Modeling mapping spaces with short hammocks

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Abstract

We construct a category of short hammocks and show that it has the weak homotopy type of mapping spaces. In doing so we tackle the problem of applying the nerve to large categories without the use of multiple universes. We also explore what the mapping space is. The main tool in showing the connection between hammocks and mapping spaces will be the use of homotopy groupoids, homotopy groupoid actions and the homotopy fiber of their corresponding bar constructions.

Sammanfattning på svenska/*Summary in Swedish*

Låt $\mathcal{M}$ vara en modellkategori och låt $X, Y \in \mathcal{M}$ med Y fibrant. Kategorin $\mathrm{Ham}(X, Y)$ utgörs då av objekten, kallade *korta hängmattor* (eng. *short hammocks*), som är diagram på formen

$$X \xleftarrow{\sim} Z \longrightarrow Y$$

där den indikerade morfin är en svag ekvivalens. En morfi från $X \xleftarrow{\sim} Z \longrightarrow Y$ till $X \xleftarrow{\sim} Z' \longrightarrow Y$ ges av en morfi $h \colon Z \to Z'$ så att följande diagram kommuterar (notera att ett sådant h nödvändigtvis måste vara en svag ekvivalens)

$$\begin{tikzcd} & Z \arrow[dl, "\sim"'] \arrow[d, "h"] \arrow[dr] & \\ X & & Y \\ & Z' \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd} .$$

I [**10**] och [**11**] visade Dwyer och Kan att det finns ett starkt sammanband mellan kategorin $\mathrm{Ham}(X, Y)$ och avbildningsrummet (eng. *mapping space*) $\mathrm{map}(X, Y)$, nämligen att nerven av $\mathrm{Ham}(X, Y)$ har samma svaga homotopityp som $\mathrm{map}(X, Y)$. Eftersom nerven av en kategori enbart är definerad om kategorin är liten, vilket kategorin $\mathrm{Ham}(X, Y)$ inte är, så innebar detta resultat mängdteoretiska problem. Dessa problem behandlades sparsamt i [**11**, 2.2]. I denna uppsats återbevisar vi detta klassiska resultat genom att utveckla de mängdteoretiska argument som antyddes i [**11**, 2.2]. Detta görs genom att använda konceptet *väsentligt små* kategorier (eng. *essentially small*) som introducerades av Blomgren and Chachólski i [**3**]. Vi bevisar följande sats.

Theorem 4.3.6. *Kategorin* $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ *är väsentligt liten och dess svaga homotopityp är* $\mathrm{map}(X, Y)$.

För att bevisa detta teorem så definierar vi homotopigruppoider (eng. *homotopy groupoids*), deras verkan samt deras respektive Borel-konstruktioner. Slutligen så applicerar vi detta teorem för att återbevisa Retakhs teorem från [**23**] på ett mängdteoretiskt korrekt sätt.

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Chapter 1

Introduction

Let $\mathcal{M}$ be a model category and let $X, Y \in \mathcal{M}$ with Y being fibrant. Consider the category $\mathrm{Ham}(X, Y)$ whose objects, called *short hammocks*, are zig-zags of the form

$$X \xleftarrow{\sim} Z \longrightarrow Y$$

where the indicated morphism is a weak equivalence. A morphism from $X \xleftarrow{\sim} Z \longrightarrow Y$ to $X \xleftarrow{\sim} Z' \longrightarrow Y$ is given by $h\colon Z \to Z'$ such that the following diagram commutes (note that such an h necessarily is a weak equivalence).

$$\begin{tikzcd} & Z \arrow[dl, "\sim"'] \arrow[dr] \arrow[dd, "h"] & \\ X & & Y \\ & Z' \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

The notion of hammocks was first introduced by Dwyer and Kan in [**10**] and [**11**]. In [**10**] they showed that taking the nerve of a category of hammocks would yield the correct weak homotopy type for a mapping space from X to Y, or $\mathrm{map}(X, Y)$. This result was made more precise in [**11**]. Only considering short hammocks, or objects of $\mathrm{Ham}(X, Y)$, when Y is fibrant was something that was suggested in [**2**, Remark 2.7]. There they claimed that a proof for the corresponding statement, that the nerve of $\mathrm{Ham}(X, Y)$ has weak homotopy type of $\mathrm{map}(X, Y)$, could be found in [**11**, 7.2]. However in [**8**] Dugger argued that there was a mistake in [**11**, 7.2] and gave a alternate proof of the statement. A alternate proof was also given by Mandell in [**20**]. Thus there exists a well studied connection between short hammocks and mapping spaces.

Something that was not well studied in these papers is the problematic treatment of the nerve of large categories. In particular we have that the category $\mathrm{Ham}(X, Y)$ is a large category and taking its nerve is therefore set theoretically impossible. This problem is tackled in [**11**, 2.2] by briefly mentioning the property of being a *homotopically small simplicial set* and then assuming that the objects studied satisfy this property. In [**3**] Blomgren and Chachólski expanded on this idea and defined the notion of *essential smallness*. This notion made it possible to make sense of statements regarding the nerve of some large categories. In this paper we use this notion to properly prove what was hinted at in [**2**, Remark 2.7]. We prove the following theorem.

Theorem 4.3.6. *The category* $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ *is essentially small and has weak homotopy type of* $\mathrm{map}(X, Y)$*.*

To prove this theorem we carefully set up what we mean by homotopy groupoids, their actions and corresponding Borel constructions.

Lastly, we apply this theorem to make a set-theoretically correct re-proving of Retakh's theorem from [**23**]. There has also been recent interest for the category $\mathrm{Ham}(X, Y)$ for other reasons than mapping spaces in e.g. [**2**], [**9**] and [**16**].

Outline of thesis. In the first chapter we recall some classical definitions and set notations that we use throughout the paper. The reader might skip the rest of this chapter and refer to it when needed. Chapter 2 is devoted to providing the set-theoretical and categorical framework for the thesis. We start by surveying the problems with large categories in section 2.1 and continue with explaining why essential smallness solves these problems in section 2.2. In section 2.3 we remind ourselves of some classical homotopy notions like homotopy colimits/limits. We define, in section 2.4, what a homotopy groupoid and its actions are and we construct their corresponding bar constructions, which are homotopy versions of the classifying space and the Borel construction. In chapter 3 we recall what the mapping space is by roughly summarizing the paper [**7**]. In section 3.1 we define bounded functors so we can use this concept in section 3.2 to define the mapping space functor. In section 3.3 we define the space of weak equivalences for bounded functors and show that it is a homotopy groupoid. We then explain how mapping spaces are acted upon by this homotopy groupoid. Chapter 4 is were we prove our main results. We start by proving some basic properties of the hammock category in section 4.1 and continue to show that the hammock category is essentially small in section 4.2. We are finally able to prove our main result in section 4.3, that the hammock category has the weak homotopy type of the mapping space. Lastly, in section 4.4, we apply our result to re-prove Retakh's theorem.

1.1. Notation and set up

1.1.1. For us, a category, $\mathcal{C}$, means a category where the class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a *small set* (see Definition 2.0.4) for any $X, Y \in \mathrm{ob}\, \mathcal{C}$. By a model category we mean a category that not only satisfies the standard axioms of model categories **MC1-MC5** (see e.g. [**12**]), but also is closed under arbitrary colimits and limits and has a functorial fibrant and functorial cofibrant replacement. The symbol $\mathcal{M}$ is used to denote such a model category. Given a model structure we use the symbols $\xrightarrow{\sim}, \twoheadrightarrow$ and $\hookrightarrow$ to denote weak equivalences, fibrations and cofibrations respectively. We denote by $\gamma_{\mathcal{M}} \colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ the localization functor of $\mathcal{M}$. If $X, Y \in \mathcal{M}$ the morphism set in the homotopy category $\mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$ is denoted by $[X, Y]_{\mathcal{M}}$. Further, if I is a small category and $\mathcal{M}$ is a model category $\mathrm{Ho}(\mathrm{Fun}(I, \mathcal{M}))$ denotes the localization of $\mathrm{Fun}(I, \mathcal{M})$ with respect to objectwise weak equivalences which exists by [**6**, Corollary 15.3].

1.1.2. The category of sets is denoted Sets. The category of (pointed) simplicial sets with the standard model structure (see e.g. [**13**]) is denoted by sSets (sSets_*).

