Influence of the Non-linear Effects in the Design of Viscous Dampers for Bridge Cables

YALDA ACAR AND PONTUS JINGSTÅL
Influence of the Non-linear Effects in the Design of Viscous Dampers for Bridge Cables

YALDA ACAR
PONTUS JINGSTÅL

Master of Science Thesis
Stockholm, Sweden 2014
Abstract

In this master thesis the performance of external viscous dampers attached to cables in cable-stayed bridges have been studied. A comparison has been performed between a linear and a non-linear cable model. The comparison was carried out for two bridge cables, one from the Dubrovnik Bridge and the other from the Normandie Bridge. The performance of the dampers have been measured in terms of maximum achieved damping ratio and minimum amplitude of vibration.

The analysis was performed using the finite element method. The damping ratio was measured using both the half-power bandwidth method and by calculating the loss factor. The half-power bandwidth method can only be applied to a linear system. Therefore, the loss factor was evaluated for the linear model and compared to the results obtained using the half-power bandwidth method. From the comparison, it was concluded that the damping ratio evaluated using the loss factor was similar to the results obtained when using the half-power bandwidth method. However, when calculating the loss factor, it was of great importance that the resonance frequency of the system was accurately determined. The loss factor was then calculated for the non-linear model and compared to the results obtained for the linear model. Since the loss factor measures the energy dissipated in a system, it could be utilised for the non-linear model. When computing the strain energy for the non-linear model an approximate method was used to take into consideration the strain energy caused by the static deformation of the cable.

From the comparison between the linear and non-linear cable models, it was concluded that the optimal damper coefficients obtained by both models are not significantly different. However, there is an uncertainty in the results due to the fact that an approximate method was used when calculating the strain energy for the non-linear model. It was also observed that a very accurate evaluation of the system’s resonance frequency was needed to calculate the loss factor. It was also observed that the variation in amplitude of vibration for varying damper coefficient was small for all modes of vibration for the Dubrovnik Bridge Cable as well as for the first mode of vibration for the Normandie Bridge Cable. The difference in the results between the two bridge cables needs to be investigated further in order to get a better understanding of the results.

Keywords: Bridge cables, external viscous damper, half-power bandwidth method, loss factor, non-linear vibration.
Sammanfattning


Nyckelord: Brokabel, externa viskösa dämpare, half-power bandwidth metoden, förlustfaktor, icke-linjär vibrationer.
Preface

The research presented in this master thesis was initiated by the Department of Civil and Architectural Engineering at the Royal Institute of Technology, KTH. It was carried out at the division of Structural Engineering and Bridges from February to June 2014 under the supervision of Associate Prof. Dr. Jean-Marc Battini.

We would like to thank our supervisor Associate Prof. Dr. Jean-Marc Battini for his support and enthusiasm during our work on this thesis.

Stockholm, June 2014

Yalda Acar  
Pontus Jingstål
Contents

Abstract i

Sammanfattning iii

Preface v

1 Introduction 1
1.1 Background .................................................. 1
1.2 Aims and Scope .............................................. 2
1.3 Previous Research ........................................... 2
1.4 Description of the Models ................................. 5

2 Theoretical Background 9
2.1 Forced Vibration of Damped Linear System ............... 9
2.2 Forced Vibration of Damped Non-linear System ............ 9
2.3 Co-rotational beam element ................................. 10
2.4 The Newmark Beta Method ................................ 10
2.5 Half-Power Bandwidth Method ............................ 11
2.6 Loss factor ..................................................... 12

3 Method 15
3.1 Linear FEM Model ........................................... 15
3.2 Non-linear FEM Model ....................................... 17
3.3 Calculations of the Initial Shape of the Cables .......... 18
   3.3.1 FEM .................................................. 18
3.3.2 Analytical Solution ............................................. 19
3.4 Dynamic Response to Harmonic Loading ........................ 20
3.5 Damping Evaluation ................................................. 23
  3.5.1 Half-Power Bandwidth Method ................................. 23
  3.5.2 Loss Factor ....................................................... 24
3.6 Determination of the Resonance Frequency ......................... 27

4 Results .................................................................. 31
  4.1 Linear Analysis ....................................................... 31
  4.2 Non-linear Analysis ................................................ 37
    4.2.1 The Initial Shape of the Cables ........................... 37
    4.2.2 Loss Factor ....................................................... 38
    4.2.3 Cable 2 ............................................................... 40
    4.2.4 Cable 3 ............................................................... 42
    4.2.5 Comparison between two different Damper Coefficients 44

5 Conclusions ................................................................. 47
  5.1 Conclusions ........................................................... 47
  5.2 Further Research ..................................................... 48

Bibliography ................................................................ 49
Chapter 1

Introduction

1.1 Background

Cable-stayed bridges are very popular around the world due to their construction and cost effectiveness together with their attractive aesthetics. The high advances and improvements on the cable material properties and construction process bring the possibility of designing and building more cable-stayed bridges with longer span distance. However, there are still many challenges to face when designing them. One of the problems in these structures is vibration in the main stay cables induced by wind and rain. The amplitude of these vibrations can become rather large and may therefore induce fatigue in the cables as well as in the connections between the cables and the tower or bridge deck [1]. Since the cables themselves have very small internal damping this problem has to be dealt with in some way. A classic and efficient solution for suppressing cable vibrations is to mount external viscous dampers close to the stay cables’ anchorages, shown in Figure 1.1. The external viscous dampers are installed close to the supports mainly for the aesthetical quality of the bridge. It has also the benefit of reducing the installation cost [2]. The challenge in the design of these dampers is to obtain the optimal damping effect.

Figure 1.1: Viscous damper at the Fred Hartman bridge [3].
Cable vibration problems have been detected in bridges all over the world, which have motivated a vast of research on the phenomenon. Numerous research concerns the linear behaviour of the viscous damped cable. However considerations regarding the non-linear effects have been neglected. In this thesis the influence of the non-linear in the design of viscous dampers will be studied.

1.2 Aims and Scope

The aim of this thesis is to investigate the damper performance when a linear cable model respectively a non-linear cable model is used. The influence of the non-linear effects on the optimal design of the external damper will be the main focus. When comparing the results from the linear and the non-linear model, the damping ratio and the amplitude of steady state vibration will be examined. The comparison will be performed for two different bridge cables. The dynamic response of the stay-cables is calculated using a 2D FEM-program developed in the commercial software, MATLAB. The intention of these analyses is to gain more knowledge in how much the results obtained with a linear analysis differ from the one obtained with a non-linear analysis and hence judge the importance of a non-linear model.

1.3 Previous Research

In this section, the work carried out by other researchers in the field of stay cable vibration and optimization of external dampers is presented.

Estimation curve for modal damping in stay cables with viscous dampers

The authors of this article studied the free vibration of taut horizontal cable with an external viscous damper attached close to one of the supports. The equation of motion for the cable in the linear range was solved numerically and resulted in a solution consisting of complex Eigenvalues. These Eigenvalues were then used for evaluating the modal damping ratio of the cable. By plotting the normalized modal damping of the cable for various values of the normalized damping of the external damper, an estimation curve could be created relating the length, mass per unit meter, placement of the damper and first natural frequency of the cable with the damping of the external damper and the modal damping ratio of the cable-damper system. This article presents a simplified way of designing external dampers for stay cables. However when constructing the estimation curve, no consideration is taken to cable sag and the nonlinearities of the cable vibration or the fact that the optimal damping will vary when targeting different modes. The authors do however make an attempt at an analytical solution to the same problem when small cable sag is included in the calculation with the conclusion that the sag of the cable moderately reduces the damping ratio of the cable-damper system.
1.3. PREVIOUS RESEARCH

**Vibration of a taut cable with an external damper**  

This article presents an analytical formula for the numerical results obtained by [1]. The same cable-damper model and equation of motion as in [1] is used for the problem formulation. The equation of motion is then solved analytically and the solution obtained constitutes the analytical solution to the estimation curve constructed by [1]. The equation of motion is solved by studying the free damped vibration of the cable. The solution to the equation of motion yields complex eigenvalues. An asymptotic solution for the equation of motion, valid for placement of the external damper close to the support, is presented. The damping ratio for the first modes of vibration is obtained from the imaginary parts of the complex Eigenvalues.

