Introduction

A fundamental fact in finance and economics is that money has a time value, meaning that if we want to value an amount of money we get at some future date we should discount the amount from the future date back to today. When facing a stream of cash flows occurring at different times we discount each of the cash flows using suitable discount rates and then sum the contributions. This sum of discounted cash flows defines the value today of this stream. Most future cash flows that appear in models in finance and economics are assumed to be stochastic (non-defaultable bonds being a counter example). To be able to value these stochastic cash flows we also have to take expectations. In some cases even the discount rate should be modelled as a stochastic object. The purpose of the two papers in this licentiate thesis ('On the Valuation of Cash Flows – Discrete Time Models' and 'On the Valuation of Cash Flows – Continuous Time Models') is to establish general properties of the value process. As time passes two things happen. Firstly, the cash flows that are realised are no more parts of the value and secondly, the information we can use to determine the expected cash flows and discount rates increases.

The two papers consider discrete time models and continuous time models respectively. Of course any continuous time model is necessarily an idealisation. Thus one could argue from a modelling point of view that we should use discrete time models. The main reason for using continuous time models is that we have the powerful machinery of stochastic calculus at hand. Discrete time models are mostly used in practice when valuing a firm or a project, while the continuous time setting is more frequently used in thoretical approaches to valuation. Most of the results are parallelled in the two papers. A difference is that we discuss some convergence results for the value in discrete time which do not occur in the continuous time paper. The reason for not including this in the continuous time paper is because we find it a more important question in discrete time. On the other hand the Brownian models in continuous time, where the Martingale Representation Theorem is an important tool, make the analysis much more transparent.

In both papers we first define the underlying objects: the discount process and the cash flow process. We then define, using these two processes, the value process (i.e. the expected discounted value of the cash flow stream). We show that the discounted value tends to zero almost surely, and that there are three equivalent ways of writing the value process, each of which has its own merits. We also extend this result to the case when the cash flow process and the value process are evaluated at a stopping time. The first paper, on discrete time models, then continues by showing examples from finance, economics, and insurance where the discounted value process plays an important role. Finally we present two propositions with necessary conditions for the value process to converge almost surely. The second paper, on continuous time models, discusses some properties of the local dynamics of the value process and then continues with Brownian models. We show that the value process can equivalently be expressed as a solution to a forward-backward stochastic differential equation. Finally we show that under some additional assumptions there is a one-to-one correspondance between the cash flow process and the value process. We also investigate the inverse problem of finding a cash flow process generating a given value process and discuss applications to real options.

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On the Valuation of Cash Flows – Discrete Time Models

Fredrik Armerin

Abstract

Discounted cash flow models in discrete time are considered. Under some general assumptions we show that the value of the cash flow stream can be written in three equivalent ways. We show that the discounted value tends to zero a.s. and give two cases of necessary conditions for the value process to converge a.s. Applications include topics from finance, economics, and life insurance.

1 Introduction

Assume that a firm or individual is facing a stream of cash flows. These could be dividends from a stock, cash flows generated by an investment or project, or claims faced by an insurance company. But what is the value today of this cash flow stream? To find the value we discount the cash flows using a suitable discount rate, take expectations and sum over time. If we call the cash flows $C_1, C_2, \ldots$ and assume that the discount rate is deterministic, given by r, the discounted value at time zero is

$$V_0 = E\left[\sum_{k=1}^{\infty} \frac{C_k}{(1+r)^k}\right]$$

To make this into a dynamic model, introduce the value at time $t \geq 0$ as

$$V_t = E_t\left[\sum_{k=t+1}^{\infty} \frac{C_k}{(1+r)^{(k-t)}}\right], \tag{1}$$

where we let $E_t[\cdot]$ denote the expectation given information up to and including time t. By multiplying this expression with $(1+r)^{-t}$ and splitting the expectation into two parts we get

$$\frac{V_t}{(1+r)^t} = E_t\left[\sum_{k=0}^{\infty} \frac{C_k}{(1+r)^k}\right] - \sum_{k=0}^{t} \frac{C_k}{(1+r)^k}. \tag{2}$$

If $E\left|\sum_{k=1}^{\infty} \frac{C_k}{1+r)^k}\right| < \infty$ then the first term on the RHS is a martingale and the value of the second one is known at time t. Iterating Equation (1) gives the relation

$$V_t = E_t\left[\frac{C_{t+1} + V_{t+1}}{1+r}\right],$$

saying that the value today is the expected discounted value of what we get tomorrow (C_{t+1}) plus the expected discounted value of having the right to the cash flow stream $C_{t+2}, C_{t+3}, \ldots$ (which is the definition of V_{t+1}). By continued iterations we get for any $T > t$

$$V_t = E_t\left[\sum_{k=t+1}^{T} \frac{C_k}{(1+r)^k}\right] + E_t\left[\frac{C_T}{(1+r)^T}\right].$$

If we impose the condition that the last term in the RHS of the previous eqution goes to zero as T goes to infinity we are back to Eq. (1) We see from Eq. (2) that if we let t go to infinity, then the discounted value $V_t/(1+r)^t$ tends to 0 a.s. A subsequent question is now what will happen to the value V_t when we let time go to infinity. It turns out that this convergence depends on the behaviour of both the discount factors and the cash flows. The idea of rewritng the value equation (1) as to identify the martingale embedded within comes from life insurance. There the expected discounted value of the cash flows is known as the retroperspective reserve. The fact that we can decompose the discounted value as the difference of a martingale and an adapted process give us a way to prove Hattendorff's Theorem. Recently valuation using real options has gained increasing interest. In these models either the value or the underlying cash flow is modelled as a stochastic process. In the latter case the question of how the dynamical properties influence the dynamics of the value process is important. For references and more concrete examples see Section 3.1. The purpose of this paper is to prove the results indicated above in a more general setting. In Section 2 we define the cash flow process as any a.s.

finite adapted process and the discount process as an adapted process, fulfilling a consistency relation connected to the absence of arbitrage. While we do not comment much upon the cash flows, the discount process and its equivalent forms, is discussed in some detail. In Section 3 we define the value process. We discuss some properties of it and then state and prove that there are three equivalent forms in which we can express the value process. These facts are known previously, at least in some special cases. We then turn to the problem of convergence of the value process. Although the discounted value tends to 0, the convergence of the value itself depends on both the the cash flowes and the discount rates. We give two propositions containing necessary condition for the convergence of the value process. The last part of Section 3 contains the case when the cash flows and/or the value process is evaluated at a stopping time. We find that the earlier result easliy also extends to this situation.

2 General definitions

Let $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t\in\mathbb{N}})$ be a complete filtered probability space. We will assume that $\mathscr{F}_0$ is the trivial σ-algebra augumented with all null sets and that $\mathscr{F}_\infty = \mathscr{F}$, where $\mathscr{F}_\infty = \bigvee_{t\geq 0} \mathscr{F}_t$. The fundamental objects are the cash flow process, the discount process and the value process. The two first of these process are used to to define the value proces. We will use the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$ and also the standard notations $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

2.1 The cash flow process

This subsection contains nothing but the defintion of the cash flow process. This is due to that we impose very mild restrictions.

Definition 2.1 A *cash flow process* $(C_t)_{t\in\mathbb{N}}$ is a process adapted to the filtration $(\mathscr{F}_t)$ and such that for each $t \in \mathbb{N}$, $|C_t| < \infty$ a.s. A cash flow process that is non-negative a.s. will be referred to as a *dividend process*.

2.2 The discount process

The discount process tells us how to discount future payments.

Definition 2.2 A *discount process* is a process $m : \mathbb{N} \times \mathbb{N} \times \Omega \to \mathbb{R}$ satisfying

(i) $0 < m(s,t) < \infty$ a.s. for every $s, t \in \mathbb{N}$,

(ii) $m(s,t,\omega)$ is $\mathscr{F}_{\max(s,t)}$-measurable for every $s, t \in \mathbb{N}$, and

(iii) $m(s,t) = m(s,u)m(u,t)$ a.s for every $s, u, t \in \mathbb{N}$.

A discount process fulfilling

(i′) $0 < m(s,t) \leq 1$ a.s. for every $s, t \in \mathbb{N}$

will be referred to as a *normal* discount process.

As a short hand notation we will write $m(t) \equiv m(0,t)$, $t \in \mathbb{N}$. Implied by the assumptions on the discount factors is the fact that $m(t,t) = 1$. This is seen by letting $s = u = t$ in (iii) together with the fact that $m > 0$ a.s. Now let $s < t$. We interpret $m(s,t)$ as the (stochastic) value at time s of getting one unit of currency at t, and analogously we interpret $m(s,t)$ as the growth of one unit of currency, invested at time t, at time s. In this latter case we should rather call m an accumulation factor. That we allow $m(s,t)$ with $s > t$ is because we want to incorporate insurance models into our framework. In life insurance applications we need to be able to both discount and accumulate cash flows. Condition (iii) in the definition could be seen as a consistency or no arbitrage condition, see Norberg [14]. The following two examples of discount factors are 'typical' (see Lemma 2.5 below).

Example 2.3 Let $r \in \mathbb{R}$. Then it is easy to verify that

$$m(s,t) = e^{-r(t-s)}$$

is a (deterministic) discount process. It is not difficult to see that m is a normal discount process if and only if $r \geq 0$. □

We can generalise this example to get a stochastic discount process.

Example 2.4 Let

$$m(s,t,\omega) = \exp\left(-\sum_{k=s+1}^{t} f_k\right),$$

with $(f_k)_{k\in\mathbb{N}}$ an adapted process that is finite a.s. As in the previous example it is immediate that m fulfills the requirements of a discount process. The requirement $f_k \geq 0$ a.s. will make m a normal discount process. □

Assumption (iii) in the definition of the discount process gives plenty of structure to it, as is seen in the following lemma.

Lemma 2.5 *Any discount process m can be written*

$$m(s,t) = \frac{\Lambda(t)}{\Lambda(s)}, \text{ a.s. for all } s,t \in \mathbb{N}, \tag{3}$$

where $(\Lambda(t))_{t\in\mathbb{N}}$ is an a.s. strictly positive and finite adapted process

Proof. We begin with the 'if' part. Obviously

$$m(s,u)m(u,t) = \frac{\Lambda(u)}{\Lambda(s)} \cdot \frac{\Lambda(t)}{\Lambda(u)} = \frac{\Lambda(t)}{\Lambda(s)} = m(s,t) \text{ a.s. for all } s,u,t \in \mathbb{N}.$$

The fact that $\Lambda(t) > 0$ a.s. implies that $m(s,t) > 0$ a.s. Since $\Lambda(t)$ is $\mathscr{F}_t$-measurable for every $t \in \mathbb{N}$, $m(s,t)$ will be $\mathscr{F}_{\max(s,t)}$-measurable for all s,t in $\mathbb{N}$. For the 'only if' part we begin by noting that since $m(0,t) > 0$ a.s. for every $t \in \mathbb{N}$ we have

$$m(0,t) = m(0,s)m(s,t) \text{ a.s. if and only if } m(s,t) = \frac{m(0,t)}{m(0,s)} \text{ a.s.}$$

Now let $\Lambda(t) := m(0,t)$. It is easily seen that this choice of $\Lambda(t)$ fulfills the desired requirements. □

The connection to Example 2.4 above becomes more transparent if we write (3) as

$$m(s,t) = \exp\left(-(\ln\Lambda_s - \ln\Lambda_t)\right) = \exp\left(-\sum_{k=s}^{t-1} \ln\frac{\Lambda_k}{\Lambda_{k+1}}\right).$$

The process Λ is known as a deflator. If it is the price of a traded asset, it is called a numeraire. In the theory of no arbitrage pricing one can show that the existence of a discount factor is equivalent to a condition ruling out arbitrage strategies. The exact condition is that the stock price process should satisfy the condition of 'no free lunch with bounded risk' (NFLBR). Intuitively this means that there is no possibility of having strategies such that the profit of the strategy can be arbitrarily large while the maximium loss of using the strategy is resticted to 1 monetary unit. The definition of (NFLBR) and the fact that it is equvivalent with the existence of an equivalent martingale measure is discussed in Schachermayer [18]. See also Section 4.C in Duffie [9] and Chapter 7 in Pliska [17] for no arbitrage pricing with an infinite discrete time horizon. Since the cash flows generated by a project or the claims in life insurance are typically not traded, we do not find it reasonable to model the value of a project or the claims as an ordinary financial asset. Thus conditions for the existence of a martingale measure are not a relevant question for us.

