. Let us denote the number of bosonic states with zero-energy by $n^{E=0}_B$ and $n^{E=0}_F$ as for fermionic states. One direct consequence is that $n^{E=0}_B - n^{E=0}_F \neq 0$ means supersymmetry is not broken.

More important, $n^{E=0}_B - n^{E=0}_F$ is an invariant in the sense that it is independent of parameters of the theory, which are the volume$^1$, the mass and the coupling constant. There are two cases when changing the parameters caused by energy changes of states.

One possible case is some pairs of Boson-Fermion states with $E \neq 0$ will down to be ground states. $n^{E=0}_B$ and $n^{E=0}_F$ will both increase by the number of pairs, thus $n^{E=0}_B - n^{E=0}_F$ is unchanged. On the other hand, some pairs of ground states might gain non-zero-energy and become pairs of Boson-Fermion states with $E \neq 0$. That is $n^{E=0}_B$ and $n^{E=0}_F$ will both decrease by the number of pairs, thus $n^{E=0}_B - n^{E=0}_F$ is again unchanged.

It is readily to see that we can identify $\text{Tr}(-1)^F$ with $n^{E=0}_B - n^{E=0}_F$. States with $E \neq 0$ are paired up. For each pair, bosonic state contributes $+1$ to the trace whilst fermionic state cancels it by contributing $-1$. Therefore we can write

$$\text{Tr}(-1)^F = n^{E=0}_B - n^{E=0}_F.$$

(1.9)

One must be caution that proper regularization is needed when sum over infinite number of states in Hilbert space.

How is this invariant related to topology? The splitting of the Hilbert space into two subspaces enable us to write $Q$ as

$$Q = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}$$

(1.10)

and accordingly states are in the form

$$\begin{pmatrix} B \\ F \end{pmatrix},$$

(1.11)

where $K^\dagger$ is the Hermitian conjugate of $K$. In language of operator theory, supersymmetric ground states are kernels of operators $K$ and $K^\dagger$. Therefore $n^{E=0}_B - n^{E=0}_F$ is the difference of the two kernels. This quantity is called the index of operator $K$ which is a topological invariant.

$^1$We are analysing supersymmetry in a finite spatial volume with periodic boundary condition. The reason for this is that if it is unbroken, it will not break by setting the limit to infinity. The periodicity is added to ensure translation invariance.
This connection between physics and topology is so beautiful that there must be profound reasons behind it. However, before unveil it, we still can do more about the supersymmetric ground states. Whether $n_{\mathbb{B}}^{E=0}$ and $n_{\mathbb{F}}^{E=0}$ are invariants themselves? If so, one simple constrain is needed, that is $n_{\mathbb{B}}^{E=0} + n_{\mathbb{F}}^{E=0}$ must be an invariant too. Let us see how this is possible. Consider now the theory is with two supercharges $Q_1$ and $Q_2$, define a linear combination of them as

$$Q_{\pm} = \frac{1}{2}(Q_1 \pm iQ_2).$$

It is readily to see the superalgebra

$$Q_+^2 = Q_-^2 = 0, \quad Q_+ Q_- + Q_- Q_+ = H.$$  

We have recovered the exterior algebra introduced earlier, check Eq.(1.4). $Q_{\pm}$ are exactly the exterior derivative $d$ and the co-differential $d^\dagger$. We see that the supersymmetric ground states, $E = 0$, are exactly the non-trivial elements of the cohomology $H^\bullet(M,d)$. However, equivalent cohomologies are classified by homotopy class of the manifold.

Let us consider an one to one map $U$ between supersymmetric states of different systems, that is $|\tilde{b}\rangle = U^{-1} |b\rangle$. This is same as defining new operators

$$\tilde{Q}_+ = U^{-1} Q_+ U, \quad \tilde{Q}_- = U Q_- U^{-1}, \quad \tilde{H} = \tilde{Q}_+ \tilde{Q}_- + \tilde{Q}_- \tilde{Q}_+.$$  

Consequently, for any state $|\Omega\rangle$ annihilated by $Q_+$ or $Q_-$ there is a state $|\tilde{\Omega}\rangle$ annihilated by $\tilde{Q}_+$ or $\tilde{Q}_-$. Furthermore, if $U$ is unitary, $U^\dagger = U^{-1}$, this corresponds to a change of basis in Hilbert space and the two systems are the same. If it is not unitary, the two systems are different whilst the number of zero-energy states are same. Therefore changes of parameters that can be brought by conjugation will not affect $n_{\mathbb{B}}^{E=0} + n_{\mathbb{F}}^{E=0}$ and the numbers can be invariants independently.

By using these ideas Witten found that $\text{Tr}(-1)^F$ of supersymmetric non-linear sigma model is a topological invariant which equals to the Euler characteristic of manifold $M$,

$$\text{Tr}(-1)^F = \chi(M).$$  

This might probably be the reason Witten discovered the relation of supersymmetry and Morse theory[38].

### 1.3 Morse theory

Morse theory enables us to study the topology of manifold $M$ by analyse differentiable functions on it. We will first give a brief introduction to Morse theory and then see its connection with supersymmetry.
Let \( h : M \to \mathbb{R} \) be a generic function on \( M \) with isolated critical points \( P_1, P_2, \ldots, P_s \). Critical points of \( h \) are defined as zeros of the gradient vector field, \( \partial h(P_s)/\partial \phi^k = 0 \) (\( \phi_i \) are local coordinates and \( k = \dim M \)). Moreover, we require the Hessian, which is the matrix \( \partial^2 h(P_s)/\partial \phi^i \partial \phi^k \), be non-degenerate, thus it is a quadratic form on the tangent space at \( P_s \) with diagonal entries. Let \( \mu^+(P_s) \) and \( \mu^-(P_s) \) being the numbers of positive and negative diagonal entries, the diagonalization has the form \( (\mu^+(P_s), \mu^-(P_s)) \) with \( \mu^+(P_s) + \mu^-(P_s) = \dim M \). The Morse index is given by \( \mu^-(P_s) \). Let \( M_q \) be the number of critical points with the Morse index \( \mu^--q \), then we can write the Euler characteristic as

\[
\chi(M) = \sum_q (-1)^q M_q. \tag{1.16}
\]

A simple example is illustrated by Fig.(1.1), where the Euler characteristic is calculated as \( \sum_q (-1)^q M_q = M_0 - M_1 + M_2 = 1 - 4 + 1 = -2 \) and easily verified by \( \chi = 2 - 2g = -2 \) (the torus is with two holes).

Figure 1.1. An illustration of Morse theory, where \( h \) is the Morse function. One can easily check the Morse index \( \mu^- \) at each critical point.
1.4 Instanton and Witten complex

Recall the definition of supercharges by linear combination of $d$ and $d^\dagger$, Eq.(1.5), we can generalize it by defining

$$ d_t = e^{-ht}de^{ht}, \quad d_t^\dagger = e^{ht}d^\dagger e^{-ht}, \quad (1.17) $$

where $h$ is the Morse function. Since $d_t^2 = d_t^\dagger 2 = 0$, we define

$$ Q_{1t} = d_t + d_t^\dagger, \quad Q_{2t} = i(d_t - d_t^\dagger), \quad H_t = d_t d_t^\dagger + d_t^\dagger d_t \quad (1.18) $$

and the superalgebra is still hold for any $t$. The ground states will not change by this conjugation as we argued before. That is, if we denote the Betti number as $b_q(t)$, it is independent of $t$ and equals to $b_q$ of $M$.

To write $H_t$ in detail, we first note that

$$ d_t = e^{-ht}de^{ht} = d + ta^i\partial_i h \quad (1.19) $$

and for a $q$-form $\omega$

$$ d(a^i\partial_i h \omega) = a^j\partial_j(a^i\partial_i h)\omega - a^i\partial_i h d\omega \\
(d^\dagger a^i\partial_i h \omega) = a^i\partial_i h(a^j\partial_j h \omega) - a^j\partial_j h d\omega. \quad (1.20) $$

Therefore, we obtain

$$ H_t = dd^\dagger + d^\dagger d + t^2(dh)^2 + \sum_{i,j} t \frac{D^2 h}{D\phi_i D\phi_j} [a^i, a^j], \quad (1.21) $$

where $D$ is the covariant derivative that $D_i h = \partial_i h$ and $\partial_j(a^i D_i h) = a^i D_j D_i h$ with $D_j D_i h = (\partial_j \partial_i - \Gamma^k_{ij} \partial_k) h$, and $(dh)^2 = \gamma^{ij} \partial_i h \partial_j h$ is the square of the gradient of $h$. Notice that we are working on Riemannian manifold $(M, \gamma)$ introduced earlier.

Now we can see how Morse theory is related with supersymmetry. For large $t$ limit, the potential energy $V(\phi) = t^2(dh)^2$ becomes very large except near each critical point, at which $dh = 0$. Therefore minimal of eigenvalues of $H_t$ are localized near the critical points and can be calculated by expanding about them. Near each critical point $P_q$, the coordinates $\phi_i$ are Euclidean, and if we set $\phi_i = 0$ to be $P_q$ then the metric tensor $\gamma$ is Euclidean up to terms of order $\phi^2$. Thus the Morse function is approximated as $h(\phi_i) = h(0) + \frac{1}{2} \sum \lambda_i \phi_i^2 + O(\phi^3)$ for some $\lambda_i$. We have $H_t$ being approximated as

$$ H_t = \sum_i (-\frac{\partial^2}{\partial \phi_i^2} + t^2 \lambda_i^2 \phi_i^2 + t\lambda_i [a^i, a^i]). \quad (1.22) $$

The corrections to this formula are determined by higher order terms of $\phi$, but for ground states they will not enter and thus reasonable to study ground states with it.
Let us denote
\[ H_i = -\frac{\partial^2}{\partial \phi^2_i} + t^2 \lambda_i^2 \phi^2_i, \quad J_i = [a_i^\dagger, a_i]. \] (1.23)

It is readily to see that \( H_i \) and \( J_j \) mutually commute and can be simultaneously
diagonalized. Moreover, we recognize \( H_i \) as the Hamiltonian of harmonic oscillator,
which has the eigenvalues \( t|\lambda_i| (1 + 2N_i) \), \( N_i = 0, 1, 2, \ldots \) For \( J_j = a_j^\dagger a_j - a_j a_j^\dagger \)
acting on a \( q \)-form \( \omega \), \( a_j^\dagger a_j \) or \( a_j a_j^\dagger \) will contribute +1 or −1. Hence the eigenvalues
of \( H_i \) are given by
\[ t \sum_i ((|\lambda_i| (1 + 2N_i) + \lambda_i n_i), \quad N_i = 0, 1, 2, \ldots, \text{ and } n_i = \pm 1. \] (1.24)

When we restrict \( H_t \) to a \( q \)-form \( \omega \), there are exactly \( q \) positive contributions from \( J_j \). Thus for Eq.(1.24) to vanish, we need \( N_i = 0 \) for all \( i \) and \( n_i = +1 \) if and only if \( \lambda_i \) is negative. Since the number of negative \( \lambda_i \) is precisely the Morse index \( \mu_-(P_a) \),
at each critical point \( H_t \) has exactly one zero eigenvalue and the eigenfunction is a
\( \mu_-(P_a) \)-form. All other eigenvalues are proportional to \( t \) and are positive.

From above analysis, we see that for every critical point \( P \), \( H_t \) has just one
eigenstate \( |\Omega\rangle \) whose energy does not diverge with \( t \) and it is a \( q \)-form if \( \mu_-(P) = q \).
Even through we have only shown that the leading contributions in perturbation
theory vanish, \( H_t \) does not annihilate any other states whose energy is proportional
to \( t \) at large \( t \). Hence at most, the number of zero-energy \( q \)-forms equals the number
of critical points of \( \mu_- = q \), and this is the weak Morse inequalities \( M_q \geq b_q \).

To derive Eq.(1.15), we need the strong Morse inequalities
\[ \sum_q M_q t^q - \sum_q b_q t^q = (1 + t) \sum_l l_q t^q, \] (1.25)
where \( l_q \) are nonnegative integers. Eq.(1.15) is arrived by setting \( t = -1 \). If we
consider the strong Morse inequalities as a better upper bound on the number of
zero eigenvalues, the refinement will not be from the higher order terms in pertur-
bation. We cannot tell which critical points are degenerate by using just local data.
Instead, we will consider tunnelling between critical points to distinguish spurious
degeneracies.

One simple example for illustration is considering \( M = S^2 \). There are two
critical points on sphere with Morse index equal to zero and two. Thus we have
two ground states represented by a 0-form and a 2-form. The Witten index for
them are same, \((-1)^P = 1\), they are both bosonic states. Therefore cannot become
non-zero energy states, and are the true ground states. However, we can deform
\( S^2 \) into other shape, a bended cigar for example, the number of critical points will
change. This must not mean we have more ground states, thus the true ground
states might be the linear combinations of them.
If so, let \( |\Omega_q^1\rangle \) and \( |\Omega_q^2\rangle \) be two zero-energy states in leading order of perturbation, their linear combination is the true ground state such that

\[
Q_t(|\Omega_q^1\rangle - |\Omega_q^2\rangle) = 0. \tag{1.26}
\]

Since none of them is annihilated by \( Q_t \), we write

\[
Q_t|\Omega_q^2\rangle = |\Omega_q^+\rangle. \tag{1.27}
\]

Thus if

\[
\langle \Omega_q^+|Q_t|\Omega_q^1\rangle = c_{ij}, \quad c_{ij} \neq 0,
\]

neither of them are the true ground states. In this way, we consider expansion

\[
Q_t|\Omega_q^i\rangle = \sum_j |\Omega_q^+|\langle \Omega_q^+|Q_t|\Omega_q^i\rangle. \tag{1.28}
\]

Notice that in differential form representation, this amplitude can be written as

\[
\langle \Omega_q^+|Q_t|\Omega_q^1\rangle = \int_M \Omega_q^+ \wedge * Q_t \Omega_q^1. \tag{1.29}
\]

This means the tunnelling corrections are only from states with the Morse indexes differ by one. Also check arguments in the footnote of next page.

To calculate this non-perturbative corrections to the matrix elements of \( Q_t \), we first need to know the Lagrangian of the system. The Hamiltonian \( H_t \) can be obtained by canonical quantization of supersymmetric non-linear sigma model, see details in Appendix A. The simplified action is written as

\[
\mathcal{L} = \frac{1}{2} \int d\tau \gamma_{ij} \left( \frac{d\phi^i}{d\tau} \frac{d\phi^j}{d\tau} + i \bar{\psi}^i \gamma_0 \gamma^0 \frac{D\psi^j}{d\tau} \right) + \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l - t^2 \gamma_{ij} \frac{\partial h}{\partial \phi^i} \frac{\partial h}{\partial \phi^j} - t \frac{D^2 h}{D\phi^i D\phi^j} \bar{\psi}^i \psi^j, \tag{1.30}
\]

where all fields in the model are functions of \( \tau \) only which can be interpreted as “time”, \( \gamma_{ij} \) and \( R_{ijkl} \) are the metric and curvature tensors of \( M \), and \( \psi^i \) are anticommuting fields tangent to \( M \). To calculate tunnelling paths, i.e. instanton solutions, of the theory, we can discard the fermionic fields and write the Lagrangian with a Euclidean metric(by Wick rotation \( \tau = -i\tau \)), the resulting action is

\[
\mathcal{L}_E^B = \frac{1}{2} \int d\tau \left( \gamma_{ij} \frac{d\phi^i}{d\tau} \frac{d\phi^j}{d\tau} + t^2 \gamma_{ij} \frac{\partial h}{\partial \phi^i} \frac{\partial h}{\partial \phi^j} \right) \tag{1.31}
\]

We see that for any \( \phi^i \) satisfies

\[
\frac{d\phi^i}{d\tau} + t \gamma_{ij} \frac{\partial h}{\partial \phi^j} = 0, \tag{1.32}
\]
which is the gradient flow of $h$ flowing between critical point $P_i$ with $\mu_-(P_i) = q+1$ and critical point $P_j$ with $\mu_-(P_j) = q^2$, Eq.(1.31) gives a minimal
\[ \mathcal{L} = t|h(\tau = \infty) - h(\tau = -\infty)|. \] (1.35)

Or say for any path, we have
\[ \mathcal{L}^B_E \geq t|h(\tau = \infty) - h(\tau = -\infty)|. \] (1.36)

Configurations that satisfy Eq.(1.32) are called instantons\(^3\). The deformation of instantons can be obtained by first order variation of this equation as
\[
\frac{d}{de} \left( \frac{d(\delta^i + \epsilon \delta^i)}{dt} \right) \pm t^{2j}(\delta^i + \epsilon \delta^i) \partial_j h(\delta^i + \epsilon \delta^i) \bigg|_{e=0}^= D^\pm \delta^i := \pm t^{2j}D_j \partial_k h \delta^k = 0.
\] (1.37)

Therefore the number of possible deformations is determined by \(\dim \ker D^\pm\). Apparently, the simplest deformation is given by time translation $\phi'(\tau) = \phi(\tau + \delta \tau)$.

Notice that the eigenfunctions are localized and peaked at corresponding critical points and rapidly decay upon departing them. However, for two related critical points\(^4\), their eigenfunctions’ overlap is greatest along the path, solutions of Eq.(1.32), connecting them. Thus if we consider the Morse function $h$ as an operator which gives the value of $h$ at the corresponding critical point, the desired amplitude, to the leading order, can be rewritten as path integral
\[
\langle \Omega^q_{P_i} | Q_i | \Omega^q_{P_j} \rangle = \frac{1}{h(P_i) - h(P_j) + O(1/t)} \lim_{T \rightarrow \infty} \langle \Omega^q_{P_j} | e^{TH} | Q_i | e^{-TH} | \Omega^q_{P_i} \rangle
\]
\[
= \frac{1}{h(P_i) - h(P_j)} \int_{\phi_i(-\infty) = P_i}^{\phi_i(\infty) = P_j} D\delta \bar{\Psi} \bar{\Psi} D\psi \psi dug \bar{h}.
\] (1.38)

---

\(^2\)The critical points satisfy boundary conditions $\phi_i(-\infty) = P_i$ and $\phi_j(\infty) = P_j$. The gradient flow is said to be downward flow if the sign in Eq.(1.32) is plus and to be the upward gradient flow if the sign is minus. Moreover, as mentioned earlier, the non-degenerate Hessian makes us to separate the tangent space at each critical point into direct sum of two subspaces
\[ T_{P_i} M = V^+_i \oplus V^-_i. \] (1.33)

By definition $\mu_\pm(P_i) = \dim V^\pm_i$. Accordingly, we can define stable and unstable submanifolds of $M$ at each critical point $P_i$ as
\[
S_{P_i} = \{ \phi_i | \lim_{\tau \rightarrow -\infty} \phi_i(\tau) = P_i \}
\]
\[
U_{P_i} = \{ \phi_i | \lim_{\tau \rightarrow \infty} \phi_i(\tau) = P_i \}.
\] (1.34)

These submanifolds are tangent to $V^+_i$ and $V^-_i$, respectively at $P_i$. The intersection, $U_{P_i} \cap S_{P_i}$, is transverse if the gradient flow of $h$ satisfies the Morse-Smale condition. In this case, if $\mu_-(P_i) - \mu_-(P_j) = r$, then $U_{P_i} \cap S_{P_j} = r$ since $\mu_+(P_j) = \dim M - \mu_-(P_j) = \dim M - \mu_-(P_i) + r$.

---

\(^3\)The name instanton literally infers the transition time is instantly short compared to the infinite interval of Euclidean time $-\infty < \tau < \infty$.

---

\(^4\)As mentioned earlier, only critical points with the Morse indexes differ by one.
The operator $e^{-TH}$ projects states to the perturbative ground states at critical points $P$. The integral in the second line is obtained by computing $[Q_t, h]$ as

$$[Q_t, h] \omega = [d + dh \wedge, h] \omega = dh \wedge \omega = \psi^{ij} \partial_k h \omega. \quad (1.39)$$

$L_E$ in the second line is the Euclidean action. It is combination of the bosonic part $L_E^B$ and the fermionic part $L_E^F$, the later is given by

$$L_E^F = \int_{-\infty}^{\infty} d\tau \gamma_{ij} \psi^{\dagger i} D_+ \psi^j = -\int_{-\infty}^{\infty} d\tau \gamma_{ij} D_- \psi^{\dagger i} \psi^j. \quad (1.40)$$

Since the integrand of Eq.(1.38) contains a term $\psi^{\dagger i} \partial_k h$, the amplitude is non-vanishing only if the number of zero modes of $\psi^j$ is larger than the number of zero modes of $\psi^i$ by exactly one. Mathematically, this means that the path integral is non-vanishing only if

$$\text{Index } D_- = \dim \ker D_- - \dim \ker D_+ = 1. \quad (1.41)$$

One can prove $\text{Index } D_- = \Delta \mu := \mu_-(P_j) - \mu_-(P_i)$, since simply speaking the tunnelling path connects states differ by one fermion only, see more detail in [23]. This again confirms conclusion we had after Eq.(1.29). Importantly, $\text{Index } D_- = 1$ ensures that only one instanton deformation exists, which is the shift in $\tau$. This is the bosonic zero mode. By supersymmetry, we can have only one fermionic zero mode.

Furthermore, with localization principle, see details in Appendix B, the path integral localizes to regions where the integral is invariant under some supersymmetries. Since the Euclidean action is invariant under

$$\delta \phi^i = \epsilon \psi^{\dagger i} - \bar{\epsilon}$$

$$\delta \psi^i = \epsilon (-\frac{d\phi^i}{d\tau} + t\gamma^{ij} \partial_j h - \Gamma^{ij}_k \psi^{\dagger j} \psi^k)$$

$$\delta \psi^{\dagger i} = \bar{\epsilon} (\frac{d\phi^i}{d\tau} + t\gamma^{ij} \partial_j h - \Gamma^{ij}_k \psi^{\dagger j} \psi^k), \quad (1.42)$$

it is natural to think set all R.H.S. equal to zero. However, we notice that the integrand is invariant under the $\epsilon$-supersymmetry (generated by $Q$)

$$\delta_\epsilon (\psi^{\dagger i} \partial_i h) = 0. \quad (1.43)$$

Thus we only left with setting the R.H.S. of supersymmetries

$$\delta \phi^i = \epsilon \psi^{\dagger i}$$

$$\delta \psi^i = \epsilon (-\frac{d\phi^i}{d\tau} + t\gamma^{ij} \partial_j h - \Gamma^{ij}_k \psi^{\dagger j} \psi^k)$$

$$\delta \psi^{\dagger i} = 0 \quad (1.44)$$
equal to zero, and this leads to configurations
\[ \psi^\dagger_i = 0, \quad \frac{d\phi^i}{d\tau} = t\gamma^{ij}\partial_j h. \] (1.45)

This gives us a specific choice of instanton solutions Eq. (1.32), the path integral is localized to the steepest ascent \( h(P_j) > h(P_i) \).