**Design of viscous dampers targeting multiple cable modes**  

In this paper the authors present a systematic design procedure for viscous dampers targeting multiple vibration modes with respect to Irwin’s criterion according to [5]. The distance from the cable support to the damper and the variance between the different modal damping ratios are minimized. When evaluating the modal damping ratio of the system the authors use an approximate method derived in [4]. The design procedure is derived by equating the modal damping ratios of the lowest targeted mode with the modal damping of the highest targeted mode. This results in equal damping ratios for the lowest and highest targeted modes and higher damping ratios for the intermediate modes.

**Optimal design of viscous dampers for multi-mode vibration control of bridge cables**  

In this paper the authors present a method of designing external viscous dampers with optimal damping when targeting multiple vibration modes. The model used is an inclined cable with axial force and sag due to its dead weight. The problem is solved analytically taking into account the variation of the axial force during the vibration of the cable. The vibration of the cable is excited by a time varying external distributed force. The equation of motion is formulated under the assumptions that the static profile of the cable is a second order parabola and that the sag to span ratio is sufficiently small. The parabola describing the static shape of the cable and cable sag was calculated according to [7]. Using the solution to equation of motion, a case study was carried out for one of the stay cables on the Donting Lake Bridge in China. The damping ratio for the three first modes of vibration were calculated when first optimizing only for the first mode, secondly optimizing for both the second and first mode and thirdly when optimizing for all three modes of vibration. It was shown that when optimizing only for the first or for the first and second modes of vibration the damping ratio for the higher order modes did not meet Irwin’s criterion of a Scruton number of at least ten, according to [5]. However,
when optimizing the damping ratio for all three modes the damping will be sufficient to fulfil Irwin’s criterion for all three modes of vibration. The damping ratio for the three modes of vibration when optimizing for all three modes was quite similar and indicating almost equal damping for all three modes. The case study was performed for two placements of the damper, at 2% of the cable length and at 1% of the cable length. When comparing the damping ratio for the different modes of vibration for the two different damper placements it was evident that the placement of the damper at 2% of the cable length resulted in a much greater damping ratio. This would suggest that for long cable spans, the instalment of external viscous damper will have insufficient effect on mitigating rain-wind-induced vibrations.

Mitigation of three-dimensional vibration of inclined sag cable using discrete oil dampers - I. Formulation and II. Application

[8] Authors: Z. Yu and Y. L. Xu

In the first part of this article the authors formulate a technique called the hybrid method for solving the dynamic response of the three-dimensional small amplitude free and forced vibration of an inclined cable with sag equipped with external oil dampers. The hybrid method combines discretization of the cable into several elements, with the dampers located at one of the element nodes with an analytical solution to the equation of motion of the system. The motion of the cable is expressed using three partial differential equations. These equations take into consideration initial tension in cable as well as the dynamic cable tension. The equations also take into consideration external distributed dynamic loading in three dimensions, damper forces, internal damping of the system and cable sag. The formulation also allows varying of the inclination of the external dampers in two directions. The in-plane equations of motion for the elements are decoupled and solved for each discretized element. These local solutions are then assembled into a system matrix taking the connective conditions between any two segments into consideration. The system matrix and the boundary conditions of the cable are then used to obtain the complex Eigenvalues and dynamic response of the cable. The modal damping ratio of the system is evaluated from the complex Eigenvalues obtained in the global solution.

In the second part of the article the method developed in the first part is applied to two cables in a long span bridge. A multitude of parameters and their effect on the modal damping ratio of the cable were studied. By varying the sag parameter defined in [7] the effects of the cable sag on the modal damping ratio was investigated. The cable sag was changed by varying the tension force in the cable. It was shown that the damping ratio was greatly reduced with increasing sag of the cable. By varying the inclination of the cable while keeping all other parameters constant the influence of the cable inclination on the maximum achievable modal damping and optimum damper size was studied. It was shown that for the first mode of in-plane vibration the maximum modal damping ratio increased with increasing inclination of the cable. While for higher modes of vibration the inclination of the cable had negligible effect on the maximum damping ratio and optimum damper size. The ef-
fects of the positioning of the external damper were also studied with the conclusion that the maximum achievable damping ratio increased when the external damper was placed farther from the support. It was also shown that while the maximum achievable modal damping ratio was almost the same for the first five modes of vibrations for placements of the damper within 2% of the cable length, the maximum modal damping ratio increased more rapidly for the higher modes when the damper was placed farther from the support.

**Evaluation of viscous dampers for stay-cable mitigation**


This paper presents results from long-term field measurements of the vibration of the stay cables of the Fred Hartman Bridge in the United States. Measurement equipment was installed at two stay cables and the measurements were carried out during three years. During the first year of measurements, there were no external dampers present. After the first year of measurements, external viscous dampers were installed. The acceleration of the cables due to vibrations was measured and presented as functions of wind speed and wind direction. The damper force was also measured and plotted against wind speed and wind direction. The results were compared for the damped and undamped cases. The conclusions drawn from these comparisons were that the dampers in an effective way reduced the high amplitude oscillation observed for the undamped cases.

### 1.4 Description of the Models

Here the utilised cable models are presented. In both cases an external damper is placed 2% of the cable length ($L_c$) and a harmonic load ($P$) is applied at 40% of the cable length. The damping coefficient is denoted $c$ and the axial stiffness is denoted ($EA$). The bending stiffness ($EI$) is considered small enough to have no effect on the result.

The linear cable model consists of a taut, horizontal cable suspended between two supports, illustrated in Figure 1.2. During the vibration of the cable the tension force ($T$) will remain constant. The stiffness of the cable will also be constant since it is dependent on the tension force.

![Figure 1.2: The linear cable model.](image-url)
The non-linear cable model is an inclined cable suspended between two supports at different levels with initial sag due to its self-weight. The non-linear model is shown in Figure 1.3. The inclination of the bridge cable is denoted $\alpha$. The tension force in the non-linear model varies as a result of the vibration of the cable. As the cable vibrates due to the applied harmonic load, the displacement caused by the vibration will change the tension force in the cable. The stiffness of the cable which is dependent on the tension force will also change during the vibration. The displacement of the cable is in turn dependent on its stiffness. This connection between the tension force, displacement and stiffness of the cable is what causes the non-linear dynamic response to the harmonic load. The static tension force in the cable will also vary along the cable length, due to the inclination and the initial deformation.

![Figure 1.3: Non-linear cable model.](image)

The dynamic analysis has been performed on three different cables. Table 1.1 shows the different parameters used in the analysis of the cables. Cable 1, presented in [10], is used for the verification of the FEM-program developed for this thesis, which is used to carry out the dynamic analysis. Cable 2 is one of the stay cables in Dubrovnik Bridge located in Croatia. It is a single deck bridge with a main span of 244 m. The data for cable 2 is obtained from [11]. Cable 3 is one of the stay cables in Normandie Bridge located in France. The bridge has a longest span of 856 m. The data for cable 3 is obtained from [3]. The sag is measured as the ratio of the transversal displacement of the cable to the length of the cable.
Table 1.1: Data for three different bridge cables.

<table>
<thead>
<tr>
<th>Cable</th>
<th>Article authors</th>
<th>$L_c$ [m]</th>
<th>$m_c$ [kg/m]</th>
<th>$T$ [kN]</th>
<th>$\alpha$ [°]</th>
<th>$EA$ [MN]</th>
<th>Sag</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Andersson</td>
<td>93</td>
<td>114</td>
<td>5017</td>
<td>-</td>
<td>1615</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>Savor et al.</td>
<td>207.7</td>
<td>87.81</td>
<td>5501</td>
<td>23.80</td>
<td>7952.2</td>
<td>0.0037</td>
</tr>
<tr>
<td>3</td>
<td>Caetano</td>
<td>441.9</td>
<td>133.0</td>
<td>6850.5</td>
<td>17.7</td>
<td>2907</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Chapter 2

Theoretical Background

In this section, the theory used in this thesis is described. A brief overview of the structural dynamics and damping of structures is given. Thereafter the methods used in this thesis will be presented. All sections except for sections 2.3 have been written using [12] as reference. Section 2.3 have been written using [13]. Hence, no further references to these sources will be made.