Definition 2.6 The *discount rate* or the *instantanous rate* at time t implied by the discount process, denoted $r(t)$ for $t = 1, 2, \ldots$, is defined as

$$r(t) = \frac{1}{m(t-1,t)} - 1 = \frac{m(t-1)}{m(t)} - 1 = \frac{\Lambda(t-1)}{\Lambda(t)} - 1, \ t = 1, 2, \ldots$$

where Λ is the deflator associated with m.

The advantage of using the instantanous rates, which uniquely determines the discount process, is that a requirement on the rates is often more easy to interpret economically than a requirement put on the discount process. The following lemma contains some facts relating the rate process and discount process.

Lemma 2.7 *Let m be a discount process and let r be the discount rate implied by m. Then the following holds:*

(i) $-1 < r(t) < \infty$, $t \in \mathbb{N}$

(ii) $r \geq 0$ *if and only if m is a normal discount process.*

(iii) *For any given $\lambda > 0$ we have for $t \in \mathbb{N}$*

$$0 < \lambda \leq r(t) \quad \textit{if and only if} \quad 0 < m(t) \leq e^{-t\ln(1+\lambda)}.$$

(iv) *The instantanuous rate process and the discount process uniquely determine each other.*

Proof. Facts (i) and (ii) are immediate from the defintion. To get (iii) we have the following implications for any $\lambda > 0$ and $t \in \mathbb{N}$:

$$\lambda \leq r(t) \quad \Rightarrow \quad \lambda \leq \frac{m(t-1)}{m(t)} - 1 \quad \Rightarrow \quad m(t) \leq \frac{1}{1+\lambda} m(t-1).$$

and Gerber [Ref!].Iterating this gives

$$m(t) \leq \left(\frac{1}{1+\lambda}\right)^t = e^{-t\ln(1+\lambda)}.$$

To go in the other direction we see that using the definition of r together with the fact that $\lambda > 0$ gives the desired result. For (iv) finally we see that given m the discount rate process r is determined uniquely. The opposite conclusion is clear from the following:

$$m(t,k) = \prod_{\ell=t+1}^{k} m(\ell-1,\ell) = \prod_{\ell=t+1}^{k} \frac{1}{1+r(\ell)}.$$

$\square$

3 Valuation

Definition 3.1 Given a cash flow process $(C_t)_{t\in\mathbb{N}}$ and a discount process $(m(s,t) : s,t \in \mathbb{N})$ we define for $t \in \mathbb{N}$ the *value process* as

$$V(t) = E\left[\sum_{k=t+1}^{\infty} C(k)m(t,k)\middle|\mathscr{F}_t\right].$$

The value process is defined *ex dividend*, meaning that we include cash flows from time $t+1$ and onwards in the value at time t. It would be possible to define it *cum dividend*, thus also including the cash-flow at time t, but since the *ex dividend* version is the most usual in financial texts we prefer it. See e.g. Campbell et al [3] or Cuthbertson [7] for more details on this issue.

Recall that the only conditions we have put on the cash flow process is that $|C_t| < \infty$ a.s. for $t \in \mathbb{N}$. A natural question to ask now is when the value process will be finite a.s. The following lemma offers a sufficient condition for this.

Lemma 3.2 *If C is a cash-flow process and $E\left[\left|\sum_{k=1}^{\infty} C_k m_k\right|\right] < \infty$ a.s. then $|V_t| < \infty$ a.s. for all $t \in \mathbb{N}$.*

Proof. Since $E\left[\left|\sum_{k=1}^{\infty} C_k m_k\right|\right] < \infty$, the following conditional expectations are well defined: For $t \in \mathbb{N}$

$$\begin{aligned}
|V_t| = \left|E\left[\sum_{k=t+1}^{\infty} C_k m(t,k)\middle|\mathscr{F}_t\right]\right| &\le \frac{1}{m(0,t)}E\left[\left|\sum_{k=1}^{\infty} C_k m(0,k) - \sum_{k=1}^{t} C_k m(0,k)\right|\middle|\mathscr{F}_t\right] \\
&\le \frac{1}{m_t}\left(E\left[\left|\sum_{k=1}^{\infty} C_k m_k\right|\middle|\mathscr{F}_t\right] + \left|\sum_{k=1}^{t} C_k m_k\right|\right) \\
&< \infty \text{ a.s.}
\end{aligned}$$

□

We immediately get the following corollary for a dividend process.

Corollary 3.3 *If C is a discount process such that $V_0 < \infty$, then $V_t < \infty$ a.s. for every $t \in \mathbb{N}$.*

Proof. Since $C_t \ge 0$ a.s. for every $t \in \mathbb{N}$ when C is a cash-flow process and $\mathscr{F}_0$ is the trivial σ-algebra augumented with the null sets

$$E\left[\left|\sum_{k=1}^{\infty} C_k m_k\right|\right] = E\left[\sum_{k=1}^{\infty} C_k m_k\right] = V_0 < \infty,$$

and the previous lemma applies. □

We now proceed by rewriting the value process. Note that since m_t is $\mathscr{F}_t$-measurable for all $t \in \mathbb{N}$ we have

$$V_t = E\left[\sum_{k=t+1}^{\infty} C_k m(t,k)\middle|\mathscr{F}_t\right] = \frac{1}{m_t}E\left[\sum_{k=t+1}^{\infty} C_k m_k\middle|\mathscr{F}_t\right].$$

By multiplying the expression for V_t by m_t we get

$$V_t m_t = E\left[\sum_{k=0}^{\infty} C_k m_k\middle|\mathscr{F}_t\right] - \sum_{k=0}^{t} C_k m_k.$$

Note that $V_t m_t$ is the value at time t discounted back to time 0. It is well known that if X is a random variable with $E|X| < \infty$ then $E[X|\mathscr{F}_t]$, $t = 1, 2, \ldots$ is a uniformly integrable (UI) martingale. Thus, if $E[|\sum_{k=0}^{\infty} C_k m_k|] < \infty$ then $E[\sum_{k=0}^{\infty} C_k m_k|\mathscr{F}_t]$ is a UI martingale. This and other facts chacterising the discounted value process are summarised in Proposition 3.5 below. Before its presentation we first recall the following result from Neveu ([13] p. 172). In this proposition, by an increasing process we mean a predictable sequence A of finite random variables such that $0 \leq A_0 \leq A_1 \leq \ldots$ a.s.

Proposition 3.4 *For every increasing process $(A_t)_{t\in\mathbb{N}}$ such that $E[A_\infty] < \infty$ a.s., the formula*

$$X_t = E[A_\infty|\mathscr{F}_t] - A_t, \quad t \in \mathbb{N}$$

defines a finite positive supermartingale $(X_t)_{t\in\mathbb{N}}$ which is called the potential of the increasing process A. This potential X determines the increasing process A uniquely.

For a finite positive supermartingale X to be the potential of an increasing process A such that $E[A_\infty|\mathscr{F}_n] <$ a.s., it is necessary and sufficient that

$$\lim_{n\to\infty} E[X_n] \downarrow 0 \quad a.s.$$

Proposition 3.5 *Let C and m be a cash flow and discount process respectively. If $E|\sum_{k=1}^{\infty} C_k m_k| < \infty$ then the discounted value process $(V_t m_t)$ can be written*

$$V_t m_t = M_t - A_t, \; t \in \mathbb{N},$$

where M is a UI martingale and A is an adapted process. Furthermore $\lim_{t\to\infty} V_t m_t = 0$ a.s. If the cash flow process is a dividend process, then Vm is the potential of the increasing process A. The decomposition given in the proposition is then the Riesz decomposition of a potential into a martingale and an increasing process.

Proof. We notice that $|\sum_{k=0}^{\infty} C_k m_k| < \infty$ a.s. since $E[|\sum_{k=0}^{\infty} C_k m_k|] < \infty$. Now let

$$\begin{aligned} M_t &= E\left[\sum_{k=1}^{\infty} C_k m_k \middle| \mathscr{F}_t\right], \; t \in \mathbb{N}, \\ A_t &= \sum_{k=1}^{t} C_k m_k, \; t \in \mathbb{N}. \end{aligned}$$

It is then immediate that $V_t m_t = M_t - A_t$. Since $E[|\sum_{k=1}^{\infty} C_k m_k|] < \infty$ M is a UI martingale and we see that A is adapted. We know that (Williams [19] p. 134) the UI martingale will converge to $E[\sum_{k=0}^{\infty} C_k m_k|\mathscr{F}_\infty] = \sum_{k=0}^{\infty} C_k m_k$ a.s. as $t \to \infty$. This yields

$$\lim_{t\to\infty} V_t m_t = \lim_{t\to\infty} M_t - \lim_{t\to\infty} A_t = 0,$$

since $M_\infty = A_\infty = \sum_{k=0}^{\infty} C_k m_k$ is finite a.s. Now let C be a dividend process. Then A is an increasing process and $E[A_\infty] = E[\sum_{k=0}^{\infty} C_k m_k] < \infty$ by assumption. Using Proposition 3.4 we see that

$$V_t m_t = E\left[\sum_{k=t+1}^{\infty} C_k m_k \middle| \mathscr{F}_t\right] = E[A_\infty|\mathscr{F}_t] - A_t.$$

is a potential. □

The following theorem characterises the relation between C, m and V in terms of their values and differences, giving three equivalent forms of defining the value process.

Theorem 3.6 *Let C be a cash flow process and m a discount process such that $E\left[\sum_{k=1}^{\infty} |C_k| m_k\right] < \infty$. Then the following three statements are equivalent.*

(i) *For every $t \in \mathbb{N}$*
$$V_t = E\left[\sum_{k=t+1}^{\infty} C_k m(t,k) \middle| \mathscr{F}_t\right].$$

(ii) (a) *For every $t \in \mathbb{N}$*
$$M_t = V_t m_t + \sum_{k=1}^{t} C_k m_k$$
is a UI martingale, and

(b) *$V_t m_t \to 0$ a.s. when $t \to \infty$.*

(iii) *For every $t \in \mathbb{N}$*

(a) *$V_t = E\left[m(t,t+1)(C_{t+1} + V_{t+1}) | \mathscr{F}_t\right]$, and*

(b) *$\lim_{T\to\infty} E\left[m(t,T)V_T | \mathscr{F}_t\right] = 0$.*

Proof. First of all we note that $E\left|\sum_{k=1}^{\infty} C_k m_k\right| \leq E\left[\sum_{k=1}^{\infty} |C_k| m_k\right] < \infty$, so $\left|\sum_{k=1}^{\infty} C_k m_k\right| < \infty$ a.s. We will show (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii)

$\boxed{\text{(i)} \Leftrightarrow \text{(ii):}}$ The 'if' part follows from Proposition 3.5. For the 'only if' part write the expression in (ii) (a) as $-m_{k+1}C_{k+1} = m_{k+1}V_{k+1} - m_k V_k - M_{k+1} + M_k$ and sum from t to $T-1$:
$$-\sum_{k=t+1}^{T} m_k C_k = m_T V_T - m_t V_t - M_T + M_t.$$
Letting $T \to \infty$ the term $m_T V_T \to 0$ a.s. by the assumption and $M_T \to M_\infty$ a.s. from the convergence result of UI martingales (Williams [19] p. 134). Thus we have
$$V_t m_t = \sum_{k=t+1}^{\infty} C_k m_k - M_\infty + M_t \text{ a.s.}$$
The convergence result concerning UI martingales also ensures the relation $E\left[M_\infty | \mathscr{F}_t\right] = M_t$ a.s. Taking conditional expectations with respect to $\mathscr{F}_t$ and using the definition of discount factors yields
$$V_t = E\left[\sum_{k=t+1}^{\infty} C_k m(t,k) \middle| \mathscr{F}_t\right].$$

$\boxed{\text{(i)} \Leftrightarrow \text{(iii):}}$ We begin with the 'if' part. Fix a $t \in \mathbb{N}$. We get
$$\begin{aligned}
V_t &= E\left[m(t,t+1)C_{t+1} + \sum_{k=t+2}^{\infty} C_k m(t,k) \middle| \mathscr{F}_t\right] \\
&= E\left[m(t,t+1)C_t + m(t,t+1)\sum_{k=t+2}^{\infty} C_k m(t+1,k) \middle| \mathscr{F}_t\right] \\
&= E\left[m(t,t+1)(C_{t+1} + V_{t+1}) | \mathscr{F}_t\right].
\end{aligned}$$