Let \( \varrho^i(\tau) \) denotes each steepest ascent. Invariance under shift in \( \tau \) makes it a moduli space
\[ \varrho^i(\tau_1) = \varrho^i(\tau + \tau_1), \] (1.46)

where \( \tau_1 \) is parameter of the moduli space and we can consider it as the “position” of instanton in \(-\infty < \tau < \infty\). Now we can consider contributions from the transverse fluctuations of both bosonic and fermionic fields. Changing the variables by
\[ \phi^i = \varrho^i + \frac{1}{\sqrt{t}}\xi^i, \] the action is only left with quadratic parts
\[ L_E = t(h(P_j) - h(P_i)) + \int d\tau \left( \frac{1}{2}D_-|\xi|^2 - D_- \psi^\dagger \psi \right), \] (1.47)

where \( D_- \) is defined by Eq. (1.37). Since the only zero mode of \( D_- \) is parametrized by \( \tau_1 \), we denote the integration variable for bosonic zero mode is \( \tau_1 \). Let \( \psi^\dagger_0 \) denotes the integration variable for fermionic zero mode. Thus we have
\[ \int_{\phi^i(\infty) = P_j} D\xi D\psi D\psi^\dagger e^{-\int L_E} \psi^\dagger k \partial_k h \]
\[ = \sum_v \int D\xi D\psi D\psi^\dagger \int_{-\infty}^{\infty} d\tau_1 \frac{d\varrho_1}{d\tau_1} \int d\psi^\dagger_0 \psi_0^\dagger k \partial_k h \]
\[ \times e^{-t(h(P_j) - h(P_i)) - \int d\tau_1 (\frac{1}{2}D_-|\xi|^2 - D_- \psi^\dagger \psi)} \]
\[ = \sum_v e^{-t(h(P_j) - h(P_i))(h(P_j) - h(P_i))} \frac{\text{det} D_-}{|\text{det} D_-|}. \] (1.48)

For each tunnelling path, \( \frac{\text{det} D_-}{|\text{det} D_-|} \) is equal to \( \pm 1 \). To determine the sign, let \( v \) be the tangent vector to \( \varrho \) at \( P_j \), and \( V^1 \) the subspace of \( V_{P_j} \) orthogonal to \( v \), and in generic case \( \text{dim} V^1 = \mu_-(P_j) = 1 = q \). The orientation of \( V^1 \) is given by \( \Omega^{q+1} \).

This will induce an orientation of \( V^1 \) by interior multiplication of \( v \) with \( \Omega^{q+1} \). After parallel transporting \( V^1 \) along \( \varrho \), we compare its orientation with \( V^1 \) and define \( n_\varrho \) to be \( +1 \) or \( -1 \) if their orientations agree or not. Finally, we have
\[ Q_t |\Omega^q \rangle = \sum_j \sum_{\varrho} n_\varrho e^{-t(h(P_j) - h(P_i))} |\Omega^{q+1} \rangle. \] (1.49)

Notice that, by definition, \( \sum_{\varrho} n_\varrho \) is an integer. To get rid of the exponential factor, we rescale the states as \( e^{-\frac{1}{t h} |\Omega|} \). Thus one can define the Witten complex and prove the strong Morse inequalities.

\[ ^5 \]This is also required by localization principle.
1.5 Topological quantum field theory

Witten’s understanding of relation between supersymmetry and Morse theory is of great profoundness. It might be the start of golden epoch for modern mathematical physics. On the other hand, gauge symmetry’s dominating role in particle physics naturally suggests to find an analogue relation between Morse theory and itself[34]. However, the picture is quite different when we consider the Yang-Mills gauge theory, the infinite-dimensional space will make the Morse index being ill defined and also the Morse function itself does not take values in \( \mathbb{R} \) but \( \mathbb{R}/\mathbb{Z} \) instead. Thanks for Floer, the work of him on three manifolds[16] significantly changed the situation. Atiyah first recognized the importance of the work and conjectured its crucial relations with low-dimensional topologies[1]. In Atiyah’s point of view, elements of Floer homology are the ground states for a non-relativistic Hamiltonian in 3-dimensional quantum field theory. Since it is closely related with Donaldson’s theory of four manifolds[12, 13, 11], the conjecture was concerning a relativistic generalization. Witten, once again, solved the problem[40] and established the BRST formulation of topological quantum field theory. We will not get into details about Floer and Donaldson, but only show the BRST argument of gauge fixing of CS action.

The construction of Floer is similar to what we have done earlier. In Yang-Mills gauge theory, the Morse function is called the Chern-Simons function, which we will start talking in detail from next section. Then one can have the Hamiltonian analogous to Eq.(1.21). Witten then found that the relativistic Lagrangian corresponds to the Hamiltonian possesses a fermionic symmetry(supersymmetry generated by \( Q \)), which is quite similar to the BRST symmetry in string theory. The result of it is of much surprise. Since BRST gauge fixed string action is metric independent, relativistic generalization of Floer’s theory is a topological quantum field theory.

Let us summarize up. We started to look at supersymmetry as its prominence of being future particle physics. Its relation to topology, by adding the Morse function as potential, leads us to topological quantum field theory. The result surprised not only physicists, it also lead mathematicians to relate low dimensional topologies, especially 3- and 4-dimensions. More recently, mathematical side pays back that knot invariants give novel understanding of quantization[20, 21]. Moreover, since the CS theory is related to 2-dimensional conformal theories, 2-,3- and 4-dimensional theories are closely related. There are many other interesting facts following. For example, the localization principle we used in instanton calculation, it reduces a great ideal of number of integration dimension.
Chapter 2

Chern-Simons

We have skipped a great ideal in last section about topological quantum field theory as the contents are much beyond the purpose of this thesis. Somehow, it is good to keep them in mind. Our concentration will now on the Morse function of Floer’s theory, the Chern-Simons function, as itself is a simple example of topological theory and the partition function constructed from it can be used to compute knot invariants, which has deep relation to Floer homology.

Let $M$ be a smooth $n$-manifold and $B$ be a smooth $SU(2)$ principle bundle over $M$ with projection $\pi: B \to M$. The bundle is trivial as $B = M \times SU(2)$. The connection on $M$ is an Lie algebra valued 1-form $A$. Let $\mathcal{A}(B)$ denotes the space of connections. Since the bundle is trivial, $\mathcal{A}(B)$ is isomorphic to $\Omega^1(M, su(2))$, the space of 1-forms with $su(2)$ coefficients. Let $\mathcal{G}(B) = C^\infty(M, SU(2))$ denote the gauge group, as smooth functions from $M \to SU(2)$. Under gauge transformation $g \in \mathcal{G}(B)$, the connections $A$ transform according to formula

$$A \rightarrow A_g = g^{-1}Ag + g^{-1}dg.$$  \hspace{1cm} (2.1)

Thus we can define quotient space

$$\mathcal{C} = \mathcal{A}/\mathcal{G},$$  \hspace{1cm} (2.2)

as space of gauge equivalent connections.

The Chern-Simons(CS) function is introduced by Chern and Simons. Consider 1-form $F = dA + A \wedge A$ on $\mathcal{C}$ given by the curvature. Since it is closed, from the Bianchi identity, locally it can be written as derivative of some function, that is the CS function. The definition is as follow, let $A_0$ be the trivial connection and put $A_t = (1-t)A + tA_0$ for $0 \leq t \leq 1$. This is a connection on $M \times [0, 1]$. Now define

$$CS(A) = \frac{1}{8\pi} \int_{M \times [0,1]} \text{Tr} F^2,$$  \hspace{1cm} (2.3)

where $F$ is the curvature on $M \times [0, 1]$. Notice that the integral on a closed 4-manifold gives $\pi$ times the second Chern class. Moreover, the CS function is a
map from $\mathcal{C} \to \mathbb{R}/\mathbb{Z}$. Let give another straightforward definition of CS function. Define CS function as map from $\mathcal{A} \to \mathbb{R}$ by

$$CS(A) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

(2.4)

The factor $k$, the level, is putted in only for physical interests, mathematically one just put 1 instead. Since we are dealing with gauge fields, first thing to check is whether this function as a physical action is gauge invariant or not. By using Eq.(2.4), we have the gauge transformation of $CS(A)$ as

$$CS(A_g) = CS(A) - \frac{k}{4\pi} \int_M d(\text{Tr}(gA \wedge dg^{-1})) + \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg).$$

(2.5)

If we consider closed manifold $M$, then the second term vanishes. The last term is the Wess-Zumino term and gives $2\pi m$, $m \in \mathbb{Z}$ which is winding number of the gauge group. Therefore $CS(A)$ is not invariant under gauge transformation. However if we consider infinitesimal gauge transformation around identity, $CS(A)$ is invariant. From this one can consider to construct a homotopy between trivial gauge transformation, $g_0$, at identity and other non-trivial gauge transformations, $g$. However non-trivial elements in the group cohomology $H^3(G, \pi_3(G))$ prevent one constructing such homotopic relation. For compact simple Lie groups $G$, $\pi_3(G) \cong \mathbb{Z}$. Therefore one cannot always possible to define a homotopy between $g_0$ and $g$, this non-triviality is characterized by the Wess-Zumino term, i.e. by a non-zero winding number. In conclusion, the CS Lagrangian transforms under a gauge transformation with winding number $m$(or in fancy words the trivialization of the tangent bundle of $M$ is twisted by $m$ units) as

$$CS(A) \longrightarrow CS(A) + 2\pi km.$$  

(2.6)

Incidentally, since we will use CS action to construct partition function of physical theory, the level must be an integer, $k \in \mathbb{Z}$, to make the partition function being single valued.

Accordingly we can define a partition function by CS action as

$$Z = \int \mathcal{D}A e^{i \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)}.$$ 

(2.7)

This enable us to analyse CS term as an independent theory. One thing similar to Morse theory is that for large $k$ we have to consider the stationary configurations

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1When considering Morse function to Yang-Mills gauge theories, the problems are that CS function as our choice is a map from $\mathcal{C} \to \mathbb{R}/\mathbb{Z}$ and that since $\mathcal{C}$ is infinite dimensional the Morse index might be ill defined.
of $CS(A)$ as the integrand of Eq. (2.7) is wildly oscillatory. The stationary points are the saddle points of the CS action and can be obtained by solving

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = 0,$$

(2.8)

the solutions are thus “flat connections”.

Let us have a better look at those flat solutions. Let $x$ on $M$ be a base point and $g : t \to x(t)$ be a closed loop such that $x(0) = x(1) = x$. For each connection $A$, we can parallel transport any tangent vector at $x(0)$ along the loop. The resulting vector might not be the same, their difference is given by $g_\epsilon = g(1) \in SU(2)$, where $g : t \to g(t) \in SU(2)$ is gauge transformation at $t$, and the difference is called holonomy. For each $g$, $g_\epsilon$ is given by solving equation

$$\frac{dg(t)}{dt} + A(\dot{x}(t))g(t) = 0$$

(2.9)

with initial condition $g(0) = id$. Since we are dealing with flat connection, there is no curvature correction. Consider a infinitesimal deformation of $g$ by attaching a small loop $\epsilon$ as $g \to g + \epsilon$, solution to Eq. (2.9) will change as

$$g_{\epsilon} = (1 - \int_\epsilon A)g_{\epsilon} = (1 - \int_S F)g_{\epsilon} = g_{\epsilon},$$

(2.10)

where $S$ is a small disk with boundary $\epsilon$. Thus the holonomy at $x$ is independent of loops that can be continuously deformed to identity, and it is given by a representation of the fundamental group $\pi_1(M, x) \to SU(2)$. This is the monodromy representation.

In conclusion, gauge equivalence classes of such flat connections correspond to homomorphisms

$$\phi : \pi_1(M) \to SU(2),$$

(2.11)

up to conjugation. For a flat connection $A^{(\alpha)}$, expanding Eq. (2.7) around it gives a function $\mu(A^{(\alpha)})$. For a 3-manifold $M$ with a topology that there are only finitely many classes of homomorphisms (2.11), we can write Eq. (2.7) as a sum, in large $k$ behavior

$$Z = \sum_\alpha \mu(A^{(\alpha)}).$$

(2.12)

In this case, the flat connection $A^{(\alpha)}$ must be isolated, i.e. $H^1(M, D_{A^{(\alpha)}}) = 0$, since a non-trivial element of $H^1(M, D_{A^{(\alpha)}})$ will continuously deform one equivalence classes of flat connection to the other. In that case the discrete sum in Eq. (2.12) will be an integral over a moduli space of gauge inequivalent flat connections.
2.1 A finite dimensional analogue

Before evaluate our infinite dimensional degenerate path integral, we first look at a finite dimensional analogue (Gaussian integrals of degenerate quadratic functional).

It is helpful to understand the Faddeev-Povov procedure by review classical Gaussian integrals. Instead of introducing ghosts fields, we will see how can one remove unphysical modes, that is how to deal with the degeneracy of quadratic functional integrals. For one-dimensional integral

\[ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-\mu x^2} = \frac{1}{\sqrt{\mu}}, \]  

(2.13)

after analytic continuation, we have

\[ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{i\lambda x^2} = |\lambda|^{-\frac{1}{2}} e^{i\frac{\pi}{4} \text{sgn} \lambda}. \]  

(2.14)

In general \( n \)-dimensional integral of a non-degenerate\(^2 \) quadratic form \( Q \), we have

\[ \int e^{iQ(x)} \frac{dx}{\pi^{n/2}} = (\text{det } W^{-1} W)^{-\frac{1}{2}} e^{i\frac{\pi}{4} \text{sgn } Q}. \]  

(2.15)

In above Gaussian integral, we have assumed \( Q \) is bijective. Let \( X \) be Euclidean space with a action of compact group \( G \) and \( Q'(x) \) be a non-degenerate \( G \)-invariant quadratic form. In this case we have a gauge symmetry or say \( Q'(x) \) is not injective. To prevent over-counting, we need gauge fixing the integral by taking a transversal slice of the space for the \( G \) action. To make this happen, a Jacobian(normalised) \((\text{det } W^{-1} W)^{-\frac{1}{2}} = V(\text{orbit}) / V(G)\) is needed. This Jacobian arises when we consider \( W \) as map from the tangent space of \( G \) at the identity to the tangent space of the orbit at \( x \)(the infinitesimal map from Lie algebra to tangent space of manifold).

We have

\[ \int e^{iQ'(x)} \frac{dx}{\pi^{n/2}} = \frac{(\text{det } W^{-1} W)^{-\frac{1}{2}}}{|\text{det } Q'|} e^{i\frac{\pi}{4} \text{sgn } Q'}. \]  

(2.16)

It is tricky here that gauge fixing condition is implicitly added by changing the integration over all space to only one transversal direction. In case of Faddeev-Popov procedure, one uses Hodge decomposition to separate the space into transversal and longitudinal directions and adds gauge fixing constrain on longitudinal direction to reduce over-counting of physical modes along gauge orbits.

2.2 Twisted de Rham cohomology

Before evaluating partition function Eq.(2.7) in 3-manifold \( M \), several points of importance are needed to be clear. We will first use standard Faddeev-Povov procedure to fix the gauge of our theory, see details in Appendix E. Then we will

\(^2\text{Non-degeneracy requires } Q \text{ has no zero eigenvalues.}\)
perturb around the flat connection by rewriting $A = A^{(\alpha)} + B$. Therefore the differential operator will be a covariant derivative $D_{A^{(\alpha)}}$. It acts on $B$ as

$$D_{A^{(\alpha)}} B = dB + [A^{(\alpha)}, B]. \quad (2.17)$$

Since the covariant derivative $D_{A^{(\alpha)}}$ is respect to the flat connection $A^{(\alpha)}$, $D_{A^{(\alpha)}}^2 = F_{A^{(\alpha)}} = 0$. We can write down a complex

$$0 \to \Omega^0(M, su(2)) \to \Omega^1(M, su(2)) \to \Omega^2(M, su(2)) \to 0.$$

Notice that $\Omega^0(M, su(2))$ is the Lie algebra of $SU(2)$ gauge transformations, that is if we write the gauge transformation as $g = e^{i\varphi}$ then $\varphi \in \Omega^0(M, su(2))$. They generate infinitesimal gauge transformations.

With a metric on $M$, we can define the Hodge star as an operator maps from $\ast : \Omega^q(M, su(2)) \to \Omega^{n-q}(M, su(2))$. This enable us to have a pairing on $\Omega^q(M, su(2))$ as

$$\langle b, b \rangle \equiv -\int_M \text{Tr}(b \wedge *b), \quad (2.18)$$

where $b$ is Lie algebra valued $q$-form. With pairing we can define the dual operator $D_{A^{(\alpha)}}^\dagger : \Omega^q(M, su(2)) \to \Omega^{q-1}(M, su(2))$ by equation

$$D_{A^{(\alpha)}}^\dagger = (-1)^{n(p+1)+1} \ast D_{A^{(\alpha)}} \ast, \quad (2.19)$$

where $n = 3$ in our case.

Consider $a \in \text{Ker}D_{A^{(\alpha)}}$, then one has

$$\langle D_{A^{(\alpha)}} a, \phi \rangle = \langle a, D_{A^{(\alpha)}}^\dagger \phi \rangle = 0, \forall \phi \in \Omega^1(M, su(2)), \quad (2.20)$$

where $D_{A^{(\alpha)}}^\dagger = - \ast D_{A^{(\alpha)}} \ast$, thus the equality

$$(\text{Ker}D_{A^{(\alpha)}})^\perp = \text{Im}D_{A^{(\alpha)}}^\dagger. \quad (2.21)$$

Therefore we have the basic orthogonal decomposition$^3$,

$$\Omega^0(M, su(2)) = \text{Ker}D_{A^{(\alpha)}} \oplus \text{Im}D_{A^{(\alpha)}}^\dagger, \quad (2.22)$$

similarly, it is readily to obtain

$$\Omega^1(M, su(2)) = \text{Ker}D_{A^{(\alpha)}}^\dagger \oplus \text{Im}D_{A^{(\alpha)}}. \quad (2.23)$$

$^3$The Hodge decomposition.
And the Laplace-Beltrami operator acting on $q$-forms
\[ \Delta_q = D^\dagger_{A^{(\alpha)}} D_{A^{(\alpha)}} + D_{A^{(\alpha)}} D^\dagger_{A^{(\alpha)}}. \] (2.24)

Since we need the integral reduces to summation, Eq.(2.12), $A^{(\alpha)}$ must be isolated, i.e. $H^1(M, D_{A^{(\alpha)}}) = 0$. However, there can be a non-trivial isotropy group $H_\alpha$ which is the subgroup of gauge transformations that leave $A^{(\alpha)}$ invariant,
\[ H_\alpha = \{ \varphi \in G | \varphi A^{(\alpha)} = A^{(\alpha)} \}. \] (2.25)

Non-trivial isotropy group $H_\alpha$ ensures generalizations of all flat gauge fields on some 3-manifolds, e.g. $S^3$ and the lens spaces. The zero-cohomology can represent the Lie algebra of $H_\alpha$
\[ \text{Lie}(H_\alpha) = H^0(M, D_{A^{(\alpha)}}) = \text{Ker} D_{A^{(\alpha)}}. \] (2.26)

Therefore irreducibility of $A^{(\alpha)}$ is determined by whether $H^0(M, D_{A^{(\alpha)}})$ vanishes or not. We denote the subgroup $H_\alpha$ of $SU(2)$ which is isomorphic to $H_\alpha$. These considerations mean that the line of our gauge fixing intersects each orbit only once is not true. We will consider only the isolated case, and if the connection is reducible one has to add a factor $1/\text{Vol} H_\alpha$.

### 2.3 First loop correction

The large $k$ limit is enforced to have a semiclassical evaluation of CS that one can add perturbations around flat connection to mimic quantum fluctuations around ground states. Before adding perturbations, we have to first do gauge fixing of CS as it is invariant under trivial gauge transformations, i.e. gauge transformations with zero winding number or say those can be continuously deformed to identity.

To begin with, we choose the covariant Feynman gauge $D^\dagger_{A^{(\alpha)}} A = 0^4$. Using Eq.(E.4) and Eq.(E.5), we have
\[ \left| \det \delta \mathcal{F}(A_{\alpha \beta}) \right|_{g_{\beta}=0} = \det D^\dagger_{A^{(\alpha)}} D_A. \] (2.27)

This determinant can be lifted into exponent by considering Gaussian Grassmann integral
\[ \det D^\dagger_{A^{(\alpha)}} D_A = \int DCD\bar{C} e^{\langle \bar{C}, D^\dagger_{A^{(\alpha)}} D_{A^{(\alpha)}} + B \bar{C} \rangle}. \] (2.28)

Therefore we have an additional term to the Lagrangian, denote it as ghosts
\[ \mathcal{L}_{\text{ghosts}} = - \langle \bar{C}, D^\dagger_{A^{(\alpha)}} D_{A^{(\alpha)}} + B \bar{C} \rangle, \] (2.29)

\footnote{As mentioned before, we want to pick a transversal section intersects each orbit only once. By using Hodge decomposition, one can see this explicitly for our choice.}
where $C, \tilde{C} \in (\text{Ker} D_{A^{(a)}})^\perp$. It can be divided into a kinetic term of ghosts plus an interaction term between the ghost fields and the fluctuation $B$ as

\[
\langle \tilde{C}, D^\dagger_{A^{(a)}} D_{A^{(a)}} + B C \rangle = \langle \tilde{C}, D^\dagger_{A^{(a)}} D_{A^{(a)}} C \rangle + \langle \tilde{C}, D^\dagger_{A^{(a)}} [B, C] \rangle = \langle \tilde{C}, D^\dagger_{A^{(a)}} D_{A^{(a)}} C \rangle + \langle \tilde{C}, D^\dagger_{A^{(a)}} B C \rangle - \langle \tilde{C}, D^\dagger_{A^{(a)}} C B \rangle = \langle \tilde{C}, D^\dagger_{A^{(a)}} D_{A^{(a)}} C \rangle - \langle \tilde{C}, D^\dagger_{A^{(a)}} C B \rangle.
\] (2.30)

On the other hand the delta constraint can also be expressed by integral

\[
\delta(D^\dagger_{A^{(a)}} A) = \int \mathcal{D} \phi e^{-i \int \text{Tr}(\phi \wedge D^\dagger_{A^{(a)}} A)} = \int \mathcal{D} \phi e^{i \int \text{Tr}(\phi \wedge D_{A^{(a)}*} A)} = \int \mathcal{D} \phi e^{-i \langle \phi, D_{A^{(a)}*} A \rangle},
\] (2.31)

where $\phi$ is a 3-form, bosonic field.