1.1.3. The simplex category, consisting of non-empty finite ordinals and order preserving maps, is denoted by Δ. Its objects are denoted by $[n]$ for $n \in \mathbb{N}$. Functors indexed by Δ^{op} are called simplicial objects. The *standard n-simplex* is the simplicial set given by the functor $\mathrm{Hom}_{\Delta}(-, [n])$ and is denoted by $\Delta[n]$. Given a simplicial set $K \in \mathrm{sSets}$, the *simplex category of K* is the category whose objects are simplicies of K, i.e. maps $\sigma \colon \Delta[n] \to K$, and whose morphisms are commutative triangles

$$\begin{tikzcd} \Delta[m] \arrow[rr, "\alpha"] \arrow[dr, "\tau"'] & & \Delta[n] \arrow[dl, "\sigma"] \\ & K & \end{tikzcd} .$$

The simplex category of K is also denoted by K. It will be clear from the context if we are talking about the simplicial set or its corresponding simplex category.

1.1.4. A simplicial object with values in sSets is also called a bisimplicial set. Given a bisimplicial set $F \colon \Delta^{\mathrm{op}} \to \mathrm{sSets}$ we denote by $\mathrm{diag}(F)$ the simplicial set defined by having n-simplicies

$$\mathrm{diag}(F)_n := (F_n)_n$$

and with face and degeneracy maps being the induced ones.

1.1.5. Let $f \colon \mathcal{B} \to \mathcal{A}$ be a functor and let $a \in \mathcal{A}$ be any object. The *over category* $f \downarrow a$ is the category with objects being morphism of the form

$f(b) \to a$, for some $b \in \mathcal{B}$, and morphisms being commutative diagrams

$$\begin{tikzcd} f(b) \arrow[rr, "f(\beta)"] \arrow[dr] & & f(b') \arrow[dl] \\ & a & \end{tikzcd}$$

where $\beta \colon b \to b'$ is a morphism in $\mathcal{B}$. The category $\mathrm{id}_{\mathcal{A}} \downarrow a$ is also denoted by $\mathcal{A} \downarrow a$ and has an obvious forgetful functor $\mathcal{A} \downarrow a \to \mathcal{A}$. Given a model category $\mathcal{M}$ and a object $\mathcal{M} \in a$ the over category $\mathcal{M} \downarrow a$ can be made into a model category by choosing fibrations, cofibrations and weak equivalences to be exactly those morphisms that become fibrations, cofibrations and weak equivalences in $\mathcal{M}$ under the forgetful functor (see [**15**, Theorem 7.6.5]).

1.1.6. Let $f \colon I \to J$ be a functor of small categories. The left adjoint of the functor $f^* \colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$, $F \mapsto F \circ f$ is called the *left Kan extension of* f and is denoted

$$f^k \colon \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(J, \mathcal{C}).$$

Similarly the right adjoint is called the *right Kan extension of* f and is denoted f_k. As long as the category $\mathcal{C}$ is closed under colimits then the left Kan extensions are guaranteed to exist (see e.g. [**6**]). This is the case for model categories.

1.1.7. Let $\mathcal{C}$ be a category. Given two objects $c, d \in \mathcal{C}$ we say that c and d are *connected* if there exists a finite sequence of morphisms

$$c = x_0 \to x_1 \leftarrow \cdots \leftarrow x_n = d$$

in $\mathcal{C}$.

A *connected component of* $\mathcal{C}$ is an equivalence class under the equivalence relation of being connected. The symbol $\pi_0(\mathcal{C})$ denotes the discrete category whose objects are connected components.

Chapter 2

Category theory and set theory

If you use a trick in logic, whom can you be tricking other than yourself?

Ludvig Wittgenstein

When working with category theory we often come in contact with statements like "consider all topological spaces" or "consider all groups" and so on. From a set-theoretical viewpoint this becomes very problematic, since these statements are often not realizable in conventional set theory. This is due to a well-known paradox which we show later in Proposition 2.0.1. Therefore we need to make it more precise what we mean in category theory when we are talking about "all groups" or even "all sets". There are a lot of different ways of making set-theoretical statements precise in category theory. See e.g. [**21**] for a nice exposition of this vast subject. In this article we are going to use the most conventional and seemingly most used set-theoretical framework, closely following the outline of the books [**19**] and [**4**].

The first thing needed is an axiomatic set theory to build our category theory upon. The conventional choice is the Zermelo-Fraenkel set theory with the Axiom of Choice, abbreviated *ZFC*. To avoid going to much into detail we are going to assume the reader is familiar with ZFC and some basic set-theory in the following discussion, allowing our exposition to be somewhat informal, but still precise enough to get an understanding of the foundation of category theory. Working in ZFC we are going to show that there exists no "set of all sets"

Proposition 2.0.1. *There exists no set S such that*

$$x \in S \Leftrightarrow x \textit{ is a set}$$

Proof. Assume such a set S exists. Construct the collection

$$T = \{x | x \in S \text{ and } x \notin x\}.$$

T defines a subset of S and, in particular, T is a set by the axioms of ZFC. We have that

$$T \in T \text{ or } T \notin T$$

But we also have

$$T \in T \Rightarrow T \notin T$$

and

$$T \notin T \Rightarrow T \in T.$$

In both cases we get a contradiction. Thus a set like S cannot exist.

□

Thus a "set of all sets" cannot exist and in the same way a "set of all groups" cannot exist. To tackle this problem we add the notion of a *universe* (or *Grothendieck-universe*).

Definition 2.0.2. A *universe* is a non-empty set, $\mathcal{U}$, with the following properties

(1) $x \in y$ and $y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$,

(2) $I \in \mathcal{U}$ and $\forall i \in I$ such that $x_i \in \mathcal{U} \Rightarrow \bigcup_{i \in I} x_i \in \mathcal{U}$,

(3) $x \in \mathcal{U} \Rightarrow \mathcal{P}(x) \in \mathcal{U}$,

(4) $x \in \mathcal{U}$ and $f : x \to y$ surjective function $\Rightarrow y \in \mathcal{U}$,

(5) $\mathbb{N} \in \mathcal{U}$

where $\mathbb{N}$ denotes the set of finite ordinals and $\mathcal{P}(x)$ denotes the set of subsets of x.

The definition of universes does not, however, guarantee their existence. For this we need an axiom of existence.

Axiom 2.0.3. *Every set belongs to some universe.*

Equipped with universes we can finally formalize the foundation of category theory.

Definition 2.0.4. Fix a universe $\mathcal{U}_0$. The elements of $\mathcal{U}_0$ are called *small sets* or simply *sets* and the subsets of $\mathcal{U}_0$ are called *classes*.

Note that by construction we have that small sets are classes, but classes are in general not small sets. Applying our tools on categories it makes sense to do the following definition.

Definition 2.0.5. A category, $\mathcal{C}$, is called *small* if ob $\mathcal{C}$ is a small set. A category that is not small is called *large*.

Note that there is no category of categories by Proposition 2.0.1. There is however a category of small categories.

Definition 2.0.6. The category whose objects are small categories and morphisms are functors between small categories is denoted Cat.

2.1. Problems with large categories

Since the aim of this paper is to apply homotopy theory on categories we want to use some tools which only make sense if the category is small. The nerve of a category for example is one such tool which is used extensively.

Definition 2.1.1. Let I be a small category. Then the *nerve* is the functor $N\colon \mathrm{Cat} \to \mathrm{sSets}$ defined by

$$N(I) := \mathrm{mor}_{\mathrm{Cat}^{\mathrm{op}}}([-], I)$$

where we view $[n] \in \Delta$ as a small category for every $n \in \mathbb{N}$.

Unfortunately very seldom the categories we are interested in are small, which makes using these tools impossible. For this reason a canonical way of handling this exact problem has been developed.
The technique is explained in the case of the category of all topological spaces, which is not small, in [**14**, page 655] like this:

> We begin by fixing a universe $\mathcal{U}$ of sets. Now by topological spaces we mean those whose point sets are sets in this strict sense. The category of such spaces is not a small category, any more than the set of all ($\mathcal{U}$-)sets is a ($\mathcal{U}$-)set. Nevertheless, if we are willing to work in two universes, we are not in such bad shape. Introduce a larger universe $\mathcal{U}_0$ in which the set of all $\mathcal{U}$-sets is a set. The category of all ($\mathcal{U}$-)spaces is then $\mathcal{U}_0$-small.