Now let $T \geq t$. From $V_T = E\left[\sum_{k=T+1}^{\infty} C_k m(T,k) | \mathscr{F}_T\right]$ we get

$$E\left[m(t,T)V_T|\mathscr{F}_t\right] = E\left[\sum_{k=T+1}^{\infty} C_k m(t,k) \middle| \mathscr{F}_t\right] = \frac{1}{m_t} E\left[\sum_{k=T+1}^{\infty} C_k m_k \middle| \mathscr{F}_t\right].$$

Since

$$\left|\sum_{k=T+1}^{\infty} C_k m_k\right| \leq \sum_{k=1}^{\infty} |C_k| m_k$$

and the last random variable is integrable by assumption we get, for every $t \in \mathbb{N}$ and $A \in \mathscr{F}_t$,

$$\lim_{T\to\infty} E\left[m(t,T)V_T \mathbf{1}_A\right] = E\left[\lim_{T\to\infty} m(t,T)V_T \mathbf{1}_A\right] = 0.$$

To prove the 'only if' part we iterate (iii) (a) to get

$$\begin{aligned} V_t &= E\left[\sum_{k=t+1}^{T} C_k m(t,k) + m(t,T)V_T \middle| \mathscr{F}_t\right] \\ &= \frac{1}{m_t} E\left[\sum_{k=t+1}^{T} C_k m_k \middle| \mathscr{F}_t\right] + E\left[m(t,T)V_T|\mathscr{F}_t\right]. \end{aligned}$$

When we let $T \to \infty$ the the last term tend to 0 a.s. from (iii) (b). Since

$$\left|\sum_{k=t+1}^{T} C_k m_k\right| \leq \sum_{k=1}^{\infty} |C_k| m_k$$

and the last random variable is integrable by assumption we get

$$V_t = E\left[\sum_{k=t+1}^{\infty} C_k m(t,k) \middle| \mathscr{F}_t\right]$$

by using the Theorem of Dominated Convergence. □

Remark 3.7 *We have written conditions (ii) (a) and (iii) (b) on the form in the theorem because of its convenient form. A more intuitive way of writing it, from an economical/financial point of view, would be to write condition (ii) (a) as*

$$\Delta V_t = r_t V_{t-1} - C_t + \frac{1}{m_t} \Delta M_t,$$

where $r_t = m_{t-1}/m_t - 1$ is the instantanuous rate, and Condition (iii) (a) as

$$V_t = E\left[\frac{V_{t+1} + C_{t+1}}{1 + r_{t+1}} \middle| \mathscr{F}_t\right].$$

Note that if m_t is predictable, then $(1/m_t)\Delta M_t$ is a martingale difference, and we have $E\left[\Delta V_t|\mathscr{F}_{t-1}\right] = r_t V_{t-1} - E\left[C_t|\mathscr{F}_t\right]$.

3.1 Exemples

We will now discuss well known relations from finance, economics and insurance where the use of Theorem 3.6 is needed. In these applications often some assumptions on the cash flows and/or the discount processes are usually made. Theorem 3.6 however shows that the reasoning can be made under quite mild assumptions.

It is a well-known fact from arbitrage pricing that the discounted gains process should be a martingale under an equivalent martingale measure In our setting the UI martingale M represents the discounted gains process. See Duffie [9] and Pliska [17] for theory and applications of no arbitrage pricing in discrete time.

If we define

$$L_t = M_t - M_{t-1} = V_t m_t - V_{t-1} m_{t-1} + C_t m_t,$$

then L will be a sequence of martingale differences and we will especially have $E[L_t L_s] = 0$ for all $s, t \in \mathbb{N}$. If the cash flows are interpreted as losses faced by an insurance company, then L_t is the discounted annual loss in the time period $(t-1, t]$. The fact that the discounted annual losses are uncorrelated is in life insurance known as Hattendorff's Theorem. See Papatriandafylou & Waters [16] for this result and more on the same theme. One the first to prove Hattendorff's Theorem using martingale methods seems to be Bühlmann [1]. We remark that in life insurance applications the value process V is known as the prospective reserve. The *value* at time t of a cash flow stream C of insurance claims is then defined to be

$$Q_t = \frac{1}{m_t} E\left[\sum_{k=1}^{\infty} C_k m_k \middle| \mathscr{F}_t\right] = \sum_{k=1}^{t} m(t,k) C_k + E\left[\sum_{k=t+1}^{\infty} C_k m_k \middle| \mathscr{F}_t\right] =: A_t + R_t,$$

where A_t is the accumulated payments and R_t is the prospective reserve (i.e. what we call the value process). See Bühlmann [2] and Bühlmann's contribution in [15].

In financial economics and econometrics models, the starting point is often the Relation (ii) (a) in Theorem 3.6. The return of a stock from time t to time $t+1$ is defined as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1,$$

where P_t and P_{t+1} is the price of the stock at time t and $t+1$ respectively and D_{t+1} is the dividend per share at $t+1$. Taking the conditional expectation with respect to $\mathscr{F}_t$ gives

$$P_t = E\left[\frac{P_{t+1} + D_{t+1}}{1 + R_{t+1}} \middle| \mathscr{F}_t\right]; \tag{4}$$

which is (ii) (a) with renamed processes. By iterating this we get

$$P_t = E\left[\sum_{k=t+1}^{T} D_k \prod_{\ell=t+1}^{k} \left(\frac{1}{1+R_\ell}\right) \middle| \mathscr{F}_t\right] + E\left[P_T \prod_{\ell=t+1}^{k} \left(\frac{1}{1+R_\ell}\right) \middle| \mathscr{F}_t\right].$$

To be able to write the stock price at time t as the disounted sum of all future dividends the second term in the equation above has to go to zero a.s. This condition, (ii) (b) in Theorem 3.6, is known as the transversality condition. Now let us look for solutions to Eq. (4), dropping all other assumptions on the behavior of the solution. In this case there is no longer a unique solution. Following Campbell et al [3] we call the solution with the transversality condition imposed P_D. Obviously this will be a solution even when we look for solutions only to (4). Now we have the following fact: Any solution P to Eq. (4) can be written

$$P_t = P_{Dt} + \frac{Z_t}{m_t}, \ t \in \mathbb{N},$$

where Z has the martingale property and $m_t = \prod_{k=1}^{t} \left(\frac{1}{1+R_k}\right)$. To see this, let P be any solution to (4). Then

$$P_t - P_{Dt} = E\left[\frac{P_t - P_{D(t+1)}}{1 + R_{t+1}} \middle| \mathscr{F}_t\right]$$

if and only if

$$(P_t - P_{Dt})m_t = E\left[\left(P_{t+1} - P_{D(t+1)}\right) m_{t+1}\middle|\mathscr{F}_t\right],$$

implying that $(P - P_D)m$ has the martingale property. The solution P_D is known as the fundamental value or the bubble free solution (since $B \equiv 0$ in this case) and Z/m is called a rational bubble. The process Z/m is called a bubble since its presence yields prices that are higher than the fundamental value, and it is 'rational' in the sense that it is not inconsistent with rational expectations. Campbell et al [3] and Cuthbertson [7] discuss rational bubbles from both a theoretical and empirical point of view.

Finally we mention the important subclass of Markov models. By assuming an underlying Markov process driving the cash flows and the discount rate the general formula for the value process can be further simplified. Much of this can be found and is commented on in Duffie [10]. There the close connection between Markov pricing and semigroups is pointed out. For the semigroup approach see also Garman [12] and references therein. See also the general texts in Duffie [9] and Pliska [17].

3.2 Asymptotic behavior of the value process

We know that $V_t m_t \to 0$ a.s., but what will happen to V_t when $t \to \infty$? We will present two results showing that V_t can, given some conditions, converge to 'almost anything' in ways which will be precised below. The essential assumption is that we have a strong law of large numbers for the sequence $\log(1 + r_t)$. Roughly this means that the discount process behaves like $m_t \sim e^{-\lambda t}$ for some $\lambda > 0$ when t is large.

Proposition 3.8 *Let C and m be a cash flow process and discount process respectively and let X be an integrable random variable. If*

(i) *There exists a constant $\lambda > 0$ such that*

$$\frac{1}{t}\sum_{k=1}^{t} \log(1 + r(k)) \to \lambda \ \textit{a.s. as } t \to \infty,$$

(ii) *$C_t \to X$ a.s. as $t \to \infty$, and*

(iii) *there exists an integrable random variable Z s.t. for all $t \geq 0$*

$$\left|C_t \frac{m_t}{e^{-\lambda t}}\right| \leq Z \ \textit{a.s.}$$

then

$$V_t \to \frac{e^{-\lambda}}{1 - e^{-\lambda}} X \ \textit{a.s. as } t \to \infty$$

Proof. First note that condition (i) above is equivalent to $m_t \to e^{-\lambda t}$ a.s. as $t \to \infty$.

$$\begin{aligned} V_t &= E\left[\sum_{k=t+1}^{\infty} C_k m(t,k)\middle|\mathscr{F}_t\right] = \frac{1}{m_t} E\left[\sum_{k=t+1}^{\infty} C_k m_k\middle|\mathscr{F}_t\right] \\ &= \frac{e^{-\lambda t}}{m_t} E\left[\sum_{k=1}^{\infty} e^{-\lambda k} C_{k+t} \frac{m_{k+t}}{e^{-\lambda(k+t)}}\middle|\mathscr{F}_t\right] \qquad (5) \end{aligned}$$

Now from (iii) above

$$\left|\sum_{k=1}^{\infty} e^{-\lambda k} C_{k+t} \frac{m_{k+t}}{e^{-\lambda(k+t)}}\right| \leq \sum_{k=1}^{\infty} e^{-\lambda k}\left|C_{k+t} \frac{m_{k+t}}{e^{-\lambda(k+t)}}\right| \leq Z \sum_{k=1}^{\infty} e^{-\lambda k} = Z \frac{e^{-\lambda}}{1-e^{-\lambda}}$$

implying that

$$E\left[\left|\sum_{k=1}^{\infty} e^{-\lambda k} C_{k+t} \frac{m_{k+t}}{e^{-\lambda(k+t)}}\right|\right] \leq E[Z] \frac{e^{-\lambda}}{1-e^{-\lambda}} < \infty$$

We now use the Dominated Convergence Theorem for conditional expectations (see for instance Durrett [11] p. 264). To do this, first note that

$$\lim_{t\to\infty} \sum_{k=1}^{\infty} e^{-\lambda k} C_{k+t} \frac{m_{k+t}}{e^{-\lambda(k+t)}} = \sum_{k=1}^{\infty} e^{-\lambda k} X = \frac{e^{-\lambda}}{1-e^{-\lambda}} X,$$

where we have used the Dominated Convergence Theorem. Now it follows from the theorem of dominated convergence for conditional expectations that when $t \to \infty$,

$$E\left[\sum_{k=1}^{\infty} e^{-\lambda k} C_{k+t} \frac{m_{k+t}}{e^{-\lambda(k+t)}} \middle| \mathscr{F}_t\right] \to E\left[\frac{e^{-\lambda}}{1-e^{-\lambda}} X \middle| \mathscr{F}_\infty\right] = \frac{e^{-\lambda}}{1-e^{-\lambda}} X \text{ a.s.}$$

Now let $t \to \infty$ in Eq. (5). Since $\frac{e^{-\lambda t}}{m_t} \to 1$ a.s. it follows that $V_t \to \frac{e^{-\lambda}}{1-e^{-\lambda}} X$ a.s. as $t \to \infty$, and the proposition is proved. □

Corollary 3.9 *Let C and m be a dividend and discount process respectively, and let X be an integrable random variable. If $0 \leq C_t \uparrow X$ a.s. as $t \to \infty$ and there exists a constant $\lambda > 0$ such that*

$$\frac{1}{t} \sum_{k=1}^{t} \log(1 + r(k)) \downarrow \lambda \text{ a.s. as } t \to \infty,$$

then $V_t \to X$ a.s. as $t \to \infty$.

Proof. That $\frac{1}{t} \sum_{k=1}^{t} (\log(1 + r(k))$ decreases to λ implies that $m_t \leq e^{-\lambda t}$ a.s. for all $t \in \mathbb{N}$. Thus, $\left|C_t \frac{m_t}{e^{-\lambda t}}\right| \leq X$, and since X is integrable the previous proposition applies. □

Proposition 3.8 has the unsatisfactory integrability condition (iii). The following result does not need this, but is on the other hand another kind of result. It says that given an integrable random variable X, there exists a cash flow process such that associated value processes converges to X a.s. Thus we can choose the cash flow process so that it will suit our purposes.