Now we are ready to check the total gauged Lagrangian. Put Eq.(2.27) together with gauge constrain $\int \mathcal{D} \phi \delta(D^\dagger_{A^{(a)}})$ into $\mathcal{L}(A)$, and if we consider change of variable $A = A^{(a)} + B$, in which $B$ is perturbation around flat connection $A^{(a)}$, expanded $\mathcal{L}_{\text{gauged}}(A)^5$ reads

\[
\mathcal{L}_{\text{gauged}}(A) = \mathcal{L}(A^{(a)}) + \frac{k}{4 \pi} \int_M \text{Tr}(B \wedge D_{A^{(a)}}, B + \phi \wedge \frac{4 \pi}{k} D^\dagger_{A^{(a)}}, B + \tilde{C} \wedge \star \frac{4 \pi}{ik} D^\dagger_{A^{(a)}}, D_{A^{(a)}}, C - \tilde{C} \wedge \star \frac{4 \pi}{ik} D^\dagger_{A^{(a)}}, C B).
\] (2.32)

Combining the dynamical term of $B$, we have the integrand written as

\[
e^{-i \frac{4 \pi}{k} \langle (B, \star D_{A^{(a)}} B) + \langle \phi, \frac{4 \pi}{k} D_{A^{(a)}*} B \rangle + \langle \tilde{C}, \frac{4 \pi}{k} D^\dagger_{A^{(a)}}, D_{A^{(a)}}, C \rangle - \langle \tilde{C}, \frac{4 \pi}{k} D^\dagger_{A^{(a)}}, C B \rangle \rangle} \]

\[
e^{-i \frac{4 \pi}{k} \langle \langle B, \phi \rangle, L_- (B, \phi) \rangle + \langle \tilde{C}, \frac{4 \pi}{k} D^\dagger_{A^{(a)}}, D_{A^{(a)}}, C \rangle - \langle \tilde{C}, \frac{4 \pi}{k} D^\dagger_{A^{(a)}}, C B \rangle \rangle},
\] (2.33)

where we have write the pair $(B, \phi)$ as a two component vector, and $L_- = (D_{A^{(a)}} \ast + \ast D_{A^{(a)})})$, where $e\psi = \psi$ for $\psi \in \Omega^1(M, su(2))$ and $e\psi = -\psi$ for $\psi \in \Omega^1(M, su(2))$, is a self-dual operator, maps within forms of odd order, given by

\[
L_- = \left( \begin{array}{cc} D_{A^{(a)}} \ast & -\frac{2 \pi}{k} D_{A^{(a)}} \ast \\ \frac{2 \pi}{k} D_{A^{(a)}} \ast & 0 \end{array} \right)
\] (2.34)

---

5The gauged Lagrangian possesses BRST symmetry, and it is metric independent. See details in Appendix F.
We see that its better to have a rescaled gauge fixing function $\frac{k}{2\pi} D_{A}^\dagger A = 0$, and rescale all fields as $B' = \sqrt{\frac{k}{2\pi}} B$, same for $C$, $\bar{C}$ and $\phi$. Now Eq.(2.33) is read, with prime suppressed

$$\int DBD\phi e^{-i\left(\frac{1}{2}\left(B,\phi\right)_{L-} + \left(\bar{C},D_{A}^\dagger D_{A} C\right) - \frac{i}{2\pi} \left(\bar{C}, D_{A}^\dagger C B\right)\right)}.$$

We see that at large $k$ limit, the last term is not contributing, same reason as the cubic term in $B$. After integrating over gauge field $B$ and the ghost fields $C$ and $C$, $\mu(A)$ in partition function Eq.(2.12) is proportional to

$$\frac{\Delta_0}{\sqrt{\det L_\omega}}.$$

The denominator, square root of determinant $\det L_\omega$, is not that obvious from integration, and one should notice that the result is $k$ dependent(after rescaling the field there is powers of $k$ hinding in the mesure). One can get this determinant by constructing eigenfunctions of $L_\omega$, then Gauss integration gives the result. However, we can make it more explicitly. Recall the Hodge decomposition of $\Omega^1(M, su(2))$, Eq.(2.23). We can write $B$ as

$$B = D_{A}^\dagger \rho + D_{A}^\dagger \beta.$$

For 3-form $\phi$, we consider it as exact form only, the harmonic part vanishes as we consider the irreducible isolated flat connection. Continue writing the Bosonic terms in the first line of Eq.(2.33) as

$$\int DBD\phi e^{-i\left(\frac{1}{2}\left(B,\phi\right)_{L-} + \left(\bar{C},D_{A}^\dagger D_{A} C\right) - \frac{i}{2\pi} \left(\bar{C}, D_{A}^\dagger C B\right)\right)}$$

$$= \int D\rho D\beta D\phi (\det D_{A}^\dagger D_{A})^{1/2} \left(\det D_{A}^\dagger D_{A}\right)^{1/2}$$

$$\times e^{-\frac{i}{2}\left(D_{A}^\dagger \rho + D_{A}^\dagger \beta\right)^2 + \frac{i}{2}\left(D_{A}^\dagger \rho + D_{A}^\dagger \beta\right) - i\left(\phi, D_{A}^\dagger \rho + D_{A}^\dagger \beta\right)}$$

$$= \int D\rho D\beta D\phi (\det D_{A}^\dagger D_{A})^{1/2} \left(\det D_{A}^\dagger D_{A}\right)^{1/2}$$

$$\times e^{-\frac{i}{2}\left(D_{A}^\dagger \rho + D_{A}^\dagger \beta\right)^2 + \frac{i}{2}\left(D_{A}^\dagger \rho + D_{A}^\dagger \beta\right) - i\left(\phi, D_{A}^\dagger \rho + D_{A}^\dagger \beta\right)}$$

$$= (\det D_{A}^\dagger D_{A})^{1/2} (\det D_{A}^\dagger D_{A\dagger})^{1/2}$$

$$\times (\det D_{A}^\dagger \rho \star D_{A}^\dagger D_{A\dagger})^{1/2} (\det D_{A}^\dagger \rho \star D_{A}^\dagger D_{A\dagger})^{1/2} e^{-\frac{i}{2\pi} \left(\phi, D_{A}^\dagger \rho + D_{A}^\dagger \beta\right)}.$$
To see the \( k \) dependence, we consider not to rescale all fields, and integral (2.35) reads

\[
\int DBD\phi DCD\bar{C}e^{-\frac{i}{\pi}(\langle B,\phi\rangle,L_-(B,\phi))-\langle \bar{C},\frac{k}{4\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}C\rangle}
\]

\[
= \int DBD\phi DCD\bar{C}(\det D_{A(\alpha)}D_{A(\alpha)}^\dagger)^{1/2}(\det D_{A(\alpha)}D_{A(\alpha)})_{0}^{1/2}
\times e^{-\frac{i}{2}(\beta,\rho,\phi)D_{A(\alpha)}A_{\beta}D_{A(\alpha)}^\dagger-i(\phi,\rho,\rho)D_{A(\alpha)}\epsilon_{A(\alpha)}^\epsilon_{A(\alpha)}^\dagger-(\bar{C},\bar{C})D_{A(\alpha)}D_{A(\alpha)}C)}
\]

\[
= \int DBD\phi DCD\bar{C}(\det D_{A(\alpha)}D_{A(\alpha)}^\dagger)^{1/2}(\det D_{A(\alpha)}D_{A(\alpha)})_{0}^{1/2}
\times e^{-((\beta,\rho,\phi),L_-(\beta,\rho,\phi))-\langle \bar{C},\frac{k}{4\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}C\rangle}
\]

\[
=(\det D_{A(\alpha)}D_{A(\alpha)}^\dagger)^{1/2}(\det D_{A(\alpha)}D_{A(\alpha)})_{0}^{1/2}(\det \frac{k}{4\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}^\dagger \epsilon_{A(\alpha)}D_{A(\alpha)}^\dagger )^{-1/2}
\times (\det \frac{k}{2\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}\epsilon_{A(\alpha)}D_{A(\alpha)}^\dagger )_{0}^{-1}(\det \frac{k}{4\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}^\dagger \epsilon_{A(\alpha)}D_{A(\alpha)}^\dagger )e^{-\frac{i\pi}{4}}
\]

the phase factor will be explicitly evaluated later on. The self-dual operator \( L_− \) now is a \( 3 \times 3 \) matrix

\[
L_− = \begin{pmatrix}
\frac{i k}{2\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)} & 0 & 0 \\
0 & 0 & \frac{i k}{2\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)} \\
0 & \frac{i k}{2\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)} & 0
\end{pmatrix}
\]

acts on \((\beta,\rho,\phi)\).

Finally the result of integral (2.39) can be rewritten as

\[
(\det D_{A(\alpha)}D_{A(\alpha)}^\dagger)^{1/2}(\det D_{A(\alpha)}D_{A(\alpha)})_{0}^{1/2}(\det \frac{k}{4\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}^\dagger \epsilon_{A(\alpha)}D_{A(\alpha)}^\dagger )^{-1/2}
\times (\det \frac{k}{2\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}\epsilon_{A(\alpha)}D_{A(\alpha)}^\dagger )_{0}^{-1}(\det \frac{k}{4\pi}D_{A(\alpha)}^\dagger D_{A(\alpha)}^\dagger \epsilon_{A(\alpha)}D_{A(\alpha)}^\dagger )e^{-\frac{i\pi}{4}}
\]

\[
=(\det \frac{k}{4\pi}D_{A(\alpha)}D_{A(\alpha)}^\dagger)^{-1/4}(\det \Delta_{0})^{1/2}e^{-\frac{i\pi}{4}}
\]

\[
=(\det \frac{k}{4\pi}D_{A(\alpha)}D_{A(\alpha)}^\dagger)^{-1/4}(\det \Delta_{0})^{1/2}e^{-\frac{i\pi}{4}}
\]

\[
=(\frac{k}{4\pi})^{-1/2}(\zeta(0,\Delta_0)-\zeta(0,\Delta_1))(\det \Delta_{0})^{1/4}e^{-\frac{i\pi}{4}}
\]
We have used identities. This agrees with Rozansky. Denote the Ray-Singer torsion at level \( k \) as 
\[ \tau = \left( \frac{k}{4\pi^2} \right)^{-1/2} (\zeta(0, \Delta_0) - \zeta(0, \Delta_1)) \left( \frac{\det \Delta_0}{\det \Delta_1} \right)^{3/4} \]
Recall that there is \( k \) dependence hidden in the integration measure after rescaled all fields, and the results of rescaled and non-rescaled cases must be consistent. As zeta regularization removes all zero modes, a factor \( \left( \frac{k}{4\pi^2} \right)^{-1/2} (\dim H^0(M, D_A) - \dim H^1(M, D_A)) \) is needed to account for the \( k \) dependence of zero modes.

Later in this section, we will see this ratio is exactly the analytic Ray-Singer torsion \( D_A \), after zeta regularization of these determinants, see details about Ray-Singer torsion in Appendix C. This torsion is topologically invariant which means the gauged Lagrangian must be metric independent. On the other hand, a similar problem appears that \( \mathcal{L}_{\text{gauge}} \)'s dependence on gauge fixing functional may ruin gauge invariance of the theory. Fortunately, a continues symmetry, which related to gauge invariance, of \( \mathcal{L}_{\text{gauge}} \) was discovered by Becchi, Rouet, and Stora in 1975, and is known in honor of its discoverers as BRST symmetry. BRST symmetry was originally found in string theory, it features general covariance, thus topological. In Appendix F, we use BRST symmetry to demonstrate the topological feature of gauge CS action.

\footnote{Here we need three identities. If one has self-dual operator \( K \), the relation \( \det |K| = (\det K^2)^{1/2} = (\det K)^{1/2} \) is trivial and we can write

\[ (\det \frac{k}{4\pi^2} D_A^{A(\alpha)} * D_A^{A(\alpha)} \epsilon D_A^{1(\alpha)} \epsilon D_A^{1(\alpha)})_{2}^{1/2} \]

\[ = (\det(\frac{k}{4\pi^2})^2 D_A^{A(\alpha)} * D_A^{A(\alpha)} \epsilon D_A^{1(\alpha)} D_A^{A(\alpha)} + D_A^{A(\alpha)} \epsilon D_A^{1(\alpha)} \epsilon D_A^{1(\alpha)} \epsilon D_A^{1(\alpha)})_{2}^{-1/4} \]

\[ = (\det(\frac{k}{4\pi^2})^{2/3} D_A^{A(\alpha)} D_A^{1(\alpha)} D_A^{1(\alpha)} \epsilon D_A^{1(\alpha)} \epsilon D_A^{1(\alpha)} \epsilon D_A^{1(\alpha)})_{2}^{3/4} \]}

Recall \( [\Delta, D] = 0 \), also true for the twisted operator \( D_A^{(\alpha)} \), and act it on a \( q \) -1-form \( \omega_{q-1} \) from both side, we have

\[ D_{q-1}(\Delta_{q-1}(D_{q-1}^{1(\alpha)} \omega_q + D_{q-1}^{2(\alpha)} \omega_q)) = \Delta_{q-1}(D_{q-1}^{1(\alpha)} \omega_q + D_{q-1}^{2(\alpha)} \omega_q). \]

We see that the eigenvalues of \( \Delta_{q-1} \) restricted to co-exact eigenmodes are same as the eigenvalues of \( \Delta_q \) restricted to exact eigenmodes. This means \( \det D_{q-1} \Delta_{q-1} = \det D_{q-1} \Delta_q \). Furthermore, orthogonality of the Hodge decomposition leads to the relation \( \det \Delta_q = (\det D_{q+1} \Delta_q)(\det D_{q-1} \Delta_q) \). One can prove this by considering the zeta invariant of \( \Delta \) given by

\[ \zeta(s, \Delta) = \zeta(s, D_{q+1}^{1(\alpha)} \Delta_q) + \zeta(s, D_{q-1}^{1(\alpha)} \Delta_q), \]

and the determinant of Laplacian is

\[ \det \Delta_q = e^{-\zeta(0, \Delta_q)} \]

\[ = e^{-\zeta(0, D_{q+1}^{1(\alpha)} \Delta_q)} + e^{-\zeta(0, D_{q-1}^{1(\alpha)} \Delta_q)} \]

\[ (\det D_{q+1} \Delta_q)(\det D_{q-1} \Delta_q) \]}

\footnote{This factor is indeed for general cases. In our situation, isolated connection indicates \( \dim H^1(M, D_{q(\alpha)}) = 0 \). In most simple case, irreducible isolated connection Witten first considered, there is no such factor.}
2.4 Regularization

The determinants in Eq.(2.75) might not be well defined as there are infinite many eigenvalues. An appropriate regularization is needed. Consider a Laplace operator $\Delta$ with positive eigenvalues $\lambda$, one can define the zeta function

$$\zeta(s, \Delta) = \sum_{\lambda} \lambda^{-s} = \text{Tr} \Delta^{-s}. \quad (2.46)$$

When $\text{Re}(s)$ is sufficiently large, zeta function is meromorphic. Schwarz showed in [32] that it can be analytically continued to the whole complex plane, in which $s = 0$ is not a pole and $\zeta(0, \Delta)$ and $\zeta'(0, \Delta)$ are well defined. From this definition, $\zeta(0, \Delta)$ is the dimension of the Hilbert space, only for even dimensions. Ray and Singer[30] showed that one can define

$$\det \Delta = e^{-\zeta'(0, \Delta)}, \quad (2.47)$$

since

$$\zeta'(0, \Delta) = \sum_{\lambda} \frac{d}{ds} \lambda^{-s}|_{s=0} = \sum_{\lambda} - \log \lambda. \quad (2.48)$$

This definition results $\det \Delta$ as a real number, and is exactly what we want from a Gaussian integration. On the other hand the sign of operator in the phase term of Eq.(2.16) can be given by the eta-function which for a self-adjoint linear operator $K$ (may have both positive and negative eigenvalues) is defined as

$$\eta(s, K) = \sum_{k} \frac{1}{(\lambda_k^+)^s} - \sum_{l} \frac{1}{(\lambda_l^-)^s}, \quad (2.49)$$

where $\lambda_k^+$ and $\lambda_l^-$ are the strictly positive and negative eigenvalues of $K$, respectively. With this regularization scheme, the determinants are well defined. We actually already used this regularization in deriving the result, as one can see the $k$ dependent factor is coming from this procedure$^8$, Eq.(2.75). Moreover, Ray and Singer[30] proved that

$$T_{A(\alpha)} = \left( \frac{\det \Delta_{A(\alpha)}^0}{\det \Delta_{A(\alpha)}^1} \right)^{\frac{1}{4}}, \quad (2.50)$$

is topological invariant. Therefore the 1-loop correction of the CS theory is topological invariant. See more details about how to define Ray-Singer torsion in Appendix C.

$^8$In literatures, some authors intend to use $\det' \Delta$ to denote the regularized determinant.
2.5 The eta invariant

The eta invariant $\eta(0, L_{-})$ is defined to see the spectral asymmetry of operators. We want to know its dependence on the background field $A^{(\alpha)}$, and write $\eta(A^{(\alpha)})$ instead later on. Before getting complicated we start with a simple example. Suppose we have a particular operator with eigenvalues $\lambda_m = m + a$, $m \in \mathbb{Z}$ and $0 < a < 1$. With zeta regularization we have the spectral asymmetry as

$$
\eta_a = \lim_{\epsilon \to 0} \frac{\sum_{\lambda_m > 0} e^{-\lambda_m \epsilon} - \sum_{\lambda_m < 0} e^{\lambda_m \epsilon}}{e^{-a \epsilon} - e^{-(1-a) \epsilon}}
= \lim_{\epsilon \to 0} \left( \frac{e^{-a \epsilon}}{1 - e^{-\epsilon}} - \frac{e^{-(1-a) \epsilon}}{1 - e^{-\epsilon}} \right)
= 1 - 2a.
$$

(2.51)

We see this non-trivial dependence of $\eta_a$ on $a$ since numbers of positive and negative $\lambda$ of different $a \in (0, 1)$ are not same. Moreover, since $m \in \mathbb{Z}$, $\eta_a$ is periodic, i.e. $\eta_a = \eta_{a+n}$, $n \in \mathbb{Z}$. We can consider variation of $a$ within $(0, 1)$ as continuous dependence of $\eta_a$ on $a$. When $a$ passes through an integer $n$, $\lambda_{-n}$ changes sign and we recognize this as discontinuous dependence of $\eta_a$ on $a$. Then we can write $\eta_a$ for any $a$ as

$$
\eta_a = 1 - 2a - 2SF_a,
$$

(2.52)

where $SF_a$ is the discontinuous jump called spectral flow and the factor 2 comes as once an eigenvalue changes sign $\eta_a$ jumps by 2 units. If we define zero eigenvalues as positive, $\eta_0 = 1$ as number of positive eigenvalues is the same as of negative eigenvalues and we have

$$
\eta_a = \eta_0 - 1 + (1 - 2a) - 2SF_a
\Rightarrow \eta_a - \eta_0 = \text{cont.}_{0 \to a} - 2SF_a.
$$

(2.53)

Now we can proceed to analyse $\eta(A^{(\alpha)})$. For trivial connection, the zero modes are easy to count. Recall operator $L_{-}$ is acting on zero and 1-forms, thus zero modes are constant Lie algebra valued functions and Lie algebra valued closed 1-forms, i.e. $(1 + b^1(M)) \dim G$. However, from the second line of Eq.(2.53) we see that it is not important to know the zero modes, especially in simplest case there is no zero modes. The crucial observation is their difference compares to “$1 - 2a$”. This difference is the continuous dependence of $A^{(\alpha)}$, and it can be written by a local expression. To see this explicitly, a rigorous derivation refers to [9], we will start with an observation from Mellin transformation

$$
\frac{1}{2} \Gamma \left( \frac{s + 1}{2} \right) t^{-(s+1)/2} = \int_0^\infty dy y^{s-1} e^{-ty^2}, \quad t > 0.
$$

(2.54)

9This phase term is induced by existing of both positive and negative eigenvalues.
Notice that if we change variable $t = x^2$, it becomes

$$x^{-s} = \frac{2}{\Gamma(\frac{s+1}{2})} \int_0^\infty dy y^s e^{-x^2 y^2}$$

$$\rightarrow |x|^{-s} \text{sign } x = \frac{2}{\Gamma(\frac{s+1}{2})} \int_0^\infty dy y^s e^{-x^2 y^2},$$

(2.55)

where the first line obviously detects the sign of $x$, and thus the second line is valid.

The eta invariant of operator $L_-$ with arbitrary gauge field $A$ can be written as

$$\eta(s, L_-) = \frac{2}{\Gamma(\frac{s+1}{2})} \int_0^\infty dyy^s \text{Tr}(L_- e^{-y^2 L_-^2}),$$

(2.56)

where the trace is over all eigenvalues of $L_-$. The eta invariant is well defined at $s = 0$ in our case, and one can write

$$\eta(L_-) = \frac{2}{\sqrt{\pi}} \lim_{s \to 0} \int_0^\infty dyy^s \text{Tr}(L_- e^{-y^2 L_-^2}).$$

(2.57)

Under an arbitrary variation of $A$, we have

$$\delta \eta(L_-) = \frac{2}{\sqrt{\pi}} \lim_{s \to 0} \int_0^\infty dyy^s \text{Tr}((\delta L_- - 2y^2 \delta L_- L_-) e^{-y^2 L_-^2})$$

$$= \frac{2}{\sqrt{\pi}} \lim_{y \to 0} \int_0^\infty dyy^s \frac{dy}{dy} (y \text{Tr}(\delta L_- e^{-y^2 L_-^2})).$$

(2.58)

Integrating by parts, setting $s \to 0$ and evaluating boundary term at $y = 0$, we have

$$\delta \eta(L_-) = \frac{2}{\sqrt{\pi}} \lim_{y \to 0} (y \text{Tr}(\delta L_- e^{-y^2 L_-^2})), $$

(2.59)

such limit might not be rigorous. In general case, one has to picking out the coefficient of $y^{-1}$ in the small $y$ expansion of $\text{Tr}(e^{-y^2 L_-^2})[9]$. 