2.1 Forced Vibration of Damped Linear System

Consider a viscously damped linear system, subjected to a harmonic load \( P(t) = P_0 \sin(ft) \), where \( P_0 \) is the force amplitude and \( f \) is the frequency of the forcing function. The linear equation of motion for such a system is as follows:

\[
m\ddot{u} + c\dot{u} + ku = P(t)
\]

where,
- \( P(t) \) is the harmonic load.
- \( m \) is the mass of the system.
- \( c \) is the damping of the system.
- \( k \) is the stiffness of the system.
- \( u \) is the displacement.
- \( \dot{u} \) and \( \ddot{u} \) is the first and second derivative of \( u \) with respect to time.

2.2 Forced Vibration of Damped Non-linear System

The non-linear dynamic response of the cable-damper system is caused by the fact that the tension force in the cable will be dependent on the displacement of the cable due to the vibration, which is dependent on the stiffness of the cable, which in turn is dependent on the tension force in the cable. The dynamic response is obtained by solving the non-linear equation of motion, which is defined in Equation (2.2)
\[ m\ddot{u} + c\dot{u} + (f_g) = p \]  

(2.2)

where,

\((f_g)\) is the internal force vector.

The internal force vector \((f_g)\) would in the linear case be \((f_g) = ku\), with \(k\) being the stiffness matrix of the system. In the non-linear case, the internal force vector will instead be dependent on the tangent stiffness matrix \((k_g)\). The tangent stiffness matrix describes the stiffness of the system in response to a small change in its configuration. The tangent stiffness matrix and the internal force vector \((f_g)\) are computed using the co-rotational method.

2.3 Co-rotational beam element

In the non-linear analysis of the dynamic response of the cable-damper system a co-rotational formulation of a beam element developed by [13] have been used. The main idea of the co-rotational method is to divide the motion of the element into two parts; rigid rotation and translation and local deformations. This is done by introducing a local coordinate system which rotates and translates as the beam is deformed in the global coordinate system. The local deformation is expressed in the local coordinate system and is coupled to the global deformations by a transformation matrix which is derived using geometrical relationship between the local and global deformations.

By first computing the local internal force vector and the local tangent stiffness matrix the global internal force vector and global tangent stiffness matrix can then be calculated using the transformation matrix. The global tangent stiffness matrix can then be used to compute the displacement vector, \(u\) and its derivatives.

2.4 The Newmark Beta Method

The Newmark Beta Method includes, in its formulation, several time-step methods used for the solution of linear or nonlinear equations. The method was proposed by Newmark and the equations for the velocity and displacement at time step \(i + 1\) are defined as:

\[ \dot{u}_{i+1} = \dot{u}_i + (1 - \gamma)\Delta t\ddot{u}_i + \gamma\Delta t\dddot{u}_{i+1} \]  

(2.3)

\[ \ddot{u}_{i+1} = \ddot{u}_i + \Delta t\dot{u}_i + (0.5 - \beta)\Delta t^2\dddot{u}_i + \beta\Delta t^2\dddot{u}_{i+1} \]  

(2.4)

The parameters \(\gamma\) provides a linearly weighting between the influence of the initial and the final accelerations on the change of velocity and \(\beta\) provides the same
weighting between the initial and the final accelerations for the displacement. The
parameters $\gamma$ and $\beta$ control the stability and accuracy of the method. Particular
numerical value for these parameters leads to well-known methods for the solution
of the differential equation of motion, Equation (2.1). The methods are called the
constant acceleration method and the linear acceleration method. It has been found
that for values of $\gamma$ different than $1/2$, the method introduced a superfluous damp-
ing in this system. For this reason, this parameter is generally set as $\gamma = 1/2$. The
choice of $\beta = 1/4$ is known as the constant acceleration method and the choice
of $\beta = 1/6$ is known as the linear acceleration method. In the present thesis the
constant acceleration method is utilised.

Using Newmark’s method, the non-linear dynamic response is solved at each time
step $t = t_i$. The dynamic response is obtained by solving Equation (2.2), which can
be rewritten into:

$$ r(u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1}) = m\ddot{u}_i + c\dot{u}_i + (f_g)_i - p_i = 0 $$ (2.5)

$r$ is called the dynamic residual. The solution is obtained by using a Newton type
method where at each iteration the dynamic residual is calculated and if the residual
is close enough to zero a solution is considered reached.

### 2.5 Half-Power Bandwidth Method

The most commonly used method to determine the damping in structures is the
Half Power Bandwidth method. This method is used in the frequency domain and is
only valid for linear systems. The amplitude of vibration of a system is obtained
from varying forcing frequency. The method consists of determining the frequencies
at which the amplitude is $A_1/\sqrt{2}$, where the amplitude at the peak is denoted $A_1$. The
frequencies $f_a$ and $f_b$ associated with the half power points on either side of the peak
are obtained, shown in Figure 2.1.

![Figure 2.1: Half-Power Bandwidth Method](image)

The damping ratio ($\zeta$) is then estimated using the following formula:
\[
\zeta = f_b - f_a \\
\frac{f_b + f_a}
\]

(2.6)

### 2.6 Loss factor

The loss factor of a system is defined as the part of the maximum strain energy over one cycle which is dissipated by viscous damping. The definition of the loss factor is given as follows:

\[
\xi = \frac{1}{2\pi} \frac{E_D}{E_{S0}}
\]

(2.7)

where,

- \(E_D\) is the dissipated energy.
- \(E_{S0}\) is the maximum strain energy during one cycle of vibration.

The damping ratio, for a linear system, is related to the loss factor by Equation (2.8), which can be rewritten into Equation (2.9).

\[
\zeta = \frac{1}{4\pi} \frac{1}{f_r} \frac{E_D}{E_{S0}}
\]

(2.8)

\[
\zeta = \frac{1}{2} \frac{1}{f_r} \xi
\]

(2.9)

where,

- \(f\) is the forcing frequency of the applied load.
- \(f_r\) is the resonance frequency of the system.

The strain energy and the kinetic energy of the system is, for a linear FE-model, expressed according to Equations (2.10) and (2.11).

\[
E_S = \{u\}^T[K]\{u\}
\]

(2.10)

\[
E_K = \{\dot{u}\}^T[M]\{\dot{u}\}
\]

(2.11)

Where,

- \(E_S\) is the strain energy.
- \(E_K\) is the kinetic energy.
- \(\{u\}\) is the displacement vector.
- \(\{u\}\) is the first derivative of \(u\) with respect to time.
- \([K]\) is the stiffness matrix of the system.
- \([M]\) is the mass matrix of the system.
For a non-linear system, the co-rotational beam element can be used to compute the strain energy for each element in the multiple degree of freedom system, using Equation (2.12).

\[ e_s = \frac{1}{2} \frac{EA}{l_0} ((l_n - l_0) + \frac{l_0}{15} (\theta_1^2 - \theta_1 \frac{\theta_2}{2} + \theta_2^2)) + 2 \frac{EI}{l_0} (\theta_1^2 + \theta_1 \theta_2 + \theta_2^2) \]  

(2.12)

Where,

- \( e_s \) is the strain energy of one element.
- \( l_0 \) is the undeformed length of the element.
- \( l_n \) is the deformed length of the element.
- \( \theta_1 \) and \( \theta_2 \) are the local nodal rotations of the element.

