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I. INTRODUCTION

In Ref. 3 a first-order phase-integral solution of the radial Dirac equation was given. The important properties of that approximation are that no singularities appears except at the transition points, that the Dirac current is exactly conserved, and the the approximate wavefunctions have the correct behaviour at the origin.

In the present paper the asymptotic expansion is pursued until the third order of approximation. It is found that an approximate wavefunction satisfying all the above-mentioned conditions can be constructed. Error estimates, exactly analogous to the basic estimates for the Fröman- Fröman phase-integral approximations, are derived. Hence virtually all the phase-integral theory of Schrödinger-like equations can be taken over for radial Dirac problems. Specifically, connection formulas across a simple turning point are given along with the phase shift formula for single-turning-point problems and the quantization condition for single well problem are given in this work.

A. The radial Dirac equation

The Dirac equation for a particle interacting with a static, spherically symmetric, field can be written

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta + V)\psi = E\psi, \quad (1)$$

where V is the interaction term. Disregarding parity breaking terms, the most general form of the potential V is

$$V(r) = \beta V_s(r) + V_v(r) + i\beta\gamma_5\sigma_r V_t(r), \quad (2)$$

where V_s is a Lorentz scalar, V_v is the time component of a Lorentz vector, and V_t defines an antisymmetric tensor with components

$$\{V_t^{\mu\nu}\} = \frac{V_t(r)}{2r} \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

For given values of the quantum numbers j and m as well as of parity, two linearly independent spinor-valued angular eigenfunctions $\psi_{\kappa,m}^{\pm}$ exist. Here the non-zero integer κ gives

both the j -value as

$$j = |\kappa| - \frac{1}{2}. \quad (4)$$

and the parity according to

$$\psi_{\kappa,m}^{\pm}(-\mathbf{r}) = (-1)^{\kappa+1} \text{sgn}(\kappa) \beta \psi_{\kappa,m}^{\pm}(\mathbf{r}). \quad (5)$$

The choice of the two eigenfunctions is in principle arbitrary, but most common is to use eigenfunctions of the β operator so that

$$\beta \psi_{\kappa,m}^{\pm} = \pm \psi_{\kappa,m}^{\pm}. \quad (6)$$

Solutions of (1) are asserted on the form

$$\psi(\mathbf{r}) = iF_{\kappa}(r)\psi_{\kappa,m}^{+}(\theta, \phi) + G_{\kappa}(r)\psi_{\kappa,m}^{-}(\theta, \phi). \quad (7)$$

This *ansatz* into (1) gives two coupled equations of the form

$$\frac{d}{dr} \begin{pmatrix} F_{\kappa} \\ G_{\kappa} \end{pmatrix} = \begin{pmatrix} A & B \\ -C & -A \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}. \quad (8)$$

$$A(z) = - \left(\frac{\kappa}{r} + \frac{V_t(r)}{\hbar c} \right) \quad (9)$$

$$B(z) = \frac{E - V_v(r) + mc^2 + V_s(r)}{\hbar c} \quad (10)$$

$$C(z) = \frac{E - V_v(r) - mc^2 - V_s(r)}{\hbar c}. \quad (11)$$

If another pair of angular eigenfunctions are used, another pair of coupled equations is obtained, but the equation system always conforms to (8). The tracelessness of the matrix in (8) reflects the conservation of the probability current.

II. SEMICLASSICAL SOLUTION

A. Transformation to a second-order equation

To prepare for the asymptotic expansion, the right-hand member of (9) will now be divided by a formal expansion parameter λ ; cf. Ref. 2. This parameter will be used to keep track of the orders of magnitude of different terms during the expansion process, and in the end

it will be put equal to unity. The independent variable will be called z rather than r , to suggest a possible rescaling of the problem. Furthermore, the index κ on the functions F and G will be dropped in order to facilitate the notation. Thus the general problem to be treated is written

$$\frac{d}{dz} \begin{pmatrix} F \\ G \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} A & B \\ -C & -A \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \quad (12)$$

For the sake of the phase-integral treatment², the equation pair (12) must be rewritten as a single second-order equation of the form

$$u''(z) + \frac{R(z)}{\lambda^2} u(z) = 0. \quad (13)$$

The function u is supposed to give both F and G as

$$\begin{aligned} F(z) &= \alpha(z)u(z) + \lambda\beta(z)u'(z) \\ G(z) &= \gamma(z)u(z) + \lambda\delta(z)u'(z), \end{aligned} \quad (14)$$

where α , β , γ , and δ are suitably chosen functions, called the *transformation coefficients*. In order to obtain (13), three conditions must be fulfilled³:

$$\alpha = A\beta + B\delta - \lambda\beta' \quad (15)$$

$$\gamma = -C\beta - A\delta - \lambda\delta'. \quad (16)$$

$$\frac{d}{dz}(\alpha\delta - \beta\gamma) = 0. \quad (17)$$

By means of these conditions, all the transformation coefficients can be expressed in terms of one arbitrarily chosen function, the *transformation generator*

$$\chi = \delta/\beta. \quad (18)$$

From (18), (15), (16), and (17) one obtains³

$$\begin{aligned} \alpha &= k[(A + B\chi)U^{-1/2} + \lambda U^{-3/2}U'/2] \\ \beta &= kU^{-1/2} \\ \gamma &= k[-(C + A\chi)U^{-1/2} + \lambda(U^{-3/2}U'\chi/2 - U^{-1/2}\chi')] \\ \delta &= kU^{-1/2}\chi, \end{aligned} \quad (19)$$

where k is a constant and

$$U(z) = B\chi^2 + 2A\chi + C + \lambda\chi', \quad (20)$$

and k is a constant.

The function R in (13) will depend on the transformation generator χ as follows³:

$$R(z) = BC - A^2 + \lambda \left[A' + B'\chi + 2B\chi' - (A + B\chi)\frac{U'}{U} \right] + \lambda^2 \left[-\frac{3}{4} \left(\frac{U'}{U} \right)^2 + \frac{1}{2} \frac{U''}{U} \right]. \quad (21)$$

The freedom of choosing χ will be used in order to avoid singularities in R as well as in the transformation coefficients.