To see how the variation works, we first separate the operator into components. For local coordinates $x^i$ on $M$, a differential operator $D_A$ coupled with gauge field $A$ acts on a $q$-form $\omega$ as

$$D_A \omega = d\omega + A \wedge \omega$$

$$= \partial_i \omega \wedge dx^i + A_i dx^i \wedge \omega$$

$$= \psi^i \partial_i \omega + \psi^i A_i \omega$$

$$= \psi^i D_i \omega$$

(2.60)

One can see the operator $\psi^i$ maps $a \to a \wedge dx^i$. Also there is a dual operation associated with it

$$\ast \psi^i = (-1)^F \chi^i, \quad \ast \chi^i = \psi^i (-1)^F,$$

(2.61)
and they obey anti-commutation relations
\[ \{ \psi^i, \psi^j \} = \{ \chi^i, \chi^j \} = 0, \quad \{ \psi^i, \chi^j \} = g^{ij}. \tag{2.62} \]

Now we can compute the local expression of self-dual operator \( L_- = (DA + D_A*)\epsilon, \epsilon = 1 \) when acts on 1- and 2-forms, \( \epsilon = -1 \) when acts on 0- and 3-forms. Since \( L_-^2 \) can be computed separately in three parts
\[
D_A^* \epsilon * D_A^* \epsilon = -\frac{1}{2} \psi^i \psi^j [D_i, D_j](-1)^F
\]
\[
*D_A \epsilon D_A^* \epsilon = -\frac{1}{2} \chi^i \chi^j [D_i, D_j](-1)^F
\]
we can write it as \( L_-^2 = -(X - \Delta) \) with
\[
X = \frac{1}{2} (\psi^i \psi^j [D_i, D_j](-1)^F + \chi^i \chi^j [D_i, D_j](-1)^F). \tag{2.64} \]

Now we can use expansion
\[
e^{-y^2 \Delta} e^{y^2 X} = e^{-y^2 \Delta} + \int_0^{y^2} ds e^{-\Delta} X e^{-(y^2 - s)\Delta} + \ldots, \tag{2.65} \]
and the Zassenhaus formula
\[
e^{y^2(-\Delta + X)} = e^{-y^2 \Delta} e^{y^2 X} e^{-\frac{y^4}{4}(\Delta, X)} e^{\frac{y^4}{4}(\Delta, X)} \ldots \tag{2.66} \]
to rewrite
\[
e^{y^2(-\Delta + X)} \to y^2 X e^{-y^2 \Delta}. \tag{2.67} \]

The fact that we can do this is because when taking the limit \( y \to 0 \), only the second term in expansion Eq.(2.65) survives. Moreover, we can see from the Zassenhaus formula that non-commutative part of \( X \) and \( \Delta \) have higher orders in \( y^2 \), thus the assumption that non-commutative part does not contribute ensure this rewrite is legitimate. The assumption will be assured by evaluating \( e^{-y^2 \Delta} \) with help of the heat kernel method. With the standard heat kernel expansion
\[
e^{-y^2 \Delta} \sim (4\pi)^{-3/2} \sum_s y^{s-3} b_s, \tag{2.68} \]
we see that the lowest power in \( y \) is \(-3\). Therefore any higher order in \( y^2 \) will be zero within the limit \( y \to 0 \). On the other hand, other terms not vanishing after the limit will be zero by taking trace in Eq.(2.59). Gather all the steps above, Eq.(2.59) can be written as
\[
\delta \eta = \frac{1}{4\pi^2} \text{Tr}(\delta L_\cdot X). \tag{2.69} \]

Since the Hodge star \( * \) maps odd forms to even forms and even forms to odd forms in 3-dimension and it does not change the eigenvalues, we can consider
2.5. The eta invariant

where acts on $\Omega^1(M, su(2)) \oplus \Omega^3(M, su(2))$ as $*D_A \epsilon$ acts on the total space $\Omega^\bullet(M, su(2))$. Thus

$$\delta \eta = \frac{1}{4\pi^2} \int_M \bar{\text{Tr}}(*D_A \epsilon \cdot X),$$

where the trace is over $\Omega^\bullet(M, su(2))$. Notice that every term in $X$ is quadratic in $\chi^i$ and $\psi^j$, and $\delta D_A = \psi^i \delta A_i$, hence if we denote $\chi^i$ and $\psi^j$ both as $\gamma^i$ we have

$$\bar{\text{Tr}} * \psi^i \epsilon \gamma^j \gamma^k = \epsilon^{ijk},$$

where $\epsilon^{ijk}$ is the Levi-Civita antisymmetric tensor. Since the space $\Omega^\bullet(M, su(2))$ is Lie algebra valued, we still have the trace over $su(2)$ in the adjoint representation

$$\delta \eta = \frac{1}{8\pi^2} \int_M \epsilon^{ijk} \text{Tr}_{su(2)}(4\delta A_i[D_j, D_k])$$

$$= -\frac{1}{\pi^2} \int_M \text{Tr}_{su(2)}(\delta A \wedge dA + \delta A \wedge A \wedge A).$$

Notice that the minus sign in second line is caused by the pairing eq.(2.18), thus one should bear in mind the eta invariant is respect to this pairing. We can take $A$ as our background field $A^{(\alpha)}$. After integration, the continuous dependence of $\eta(A^{(\alpha)})$ is given by

$$\tilde{\eta}(A^{(\alpha)}) = -\frac{1}{2\pi^2} \int_M \text{Tr}_{su(2)}(A^{(\alpha)} \wedge dA^{(\alpha)} + \frac{2}{3} A^{(\alpha)} \wedge A^{(\alpha)} \wedge A^{(\alpha)})$$

$$= -\frac{2c_2(SU(2))}{\pi} CS(A^{(\alpha)}),$$

where $c_2(G)$ is the value of the quadratic Casimir operator of $G$ in the adjoint representation $(c_2(SU(N)) = 2N$, see Eq.(3.63) and Eq.(3.69) for detail), $CS(A^{(\alpha)})$ is the CS invariant of flat connection $A^{(\alpha)}$.

In conclusion, $\eta(A^{(\alpha)})$ in the phase term of the one-loop correction can be written as

$$\eta(A^{(\alpha)}) = \eta(0) - \frac{2c_2(G)}{\pi} CS(A^{(\alpha)}) - 2SF_{A^{(\alpha)}}.$$  

(2.74)

Again, this is the formula for general cases. The problem of this formula is that only $\eta(0)$ is not topological. Now we can write the partition function at 1-loop correction of flat connection $A^{(\alpha)}$ as

$$Z_\alpha = \frac{1}{\text{Vol} H_\alpha} e^{i\pi \eta(0)/4 - i\pi SF_{A^{(\alpha)}}/2} \left( \frac{k}{4\pi^2} \right)^{-1/2 \dim H^0(M, D_A^{(\alpha)})}$$

$$e^{i(k + \frac{c_2(SU(2))}{2} CS(A^{(\alpha)})/T A^{(\alpha)}).}$$

(2.75)

Notice that the path integral Eq.(2.33) is over perturbation $B$ which transforms in adjoint representation under gauge symmetry. Thus the trace left with is in adjoint representation instead of the fundamental one. However they differ by a quadratic Casimir of the $SU(2)$ adjoint representation.
Chapter 2. Chern-Simons

For isolated irreducible flat connection $(\frac{k}{2\pi})^{-1/2 \dim H^0(M, D_{\alpha})) = 1$, Vol$H_n = 1$ and $SF_{\alpha} = 0$. To get rid of the metric dependence for $\eta(0)$, Witten in [42] added a gravitational CS functional (definition in Appendix D) to cancel it. This operation requires a trivialization of the tangent bundle of $M$. Different trivializations only differ in a number of “twists”, which is the framing dependence. Thus there is an overall phase factor

$$e^{\pi d(\eta_{grav}/4 - CS(\omega)/24\pi)} = e^{-\frac{\pi d}{8}(\mu)},$$

(2.76)

where $d = \dim SU(2)$, $d\eta_{grav} = \eta(0)$, and define $\sigma(\mu)$ as the overall factor depends on framing. The theory finally gives topologically invariant result with a specific framing choice. In addition, Atiyah showed that there always exists a canonical 2-framing $\mu$ such that $\sigma(\mu) = 0$ [2].

2.6 Metric dependence of $\eta(0)$

To understand the detail about the framing dependence of the CS theory, we first start with Atiyah-Patodi-Singer theorem and see why $\eta(0)$ is metric dependent. The original purpose of the theorem is to have a generalization of Hirzebruch’s signature theorem of manifolds with boundary. Let $W$ be a compact manifold with boundary $Y$ and $D : C^\infty(W, E) \to C^\infty(W, F)$ be a linear first order elliptic differential operator acting from vector bundle $E$ to vector bundle $F$ over $W$. Assuming that, in a neighbourhood $Y \times [0, 1]$ of the boundary, $D$ has the form $D = \partial_u + A$, where $u$ is the inward normal coordinate to the boundary and $A : E|Y \to F|Y$ is elliptic self-adjoint operator on $Y$. Then one can write a finite index of $D$ as

$$\text{index } D = \int_W \alpha_0 dx - \frac{h + \eta(A)}{2},$$

(2.77)

where $\alpha_0$ is some characteristic polynomial obtained from the constant term in the asymptotic expansion of the heat kernel $e^{-tD^*D} - e^{-tDD^*}$. In the case of operator $L$, recall $L_-$ is the restriction to odd forms, $\alpha_0$ is given by [3]

$$\alpha_0 = -\pi^{-2} \text{Tr}[F^2] + \frac{2}{3}p_1,$$

(2.78)

and we can consider it as twice the continuous part of $\eta(A^{(\alpha)})$. The second term comes from boundary condition. An interesting point is that this term only depends on $Y$ as one can glue $W'$ with opposite orientation to $W$ along $Y$. With suitable example of lens space, this indicates the boundary term must be a global invariant of metric on $Y$ [4]. Surprisingly $\eta$ invariant separates the spectrum of $A$ into positive and negative parts and such non-local character is exactly what we want for the boundary condition. This also explain the reason that $\eta(0)$ is metric
dependent. \( h \) in the second term is the multiplicity of zero eigenvalues of \( D \) on \( Y \),

\[
h = (\dim H^0(Y, D) + \dim H^1(Y, D) + \dim H^2(Y, D) + \dim H^3(Y, D)).
\]

Freed and Gompf used this theorem to calculate \( \eta(A^{(\alpha)}) \)[17]. By putting \( A^{(\alpha)} \) and trivial connection to the ends of \( Y \times [0, 1] \), separately, the APS theorem gives exactly same result as Eq.(2.74). The only thing one has to take care is that \( D \) is not restricted to odd forms, i.e \( \eta(A) = 2\eta(A^{(\alpha)}) \), when applying the APS theorem.

### 2.7 2-framing of 3-manifold

Given a manifold, we are wondering whether it is plausible to have a same basis at each point, that is whether one can find a global section to trivialize the tangent bundle. For this reason one constructs a frame bundle over the base manifold, and at each point the bundle consists all local bases. Suppose we have a section of the frame bundle, it gives a trivialization of the tangent bundle. To ensure the trivialization is globally defined, we have to first make sure it is extendible from 0-skeleton to 1-skeleton. This is indeed the definition of orientation, that is for an oriented 3-manifold \( Y \), the extension is trivial. An orientation is specified by its homotopy class. Moreover, analogous to orientation we define spin structure. In this scenario \( Y \) admits spin structure if the trivialization can be extended over the 2-skeleton. For oriented 3-manifold, one can always find such globally defined trivialization and the manifold is called parallelizable. However, the choice of trivialization of its tangent bundle is not unique. This arouses the definition of framing of 3-manifold, a choice of trivialization of the tangent bundle of \( Y \). Two different trivializations are related by the transition function, thus it can be characterized by the map \( Y \to SO(3) \). Since \( Y \) is oriented, the first Stiefel-Whitney class vanishes, \( \omega_1(Y) = 0 \). Since the tangent bundle of \( Y \) has structure group \( SO(3) \) and the Stiefel-Whitney classes \( \omega_1(Y) = 0 \), we can lift map \( Y \to SO(3) \) to \( Y \to SU(2) \) if and only if \( \omega_2(Y) = 0 \), i.e. \( Y \) admits a spin structure. Different framings of \( Y \) is thus characterized by homotopy classes of this map\(^{11}\). However, there is another more interesting integer we can define, called the 2-framing of \( Y \), which is introduced by Atiyah in [2].

Let \( T_Y \) denotes tangent bundle of \( Y \), we can define a direct sum \( 2T_Y = T_Y \oplus T_Y \). Consider a diagonal embedding map

\[
SO(3) \to SO(3) \times SO(3) \to SO(6),
\]

\(^{11}\)From standard obstruction theory when one wishes to expend a trivialization from 0-skeleton over 1-skeleton, the obstruction is coming from \( \omega_1 \in H^1(Y, \pi_0(SO(3))) \). To extend over 2-skeleton, the obstruction is coming from \( \omega_2 \in H^2(Y, \pi_1(SO(3))) \). Thus if both \( \omega_1 \) and \( \omega_2 \) are zero, one can change the \( SO(3) \) bundle to \( SU(2) \) bundle which is a double over of \( SO(3) \). It is readily seen that \( H^1(Y, \pi_0(SU(2))) \) and \( H^2(Y, \pi_1(SU(2))) \) vanish trivially as \( SU(2) \) is 2-connected, thus a globally defined trivialization of \( SU(2) \) bundle exists. Different trivialization is within homotopy class of map \( Y \to SU(2) \). Since we are within 3-dimension, this map is indeed the degree map that different homotopy classes are characterized by the number of times the map wraps over \( Y \).
we have $SO(6)$ as the structure group of $2T_Y$. By the sum-rule for Stiefel-Whitney classes

$$\omega_2(2T_Y) = 2\omega_0(T_Y) \cup \omega_2(T_Y) + \omega_1(T_Y) \cup \omega_1(T_Y) = 0,$$

thus we can lift the map $SO(6)$ to $Spin(6) = SU(4)$ as before. Define 2-framing of $Y$ as homotopy classes of trivializations of $2T_Y$. Two different 2-framings differ by a homotopy class of map $\theta_{\mu\nu} : Y \to Spin(6)$. This map can be deformed into $Spin(3) \subset Spin(6)$ as the successive quotients $SU(4)/SU(3)$, $SU(3)/SU(2)$ are all 3-connected\(^\text{12}\). Thus the 2-framing is also characterized by the degree map and is an integer. To compute this integer we first lift $Y$ to $W = Y \times I$. Put two different trivializations $\mu$ and $\nu$ at the two ends $Y \times \{0\}$ and $Y \times \{1\}$, respectively, check Fig.(2.1(a)). The bundle $2T_Y$ is not trivial on $Y \times I$, we can define the relative Pontrjagin class $\hat{p}_1(2T_W) \in H^4(Y \times I, Y \times \{0, 1\})$. (2.81)

The degree of $\theta_{\mu\nu}$ is given by

$$\deg(\theta_{\mu\nu}) = \frac{1}{2} \int_{Y \times I} \hat{p}_1(2T_W),$$

since $\hat{p}_1(2T_W)$ is even for spin bundles, i.e. $\hat{p}_1(2T_W) \equiv \omega^2 \mod 2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.1.png}
\caption{(a): 4-manifold $W$ constructed by gluing $M_1$ and $M_2$ along boundary $Y$. (b): $W$ can be also represented as product manifold $Y \times I$ near $Y$.}
\end{figure}

\(^{12}\)To see this, typically we start with the exact sequence

$$\pi_3(SU(3)) \to \pi_3(SU(4)) \to \pi_3(SU(4)/SU(3)) \to \pi_2(SU(4)) \to \pi_2(SU(4)/SU(3)) \to \pi_1(SU(3)) \to \pi_1(SU(4)/SU(3)) \to \pi_0(SU(3)) \to \pi_0(SU(4)/SU(3)) \to 0.$$ 

Then, for example, since $\pi_2(SU(4))$ is trivial the kernel of map $\pi_2(SU(4)/SU(3)) \to \pi_1(SU(3))$ is zero. Therefore $\pi_2(SU(4)/SU(3))$ is isomorphic to its image in $\pi_1(SU(3))$ which is equal to the kernel of map $\pi_1(SU(3)) \to \pi_0(SU(4))$. Accordingly, we have $\pi_2(SU(4)/SU(3)) = \pi_1(SU(3))/\pi_0(SU(4)) = 0$. Similarly, one can prove $SU(4)/SU(3)$, $SU(3)/SU(2)$ are all 3-connected.
2.7. 2-framing of 3-manifold

Naively, by using Stokes theorem
\[ \int_{Y \times I} \text{Tr} F^2 = \int_Y \text{Tr}(\text{Ad}A + \frac{2}{3}A^3), \]  
we can consider the degree of \( \theta_{\mu\nu} \) as the difference between non-trivial gauge transformations of the CS functional, i.e. the last term of Eq.(2.5). Indeed it is an integer. To be precise, we first cover \( W \) with coordinates patches \( U_i \), on which \( 2T_Y \) is trivialized. Locally, we can always define connection over \( U_i \), and glue them together with the partition of unity \( \{ \rho_i \} \) as
\[ A_i = \sum_j \rho_j t_{ji}^{-1} dt_{ji}, \]  
(2.84)
where \( t_{ij} \) is the transition function. If we write \( t_{ij} = t_{ik} t_{kj} \), the formula of gauge transformation is recovered
\[ A_i = \rho_2 t_{i1}^{-1} dt, \quad A_2 = \rho_1 t_{21}^{-1}, \quad \rho_1 + \rho_2 = 1. \]  
(2.86)
The curvature is given by
\[ F_1 = dA_1 + A_1 \wedge A_1 = d\rho_2 t_{i1}^{-1} dt + (\rho_2^2 - \rho_2)(t_{i1}^{-1} dt)^2. \]  
(2.87)
Now the degree of \( \theta_{\mu\nu} \) is calculated as
\[ \deg(\theta_{\mu\nu}) = \frac{1}{2} \int_{Y \times I} \hat{p}_1(2T_W) \]
\[ = \frac{1}{8\pi^2} \int_{Y \times I} 2d\rho_2(\rho_2^2 - \rho_2^3) \text{Tr}[(t_{i1}^{-1} dt)^3] \]
\[ = \frac{1}{24\pi^2} \int_Y \text{Tr}[(\theta_{\mu\nu}^{-1} d\theta_{\mu\nu})^3]. \]  
(2.88)
Now consider Fig.(2.1(b)), in this scenario \( M_1 \) is an oriented 4-manifold with boundary \( Y \). Given any 2-framing \( \mu \) of \( Y \) we can define
\[ \sigma(\mu) = \text{Sign}M_1 - \frac{1}{6} \int_{M_1} \hat{p}_1(2T_{M_1}, \mu), \]  
(2.89)
where \( \text{Sign}M_1 \) is the Hirzebruch signature. The definition here makes sense since we can consider \( \sigma(\mu) \) as the boundary condition to the generalized Hirzebruch signature.
of 4-manifold with boundary or simply consider it as the Hirzebruch defect. We have a same equation for \( M_2 \) as they share a same boundary. Since the signature and the relative Pontryagin class are additive, we can write the Hirzebruch signature of \( W \) as

\[
\text{Sign} \, W = \frac{1}{3} \int_W p_1(T_W) = \frac{1}{6} \int_W p_1(2T_W).
\]

This shows \( \sigma(\mu) \) is indeed only depend on the choice of 2-framing. Different 2-framings differ by an integer, hence it is plausible to define the canonical 2-framing \( \mu \) such that \( \sigma(\mu) = 0 \).

Consider using a Levi-Civita connection on \( 2T_{M_1} \) over all \( M_1 \) except at the final cylinder, which is the neighbourhood of boundary \( Y \). This scenario can be identified with a 4-manifold \( M_1 \) with boundary \( Y \), then the APS theorem indicates that

\[
\text{Sign} \, M_1 = \frac{1}{3} \int_{M_1} p_1(\omega) - \eta_{\text{grav}}(0) = -\frac{1}{6\pi} CS(\omega) - \eta_{\text{grav}}(0),
\]

where \( CS(\omega) \) is gravitational CS term\(^{13} \). This equation is similar to Eq.(2.89), here the eta invariant represents the boundary correction. On the other hand, the relative Pontryagin class on \( M_1 \) can be calculated as

\[
\int_{M_1} \hat{p}_1(2T_{M_1}, \mu) = 2 \int_{M_1} p_1(\omega) + \int_{Y \times I} \hat{p}_1(2T_{M_1}, \mu)
= 2 \int_{M_1} p_1(\omega) - \frac{1}{\pi} CS_\mu(\omega).
\]

Together with Eq.(2.89) and Eq.(2.91), we finally arrive

\[
CS_\mu(\omega) = 6\pi \eta_{\text{grav}}(0).
\]

We see that this formula ensures one to subtract the metric dependence of \( \eta(0) \) by adding a gravitational CS term. Moreover, we see that there is always a canonical 2-framing that the anomaly phase in Eq.(2.76) is equal to zero.

\(^{13}\text{In Appendix D, we give the original definition of gravitational CS term.}\)
Chapter 3

Non-perturbative comparison

We will compare the 1-loop correction with exact result in this section. The reason besides evaluating the validity of perturbative method are that this leads to the first example of AdS$_3$/CFT$_2$. The main tool to compute the CS partition function is the affine Lie algebra introduced by Kac[24]. We will first give a brief review of representation theory of Lie algebra and see how to calculate the character of it. Then extend to the case of affine Lie algebra and we will see its connection to conformal field theory.

3.1 Weyl character formula

Let $G$ be a Lie group and $t_a$ ($a = 0, \ldots, \dim G$) be its group generators. The commutation relations of $t_a$ define its Lie algebra $\mathfrak{g}$. We will focus on semi-simple Lie algebra which is a direct sum of simple Lie algebra$^1$. Most simple example of non-semisimple group is $U(n) = U(1) \times SU(n)$ since $U(1)$ is the non-trivial Abelian subalgebra, and we see that $SU(n)$ is semisimple.

Let $V$ be representations of $G$ on complex vector spaces, its characters are defined as functions

$$\chi_V : g \in G \rightarrow \text{Tr}(V(g)) \in \mathbb{C},$$

and they are invariant under conjugation

$$\chi(g'gg^{-1}) = \chi(g) \text{ } g' \in G.$$
Moreover, the character characterizes the associated representation up to isomorphism. Consider $G = U(1)$, the irreducible representations are one dimensional with integer label $n$. The character is given by

$$\chi(e^{i\theta}) = e^{i n \theta}.$$  \hspace{1cm} (3.3)

Let $H^a$ ($a = 0, \ldots, \text{rank } G$) be linearly independent generators of $\mathfrak{g}$ that commute within themselves

$$[H^a, H^b] = 0, \ a, b = 1, \ldots, \text{rank } G.$$  \hspace{1cm} (3.4)

$\{H^a\}$ forms a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, i.e. the Cartan subalgebra. The Lie group of $\mathfrak{h}$ is denoted as $T$, a maximal torus of $G$. We can consider the maximal torus as a finite product of $k$ copies of $U(1)$ and its irreducible representations are labelled by $k$ integers. The importance of maximal torus is that any element $g \in G$ can be conjugated into an element of $T^2$. Therefore a conjugation invariant function, e.g. $\chi_V$, is determined by its values on $T$. Since the characters of irreducible representations of $T$ are known, it is plausible to determine that of $G$.