During steady state vibration, the system is vibrating at a constant frequency and with constant amplitude. This means that the total energy in the system is constant over one cycle of vibration. This in turn means that the input energy must be equal to the energy dissipated through damping. The input energy is the work done by the harmonic load and can be expressed as:

\[ E_I = \int P \, du = \int P \dot{u} \, dt \]  

(2.13)

where,

- \( E_I \) is the input energy.
- \( P \) is the external harmonic load.
- \( u \) is the displacement at the degree of freedom where the external force is applied.
- \( \dot{u} \) is the first derivative of \( u \) with respect to time of \( u \).
Chapter 3

Method

3.1 Linear FEM Model

The linear analysis was carried out using the finite element method, with classical Bernoulli beam elements. The cables were discretized into 100 elements, each element having four degrees of freedom; vertical translation and rotation at both nodes. Figure 3.1 shows the cable element and its degrees of freedom, where $L_e$ is the length of the element. The element stiffness matrix ($k_e$) which takes into consideration the stiffness caused by the axial force in the cable is derived in [14] and shown in Equation (3.1).

![Figure 3.1: The cable element and its degrees of freedom.](image)

$$k_e = \begin{bmatrix}
\frac{2(S+t)}{L_e^2} + \frac{T}{L_e} & S & Symmetrical \\
\frac{S+t}{L_e} & -\frac{S+t}{L_e} & \frac{2(S+t)}{L_e^2} + \frac{T}{L_e} \\
-\frac{2(S+t)}{L_e^2} & -\frac{S+t}{L_e} & \frac{S+t}{L_e} \\
\frac{S+t}{L_e} & t & -\frac{S+t}{L_e}
\end{bmatrix} \quad (3.1)$$

where,
CHAPTER 3. METHOD

\[ S = \frac{u(u \cosh u - \sinh u)}{2 - 2 \cosh u + u \sinh u} \frac{EI}{L_e} \]  
(3.2)

\[ t = \frac{u(\sinh u - u)}{2 - 2 \cosh u + u \sinh u} \frac{EI}{L_e} \]  
(3.3)

\[ u = L_e \sqrt{\frac{T}{EI}} \]  
(3.4)

The element mass matrix, \( m_e \), used in the calculations was a consistent mass matrix and its definition is shown in Equation (3.5).

\[
\begin{bmatrix}
156 & 22L_e & 54 & -13L_e \\
22L_e & 4L_e^2 & 13L_e & -3L_e^2 \\
54 & 13L_e & 156 & -22L_e \\
-13L_e & -3L_e^2 & -22L_e & 4L_e^2
\end{bmatrix}
\]  
(3.5)

Using the element stiffness matrix, \( k_e \), and the element mass matrix, \( m_e \), a global mass matrix, \( M \), and a global stiffness matrix, \( K \) was assembled. The boundary conditions were implemented by removing the rows and vectors in the global stiffness and mass matrices corresponding to the degrees of freedom associated with vertical translation at the supports. In order to obtain the dynamic properties of the un-damped cable, Equation (3.6) was solved.

\[(K - \omega^2 M)u = 0\]  
(3.6)

where,

\( \omega \) is the natural circular frequency of the un-damped cable.

\( u \) is the vector of displacements.

The damping of the system provided by the external damper was taken into consideration by introducing the damping matrix \( C \), which is of the same size as \( M \) and \( K \). \( C \) is a null matrix except for the element corresponding to the degree of freedom where the external damper acts, where the value of the damping coefficient is inserted. The definition of the damping matrix, \( C \), is shown in Equation (3.7), where \( ndof \) is the number of degrees of freedom.
3.2 Non-linear FEM Model

The FEM-element used in the non-linear analysis is a co-rotational beam element derived in [13]. Its degrees of freedom is shown in figure 3.2, where \( L_e \) denotes the element length. Each element has six degrees of freedom; vertical translation, horizontal translation and rotation at both nodes.

![Figure 3.2: The cable element used in the non-linear analysis.](image)

The element mass matrix used, was a consistent mass matrix and is the same as the one used in the linear analysis in section 3.1 but extended to include six degrees of freedom. Equation (3.8) defines the element mass matrix.

\[
m_e = m_c \frac{L_e}{420} \begin{bmatrix}
140 & 0 & 0 & 70 & 0 & 0 \\
0 & 156 & 22L_e & 0 & 54 & -13L_e \\
0 & 22L_e & 4L_e^2 & 0 & 13L_e & -3L_e^2 \\
70 & 0 & 0 & 140 & 0 & 0 \\
0 & 54 & 13L_e & 0 & 156 & -22L_e \\
0 & -13L_e & -3L_e^2 & 0 & -22L_e & 4L_e^2
\end{bmatrix}
\]  

(3.8)
Since the vibration is non-linear, the stiffness of the system will change during the vibration. The dynamic response of the system was solved using the theories described in section 2.4. The External damper was taken into consideration by using a constant damping matrix as in section 3.1.

3.3 Calculations of the Initial Shape of the Cables

The static shape of the cables were determined using a FEM-program and an analytical solution.

3.3.1 FEM

To solve the static deformation of the cable caused by its self weight in the non-linear model, a non-linear finite element program was used. The cable was discretized into 100 elements. Figure 3.3 shows a schematic representation of a discretized cable. At each node the x- and z-component of the self-weight, $mg$, of one element was applied except for the nodes at the support were half the dead weight of one element was applied. At the node at the right support, the tension force, $T$ was applied in the degree of freedom corresponding to displacement in the x-direction in that node. At the right support a roller bearing was introduced, which would allow the cable to deform.

![Figure 3.3: Schematic representation of the non-linear FEM-model.](image)

The static response of the cable due to the dead weight, $mg$ and the tension force $T$ is a non-linear problem since the stiffness, tension force and displacement is dependent of each other. To solve the problem a Newton-type procedure was used. For each iteration the residual, defined in Equation 3.9 was computed.

$$r = f_i - p$$

where,
$r$ is the residual.
3.3. **CALCULATIONS OF THE INITIAL SHAPE OF THE CABLES**

$f_i$ is the vector of internal forces calculated based on the deformation from the previous iteration step.

$p$ is the vector of applied load.

For a cable, which lacks any significant bending stiffness, the model shown in Figure 3.3 would have no initial stiffness. This would mean that the system would collapse when applying external loads. To make sure that the calculations converged to a solution, an initial bending stiffness was introduced. When the external load had been applied, the bending stiffness was reduced gradually from a very large value. The bending stiffness of the cable elements was decreased from a value of $10^6$ N to 300 N. A bending stiffness of 300 N was considered small enough to have no significant effect on the stiffness of the system. When the static shape of the cable had been calculated the tension force in each element could be computed based on the axial displacements of each element.

When the initial shape of the cable had been calculated, the displacement vector was imported to the FEM-program used for calculating the dynamic response. The roller bearing was then replaced with a fixed support.

### 3.3.2 Analytical Solution

To verify that the initial deformation and tension force calculated according to section 3.3.1 were correct, the analytical solution to the shape and tension force was calculated for the two cables studied using the non-linear model. The analytical solutions were compared with the FEM-calculations. The analytical solution to a suspended cable has been derived in [7] where the model shown in Figure 3.4 was used in the derivation.