By working in multiple universes the problem seemingly disappears. Unfortunately for us this trick does not always work as one would want, since some categorical tools depend on the chosen universe (see e.g. [**29**]). So then the question becomes: Do we really need large categories? Could not we focus on working with just small categories? We shall see in Theorem 2.1.4 that largeness is actually something to strive for as long as we want our categories to be complete. But first we need to define what we mean by a complete category.

Definition 2.1.2. A category is *complete* if it contains all its small limits and *cocomplete* if it contains all its small colimits.

We also need the following definition.

Definition 2.1.3. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a *preorder* if, for any pair of objects X, Y in $\mathcal{C}$, there exists at most one morphism $X \to Y$.

The motivating theorem for why large categories are needed can then be formulated as follows.

Theorem 2.1.4. *Any complete or cocomplete small category is a preorder.*

Proof. Let us assume, for contradiction, that D is a complete small category but not a preorder. Then there exists at least two different morphisms $x \rightrightarrows y$. Since D is complete and small it contains the product

$$\prod_{f \in \mathrm{mor}(D)} y.$$

But then there exists at least $2^{|\mathrm{mor}(D)|}$ different morphisms of the form

$$x \to \prod_{f \in \mathrm{mor}(D)} y.$$

Since $2^{|\mathrm{mor}(D)|} > |\mathrm{mor}(D)|$ we get a contradiction. The cocomplete case follows from dualization or by using [**1**, Theorem 12.7]. $\square$

Remark 2.1.5. For more discussion on the problems of large categories the interested reader is encouraged to read the preprints [**27**] and [**17**].

2.2. Essentially small categories

What Theorem 2.1.4 tells us is that if we are interested in complete categories we really want them to be large for them to be interesting. In the case of complete categories size does actually matter. And since working with multiple universes was not desirable we need to find a way of representing our large categories with smaller ones for us to be able to talk about the nerve of these categories and more specifically their weak homotopy type. But before we get there let us remind ourselves of the basic concepts of homotopy theory for categories.

Definition 2.2.1. Let $f, g \colon \mathcal{B} \to \mathcal{A}$ be two functors. Then we say that f and g are *homotopic* if there is a finite sequence of functors $\{h_k \colon \mathcal{B} \to \mathcal{A}\}_{0 \leq k \leq n}$ and natural transformations $f = h_0 \to h_1 \leftarrow \cdots \to h_{n-1} \leftarrow h_n = g$,

connecting f and g.
A functor $f\colon \mathcal{B} \to \mathcal{A}$ is called a *homotopy equivalence* if there is a functor $g\colon \mathcal{A} \to \mathcal{B}$ such that gf is homotopic to $\mathrm{id}_{\mathcal{B}}$ and fg is homotopic to $\mathrm{id}_{\mathcal{A}}$. If $f\colon \mathcal{B} \to \mathcal{A}$ is a homotopy equivalence then we say that $\mathcal{A}$ and $\mathcal{B}$ are homotopically equivalent.

Homotopy equivalence is the fundamental concept in homotopy theory for categories and the astute reader has noticed that its definition does not involve the concept of smallness. But unfortunately it is often to harsh to ask two categories to be homotopy equivalent. Often we instead ask categories to be *weakly equivalent* to each other.

Definition 2.2.2. Let $f\colon J \to I$ be a functor between small categories. Then we say that f is a *weak equivalence* if $N(f)\colon N(J) \to N(I)$ is a weak equivalence of simplicial sets.

Here we see the importance of smallness and the nerve construction. As we mentioned earlier we want to find a way to represent large categories by smaller ones to be able to talk about weak equivalences of large categories. One such representation is the *core*.

Definition 2.2.3. Let $\mathcal{C}$ be a category and $I \subset \mathcal{C}$ a subcategory. Then we say that I is a *core* of $\mathcal{C}$ if I is small and for any other small subcategory $J \subset \mathcal{C}$ with $I \subset J$ there exists a small subcategory $K \subset \mathcal{C}$ for which $J \subset K$ and the inclusion $I \subset K$ is a weak equivalence. We say that $\mathcal{C}$ is *essentially small* if it has a core.

Remark 2.2.4. The intuitive idea behind the above definition is that given a sequence of small categories, $I_0 \subset I_1 \subset \cdots$, where each inclusion is a weak equivalence, then $I_0 \subset \mathrm{colim} I_n = \cup_{n\geq 0} I_n$ is also a weak equivalence. Note that the definition of essentially small can be viewed as an expansion of the concept of *homotopically small* as it was introduced by Dwyer and Kan in [**11**, 2.2]

Example 2.2.5. If a category admits a small skeleton then this skeleton is its core.

Unfortunately not every category has a core, as illustrated in the following example.

Example 2.2.6. Let $\mathcal{C}$ be a discrete category where the objects of $\mathcal{C}$ is a proper class (not a set). Then $\mathcal{C}$ does not have a core. This is because if it had one, say I, for every J such that $I \subset J$ the core I would have to be weakly equivalent to some larger subcategory K such that $J \subset K$. By induction this would imply that I would have to contain all objects of $\mathcal{C}$ making I large.

Essentially smallness would be useless if it was not preserved by homotopy equivalences.

Proposition 2.2.7 ([**3**, Corollary 5.9])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be homotopically equivalent. Then $\mathcal{A}$ is essentially small if and only if $\mathcal{B}$ is.*

With the core defined we can finally allow us to talk about the weak homotopy type of large categories fortunate enough to have a core.

Definition 2.2.8. Let $\mathcal{C}$ be a essentially small category and let I be a core. The *weak homotopy type* of $\mathcal{C}$ is defined to be the isomorphism class of $N(I)$ in Ho(sSets). Since it does not depend on the choice of I we denote it by $N(\mathcal{C})$.

The notion of weak homotopy types suggest the existence of weak equivalences for essentially small categories. To define such a notion we are going to slightly alter a classical construction to work with any category regardless of size.

Definition 2.2.9. A *system of categories*, $\mathcal{F}$, indexed by a category $\mathcal{C}$ consists of

- a category $\mathcal{F}_c$ for any object c in $\mathcal{C}$,
- a functor $\mathcal{F}_\alpha \colon \mathcal{F}_{c_0} \to \mathcal{F}_{c_1}$ for any morphism $\alpha : c_1 \to c_0$ in $\mathcal{C}$ such that the two following conditions hold:
  (i) $\mathcal{F}_{\mathrm{id}_c} = \mathrm{id}_{\mathcal{F}_c}$ for any object c in $\mathcal{C}$,
  (ii) $\mathcal{F}_{\alpha\alpha'} = \mathcal{F}_{\alpha'}\mathcal{F}_\alpha$ for any morphisms $\alpha' : c_2 \to c_1$ and $\alpha : c_1 \to c_0$ in $\mathcal{C}$.

Let $\mathcal{F}$ be a system of categories. A *subsystem of* $\mathcal{F}$ consists of a system of categories $\mathcal{G}$ such that $\mathcal{G}_c \subset \mathcal{F}_c$ is a subcategory for any $c \in \mathcal{C}$ and for any morphism $\alpha \colon c_1 \to c_0$ in $\mathcal{C}$ the functor $\mathcal{F}_\alpha \colon \mathcal{F}_{c_0} \to \mathcal{F}_{c_1}$ restricted to $\mathcal{G}_{c_0}$ is $\mathcal{G}_\alpha \colon \mathcal{G}_{c_0} \to \mathcal{G}_{c_1}$. Such a subsystem is denoted by $\mathcal{G} \subset \mathcal{F}$.

Remark 2.2.10. Note that a system of categories can be seen as a generalization of the concept of a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \mathrm{Cat}$$

where we allow large categories in the "image" of $\mathcal{F}$.

Example 2.2.11. A functor $f \colon \mathcal{F}_1 \to \mathcal{F}_0$ can be viewed as a system of categories indexed by the category $0 \to 1$. Similarly a commuting square of functors

$$\begin{tikzcd} \mathcal{F}_{\{0,1\}} \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[d] \\ \mathcal{F}_0 \arrow[r] & \mathcal{F}_\emptyset \end{tikzcd}$$

can be viewed as a system of categories indexed by the poset category of all the subsets of $\{0,1\}$.