To achieve this goal, we begin with denoting other generators of $\mathfrak{g}$ as $E^a$ and their commutation relations with $H^a$

$$[H^a, E^a] = \alpha^a E^a, \ a = 1, \ldots, \text{rank } G = r.$$  \hspace{1cm} (3.5)

The eigenvalues $\alpha^a$, the Dynkin labels, form a $r$-dimensional vector space and is called a root of $\mathfrak{g}$. The root $\alpha$ is called a positive root if its first component is positive. We can choose a basis for this $r$-dimensional vector space as $\{\alpha^a\}$ and each $\alpha^a$ is called a simple root. With this set up, we have the Cartan decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$  \hspace{1cm} (3.6)

For semi-simple Lie groups, their Killing forms are non-degenerate. Thus it can help us to define an inner product $\langle \cdot, \cdot \rangle$ on Cartan subalgebra $\mathfrak{h}$

$$K(H^a, H^b) = 2 \sum_c \frac{\langle \alpha^{(c)}, \alpha^{(a)} \rangle \langle \alpha^{(c)}, \alpha^{(b)} \rangle}{\langle \alpha^{(a)}, \alpha^{(a)} \rangle \langle \alpha^{(b)}, \alpha^{(b)} \rangle}. \hspace{1cm} (3.7)$$

In addition, we can define the $r \times r$ Cartan matrix and its entries are given by

$$A^{ab} = \frac{\langle \alpha^{(a)}, \alpha^{(b)} \rangle}{\langle \alpha^{(b)}, \alpha^{(b)} \rangle}. \hspace{1cm} (3.8)$$

If we define the dual root or coroot of $\alpha$ by

$$\alpha^\vee = \frac{2 \alpha}{\langle \alpha, \alpha \rangle}, \hspace{1cm} (3.9)$$

\footnote{Indeed, this is the main theorem of maximal torus. For detail proof, we refer interested reader to look into formal text books on Lie group representation theory, e.g. [10].}
we have
\[ A^{ab} = \langle \alpha^{(a)}, \alpha^{(b)\lor} \rangle. \] (3.10)

We can now switch the basis \( \{ \alpha^{(a)} \} \) to a basis constructed by simple coroots as
\[ \{ \alpha^{(a)\lor} | a = 1, \ldots, r \}. \] (3.11)

The fundamental weights \( \Lambda^{(a)} \) are defined to satisfy relation
\[ \langle \Lambda^{(a)}, \alpha^{(a)\lor} \rangle = \delta_a^b, \] (3.12)

and they form the basis of the weight space
\[ \{ \Lambda^{(a)} | a = 1, \ldots, r \}. \] (3.13)

From Eq.(3.12), the weight space is dual to root space, thus they can be considered as \( r \) 1-forms. It is called the Dynkin basis and the components of a weight in the Dynkin basis are called Dynkin labels. Therefore, for a given weight \( \lambda \), we can write it in two bases as
\[ \lambda = \lambda_a \alpha^{(a)\lor} = \lambda_a \Lambda^{(a)}. \] (3.14)

\( \lambda \) is the highest weight if \( \langle \lambda, \alpha^{(a)} \rangle \geq 0 \) for any simple root \( \alpha^{(a)} \). The Weyl vector is a highest weight constructed by half of the sum of all positive roots.

The inner product helps us to construct a map \( S_\alpha \) between all simple roots
\[ S_\alpha(\beta) = \beta - \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha, \] (3.15)

the reflection of \( \beta \) through the hyperplane which is perpendicular to \( \alpha \). If we consider such maps as permutation between simple roots, \( S_\alpha \) must form a permutation group. Indeed, let \( N(T) = \{ g \in G | gT = Tg \} \) be the normalizer of \( T \) and it is a normal subgroup of \( G \). The Weyl group of \( G \) is defined as a quotient group \( W(G, T) = N(T)/T \). \( S_\alpha \) generate the entire Weyl group. The fundamental Weyl chamber one can be considered as the equivalence class of roots under action of the Weyl group.

In short, we classify simple Lie algebra by the eigenvalues of Cartan subalgebra associated with it. All information is encoded in the Cartan matrix. In general it satisfies

1. \( A^{ii} = 2 \)
2. \( A^{ij} = 0 \) if and only if \( A^{ji} = 0 \),
3. \( A^{ij} \in \mathbb{Z} \leq 0 \) for \( i \neq j \),
4. \( \det A > 0 \),
5. \( A \) is indecomposable.

Without the last condition, we have the properties of Cartan matrix for semi-simple Lie algebra.
To be more concrete, we give an explicit example. Let the group be $SU(3)$ and pick a set of orthonormal basis $\{e_i | i = 1, 2, 3\}$ for the Euclidean space $\mathbb{R}^3$. The positive roots are

$$
\begin{align*}
\alpha_{(1,2)} &= e_1 - e_2 \\
\alpha_{(2,3)} &= e_2 - e_3 \\
\alpha_{(1,3)} &= e_1 - e_3.
\end{align*}
$$

The simple roots are linearly independent ones, let

$$
\begin{align*}
\alpha^{(1)} &= \alpha_{(1,2)} \\
\alpha^{(2)} &= \alpha_{(2,3)}
\end{align*}
$$

be the basis. It is readily see that the angle between them is $\angle(\alpha^{(1)}, \alpha^{(2)}) = \frac{2\pi}{3}$. More explicitly, Fig.(3.1) shows the fundamental chambers of $SU(3)$. Any elements can be arrived from elements inside the first chamber by the action of Weyl group or reflections of $\alpha_{(i,j)}$, Eq.(3.15).

![Diagram of Weyl group and Weyl fundamental chambers](image)

**Figure 3.1.** The Weyl group and Weyl fundamental chambers

The two generators $H^1, H^2$ of $\mathfrak{h}$ are diagonal $3 \times 3$ matrices

$$
H^1 = \begin{pmatrix}
   l_1 & 0 & 0 \\
   0 & l_2 & 0 \\
   0 & 0 & l_3
\end{pmatrix},
H^2 = \begin{pmatrix}
   l_1^2 & 0 & 0 \\
   0 & l_2^2 & 0 \\
   0 & 0 & l_3^2
\end{pmatrix}.
$$

Other generators $E^{\alpha_{(i,j)}}$ of $\mathfrak{h}$ are $3 \times 3$ matrices with all entries vanish except the $i,j$-th which equal to one. The commutation relations given by Eq.(3.5) are now
written as,

\[ [H^1, E^{\alpha_{(i,j)}}] = (l^1_i - l^1_j)E^{\alpha_{(i,j)}}, [H^2, E^{\alpha_{(i,j)}}] = (l^2_i - l^2_j)E^{\alpha_{(i,j)}}. \] (3.21)

Now the components of root \( \alpha_{(i,j)} \) in the basis of fundamental weights \( \Lambda_{(i)} \) are given by the eigenvalues, the Dynkin labels, of these commutation relations

\[ \alpha_{(i,j)} = (l^a_i - l^a_j)\Lambda_{(a)} = \alpha_{(i,j)}(H^a)\Lambda_{(a)}, \] (3.22)

where \( \alpha_{(i,j)}(H^a) \) denotes the inner product of \( \langle \alpha_{(i,j)}, H^a \rangle \) in the Euclidean basis \( e_i \).

For example,

\[ \alpha^{(1)} = 2\Lambda_{(1)} - \Lambda_{(2)}, \alpha^{(2)} = -\Lambda_{(1)} + 2\Lambda_{(2)}. \] (3.23)

Moreover, the Cartan matrix is computed as

\[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \] (3.24)

by Eq.(3.8).

One can see these relation explicitly in Fig.(3.2). The highest weights are living in the fundamental Weyl chamber one. We point out three of them, in \( \Lambda_{(i)} \) basis, and how they transform under the Weyl group. The red line represents the \((1,0)\) fundamental representation of \( SU(3) \) which is the defining representation on \( \mathbb{R}^3 \), the highest weight is \( \Lambda_{(1)} \). The blue line represents the \((0,1)\) fundamental representation of \( SU(3) \) which is the wedge product representation on \( \wedge^2(\mathbb{R}^3) \), the highest weight is \( \Lambda_{(2)} \). The green line represents the adjoint representation of \( SU(3) \), the highest weight is the Weyl vector

\[ \rho = \frac{1}{2}(\alpha_{(1,2)} + \alpha_{(2,3)} + \alpha_{(1,3)}). \] (3.25)
Chapter 3. Non-perturbative comparison

Positive roots:
\[ \alpha_{(1,2)} = e_1 - e_2 \]
\[ \alpha_{(2,3)} = e_2 - e_3 \]
\[ \alpha_{(1,3)} = e_1 - e_3 \]

Fundamental weights:
\[ \Lambda_1 = e_1 \]
\[ \Lambda_2 = e_1 + e_2 \]

The Weyl vector:
\[ \rho = \frac{1}{3}(\alpha_{(1,2)} + \alpha_{(2,3)} + \alpha_{(1,3)}) \]

Figure 3.2. Root and weight lattice of SU(3). The dash lines correspond to the different representations.

Let us back to the adjoint invariant function \( \chi(g) \) and label the character of irreducible representations with \( ij \). The orthogonality of characters is expressed by integral
\[
\int \chi_i(g)\chi_j(g)dg = \delta_{ij},
\]
where \( dg \) is the standard Haar measure normalized that the volume of \( G \) is one. Any representation \( V \) now can be decomposed into direct sum of irreducible representations as
\[
V = \bigoplus_i n_i V_i.
\]
The multiplicities \( n_i \) of \( i \)-th irreducible representation is obtained by formula
\[
n_i = \int \chi_i(g)\chi(g)dg.
\]
The characters form a ring structure \( R(G) \), a vector space with an inner product \( \langle \cdot, \cdot \rangle_G \) and \( \chi_i(g) \) form a orthonormal basis. Accordingly our goal is to rewrite an integral over \( R(G) \) to an integral over \( R(T) \).

---

\(^3\)A ring is an abelian group with addition as group action and a second binary operation, multiplication, which is distributive and associative.
3.1. Weyl character formula

The key is to notice that the map \( \tilde{\varphi} : G \times T \to G \) defined by action
\[
\tilde{\varphi}(g, t) = gtg^{-1}
\] (3.29)
induces another map \( \varphi : G/T \times T \to G \) with action
\[
\varphi(gT, t) = gtg^{-1}.
\] (3.30)

The volume form on \( G \) naturally induces a volume form on \( G/T \times T \). The change of coordinates is based on the following diagram

\[
\begin{array}{c}
G \ni \pi^{-1}(U) \xrightarrow{\varphi} U \times T \xrightarrow{pr_2} T \\
\downarrow \pi \quad \downarrow \pi \\
G/T \ni U \\
\end{array}
\]

where \( pr_i \) is projection of the \( i \)-th component of \( U \times T \) and \( \pi \) is the projection from \( G \to G/T \). Let \( \Phi \) be a non-zero positive function maps from \( G \to \mathbb{R} \). First notice that we have relation
\[
0 \neq \int_G \Phi dg = \int_{G/T} \left( \int_T \Phi(gt) dt \right) d(gT) = \int_U \left( \int_T \Phi(gt) dt \right) d(gT). \quad (3.31)
\]
On the other hand, \( \pi \) induces maps on the Lie algebra from \( g = \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{g}/\mathfrak{h} \). Thus one can write a volume form on \( G \) as \( \pi^* d(gT) \wedge d\tau \) with \( d\tau \big|_T = dt \). It lefts to determine the Jacobian, and with the above diagram we have
\[
\int_U \left( \int_T \Phi(gt) dt \right) d(gT) = \int_U \left( \int_T \Phi\varphi^{-1}(gT, t) pr_2^* dt \right) pr_1^* d(gT)
\]
\[
= \int_{U \times T} \Phi\varphi^{-1}(gT, t) pr_2^* d(gT) \wedge pr_1^* dT
\]
\[
= \int_{U \times T} \Phi\varphi^{-1}(\varphi^{-1})^*(\pi^* d(gT) \wedge d\tau)
\]
\[
= J \int_{U \times T} \Phi\varphi^{-1}(\varphi^{-1})^* dg
\]
\[
= J \int_G \Phi dg,
\]
where \( J \) is the Jacobian and notice that there is no sign change when switching \( pr_1^* d(gT) \) and \( pr_2^* dt \) since the quotient space \( G/T \) is even dimensional\(^4\) and \( pr_1^* d(gT) \).

\(^4\)This is in general true and we will not prove it. However, one can have a simple example with \( G = SU(2) \). In this case, locally we have \( S^3 = S^2 \times S^1 \), the Hopf fibration, and \( G/T = S^2 \).
is an even form. Eq.(3.31) indicates $J = 1$, and we can write the volume forms

$$dg = \pi^*d(gT) \wedge d\tau, \left. d\tau \right|_T = dt \quad (3.33)$$
on G and

$$\varpi = pr_1^*d(gT) \wedge pr_2^*dt \quad (3.34)$$
on $G/T \times T$.

Now back to the map $q$ defined by Eq.(3.30). Since the maximal torus is invariant under adjoint action of elements of the Weyl group $W(G,T)$, for each $t$ any $w \in W(G,T)$ ensures $q(gT,t) = gtg^{-1} = gw^{-1}g^{-1}$. Denote number of elements in $W(G,T)$ by $|W|$, then $q$ is a $|W|$ folds cover map. We can use this map pushes the volume form on $G$ back to $G/T \times T$, the coordinate transformation requires a Jacobian $\det q$ defined by

$$q^*dg = (\det q)\varpi. \quad (3.35)$$

Let $f_t : G/T \rightarrow \mathbb{R}$ be defined by $f_t(g) = f(gtg^{-1})$ where $f$ is a continuous conjugate invariant function on $G$. We can then write $f(gtg^{-1}) = f_t \circ \pi(g)$. It follows that

$$\int_G f(g) dg = |W|^{-1} \int_{G/T \times T} q^*(f dg)$$

$$= |W|^{-1} \int_{G/T \times T} f \circ q(gT,t)(\det q)pr_1^*d(gT) \wedge pr_2^*dt$$

$$= |W|^{-1} \int_T \int_{G/T} f \circ q(gT,t)(\det q)d(gT)dt$$

$$= |W|^{-1} \int_T \int_{G/T} (\det q) \pi^*(f_t(g))dgdt$$

$$= |W|^{-1} \int_T (\det q) \int_G f(gtg^{-1})dgdt$$

$$= \frac{|G|}{|W|} \int_T (\det q) f(t)dt. \quad (3.36)$$

The determinant $(\det q)$ can be determined as follow. First notice that the integral is left-invariant, we can free to shift the points before integral. Let $L_{(g,t)}$ left invariant action on $G \times T$, if $(x,y) \in G \times T$ define $\tilde{F}(G,T)$ as a map from

$$(x,y) \xrightarrow{L_{(g,t)}} (gx,ty) \xrightarrow{\tilde{q}} gxtyx^{-1}g^{-1} \xrightarrow{L_{st^{-1}s^{-1}}} gt^{-1}txx^{-1}g^{-1}.$$    

Notice that $\tilde{F}(e,e) = e$, $\tilde{F}(x,e) = gt^{-1}txx^{-1}g^{-1}$ and $\tilde{F}(e,y) = gyy^{-1}$. The differential of $\tilde{F}(G,T)$ evaluated at identity is a map from $g \oplus h \rightarrow g$. More explicitly,
let $X \in \mathfrak{g}$ and $x(s)$ be a curve in $G$ such that $x(0) = e$ and $x'(0) = X$. Let $Y \in \mathfrak{h}$ and $y(s)$ be a curve in $T$ such that $y(0) = e$ and $y'(0) = Y$. Denote $D_i F(e,e)$ as differentiation with respect to the $i$-th components at identity. Then we can write

\[
D\tilde{F}(e,e)(X,Y) = D_1\tilde{F}(e,e)X + D_2\tilde{F}(e,e)Y
= \frac{d}{ds}gt^{-1}x(s)tx^{-1}(s)g^{-1}|_{s=0} + \frac{d}{ds}gy(s)g^{-1}|_{s=0}
= Ad_g(t^{-1}Xt - X) + Ad_gY
= Ad_g(Ad_{t^{-1}} - I)X + Ad_gY.
\tag{3.37}
\]

Again $\tilde{F}(G,T)$ induces $F(G,T)$ with $\tilde{q}$ replaced by $q$. For $Z \in \mathfrak{g}/\mathfrak{h}$, we have

\[
DF(e,e)Z = Ad_g((Ad_{t^{-1}} - I)Z).
\tag{3.38}
\]

Since $F$ is same to $q$, the Jacobian $\det q$ is indeed the determinant of $DF(e,e)$

\[
\det q = \det(Ad_{t^{-1}} - I)_{\mathfrak{g}/\mathfrak{h}},
\tag{3.39}
\]

where $\det Ad_g = 1$ as $G$ is connected. Eq.(3.36) now reads, the Weyl integration formula

\[
\int_G f(g) dg = |W|^{-1} \int_T (\det(Ad_{t^{-1}} - I)_{\mathfrak{g}/\mathfrak{h}}) \int_G f(gtg^{-1}) dgd\tau.
\tag{3.40}
\]

Let us denote the Jacobian as

\[
J(t) = \det(Ad_{t^{-1}} - I)_{\mathfrak{g}/\mathfrak{h}}.
\tag{3.41}
\]

To write it in terms of root, let $t = e^H$ and $R$ be the set of all roots, we have

\[
J(e^H) = \prod_{\alpha \in R} (1 - e^{-\alpha(H)})
= \prod_{\alpha \in R^+} (1 - e^{\alpha(H)})(1 - e^{-\alpha(H)}).
\tag{3.42}
\]

If we define

\[
\varsigma(e^H) = e^{\rho(H)} \prod_{\alpha \in R^+} (1 - e^{-\alpha(H)}),
= \prod_{\alpha \in R^+} (e^{\frac{1}{2} \alpha(H)} - e^{-\frac{1}{2} \alpha(H)}).
\tag{3.43}
\]

the Jacobian is then written as

\[
J(t) = \bar{\varsigma}\varsigma.
\tag{3.44}
\]

The Weyl integration formula is crucial to find the character functions. The original idea of Weyl is to focus on the antisymmetric character functions in $R(T)$.
Chapter 3. Non-perturbative comparison

The antisymmetric property is respect to one simple Weyl reflection. These functions transform under Weyl group as

\[ f(wt) = \text{sgn}(w)f(t), \quad (3.45) \]

where \( w \in W(G,T) \) and the \( \text{sgn}(w) \) is determined by the lowest number, \( n_w \), of Weyl reflection needed for \( w \) that \( \text{sgn}(w) = (-1)^{n_w} \). For any dominant weight \( \lambda \), we define the antisymmetric character function by

\[ \mathcal{A}_\lambda(e^H) = \sum_{w \in W(G,T)} \text{sgn}(w)e^{w\lambda(H)}. \quad (3.46) \]

These functions form an orthogonal basis for all antisymmetric character functions since

\[ \langle \mathcal{A}_\omega, \mathcal{A}_\lambda \rangle_T = \int_T \mathcal{A}_\omega \mathcal{A}_\lambda dt = \pm |W| \delta_{\omega\lambda}, \mathcal{A}_\omega = \pm \mathcal{A}_\lambda. \quad (3.47) \]

Now the Weyl integration formula gives

\[ \langle \chi_V(g), \chi_V(g) \rangle_G = |W|^{-1} \int_T (\chi_V)(\chi_V) dt = 1 \quad (3.48) \]

which means \( \chi_V \) is antisymmetric. On the other hand, by direct computation it is easy to see that

\[ \varsigma(e^H) = e^{\rho(H)} \prod_{\alpha \in R^+} (1 - e^{-\alpha(H)}) = \sum_{w \in W(G,T)} \text{sgn}(w)e^{w\rho(H)}. \quad (3.49) \]

Therefore the character \( \chi_V \) is a ratio of antisymmetric function. The Weyl character formula for irreducible representation \( V_\lambda \) with highest weight \( \lambda \) is given by

\[ \chi_{V_\lambda}(\mu) = \frac{\sum_{w \in W(G,T)} \text{sgn}(w)e^{w(\lambda+\rho)(\mu)}}{\sum_{w \in W(G,T)} \text{sgn}(w)e^{w\rho(\mu)}}, \quad (3.50) \]

where \( \mu \in \mathfrak{h} \). One simple application of the Weyl character formula is to compute the dimension of irreducible representation with highest weight \( \lambda \). By evaluating Eq.(3.50) at identity, \( \mu = 0 \), with Eq.(3.43) and Eq.(3.49) one immediately arrives the Weyl dimension formula

\[ \dim V_\lambda = \frac{\prod_{\alpha \in R^+} \langle \alpha, \lambda + \rho \rangle}{\prod_{\alpha \in R^+} \langle \alpha, \rho \rangle}. \quad (3.51) \]

Dominant weights are living in the fundamental Weyl chamber one.
### 3.2 Affine Lie algebra

The significance of Weyl character formula is that affine Lie algebra character functions can be regarded as partition functions of physics models. To see this, we first need to learn several basic aspects of affine Lie algebra. It is a subclass of infinite dimensional Lie algebra, or the Kac-Moody algebra which is a generalization of finite Lie algebra to infinite dimension. The affine generalization is done by soften the positive definite restriction of Cartan matrix, the 4-th property of Eq.(3.17), to be

\[ \det A\{i\} > 0 \text{ for all } i = 0, \ldots, r, \]

(3.52)

where \(\{i\}\) means removing the \(i\)-th row and column of \(A\). It turns out that this algebra can be obtained by a non-trivial central extension\(^6\) of loop algebra which is a set of maps from \(S^1\) into some finite dimensional Lie algebra \(\mathfrak{g}\).