![Figure 3.4: Definitions used in the derivation of the analytical solution to a suspended cable](image)

The cable is suspended between supports $A$ and $B$. $A$ has the Cartesian coordinates $(0,0)$ and $B$ $(l,h)$. The cable has the unstrained length of $L_c$, the Lagrangian coordinate along the unstrained cable is denoted $s$. The self-weight of the cable is $W = mgL_c$, where $m$ is the mass per unit length and $g$ is the gravitational constant. The horizontal reaction force at support $A$ is denoted $H$ and the vertical reaction force at support $A$ is denoted $V$. The axial stiffness of the cable is denoted $EA$. Equations 3.10, 3.11 and 3.12 shows the tension force, x-coordinate and z-coordinate for the deformed suspended cable as a function of $s$. 19
\[ T(s) = \sqrt{H^2 + \left( V - \frac{W s}{L_c} \right)^2} \]  
(3.10)

\[ x(s) = \frac{H s}{EA} + \frac{H L_c}{W} \left( \sinh^{-1}\left( \frac{V}{H} \right) - \sinh^{-1}\left( \frac{V - W s/L_c}{H} \right) \right) \]  
(3.11)

\[ z(s) = \frac{W s}{EA} \left( V \frac{W}{2L_c} \right) + \frac{H L_c}{W} \left( \sqrt{1 + \left( \frac{V}{H} \right)^2} - \sqrt{1 + \left( \frac{V - W s/L_c}{H} \right)^2} \right) \]  
(3.12)

Equations (3.11) and (3.12) give the coordinates for point B as functions of the reaction forces \( H \) and \( V \). By solving these equation for \( H \) and \( V \) for given values of \( l \) and \( h \) the solutions to Equations (3.10), (3.11) and (3.12) was obtained as follows:

\[ l = \frac{H L_c}{EA} + \frac{H L_c}{W} \left( \sinh^{-1}\left( \frac{V}{H} \right) - \sinh^{-1}\left( \frac{V - W s/L_c}{H} \right) \right) \]  
(3.13)

\[ h = \frac{W L_c}{EA} \left( V \frac{W}{2L_c} - \frac{1}{2} \right) + \frac{H L_c}{W} \left( \sqrt{1 + \left( \frac{V}{H} \right)^2} - \sqrt{1 + \left( \frac{V - W s/L_c}{H} \right)^2} \right) \]  
(3.14)

The non-linear equation system consisting of equations 3.13 and 3.14 was solved for \( H \) and \( V \) for cables 2 and 3 using Newton-Raphson’s method for various values of \( l \) and \( h \). When \( H \) and \( V \) had been computed the shape of the cables and the cable tensions force were obtained using Equations 3.10, 3.11 and 3.12.

### 3.4 Dynamic Response to Harmonic Loading

The dynamic response of the cables to the external harmonic force, \( P \), was calculated using Newmark’s method with constant acceleration, as described in section 2.4. The external force had an amplitude of \( P_0 = 10^4 \) N. Figures 3.5 and 3.6 shows the displacement of the point where the load was applied, for cable 3 with a damper coefficient of \( c = 350 \) kN·s/m and a forcing frequency of 0.3 Hz. The figures show the response both for the linear cable model and for the non-linear cable model.
3.4. DYNAMIC RESPONSE TO HARMONIC LOADING

Figure 3.5: Example of the dynamic response of a cable. Cable 3, $f = 0.3$ Hz, $c = 350$ kN/m.

Figure 3.6: The steady state vibration of the signal shown in figure 3.5.

It can be seen in Figures 3.5 and 3.6 that the response differs between the two
models. The amplitude of vibration is different due to the difference in resonance frequency as well as the difference in stiffness between the models. In the linear case the vibration is symmetrical around zero displacement but in the non-linear case the, due to the initial sag, the displacement does not reach above zero displacement in this case.

By varying the forcing frequency, the steady state amplitude of vibration could be obtained for different frequencies in order to investigate which forcing frequency gave the largest response. Figure 3.7 shows the response (steady state amplitude) of cable 3 with a damper coefficient of $c = 350$ kNs/m for varying forcing frequency, when using the linear and the non-linear cable models. These curves are called response curves and show the amplitude of vibration as a function of the forcing frequency. The amplitude of vibration was based on the displacements of the point where the external load was applied.

![Figure 3.7: Example of the dynamic response for varying forcing frequency for the linear and non-linear cable model. Cable 3, $c = 350$ kNs/m.](image)

From the response curves, the resonance frequency, $f_r$, of the cable-damper system could be obtained. The resonance frequency corresponds to the frequency which gives the largest response. In this example the resonance frequency for the linear model is 0.259 Hz and 0.288 Hz for the non-linear model.

There is a significant difference in response between the two models. In the linear model, the response curve is symmetrical around the resonance frequency. In the non-linear case however, it can be seen that a small change in forcing frequency will give a drastic change in response.
3.5 Damping Evaluation

Two different methods of evaluating the damping of the cable-damper system have been used. For the linear model, the half-power bandwidth method was used. However, due to the asymmetrical shape of the response curve and due also to the variation in the stiffness of the structure during the vibration for the non-linear model, the half-power bandwidth method could not be used. The second method of evaluating the damping was to calculate the loss factor. Since the loss factor is based on the energy in a vibrating system, the method can be used for the non-linear model.

In order to verify that the damping ratio calculated using the loss factor gave good results, the damping ratio for the linear model was evaluated using both the half-power bandwidth method and the loss factor. The loss factor was then calculated for the non-linear model and compared to the results from the linear model.

3.5.1 Half-Power Bandwidth Method

Response curves were created as described in section 3.4 for various values of the damper coefficient, $c$, and for forcing frequencies varying around the first four Eigen-frequencies of the un-damped cables. The damping ratio was calculated for values of the damping coefficient, $c$, of 50 to 600 kNs/m for cable 1, 50 to 500 kNs/m for cable 2 and 100 to 550 kNs/m for cable 3. The frequency response curves were created using a forcing frequency varying from 0.95 times the natural frequency of the un-damped cable to 1.05 times the natural frequency. The analysis time was 300 seconds for cable 1, 600 seconds for cable 2 and 700 seconds for cable 3. The time step used in the calculations was 0.001 seconds for cables 1 and 2 and 0.005 seconds for cable 3. Using these response curves the half-power bandwidth method was used to evaluate the damping ratio for the first four modes of vibration.

In order to get a good representation of the response curves, a polynomial was fitted to the data points. For a linear system, the maximum response of a damped system to a harmonic load can be calculated using the following equation:

$$ u_0 = \frac{(u_{stat})_0}{\sqrt{(1 - f^2)^2 + (2f\zeta)^2}} $$

(3.15)

Where

$u_0$ is the maximum steady state response.
$(u_{stat})_0$ is the static response caused by the amplitude of the harmonic force.
$f$ is the forcing frequency.

Equation (3.15) can be rewritten as:
\[
\frac{1}{u_0^2} = \frac{1}{(u_{stat})^2} \left( (1 - f^2)^2 + (2f\zeta)^2 \right) \tag{3.16}
\]

This is suitable for a polynomial approximation. The calculated values of the maximum steady state response, \(u_0\) of the system was thus rewritten as:

\[
A = \frac{1}{u_0^2} \tag{3.17}
\]

A polynomial was then fitted to the values of \(A\) and the values of \(f\). Using the polynomial, new values was calculated and then recalculated according to:

\[
u_0 = \frac{1}{\sqrt{A}} \tag{3.18}
\]

Figure 3.8 shows an example of a polynomial approximation to the data points.

Figure 3.8: Example of a polynomial approximation of a response curve.

### 3.5.2 Loss Factor

The damping of the cable-damper system was also evaluated by calculating the loss factor of the systems. This was done by first calculating the dynamic response of the systems for various values of \(c\), for the first four modes of vibration, when the external harmonic load was applied with a frequency equal to the resonance frequency of that system. The definition of the loss factor is given in section 2.6. If the harmonic load is applied with a forcing frequency equal to the resonance frequency of the system, Equation (2.8), which is valid for a linear system, can be reduced to:

\[
\zeta = \frac{1}{2}\xi \tag{3.19}
\]
The dynamic response was calculated using Newmark’s method. Using the displacements and velocities obtained with Newmark’s method, the strain energy and kinetic energy could be calculated for each time step in the calculations using the Equations (2.10) and (2.11). Figure 3.9 shows the displacement and corresponding total energy of cable 3 using both the linear and non-linear model, when the damping coefficient was 350 kNs/m and the external force was applied with a forcing frequency equal to the resonance frequency corresponding to the first mode of vibration. The total energy is the sum of kinetic and strain energy.