The notion of essential smallness can be expanded to apply to systems of categories.

Definition 2.2.12. Let $\mathcal{F}$ be a system of categories indexed by a small category I. A *core* of $\mathcal{F}$ is a subsystem $F \subset \mathcal{F}$ such that $F_i \subset \mathcal{F}_i$ is a core for all $i \in I$. We say that a system of categories $\mathcal{F}$ is *essentially small* if it has a core.

Remark 2.2.13. If $\mathcal{F}$ is an essentially small system of categories with core $F \subset \mathcal{F}$ then F is by construction a contravariant functor

$$F\colon I \to \mathrm{Cat}$$

where I is the indexing category of $\mathcal{F}$.

This gives us the following compact way of saying that a functor of essentially small categories is a weak equivalence.

Definition 2.2.14. Let $f\colon \mathcal{B} \to \mathcal{A}$ be a functor between essentially small categories. Then we say that f is a *weak equivalence* if, when viewed as a system of categories (see Example 2.2.11), it has a core which is a weak equivalence.

Weak equivalences of essentially small categories can be seen as generalizations of homotopy equivalences which the following proposition shows.

Proposition 2.2.15 ([**3**, Proposition 6.2])**.** *Let $f : \mathcal{B} \to \mathcal{A}$ be a homotopy equivalence between essentially small categories. Then f is a weak equivalence.*

Another important reason of defining systems of categories is because it allows us to work with "large" Grothendieck constructions. These constructions are crucial in the proof of our main theorem.

Definition 2.2.16. Let $\mathcal{F}$ be a system of categories indexed by $\mathcal{C}$. We define the *Grothendieck construction of* $\mathcal{F}$, denoted by $\mathbf{Gr}_{\mathcal{C}}\mathcal{F}$, to be the category whose

- objects are pairs (c,x) where $c \in \mathrm{ob}\,\mathcal{C}$, and $x \in \mathrm{ob}\,\mathcal{F}_c$.
- morphisms from (c_0,x_0) to (c_1,x_1) are pairs (α,β) where $\alpha \in \mathrm{Hom}_{\mathcal{C}}(c_0,c_1)$ and $\beta \in \mathrm{Hom}_{\mathcal{F}_{c_0}}(x_0,\mathcal{F}_\alpha(x_1))$.
- composition of two morphisms, $(\alpha,\beta) \in \mathrm{Hom}_{\mathbf{Gr}_{\mathcal{C}}\mathcal{F}}\Big((c_1,x_1),(c_2,x_2)\Big)$ and $(\alpha',\beta') \in \mathrm{Hom}_{\mathbf{Gr}_{\mathcal{C}}\mathcal{F}}\Big((c_0,x_0),(c_1,x_1)\Big)$, is given by
$$(\alpha,\beta)\circ(\alpha',\beta') := (\alpha\circ\alpha', \mathcal{F}_{\alpha'}(\beta)\circ\beta').$$

For any object $c \in \mathcal{C}$ the functor $\mathcal{F}_c \to \mathbf{Gr}_{\mathcal{C}}\mathcal{F}$ which assigns to an object x the pair (c, x) and to a morphism $\beta : x \to y$ the pair (id_c, β) is called the *standard inclusion.*

Lastly we give a non-trivial example of an essentially small category which comes in handy later on when dealing with mapping spaces.

Definition 2.2.17. Let $\mathcal{M}$ be a model category. Then we define $\mathcal{M}_{\text{we}}$ to be the category with the same objects as $\mathcal{M}$ and with morphisms being all the weak equivalences in $\mathcal{M}$.
Further let $X \in \mathcal{M}$ be any object. Then we define the category X_{we} to be the full subcategory of $\mathcal{M}_{\text{we}}$ with objects being weakly equivalent to X.

We have the following result.

Theorem 2.2.18 ([**3**, Theorem 14.1])**.** *Let $\mathcal{M}$ be a model category and let $X \in \mathcal{M}$ be any object. Then X_{we} is essentially small.*

In [**3**] it was shown that not only is X_{we} essentially small but its core is weakly equivalent to the classifying space of weak equivalences of X. The interested reader is encouraged to read sections 13-14 of [**3**]. In this paper we use the above result to prove essential smallness when we have situations matching the following lemma.

Lemma 2.2.19. *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a full subcategory of $\mathcal{M}_{\text{we}}$ such that $\pi_0\mathcal{C}$ is a set. Then $\mathcal{C}$ is essentially small.*

Proof. Let $\sigma \in \mathcal{C}$. By definition $\tau \in \sigma_{\text{we}}$ if we can find a finite sequence of weak equivalences

$$\tau = x_0 \xrightarrow{\sim} x_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} x_n = \sigma$$

in $\mathcal{M}$. But this is the condition for belonging to the same connected component with the added condition of having all the connecting morphisms being weak equivalences. Thus $\tau \in \sigma_{\text{we}}$ if and only if $[\tau] = [\sigma] \in \pi_0\mathcal{C}$, since $\mathcal{C}$ is a full subcategory of $\mathcal{M}_{\text{we}}$. We get that

$$\mathcal{C} = \coprod_{[\sigma] \in \pi_0\mathcal{C}} \sigma_{\text{we}}.$$

σ_{we} is essentially small by Theorem 2.2.18 which makes $\mathcal{C}$ into a coproduct of essentially small categories. This is also essentially small since we can take the core to be the coproduct of the cores for the factors in the coproduct. $\square$

2.3. Homotopy limits and colimits

As we have seen the core allowed us to talk about weak equivalences for a large range of categories, namely the essentially small categories. In this section we focus on developing this subject further. Before we do this we need to go through some standard constructions and concepts. One of the first things to look at when working with homotopy theory of categories are the homotopy colimits and homotopy limits. Remember that if I is a small category and $\mathcal{M}$ is a model category then $\mathrm{Ho}(\mathrm{Fun}(I, \mathcal{M}))$ denotes the localization of $\mathrm{Fun}(I, \mathcal{M})$ with respect to objectwise weak equivalences (which exists by [**6**, Corollary 15.3]).

Definition 2.3.1. Let $\mathcal{M}$ be a model category and I a small category. The *homotopy colimit* (resp. *limit*)

$$\mathrm{hocolim}_I \colon \mathrm{Ho}(\mathrm{Fun}(I, \mathcal{M})) \to \mathrm{Ho}(\mathcal{M})$$
$$(\text{resp. } \mathrm{holim}_I \colon \mathrm{Ho}(\mathrm{Fun}(I, \mathcal{M})) \to \mathrm{Ho}(\mathcal{M}))$$

is the total left (resp. right) derived functor of $\mathrm{colim}_I \colon \mathrm{Fun}(I, \mathcal{M}) \to \mathcal{M}$ (resp. $\lim_I \colon \mathrm{Fun}(I, \mathcal{M}) \to \mathcal{M}$) (for a remainder of what a derived functor is see [**12**, Section 9]). If D is the category $\{a \to b \leftarrow c\}$, with three objects and two indicated nonidentity morphisms, and $F \colon D \to \mathcal{M}$ is any functor, then we call $\mathrm{holim}_D F$ the *homotopy pullback of* F.

Existence of homotopy colimits and homotopy limits are guaranteed by the following proposition.

Proposition 2.3.2 ([**6**, Corollary 16.2 & Section 31])**.** *Let* $\mathcal{M}$ *be a model category and* I *a small category. Then* $\mathrm{hocolim}_I$ *and* holim_I *exists.*

The model category sSets have certain homotopy pullbacks which are of particular interest.

Definition 2.3.3. Let $f \colon Y \to X$ be a morphism in sSets. Then the homotopy pullback

$$\mathrm{holim}\left(\Delta[0] \overset{x_0}{\to} X \overset{f}{\longleftarrow} Y\right)$$

is called the *homotopy fiber of* f *over* x_0 which we denote by $\mathrm{hofib}_{x_0}(f)$.

As is usual in category theory we see that the homotopy colimit/limit is defined by having a certain universal property, namely being a total derived functor. For practical purposes however one usually wants to work with concrete constructions that take place in $\mathcal{M}$ and not in $\mathrm{Ho}(\mathcal{M})$. Such rigidfications are called models for hocolim/holim.