For a simple Lie algebra \(\mathfrak{g}\) with generators \(\{t^a\}\) and the unit circle \(S^1\) in the complex plane with coordinate \(z\). Let us define \(t^a_m = t^a \otimes z^m\) as basis for analytic maps from \(S^1 \to \mathfrak{g}\), and the new commutation relations are given by

\[ [t^a_m, t^b_n] = f^{ab}_c t^c_{m+n}, \]

(3.53)

where \(f^{ab}_c\) are structure constants of \(\mathfrak{g}\). Denote this loop algebra as \(\mathfrak{g}_{\text{loop}}\). There is a non-trivial central extension \(\tilde{\mathfrak{g}}\) of \(\mathfrak{g}_{\text{loop}}\) with commutation relations

\[ [t^a_m, t^b_n] = f^{ab}_c t^c_{m+n} + m \delta_{m+n,0} \kappa^{ab} K, \]

\[ [K, t^a_m] = 0, \]

(3.54)

where \(K\) is the additional generator of \(\tilde{\mathfrak{g}}\) and \(\kappa^{ab}\) is the Killing form of \(\mathfrak{g}\) which also called the Cartan metric. Notice that \(t^a_0\) generate the finite \(\mathfrak{g}\). Since the central element \(K\) commute with all the other generators, there exists degeneracy. By introducing one more generator \(D\) to lift the degeneracy with commutation relations

\[ [D, t^a_m] = mt^a_m, \]

\[ [D, K] = 0, \]

(3.55)

we arrive the affine Lie algebra \(\hat{\mathfrak{g}}\). The eigenvalues of \(D\) defines gradation of the algebra and the eigenvalues of \(K\) is the level of the algebra. One way to consider this is that the root lattice of affine Lie algebra contains infinite copies of root lattice of \(\mathfrak{g}\) with a ladder-like structure such that \(t^a_{m \neq 0}\) shifts one copy to the other. The scenario is same as a torus, and one would expect the module invariance to occur.

---

\(^6\)A central extension of group \(C\) is an exact sequence

\[ 1 \to A \to B \to C \to 1 \]

such that \(A\) is a normal subgroup of \(B\) which lies in the center of \(B\).
The Cartan subalgebra of affine Lie algebra has two additional generators compared to associated Lie algebra, thus we can write it as $\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{d}$ with $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{d}$ generated by $H^a$, $K$ and $D$, respectively. The roots are written as $(\alpha, k, d)$ with $\alpha$ being the root of $g$, $k$ and $d$ are eigenvalues of $K$ and $D$. Define two important roots

$$\Lambda_0 = (0, 1, 0), \ \iota = (0, 0, 1).$$  \hspace{1cm} (3.56)$$

The inner product defined by Killing form, Eq.(3.7), can be extended to $(\cdot, \cdot)$ on $\mathfrak{h}^*$ such that

$$\left( \sum_{i=1}^{r} \alpha^{(i)} k^i + d \Lambda_0 \right) = 0, \ (\iota, \iota) = (\Lambda_0, \Lambda_0) = 0, \ (\iota, \Lambda_0) = 1. \hspace{1cm} (3.57)$$

and

$$(\hat{\alpha}, \hat{\beta}) = (\langle \alpha, k, d \rangle, \langle \beta, k', d' \rangle) = \langle \alpha, \beta \rangle + kd' + dk'. \hspace{1cm} (3.58)$$

The roots with zero norm, $(\hat{\beta}, \hat{\beta}) = 0$, are called imaginary roots and are some integer times $\iota$.

The simple roots are

$$\alpha^i = (\alpha^{(i)}, 0, 0), \ \text{for } i = 1, \ldots, r, \hspace{1cm} (3.59)$$

and

$$\alpha^0 = \iota - \theta = (-\theta, 0, 1), \hspace{1cm} (3.60)$$

where $\theta$ is the highest root\footnote{The highest root $\theta$ is a unique root such that $\theta - \alpha$ is positive for any $\alpha \in R \setminus \theta$. Within our normalization, its norm $\theta^2 = 2$.} of $g$. The fundamental weights satisfy the condition Eq.(3.12) and thus we have

$$\Lambda_i = (\Lambda^{(i)}, m_i, 0), \ \text{for } i = 1, \ldots, r, \hspace{1cm} (3.61)$$

and

$$\Lambda_0 = (0, m_0, 0), \ m_0 = 1, \hspace{1cm} (3.62)$$

where $m_i$ is components of $\theta$ in the coroot space. The sum of them is the dual Coxeter number denoted as $c_v$ which is one-half of the quadratic Casimir in the adjoint representation\footnote{Give the Cartan metric $\kappa_{ab}$, the quadratic Casimir is given by

$$c_2 = \kappa_{ab} t^a t^b = \sum_{a,b} A_{ab} H^a H^b + \sum_{\alpha \in \mathbb{R}^+} (E^{-\alpha} E^{\alpha} + E^{-\alpha} E^\alpha).$$

(3.63)
The Weyl vector is half of sum of all positive roots, for infinite \(\hat{g}\) it is not well defined. We have to change the definition with a certain regularization of the infinite sum. By requiring

\[
(\hat{\rho}, \alpha^i) = \frac{1}{2} (\alpha^i, \alpha^i),
\]

we have the affine Weyl vector \(\hat{\rho} = (\rho, c_v, 0)\).

The generalization of Weyl character formula from finite algebra to infinite algebra was done by Kac\[24, 25\]. With appropriate choice of weights \(\hat{\mu}\), we have the well defined Weyl-Kac character formula, for irreducible representation \(\hat{V}_{\lambda}\) with highest weight \(\lambda\),

\[
\chi_{\hat{V}_{\lambda}}(\hat{\mu}) = \sum_{\hat{w} \in \hat{W}} \text{sgn}(\hat{w}) e^{\hat{w}(\hat{\lambda}+\hat{\rho})(\hat{\mu})} / \sum_{\hat{w} \in \hat{W}} \text{sgn}(\hat{w}) e^{\hat{w}\hat{\rho}(\hat{\mu})}, \tag{3.66}
\]

Similar to Eq.(3.49), the Weyl denominator identity, we have

\[
\sum_{\hat{w} \in \hat{W}} \text{sgn}(\hat{w}) e^{\hat{w}\hat{\rho}(\hat{\mu})} = e^{\hat{\rho}(\hat{\mu})} \prod_{\hat{\alpha} \in \hat{R}^+} (1 - e^{-\hat{\alpha}(\hat{\mu})})^{\text{mult}(\hat{\alpha})}, \tag{3.67}
\]

where mult(\(\hat{\alpha}\)) appears to count degeneracies of positive roots, and apparently it is one for real roots and \(r\) for imaginary roots. Then we can write the formula as

\[
\chi_{\hat{V}_{\lambda}}(\hat{\mu}) = \sum_{\hat{w} \in \hat{W}} \text{sgn}(\hat{w}) e^{\hat{w}(\hat{\lambda}+\hat{\rho})-\hat{\rho}(\hat{\mu})} / \prod_{\hat{\alpha} \in \hat{R}^+} (1 - e^{-\hat{\alpha}(\hat{\mu})})^{\text{mult}(\hat{\alpha})}. \tag{3.68}
\]

The importance of this formula is its underlining modular invariance which is generic for partition functions of string theory. To see this we first look at the affine Weyl group \(\hat{W}\). Given a root \(\hat{\alpha} = (\alpha, 0, 1)\), the action of Weyl reflection \(\hat{S}_{\hat{\alpha}}\) on \(\hat{\lambda} = (\lambda, k, d)\), check Eq.(3.15 for finite case, is given by

\[
\hat{S}_{\hat{\alpha}}(\hat{\lambda}) = \hat{\lambda} - 2 (\hat{\alpha}, \hat{\lambda}) \hat{\alpha} = (S_{\alpha}(\lambda + 2k\alpha^\vee), k, d + 1/2k[\lambda^2 - (\lambda + 2k\alpha^\vee)^2]). \tag{3.69}
\]

In terms of highest root \(\theta\)

\[
c_2(\theta) = \sum_{a,b} A_{ab}(\theta, (\alpha^a)\vee)(\theta, (\alpha^b)\vee) + \sum_{\alpha \in \hat{R}^+} (\theta, \alpha) \\
= (\theta + 2\rho, \theta) \\
= \theta^2 + 2\rho \cdot \sum_{i} m_i \alpha^{(i)\vee} \\
= 2c_v. \tag{3.64}
\]
We see that the affine Weyl group contains two parts, the finite Weyl group $W$ and a translation $t_{\alpha^\vee}$ defined by
\[
t_{\alpha^\vee}(\hat{\lambda}) = (\lambda + k\alpha^\vee, k, d + \frac{1}{2k}[\lambda^2 - (\lambda + k\alpha^\vee)^2]).
\] (3.70)

One can readily check that $t_{\alpha^\vee}t_{\beta^\vee} = t_{\alpha^\vee+\beta^\vee}$ and $t_{\hat{\alpha^\vee}} = \hat{\alpha}t_{\alpha^\vee}\hat{\alpha}^{-1}$. Moreover, all possible translations are associated with simple root $\hat{\alpha}_0 = (-\theta, 0, 1)$ and its reflections under $W$. Therefore we see that the translation subgroup corresponds to a lattice generated by $\hat{\alpha}^{\theta}$, denote the lattice by $M^{10}$. Let $\mathfrak{T}$ denotes the translation group. It is a normal subgroup of $W$ from $t_{\hat{\alpha^\vee}} = \hat{\alpha}t_{\alpha^\vee}\hat{\alpha}^{-1}$, and any element is composed as $\hat{\alpha} = \alpha t_{\alpha^\vee}$. In addition, $W \cap \mathfrak{T} = 1$, therefore $\hat{W}$ is a semidirect product of the finite Weyl group and the translation group
\[
\hat{W} = W \ltimes \mathfrak{T}.
\] (3.71)

Same to the finite case, the character of affine Lie algebra is a ratio of antisymmetric functions, we choose them as the theta functions. Given $\hat{\lambda} = (\lambda, k, d)$, define the theta function as
\[
\Theta_{\hat{\lambda}} = e^{-\hat{\lambda}^2i/2k} \sum_{t \in \mathfrak{T}} e^{t(\hat{\lambda})}.
\] (3.72)

Notice that with Eq.(3.70), if we write $k\beta = \lambda + k\alpha^\vee$ the summation over $\mathfrak{T}$ in above equation can be replaced by a summation over $\beta \in M + k^{-1}\lambda$ as
\[
\Theta_{\hat{\lambda}} = e^{k\Lambda_0} \sum_{\beta \in M + k^{-1}\lambda} e^{-k\beta^2i/2 + k\beta}.
\] (3.73)

We can write this equation more explicitly with appropriately chosen basis. Let $\nu_1, \ldots, \nu_r$ be a orthonormal basis for the root space of $\hat{g}$, the additional “dimensions” for $\hat{g}$ are based by $\iota$ and $\Lambda_0$. Any vector in the root space can be written as
\[
v = -2\pi i\left(\sum_{s=1}^{r} z_s \nu_s + \tau \Lambda_0 + u\iota\right),
\] (3.74)

where $z_s$, $\tau$ and $u$ are arbitrary complex numbers. We have a classical theta function at level $k$ being explicitly written as
\[
\Theta_{\hat{\lambda}}(z, \tau, u) = e^{-2\pi iku} \sum_{\beta \in M + k^{-1}\lambda} e^{\pi k\tau \beta^2 - 2\pi ik(z_1\beta_1 + \cdots + z_r\beta_r)}.
\] (3.75)

The affine Weyl-Kac character formula Eq.(3.66) can be expressed in terms of the theta functions. The summation over the affine Weyl group is replaced by a

\footnote{In the case of $SU(N)$, $M$ is simply the root lattice.}
sum over the finite Weyl group and the translation group. For the denominator we have
\[ \sum_{\hat{w} \in \hat{W}} \text{sgn}(\hat{w}) e^{\hat{w}\hat{\rho} - \hat{\rho}} = e^{-\hat{\rho}} \sum_{\hat{w} \in \hat{W}} \text{sgn}(\hat{w}) e^{\hat{w}\lambda(\hat{\rho})} \]
\[ = e^{-\hat{\rho} + \hat{\rho}^2/2c_v} \sum_{w \in W} \text{sgn}(w) \Theta_{w\hat{\rho}}. \]  
(3.76)

On the other hand, we have the numerator
\[ \sum_{\hat{w} \in W} \text{sgn}(\hat{w}) e^{\hat{w}(\hat{\lambda} + \hat{\rho}) - \hat{\rho}} = \exp(-\hat{\rho} + (\hat{\lambda} + \hat{\rho})^2/2(k + c_v)) \sum_{w \in W} \text{sgn}(w) \Theta_{w(\hat{\lambda} + \hat{\rho})}, \]  
(3.77)

where \( \lambda \) associated to \( \hat{\lambda} \) is highest weight. Therefore the Weyl-Kac character formula is of the form
\[ e^{-s_{\hat{\lambda}\hat{\chi}}} = \frac{\sum_{w \in W} \text{sgn}(w) \Theta_{w(\hat{\lambda} + \hat{\rho})}}{\sum_{w \in W} \text{sgn}(w) \Theta_{w\hat{\rho}}}, \]  
(3.78)

with \( s_{\hat{\lambda}} \) written as
\[ s_{\hat{\lambda}} = (\hat{\lambda} + \hat{\rho})^2/2(k + c_v) - \hat{\rho}^2/2c_v. \]  
(3.79)

The above formula is of significance in string theory as it can be used to construct the partition function explained by Gepner and Witten[19]. The reason at early times using affine Lie algebra as nonperturbative tool was mainly about the modular invariance of theta functions, a generic symmetry of string models. As a modular form, the theta function appears in almost every branch of mathematics, thus its appearance is not quite a surprise. However, it might be crucial to unveil the mysteriousness and profoundness in connection between physics and number theory.

Even through its importance, we will give a literal sketch how such symmetry arises.

The story from number theory side begins in finding \( p(n) \), the number of ways to write integer \( n \) as sum of positive integers, also called the partition function of \( n \). Euler found that one can write the generating function as
\[ \sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}. \]  
(3.80)

The left hand side is the inverse of Euler function defined as \( \phi(q) = \prod_{k=1}^{\infty} 1 - q^k \).

Euler also proved that
\[ \phi(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}. \]  
(3.81)
Based on this identity, Jacobi later proved the triple identity

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2\pi i z})(1 + q^{2n-1}e^{-2\pi i z}) = \sum_{n=-\infty}^{\infty} q^n e^{2\pi i z n}. \quad (3.82)$$

Now write \( q = e^{\pi i \tau} \), then we arrive at the Jacobi theta function

$$\vartheta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in^2 \tau} e^{2\pi inz}, \quad (3.83)$$

where the series converges for all \( z \in \mathbb{C} \), and \( \tau \) in the upper half-plane. In spite of the difference between above definition of theta function and Eq. (3.75), one can already expect the modular invariance by applying the Poisson summation formula to both case. For example,

$$\vartheta_3(0|\tau) = \sum_{n=0}^{\infty} e^{\pi in^2 \tau}$$

$$= \sum_{m=0} \int e^{\pi in^2 \tau} e^{-2\pi i mn} dn$$

$$= \sum_{m=0} \int e^{-\pi in^2/m \tau} e^{\pi i \tau (n-\frac{m}{\tau})^2} dn$$

$$= \sqrt{\frac{i}{\tau}} \vartheta_3(0|\tau) \vartheta_3(0|\tau) = \sqrt{\frac{i}{\tau}} \vartheta_3(0|\tau) - 1/\tau,$$

where the second line is the Poisson summation formula. The modular invariance thus signals the duality in terms of Fourier transformation of Gaussian(\( e^{-x^2} \)). To relate the different summation, we first notice that

$$\vartheta_3^2(0|\tau) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{\pi in_1^2 \tau} e^{\pi in_2^2 \tau}$$

$$= \sum_{(n_1,n_2) \in \mathbb{Z} \times \mathbb{Z}} e^{\pi i \tau (n_1^2+n_2^2)}$$

$$= \sum_{m=0}^\infty \sum_{n_2=0}^\infty r_2(m) e^{\pi im\tau}, \quad (3.85)$$

where \( r_2(m) \) is the number of ways writing \( m \) as sum of squares. Together with \( p(n) \) these two factors make great importance in number theory. Notice that the summation in second line is over a \( \mathbb{Z} \times \mathbb{Z} \) lattice which is similar to \( M + k^{-1}l \) in Eq.(3.75). More details about theta function and modular invariance related to affine Lie algebra can be found in [25].
Let us go back to details on the modular properties of Eq.(3.75). Write $T$ and $S$ as generators of modular group $PSL(2, \mathbb{Z})$, they are $2 \times 2$ matrices
\begin{align*}
T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\
S &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
(3.86)

Denote the action of $X \in PSL(2, \mathbb{Z})$ on the theta function, Eq.(3.75), as $\hat{\Theta}_{\lambda} |_{X}$, which is defined by its action on the affine root space. Let us write $X$ as
\begin{equation}
\begin{pmatrix} a & b \\ c & d \end{pmatrix},
\end{equation}
(3.87)
we define
\begin{equation}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z + \tau \Lambda_0 + u \iota \\ c \tau + d \end{pmatrix} = \frac{1}{c \tau + d} z^2 + \frac{a \tau + b}{c \tau + d} \Lambda_0 + \left( u + \frac{cz^2}{2(c \tau + d)} \right) \iota.
\end{equation}
(3.88)

We see that the modular group element $X$ acting on parameter $\tau$ obeys
\begin{equation}
\tau \rightarrow \frac{a \tau + b}{c \tau + d},
\end{equation}
(3.89)
and as usual the two generators acting on it as
\begin{align*}
T : \tau & \rightarrow \tau + 1 \\
S : \tau & \rightarrow -\frac{1}{\tau}.
\end{align*}

As mentioned before, the modular invariance of theta function can be showed by applying the Poisson summation formula which involves the dual space. Let $M^*$ be the dual space of $M$, since in our case, $SU(N)$, $M$ is the root lattice, we identify $M^*$ as the weight lattice. Let $P_k$ denotes the weights which can be obtained by affine Weyl group acting on dominant weights at level $k$, i.e. $\hat{\lambda} = \hat{w}(\Lambda)$ and $(\Lambda, \iota) = k$. Let $P_k^+$ be the set of dominant weights. We can write down explicitly the transformation of $\hat{\Theta}_{\lambda}$ under $S$ as
\begin{equation}
\hat{\Theta}_{\lambda} |_{S} = (-i\tau)^{\tau/2} |M^*/kM|^{-1/2} \sum_{\hat{\mu} \in P_{k \text{mod } kM}} e^{-2\pi i / k(\lambda, \mu)} \Theta_{\hat{\mu}},
\end{equation}
(3.90)
where $|M|$ denotes the volume of the basic cell of the lattice. Again the proof of this is done by Poisson summation formula.

Since the character of affine Lie algebra is quotient of above theta function, it possesses this symmetry as well. Without proving we will just state some results, details can be found in [25]. The modular transformation of them are given by
\begin{align*}
\chi_{\lambda}(z, \tau + 1, u) &= \sum_{\hat{\mu} \in P_k^+} T_{\lambda \hat{\mu}} \chi_{\hat{\mu}}(z, \tau, u) \\
\chi_{\lambda}(z/\tau, -1/\tau, u + z^2/2\tau) &= \sum_{\hat{\mu} \in P_k^+} S_{\lambda \hat{\mu}} \chi_{\hat{\mu}}(z, \tau, u).
\end{align*}
(3.91)
We see that the characters of different representations at level $k$ transform into each other under the modular group. We can write the explicit form of $S_{\lambda \mu}$ as

$$S_{\lambda \mu} = i^{[\hat{R}^+]} |M/M^*|^{-1/2}(k + c_v)^{-r/2} \sum_{w \in W} \text{sgn}(w)e^{-2\pi i/(k+c_v)w(\lambda+\rho)(\mu+\rho)}, \quad (3.93)$$

where $[\hat{R}^+]$ is the number of positive roots. Notice that $|M/M^*| = \det A^{ij}$, the determinant of Cartan matrix. The matrix $S$ is unitary and symmetric such that $S^{-1} = S^\dagger = S^\ast$.

The non-perturbative method needs to use this matrix of $SU(2)$. For this case, $[\hat{R}^+] = 1$, $|M/M^*| = 2$, $c_v = 2$ and the weight lattice is 1-dimensional with $\Lambda_1^2 = 1^2$. we have the $S$ matrix written as

$$S_{\lambda \mu} = (\frac{2}{k+2})^{1/2} \sin\left(\frac{2\pi(n+1)(m+1)}{k+2}\right) \quad (\text{3.94})$$

where $n$ and $m$ are integers. This will be the essential gradient of non-perturbative method.

### 3.3 Canonical quantization of Chern-Simons theory

Our goal is to illustrate how does Eq.(3.94) help us to compute the CS partition function on $S^3$ non-perturbatively. Let start with closed 3-manifold $W$, and split it into two manifolds $M_1$ and $M_2$ with common boundary $Y$, e.g. Fig.(2.1(a)) for the case of 4-dimensional. Such splitting is called the Heegaard split. After splitting $W$, we can act diffeomorphisms on the boundary before gluing them back. With this method we can construct any 3-dimensional differentiable manifold.

As example, let us consider the case of $W = S^3$. It might be a little hard to imagine $S^3$ from gluing of two solid toruses. But if denote them as $T^2_{ab}$ and $T^2_{ba}$ then glue the non-contractible cycles to contractible cycles of the other, one readily see the resulting manifold is simply connected. This is also called the Hopf fibration, which locally is a product map $S^2 \times S^1$. Notice that the two toruses are related by the S-duality. If we act $S$ on one torus before gluing them back, we will have the trivial product manifold $S^2 \times S^1$. Here we already see the partition functions of $S^3$ and $S^2 \times S^1$ are related by $S$-duality.

To quantize CS theory on $W$, we first Heegaard split it and near the boundary $\Sigma$ we have product manifold $\Sigma \times R$ with $R$ serves as the time. Canonical quantization

---

\footnote{In the case of $SU(2)$, the fundamental basis consists one vector $e_1 = (1, -1)$. The weights in this basis are $\frac{2}{3}e_1$ with $n \in \mathbb{Z}$. The Weyl vector is $\rho = \frac{1}{3}e_1$}
then gives the Hilbert space $\mathcal{H}_\Sigma$. Let us denote the components of gauge field as $A_0$ in the time direction and $A_i$ in $\Sigma$. The CS action Eq.(2.4) is then written as

$$
CS(A) = \frac{k}{4\pi} \int dt \int_\Sigma d\mu_\Sigma \text{Tr}[e^{ij}(A_i \partial_0 A_j + A_0 F_{ij})],
$$

(3.94)

where $\mu_\Sigma$ is measure on $\Sigma$. We see that by integrating over $A_0$, we get constraint on the gauge field $F_{ij} = 0$. To quantize the system, we write the canonical commutation relations of the gauge fields

$$
[A_i^a(x), A_j^b(y)] = \frac{2\pi i}{k} \varepsilon_{ij} \delta^{ab} (x - y),
$$

(3.95)

where $x$ and $y$ are points on $\Sigma$. There are two ways to quantize the system. One can first impose the constraint. Resulting a space of flat vector bundles over $\Sigma$. Then do symplectic reduction and quantize with the reduced phase space. Or one can first impose the commutation relations and then add $F_{ij}$ as constraints on wave functions. One should get same answers from these two procedures. We will not get into any detail about this, interested readers may find everything in [14, 33, 29].