B. General form of the asymptotic solution

According to the phase-integral method², two solutions to the Schrödinger-like equation (13) are searched of the form²

$$u_{\pm}(z) = q^{-1/2}(z) \exp \left(\pm \frac{i}{\lambda} \int q(z) dz \right). \quad (22)$$

The function q is expanded in powers of λ . In the original Fröman-Fröman method, R is assumed to have a simple λ -dependence, and only odd-order approximations exist. Here, however, because of (21), R is an infinite series

$$R = R_0 + \lambda R_1 + \lambda^2 R_2 + \dots. \quad (23)$$

Therefore, the phase-integral approximations become somewhat more complicated, and even-order as well as odd-order approximations must be considered. The N -th order of approximation is given by

$$q = Q \sum_{n=0}^N \lambda^n Y_n, \quad (24)$$

where the *base function* Q is given by

$$Q = R_0^{1/2} \quad (25)$$

and the first few functions Y_{2n} are

$$Y_0 = 1 \quad (26)$$

$$Y_1 = \frac{R_1}{2Q^2} \quad (27)$$

$$Y_2 = \frac{1}{2} Q^{-3/2} \frac{d^2}{dz^2} Q^{-1/2} - \frac{Y_1^2}{2} + \frac{R_2}{2Q^2} \quad (28)$$

$$Y_3 = \frac{1}{Q} \frac{d}{dz} \left(\frac{1}{4Q} \frac{dY_1}{dz} \right) - Y_1 Y_2 + \frac{R_3}{2Q^2}. \quad (29)$$

The coefficients of R are calculated from (21). It is thereby assumed that A , B , and C are independent of λ while χ is represented by the series

$$\chi = \chi_0 + \lambda\chi_1 + \lambda^2\chi_2 + \cdots. \quad (30)$$

Explicit expressions for R_n and Y_n will be derived later on in each order of approximation.

From (14) and (22) we get solutions of (12)

$$\begin{aligned} F^\pm &= q^{-1/2} \left(\alpha - \frac{q'}{2q}\beta \pm iq\beta \right) \exp \left(\pm \frac{i}{\lambda} \int q(z)dz \right) \\ G^\pm &= q^{-1/2} \left(\gamma - \frac{q'}{2q}\delta \pm iq\delta \right) \exp \left(\pm \frac{i}{\lambda} \int q(z)dz \right). \end{aligned} \quad (31)$$

In the final semiclassical solution, the functions in front of the exponent in (31) are to be expanded in powers of λ , retaining only terms proportional to λ^{N-1} in N th order of approximation. Thus the outgoing and ingoing solutions take the form

$$\begin{pmatrix} F^\pm \\ G^\pm \end{pmatrix} = \begin{pmatrix} \xi^\pm \\ \eta^\pm \end{pmatrix} \exp \left(\pm \frac{iw}{\lambda} \right), \quad (32)$$

where, in N -th order of approximation,

$$\xi^\pm = \sum_{n=0}^{N-1} \lambda^n \xi_n^\pm, \quad (33)$$

$$\eta^\pm = \sum_{n=0}^{N-1} \lambda^n \eta_n^\pm, \quad (34)$$

and

$$w = \int Q \sum_{n=0}^N \lambda^n Y_n dz. \quad (35)$$

C. Symmetry and regularity considerations

The zeros and poles of Q^2 are the *transition points* where the phase-integral approximation breaks down, as is well known from classical WKB theory. It should be noted that the transition points do not depend on the transformation generator.

Depending on the choice of the transformation generator, singularities may also appear in the higher-order terms in the λ -expansion of the function R in (13). For example, putting $\chi = 0$ in (21) would make R singular when $C = 0$. These auxiliary singularities can be

avoided by making R invariant under constant, linear transformations in (F, G) -space. Such transformations have the form

$$\begin{pmatrix} F \\ G \end{pmatrix} \rightarrow \mathbf{P} \begin{pmatrix} F \\ G \end{pmatrix}, \quad (36)$$

where $\mathbf{P}$ is a constant, non-singular, 2×2 matrix. The transformed wavefunction satisfies an equation similar to (12) but with the matrix changed according to

$$\begin{pmatrix} A & B \\ -C & -A \end{pmatrix} \rightarrow \mathbf{P} \begin{pmatrix} A & B \\ -C & -A \end{pmatrix} \mathbf{P}^{-1}. \quad (37)$$

Once the function R in (13) is invariant under (37), the semiclassical wavefunctions (32) will transform precisely as the exact wavefunctions, *i.e.* according to (36). (This is because the coefficients η and ξ are uniquely given by the transformation generator, and the result is therefore the same either the semiclassical expansion is performed before or after the transformation.) Thus, singularities in the approximations can appear only under conditions invariant under the transformation (37), effectively only at the transition points.

When choosing the expansion coefficients R_n of R , one must also see that the transformation generator gets suitable analytic properties. This will be made more specific in the following subsections.

D. Decomposition of the phase integral

The phase-integral solution depends on the choice of the transformation to a second-order equation. However, it will now be shown that the phase integral w can be decomposed into one part that doesn't change and another part that can be calculated without integration. Differently stated, the integrand q is the sum of an unchanging function and the derivative of another function.

Let us first regard the case of a λ -independent transformation generator. The theorem then states that there exist two functions Φ and Ψ such that

$$q = \Phi(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}) + \frac{d}{dz} \Psi(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}, \{\chi^{(i)}\}). \quad (38)$$

Here $\{A^{(i)}\}$ is the set of A and its first $N - 1$ derivatives, and similarly for the other functions. To prove the theorem, compare the two semiclassical solutions of the same radial Dirac problem, obtained by means of two different choices of the transformation generator.