Proposition 3.10 *Let X be an integrable random variable. If there exists a constant $\lambda > 0$ such that*

$$\frac{1}{t} \sum_{k=1}^{t} \log(1 + r(k)) \to \lambda \text{ a.s. as } t \to \infty$$

then there exists a cash-flow process such that

$$V_t \to X \quad \text{a.s.}$$

Proof. Take $\lambda > 0$ such that $m_t \to e^{-\lambda t}$ a.s. and fix $t \geq 0$. For $k \geq t$ let

$$C_k = \frac{E\left[X|\mathscr{F}_k\right]e^{-\lambda k}(1-e^{-\lambda})}{m(k)e^{-\lambda}}.$$

Now,

$$\begin{aligned}
V_t &= E\left[\sum_{k=t+1}^{\infty} \frac{E\left[X|\mathscr{F}_k\right]e^{-\lambda k}(1-e^{-\lambda})}{m(k)e^{-\lambda}} m(t,k)\middle|\mathscr{F}_t\right] \\
&= \frac{1-e^{-\lambda}}{m(t)e^{-\lambda}} E\left[\sum_{k=1}^{\infty} E\left[X|\mathscr{F}_{t+k}\right]e^{-\lambda(t+k)}\middle|\mathscr{F}_t\right] \\
&= \frac{1-e^{-\lambda}}{m(t)e^{-\lambda}} E\left[X|\mathscr{F}_t\right]\sum_{k=1}^{\infty} e^{-\lambda(t+k)} \\
&= \frac{e^{-\lambda t}}{m(t)} E\left[X|\mathscr{F}_t\right] \to X \ a.s.
\end{aligned}$$

as $t \to \infty$ since $\frac{e^{-\lambda t}}{m(t)} \to 1$ a.s. and $E\left[X|\mathscr{F}_t\right] \to E\left[X|\mathscr{F}_\infty\right] = X$ a.s. when $t \to \infty$. The interchange of summation and conditional expectation is justified by the Fubini theorem. To see this first note that for $A \in \mathscr{F}_t$

$$\begin{aligned}
E\left|\mathbf{1}_A E\left[X|\mathscr{F}_{t+k}\right]\right| &= E\left[E\left[\mathbf{1}_A|E\left[X|\mathscr{F}_{t+k}\right]|\middle|\mathscr{F}_t\right]\right] \\
&= E\left[\mathbf{1}_A E\left[|E\left[X|\mathscr{F}_{t+k}\right]|\middle|\mathscr{F}_t\right]\right] \\
&\leq E\left[\mathbf{1}_A E\left[E\left[|X|\middle|\mathscr{F}_{t+k}\right]\middle|\mathscr{F}_t\right]\right] \\
&= E\left[\mathbf{1}_A E\left[|X|\middle|\mathscr{F}_t\right]\right].
\end{aligned}$$

Thus for any $A \in \mathscr{F}_t$ we get

$$\begin{aligned}
E\left[\sum_{k=0}^{\infty}\left|\mathbf{1}_A E\left[X|\mathscr{F}_{t+k}\right]e^{-\lambda(t+k)}\right|\right] &= \sum_{k=0}^{\infty} E\left|\mathbf{1}_A E\left[X|\mathscr{F}_{t+k}\right]\right| e^{-\lambda(t+k)} \\
&\leq \sum_{k=0}^{\infty} e^{-\lambda(t+k)} E\left[\mathbf{1}_A E\left[|X|\middle|\mathscr{F}_t\right]\right] \\
&= e^{-\lambda t} E\left[\mathbf{1}_A E\left[|X|\middle|\mathscr{F}_t\right]\right]\sum_{k=0}^{\infty} e^{-\lambda k} \\
&= \frac{e^{-\lambda t}}{1-e^{-\lambda}} E\left[\mathbf{1}_A E\left[|X|\middle|\mathscr{F}_t\right]\right] < \infty,
\end{aligned}$$

which justifies the interchange of expectation and summation. $\square$

3.3 Stopping the cash flow and value process

Theorem 3.6 on the three equivalent representations of the value process concerns the value at deterministic times. It also assumes that the cash flow stream is defined for all $t \geq 0$. In some cases we would like to consider the value at a stopping time and/or the cash flow process stopped at some stopping time. Before we proceed we recall the definition and some basic facts regarding stopping times, see e.g. Durrett [11], or Neveu [13] for more on stopping times. By utilising the fact that the martingale $M_t = V_t m_t + \sum_{k=1}^{t} C_k m_k$ from Theorem 3.6 is uniformly integrable we can get the following result.

Proposition 3.11 *Let C be a cash flow process and let m be a discount process such that $E\left[\sum_{k=t+1}^{\infty} |C_k| m(t,k)\right] < \infty$ for every $t \in \mathbb{N}$. Further let τ and σ be $(\mathscr{F}_t)$-stopping times such that $\sigma \leq \tau$ a.s. Then the following two statements are equivalent*

(i) *We have*

$$V_\sigma = E\left[\sum_{k=\sigma+1}^{\tau} C_k m(\sigma,k) + V_\tau m(\sigma,\tau)\mathbf{1}_{\tau<\infty} \middle| \mathscr{F}_\sigma\right] \quad \text{on } \{\sigma < \infty\}.$$

(ii) (a) *For every $t \in \mathbb{N}$*

$$M_t = V_t m_t + \sum_{k=1}^{t} C_k m_k$$

is a UI martingale, and

(b) $V_t m_t \to 0$ *a.s. when* $t \to \infty$.

Proof. We begin with the implication (ii) $\Rightarrow$ (i). The stopping time τ may be unbounded so we consider the stopping times $\tau \wedge n$, where $n \in \mathbb{N}$. We get

$$M_{\tau\wedge n} = V_{\tau\wedge n} m_{\tau\wedge n} + \sum_{k=1}^{\tau\wedge n} C_k m_k. \tag{6}$$

Now,

$$V_{\tau\wedge n} m_{\tau\wedge n} \xrightarrow{\text{a.s.}} V_\tau m_\tau \mathbf{1}\tau < \infty,$$

as $n \to \infty$ since $V_n m_n \mathbf{1}_{\tau=\infty} \to 0$ a.s. By letting $n \to \infty$ in Equation (6) we thus get

$$M_\tau = \sum_{k=1}^{\tau} C_k m_k + V_\tau m_\tau \mathbf{1}_{\tau<\infty}.$$

Since M is uniformly integrable we can take the conditional expectation of M_τ with respect to the σ-algebra $\mathscr{F}_\sigma$ to get on $\{\sigma < \infty\}$

$$\begin{aligned}
\sum_{k=1}^{\sigma} C_k m_k + V_\sigma m_\sigma &= M_\sigma = E\left[M_\tau | \mathscr{F}_\sigma\right] \\
&= E\left[\sum_{k=1}^{\tau} C_k m_k + V_\tau m_\tau \mathbf{1}_{\tau<\infty} \middle| \mathscr{F}_\sigma\right] \\
&= \sum_{k=1}^{\sigma} C_k m_k + E\left[\sum_{k=\sigma+1}^{\tau} C_k m_k + V_\tau m_\tau \mathbf{1}_{\tau<\infty} \middle| \mathscr{F}_\sigma\right].
\end{aligned}$$

Since $|\sum_{k=1}^{\sigma} C_k m_k| \leq |\sum_{k=1}^{\infty} C_k m_k| < \infty$ a.s. we can cancel the sum $\sum_{k=1}^{\sigma} C_k m_k$ from both sides. Dividing by m_σ gives the desired result. To prove (i) $\Rightarrow$ (ii) we let

$\tau = \infty$ and $\sigma = t$, for $t \in \mathbb{N}$. We are now back to Theorem 3.6 and the proof found there. $\square$

We know from Theorem 3.6 that (ii) in the previous proposition is equivalent to the fact that the value process has the form $V_t = E\left[\sum_{k=t+1}^{\infty} m(t,k)C_k \middle| \mathscr{F}_t\right]$. Thus if we replace the infinite horizon and the time t with two stopping times, we still have the equivalences of Theorem 3.6. We finally remark that the stopping times τ and σ may be unbounded. For τ this is necessary since we want to generalise the infinite horizon by replacing it with a stopping time.

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On the Valuation of Cash Flows – Continuous Time Models

Fredrik Armerin

Abstract

Valuation models where the value at a time is defined as the expected discounted value of a stream of cash flows are considered. We establish three equivalent formulations of this value process, each of which has its own merits. When considering Brownian models, it is possible to write the value process as a solution to a forward-backward stochastic differential equation. Applications include real options and the question of recovering cash flows from a given value process.

1 Introduction

When an individual or firm is faced with a stream of future cash flows the immediate question is: What is the present value of these cash flows? The natural way to value the cash flows is to discount them using some suitable discount rate and then sum them up. If the cash flows and/or the discount rates are stochastic we also have to take expectations. See Brealey & Myers [4] for the basics on valuation of cash flows Copeland et al [7] for an introduction to corporate valuation. In life insurance the prospective reserve is the discounted value of future cash flows. Martin-Löf [19] and Norberg [21] discuss properties of the reserves (prospective and retrospective), and Norberg [20], with applications to insurance in mind, gives an axomatic approach to valuation. Norberg [22] gives a general introduction to life insurance. In the approach of no arbitrage pricing it is a well known fact that absence of arbitrage will imply the existence of an equivalent martingale measure under which the expectations are to be taken. The discount rate in this case should then be taken as the risk-free rate. Björk [3] and Duffie [10] are standard text books and Delbaen & Schachermayer [8] presents the general theory when the stock prices are semimartingales. If there is no capital market generating the cash flows we can not rely on no arbitrage pricing and we have to choose some probabilities together with a risk-adjusted rate to try to value the cash flows. It could even be that different individuals have different perceptions of the probability laws ruling the cash flows and the discount rates. Whatever route we take, the same structure applies: the value is an expected sum of the discounted cash flows. Recently the theory of real options has gained interest in the valuation problems. The idea is to identify an emedded option in the investment and adding this value to the net present value (calculated as descibred above). There are two general ways of doing the modelling underlying the real option valuation. Either one models the value directly, or one models the cash flows generating the value and then uses the this calculated value as the underlying process in the option valuation. In the latter case we need to understand how the dynamics of the cash flow process influence the dynamics of the value process. We will approach this problem as an application in the Brownian models treated below. In Dixit & Pindyck [9] many examples of the theory of real options are presented, while Copeland & Antikarov [6] focuse more on how to apply the theory in practice. Schachermayer & Hubalek [14] discuss the connection of real options to the theory of no arbitrage pricing.

Although one could argue that the cash flows arrive at discrete times, in this paper we choose to work in continuous time. The advantage of this approach is that we can rely on the stochastic calculus of semimartingales, and especially on the Itô-diffusion models. Assume that a firm is facing the (stochastic) cashflows $(C_t)_{t\geq 0}$. We define the value at time t as

$$V_t = E_t\left[\int_t^\infty C_s e^{-r(s-t)} ds\right],$$

where $E_t[\cdot]$ denotes that the expectations should be taken with respect to all known information up to time t, and r is some constant discount rate. We rewrite this expression as

$$V_t e^{-rt} = E_t\left[\int_0^\infty C_s e^{-rs} ds\right] - \int_0^t C_s e^{-rs} ds.$$

If we assume that $E\left|\int_0^\infty C_s e^{-rs} ds\right| < \infty$, then this is a decompostion where the discounted present value is the sum of a uniformly martingale and a predictable process. If we denote the martingale by M_t, it is easy to see that the dynamics of the present value V_t is given by

$$dV_t = (rV_t - C_t)dt + e^{rt} dM_t.$$

That is, the value process V_t is also a semimartingale. We can also follow Cochrane [5], who offers the following heuristic analysis. Define $\Lambda_t = e^{-rt}$, which we call a deflator, and start with

$$V_t\Lambda_t = E_t\left[\int_t^\infty C_s\Lambda_s ds\right] = E_t\left[\int_t^{t+h} C_s\Lambda_s ds + V_{t+h}\Lambda_{t+h}\right].$$

Moving $V_t\Lambda_t$ to the RHS and letting $h \downarrow 0$ we get $0 = C_t\Lambda_t dt + E_t\left[d(V_t\Lambda_t)\right]$. The idea when introducing Λ_t is of course to allow for more general discount factors, especially stochastic ones. We also want to generalise the cash flows, allowing processes of finite variation as integrators with which we integrate the deflator.