As can be seen in Figure 3.9, the total energy for the linear model, reaches a near constant value when the vibration of the system reaches the steady state. In the non-linear model, the total energy varies during the vibration. This can be explained by studying figure 3.10, which shows the displacement and energy for the last four cycles of vibration as well as the product $P\dot{u}$, which is used to calculate the input energy as described in section 2.6.
Figure 3.10: The displacement, energy and input energy for the last four cycles of vibration of the signals presented in figure 3.9.

As can be seen in Figure 3.10, the total energy in the system is constant over each cycle of vibration for the linear model during steady state vibration. The strain energy reaches peak values when the displacement of the cable reaches its maximum and minimum values and the kinetic energy reaches peak values when the displacement of the cable is equal to zero.

For the non-linear model, the strain energy has a greater value than in the linear model. This is due to fact that the cable has an initial deformation due to its sag. This deformation will add to the strain energy caused by the deformation during vibration. When examining Figure 3.10 it can be seen that strain energy for the non-linear model will have two types of peaks. The peaks in strain energy correspond to the maximum and minimum displacement of the cable. However, since the cable has an initial deformation the positive displacement will give much lower strain energy than the downwards, negative displacements, since those will have a much greater absolute value.

The input energy was calculated by numerical integration of the two graphs presented at the bottom of Figure 3.10. To minimize numerical errors the integration was carried out over the last four cycles to get an average of the input energy during one cycle.
3.6 Determination of the Resonance Frequency

The term $E_{S0}$ in Equation (3.19) was taken as the maximum strain energy over the last four cycles.

To compensate for the fact that the initial deformation of the cable adds to the strain energy caused by the deformation during vibration, a second method was used when calculating the loss factor for the non-linear model.

$$\xi = \frac{1}{2\pi} \frac{E_t}{E_{S0} - E_S(t = 0)}$$  \hspace{1cm} (3.20)

where, $E_{S}(t = 0)$ is the strain energy at time $t = 0$, i.e. the strain energy caused by the initial deformation.

This is an approximate method. But it provides an easy way of obtaining the strain energy caused by the actual vibration.

3.6 Determination of the Resonance Frequency

In order to evaluate the damping ratio using the loss factor, the external harmonic load has to be applied with a forcing frequency equal to the resonance frequency of the system. To find the resonance frequency for the various cable-damper systems two different methods were used. The first method was to study the frequency response curves of the systems. The resonance frequency will be the frequency corresponding to the peak value of the curve as described in section 3.4.

The second method used to evaluate the resonance frequencies was to apply the external harmonic load with a linearly increasing frequency, $f(t)$, and calculating the dynamic response of the systems. The forcing frequency increased linearly so that at time, $t = 0$, the forcing frequency was 0.9 times the natural frequency of the un-damped cable-damper system of the mode studied and at time $t = t_{max}$, the forcing frequency was 1.1 times the natural frequency of the un-damped cable-damper system of the mode studied. Figure 3.11 shows an example of the dynamic response to a harmonic force with linearly increasing frequency.
After a certain amount of time the amplitude of vibration increases rapidly. When the amplitude of vibration reaches its maximum value, the system is vibrating at resonance. By making a fast Fourier transform on the signal in the time domain, the signal can be studied in the frequency domain and the resonance frequency can be evaluated from the peak value in the frequency domain. Figure 3.12 shows the frequency spectrum of the signal shown in Figure 3.11.

A peak is clearly visible at the frequency 1.13 Hz, which in this case would be the resonance frequency. There is a discrepancy between the two methods used when determining the resonance frequency. The resonance frequency obtained using FFT-analysis, was not accurate enough for the non-linear cable model. Hence, when
determining the resonance frequency for the non-linear cable model, the resonance frequency obtained from response curves was used.
Chapter 4

Results

4.1 Linear Analysis

Firstly, the damping ratio and the maximum steady state response was calculated for the three cables presented in chapter 1.4, using the linear cable model. The damping ratio for the different cables was evaluated using the half-power bandwidth method as well as calculating the loss factor. When calculating the loss factor, the forcing frequency was set equal to the resonance frequency. The difference in result when using the resonance frequency obtained from the FFT-analysis and when using the resonance frequency obtained from response curves was also examined and the results are presented here.

The analysis carried out on cable 1 was done in order to verify that the FEM-program, developed for this thesis, performed correctly. The damping ratio for various values of the damper coefficient, $c$, was calculated and compared to the results in [10].

The damping ratio for the three cables are presented in Figures 4.1, 4.2 respectively 4.9. As can be seen in the figures, the damping ratio calculated using the half-power bandwidth method and the damping ratio obtained when calculating the loss factor with resonance frequencies obtained from response curves agrees very well with each other. These two methods also gave the same results as in [10] for cable 1 and therefore it was concluded that the FEM-program performed correctly. When calculating the loss factor with resonance frequencies obtained from the FFT analysis, a discrepancy was observed for modes 2 to 4 for the three cables.

Tables 4.1, 4.2 and 4.3 presents the resonance frequencies obtained from response curves for cables 1, 2 respectively 3. The tables also present the difference between the resonance frequency obtained from response curves and FFT-analysis. As can be observed in the tables, the difference in resonance frequency between the two methods is greater for higher modes of vibration.

As can be observed in the figures and tables presented in this section, a small change in the forcing frequency can give a rather large change in the damping ratio.
calculated using the loss factor. Hence when calculating the loss factor, it is of great importance that the resonance frequency of the system is determined accurately. The change in loss factor for different forcing frequencies can be explained by the fact the cable-damper system does not behave as a single degree of freedom system.

Figure 4.1: Cable 1: the damping ratio for the first four modes of vibration using loss factor with resonance frequencies obtained from response curves (⋯) and FFT-analysis (- - -) as well as damping ratios evaluated using half-power bandwidth method (—).
4.1. LINEAR ANALYSIS

Table 4.1: Cable 1: the resonance frequencies, \( f_r \), evaluated using response curves and the difference in resonance frequency when using FFT-analysis. \( f_i \) [Hz] is the resonance frequency for the \( i \)th mode of vibration.

<table>
<thead>
<tr>
<th>( c ) [kNs/m]</th>
<th>( f_r ) from response curves</th>
<th>Difference [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( f_1 ) ( f_2 ) ( f_3 ) ( f_4 )</td>
<td>Mode 1 Mode 2 Mode 3 Mode 4</td>
</tr>
<tr>
<td>50</td>
<td>1,128 2,258 3,390 4,527</td>
<td>0,029 0,196 0,315 0,665</td>
</tr>
<tr>
<td>75</td>
<td>1,129 2,261 3,398 4,542</td>
<td>0,061 0,158 0,313 0,674</td>
</tr>
<tr>
<td>100</td>
<td>1,129 2,265 3,406 4,555</td>
<td>0,003 0,150 0,293 0,681</td>
</tr>
<tr>
<td>125</td>
<td>1,130 2,269 3,414 4,566</td>
<td>0,059 0,154 0,296 0,685</td>
</tr>
<tr>
<td>150</td>
<td>1,131 2,273 3,420 4,573</td>
<td>0,015 0,157 0,271 0,687</td>
</tr>
<tr>
<td>175</td>
<td>1,132 2,276 3,426 4,579</td>
<td>0,086 0,201 0,286 0,674</td>
</tr>
<tr>
<td>200</td>
<td>1,133 2,280 3,430 4,583</td>
<td>0,060 0,177 0,305 0,683</td>
</tr>
<tr>
<td>225</td>
<td>1,133 2,282 3,433 4,586</td>
<td>0,029 0,190 0,299 0,670</td>
</tr>
<tr>
<td>250</td>
<td>1,134 2,285 3,436 4,589</td>
<td>0,005 0,189 0,310 0,669</td>
</tr>
<tr>
<td>275</td>
<td>1,135 2,287 3,438 4,590</td>
<td>0,089 0,173 0,340 0,682</td>
</tr>
<tr>
<td>300</td>
<td>1,136 2,288 3,440 4,592</td>
<td>0,056 0,198 0,322 0,688</td>
</tr>
<tr>
<td>325</td>
<td>1,137 2,290 3,442 4,593</td>
<td>0,026 0,157 0,331 0,686</td>
</tr>
<tr>
<td>350</td>
<td>1,138 2,291 3,443 4,594</td>
<td>0,007 0,162 0,332 0,680</td>
</tr>
<tr>
<td>375</td>
<td>1,139 2,292 3,444 4,595</td>
<td>0,064 0,158 0,362 0,697</td>
</tr>
<tr>
<td>400</td>
<td>1,140 2,293 3,445 4,595</td>
<td>0,029 0,199 0,352 0,683</td>
</tr>
<tr>
<td>425</td>
<td>1,140 2,294 3,445 4,596</td>
<td>0,085 0,184 0,373 0,696</td>
</tr>
<tr>
<td>450</td>
<td>1,141 2,295 3,446 4,596</td>
<td>0,042 0,164 0,357 0,679</td>
</tr>
<tr>
<td>475</td>
<td>1,142 2,296 3,447 4,597</td>
<td>0,096 0,190 0,372 0,688</td>
</tr>
<tr>
<td>500</td>
<td>1,142 2,296 3,447 4,597</td>
<td>0,038 0,163 0,385 0,695</td>
</tr>
<tr>
<td>525</td>
<td>1,143 2,296 3,447 4,597</td>
<td>0,084 0,184 0,363 0,676</td>
</tr>
<tr>
<td>550</td>
<td>1,143 2,297 3,448 4,597</td>
<td>0,026 0,151 0,373 0,682</td>
</tr>
<tr>
<td>575</td>
<td>1,144 2,297 3,448 4,598</td>
<td>0,066 0,168 0,382 0,684</td>
</tr>
<tr>
<td>600</td>
<td>1,144 2,297 3,448 4,598</td>
<td>0,002 0,182 0,390 0,689</td>
</tr>
</tbody>
</table>
Figure 4.2: Cable 2: the damping ratio for the first four modes of vibration using loss factor with resonance frequencies obtained from response curves (···) and FFT-analysis (- - -) as well as damping ratios evaluated using half-power bandwidth method (—).
4.1. LINEAR ANALYSIS