Chose a point z such that $\text{Re } iw_0(z_0) > 0$. Then the solutions with a plus sign in the exponent are dominant. Consider the upper component represented by $\xi^+ \exp(iw/\lambda)$ and $\tilde{\xi}^+ \exp(i\tilde{w}/\lambda)$, respectively, in the two approximations. Since both approximations must agree with an exact solution up to λ^N , we must have

$$\xi^+ \exp(iw/\lambda)[1 + \lambda^N O(\lambda)] = \tilde{\xi}^+ \exp(i\tilde{w}/\lambda)[1 + O(\lambda^N)]. \quad (39)$$

That gives

$$w - \tilde{w} = -i\lambda \ln(\xi^+/\tilde{\xi}^+)[1 + O(\lambda^N)]. \quad (40)$$

Here the left-hand member is a truncated power series in λ , the last term being proportional to λ^N . All the terms of that series can be identified by expanding the right-hand member in powers of λ . That will yield only local functions of A , B , C , χ , and their derivatives. Taking the derivative of both members, we get

$$q - \tilde{q} = \frac{d}{dz} \Psi(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}, \{\chi^{(i)}\}), \quad (41)$$

proving the theorem. It should be noted that the functions Φ and Ψ are not uniquely defined.

Next, χ is assumed to be λ -dependent. Inserting the expansion (30) into (38) and expanding each member in powers of λ gives

$$QY_n = \Phi_n(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}) + \frac{d}{dz} \Psi_n(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}, \{\chi_j^{(i)}\}), \quad (42)$$

where $\{\chi_j^{(i)}\}$ is the set of the n first coefficients in (30) and their first $n - 1$ derivatives. (In practise, only terms with $i + j < n$ will appear.)

The phase decomposition theorem is useful to find a suitable transformation generator, as will be seen in the following subsections.

E. First-order approximation

Inserting (30) into (21) we get the first terms in the λ -expansion of R :

$$R_0 \equiv Q^2 = BC - A^2 \quad (43)$$

$$R_1 = A' + B'\chi_0 + 2B\chi'_0 - (A + B\chi_0)\frac{U'_0}{U_0} \quad (44)$$

with

$$U_0 = B\chi_0^2 + 2A\chi_0 + C. \quad (45)$$

From (27), in accordance with (42), one finds that

$$QY_1 = \frac{-A' + AB'/B}{2Q} + \frac{d}{dz} \arctan \frac{A + B\chi_0}{Q}. \quad (46)$$

In order to ensure that the first order approximation is invariant under (36), χ_0 is determined such that

$$R_1 = 0. \quad (47)$$

Using (27) and (46), we find the solution

$$\chi_0 = \frac{1}{B} \left(Q \cot \int \frac{-A' + AB'/B}{2Q} dz - A \right). \quad (48)$$

(The use of cotangent instead of tangent here is for later convenience.) Inserting this into (45) we get

$$U_0 = \frac{Q^2}{B} \sin^{-2} \int \frac{-A' + AB'/B}{2Q} dz. \quad (49)$$

In order that the transformation coefficients be regular at a turning point z_1 , one has to choose that turning point as the lower integration limit. Thus from (19) we obtain the transformation coefficients in the first order of approximation:

$$\begin{aligned} \alpha_0 &= kB^{1/2} \cos s_1, \\ \beta_0 &= kB^{1/2} Q^{-1} \sin s_1, \\ \gamma_0 &= -kB^{-1/2} (A \cos s_1 + Q \sin s), \\ \delta_0 &= kB^{-1/2} Q^{-1} (Q \cos s - A \sin s_1), \end{aligned} \quad (50)$$

where k is a constant,

$$s_1 = \int_{z_1}^z \frac{-A' + AB'/B}{2Q} d\zeta, \quad (51)$$

the subscript 1 being added to indicate the lower integration limit.

Using (22) (50) with $k = 1$ and performing appropriate expansions in powers of λ we get the first-order wave functions

$$\begin{pmatrix} F_{\pm} \\ G_{\pm} \end{pmatrix} = \begin{pmatrix} B \\ -A \pm iQ \end{pmatrix} (BQ)^{-1/2} \exp \left[\pm i \left(\frac{w}{\lambda} + s \right) \right] \quad (52)$$

Although it is not immediately evident, these solutions in fact fulfil the transformation property (36) and are regular everywhere except at the transition points.

F. Second-order approximation

1. Decomposition of the second-order phase

For an unspecified transformation generator, the second-order contribution to R according to (21) is

$$R_2 = \frac{\partial R_1}{\partial \chi_0} \chi_1 + (A + B\chi_0) \frac{U'_0 \chi'_0 - U_0 \chi''_0}{U_0^2} - \frac{3}{4} \left(\frac{U'_0}{U_0} \right)^2 + \frac{1}{2} \frac{U''_0}{U_0}, \quad (53)$$

where R_1 is given by (44). By means of (27) and (46), one rewrite this expression as

$$R_2 = 2Q \frac{d}{dz} \frac{Q\chi_1}{U_0} + (A + B\chi_0) \frac{U'_0 \chi'_0 - U_0 \chi''_0}{U_0^2} - \frac{3}{4} \left(\frac{U'_0}{U_0} \right)^2 + \frac{1}{2} \frac{U''_0}{U_0}. \quad (54)$$

Inserting (44) and (54) into (28) gives a quite lengthy expression for Y_2 :

$$\begin{aligned} Y_2 = & \frac{1}{2} Q^{-3/2} \frac{d^2}{dz^2} Q^{-1/2} + \frac{1}{Q} \frac{d}{dz} \frac{Q\chi_1}{U_0} + \left(\frac{1}{2Q^2 U_0} \Theta - \frac{A + B\chi_0}{2Q^2 U_0^2} \Xi \right) \chi'_0 \\ & - \frac{1}{8Q^4} \Theta^2 + \frac{A + B\chi_0}{4Q^4 U_0} \Theta \Xi + \frac{3Q^2 + (A + B\chi_0)^2}{8Q^4 U_0^2} \Xi^2 \\ & + \frac{B''\chi_0^2 + 2A''\chi_0 + C''}{4Q^2 U_0}, \end{aligned} \quad (55)$$

where

$$\Theta = A' + B'\chi_0, \quad (56)$$

and

$$\Xi = B'\chi_0^2 + 2A'\chi_0 + C'. \quad (57)$$

This result can however be rewritten in a considerably more tractable form by means of the decomposition theorem (42). Since χ'_0 is the highest derivative of χ_0 appearing in (54), the function Ψ_2 in (42) will not contain any derivatives of χ_0 and must fulfil

$$\frac{\partial \Psi_2}{\partial \chi_0} = \frac{\partial(QY_2)}{\partial \chi'_0}. \quad (58)$$