Definition 2.3.4. Any functor $G \colon \mathrm{Fun}(I, \mathcal{M}) \to \mathcal{M}$ that fits into the following commuting diagram

$$\begin{tikzcd}
\mathrm{Fun}(I, \mathcal{M}) \arrow[r, "G"] \arrow[d, "\gamma"] & \mathcal{M} \arrow[d, "\gamma_{\mathcal{M}}"] \\
\mathrm{Ho}(\mathrm{Fun}(I, \mathcal{M})) \arrow[r, "\mathrm{hocolim}_I"] & \mathrm{Ho}(\mathcal{M})
\end{tikzcd}$$

is called a *model* for $\mathrm{hocolim}_I$. Similarly any functor that fits into the following commuting diagram

$$\begin{tikzcd} \mathrm{Fun}(I,\mathcal{M}) \arrow[r, "G"] \arrow[d, "\gamma"] & \mathcal{M} \arrow[d, "\gamma_{\mathcal{M}}"] \\ \mathrm{Ho}(\mathrm{Fun}(I,\mathcal{M})) \arrow[r, "\mathrm{holim}_I"] & \mathrm{Ho}(\mathcal{M}) \end{tikzcd}$$

is called a *model* for holim_I.

There is also a concrete way to talk about homotopy pullbacks.

Definition 2.3.5. A commuting square in $\mathcal{M}$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

is called a *homotopy pullback* if the induced map $P \to \mathrm{holim}(C \to B \leftarrow A)$ is an isomorphism in $\mathrm{Ho}(\mathcal{M})$. If further $\mathcal{M}$ is sSets, C is contractible and c is a vertex in the image of g then we say that P is the *homotopy fiber of f over c* and we denote it by $\mathrm{hofib}_c(f)$.

In this paper we see two examples of models for the homotopy colimit. One of these constructions arises in the case of bisimplicial sets while the other is Thomason's theorem.

Lemma 2.3.6 ([**15**, Corollary 18.7.7])**.** *Let $F\colon \Delta^{\mathrm{op}} \to \mathrm{sSets}$ be a bisimplicial set. Then $\mathrm{diag}(F)$ is a model for $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} F$.*

Theorem 2.3.7 ([**28**, Theorem 1.2])**. (Thomason's theorem)**
Let I be a small category and $F\colon I^{\mathrm{op}} \to \mathrm{Cat}$ be a functor. Then $N(\mathbf{Gr}_I F)$ is a model for $\mathrm{hocolim}_{I^{\mathrm{op}}} N(F)$.

We saw in Definition 2.2.2 that we could define weak equivalences for small categories by using the weak equivalences of simplicial sets. This led us to a definition of weak equivalences of essentially small categories in Definition 2.2.14. In the same way we can expand the notions of homotopy pullbacks and homotopy fibers.

Definition 2.3.8. We say that a commuting square of small categories is a *homotopy pullback* if after applying the nerve we get a homotopy pullback of simplicial sets. Similarly we say that a commuting square of essentially small categories is a *homotopy pullback* if it, when viewed as a system of categories (see Example 2.2.11), has a core that is a homotopy pullback. Further let $f\colon \mathcal{B} \to \mathcal{A}$ be a functor of essentially small categories and $a \in \pi_0(\mathcal{A})$ be a component of $\mathcal{A}$. Then we say that a essentially small category $\mathcal{D}$ is the *homotopy fiber of f over a*, denoted $\mathrm{hofib}_a(f)$, if

$$N(\mathcal{D}) = \mathrm{hofib}_{N(a)}(N(f))$$

where it is understood that we take the nerve of the respective cores.

Precisely as in the case of the homotopy colimits we would want to work with something tangible rather than working with a universal property when deciding if a diagram is a homotopy pullback. One such way is to check if the diagram induces a weak equivalence on homotopy fibers.

Lemma 2.3.9 ([**5**, 5.2])**.** *A commuting square*

$$\begin{tikzcd} L \arrow[r] \arrow[d, "g"] & J \arrow[d, "f"] \\ K \arrow[r, "p"] & I \end{tikzcd}$$

in sSets *is a homotopy pullback if and only if the induced maps of homotopy fibers*

$$\mathrm{hofib}_k(g) \xrightarrow{\sim} \mathrm{hofib}_{p(k)}(f)$$

are weak equivalences for all $k \in K$.

Example 2.3.10. Let

$$\begin{tikzcd} L \arrow[r] \arrow[d] & J \arrow[d, two heads, "f"] \\ K \arrow[r] & I \end{tikzcd}$$

be a pullback square in sSets where f is a fibration. Then by Lemma 2.3.9 it is also a homotopy pullback. Further if K is contractible then L is weakly equivalent to $\mathrm{hofib}_K(f)$.

The above example illustrates the importance of fibrations in homotopy pullbacks and we would like to extend this example to essentially small categories. For that we need the notion of a strong fibration.

Definition 2.3.11. A functor $f\colon \mathcal{B} \to \mathcal{A}$ is called a *strong fibration* if $f \downarrow \alpha\colon f \downarrow a_1 \to f \downarrow a_0$ is a homotopy equivalence for any morphism $\alpha\colon a_1 \to a_0$ in $\mathcal{A}$.

Remark 2.3.12. It should be noted that our definition of a strong fibration is dual to the corresponding definition in [**3**]. Thus we need to dualize some of the propositions we get from there.

Lemma 2.3.13. *Let $f\colon \mathcal{B} \to \mathcal{A}$ be a strong fibration between essentially small categories. Then $f \downarrow a$ is essentially small for any object $a \in \mathcal{A}$.*

Proof. This follows from dualizing [**3**, Proposition 6.5]. The map $f\colon \mathcal{B} \to \mathcal{A}$ induces a map $f^\vee\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$. This gives us $a \uparrow f^\vee = (f \downarrow a)^{\mathrm{op}}$ and the results follow by observing that $N(A)$ is weakly equivalent to $N(A^{\mathrm{op}})$. □

The analogue of Example 2.3.10 can now be stated as the following theorem, known as Quillen's theorem.

Theorem 2.3.14. (Quillen's theorem)
Let $f\colon \mathcal{B} \to \mathcal{A}$ be a strong fibration between essentially small categories. Then, for any object $a \in \mathcal{A}$, the following is a homotopy pullback square:

$$\begin{tikzcd} f \downarrow a \arrow[r, "\text{forget}"] \arrow[d] & \mathcal{B} \arrow[d, "f"] \\ \mathcal{A} \downarrow a \arrow[r, "a"] & \mathcal{A} \end{tikzcd}$$

Proof. Let $B \subset \mathcal{B}$ and $A \subset \mathcal{A}$ be cores. By Lemma 2.3.13, the following is a core for the square:

$$\begin{tikzcd} f \downarrow a \arrow[r, "\text{forget}"] \arrow[d] & B \arrow[d, "f"] \\ A \downarrow a \arrow[r, "a"] & A \end{tikzcd}$$

By dualizing [**3**, Proposition 4.2 (4)], using the same argument as in Lemma 2.3.13, we get that this is a homotopy pullback square since $A \downarrow a$ has a terminal object, namely $\mathrm{id} : a \to a$, and thus is contractible. □

2.4. Homotopy groupoids and actions

One important tool that we need in our proof of our main theorem is the notion of a homotopy groupoid. This notion is an generalization of the more familiar notion of an enrichment.