The problem of valuation, defined as determining the value process, has connections to forward–backward stochastic differential equations (FBSDE), especially to the so called Black's consol rate conjecture. This conjucture is about the relation between the value of a bond and the disount rate. The price (value) Y_t of the bond is assumed to be $Y_t = E\left[\int_t^\infty e^{-\int_t^s r_u du} ds \middle| \mathscr{F}_t\right]$ and the rate (in this context called consol rate) is modelled as $dr_t = \mu(r_t, Y_t)dt + \alpha(r_t, Y_t)dB_t$. The question in the consol rate problem is if, given the dynamics of the underlying rate, it is always possible to find a diffusion term of the price of the bond that is consistent with the dynamics of the rate. This problem was solved by Duffie et al [11] by using FBSDE techniques. The idea is to write the value process on differential form and then using the Martingale Representation Theorem. It can be shown that the consol rate problem can be formulated as follows: Find a solution (X_t, Y_t, Z_t) to the following system of equations:

$$\begin{cases} dX_t &= b(X_t, Y_t)dt + \sigma(X_t, Y_t)dB_t, \ t \in [0, \infty) \\ dY_t &= (h(X_t)Y_t - 1)dt - Z_t dB_t \ t \in [0, \infty), \\ X(0) &= x_0, \\ Y_t & \quad \text{bounded a.s., uniformly in } t \in [0, \infty). \end{cases}$$

B is here a Brownian motion. We show that the general valuation problem, in the Brownian model, can equivalently be written as an FBSDE.

The rest of the paper is organized as follows. In Section 2 we precise what we mean by a cash flow process and deflator. Section 3 contains the definition and the basic properties of the valuation process. We show that there exists three equivalent forms on which we can state that the value process is generated by the cash flows and deflator as discussed above. Some applications are then discussed. Finally, Section 4 contains the case of Brownian models, where we focus on two questions. Firstly the connection the valuation problem has to FBSDE. Secondly we investigate how the dynamics of the cash flow process and the dynamics of the value process depend on each other. This is then applied to real options. This is a continuation of the paper Armerin [1] where discrete time models are discussed. Sections 2 and 3 has counterparts in discrete time models, see Armerin [1].

2 Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space equipped with a filtration $(\mathscr{F}_t)_{t\geq 0}$. The filtration is assumed to be right continuous and complete. Any adapted process will be adapted with respect to this filtration. We also let $\mathscr{F}_\infty$ denote the σ-algebra $\bigvee_{t\geq 0}\mathscr{F}_t$, and assume that $\mathscr{F}_\infty = \mathscr{F}$. We say that a process is cadlag if almost every sample path of the process is right continuous with left limits.

By an increasing process we mean a process which paths a.s. are positive, increasing and right continuous. An increasing function has left limits, and thus any increasing process is cadlag. We will use the convention $A_{0-} = 0$ a.s. for every increasing process A, but we do not require that $A_0 = 0$ a.s. A process is a finite variation process (or an FV process) if it is cadlag and adapted and if almost every sample path is of finite variation on each compact subset of $[0, \infty)$. A process X is said to be optional if the mapping $X : [0, \infty)\times\Omega \to \mathbb{R}$ is measurable when $[0, \infty)\times\Omega$ is given the optional σ-algebra. Since the optional σ-algebra is generated by the family of all adapted process which are cadlag (Elliot [13], Theorem 6.35) every FV process A is optional. If A is an FV process and X is a real valued process on $[0, \infty) \times \Omega$ that is $\mathscr{B} \times \mathscr{F}$-measurable (here $\mathscr{B}$ denotes the Borel σ-algebra on $[0, \infty)$) we define the Stieltjes integral

$$(X \cdot A)_t(\omega) = \int_{[0,t]} X_s(\omega)dA_s(\omega), \tag{1}$$

whenever it exists. If $X \cdot A$ exists for all $t \in [0, \infty)$ and almost all $\omega \in \Omega$ then $(X \cdot A)_t$ defines a process with $(X \cdot A)_0 = X_0A_0$. If X is optional then there is an optional version of $(X \cdot A)$. If A is an FV process then $|A|_t(\omega) = \int_0^t |dA_s(\omega)|$ denotes the total variation of A. $|A|$ is adapted and cadlag and $|A|_0 = |A_0|$. If A is an FV process and X is a measurable process then the integral $\int_0^t X_s(\omega)|dA_s(\omega)|$ denotes integration with respect to $d|A|$. A semimartingale is an adapted and cadlag process $(X_t)_{t\geq 0}$ having a decomposition $X_t = X_0 + M_t + A_t$, where M is a local martingale and A is an FV process.[1] Assuming that the involved processes are semimartingales is often general enough. Norberg [22] argues that the generality needed is often gained if we make the stronger assumption that the cash flow process is piecewise differentiable. A property is said to hold piecewise if it holds everywhere except possibly at a finite number of points in every finite interval. Thus, if the set of jump points is not empty it must be on the form $\{t_0, t_1, \ldots\}$ with $t_0 < t_1 < \ldots$, and in the case it is infinite, $\lim_{j\to\infty} t_j = \infty$. If the piecewise continuous process X, defined on $[0, \infty)$, can be written

$$X_t(\omega) = X_t^c(\omega) + X_t^d(\omega) = \int_0^t x_s(\omega)ds + \sum_{0<s\leq t} \left[X_s(\omega) - X_{s-}(\omega)\right], \tag{2}$$

where x is a piecewise continuous process, then X is also piecewise differentiable. The integral $\int_0^t$ is interpreted as $\int_{(0,t]}$. Eq. (2) can equivalently be written on differential form:

$$dX_t(\omega) = x_t(\omega)dt + X_t(\omega) - X_{t-}(\omega).$$

2.1 Cash flows and deflators

Definition 2.1 A *cash-flow process* $(C_t(\omega))_{t\geq 0}$ is an FV process.

[1] We prefer to use the 'classical' definition of semimartingales. There are several equivalent defintions in the literature, see Protter [24], Chapter III, Theorem 1.

This definition of the cash flow process makes it trivially a semimartingale. We can also use it as an integrator, thus making it possible to define processes of the type as in Eq. (1).

Definition 2.2 A *deflator* is a strictly positive semimartingale that is finite a.s.

This is a generalisation of the definition of Duffie [10], who defines a deflator to be a strictly positive Itô process. The reason for demanding the deflator to be a semimartingale, and not a more general process, is that we want to use the differentiation rule valid for semimartingales. We note here that if Λ is a deflator then both $1/\Lambda$ and $\ln \Lambda$ are well defined, and since $1/x$ and $\ln x$ are twice continuously differentiable on $(0, \infty)$, both $1/\Lambda$ and $\ln \Lambda$ are semimartingales (this follows from Theorem 32 of Chapter II in Protter [24]).

Definition 2.3 Given a deflator Λ, the *discount process* implied by Λ is defined by

$$m(s,t) = \frac{\Lambda(t)}{\Lambda(s)}, \quad s, t \geq 0.$$

The following proposition, which proof is an immediate consequence of the definition of deflator, presents some important properties of the discount process. In Armerin [1] the properties of m proved in the following proposition were taken as the definition of the discount process. The reason for this change is that the definition given in Armerin [1] is the more natural one and works well in dicrete time. In continuous time, however, it is easier to work with the deflator as the defining object.

Proposition 2.4 *Let m be a discount process implied by the deflator Λ. Then*

(i) *$m(s,t,\omega)$ is $\mathscr{F}_{\max(s,t)}$-measurable for every $s, t \in [0, \infty)$.*

(ii) *$0 < m(s,t) < \infty$ a.s. for every $s, t \in [0, \infty)$.*

(iii) *$m(s,t) = m(s,u)m(u,t)$ a.s for every $0 \leq s \leq u \leq t$.*

A discount process fulfilling $0 < m(s,t) \leq 1$ a.s. for every $s, t \in [0, \infty)$ will be referred to as a *normal* discount process. We see that m is normal if and only if Λ is nondecreasing.

Proposition 2.5 *A discount process m, with deflator Λ, can be written in the form*

$$m(s,t,\omega) = \exp\left(- \int_s^t \lambda(u,\omega) du \right)$$

if and only if $\ln \Lambda(t,\omega)$ *is absolutely continuous in t for almost every $\omega \in \Omega$, with density $-\lambda(t,\omega)$.*

Proof. Since $\ln \Lambda(t,\omega)$ is absolutely continuous if and only if it can be written

$$\ln \Lambda(t,\omega) = \ln \Lambda(s,\omega) - \int_s^t \lambda(u,\omega) du,$$

the proposition follows from the defining identity $\Lambda(t,\omega) = \Lambda(s,\omega) m(t,s,\omega)$. □

3 Valuation

Definition 3.1 Given a cash flow process C and a deflator Λ such that $E\left[\int_{[0,\infty)} \Lambda_s |dC_s|\right] < \infty$, the *value process* is defined for $t \in [0,\infty)$ as

$$V_t = \frac{1}{\Lambda_t} E\left[\int_{(t,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t\right].$$

By noting that $\int_{[0,\infty)} = \int_{[0,t]} + \int_{(t,\infty)}$ and using the fact that every optional process is adapted (Jacod & Shiryaev [15], Proposition 1.21) we get

$$V_t \Lambda_t = E\left[\int_{[0,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t\right] - \int_{[0,t]} \Lambda_s dC_s = M_t - (\Lambda \cdot C)_t. \tag{3}$$

Since

$$E|M_t| = E\left|E\left[\int_{[0,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t\right]\right| \leq E\left[\int_{[0,\infty)} \Lambda_s |dC_s|\right] < \infty$$

for every $t \in [0,\infty)$, $M_t = E\left[\int_{[0,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t\right]$ is a uniformly integrable martingale. The filtration $(\mathscr{F}_t)$ is right continuous, thus there exists a modification of M that is right continuous. We also remark here that $M_\infty = \lim_{t\to\infty} M_t = \int_0^\infty \Lambda dC_s$ a.s. These facts immediately follow from Elliot [13], Theorem 4.11. Now, since both M and $\Lambda \cdot C$ are right continuous and adapted the value process is also continuous and adapted. From this it follows that the value process is optional. Eq. (3) implies that

$$V_t = \frac{M_t}{\Lambda_t} - \frac{(\Lambda \cdot C)_t}{\Lambda_t}.$$

Since M is a (true) martingale it is especially a semimartingale. $C \cdot \Lambda$ is a process of finite variation, and is thus also a semimartingale. Since Λ is a strictly positive semimartingale it follows that $1/\Lambda$ is a stricly positive semimartingale, and since the product of two semimartingales is again a semimartingale we see finally that V is a semimartingale. We remark here the fact that Delbaen & Schachermayer [8] show that it is reasonable to model the price process of an financial asset as a semimartingale. Norberg [21] defines the prospective reserve of a life insurance company in the same way as we have defined the value here. In life insurance the *value* at time t of a cash flow stream is defined as, using our definitions, $E\left[\frac{1}{\Lambda_t}\int_{[0,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t\right]$. For more on the reserves in life insurance see Norberg [21] and references therein and Norberg [22].

As in the discrete time case (Armerin [1], Theorem 3.6) there exist three equivalent representations of the value processes. The only general assumptions made here are measurability conditions on C and Λ and the integrability condition making M into a uniformly integrable martingale. We also need a condition essentially stating that the discounted value goes to zero as t tends to infinity (see the discussion in Armerin [1] on rational bubbles, what happens if we disgard this condition).

Theorem 3.2 *Let C and Λ be a cash flow process and a deflator respectively, such that $E\left[\int_{[0,\infty)} \Lambda_s |dC_s|\right] < \infty$. Then the following three statements are equivalent.*

(i) *For every $t \in [0,\infty)$*

$$V_t = \frac{1}{\Lambda_t} E\left[\int_{(t,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t\right]. \tag{4}$$

(ii) (a) *For every* $t \in [0, \infty)$

$$M_t = V_t \Lambda_t + \int_{[0,t]} \Lambda_s dC_s \tag{5}$$

is a uniformly integrable martingale, and

(b) $V_t \Lambda_t \to 0$ *a.s. when* $t \to \infty$.

(iii) *For each* $t \in [0, \infty)$ *we have*

(a) *For every* $h > 0$

$$V_t \Lambda_t = E\left[V_{t+h}\Lambda_{t+h} + \int_{(t,t+h]} \Lambda_s dC_s \middle| \mathscr{F}_t \right], \tag{6}$$

and

(b) $\lim_{T \to \infty} E\left[V_{t+T}\Lambda_{t+T} | \mathscr{F}_t\right] = 0$.

Proof. We will show (i)⇔(ii) and (i)⇔(iii).