### 3.4 Calculation with simple surgery

If we let $\Sigma = S^2$, the only flat connection is the trivial one. Therefore the Hilbert space $\mathcal{H}_{S^2}$ is 1-dimensional. In this case, $W$ is obtained by gluing $M_1$ and $M_2$ along common boundary $S^2$. The partition function on $W$ is given by

$$
Z_W = \langle \psi_{M_1} | \psi_{M_2} \rangle, \quad \psi_{M_i} \in \mathcal{H}_{S^2}.
$$

(3.96)

On the other hand, $M_1$ and $M_2$ are obtained by removing a solid ball $B^3$ from $M'$ and $M''$, and two $B^3$ can be glued to be a $S^3$. Therefore we can write

$$
Z_W = \frac{\langle \psi_{M_1} | \psi_{B^3} \rangle \langle \psi_{B^3} | \psi_{M_2} \rangle}{\langle \psi_{B^3} | \psi_{B^3} \rangle} = \frac{Z_{M'} Z_{M''}}{Z_{S^3}},
$$

(3.97)

which is simply 1-dimensional linear algebra. If we embed unknots $C_i$, Wilson loops, in $S^3$, we can write

$$
\frac{Z_{S^3}(C_1, \ldots, C_s)}{Z_{S^3}} = \prod_i \frac{Z_{S^3}(C_i)}{Z_{S^3}}.
$$

(3.98)

When we have knots or links embedded in $S^3$, the Hilbert spaces are not simply 1-dimensional any more. For example, on the right side of Fig.(3.3) we have a Wilson line encircled by a Wilson loop. The quantization respects to $S^2 \times R$ with 3 points on different representations inserted on $S^2$ will give a 1-dimensional Hilbert space. If a Hopf link is embedded, then there will be 4 points with different representations inserted on $S^2$ and the resulting Hilbert space will be 2-dimensional.
Chapter 3. Non-perturbative comparison

The later one can be easily understood by regarding the Hopf link on a torus, non-trivial flat connections then respect to the generator cycles. The former one is not that obvious, however both cases and even more general ones are described by the Verlinder’s formula\[^{[36]}\]. The details of the two examples essentially signal the origin of AdS\(_3\)/CFT\(_2\) correspondence. One can find a brief explanation of relation between CS and CFT in Appendix G. Without any more elaborations, we continue with Fig.(3.3) to fully illustrate the use of Eq.(3.94) on computing CS over S\(^3\).

\[ \alpha \]
\[ \beta \]
\[ \psi_\alpha \]
\[ \psi \]

**Figure 3.3.** The wave function of S\(^3\) with Wilson line embedded is proportional to one of which a additional Wilson loop encircles, since the Hilbert spaces both are 1-dimensional.

In the two sides of Fig.(3.3), the resulting Hilbert space will be of dimension 1, thus their wave functions are proportional to each others

\[ \psi = \lambda^3_\alpha \psi_\alpha, \]  \hspace{1cm} (3.99)

where \( \lambda^3_\alpha \) is a constant depending on the representations of Wilson line and Wilson loop embedded. After gluing them along a S\(^2\), we will have a Hopf link embedded in S\(^3\), write the partition function as

\[ Z_{S^3}(L_{\alpha,\beta}) = \langle \psi_\alpha | \psi \rangle = \langle \psi_\alpha | \lambda^3_\beta | \psi_\alpha \rangle. \]  \hspace{1cm} (3.100)

On the other hand, as we mentioned before S\(^3\) can be obtained by gluing two solid torus which are S-dual to each other. Therefore we start with two solid toruses with a Wilson loop embedded, and glue them by S, the resulting S\(^3\) will content a Hopf link. The partition function then can be given as

\[ Z_{S^3}(L_{\alpha,\beta}) = \langle \phi_\alpha | S | \phi_\beta \rangle = S_{\alpha,\beta}, \]  \hspace{1cm} (3.101)
3.5 Exact result from the fusion rule

In CFT, the wave functions are conformal blocks of the theory which obey OPE, thus consider the expansion only involves primary fields $\phi_\gamma$ as

$$\phi_\alpha \otimes \phi_\beta = \sum_\gamma N^\gamma_{\alpha\beta} \phi_\gamma.$$ (3.103)

One can regard this as putting two Wilson loops with different representations into a solid torus and by pushing them close to each other we have a Wilson loop with a tensor product representation, thus $N^\gamma_{\alpha\beta}$ are the tensor product coefficients. Fig.(3.4) shows this operation. First two Wilson lines are embedded in two solid toruses, with $S$ operation we have a Hopf link in $S^3$. Then by Heegaard splitting we have the two $S^3$ in Fig.(3.3). Now we have arrived the third step in Fig.(3.4). After an inverse $S$ operation on two $S^3$ with one Wilson line embedded we have two solid toruses where only one has a Wilson loop embedded. We see that the two loop fuse to be one.

If the Wilson loops embedded are in trivial representations, then Eq.(3.101) will give the partition function of $S^3$

$$Z_{S^3}(L_0, 0) = Z_{S^3} = S_{00}. \quad (3.104)$$

Together with Eq.(3.94) we have exact result of CS on $S^3$ with $SU(2)$ as gauge group, then we can check its asymptotic behaviour at large $k$. Compare the one loop

---

\footnote{A solid torus is a product manifold $D \times S^1$. Therefore with a Wilson loop of representation $V_\alpha$ embedded, $D$ is with a point inserted which one can simply regard as a source.}
computation result Eq.(3.16). The partition function reads

\[ Z(k) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \sim \sqrt{2\pi} k^{-3/2} \quad \text{for} \quad k \to \infty. \quad (3.105) \]

The only flat gauge field on the 3-sphere is the trivial field \( A = 0 \) as \( \pi_1(S^3) \) is trivial. Hence \( \dim H^0(D_A) = \dim(SU(2)) \cdot \dim H^0(S^3) = 3 \) and \( \dim H^1(D_A) = \dim(SU(2)) \cdot \dim H^1(S^3) = 0 \). Eq.(3.16) gives \( k^{-3/2} \) upto one loop at large \( k \) and agrees with non-perturbative result.
Laozi, probably the first Chinese theoretical physicist, had ended the realm by stating that the law of nature which can be found by people is not the never changing law. He surely led Chinese nature science into a “bizarre” direction whilst this conclusion might still be part of reasons young physicists of any generations are brave enough to join a realm that having been said almost done by experts. Moreover, he noticed that the law we are seeking is indeed an invariant. We are not going to talk philosophy, but one worth mentioning point is its relation to mathematics and physics. They all seem to describe some aspects of nature whilst it is the information capacity of the languages of them distinguishes their tremendous differences in reasoning. Mathematics as a more rigorous language compared to speaking language makes it better tool of talking nature whilst such rigorousness also makes it harder to translate into concrete understanding in real life. We surely desire such understandings as they are not only providing profound comprehensions of nature, the best things are predictions of unknowns or future. This is where physics enters.

The interplay of mathematics and physics has been heavy once they were born. However the fundamental problems of them diverged awhile. Their remarriage started with use of differential geometry in relativity theory during the beginning of 20-th century, and with use of low-dimensional topology in quantum theory during late 70s. Such trend on the one side has brought us grand new perspectives about the world, on the other forces physicists rethinking the tool we have been used for centuries. Until recent years people can speculate rigorous clues for axiomatically construction of quantum mechanics[26], which categories physical world as “part” of mathematical reality. One might be scared by leaving in the Platonic reality
whilst the true “bizarreness” one needs to take care are other “parts” that different from ours.

We see that both philosophy and mathematics would imply there are different representations of reality. In this thesis we have seen the power of duality which might be appropriate to regard as symmetry between representations of reality.

4.1 Discussion

Duality has a long history in mathematics, first an also simple example is between points and lines in a plane. Any two points identify a line and if regard infinity as a point then any two lines also identify a point, in this case we have perfect symmetric descriptions of point and line in projective plane. One should also remember vector spaces and their dual spaces in linear algebra. Direct generalization to linear analysis, Fourier theory signals similar kind duality. In chapter 3, we have seen that by using Poisson summation formula which duality in Fourier theory gives the modular invariance of theta function. We will now discussion what we have done in each chapter and the use of duality in them.

In chapter 1 we introduced briefly about supersymmetry which can be regarded as a duality between even and odd operators. Supersymmetry arose as the only consistent combination of internal and Poincaré symmetries from physics, and its relation to differential forms seems suggest it is also mathematically elegant. Moreover with the Morse theory we can better bound the number of ground states from perturbation theory. This is done by instaton calculation. The true ground states are isomorphic to the cohomology of Witten complex, and it is indeed the Hodge-de Rham cohomology. Moreover, during the instaton calculation we have seen how would localization principle help the path integral of supersymmetric action. Indeed localization is one elegant technic when dealing with supersymmetric perturbation theories and it has a profound origin from topology. One direct generalization of the Witten-Morse homology is the Floer homology where the Morse function is given by the CS functional of su(2)-valued connection 1-forms on oriented smooth 3-manifolds. One key observation of Floer is that the gradient flows of CS are the anti-self-duality equation for 4-manifolds, and the solutions are exactly the minima of the Yang-Mills functional, Donaldson theory. Moreover, with knots and links embedded Floer instanton homology is related to Khovanov homology which originally arose as a categorification of the Jones polynomial which can be obtained from CS partition function. This leads to quantization and categorification [20, 21]. Instead going into such wonderful direction, we started the evaluation of CS partition function.

In chapter 2 we started with first loop correction of perturbative CS theory. We have seen that the Hodge duality plays the central role during the evaluation. With standard elliptic theory we also made comments on the De Rham cohomology of flat connection which will be important for dealing with the Floer theory. After gauge fixing the action is topological by BRST argument. This topological model
is a simple example of BRST formulation of topological quantum field theory. The leading order contribution in perturbation is the Ray-Singer torsion which is a topological invariant. With APS theorem one can fix the metric dependent phase term to zero by adding a gravitational CS term, a local counterterm, and by the canonical 2-framing. Therefore we arrived that the leading order contribution is topological. For higher loop, it is always possible to add such local counterterms to cancel out metric dependence[5, 6]. One very interesting point we did not get into is the relation between surgery of 3-manifold and framing dependence. In topology theories, one can usually simplify calculations by knowing how the surgery works, and we used this simple technic to compute the exact result of CS partition function in chapter 3. If we include knots or links in $S^3$, framing dependence is just about the surgery along them. On can find more details about surgery of CS in [17]. This can be done by adding non-trivial Wilson loops and the partition function will give topological invariants of those loops, or knot invariants. Thus one can regard CS partition function as generator of knots.

In chapter 3 we used affine Lie algebra to derive the exact result of CS theory, or say the non-perturbative comparison. Started with classification of semi-simple groups we showed that how one can characterize groups with their maximum torus, which is the maximum Abelian subgroup. If we recall that the Green function on torus is related to one of the Jacobi theta functions, then it would be not surprised that we arrived a character formula with modular invariance. The generators of this special symmetry forms $PSL(2, \mathbb{Z})$ group which can be interpreted as surgery operations on 3-manifolds. A simple example is presented to illustrate how one uses the surgery to compute the surgery matrix. Then we showed that such surgery matrix can help to arrive the exact result of CS. At large $k$ limit they agree with each other. This fact is astonishing even at first sight. It is first example of AdS$_3$/CFT$_2$ correspondence, and we added more details in Appendix G about this. This “new” duality has gotten great attention. To our case, on the one side quantization of CS leads to 2-dimensional sigma model with conformal symmetry, on the other the coefficient of operator product expansion can be obtained by the surgery of 3-manifolds.

### 4.2 Outlook

Along the main stream of frontier of theoretical physics, future research of the author will probably be focused on mathematical physics in relation with low-dimensional topology and quantum mechanics. On the side of mathematics, we will continue the study of interplay of Morse theory and knot theory as the fact that the Morse theory is so effective in gaining topological information of manifold from functions on it and since knots are non-trivial 1-dimensional structures that signal a great deal of differences between dimensionality of 2 and 3. From the physics side, relativistic extension of the Floer theory to 4-dimensional spacetime leads one to TQFT. Moreover, the gradient flow on the space of complex-valued
connections is same as the Kapustin-Witten equation resulted from $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensional spacetime, and with electric-magnetic duality one will see both the Jones polynomial and the associated quantum knot invariants are as Laurent polynomials with integer coefficients. More interestingly, the Khovanov homology emerges by adding a fifth dimension and leads to non-integer-valued level $k$. The two sides are dual to each other, i.e. if we have a simple Lie group $G$ for CS then the KW equation is associated to gauge theory in 4-dimension with the Langlands dual group $G^\vee$. To arrive these statements, novel technics are needed, e.g. analytic continuation of CS, etc, which are of great interest in pure mathematical pursuing. In short, modern topology makes it possible to access relations of different dimensionality and from these relations one can probably arrive better comprehension of supersymmetry and this will be a central guideline for future research.

There are also intuitions of the author toward modern understanding of gravity. At present, theories such as eternal inflation and multiverse from string cosmology have gotten great attention from physics community not only since experimental supports, their philosophical implications are more satisfactory. On the one hand, gauge theory is related to integrable models, e.g. knots in CS project to plane can be described by IFR model(interaction round a surface) and vertex model. One direct result is holography theory. A novel idea along the line to interpret the emergence of spacetime from holographic information, especially the spatial part, is called theory of entropic gravity. My own intuition towards such idea is simply how is it possible to find a dual description of quantization as a process from quantum world to classic. It is hard to make the idea concrete, even just figure out a right direction to look at. However, back to the Jones polynomial and Khovanov homology, recent developments on the Nahm’s equation shed light on the probability. It relates not only theories in different dimensions but also different geometries in geometrization conjecture.
Appendix A

Supersymmetric non-linear sigma model

Here we give detail relations between canonical quantisation of supersymmetric non-linear sigma model and exterior algebra by following Witten[37]. The idea was when we try to count the number of zero-energy states of the model whose action is given by

\[
\mathcal{L} = \frac{1}{2} \int d^2x \gamma_{ij} \left( \partial_{\mu} \phi^i \partial^\mu \phi^j + i \bar{\psi}^i \gamma^\mu D_{\mu} \psi^j \right) + \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l, \tag{A.1}
\]

where \( \gamma_{ij} \) and \( R_{ijkl} \) are metric and curvature tensors on manifold \( M \), and \( D_{\mu} \) is a covariant derivative whose action on spinor fields is given by

\[
D_{\mu} \psi^i = \partial_{\mu} \psi^i + \Gamma^i_{jk} \partial_{\mu} \phi^j \psi^k. \tag{A.2}
\]

The classical vacuum states of the theory are degenerate, thus we turn on a perturbation that lifts the degeneracy. That is to add a generic function \( h(\phi^i) \), as a “magnetic field”, define on \( M \) to the action. The new components to the Lagrangian is

\[
\Delta \mathcal{L} = -\frac{1}{2} \int d^2x \gamma_{ij} \frac{\partial h}{\partial \phi^i} \frac{\partial h}{\partial \phi^j} + \frac{\partial^2 h}{\partial \phi^i \partial \phi^j} \bar{\psi}^i \psi^j. \tag{A.3}
\]

The potential energy \( V(\phi^i) \) lifts the degeneracy that the energy is zero only if \( \partial h / \partial \phi^i = 0 \).

On the other hand, since classically non-zero momentum modes carry non-zero energies, we can determine the spectrum of low-lying states by quantizing the zero-momentum modes. Therefore we drop the spatial dependence of \( \phi^i \) and \( \psi^i \) and
integrate out spatial components. Choosing a convenient base \( \gamma^0 = \text{diag}(1, -1) \) and consider Majorana fermion \( \tilde{\psi} = (\psi^\dagger, \psi) \) the simplified Lagrangian is written as

\[
\mathcal{L} = \frac{1}{2} L \int d\tau \gamma_{ij} \left( \frac{d\phi^i}{d\tau} \frac{d\phi^j}{d\tau} + i\psi^{\dagger i} \frac{D\psi^j}{D\tau} - i\frac{D\psi^{\dagger i}}{D\tau} \psi^j \right) + R_{ijkl} \psi^{\dagger i} \psi^k \psi^{\dagger j} \psi^l, \tag{A.4}
\]

where \( L \) is the spatial part. Under the supersymmetry transformations

\[
\begin{align*}
\delta \phi^i &= \epsilon \psi^{\dagger i} - \bar{\epsilon} \psi^i \\
\delta \psi^i &= \epsilon \left( i\dot{\phi}^i - \Gamma^i_{jk} \psi^j \psi^k \right) \\
\delta \psi^{\dagger i} &= \bar{\epsilon} \left( -i\dot{\phi}^i - \Gamma^i_{jk} \psi^j \psi^k \right),
\end{align*}
\tag{A.5}
\]

the action is invariant. By the Noether procedure, we find the corresponding supercharges

\[
Q = i\gamma_{ij} \psi^{\dagger i} \phi^j \\
\bar{Q} = -ig_{ij} \psi^i \phi^j.
\tag{A.6}
\]

The conjugate momentum of \( \phi^i \) and \( \psi^i \) are given by

\[
\begin{align*}
p_i &= \gamma_{ij} \dot{\phi}^j \\
\pi_{\psi, i} &= i\gamma_{ij} \psi^{\dagger j}.
\end{align*}
\tag{A.7}
\]

The canonical commutation and anticommutation relations are

\[
\begin{align*}
[\phi^i, p_j] &= i\delta^i_j, \\
\{\psi^i, \psi^{\dagger j}\} &= \gamma^{ij},
\end{align*}
\tag{A.8}
\]

and other relations vanish. The supercharges can be rewritten in terms of the conjugate momentum as

\[
\begin{align*}
Q &= i\psi^{\dagger i} p_i \\
Q^\dagger &= -i\psi^i p_i.
\end{align*}
\tag{A.9}
\]

Thus we see that by choosing the space of differential forms as representation of above algebra we recover Eq.(1.1)-Eq.(1.4).

Moreover, the Lagrangian (A.4) is also invariant under the phase rotation of the fermionic fields

\[
\psi^i \to e^{-i\alpha} \psi^i, \quad \psi^{\dagger i} \to e^{i\alpha} \psi^{\dagger i}.
\tag{A.10}
\]

The corresponding Noether charge is given by

\[
F = \gamma_{ij} \psi^{\dagger i} \psi^{\dagger j},
\tag{A.11}
\]

the fermion number operator. Notice that

\[
[F, Q] = Q, \quad [F, Q^\dagger] = -Q^\dagger,
\tag{A.12}
\]

thus within the differential form representation, harmonic \( q \)-form is considered as a state with \( q \) fermions and \( F \) in the Witten index \((-1)^F\) is the same as fermion number \( q \).
In conclusion, the canonical quantization of supersymmetric non-linear sigma model does lead to exterior algebra. If the "magnetic field" $h$ is switched on, we will have deformed supercharges and can be identified with $d_t$ and $d^t$ (notice that $t$ is considered as a rescaling factor of $h$.)
Appendix B

Localization principle

We will introduce a simple example to illustrate the localization principle. This principle occurs in supersymmetric theories in general. With localization we can reduce path integrals to much smaller-dimensional integral, and in ideal situations reduce to sum of contributions from certain points.

Let $X$ be maps from $M \to \mathbb{R}$ as bosonic fields and $\psi_1$ and $\psi_2$ be fermionic fields. Start with a simple supersymmetric action

$$
\mathcal{L} = \frac{1}{2} (\partial h)^2 - \partial^2 h \psi_1 \psi_2, \quad (B.1)
$$

where $h$ is a real function of $X$. This action is invariant under supersymmetric transformations

$$
\begin{align*}
\delta \psi_1 &= \epsilon^2 \partial h \\
\delta \psi_2 &= -\epsilon^1 \partial h.
\end{align*} \quad (B.2)
$$

Suppose $\partial h$ is nowhere zero, we can consider $\epsilon^1 = \epsilon^2 = -\psi_1/\partial h$. Then the supersymmetric transformations above are rewritten as

$$
\begin{align*}
\delta X &= -\frac{\psi_1 \psi_2}{\partial h} \\
\delta &= -\psi_1 \\
\delta &= \psi_1, \quad (B.3)
\end{align*}
$$

which means after the transformation the $\psi_1$ is eliminated from the action. If we denote the variables by

$$
\begin{align*}
\hat{X} &= X - \frac{\psi_1 \psi_2}{\partial h} \\
\hat{\psi}_1 &= \alpha(X) \psi_1 \\
\hat{\psi}_2 &= \psi_1 + \psi_2, \quad (B.4)
\end{align*}
$$

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Appendix B. Localization principle

where $\alpha(X)$ is an arbitrary function of $X$. We see that the supersymmetric transformations Eq.(B.3) actually means that

$$S(X,\psi_1,\psi_1) = S(\hat{X},0,\hat{\psi}_2). \tag{B.5}$$

As dealing with Grassmann integral, the integral measure becomes, after change of variables Eq.(B.4)

$$dXd\psi_1d\psi_2 = (\alpha(\hat{X}) - \frac{\partial^2 h(\hat{X})}{(\partial h(X))^2}\hat{\psi}_1\hat{\psi}_2)d\hat{X}d\hat{\psi}_1d\hat{\psi}_2. \tag{B.6}$$

Therefore the partition function can be written as

$$Z = \int d\hat{\psi}_1 \int e^{-S(\hat{X},0,\hat{\psi}_2)}\alpha(\hat{X})d\hat{\psi}_2$$

$$- \int e^{-S(\hat{X},0,\hat{\psi}_2)}\frac{\partial^2 h(\hat{X})}{(\partial h(X))^2}\hat{\psi}_1\hat{\psi}_2d\hat{X}d\hat{\psi}_1d\hat{\psi}_2 \tag{B.7}$$

$$= 0.$$  

Since there is no $\hat{\psi}_1$ in the integrand of first term, the integral over it gives zero. The second term survives after the Grassmann integration whilst still vanishes as the integrand is a total derivative in $\hat{X}$.

If we admit $\partial h$ be zero for some points $X_c$, the change variables Eq.(B.4) are singular at $X - c$. At these points the fermionic supersymmetric transformations are fixed, i.e. $\delta\psi_1 = 0$. Therefore we get contributions from the vicinity of the fixed points. The localization principle states as: The path-integral is localized at loci where the fermionic transformation under supersymmetry is zero.