Definition 2.4.1. Let $\mathcal{S}$ be a class. An *enrichment indexed by* $\mathcal{S}$, $G_{\mathcal{S}}$, consists of

- simplicial sets $G(r,t)$ for every pair $r,t \in \mathcal{S}$,
- maps $\diamond\colon G(r,s) \times G(s,t) \to G(r,t)$ for every triple $r,s,t \in \mathcal{S}$,
- maps $e_r\colon \Delta[0] \to G(r,r)$ for every $r \in \mathcal{S}$,

such that the following properties are satisfied:

(1) For every quadruple $r, s, t, v \in \mathcal{S}$ the following diagram commutes

$$\begin{tikzcd} G(r,s) \times G(s,t) \times G(t,v) \arrow[r, "\diamond \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \diamond"] & G(r,t) \times G(t,v) \arrow[d, "\diamond"] \\ G(r,s) \times G(s,v) \arrow[r, "\diamond"] & G(r,v) \end{tikzcd}$$

(2) For every pair $r, s \in \mathcal{S}$ the following diagrams commute

$$\begin{tikzcd} \Delta[0] \times G(r,s) \arrow[r, "e_r \times \mathrm{id}"] \arrow[dr, "\mathrm{pr}"'] & G(r,r) \times G(r,s) \arrow[d, "\diamond"] \\ & G(r,s) \end{tikzcd} \qquad \begin{tikzcd} G(r,s) \times G(s,s) \arrow[d, "\diamond"] & G(r,s) \times \Delta[0] \arrow[l, "id \times e_s"'] \arrow[dl, "\mathrm{pr}"] \\ G(r,s) & \end{tikzcd}$$

If for every pair $s, t \in \mathcal{S}$ the simplicial set $G(s,t)$ is non-empty we say that $G_{\mathcal{S}}$ is *connected*. Further if $\mathcal{S}$ is a set we say that $G_{\mathcal{S}}$ is *small*.

Demanding that the simplicial sets of an enrichment are fibrant and that certain commuting squares are homotopy pullbacks gives us the notion of a homotopy groupoid.

Definition 2.4.2. Let $G_{\mathcal{S}}$ be an enrichment indexed by a class $\mathcal{S}$. We say that $G_{\mathcal{S}}$ is a *homotopy groupoid indexed by* $\mathcal{S}$ if for every pair $r, t \in \mathcal{S}$ the simplicial set $G(r,t)$ is fibrant and for every triple $r, s, t \in \mathcal{S}$ the following diagrams are homotopy pullbacks

$$\begin{tikzcd} G(r,s) \times G(s,t) \arrow[r, "\diamond"] \arrow[d, "\mathrm{pr}"] & G(r,t) \arrow[d, two heads] \\ G(r,s) \arrow[r] & \Delta[0] \end{tikzcd} \qquad \begin{tikzcd} G(r,s) \times G(s,t) \arrow[r, "\diamond"] \arrow[d, "\mathrm{pr}"] & G(r,t) \arrow[d, two heads] \\ G(s,t) \arrow[r] & \Delta[0] \end{tikzcd}$$

The definition of a homotopy groupoid is a generalization of the concept of a group which the following example demonstrates.

Example 2.4.3. Let M be a monoid with a unit. Then we can create a enrichment $M_\star$ indexed by a single object, $\star$, by regarding M as a discrete simplicial set and setting $M(\star, \star) = M$. The maps $\diamond$ are given by the binary operation of M and the image of the map $e_\star$ is the unit of M. Condition (1) in Definition 2.4.1 then follows because the monoidal operation is associative and condition (2) follows because M is assumed to have a unit.
Let G be any group and construct its corresponding enrichment $G_\star$. Then $G_\star$ is a homotopy groupoid. This is because discrete simplicial sets are always fibrant and further the homotopy pullback conditions of Definition 2.4.2 translate into demanding the existence of inverses. This can be seen

by applying Lemma 2.3.9 to the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\diamond"] \arrow[d, "\mathrm{pr}"] & G \arrow[d, two heads] \\ G \arrow[r] & \Delta[0] \end{tikzcd} .$$

which tells us that $g \cdot G \simeq G$ for the diagram to be a homotopy pullback. Since G has inverses this is true.

We also need to consider an action of a homotopy groupoid.

Definition 2.4.4. Let $G_{\mathcal{S}}$ be a homotopy groupoid indexed by a class $\mathcal{S}$. A *$G_{\mathcal{S}}$-space*, $X_{\mathcal{S}}$, consists of a simplical set $X(s)$ for any $s \in \mathcal{S}$ and a map

$$*\colon G(s,t) \times X(t) \to X(s)$$

for every pair $s, t \in \mathcal{S}$ such that the following properties are satisfied:

(1) For every triple $r, s, t \in \mathcal{S}$ the following diagram commutes

$$\begin{tikzcd} G(r,s) \times G(s,t) \times X(t) \arrow[r, "\diamond\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times *"] & G(r,t) \times X(t) \arrow[d, "*"] \\ G(r,s) \times X(s) \arrow[r, "*"] & X(r) \end{tikzcd}$$

(2) For every $s \in \mathcal{S}$ the following diagram commutes

$$\begin{tikzcd} \Delta[0] \times X(s) \arrow[r, "e_s\times\mathrm{id}"] \arrow[dr, "\mathrm{pr}"'] & G(s,s) \times X(s) \arrow[d, "*"] \\ & X(s) \end{tikzcd}$$

(3) For every pair $s, t \in \mathcal{S}$ the following diagram is a homotopy pullback

$$\begin{tikzcd} G(s,t) \times X(t) \arrow[r, "*"] \arrow[d, "\mathrm{pr}"] & X(s) \arrow[d] \\ G(s,t) \arrow[r] & \Delta[0] \end{tikzcd}$$

The map $*\colon G(s,t) \times X(t) \to X(s)$ is called the *homotopy groupoid action of $G_{\mathcal{S}}$ on $X_{\mathcal{S}}$*.

The definition of a homotopy groupoid action is a generalization of the concept of a group action which the following example demonstrates.

Example 2.4.5. Let G be a group, X be a set and let $*\colon G \times X \to X$ be a group action of G on X. Define the corresponding homotopy groupoid of G, denoted $G_\star$, just like in Example 2.4.3. Then we can create a $G_\star$-space, $X_\star$, by setting $X(\star) := X$ and using $*$ as the homotopy groupoid action.

Condition (1) and (2) from Definition 2.4.4 follows because of corresponding compatibility conditions for group actions. Condition (3) of Definition 2.4.4 follows from the fact that for any $g \in G$ the map $g * -\colon X \to X$ is bijective, which translates (by applying Lemma 2.3.9) to the following square being a homotopy pullback

$$\begin{tikzcd} G \times X \arrow[r, "*"] \arrow[d, "\mathrm{pr}"] & X \arrow[d, two heads] \\ G \arrow[r] & \Delta[0] \end{tikzcd} .$$

The whole purpose of using homotopy groupoids and their actions is the possibility of taking their bar constructions. These are analogues of the Borel constructions. But before defining the bar construction we need a preliminary definition.

Definition 2.4.6. Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$ and let $X_{\mathcal{S}}$ be a $G_{\mathcal{S}}$-space. Define the products for $t_0, \ldots, t_n, t \in \mathcal{S}$ and $n \geq 1$ by

$$\begin{aligned} \mathcal{B}G_{t_0,\ldots,t_n} &:= \prod_{k=0}^{n-1} G(t_k, t_{k+1}) \ , & \mathcal{B}G_t &:= \Delta[0], \\ \mathcal{E}X_{t_0,\ldots,t_n} &:= \mathcal{B}G_{t_0,\ldots,t_n} \times X(t_n) \ , & \mathcal{E}X_t &:= X(t). \end{aligned}$$

Further we define:

$$\begin{aligned} \mathrm{pr}_0 &\colon \mathcal{B}G_{t_0,\ldots,t_n} \to \mathcal{B}G_{t_1,\ldots,t_n} \\ \mathrm{pr}_n &\colon \mathcal{B}G_{t_0,\ldots,t_n} \to \mathcal{B}G_{t_0,\ldots,t_{n-1}} \end{aligned}$$

to be the projection onto to the last n factors and respectively the n first factors when $n > 1$ or the unique maps when $n = 1$.

We can now define the bar construction.