$\boxed{\text{(i)}\Leftrightarrow\text{(ii):}}$ To prove the 'if' part we rewrite Eq. (4) as $M_t = V_t\Lambda_t + \int_{[0,t]} \Lambda_s dC_s$, $t \in [0, \infty)$. We know from above that M is a uniformly integrable martingale, and using this together with the fact that $M_t \xrightarrow{a.s.} M_\infty = \int_0^\infty \Lambda_s dC_s$ gives

$$\lim_{t\to\infty} V_t\Lambda_t = \lim_{t\to\infty} \left(M_t - \int_{[0,t]} \Lambda_s dC_s \right) = 0 \text{ a.s.}$$

Turning to the 'only if' part we let $t \to \infty$ in Eq. (5). Using (ii) (b) we get $M_\infty = \int_{[0,\infty)} \Lambda dC_s$. Taking the conditional expectation with respect to the σ-algebra $\mathscr{F}_t$ we get

$$\begin{aligned} V_t\Lambda_t + \int_{[0,t]} \Lambda_s dC_s &= M_t = E\left[M_\infty | \mathscr{F}_t\right] \\ &= E\left[\int_{[0,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t \right] \\ &= \int_{[0,t]} \Lambda_s dC_s + E\left[\int_{(t,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t \right]. \end{aligned}$$

Rearranging this relation gives the desired result.

$\boxed{\text{(i)}\Leftrightarrow\text{(iii):}}$ For the 'if' part take $h > 0$. We get, using Eq. (4),

$$\begin{aligned} V_t\Lambda_t &= E\left[\int_{(t,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t \right] \\ &= E\left[\int_{(t,t+h]} \Lambda_s dC_s + \int_{(t+h,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_t \right] \\ &= E\left[\int_{(t,t+h]} \Lambda_s dC_s + E\left[\int_{(t+h,\infty)} \Lambda_s dC_s \middle| \mathscr{F}_{t+h} \right] \middle| \mathscr{F}_t \right] \\ &= E\left[\int_{(t,t+h]} \Lambda_s dC_s + \Lambda_{t+h}V_{t+h} \middle| \mathscr{F}_t \right]. \end{aligned}$$

Since

$$\left|\int_{(t,\infty)} \Lambda_s dC_s\right| \le \int_{(t,\infty)} |\Lambda_s||dC_s| \le \int_{[0,\infty)} |\Lambda_s||dC_s|$$

and $\int_{[0,\infty)} |\Lambda_s||dC_s|$ is integrable, we use the Dominated Convergence Theorem to get for every $A \in \mathscr{F}_t$

$$\lim_{T\to\infty} E\left[V_{t+T}\Lambda_{t+T}\mathbf{1}_A\right] = \lim_{T\to\infty} E\left[\int_{(t+T,\infty)} \Lambda_s dC_s \mathbf{1}_A\right] = E\left[\lim_{T\to\infty}\int_{(t+T,\infty)} \Lambda_s dC_s \mathbf{1}_A\right] = 0,$$

where the last equality follows from the fact that $\int_{[0,\infty)} \Lambda_s dC_s$ is finite a.s. To prove the other direction of the equivalence we let $T \to \infty$ in Eq. (6):

$$\begin{aligned} V_t\Lambda_t &= \lim_{T\to\infty} E\left[V_{t+T}\Lambda_{t+T}|\mathscr{F}_t\right] + \lim_{T\to\infty} E\left[\int_{(t,t+T]} \Lambda_s dC_s \middle| \mathscr{F}_t\right] \\ &= \lim_{T\to\infty} E\left[\int_{(t,t+T]} \Lambda_s dC_s \middle| \mathscr{F}_t\right]. \end{aligned}$$

Again we used the Dominated Convergence Theorem to interchange the limit and the expectation to get the desired conclusion. □

3.1 Stopping Times

It is not difficult to see that Theorem 3.2 can be generalised to allow also for stopping times. The content of the following theorem is that we can strenghten the results of Theorem 3.2 by replacing both the infinite horizon and the time of valuation with a stopping time. For the proof we essentially only need to use the Theorem of Optional Stopping for uniformly intgrable martingales.

Theorem 3.3 *Let C and Λ be a cash flow process and a deflator respectively, and such that $E\left[\int_{[0,\infty)} \Lambda_s|dC_s|\right] < \infty$. Then the following two statements are equivalent.*

(i) *For all stopping times σ and τ such that $0 \le \sigma \le \tau$ a.s.*

$$V_\sigma = \frac{1}{\Lambda_\sigma} E\left[V_\tau \mathbf{1}_{\tau<\infty} + \int_{(\sigma,\tau]} \Lambda_s dC_s \middle| \mathscr{F}_\sigma\right] \quad \textit{on } \{\sigma < \infty\}.$$

(ii) (a) *For every $t \in [0,\infty)$*

$$M_t = V_t\Lambda_t + \int_{[0,t]} \Lambda_s dC_s$$

is a uniformly integrable martingale, and

(b) *$V_t\Lambda_t \to 0$ a.s. when $t \to \infty$.*

Proof. We first show (ii)⇒(i). Take $n \in \mathbb{N}$, then

$$\begin{aligned} M_{\tau\wedge n} &= V_{\tau\wedge n}\Lambda_{\tau\wedge n} + \int_{[0,\tau\wedge n]} \Lambda_s dC_s \\ &\xrightarrow{a.s.} V_\tau\Lambda_\tau\mathbf{1}_{\tau<\infty} + \int_{[0,\tau]} \Lambda_s dC_s \text{ as } n \to \infty \end{aligned}$$

From this and the Theorem of Optional Stopping we get

$$V_\sigma \Lambda_\sigma \mathbf{1}_{\sigma<\infty} + \int_{[0,\sigma]} \Lambda_s dC_s = M_\sigma = E\left[M_\tau | \mathscr{F}_\sigma\right] = E\left[V_\tau \Lambda_\tau \mathbf{1}_{\tau<\infty} + \int_{[0,\tau]} \Lambda_s dC_s \,\middle|\, \mathscr{F}_\sigma \right].$$

Using the fact that $\int_{[0,\sigma]} \Lambda_s dC_s$ is finite a.s. and measurable with respect to $\mathscr{F}_\sigma$ yields the desired result. To show (i)⇒(ii) we let $\tau = \infty$ and $\sigma = t$ and then use the proof of Theorem 3.2. □

3.2 On the local dynamics of the value process

In this section we will comment on the local behavior of the value process. The starting point is the relation

$$V_t \Lambda_t = M_t - (\Lambda \cdot C)_t.$$

Since all the processes in this expression are semimartingales we can use the differentiation rule for products of semimartingales (Protter [24], Chapter II, Corollary 2) to get

$$d(V_t \Lambda_t) = V_{t-} d\Lambda_t + \Lambda_{t-} dV_t + d[V, \Lambda]_t = dM_t - \Lambda_t dC_t. \tag{7}$$

To increase the economical interpretation of Eq. (7) note that if we have a cash flow given by $r_t^f dt$ for $t \in [0, \infty)$, where r_t^f is measurable and adapted and such that for a.e. $\omega \in \Omega$ we have $0 \le r_t^f(\omega)$ for every $t \in [0, \infty)$, and if the value process of this cash flow stream fulfills $V_t \equiv 1$, then we can think of r_t^f as a locally risk-free interest rate. Inserting this into Eq. (7) yields the relation

$$r_t^f dt = -\frac{d\Lambda_t}{\Lambda_t} + \frac{1}{\Lambda_t} d\widetilde{M}_t,$$

where $\widetilde{M}$ is a martingale. Thus, if the deflator Λ assigns the cash flow stream given by $r_t^f dt$ the value 1 for all $t \in [0, \infty)$, then, in using the same Λ for valuing another cash flow stream C, we can express the differential of V in terms of the risk-free rate. Assuming that Λ is a continuous process and V is continuous and strictly positive we can write

$$\frac{dV_t}{V_t} = -\frac{d\Lambda_t}{\Lambda_t} - \frac{1}{V_t} dC_t + \frac{1}{V_t \Lambda_t} dM_t - \frac{d[V, \Lambda]_t}{V_t \Lambda_t},$$

or, replacing $-d\Lambda_t / \Lambda_t$ by $r_t^f dt + (1/\Lambda_t) d\widetilde{M}_t$,

$$\frac{dV_t}{V_t} = r_t^f dt - \frac{1}{V_t} dC_t + \frac{1}{V_t \Lambda_t} dM_t - \frac{1}{\Lambda_t} d\widetilde{M}_t - \frac{d[V, \Lambda]_t}{V_t \Lambda_t}.$$

In this case we can rewrite the last equation as

$$\frac{dV_t + dC_t}{V_t} = r_t^f dt + \frac{1}{V_t \Lambda_t} dM_t - \frac{1}{\Lambda_t} d\widetilde{M}_t - \frac{d[V, \Lambda]_t}{V_t \Lambda_t}.$$

The left hand side of this equation is the instantaneous net return of the value process at t. Taking expectations conditioned on $\mathscr{F}_t$ and also write $d[V, \Lambda]_t = dV_t d\Lambda_t$ this equations can be written

$$E\left[\frac{dV_t + dC_t}{V_t} \,\middle|\, \mathscr{F}_t \right] = r_t^f dt - E\left[\frac{dV_t}{V_t} \frac{d\Lambda_t}{\Lambda_t} \,\middle|\, \mathscr{F}_t \right].$$

We have now decompsed the expected return of the value process into two parts: the risk-free part ($r_t^f dt$) and a risk premium $\left(-E\left[\frac{dV_t}{V_t}\frac{d\Lambda_t}{\Lambda_t}\middle|\mathscr{F}_t\right]\right)$. Thus if $\frac{dV_t}{V_t}$ and $\frac{d\Lambda_t}{\Lambda_t}$ are negatively correlated there is a positive risk premium, and if they are positively correlated the risk premium is negative. Since we expect the value of a risky investment giving us the cash flow c to have a return strictly greater than the risk-free rate, we see that we expect dV_t/V_t and $-d\Lambda_t/\Lambda_t$ to be positively correlated. The intuition is that a risky investment is desirable if its value is high in 'bad' states of the world (when we really need money) and low in 'good' states of the economy (when everything else is good), where we think of an element ω of the sample space Ω as a state of the world. An investment with such properties will have a high price (since demand for this desirable investment opportunity is high), and thus a low expected return. This allows for the interpretation of $d\Lambda_t/\Lambda_t$ as a measure of how 'bad' a state of the economy is. See Cochran ([5] Section 1.5 and Part III) for more on this type of asset pricing in continuous time.

4 Brownian models

We will from now on assume that the cash flow process and deflator both are driven by a (possibly multi-dimensional) Brownian motion. The model we use consists of a time-homogenuous Itô diffusion X representing some state(s) that influence the cash flows and the discount factors. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let B be an n-dimensional Brownian motion on this space. We will let $(\mathscr{F}_t)$ denote the standard Brownian filtration generated by B augumented with all null sets of $\mathscr{F}$. We will also assume that $\mathscr{F}_0$ is the trivial σ-algebra (with the null sets of $\mathscr{F}$) and that $\mathscr{F}_\infty = \mathscr{F}$. For $i = 1, \ldots, d$ and $j = 1, \ldots, n$ let b_i and σ_{ij} be Borel measurable functions from $[0, \infty) \times \mathbb{R}^d$ into $\mathbb{R}$. We write $b(t,x) = [b_i(t,x)]_{1 \le i \le d}$ and $\sigma(t,x) = [\sigma_{ij}(t,x)]_{1 \le i \le d,\ 1 \le j \le n}$ for the vector of b_i's and matrix of σ_{ij}'s respectively. We let X be given by the SDE

$$\left\{ \begin{array}{rcl} dX_t & = & b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 & = & \xi, \end{array} \right. \tag{8}$$

with ξ being a random variable independent of the Brownian motion B and with finite second moment: $E\|\xi\|^2 < \infty$. If b and σ fulfill

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le K\|x - y\|$$

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \le K^2(1 + \|x\|^2)$$

where $K > 0$ is a given constant, then it is well known (see e.g. Karatzas & Shreve [17], Theorem 5.2.9) that the SDE (8) posesses a unique strong solution. We remark that

$$\|b(t,x)\|^2 = \sum_{i=1}^{d} b_i^2(t,x) \quad \text{and} \quad \|\sigma(t,x)\|^2 = \sum_{i=1}^{d} \sum_{j=1}^{n} \sigma_{ij}^2(t,x).$$

To return to the valuation problem, the general model in this Brownian framework can be written

$$\left\{ \begin{array}{rcl} dX_t & = & b(t, X_t, C_t, \Lambda_t)dt + \sigma(t, X_t, C_t, \Lambda_t)dB_t;\ Y_0 = y \\ dC_t & = & \mu_C(t, X_t, C_t, \Lambda_t)dt + \sigma_C(t, X_t, C_t, \Lambda_t)dB_t;\ C_0 = c \\ d\Lambda_t & = & \alpha(t, X_t, C_t, \Lambda_t)\Lambda_t dt + \beta(t, X_t, C_t, \Lambda_t)\Lambda_t dB_t; \Lambda_0 = \gamma, \end{array} \right.$$

The process Y is a external process influencing the cash flows and the deflator. It could be macro economical (e.g. inflation, GDP or some exchange rate) or it could be a firm specific variable (e.g. the level of knowledge among the workers of the firm or a measure of progress in the R&D department of the firm). The drifts and diffusions are assumed to be so nice that the system of equations possesses a strong solution and such that $\Lambda > 0$ a.s. We often make simplifying assumptions, specifically we almost always assume that the cash flow process and the deflator are the only processes, and that they are independent from each other.