Table 4.2: Cable 2: the resonance frequencies, $f_r$, evaluated using response curves and the difference in resonance frequency when using FFT-analysis. $f_i$ [Hz] is the resonance frequency for the $i$th mode of vibration.

<table>
<thead>
<tr>
<th>$c$ [kNs/m]</th>
<th>$f_r$ from response curves</th>
<th>Difference [%]</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.603 1.207 1.811 2.418</td>
<td>0.066 0.017 0.061 0.141</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>0.603 1.208 1.816 2.427</td>
<td>0.050 0.058 0.099 0.186</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.603 1.211 1.821 2.435</td>
<td>0.017 0.025 0.082 0.185</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>125</td>
<td>0.604 1.213 1.825 2.440</td>
<td>0.017 0.099 0.110 0.156</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>0.604 1.215 1.828 2.444</td>
<td>0.099 0.025 0.093 0.160</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>175</td>
<td>0.605 1.217 1.831 2.446</td>
<td>0.033 0.025 0.115 0.180</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>0.605 1.219 1.833 2.448</td>
<td>0.066 0.033 0.104 0.164</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>225</td>
<td>0.606 1.220 1.835 2.450</td>
<td>0.017 0.090 0.065 0.127</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>0.607 1.221 1.836 2.451</td>
<td>0.083 0.000 0.065 0.172</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>275</td>
<td>0.607 1.222 1.837 2.452</td>
<td>0.049 0.082 0.065 0.163</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.608 1.223 1.838 2.452</td>
<td>0.033 0.049 0.109 0.139</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>325</td>
<td>0.608 1.224 1.839 2.453</td>
<td>0.115 0.016 0.076 0.159</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>350</td>
<td>0.609 1.225 1.839 2.453</td>
<td>0.033 0.065 0.054 0.135</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>375</td>
<td>0.609 1.225 1.840 2.454</td>
<td>0.033 0.024 0.076 0.151</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>0.609 1.225 1.840 2.454</td>
<td>0.066 0.057 0.098 0.159</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>425</td>
<td>0.610 1.226 1.841 2.454</td>
<td>0.000 0.016 0.120 0.171</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>450</td>
<td>0.610 1.226 1.841 2.454</td>
<td>0.066 0.016 0.082 0.180</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>475</td>
<td>0.610 1.226 1.841 2.455</td>
<td>0.098 0.033 0.092 0.135</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>0.611 1.227 1.841 2.455</td>
<td>0.049 0.057 0.103 0.139</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 4.3: Cable 3: the damping ratio for the first four modes of vibration using loss factor with resonance frequencies obtained from response curves (⋯) and FFT-analysis (- - -) as well as damping ratios evaluated using half-power bandwidth method (—).
4.2 Non-linear Analysis

4.2.1 The Initial Shape of the Cables

The static shapes of the cables analysed in this thesis are not given in the articles where the data for the cables are obtained. Therefore, the static shapes had to be calculated based on the tension force, inclination, cable length and self-weight of the cables. Equations (3.13) and (3.14) was solved for $H$ and $V$ for various values of $h$ and $l$. $h$ and $l$ were chosen so that the inclinations of the cables were not changed. The tension force in the cables were then computed using Equation (3.10) and compared to the values presented in Table 1.1.

When the analytical solution to the initial shape and tension force in the cables had been obtained, the tension force in the cables at the right hand support were used.
when determining the initial shape using the FEM-program described in section 3.3.1. Figures 4.4 and 4.5 shows the shape and tension force of cables 2 and 3 calculated using both the analytical solution and the FEM-program with 100 elements.

As can be seen in figures 4.4b and 4.5b, the tension force in cable 2 varies from 5430.4 kN to 5502.6 kN and the tension force in cable 3 varies from 6671.8 kN to 6847.0 kN. This agrees well with the values of the tension force presented in table 1.1, which was 6850.5 kN for cable 2 and 5501 kN for cable 3. Figures 4.4 and 4.5 also shows that the shapes and tension force computed using the FEM-program agrees well with the analytical solution.

4.2.2 Loss Factor

When performing the analysis for the linear model, it was observed that when calculating the loss factor, the best results were obtained when using a forcing frequency
equal to the resonance frequency obtained from the response curves. Therefore, when calculating the loss factor for the non-linear model, the forcing frequency of the harmonic load was equal to resonance frequency obtained from response curves.

When calculating the loss factor for the non-linear cable model, two different methods of were used, as described in section 3.5.2. Figure 4.7 shows the loss factor for varying damper coefficient, for cable 3 using both methods.

\[ E_{S0} - E_S(t = 0) \]

Figure 4.6: Loss factor for the first four modes of vibration for cable 3. The top graph shows the loss factor when the initial strain energy is subtracted from the maximum strain energy. The bottom graph shows the loss factor when the maximum strain energy is used.

The values of the loss factor in the bottom graph are much smaller than the values in the top graph. This is because the value of the maximum strain energy, \( E_{S0} \), is reduced by the strain energy caused by the initial deformation of the cable. The shapes of the curves in the two graphs are also different. No peak values can be seen for modes 3 and 4 in the bottom graph. The curves in the lower graph are not comparable to curves obtained for the linear cable model. Therefore, the method presented in the top graph, was used when comparing the linear and non-linear cable models.
4.2.3 Cable 2

Figure 4.7 shows the loss factor calculated using the linear and non-linear cable models for cable 2. There is a significant difference between the values of the loss factor obtained when using the linear and non-linear cable models. However, the shapes of the curves are quite similar. The optimal damper coefficient for each mode does not vary much when comparing the linear and non-linear cable models.

The difference in the amplitude of the steady state vibration between the two models was also examined. The amplitude of vibration was based on the displacement of the point where the external load was applied. Figure 4.8 shows the amplitude for varying damper coefficient, for both models. As can be seen in the figure, the variation in amplitude for varying $c$ is much smaller for the non-linear cable model. For mode 1, the amplitude is near constant for varying damper coefficient. The variation in amplitude for the non-linear model, is larger and more similar to the linear model for modes 2, 3 and 4 but still significantly smaller than in the linear model. Although the variation in amplitude is smaller for the non-linear model, the minimum amplitude of vibration is obtained for roughly the same damper coefficient for both models.