Definition 2.4.7. Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$. Then the *bar construction of* $G_{\mathcal{S}}$, denoted $\mathcal{B}G_{\mathcal{S}}$, is defined degreewise by

$$(\mathcal{B}G_{\mathcal{S}})_n := \coprod_{t_0,\ldots,t_n \in \mathcal{S}} \mathcal{B}G_{t_0,\ldots,t_n}$$

with face maps $d_i^{\mathcal{B}}\colon (\mathcal{B}G_{\mathcal{S}})_n \to (\mathcal{B}G_{\mathcal{S}})_{n-1}$ defined componentwise by

$$\begin{aligned} d_i^{\mathcal{B}} &:= \mathrm{id}^{i-1} \times \diamond \times \mathrm{id}^{n-1-i} \ , \quad 0 < i < n \\ d_0^{\mathcal{B}} &:= \mathrm{pr}_0 \ , \quad d_n^{\mathcal{B}} := \mathrm{pr}_n \end{aligned}$$

where id^k is the product of k copies of id and $\diamond$ is the binary map of $G_{\mathcal{S}}$, and with degeneracy maps $s_j^{\mathcal{B}}\colon (\mathcal{B}G_{\mathcal{S}})_n \to (\mathcal{B}G_{\mathcal{S}})_{n+1}$ defined componentwise by

$$s_j^{\mathcal{B}} := (\mathrm{id}^j \times e_{t_j} \times \mathrm{id}^{n-j}) \circ \sigma_j \ , \quad 0 \leq j \leq n$$

where σ_j is the natural isomorphism

$$\mathcal{B}G_{t_0,\ldots,t_n} \to \mathcal{B}G_{t_0,\ldots,t_j} \times \Delta[0] \times \mathcal{B}G_{t_j,\ldots,t_n}$$

and e_{t_j} is the map given from $G_{\mathcal{S}}$.

Given a homotopy groupoid action we also need a version of the Borel construction. We call this $\mathcal{E}X_{\mathcal{S}}$.

Definition 2.4.8. Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$ and $X_{\mathcal{S}}$ be a $G_{\mathcal{S}}$-space. Then the *Borel construction of* $X_{\mathcal{S}}$, denoted $\mathcal{E}X_{\mathcal{S}}$, is defined degreewise by

$$(\mathcal{E}X_{\mathcal{S}})_n := \coprod_{t_0,\ldots,t_n \in \mathcal{S}} \mathcal{E}X_{t_0,\ldots,t_n}$$

with face maps $d_i^{\mathcal{E}} \colon (\mathcal{E}X_{\mathcal{S}})_n \to (\mathcal{E}X_{\mathcal{S}})_{n-1}$ defined componentwise by

$$d_i^{\mathcal{E}} := d_i^{\mathcal{B}} \times \mathrm{id}\ , \quad i < n,$$
$$d_n^{\mathcal{E}} := \mathrm{id}^n \times *,$$

and with degeneracy maps $s_j^{\mathcal{E}} \colon (\mathcal{E}X_{\mathcal{S}})_n \to (\mathcal{E}X_{\mathcal{S}})_{n+1}$ defined componentwise by

$$s_j^{\mathcal{E}} := s_j^{\mathcal{B}} \times \mathrm{id}\ , \quad 0 \leq j \leq n.$$

Obviously we want these constructions to be bisimplicial sets.

Proposition 2.4.9. *Let $G_{\mathcal{S}}$ be a small homotopy groupoid indexed by a set $\mathcal{S}$ and $X_{\mathcal{S}}$ be a $G_{\mathcal{S}}$-space. Then $\mathcal{B}G_{\mathcal{S}}$ and $\mathcal{E}X_{\mathcal{S}}$ are both bisimplicial sets.*

Proof. We need to check that the face maps and degeneracy maps satisfies the standard simplicial identities. This is a standard argument and follows from properties (1) and (2) in Definition 2.4.1 in the case of $\mathcal{B}G_{\mathcal{S}}$ together with property (1) in Definition 2.4.4 in the case of $\mathcal{E}X_{\mathcal{S}}$. We leave these calculations to the reader. □

Given a homotopy groupoid $G_{\mathcal{S}}$ and a $G_{\mathcal{S}}$-space $X_{\mathcal{S}}$ we can use projection maps to construct a map between $\mathcal{B}G_{\mathcal{S}}$ and $\mathcal{E}X_{\mathcal{S}}$.

Definition 2.4.10. Define the map $\pi \colon \mathcal{E}X_{\mathcal{S}} \to \mathcal{B}G_{\mathcal{S}}$ degreewise by

$$\pi_n := \coprod_{t_0,\ldots,t_n \in \mathcal{S}} \pi_{t_0,\ldots,t_n}$$

where the maps $\pi_{t_0,\ldots,t_n} \colon \mathcal{B}G_{t_0,\ldots,t_n} \times X(t_n) \to \mathcal{B}G_{t_0,\ldots,t_n}$ are the projections onto the first component.

By construction we get the following result.

Proposition 2.4.11. *Let $\mathcal{S}$, $G_{\mathcal{S}}$ and $X_{\mathcal{S}}$ be as above. Then the diagrams*

$$\begin{tikzcd} (\mathcal{E}X_{\mathcal{S}})_n \arrow[r, "d_i^{\mathcal{E}}"] \arrow[d, "\pi_n"'] & (\mathcal{E}X_{\mathcal{S}})_{n-1} \arrow[d, "\pi_{n-1}"] \\ (\mathcal{B}G_{\mathcal{S}})_n \arrow[r, "d_i^{\mathcal{B}}"] & (\mathcal{B}G_{\mathcal{S}})_{n-1} \end{tikzcd} \qquad \begin{tikzcd} (\mathcal{E}X_{\mathcal{S}})_n \arrow[r, "s_j^{\mathcal{E}}"] \arrow[d, "\pi_n"'] & (\mathcal{E}X_{\mathcal{S}})_{n+1} \arrow[d, "\pi_{n+1}"] \\ (\mathcal{B}G_{\mathcal{S}})_n \arrow[r, "s_j^{\mathcal{B}}"] & (\mathcal{B}G_{\mathcal{S}})_{n+1} \end{tikzcd}$$

are both homotopy pullbacks for all i and all j.

Proof. Since everything is constructed componentwise it is enough to consider the maps componentwise.
We start with the left diagram. In the case when $i = n$ then $d_i^{\mathcal{E}}$ uses the homotopy groupoid action and the result follows from property (2) in Definition 2.4.4. When $i < n$, if we use $\hat{t_i}$ to indicate that an element has been excluded, then the homotopy fibers in the diagram

$$\begin{tikzcd} \mathcal{E}X_{t_0,\dots,t_n} \arrow[r, "d_i^{\mathcal{E}}"] \arrow[d, "\pi_{t_0,\dots,t_n}"'] & \mathcal{E}X_{t_0,\dots,\hat{t_i},\dots,t_n} \arrow[d, "\pi_{t_0,\dots,\hat{t_i},\dots,t_n}"] \\ \mathcal{B}G_{t_0,\dots,t_n} \arrow[r, "d_i^{\mathcal{B}}"] & \mathcal{B}G_{t_0,\dots,\hat{t_i},\dots,t_n} \end{tikzcd}$$

are both $X(t_n)$ and have an induced weak equivalence between them so the result follows from Lemma 2.3.9.
The right diagram can be treated exactly like the case for $i < n$ in the left diagram. $\square$

We can combine this result with the following useful lemma.

Lemma 2.4.12 ([**5**, Theorem 6.2])**.** *Let $F, G\colon I \to \mathrm{sSets}$ be two functors indexed by a small category I and let $\psi\colon F \to G$ be a natural transformation. Further assume that for any morphism $\alpha\colon i \to j$ in I the diagram*

$$\begin{tikzcd} F(i) \arrow[r, "F(\alpha)"] \arrow[d, "\psi_i"'] & F(j) \arrow[d, "\psi_j"] \\ G(i) \arrow[r, "G(\alpha)"] & G(j) \end{tikzcd}$$

is a homotopy pullback. Then for any $k \in I$ the diagram

$$\begin{tikzcd} F(k) \arrow[r] \arrow[d, "\psi_i"'] & \mathrm{hocolim}_I F \arrow[d, "\mathrm{hocolim}_I \psi"] \\ G(k) \arrow[r] & \mathrm{hocolim}_I G \end{tikzcd}$$

is also a homotopy pullback where the horizontal maps are induced by the inclusion $k \hookrightarrow I$.

This gives us the following.

Corollary 2.4.13. *For any $t \in \mathcal{S}$ the following diagrams are homotopy pullbacks*

$$\begin{tikzcd} X(t) \arrow[r] \arrow[d] & \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{E}X_{\mathcal{S}} \arrow[d, "\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \pi"] \\ \Delta[0] \arrow[r] & \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{B}G_{\mathcal{S}} \end{tikzcd} \qquad \begin{tikzcd} X(t) \arrow[r] \arrow[d] & \mathrm{diag}(\mathcal{E}X_{\mathcal{S}}) \arrow[d, "\mathrm{diag}(\pi)"] \\ \Delta[0] \arrow[r] & \mathrm{diag}(\mathcal{B}G_{\mathcal{S}}) \end{tikzcd}$$

where the horizontal maps are induced by the inclusion $\Delta[0] \hookrightarrow \Delta^{\mathrm{op}}$.