4.1 The value process as a solution to an FBSDE

The aim of this section is to show the close connection between the value process and a class of forward–backward stochastic differential equations (FBSDE). We begin by motivating why one should study backward stochastic differential equations (BSDE). Consider the problem of finding adapted solutions to equations of the type

$$\left\{ \begin{array}{rcl} dY_t & = & -f(t, Y_t)dt,\ 0 \le t < T, \\ Y_T & = & \xi, \end{array} \right.$$

where $T > 0$ is a fixed time and $\xi \in L^2(\Omega, \mathscr{F}_T)$. If $f \equiv 0$ then $Y_t = \xi$, $0 \le t \le T$, satisfies the equation but it is not adapted to the filtration. The idea is that since

ξ is square integrable, if we additionally assume that the filtration is generated by some Brownian motion, then $Y_t = E[\xi|\mathscr{F}_t]$, $0 \le t \le T$, satisfies the terminal condition, and since it is a martingale it can be represented as $Y_t = Y_0 + \int_0^t Z_s dB_s$ for some a.s. unique adapted and square integrable Z. This Y satisfies for $0 \le t \le T$

$$\begin{cases} dY_t &= Z_t dB_t \\ Y_T &= \xi. \end{cases}$$

Note that if $\xi \in \mathbb{D}^{1,2}$ (see Nualart [23] for a definition of this space) then ξ has a Malliavin derivative $D_t\xi$, $t \in [0,T]$, and we have $Z_t = E[D_t\xi|\mathscr{F}_t]$ by the Clark-Ocone formula (Nualart [23] Proposition 1.3.5). We now define a solution to this problem as a pair (Y,Z) of adapted processes. We have thus been able to find an adapted solution, not to our original problem, but to a similar one. It has been shown that this is the 'right' way to do it, see Ma & Yong [18] Chapter 1. Generally we want solve equations on the form

$$\begin{cases} dY_t &= -f(t,Y_t,Z_t)dt + Z_t dB_t, \\ Y_T &= \xi, \end{cases}$$

where B is a d-dimensional Brownian motion, $\xi \in L^2(\Omega, \mathscr{F}_T)$ and $T > 0$ is a fixed time (note that Z also is allowed to be included in f). It has turned out that there is a variety of problems that be formulated in the context of BSDE; see e.g. Ma & Yong [18] and references therein for the theory and applications, and El Karoui et al [12] and El Karoui [2] for applications to finance. The extension to forward-backward stochastic differential equations (FBSDE) is done by introducing another state variable X moving 'forward':

$$\begin{cases} dX_t &= b(t,X_t,Y_t,Z_t)dt + \sigma(t,X_t,Y_t,Z_t)dB_t, \\ dY_t &= h(t,X_t,Y_t,Z_t)dt + Z_t dB_t, \\ X(0) &= x, \\ Y(T) &= g(X(T)). \end{cases}$$

For the technical assumptions on b, σ and h see Ma & Yong [18]. We will be interested in the case when the time horizon is infinite. There is no immediate generalisation of the above equation for this case, but Ma & Yong propose the additional requirement that Y be bounded a.s. uniformly in $t \in [0,\infty)$. See also Duffie et al [11] for an application of the infinite horizon case. We will now show that there is an equivalent formulation of the definition of the value process in the form of an FBSDE. Let B be an n-dimensional Brownian motion and let $b : [0,\infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $\sigma : [0,\infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be Borel measurable function. Consider the following problems:

(P1) Find a pair of adapted, locally square integrable processes (X,V) such that for $t \in [0,\infty)$

$$\begin{cases} dX_t &= b(t,X_t,V_t)dt + \sigma(t,X_t,V_t)dB_t \\ V_t &= E\left[\int_t^\infty \exp\left(-\int_t^s \lambda(u,X_u,V_u)du\right) g(s,X_s,V_s)ds \middle| \mathscr{F}_t\right] \\ X_0 &= x, \end{cases}$$

(P2) Find a triplet (X,V,Z) of adapted process such that

$$\begin{cases} dX_t &= b(t,X_t,V_t)dt + \sigma(t,X_t,V_t)dB_t \\ dV_t &= [\lambda(t,X_t,V_t)V_t - g(t,X_t,V_t)]\,dt + Z_t dB_t \\ X_0 &= x, \\ E\left[V_T \exp\left(-\int_0^T \lambda(s,X_s,V_s)ds\right) \middle| \mathscr{F}_t\right] \to 0 \text{ a.s. as} T \to \infty \text{ for every } t \in [0,\infty). \end{cases}$$

Here X is the external process influencing the cash flows and the deflator, given by

$$\begin{aligned} dC_t &= g(t, X_t, V_t)dt \\ d\Lambda_t &= -\lambda(t, X_t, V_t)\Lambda_t dt,\ \Lambda_0 = 1, \end{aligned}$$

where V is the value process. We now precise what we mean by an adapted solution to the FBSDE **(P2)**. To begin with we let $L^2(C([0,T];\mathbb{R}^n))$ denote the set of $(\mathscr{F}_t)$-progressively measurable continuous processes X taking values in $\mathbb{R}^n$ such that $E\left[\sup_{t\in[0,T]} \|X(t)\|^2\right] < \infty$, and let $L^2(0,T;\mathbb{R}^n)$ denote the set of $(\mathscr{F}_t)$-progessively measurable processes X taking values in R^n and such that $\int_0^T E\left[\|X(t)\|^2\right] dt < \infty$. Following Ma & Yong [18] we say that (X,Y,Z) is an adapted solution to **(P2)** if $(X,Y,Z)\big|_{[0,T]} \in L^2(C([0,T];R^d)) \times L^2(C([0,T];R)) \times L^2(0,T;R^n)$. The following theorem shows the equivalence between **(P1)** and **(P2)**. It is a generalisation of Theorem 3.1 in Chapter 8 in Ma & Yong [18].

Theorem 4.1 *Assume that*

(i)

$$\inf_{(t,x,y)\in[0,\infty)\times\mathbb{R}^n\times\mathbb{R}} \lambda(t,x,y) = \delta > 0 \quad \textit{and} \quad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^n\times\mathbb{R}} \lambda(t,x,y) = \varepsilon < \infty$$

(ii) *There exists constants $K > 0$ and $0 \le \delta_1 < \delta$ such that*

$$|g(t,x,y)| \le K e^{\delta_1 t} \textit{ for every } (t,x,y) \in [0,\infty)\times\mathbb{R}^n\times\mathbb{R}$$

Under these assumptions if (X,V,Z) is an adapted solution to **(P2)**, *then (X,V) is an adapted solution to* **(P1)**. *Conversely, if (X,V) is an adapted solution to* **(P1)**, *then there exists an adapted, $\mathbb{R}^n$-valued square integrable process Z such that (X,V,Z) is an adapted solution to* **(P2)**.

Proof. To prove the first statement we assume that (X,V,Z) is an adapted solution to **(P2)** and fix a $t > 0$. Using the integration by parts formula and the property of the solution to **(P2)** we get for every $T \ge t$

$$V_T\Lambda_T = V_t\Lambda_t - \int_t^T \Lambda_s g_s ds + \int_t^T \Lambda_s Z_s dB_s. \tag{9}$$

We note that the process $(\Lambda_t Z_t)_{t\in[0,\infty)}$ is measurable and adapted, and fulfills

$$\int_0^T E\left[\Lambda_t^2 Z_t^2\right] dt \le e^{-2\delta T} \int_0^T E\left[Z_t^2\right] dt < \infty.$$

Taking conditional expectations with respect to $\mathscr{F}_t$ of Eq. (9) we get

$$E\left[V_T\Lambda_T|\mathscr{F}_t\right] = V_t\Lambda_t - E\left[\int_t^T \Lambda_s g_s ds \,\middle|\, \mathscr{F}_t\right]. \tag{10}$$

Now,

$$\begin{aligned} \left|\int_t^T \Lambda_s g_s ds\right| &\le \int_t^T |\Lambda_s g_s| ds \le \int_t^T K e^{-(\delta-\delta_1)s} ds \\ &= \frac{K}{\delta-\delta_1}\left[e^{-(\delta-\delta_1)t} - e^{-(\delta-\delta_1)T}\right] \le \frac{K}{\delta-\delta_1} \end{aligned}$$

so letting $T \to \infty$ in Eq. (10) and using the Bounded Convergence Theorem yields the desired conclusion.

For the other direction assume that (X, V) is an adapted solution to problem **(P1)**. We get

$$V_t \Lambda_t = E\left[\int_0^\infty \Lambda_s g_s ds \middle| \mathscr{F}_t \right] - \int_0^t \Lambda_s g_s ds. \tag{11}$$

From assumptions (i) and (ii) it follows that

$$\begin{aligned}\left(\int_0^\infty \Lambda_t g_t dt \right)^2 &\leq \left(\int_0^\infty \Lambda_t |g_t| dt \right)^2 \leq \left(\int_0^\infty K e^{-(\delta - \delta_1)t} dt \right)^2 \\ &= \frac{K^2}{(\delta - \delta_1)^2} < \infty \text{ a.s.,}\end{aligned}$$

thus the first processes on the right of Eq. (11) is a square integrable martingale. Defining it as M we thus have $V_t = M_t / \Lambda_t - \int_0^t \Lambda_s g_s ds / \Lambda_t$. Integrating by parts yields

$$\begin{aligned} dV_t &= \frac{1}{\Lambda_t} dM_t - \frac{M}{\Lambda_t^2} d\Lambda_t - \frac{\Lambda_t g_t}{\Lambda_t} dt + \frac{1}{\Lambda_t^2} \left(\int_0^t \Lambda_s g_s ds \right) d\Lambda_t \\ &= \{\lambda_t V_t - g_t\} dt + \frac{1}{\Lambda_t} dM_t. \end{aligned}$$

The Martingale Representation Theorem (see e.g. Theorem 4.3.4 in Øksendal [25]) implies that there exists an a.s. unique stochastic process $\varphi(s, \omega)$ such that for every $t \in [0, \infty)$ $M_t = M_0 + \int_0^t \varphi(s, \omega) dB_s$ and $E\left[\int_0^t \varphi^2(s, \omega) ds \right] < \infty$. Defining $Z_t = \varphi_t / \Lambda_t$ we see that

$$dV_t = \{\lambda_t V_t - g_t\} dt + Z_t dB_t.$$

Since X and V are locally square integrable and adapted, and

$$\int_0^T E\left[Z_t^2 \right] dt \leq e^{2\varepsilon T} \int_0^T E\left[\varphi_t^2 \right] dt < \infty$$

for every $T > 0$, (X, V, Z) is an adapted solution to **(P2)**. We only have to check that $E\left[V_T \Lambda_T | \mathscr{F}_t \right] \to 0$ a.s. For this purpose fix a $T > t$. The relation $V_t \Lambda_t = M_t - \int_0^t \Lambda_s g_s ds$ gives

$$E\left[V_T \Lambda_T | \mathscr{F}_t \right] = V_t \Lambda_t - E\left[\int_t^T \Lambda_s g_s ds \middle| \mathscr{F}_t \right].$$

Letting $T \to \infty$ and again using the Theorem of Dominated Convergence, the right hand side converges to 0 a.s. □

4.2 Recovering cash flows from their value process

It is obvious that a given cash flow process uniquely determines a value process. In this section we want to answer the opposite question: Given a value process V, does there always exist a cash flow process generating this process, and if it exists is it unique? To simplify we will make the following assumption.