Table 4.5 shows the optimal damper coefficient, when optimizing for the maximum loss factor, for each mode of vibration, when using the linear and non-linear cable models. The optimal damper coefficient, when optimizing for minimum amplitude of vibration, is presented in Table 4.4.
4.2. NON-LINEAR ANALYSIS

Figure 4.7: Cable 2: the damping ratio for the first four modes of vibration using both the linear (—) and the non-linear model ( - - -)

Figure 4.8: Cable 2: comparison of the amplitude of steady state vibration between the linear (—) and the non-linear ( - - -) model.
CHAPTER 4. RESULTS

Table 4.4: Cable2: the optimal damper coefficient, when optimizing for the minimum amplitude of vibration.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Linear model</th>
<th>Non-linear model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>350</td>
<td>475</td>
</tr>
<tr>
<td>2</td>
<td>150</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>75</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 4.5: Cable 2: the optimal damper coefficient, when optimizing for the maximum loss factor.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Linear model</th>
<th>Non-linear model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>350</td>
<td>375</td>
</tr>
<tr>
<td>2</td>
<td>150</td>
<td>175</td>
</tr>
<tr>
<td>3</td>
<td>125</td>
<td>125</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

4.2.4 Cable 3

Figure 4.7 shows the loss factor calculated using the linear and non-linear cable models for cable 3. The results for cable 3 are similar to the results for cable 2. The values of the loss factor differs between the two models but the shapes of the curves and optimal damper coefficient for each mode of vibration do not vary significantly between the two models.

Figure 4.10 shows the amplitude of steady state vibration for varying damper coefficient, for both the linear and non-linear cable models. The amplitude of vibration is based on the displacement of the point where the external load is applied. As for cable 2 the variation in the amplitude for cable 3, for varying damper coefficient for mode 1 is much smaller for the non-linear model, than for the linear model. However, for modes 2, 3 and 4, the variation in amplitude is similar for both models. The optimal damper coefficient is roughly the same for both models.

Table 4.7 shows the optimal damper coefficient, when optimizing for maximum loss factor, for each mode of vibration, when using the linear and non-linear cable models. The optimal damper coefficient, when optimizing for minimum amplitude of vibration, is presented in Table 4.6.
4.2. NON-LINEAR ANALYSIS

![Diagram showing damping ratio for the first four modes of vibration for cable 2 using both the linear model (—) and the non-linear model (---).](image)

Figure 4.9: Cable 3: the damping ratio for the first four modes of vibration for cable 2 using both the linear model (—) and the non-linear model (---).

![Diagram showing amplitude of steady state vibration for the first mode of vibration and the second, third, and fourth mode of vibration.](image)

(a) First mode of vibration.  
(b) Second, third and fourth mode of vibration.

Figure 4.10: Cable 3: comparison of the amplitude of steady state vibration between the linear (—) and the non-linear (---) model.
Table 4.6: Cable 3: the optimal damper coefficient, when optimizing for the minimum amplitude of vibration.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Linear model</th>
<th>Non-linear model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>500</td>
<td>475</td>
</tr>
<tr>
<td>2</td>
<td>225</td>
<td>225</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>125</td>
<td>125</td>
</tr>
</tbody>
</table>

Table 4.7: Cable 3: the optimal damper coefficient, when optimizing for the maximum loss factor.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Linear model</th>
<th>Non-linear model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>475</td>
<td>475</td>
</tr>
<tr>
<td>2</td>
<td>250</td>
<td>225</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>125</td>
<td>125</td>
</tr>
</tbody>
</table>

### 4.2.5 Comparison between two different Damper Coefficients

The variation in amplitude of vibration for varying damper coefficient was very small for the non-linear case. In order to investigate the results further, the dynamic response for cable 3 were plotted for a damper coefficient of \( c = 200 \) kNs/m and for a damper coefficient of \( c = 600 \) kNs/m.

![Dynamic response of cable 3, \( c = 200 \) kNs/m.](image1)

![Dynamic response of cable 3, \( c = 600 \) kNs/m.](image2)

Figure 4.11: Example of a polynomial approximation of a response curve.
As can be seen in Figure 4.11 the amplitude of steady state vibration for \( c = 200 \) kNs/m is 6.4 m and 5.7 m for \( c = 600 \) kNs/m. These are same results as presented in Figure 4.10. Figure 4.12 shows the shape of the cables when the steady state displacement reaches its maximum.

(a) Shape of the cable when the displacement in figure 4.11 (a) reaches its maximum.  
(b) Shape of the cable when the displacement in figure 4.11 (b) reaches its maximum.

Figure 4.12: Cable shapes.
Chapter 5

Conclusions

In this chapter the results will be summarized and discussed. Some suggestions for future research will also be presented.

5.1 Conclusions

Based on the results presented in this thesis, when designing external viscous dampers for stay cables, a linear cable model is sufficient. The difference in optimal damper coefficient for the first four modes of vibration between the linear and non-linear cable models was very small. There is however an uncertainty in the results due to the fact that the strain energy for the non-linear cable model is calculated using an approximate method. Another consideration which has to be taken into account is that the comparison between the two models was made separately for each mode of vibration. Other conclusion may be drawn when comparing the results for when the damper is designed to target multiple modes of vibration.

The variation in amplitude of steady state vibration for the non-linear cable model was very small for cable 2 and for mode 1 for cable 3. This would mean that for some cases, the size of the external damper does not have a great effect on the amplitudes of vibration.

It is also worth mentioning the difference when calculating the loss factor for different forcing frequencies. As was shown in section 4.1, a small change in forcing frequency gave a large change in the calculated value of the loss factor. This shows the importance of obtaining a correct value for the resonance frequency when calculating the damping ratio using the loss factor. The best way of obtaining the resonance frequency is to study the response of the system for different forcing frequencies in order to find the forcing frequency which gives the largest response. Studying the response in the frequency domain as described in section 3.6 does not give accurate enough results. The large change in loss factor could be explained by the fact that the cable-damper system does not behave as a single degree of freedom system since the damper matrix is not diagonal. For a linear system behaving as a single degree of freedom system, the expected result would have been that a change in forcing
frequency would have given a proportional change in loss factor.

When calculating the loss factor for the non-linear cable model, it was concluded that some consideration has to be taken to the fact that the initial deformation of the cable, due to its self-weight, contributes to the total strain energy. For higher modes of vibration, when the location and the magnitude of the applied load are unchanged, the initial strain energy becomes a dominant factor since the deformation caused by the self-weight will be larger than the deformation caused by the vibration. In this thesis the strain energy caused by the initial deformation was subtracted from the total strain energy. This is an approximate method and it only allowed for the comparison to be made separately for each mode of vibration.

5.2 Further Research

This thesis has examined the difference in the design of external dampers for cable-stayed bridges when using a linear cable model and when using a non-linear cable model. In order to get a better understanding of the differences between these models some suggestions of further research is presented in this chapter.

Different results might be obtained, if another method of calculating the strain energy for the non-linear model is used. When calculating the loss factor, only the strain energy caused by the vibration should be considered. If a better way of calculating the strain energy is implemented, it might be possible to compare the optimal damper coefficient, when optimizing for multiple modes of vibration.

Further investigations in the effects of the damper coefficient on the amplitude of vibration should also be carried out. The results presented in this thesis shows that the damper coefficient has little effect on the amplitudes of vibration for some cases. This result needs to be verified and investigated further.

The cable models in this thesis have been two-dimensional. The next step would be to investigate a non-linear three-dimensional cable model. It might also be worth investigating different placements of the external damper as well as the effect of multiple dampers attached to the cable. A three-dimensional model would also allow for the external load to be applied laterally and at different angles.
Bibliography