The simplest solution gives the decomposition

$$QY_2 = \frac{1}{2} Q^{-1/2} \frac{d^2}{dz^2} Q^{-1/2} + \frac{B'C' - A'^2}{8Q^3} \quad (59)$$

$$+ \frac{d}{dz} \left(\frac{Q\chi_1}{U_0} + \frac{B'\chi_0^2 + 2A'\chi_0 + C'}{4QU_0} \right) \quad (60)$$

2. Selecting the transformation generator

Comparing (59) with (28), where R_1 now is put to 0, yields

$$\frac{R_2}{2Q^2} = \frac{B'C' - A'^2}{8Q^3} + \frac{d}{dz} \left(\frac{Q\chi_1}{U_0} + \frac{B'\chi_0^2 + 2A'\chi_0 + C'}{4QU_0} \right) \quad (61)$$

In order to avoid nested integrals in the transformation generator, R_2 is chosen as

$$R_2 = \frac{B'C' - A'^2}{4Q^2}. \quad (62)$$

This expression is obviously invariant under the linear transformation (37). With an appropriate choice of the integration constant, (61) now yields

$$\chi_1 = -\frac{B'\chi_0^2 + 2A'\chi_0 + C'}{4Q^2} \quad (63)$$

3. The second-order wavefunction

Properly truncated, the outgoing and ingoing wavefunctions in second order of approximation would read

$$\begin{aligned} F_{\pm} &= Q^{-1/2} \left[\alpha_0 \pm iQ\beta_0 + \lambda \left(-\frac{1}{2}\frac{Q'}{Q}\beta_0 + \alpha_1 \pm iQ\beta_1 \right) \right] \exp(\pm i\frac{w}{\lambda}) \\ G_{\pm} &= Q^{-1/2} \left[\gamma_0 \pm iQ\delta_0 + \lambda \left(-\frac{1}{2}\frac{Q'}{Q}\delta_0 + \gamma_1 \pm iQ\delta_1 \right) \right] \exp(\pm i\frac{w}{\lambda}). \end{aligned} \quad (64)$$

Here, the second-order contributions to transformation coefficients must be calculated from (14). Expanding U in terms of λ up to the first power we get

$$U = U_0 + \lambda(2B\chi_0\chi_1 + 2A\chi_1 + \chi_0'). \quad (65)$$

After some calculation, we find that

$$\chi_0' = \frac{(B\chi_0^2 - C)A' - (A\chi_0^2 + C\chi_0)B' + (A + B\chi_0)C'}{2Q^2} \quad (66)$$

and

$$U = U_0 \left(1 - \lambda \frac{A' + B'\chi_0}{2Q^2} \right). \quad (67)$$

These results together with (48) and (63) are to be inserted into (19). Expansion in powers of λ gives the following results for the terms proportional to λ :

$$\begin{aligned} \alpha_1 &= \frac{1}{4B^{1/2}} \left(\frac{BA' - AB'}{Q^2} \cos s + \frac{-2ABA' + A^2B' + B^2C'}{Q^3} \sin s \right) \\ \beta_1 &= \frac{1}{4B^{1/2}} \left(\frac{B'}{Q^2} \cos s + \frac{BA' - AB'}{Q^3} \sin s \right) \\ \gamma_1 &= \frac{1}{4B^{1/2}} \left(\frac{CB' - AA'}{Q^2} \cos s + \frac{(A^2 + BC)A' - ACB' - ABC'}{Q^3} \sin s \right) \\ \delta_1 &= \frac{1}{4B^{1/2}} \left(-\frac{A'}{Q^2} \cos s + \frac{AA' - BC'}{Q^3} \sin s \right) \end{aligned} \quad (68)$$

Inserting (50) and (68), into (64), we get

$$F_{\pm} = (BQ)^{-1/2} \left(B + \lambda \frac{BA' - AB' \pm iQB'}{4Q^2} \right) \exp \left[\pm i \left(\frac{w}{\lambda} + s \right) \right] \quad (69)$$

$$G_{\pm} = (BQ)^{-1/2} \left(-A \pm iQ + \lambda \frac{AA' - BC' \mp iQA'}{4Q^2} \right) \exp \left[\pm i \left(\frac{w}{\lambda} + s \right) \right]. \quad (70)$$

G. Third-order approximation

The third-order term in the expansion (23) is

$$\begin{aligned} R_3 = & \left[-(A + B\chi_0) \frac{U'_0 \chi'_0}{U_0^3} + \frac{3}{2} \left(\frac{U'_0}{U_0} \right)^2 \right] \chi'_0 + \left(\frac{(A + B\chi_0) \chi'_0}{U_0^2} - \frac{3U'_0}{2U_0^2} \right) \chi''_0 \\ & - \frac{U''_0 \chi'_0}{2U_0^2} + \frac{\chi'''_0}{2U_0} + 2Q \frac{d}{dz} \left(\frac{Q\chi_2}{U_0} - \frac{Q(A + B\chi_0) \chi_1^2}{U_1^2} \right. \\ & \left. + \frac{(B'A - BA') \chi_0^2 + (B'C - BC') \chi_0 + A'C - AC'}{2QU_0^2} \chi_1 \right) \end{aligned} \quad (71)$$

Inserting this, as well as (54) and (44), into (29), we obtain the third-order contribution to the phase integrand

$$\begin{aligned} QY_3 = & \frac{d}{dz} \left(\frac{Q\chi_2}{U_0} + \frac{1}{4Q} \frac{dY_1}{dz} - \frac{Q(A + B\chi_0) \chi_1^2}{U_0^2} \right. \\ & \left. + \frac{(B'A - BA') \chi_0^2 + (B'C - BC') \chi_0 + A'C - AC'}{2QU_0^2} \chi_1 \right) + H_3, \end{aligned} \quad (72)$$

where

$$H_3(A, A', A'', B, B', B'', C, C', C'', \chi_0, \chi'_0, \chi''_0, \chi'''_0) = -Y_1 Y_2 + \frac{R_3}{2Q^2}. \quad (73)$$