Assumption 4.2 The instantanous rate is constant, i.e. the deflator is given by $\Lambda_t = e^{-\lambda t}$ for some $\lambda \in \mathbb{R}$, and the cash flow process is assumed to be absolutely continuous: $dC_t = c_t dt$. The process c is assumed to be an Itô diffusion such that $E\left[\left(\int_0^\infty e^{-\lambda t} c_t dt \right)^2 \right] < \infty$.

Given these assumptions, $M_t = E\left[\int_0^\infty e^{-\lambda s} c_s ds \middle| \mathscr{F}_t\right]$ is a square integrable martingale and from the Martingale Representation Theorem we know that there exists a measurable and adapted process Z fulfilling $E\left[\int_0^t Z_s^2 ds\right] < \infty$ for $t \in [0, \infty)$ such that $M_t = M_0 + \int_0^t Z_s dB_s$. The dynamics of the value process V under these assumptions is given by

$$dV_t = (\lambda V_t - c_t)dt + Z_t dB_t.$$

Let $\mu(x)$ and $\sigma(x)$ be two functions such that $dV_t = \mu(V_t)dt + \sigma(V_t)dB_t$ possesses a strong solution. Now, given μ and σ we see that in order for the value process dynamics to be consistent with the dynamics of the value process, in terms of the cash flow process we must have

$$\begin{cases} \mu(V_t) &= \lambda V_t - c_t \\ \sigma(V_t) &= Z_t. \end{cases}$$

Thus the cash flow process generating this value process must fulfill

$$c_t = \lambda_t V_t - \mu(V_t). \tag{12}$$

If we can invert this relation (i.e. expressing the value process as a function of the cash flow process) then the program is obvious: Apply Itô's lemma to Eq. (12) and then replace V_t by the function of c_t. Of course this demands μ to be regular enough to be an Itô diffusion, and we have to be able to invert the function $f(x) := \lambda x - \mu(x)$. In guaranteeing that the function f is invertible we can demand either $\mu'(x) > \lambda$ or $\mu'(x) < \lambda$. The first case, however, will not be interesting as is seen from the following example. Take $\alpha > 0$ and let $\mu(x) = \alpha x$ and $\sigma(x) = \sqrt{\alpha} x$. Further let λ be such that $0 < \lambda < \alpha/2$. Then $\mu'(x) - \lambda > 0$ for all $x \in \mathbb{R}$, but

$$V_t \Lambda_t = V_0 \exp\left(\left[\frac{1}{2}\alpha - \lambda\right] t + \sqrt{\alpha} B_t\right),$$

which tends to infinity a.s. as $t \to \infty$, contradicting the fact that is should converge to 0 a.s.

Theorem 4.3 *Let $\mu \in C^2(\mathbb{R};\mathbb{R})$ and let $\sigma : \mathbb{R} \to (0, \infty)$ be such that*

$$dV_t = \mu(V_t)dt + \sigma(V_t)dB_t; \quad V_0 = v \tag{13}$$

possesses a strong solution. If there exists a constant λ such that $\mu'(x) < \lambda$, then there exists a unique cash flow process generating the value process via $V_t = E\left[\int_t^\infty c_s e^{-\lambda(t-s)} ds \middle| \mathscr{F}_t\right]$. Further, letting $I(x)$ denote the inverse function of $\lambda x - \mu(x)$, the cash flow process generating V has the dynamics

$$dc_t = \left[(\lambda - \mu'(I(c_t)) \cdot \mu(I(c_t)) - \frac{1}{2}\mu''(I(c_t))\sigma^2(I(c_t))\right] dt \tag{14}$$

$$+(\lambda - \mu'(I(c_t))) \cdot \sigma(I(c_t))dB_t. \tag{15}$$

Proof. We know from the earlier disussion that $c_t = \lambda V_t - \mu(V_t)$. Introduce again $f(x) = \lambda x - \mu(x)$. Since $f'(x) = \lambda - \mu'(x)$ it follows from the assumptions that $f'(x) > 0$ for every $x \in \mathbb{R}$. The expression for the dynamics of the cash flow process follows immediately from an application of Itô's lemma. $\square$

Example 4.4 Assume that we want the value process to have a linear drift term: $dV_t = (a+bV_t)dt+\sigma(V_t)dB_t$. We let σ be unspecified so far. Using Theorem 4.3 we see that the cash flow process producing this drift must be $c_t = (\lambda - b)V_t - a$. Since the derivative of the drift term is b, we see that if we let λ be any constant discount rate strictly greater than b, then the conditions of the theorem are fulfilled. The dynamics of c becomes

$$dc_t = \{a(\lambda - b) + b(a + c_t)\}\,dt + (\lambda - b)\sigma\left(\frac{a + c_t}{\lambda - b}\right)dB_t.$$

□

Two of the most commonly used diffusions are the geometric Brownian motion and the mean reverting Ornstein-Uhlenbeck process.[2] Together with having a lognormal distribution and being strictly positive the first one also has very nice computationally properties. The Ornstein-Uhlenbeck process is the only stationary Gaussian process.

Proposition 4.5 *Let $\lambda > 0$ be a constant discount rate. If the derivative of the drift term (which in both cases below is constant) is strictly less than λ, then the following holds:*

(a) A value process is a geometric Brownian motion if and only if the cash flow process generating it is a geometric Brownian motion.

(b) A value process is an Ornstein-Uhlenbeck process if and only if the cash flow process generating it is an Ornstein-Uhlenbeck process.

Proof. For part (a) assume that c has dynamics $dc_t = \alpha c_t dt + \sigma c_t dB_t$, where $\alpha < \lambda$ and $\sigma > 0$. Then $V_t = c_t/(\lambda - \alpha)$ and $dV_t = \alpha V_t dt + \sigma V_t dB_t$. For the other direction we assume that V_t has dynamics given by $dV_t = \alpha V_t dt + \sigma V_t dB_t$, where again $\alpha < \lambda$ and $\sigma > 0$. The drift condition implies that $c_t = (\lambda - \alpha)V_t$, and we are finished with part (a). For (b) let a, b and σ be strictly positive real numbers and assume that the cash flow process solves the Ornstein-Uhlenbeck SDE $dc_t = a(b - c_t)dt + \sigma dB_t$. (To be precise is this an Ornstein-Uhlenbeck only when $b = 0$.) It is well known that the solution to this equation can be written

$$c_t = c_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) + \sigma \int_s^t e^{-a(t-u)} dB_u.$$

Since

$$E\left[c_s|\mathscr{F}_t\right] = b + e^{-a(t-s)}(c_s - b)$$

for $0 \le t \le t$ we have

$$V_t = E\left[\int_t^\infty e^{-\lambda(s-t)} c_s ds \middle| \mathscr{F}_t\right] = \frac{b}{\lambda} + \frac{c_t - b}{\lambda - a}.$$

and from this

$$dV_t = a\left(\frac{b}{\lambda} - V_t\right)dt + \frac{\sigma}{\lambda - a} dB_t.$$

Now assume that we want V to be an Ornstein-Uhlenbeck process; specifically assume that the drift of V is given by $a(b - V_t)$. The cash flow process has to fulfill $c_t = \lambda V_t - ab + aV_t$, implying that

$$dc_t = a(b\lambda - c_t)dt + \sigma(\lambda + a)dB_t.$$

[2]When we say that X is an Ornstein-Uhlenbeck process we mean that X satisfies the SDE $dX_t = a(b - X_t)dt + \sigma dB_t$ for some constants a, b and $\sigma > 0$. Strictly speaking, X is only an Ornstein-Uhlenbeck process if it has $b = 0$.

□

One of the consequences of that the cash flows must follow a geometric Brownian motion if we want the value process to do so, is that if we want to model the stock price as a geometric Brownian motion and we believe that a discounted cash flow model give the value of the stock, then the cash flows must also follow a geometric Brownian motion. Thus the cash flows of the firm must be strictly positive, a fact that is not reasonable to assume for all firms. On the other hand we could argue that the value of the firm should be the discounted value of the dividends, and since dividends are always non-negative, we could model them as a geometric Brownian motion.

4.3 Applications to real options

We end this section on Brownian models with some examples on how the methods described earlier can be specifically applied to problems arising in the valuation of real options. The idea of real options is that added to the net present value (represented by the value process as specified here) there should be a value coming from some implict option. A typical example is the case when we own a gold mine. Suppose that the gold price is as low that it is not profitable (in the sense that the value process at this instant is negative) to keep it running. There is, however, a possibility that the gold price will increase in the future, and it is possible that it eventually will become profitable (i.e. the value process becomes positive) to use the mine. Thus, we can see the mine as an option with the gold price as the underlying asset, and as any option it has a value even though it is not presently in the money. The value added to the mine in this case is the value of waiting to invest. If we have a 'no-or-never' choice to make today to deside if should close the mine down or let it run, we should of course (still assuming that the value process today is negative) shut the mine down. Examples where there exists an embedded real option are many, ranging from investment timing (when should an irrevsible inventment be done), valuation, entering and exiting markets, sequential investments (often an investment is done in stages with a possibility of interupting after the first stage if it is no longer profitable, the search for a new drug at a pharmaceutical company is a typical example) and real estate (where unexploited land can increase in value if the rents increase and/or the cost of construction decrease to purely noneconomic applications such as marriage (there is a value of waiting to marry to see if a better candidate might appear) and legal reforms (since there is a cost present, both monetary and socially, when some laws are changed, there is a value in waiting to see what the opinion among the voters is). These examples (see Dixit & Pindyck [9]) serve to show that the area to which we may apply real options is indeed vast. In the rest of this section we will focus on the problem of valuation.

There are two main routes to take when modelling the value of a project. Either we directly model the diffusion (we will assume henceforth that every stochastic process occuring in the rest of this section is a time-homogenous Itô diffusion) the value process follows, or we model the cash flow process generating the value process and then derive the properties of the value process from the cash flows.

Example 4.6 When modelling the value process directly, the geometric Brownian motion is often used. As was pointed out above it guarantees among other things that the value process is strictly positive. To expand the model the following mean-reverting models are sometimes used

- $dV_t = \eta(b - V_t)V_t dt + \sigma V_t dB_t$,

- $dV_t = \eta(b - \ln V_t)V_t dt + \sigma V_t dB_t$, and
- $dV_t = \eta(b - V_t)dt + \sigma V_t dB_t$

(the second equation describes the dynamics of an exponential Ornstein-Uhlenbeck pricess; see Sick [16] for these models.) Since $V_t > 0$ a.s. for $t \geq 0$ the cash flow process generating this process must also be strictly positive. But is it reasonable to assume that the cash flows always are positive? On the other hand must it always hold that the value process is strictly positive? We will now, using the theory presented above, find out how the cash flow process has to look like if we want a mean-reverting drift of the value process. Again we assume that the discount rate is a positive constant λ, and that we want V to look like $dV_t = a(b - V_t)dt + \sigma(V_t)dB_t$, were we wait to specify σ. We thus want to solve $\lambda V_t - c_t = a(b - V_t)$, a and b being positive constants, yielding $c_t = (\lambda + a)V_t - ab$. From this do we get

$$dc_t = a(b\lambda - c_t)dt + (\lambda + a)\sigma\left(\frac{c_t + ab}{\lambda + a}\right)dB_t.$$

Now c can take both positive and negative values. Assuming that we still want V to be strictly positive, take a continuous function $g : \mathbb{R} \to (0, \infty)$ such that $g(0) = 0$ and let $\sigma(V_t) = g(V_t)$. As an explicit example we can take $g(x) = x^\gamma$, $\gamma > 0$, yielding

$$dc_t = a(b\lambda - c_t)dt + (\lambda + a)^{1-\gamma}(c_t + ab)^\gamma dB_t.$$

□

In some cases it could be desirable to allow for the value process to take negative values. Either we let the value process continue, disregarding simply whether it is positive or negative, or we could start the cash flow process, letting it generate the value process, but consider the project bancrupt if $V_t = 0$ for some t.

Another application to real options of the relation between the cash flow process and the value process concerns estimation of the volatility. Assuming that a cash flow process is driving the value process (and not assuming the value process itself as the underlying object) we have to be able to estimate the diffusion term of c. The problem is that the cash flow process is often not directly observable; what we observe is the value process. As an example we could think of a pharmaceutical company which for its survival is dependent on the success of a new drug. Research on this drug is still done, and it is not certain it will be good enough (from an economical and/or medical point of view). We could try to value this company using real options, in which case the dynamics of the cash flow process is needed. But what we observe, as was said earlier, is the value process. We can assume that the stock price of this company is equal to value process, use the time series of the stock price to estimate parameters for V and then, using the relation $\mu(V_t) - \lambda V_t = c_t$, estimate the parameters for c.

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