Let consider $h$ as a generic polynomial of order $n$ with isolated critical points $X_c$. The number of critical points are at most $n - 1$. In the vicinity of $X_c$ we have

$$h(X) = h(X_c) + \frac{h''(X_c)}{2}(X - X_c)^2 + \cdots. \tag{B.8}$$

This is the leading contribution from the infinitesimal neighborhood of $X_c$. Near the critical points we have the partition function

$$Z = \sum_{X_c} \int \frac{dXd\psi_1\psi_2}{\sqrt{2\pi}} e^{-\frac{1}{2}h''(X_c)(X - X_c)^2 + h''\psi_1\psi_2} = \sum_{X_c} \frac{h''(X_c)}{|h''(X_c)|}. \tag{B.9}$$

This gives $\pm$ if $n$ is even and it vanishes if $n$ is odd.
Appendix C

Ray-Singer torsion

Here we will introduce Ray-Singer invariant, and derive the analytic torsion appears in 1-loop correction result.

Let $M$ be a $n$-dimensional compact oriented Riemannian manifold, and let $K$ be a simplicial complex which is a smooth triangulation of $M$. The Reidemeister-Franz torsion $\tau$ of $K$ is a function of certain representations of the fundamental group of $K$. The combinatorial invariance and the equivalence of smooth triangulations of $M$ make this torsion a manifold invariant. By following Milnor[28], give a $n$-dimensional vector space $V$ over real numbers, one can have two bases \( v = (v_1, \ldots, v_n) \) and \( w = (w_1, \ldots, w_n) \), and use \([w/v]\) to denote $|\text{determinant } T|$, where $T$ represents the transformation of bases: $v = Tw$. If \([w/v] = 0\) then we can say $w$ is equivalent to $v$. This will give an equivalence relation to free modules.

Now consider an exact sequence

\[
0 \to E \to F \to G \to 0 \quad (C.1)
\]

of free modules. Given bases of $E$ and $G$, one can construct a basis for $F$ as $e_\mathbf{g} = (e_1, \ldots, e_i, g'_i, \ldots, g'_l)$, where $e_i \in E$ and $g'_i \in F$ is lifted from $g_i \in G$. We can see the obvious dependence of $e_\mathbf{g}$ on the choice of $g'_i$, whilst the equivalent class of $e_\mathbf{g}$ only depends on $e$ and $g$. If we choose different bases $\hat{e}$ and $\hat{g}$ for modules $E$ and $G$, then

\[
[e_\mathbf{g}/e_\mathbf{g}] = [\hat{e}/e][\hat{g}/g]. \quad (C.2)
\]

If we have inclusions of free modules, eg. $F_0 \subset F_1 \subset F_2 \subset F_3$, then using the exact sequence of free quotient modules

\[
0 \to F_1/F_0 \to F_3/F_0 \to F_2/F_1 \to F_3/F_3 \to F_2/F_2 \to 0 \quad (C.3)
\]

the bases $\mathbf{b}_i$ of $F_i/F_{i-1}$ form a basis $\mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k$ for $F_k/F_0$. 

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Appendix C. Ray-Singer torsion

Now we can proceed to the definition of torsion of a complex chain. Given a chain complex

\[ C_n \to C_{n-1} \to \cdots C_1 \to C_0 \]  
(C.4)

of free modules that each \( C_i \) has a preferred basis \( c_i \). Let \( B_i \) denotes the image of boundary operator \( \partial : C_{i+1} \to C_i \), \( Z_{i+1} \) as its kernel and \( H_i \) as the \( i \)-th homology group. Choose bases \( b_i \) and \( h_i \) for \( B_i \) and \( H_i \) respectively. For each \( C_i \) we have inclusions \( 0 \subset B_i \subset Z_i \subset C_i \) and notice that \( Z_i/B_i \cong H_i \), \( C_i/Z_i \cong B_{i-1} \). This suggests that by combining bases \( b_i \), \( h_i \) and \( b_{i-1} \) we can construct a new basis \( b_i h_i b_{i-1} \) for \( C_i \). We define \( \tau(C) = \sum (-1)^i \log [b_i h_i b_{i-1}/c_i] \) as the torsion of this complex chain. The independence of \( \tau(C) \) on the choice of basis \( b_i \) is easily seen by choosing a different basis \( \tilde{b}_i \), the torsion now is written as

\[ \tau(C) = \sum (-1)^i \log [\tilde{b}_i h_i \tilde{b}_{i-1}/c_i] \]

where the last two terms sum up to zero.

To make such abstractness useful for physics, we can define an analytic analogue of the torsion above when dealing with Riemannian manifold. In that case, the metric induces a natural pairing, by the Hodge dual, that gives a preferred orthonormal basis for each \( C_i \). We can use the Hodge decomposition to split each \( C_i \) as a combination of three orthogonal parts. With the help of the Laplacian \( \Delta = -(\partial^\dagger \partial + \partial \partial^\dagger) : C_i \to C_i \), we can identify the three parts of any \( \omega_q \in C_q \) as its eigenvectors

\[ \omega_q = \psi_q + \phi_q + h_q \]
\[ \Delta \psi_q^i = -\partial \partial^\dagger \psi_q^i = \lambda_q,i \psi_q^i , i = 1, \ldots, r_q \]  
(C.5)

where we recognize \( \psi_q^i \) as the \( i \)-th component of the \( B_q \). In addition we can define the co-boundary part \( \phi_q \) as

\[ \phi_q^j = -\frac{1}{\lambda_{q-1,j}} \partial^\dagger \psi_{q-1}^j , j = 1, \ldots, r_{q-1} \]  
(C.6)

such that \( \partial \phi_q^j = \psi_q^{j-1} \). Notice that

\[ |\phi_q|^2 = \frac{1}{\lambda_{q-1,j}^2} (\partial^\dagger \psi_{q-1}^j, \partial \partial^\dagger \psi_q^j) = -\frac{1}{\lambda_{q-1,j}} \]  
(C.7)
we finally have an orthonormal basis \( \{ \sqrt{-\lambda_{q-1}} \phi_q, \psi_q, h_q \} \) for \( C_q \) and the torsion of this complex chain is

\[
\tau(C) = \sum (-1)^i \log[b_i, h, b_{i-1} / c_i] \\
= \sum (-1)^q \log[\psi_q, \partial \phi_q, h / \{ \sqrt{-\lambda_{q-1}} \phi_q, \psi_q, h_q \}] \\
= \sum (-1)^q \log \prod_{j=1}^{\lambda_{q-1,j}^{-1}/2}.
\] (C.8)

Moreover, the Laplacian is diagonal in the orthonormal basis that

\[
(\det - \Delta_q) = \prod_{j=1}^{\lambda_q} (-\lambda_{q,j}) \prod_{j=1}^{\lambda_{q-1,j}} (-\lambda_{q-1,j}).
\] (C.9)

By taking logarithm and induction of this, we have

\[
\log \prod_{j=1}^{\lambda_{q-1,j}} (-\lambda_{q-1,j}) = \sum_{k=q}^{N} (-1)^{k-q} \log(\det - \Delta_q).
\] (C.10)

Now we have the analytic R-torsion, or Ray-Singer torsion, as

\[
\tau(C) = \frac{1}{2} \sum_{q=0}^{N} (-1)^{q+1} q \log(\det - \Delta_q).
\] (C.11)

Giving a finite-dimensional chain complex \( C \) of length \( n \) over a field \( F \).
Appendix D

Gravitational Chern-Simons

People spent a lot time to formulate a gauge theory for gravity. Indeed, in 3-dimension we have one. Such profound connection between 2 + 1-gravity and the CS gauge theory is first noticed by Witten\[39\]. This is also one of the main reasons we are interested in the pure CS theory. Concisely speaking, the Einstein-Hilbert action of 2 + 1 dimensional manifold $Y$ is written as

$$S = \frac{1}{2} \int_Y \varepsilon^{ijk} \varepsilon_{abc} (e_i^a (\partial_j \omega^b_k - \partial_k \omega^b_j + \omega_j^c, \omega_k^b c)).$$ (D.1)

We can recognize the vierbein $e_i^a$ and the spin connection $\omega^a_{ib}$ as gauge field which is Lie-algebra valued 1-form

$$A_i = e_i^a P_a + \omega^a_{ib} J_a, \quad J_a = \frac{1}{2} \varepsilon^{abc} J_{bc},$$ (D.2)

where $J_{bc}$ are the Lorentz generators and $P_a$ are translations. Under this construction, Eq.(D.1) is identified with the CS invariant of $\omega$ written as

$$CS(\omega) = \frac{1}{4\pi} \int_Y \text{Tr}(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega).$$ (D.3)

This is also called CS gravitational term.
Appendix E

Faddeev-Popov gauge fixing

The idea of gauge symmetry dominates modern understanding of elementary particles in quantum field theories, as most importantly, it makes one theory renormalizable. However it leads to infinite-dimensional spaces with infinitely many local gauge transformations. The independence of gauge conditions is therefore crucial for both defining and studying physical quantities in gauge theories. The action is forced to be gauge invariant for this purpose, and the resulting Lagrangian is degenerate. To get rid of irrelevant degrees of freedom, here we introduce the standard Faddeev-Popov procedure[15].

Naively, one puts a delta function $\delta(F(A))$ in partition function as

$$Z = \int \mathcal{D}A \mathcal{G}(A) \delta(\mathcal{F}(A)), \tag{E.1}$$

where $\mathcal{G}(A)$ is gauge invariant functional. However this is a bad idea that we are dealing with non-Abelian gauge theories, and the gauge fixing function must be local, i.e. $\delta(\mathcal{F}(A_{g\alpha}(x)))$, $g\alpha$ means group elements parametrized by $\xi^\alpha_a$, $a$ is group index. The delta function makes the integral intersecting each orbits only once, and at the crossing point one has the local gauge fixing condition $g = g_{\alpha}$, which is non-gauge-invariant. Unfortunately, such delta function will ruin the gauge-invariant feature, and it gives a result that the volume of the gauge group been divided.

To solve the problem, Faddeev, Popov and De Witt found that one just need to multiply the volume of the gauge group back by writing

$$V(G)/V(G) = 1 = \int \mathcal{D}g \delta(g - g_{\alpha}), \tag{E.2}$$

and the partition function now reads

$$Z = \int \mathcal{D}A \mathcal{D}g \mathcal{G}(A) \delta(\mathcal{F}(A_{g\alpha})) \left| \det \frac{\delta \mathcal{F}(A_{g\alpha})}{\delta g^\beta} \right|, \tag{E.3}$$
where the determinant is Jacobian from $\delta (g - g_\alpha)$ to $\delta (\mathcal{F}(A_{g_\alpha}))$. Under infinitesimal gauge transformation, first order, $g_\alpha$

$$A_{g_\alpha} = A + \partial \xi^a t_a + [A, \xi_\alpha] = A + D_A \xi_\alpha,$$

(E.4)

where $D_A$ is the covariant exterior derivative respect to connection $A$ and $t_a$ are group generators, i.e. the Lie algebra of $G$. Therefore if $\mathcal{F}(A_{g_\alpha})$ is linear in $A_{g_\alpha}$, and we evaluate the determinant at identity, i.e $\xi^a = 0$,

$$\left| \frac{\delta \mathcal{F}(A_{g_\alpha})}{\delta g_\beta} \right|_{g_\beta = 0} = \left| \frac{\delta \mathcal{F}(A_\alpha)}{\delta A_{g_\alpha}} \frac{\delta A_{g_\alpha}}{\delta \xi_\beta} \frac{\delta \xi_\beta}{\delta \xi_\beta} \right|_{\xi_\beta = 0}$$

(E.5)

is independent of $A_\alpha$, thus gauge invariant. The assumption that the measure $DA$ and $\mathcal{F}(A)$ are gauge invariant makes sure that the partition function

$$Z = \int DA Dg \mathcal{F}(A) \delta (\mathcal{F}(A_{g_\alpha})) \left| \frac{\delta \mathcal{F}(A_{g_\alpha})}{\delta g_\beta} \right|$$

$$= \int DA_{g_\alpha} Dg \mathcal{F}(A_{g_\alpha}) \delta (\mathcal{F}(A_{g_\alpha})) \left| \frac{\delta \mathcal{F}(A_{g_\alpha})}{\delta g_\beta} \right|$$

$$= \int DA' Dg \mathcal{F}(A') \delta (\mathcal{F}(A')) \left| \frac{\delta \mathcal{F}(A_\alpha)}{\delta h} \right|$$

$$= \int Dg \int DA \mathcal{F}(A) \delta (\mathcal{F}(A)) \left| \frac{\delta \mathcal{F}(A_\alpha)}{\delta h} \right|$$

(E.6)

is gauge independent, we have changed the dummy variable $g$ to $h$ in the determinant. This is the case for irreducible connection $A$. We will see the reducible case in the CS theory first noted by Rozansky[31].
Appendix F

BRST symmetry and gauge fixed Chern-Simons

Following [7], we first define the BRST operator $Q$, which is odd derivation of the algebra of local functionals, by its action on $B$, $\phi$, $C$ and $\bar{C}$

$$QB_i = -(D_i^{A(\alpha)} + \text{ad}B_i)C,$$
$$Q\phi = 0,$$
$$QC = \phi,$$
$$Q\bar{C} = \frac{1}{2}[C, C] = \frac{1}{2}t_a f_{bc}^a e^b e^c,$$  \hspace{1cm} (F.1)

where $(\text{ad}B_i)c \equiv [B_i, c]$, and $f^a_{bc}$ is the structure constant of $G$, $[t_b, t_c] = f^a_{bc} t_a$.

Acting $Q$ on $L_{\text{gauge}}$,

$$Q L_{\text{gauge}} = -\frac{k}{4\pi} \int \text{Tr} \phi D^A_i (D_i^{A(\alpha)} + \text{ad}B_i)C + \text{Tr}(\phi D^A_i (D_i^{A(\alpha)} + \text{ad}B_i)C$$
$$+ \bar{C} D^{A(\alpha)}_i [D_i^{A(\alpha)} C + [B^i, C], C] - \frac{1}{2} \bar{C} D^{A(\alpha)}_i (D_i^{A(\alpha)} + \text{ad}B_i)[C, C])$$
$$= 0,$$  \hspace{1cm} (F.2)

the gauged Lagrangian is invariant under BRST transformation. We continue to investigate how the $L_{\text{gauge}}$ changes under $g^{ij} \rightarrow g^{ij} + \delta g^{ij}$. Since we are dealing with the CS theory, which is topological, $\delta CS(B) = 0$. $\delta L$ will only depends on the gauge fixing and ghosts terms as

$$\delta L_{\text{gauge-fixing}} = \frac{k}{4\pi} \int \text{Tr}(\phi D_i^{A(\alpha)} (\sqrt{g} \delta g^{ij} B_j) - \frac{1}{2} \phi D_k^{A(\alpha)} (\sqrt{g} g_{ij} \delta g^{ji} g^{kl} B_l))$$  \hspace{1cm} (F.3)
and
\[ \delta \mathcal{L}_{\text{ghosts}} = \frac{k}{4\pi} \int \text{Tr}(\bar{C} D_i^{(\alpha)}(\sqrt{g} \delta g^{ij} (D_j^{(\alpha)} + \text{ad} B_j) C) \] (F.4)
\[ - \frac{1}{2} \bar{C} D_k^{(\alpha)}(\sqrt{g} g_{ij} \delta g^{ij} g^{kl} (D_l^{(\alpha)} + \text{ad} B_l) C).] \]
one can then write
\[ \delta \mathcal{L}_{\text{gauge}} = \frac{k}{4\pi} \int \sqrt{g} \delta g^{ij} T_{ij} \] (F.5)
with
\[ T_{ij} = \text{Tr}((D_i^{(\alpha)} \phi) B_j + (D_i^{(\alpha)} \bar{C})(D_j^{(\alpha)} + \text{ad} B_j) C - \frac{1}{2} g_{ij} ((D_k^{(\alpha)} \phi) g^{kl} B_l \] (F.6)
\[ + (D_k^{(\alpha)} \bar{C}) g^{kl} (D_l^{(\alpha)} + \text{ad} B_l) C)).] \]
If one writes
\[ \lambda_{ij} = \text{Tr}((D_i^{(\alpha)} \phi) B_j - \frac{1}{2} (D_k^{(\alpha)} \bar{C}) g^{kl} B_l), \] (F.7)
then \( T_{ij} = Q \lambda_{ij} \) and
\[ \delta \mathcal{L}_{\text{gauge}} = Q\left(\frac{1}{4\pi} \int \sqrt{g} \delta g^{ij} \text{Tr}((D_i^{(\alpha)} \phi) B_j - \frac{1}{2} (D_k^{(\alpha)} \bar{C}) g^{kl} B_l)) \right) \] (F.8)
\[ \equiv QA. \]
Now we are ready to see how the BRST argument ensures \( \mathcal{L}_{\text{gauge}} \) is independent of metric. Moreover, one can consider a metric independent function \( O \) that is also BRST invariant, then
\[ \langle O \rangle \equiv \int DBD\phi DC\bar{C}O e^{i\mathcal{L}_{\text{gauge}}} \] (F.9)
is metric independent as,
\[ \delta \langle O \rangle = \delta \int DBD\phi DC\bar{C}O e^{i\mathcal{L}_{\text{gauge}}} \]
\[ = i\delta \int DBD\phi DC\bar{C}O e^{i\mathcal{L}_{\text{gauge}}} \delta \mathcal{L}_{\text{gauge}} \] (F.10)
\[ = i\delta \int DBD\phi DC\bar{C}Q(O e^{i\mathcal{L}_{\text{gauge}}} A). \]
If we consider \( Q \) as a zero divergence vector field, i.e. \( Q \) is nilpotent, the last step indicates \( \mathcal{L}_{\text{gauge}} \) is metric independent.
Appendix G

AdS$_3$/CFT$_2$ correspondence

To see the relation from CS to CFT, we start with part of holomorphic quantization of Eq.(2.4). Let $A_z = \frac{1}{2}(A_x - iA_y)$ and $A_z = \frac{1}{2}(A_x + iA_y)$, the CS action is then given by

$$CS = -\frac{ik}{2\pi} \int d^2z [\text{Tr}(A_z \partial_0 A_{\bar{z}} - A_{\bar{z}} \partial_0 A_z) + i A_0 F_{\bar{z}z}]. \quad (G.1)$$

The canonical commutation relations become

$$[A^a_z(z, \bar{z}), A^b_{\bar{z}}(\omega, \bar{\omega})] = \frac{2\pi}{k} \delta^{ab} \delta(z - \omega) \delta(\bar{z} - \bar{\omega}). \quad (G.2)$$

The Hilbert space is now the space of holomorphic functionals $\psi$ of $A_{\bar{z}}$. Meanwhile $A_z$ can be identified as the functional derivative

$$A^a_z \psi = \frac{2\pi}{k} \delta \frac{\delta}{\delta A^a_z} \psi. \quad (G.3)$$

The physical Hilbert subspace is obtained by restriction $F_{\bar{z}z} = 0$ on wave functions

$$(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + [A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}] )\psi = \frac{k}{2\pi} \partial_z A_{\bar{z}} \psi. \quad (G.4)$$

The solutions to above equation can be obtained from conformal field theory, the WZW model. Let define map $g : \Sigma \rightarrow G$ with two dimensional Riemann surface $\Sigma$ and compact Lie group $G$. For our case we choose the group as $SU(2)$. The WZW action is given by

$$S_{WZW} = -\frac{1}{8\pi} \int_{\Sigma} d^2x \sqrt{\gamma} \gamma^{ij} \text{Tr}(g^{-1} \partial_i g^{-1} \partial_j g) + i\Gamma(g), \quad (G.5)$$
where $\gamma$ is a metric on $\Sigma$ and the trace is same as in CS over the fundamental representation of $SU(2)$. Let $\partial Y = \Sigma$ and extend $g$ over $Y$. The second term in $S_{WZW}$ is defined by

$$\Gamma(g) = \frac{1}{12\pi} \int_Y d^3x \varepsilon^{ijk} \text{Tr}(g^{-1}\partial_ig^{-1}\partial_jgg^{-1}\partial_ig).$$

(G.6)

This is the WZ term which characterizes the topological winding, e.g. last term of eg.(2.5). In terms of $A_z$ and $A_{\bar{z}}$, $S_{WZW}$ is given by

$$S_{WZW} = \frac{1}{2\pi} \int_{\Sigma} \text{Tr}(\partial_z g \partial_{\bar{z}} g^{-1}) + i\Gamma(g).$$

(G.7)

The equations of motion then are given by

$$\partial_z(g^{-1}\partial_ig) = \partial_{\bar{z}}(\partial_z gg^{-1}) = 0.$$  

(G.8)

One important property of WZW action is the Polyakov-Wiegmann identity which describes how $S_{WZW}$ transform under gauge symmetry. Let $g' : \Sigma \rightarrow SU(2)$, we have

$$S_{WZW}(gg') = S_{WZW}(g) + S_{WZW}(g') - \frac{1}{\pi} \int_{\Sigma} \text{Tr}(g^{-1}\partial_z g \partial_{\bar{z}} g' g^{-1}).$$

(G.9)

The last term shows the chiral splitting. If we give an infinitesimal left translation for $g$ by a holomorphic function, $g \rightarrow (1 + f(z))g$, or a right translation with an antiholomorphic function, $g \rightarrow g(1 + f(\bar{z}))$, the action is invariant. Therefore we have the according currents

$$J_z = -\partial_z gg^{-1}$$

$$J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g.$$  

Now let consider the level $k$ WZW model which is simply $kS_{WZW}$. The wave function $\psi$ can be written as

$$\psi = \int Dg e^{ikS_{WZW}} - \frac{i\pi}{2} \int_{\Sigma} \text{Tr}(A_z \partial_z gg^{-1}).$$

(G.10)

As conformal field theory, one can write the operator product expansion(OPE) of currents Eq.(G.10). Without explicit illustration, we write an OPE

$$\partial_z J^a_z J^b_{\bar{z}} = \frac{k\pi}{2} \delta^{ab} \partial_\omega \delta^2(z - \omega) - \pi f^{abc} \delta^2(z - \omega) J^c_{\bar{z}}.$$  

(G.11)

We can regard $\psi$ as a generating functional of $J_z$, and the above OPE gives a differential equation of $\psi$ which is exactly same as the restriction of wave function from quantization of CS, Eq.(G.4). In WZW models, $\psi$ are called the conformal blocks, together with their complex conjugates one can also write the partition function of $\Sigma$ as a finite sum of their products, e.g. Eq.(3.96)

$$Z = \sum_i \psi_i \bar{\psi}_i.$$  

(G.12)
Bibliography