According to the general theorem, QY_3 can be decomposed into a function independent of the transformation and a derivative:

$$QY_3 = \Phi_3(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}) + \frac{d}{dz} \Psi_3(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}, \{\chi_0^{(i)}\}). \quad (74)$$

Here, of course, the z -derivative is to be calculated with the chain rule,

$$\frac{d\Psi_3}{dz} = \frac{\partial \Psi_3}{\partial A} A' + \dots + \frac{\partial \Psi_3}{\partial \chi_0'''} \chi_0'''. \quad (75)$$

To calculate the functions Φ_3 and Ψ_3 , first consider the function

$$H_2(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}, \chi_0, \chi'_0, \chi''_0) = H_3 - \frac{d}{dz} \int \frac{\partial H_3}{\partial \chi_0'''} d\chi_0''. \quad (76)$$

This function does not depend on χ_0''' . The procedure is repeated as follows:

$$H_1(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}, \chi_0, \chi_0') = H_2 - \frac{d}{dz} \int \frac{\partial H_2}{\partial \chi_0''} d\chi_0'. \quad (77)$$

$$H_0(\{A^{(i)}\}, \{B^{(i)}\}, \{C^{(i)}\}) = H_1 - \frac{d}{dz} \int \frac{\partial H_1}{\partial \chi_0'} d\chi_0'. \quad (78)$$

Note that only the phase decomposition theorem grants that H_0 is independent of χ_0 .

The explicit calculation of H_0 was carried out by computer algebra. The result is not unique since there is considerable freedom in choosing the primitive functions. Straightforward calculation produced an expression that is singular when $B = 0$. It was conjectured that it might be possible to construct a function Φ_3 that is invariant under linear transformations in (F, G) -space and hence free of singularities other than at the transition points. There must then exist a function h such that

$$H_0 = \Phi_3 + \frac{dh}{dz}. \quad (79)$$

By trial and error the following solution was found:

$$\Phi_3 = \frac{1}{32Q^5} \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} \quad (80)$$

Straightforward calculation confirms that Φ_3 thus defined is indeed invariant under linear transformations in (F, G) -space. The function Ψ_3 is given by

$$\Psi_3 = \frac{1}{4Q} \frac{dY_1}{dz} + \int \frac{\partial H_3}{\partial \chi_0'''} d\chi_0'' + \int \frac{\partial H_2}{\partial \chi_0''} d\chi_0' + \int \frac{\partial H_1}{\partial \chi_0'} d\chi_0 + h. \quad (81)$$

Inserting this into (72) and noting that $R_1 = 0$, we get

$$\begin{aligned} QY_3 = & \Phi_3 + \frac{d}{dz} \left(\Psi_3 + \frac{Q\chi_2}{U_0} - \frac{Q(A + B\chi_0)\chi_1^2}{U_0^2} \right. \\ & \left. + \frac{(B'A - BA')\chi_0^2 + (B'C - BC')\chi_0 + A'C - AC'}{2QU_0^2} \chi_1 \right). \end{aligned} \quad (82)$$

Using (29) and (82) one can calculate χ_2 in terms of R_3 :

$$\begin{aligned} \chi_2 = & \frac{A + B\chi_0}{U_0} \chi_1^2 - \frac{(B'A - BA')\chi_0^2 + (B'C - BC')\chi_0 + A'C - AC'}{2Q^2U_0} \chi_1 \\ & - \frac{U_0}{Q} \Psi_3 + \frac{U_0}{Q} \int \left(\frac{R_3}{2Q} - \Phi_3 \right) dz. \end{aligned} \quad (83)$$

Here the integral should be made zero as was the case in the second order of approximation. Thus the natural choice is

$$R_3 = 2Q\Phi_3 = \frac{1}{16Q^4} \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}. \quad (84)$$

H. Higher-order approximations

In order to construct the n -th order approximation for $n > 3$, one would first calculate the function QY_n with an unspecified transformation generator. Then, one must find a suitable decomposition of Q_n according to (42) such that the function Φ_n is invariant under (37). This is a crucial step: there is at present no proof that such a decomposition exists for $n > 3$, nor is there any criterion that would make that decomposition unique. Provided that the decomposition can be found, however, χ_n is chosen such that

$$\Psi_n = 0, \quad (85)$$

which always yields a trivially solvable equation. Once χ_n has been found, the semiclassical solution is calculated straightforwardly.

I. Summary

To finish the calculations, the expansion parameter λ is now put equal to unity. A correction factor is introduced in order that the semiclassical solutions fulfil the exact Wronskian condition

$$F^+G^- - F^-G^+ = -2i, \quad (86)$$

which is in fact the conservation of the Dirac probability current. Finally, the common factor $e^{\pm is}$ is broken out of the contributions of each order of approximation. The first, second, and third order semiclassical approximation can then be written

$$\begin{pmatrix} F^\pm \\ G^\pm \end{pmatrix} = W^{-1/2} \begin{pmatrix} \phi^\pm \\ \psi^\pm \end{pmatrix} e^{\pm i(w+s)}, \quad (87)$$

where

$$W = i(\phi^+\psi^- - \phi^-\psi^+)/2. \quad (88)$$

In N -th order of approximation, we have

$$\phi^\pm = \sum_{n=0}^{N-1} \phi_n^\pm, \quad (89)$$

$$\psi^\pm = \sum_{n=0}^{N-1} \psi_n^\pm, \quad (90)$$

where

$$\phi_0^\pm = B^{1/2} Q^{-1/2} \quad (91)$$

$$\phi_1^\pm = \frac{BA' - AB' \pm iQB'}{4B^{1/2}Q^{5/2}} \quad (92)$$

$$\begin{aligned} \phi_2^\pm = & \frac{1}{64(BQ^{13})^{1/2}} \{4Q^2[-2ABA'' + (2A^2 - BC)B'' + B^2C'''] \\ & + 2B(BC - 11A^2)A'^2 + 5C(BC - 2A^2)B'^2 - 5B^3C'^2 \\ & + 20A^3A'B' + 20AB^2A'C' - 2B(BC + 4A^2)B'C' \\ & \pm i2Q[4Q^2(BA'' - AB'') + 10ABA'^2 + 5CB'^2 \\ & - 5(BC + 2A^2)A'B' - 5B^2A'C' + 5ABB'C']\} \end{aligned} \quad (93)$$

$$\psi_0^\pm = (BQ)^{-1/2}(-A \pm iQ) \quad (94)$$

$$\psi_1^\pm = \frac{BC' - AA' \pm iQA'}{4B^{1/2}Q^{5/2}} \quad (95)$$

$$\begin{aligned} \psi_2^\pm = & \frac{1}{64(BQ^{13})^{1/2}} \{4Q^2(BCA'' - ACB'' - ABC''') \\ & + 2A(9BC + A^2)A'^2 + 5A(C^2B'^2 - B^2C'^2) \\ & - 10(BC + A^2)(CA'B' + BA'C') + 2A(6BC - A^2)B'C' \\ & \pm iQ[4Q^2(CB'' - BC''') + 2Q^2(A'^2 - B'C') \\ & + 5(B^2C'^2 - C^2B'^2) + 10A(CA'B' - BA'C')]\} \end{aligned} \quad (96)$$

with

$$Q = (BC - A^2)^{1/2}, \quad (97)$$

$$s = \int \frac{-A' + AB'/B}{2Q} dz, \quad (98)$$

and

$$w = \int Q \sum_{n=0}^N Y_n dz, \quad (99)$$

with

$$Y_0 = 1 \quad (100)$$

$$Y_1 = 0 \quad (101)$$

$$Y_2 = \frac{1}{2}Q^{-3/2}\frac{d^2}{dz^2}Q^{-1/2} + \frac{B'C' - A'^2}{8Q^4} \quad (102)$$

$$Y_3 = \frac{1}{32Q^6} \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}. \quad (103)$$

III. THE CONNECTION PROBLEM

A. Exact representation of the connection matrix

Let (f^+, g^+) and (f^-, g^-) be two linearly independent approximate solutions, given by (87). Any exact solution (F, G) can be represented as a variable linear combination of the approximate solutions as follows:

$$\begin{cases} F(z) = a_1(z)f^+(z) + a_2(z)f^-(z) \\ G(z) = a_1(z)g^+(z) + a_2(z)g^-(z) \end{cases} \quad (104)$$

Along a path where dw is real, the superposition coefficients $a_1(z)$ and $a_2(z)$ are approximately constants. In other cases, they may be rapidly changing. The change of the superposition coefficients as one moves from a point z_1 to another point z_2 is expressed by the *connection matrix* $\mathbf{T}(z_2, z_1)$ as follows:

$$\begin{pmatrix} a_1(z_2) \\ a_2(z_2) \end{pmatrix} = \mathbf{T}(z_2, z_1) \begin{pmatrix} a_1(z_1) \\ a_2(z_1) \end{pmatrix} \quad (105)$$

Since the approximate solutions exactly satisfy the Wronskian relation (86), the approximate solutions will then satisfy a system of equations similar to the exact radial Dirac equation:

$$\frac{d}{dz} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ -\tilde{C} & -\tilde{A} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \quad (106)$$

where

$$\tilde{A} = -(f'_1 g_2 - f'_2 g_1)/(2i) \quad (107)$$

$$\tilde{B} = (f'_1 f_2 - f'_2 f_1)/(2i) \quad (108)$$

$$\tilde{C} = (g'_1 g_2 - g'_2 g_1)/(2i). \quad (109)$$

Furthermore, the connection matrix satisfies

$$\frac{\partial}{\partial z} \mathbf{T}(z, z_0) = \mathbf{M}(z) \mathbf{T}(z), \quad (110)$$

where

$$\mathbf{M} = \frac{1}{2i} \begin{pmatrix} \epsilon_a & e^{-2iw} \epsilon_b \\ -e^{2iw} \epsilon_c & -\epsilon_a \end{pmatrix} \quad (111)$$

with

$$\epsilon_a = [(\phi^+ \psi^- + \phi^- \psi^+) \Delta A + \psi_1 \psi_2 \Delta B + \phi^+ \phi^- \Delta C]/W \quad (112)$$

$$\epsilon_b = [2\phi^- \psi^- \Delta A + (\psi^-)^2 \Delta B + (\phi^-)^2 \Delta C] e^{-2i}/W \quad (113)$$

$$\epsilon_c = [2\phi^+ \psi^+ \Delta A + (\psi^+)^2 \Delta B + (\phi^+)^2 \Delta C] e^{2i}/W \quad (114)$$

and

$$\Delta A = \tilde{A} - A \quad (115)$$

$$\Delta B = \tilde{B} - B \quad (116)$$

$$\Delta C = \tilde{C} - C. \quad (117)$$

The equation (110) is to be solved with the boundary condition

$$\mathbf{T}(z_0, z_0) = \mathbf{1} \quad (118)$$

The solution can be constructed by Picard iteration. First the differential equation is rewritten as an integral equation:

$$\mathbf{T}(z, z_0) = \mathbf{1} + \int_{z_0}^z \mathbf{M}(\zeta) \mathbf{T}(\zeta, z_0) d\zeta \quad (119)$$

Then a convergent sequence $\mathbf{T}^{(n)}$ of approximate solutions is constructed a by the iteration formula

$$\mathbf{T}^{(n+1)}(z, z_0) = \mathbf{1} + \int_{z_0}^z \mathbf{M}(\zeta) \mathbf{T}^{(n)}(\zeta, z_0) d\zeta. \quad (120)$$

This leads to a series solution on the form

$$\mathbf{T}(z, z_0) = \mathbf{1} + \sum_{n=1}^{\infty} \int_{z_0}^z dz_n \int_{z_0}^{z_n} dz_{n-1} \cdots \int_{z_0}^{z_2} dz_1 \mathbf{M}^{(n)}(z_1, z_2, \dots, z_n), \quad (121)$$

where

$$\mathbf{M}^{(n)}(z_1, z_2, \dots, z_n) = \mathbf{M}(z_n) \mathbf{M}(z_{n-1}) \cdots \mathbf{M}(z_1) \quad (122)$$

B. Basic estimates of the connection matrix elements

Convenient estimates for the elements of $\mathbf{T}(z_0, z)$ can be derived under the condition that $|e^{iw}|$ is nondecreasing on the path from z_0 to z . In that case the integrands in (121) are ordered such that

$$|e^{iw_0}| \leq |e^{iw_1}| \leq \cdots \leq |e^{iw_n}| \leq |e^{iw}| \quad (123)$$

(where w_1 is short for $w(z_1)$ and so on). It will now be shown that, provided that (123) is fulfilled, the elements of $\mathbf{M}^{(n)}(z_1, z_2, \dots, z_n)$ satisfy the inequalities

$$|M_{11}^{(n)}| \leq \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_n) / 2 \quad (124)$$

$$|M_{12}^{(n)}| \leq \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_n) |e^{-2iw_1}| / 2 \quad (125)$$

$$|M_{21}^{(n)}| \leq \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_n) |e^{2iw_n}| / 2 \quad (126)$$

$$|M_{22}^{(n)}| \leq \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_n) |e^{2i(w_n - w_1)}| / 2, \quad (127)$$

where

$$\epsilon(z) = \max(|\epsilon_a(z)|, |\epsilon_b(z)|, |\epsilon_c(z)|). \quad (128)$$

The proof of these inequalities goes by simple induction. First we note that they are trivially fulfilled for $n = 1$. Then we assume that for some number p they are proved for all $n \leq p$. From (122) it is found that

$$M_{11}^{(p+1)}(z_1, z_2, \dots, z_{p+1}) = \frac{\epsilon_a(z_{p+1}) M_{11}^{(p)} + \epsilon_b(z_{p+1}) M_{21}^{(p)} e^{-2iw_{p+1}}}{2i}. \quad (129)$$

Using (124), (126), and (123) we find that

$$|M_{11}^{(n)}| \leq \frac{1}{2} \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_{p+1}) \left(\frac{1}{2} + \frac{|e^{2i(w_p - w_{p+1})}|}{2} \right) \quad (130)$$

$$\leq \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_{p+1}) / 2, \quad (131)$$

meaning that (124) is valid also for $n = p + 1$. Similar calculations for the other matrix elements allow us to prove (125) - (127) for $n = p + 1$, and so all the estimates are proved by induction.

The multiple integrals appearing in (121) can now be estimated. From (124) it is seen that

$$\int_{z_0}^z dz_n \int_{z_0}^{z_n} dz_{n-1} \cdots \int_{z_0}^{z_2} dz_1 M_{11}^{(n)} \leq \frac{1}{2} \int_{z_0}^z |dz_n| \int_{z_0}^{z_n} |dz_{n-1}| \cdots \int_{z_0}^{z_2} |dz_1| \epsilon(z_1) \epsilon(z_2) \cdots \epsilon(z_n). \quad (132)$$

The integral in the right-hand member is unchanged under any permutation of the variables $z_1, \dots, z_n$. The sum of all $n!$ permutations yields a multiple integral where all integrals range from z_0 to z . Hence

$$\int_{z_0}^z dz_n \int_{z_0}^{z_n} dz_{n-1} \cdots \int_{z_0}^{z_2} dz_1 M_{11}^{(n)}(z_1, z_2, \dots, z_n) \leq \frac{\mu^n}{2n!}, \quad (133)$$

where

$$\mu = \int_{z_0}^z \epsilon(\zeta) |d\zeta|. \quad (134)$$

Similar estimates give

$$\int_{z_0}^z dz_n \int_{z_0}^{z_n} dz_{n-1} \cdots \int_{z_0}^{z_2} dz_1 M_{12}^{(n)}(z_1, z_2, \dots, z_n) \leq \frac{\mu^n}{2n!} |\exp[-2iw(z_0)]|, \quad (135)$$

$$\int_{z_0}^z dz_n \int_{z_0}^{z_n} dz_{n-1} \cdots \int_{z_0}^{z_2} dz_1 M_{21}^{(n)}(z_1, z_2, \dots, z_n) \leq \frac{\mu^n}{2n!} |\exp[2iw(z)]|. \quad (136)$$

$$\int_{z_0}^z dz_n \int_{z_0}^{z_n} dz_{n-1} \cdots \int_{z_0}^{z_2} dz_1 M_{22}^{(n)}(z_1, z_2, \dots, z_n) \leq \frac{\mu^n}{2n!} |\exp[2iw(z) - 2iw(z_0)]|. \quad (137)$$

In fact, the a stronger estimate can be given for the last matrix element when $n = 1$:

$$\int_{z_0}^z dz_1 M_{22}(z_1) \leq \frac{\mu}{2}. \quad (138)$$

Inserting these results into (121) we obtain the following estimates for the elements of the connection matrix along a path where $|\exp[iw(z)]|$ is nondecreasing:

$$\begin{aligned} |T_{11}(z, z_0) - 1| &\leq (e^\mu - 1)/2 \\ |T_{12}(z, z_0)| &\leq |\exp[-2iw(z_0)]|(e^\mu - 1)/2 \\ |T_{21}(z, z_0)| &\leq |\exp[2iw(z)]|(e^\mu - 1)/2 \\ |T_{22}(z, z_0) - 1| &\leq \mu/2 + |\exp[2iw(z) - 2iw(z_0)]|(e^\mu - 1 - \mu)/2 \end{aligned} \quad (139)$$

These estimates are formally identical to the corresponding estimates for phase-integral solutions to the Schrödinger equation; cf. eqs. (2.3.17a-d) in 2.

C. Connection formulas across a simple turning point

Let z_1 be a simple turning point on the real axis, *i.e.* a simple zero of Q^2 where A , B , and C are all analytic. The base function Q is then double-valued and has a branch point at z_1 . The same is true for the phase integrand q provided that the transformation generator is chosen as described in the previous section. The constants of integration in (98) and (99) can thereby be specified such that

$$s_1 = \int_{z_0}^z \frac{-A' + AB'/B}{2Q} d\zeta. \quad (140)$$

and

$$w_1 = \int_{(z_0)}^z q(\zeta) d\zeta \equiv \frac{1}{2} \int_{\Gamma(z)} q(\zeta) d\zeta, \quad (141)$$

where $\Gamma(z)$ is a contour enclosing z_1 , starting in the mirror point of z on the opposite Riemann plane and ending in z . The semiclassical solutions will then change in the following way as one moves one turn around z_1 in the positive sense:

$$\begin{pmatrix} F_{\pm} \\ G_{\pm} \end{pmatrix} \rightarrow -i \begin{pmatrix} F_{\mp} \\ G_{\mp} \end{pmatrix} \quad (142)$$

This condition together with the basic estimates (139) is enough to derive connection formulas analagous to eqs. (3.8.6) and (3.9.6) in Ref. 2. For the radial Dirac equation, these equations take the following form.

For tracing a solution *from the forbidden to the allowed* region, assuming $iw(z) > 0$ in the forbidden region and $w(z) > 0$ in the allowed region:

$$\begin{pmatrix} \phi_- \\ \psi_- \end{pmatrix} e^{-i(w+s)} \rightarrow -i \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} e^{i(w+s)} + \begin{pmatrix} \phi_- \\ \psi_- \end{pmatrix} e^{-i(w+s)}. \quad (143)$$

For tracing a solution *from the allowed to the forbidden* region, assuming $iw(z) > 0$ in the forbidden region and $w(z) > 0$ in the allowed region:

$$a \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} e^{i(w+s)} + b \begin{pmatrix} \phi_- \\ \psi_- \end{pmatrix} e^{-i(w+s)} \rightarrow (a + ib) \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} e^{i(w+s)}. \quad (144)$$

IV. APPLICATIONS

A. Phase shifts

In a usual scattering problem B and C tend to constants while A tends to zero as $z \rightarrow \infty$. The behaviour of Q^2 for large z is assumed to be given by

$$Q^2(z) \sim K^2 + 2K\Sigma/z, \quad (145)$$

where K and Σ are constants with $K > 0$. Then the exact solution to (12) behaves like⁴

$$\begin{aligned} F(z) &\sim \text{const.} \times \sin(Kz + \Sigma \ln 2z - \ell\pi/2 + \delta_\kappa) \\ G(z) &\sim \text{const.} \times \cos(Kz + \Sigma \ln 2z - \ell\pi/2 + \delta_\kappa), \end{aligned} \quad (146)$$

where the constant δ_κ is, by definition, the phase shift.

In order to derive a semiclassical formula for the phase-shift, it is assumed that there is one turning point z_1 such that $Q^2(z)$ is negative for $z < z_1$ and positive for $z > z_1$. The phase-integral solution in the allowed region is given by the connection formula (143). As $z \rightarrow \infty$, we have

$$\begin{aligned} F_1(z) &= B^{1/2}Q^{-1/2} \cos(w_1 + s_1 - \pi/4) \\ G_1(z) &= \mp C^{1/2}Q^{-1/2} \cos(w_1 + s_1 - \pi/4), \end{aligned} \quad (147)$$

where the upper sign is to be taken if A is positive for large z . The factors in front of the cosines tend to constants as $z \rightarrow \infty$ if (145) is fulfilled. Comparing (147) with (146) we thus find that the phase shift in first order of approximation is given by

$$\delta_\kappa = \lim_{z \rightarrow \infty} [w_1(z) + s_1(z) - Kz - \Sigma \ln 2z] + (2\ell + 1)\pi/4. \quad (148)$$

B. Single-well quantization condition

Now assume that there are two turning points z_1 and z_2 with an allowed region between them, forming a single well. One solution (F_1, G_1) to the Dirac equation is defined such that it is regular at the origin. That solution is traced into the allowed region by means of (143). Another solution (F_2, G_2) is defined such that it tends to zero as z tends to infinity. To construct the phase-integral representation of that solution, we must use z_2 as lower integration limit in order that the condition (142) be fulfilled. In analogy with (52) we

obtain for $z_1 < z < z_2$

$$\begin{pmatrix} \phi_- \\ \psi_- \end{pmatrix} e^{-i(w_2+s_2)} \rightarrow \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} e^{i(w_2+s_2)} - i \begin{pmatrix} \phi_- \\ \psi_- \end{pmatrix} e^{-i(w_2+s_2)}. \quad (149)$$

where w_2 and s_2 are defined like w_1 and s_1 but with z_2 as lower integration limit. For bound states the two solutions (F_1, G_1) and (F_2, G_2) must be proportional to each other, which gives the quantization condition

$$w_1 + s_1 - (w_2 + s_2) - \pi/2 = n\pi, \quad n = 0, 1, \dots \quad (150)$$

This condition can also be written

$$\frac{1}{2} \oint_{\Gamma} \left(q + \frac{-A' + AB'/B}{Q} \right) dz = \left(n + \frac{1}{2} \right) \pi, \quad (151)$$

where Γ is as contour encircling z_1 and z_2 in the negative sense.

V. CONCLUDING REMARKS

The semiclassical solution to the radial Dirac equation described here share virtually all important properties with the Fröman-Fröman phase-integral approximation for Schrödinger-like equations. Therefore most of the phase-integral theory can probably be extended to relativistic problems. The first- and second-order approximations have a strikingly simple structure. In third order of approximation, the formulas for the wavefunction are quite lengthy. However, most semiclassical formulas do not use the wavefunctions directly but are rather expressed in terms of the phase integral, that is not overly complicated even in the third order.

The complicated transformation to a second-order equation may look like an unnecessary intermediate step. Indeed, it would be preferable to insert the *ansatz*(32) directly into the equation pair (12) and equationing the successive powers of λ . However, that procedure does not give enough information to determine all the coefficients of (32). The problem of fulfilling the symmetry and regularity conditions discussed in Section II C has been solved only in the framework of a transformation to a single second-order equation.

As explained in Section II H, it is yet not known if the approximation can be pursued beyond the third order in a completely satisfactory way. For practical problems, however,

the third order of approximation is probably sufficient.

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