Proof. That the left diagram is a homotopy pullback follows from identifying $\mathcal{E}X_t$ with $X(t)$ and $\mathcal{B}G_t$ with $\Delta[0]$ and then using Lemma 2.4.12 together with the fact that the face maps and the degeneracy maps generate all the simplicial maps. It then follows that the right diagram is also a homotopy pullback by Lemma 2.3.6. □

Finally we prove a proposition which is very helpful in proving essential smallness of bar-like constructions.

Proposition 2.4.14. *Let $\mathcal{S}' \subset \mathcal{S}$ be non-empty sets, $G_{\mathcal{S}}$ a connected small homotopy groupoid and $X_{\mathcal{S}}$ a $G_{\mathcal{S}}$-space. Further let $G_{\mathcal{S}'}$ and $X_{\mathcal{S}'}$ be the restrictions of $G_{\mathcal{S}}$ and $X_{\mathcal{S}}$ to $\mathcal{S}'$ respectively. Then the commutative square of simplicial maps*

$$\begin{tikzcd} \mathcal{E}X_{\mathcal{S}'} \arrow[r, hook] \arrow[d, "\pi"'] & \mathcal{E}X_{\mathcal{S}} \arrow[d, "\pi"] \\ \mathcal{B}G_{\mathcal{S}'} \arrow[r, hook] & \mathcal{B}G_{\mathcal{S}} \end{tikzcd}$$

induces a commutative square of simplicial sets

$$\begin{tikzcd} \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{E}X_{\mathcal{S}'} \arrow[r, "\sim"] \arrow[d, "\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \pi"'] & \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{E}X_{\mathcal{S}} \arrow[d, "\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \pi"] \\ \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{B}G_{\mathcal{S}'} \arrow[r, "\sim"] & \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{B}G_{\mathcal{S}} \end{tikzcd}$$

where the horizontal maps are weak equivalences.

Proof. That $\mathcal{B}G_{\mathcal{S}'} \subset \mathcal{B}G_{\mathcal{S}}$ induces a weak equivalence between connected simplicial sets

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{B}G_{\mathcal{S}'} \xrightarrow{\sim} \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{B}G_{\mathcal{S}}$$

is proven in [**3**, Proposition 19.7 (1) & (4)]. Thus we only need to prove that $\mathcal{E}X_{\mathcal{S}'} \subset \mathcal{E}X_{\mathcal{S}}$ induces a weak equivalence between simplicial sets

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{E}X_{\mathcal{S}'} \xrightarrow{\sim} \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{E}X_{\mathcal{S}} .$$

First we note that if $X(s_0)$ is empty for some $s_0 \in \mathcal{S}$ then it is empty for all $s \in \mathcal{S}$ by property (2) in Definition 2.4.4 since $G_{\mathcal{S}}$ is connected. In that case $\mathcal{E}X = \mathcal{B}G$ and the proposition is trivially true.
Thus we can assume that all $X(s)$ are non-empty. But from Corollary 2.4.13 we then get the following diagram of homotopy fibers

$$\begin{tikzcd} X(t) \arrow[r, "\sim"] \arrow[d] & X(t) \arrow[d] \\ \operatorname{hocolim}_{\Delta^{op}} \mathcal{E}G_{\mathcal{S}'} \arrow[r] \arrow[d, "\operatorname{hocolim}_{\Delta^{op}} \pi"'] & \operatorname{hocolim}_{\Delta^{op}} \mathcal{E}G_{\mathcal{S}} \arrow[d, "\operatorname{hocolim}_{\Delta^{op}} \pi"] \\ \operatorname{hocolim}_{\Delta^{op}} \mathcal{B}G_{\mathcal{S}'} \arrow[r, "\sim"] & \operatorname{hocolim}_{\Delta^{op}} \mathcal{B}G_{\mathcal{S}} \end{tikzcd}$$

for some $t \in \mathcal{S}' \subset \mathcal{S}$. This combined with taking the long exact sequences of homotopy groups gives that the middle arrow must be a weak equivalence since $\operatorname{hocolim}_{\Delta^{op}} \mathcal{B}G_{\mathcal{S}'}$ and $\operatorname{hocolim}_{\Delta^{op}} \mathcal{B}G_{\mathcal{S}}$ were connected. □

Corollary 2.4.15. *Let $\mathcal{S}' \subset \mathcal{S}$ be non-empty sets, $G_{\mathcal{S}}$ a connected small homotopy groupoid and $X_{\mathcal{S}}$ a $G_{\mathcal{S}}$-space. Further let $G_{\mathcal{S}'}$ and $X_{\mathcal{S}'}$ be the restrictions of $G_{\mathcal{S}}$ and $X_{\mathcal{S}}$ to $\mathcal{S}'$ respectively. Then we have an induced commutative square of simplicial sets*

$$\begin{tikzcd} \operatorname{diag}(\mathcal{E}X_{\mathcal{S}'}) \arrow[r, "\sim"] \arrow[d, "\operatorname{diag}(\pi)"'] & \operatorname{diag}(\mathcal{E}X_{\mathcal{S}}) \arrow[d, "\operatorname{diag}(\pi)"] \\ \operatorname{diag}(\mathcal{B}G_{\mathcal{S}'}) \arrow[r, "\sim"] & \operatorname{diag}(\mathcal{B}G_{\mathcal{S}}) \end{tikzcd}$$

where the horizontal maps are weak equivalences.

Proof. This follows by combining Proposition 2.4.14 with Lemma 2.3.6. □

Chapter 3

The mapping space

"Space," it says, "is big. Really big. You just won't believe how vastly, hugely, mindbogglingly big it is. I mean, you may think it's a long way down the road to the chemist's, but that's just peanuts to space, listen..."

Douglas Adams
The Hitchhiker's Guide to the Galaxy

Model categories were developed by Quillen in his pivotal paper [**22**] which was published 1967. But it was not until in the '80s that Dwyer and Kan, in [**11**], noticed that model categories carried with them an inherent continuous structure for morphims, now known as mapping spaces. This structure was shown to exist constructively using so called hammocks. However the natural way of defining objects is by characterising them by a universal property. The aim of this section is to recall such a universal property for mapping spaces that was used by Chachólski and Scherer in [**7**]. We also recall the construction of the mapping spaces from [**7**] as it is convenient for our purposes.

Recall that the homotopy colimit is functorial with respect to the indexing category. Thus, for a fixed $X \in \mathcal{M}$, we obtain a functor

$$\mathrm{hocolim}_{(-)} cX \colon \mathrm{sSets} \to \mathrm{Ho}(\mathcal{M})$$

where $cX \colon \mathrm{sSets} \to \mathcal{M}$ is the constant functor in $\mathcal{M}$ with value X. In general the homotopy colimit is not homotopy invariant with respect to weak equivalences of the indexing categories. It is however true if the functors involved are constant:

Proposition 3.0.1 ([**7**, Corollary 7.2])**.** *If $f : K \to L$ is a weak equivalence of simplicial sets, then, for any $X \in \mathcal{M}$*

$$\mathrm{hocolim}_f cX \colon \mathrm{hocolim}_K cX \to \mathrm{hocolim}_L cX$$

is an isomorphism in $\mathrm{Ho}(\mathcal{M})$*.*

It follows that $\mathrm{hocolim}_{(-)} cX \colon \mathrm{sSets} \to \mathrm{Ho}(\mathcal{M})$ factors as a composition of the localization $\gamma \colon \mathrm{sSets} \to \mathrm{Ho}(\mathrm{sSets})$ and a functor that we denote by $- \otimes_h X \colon \mathrm{Ho}(\mathrm{sSets}) \to \mathrm{Ho}(\mathcal{M})$:

$$\begin{tikzcd} \mathrm{sSets} \arrow[r, "\gamma"] \arrow[rr, bend left, "\mathrm{hocolim}_{(-)} cX"] & \mathrm{Ho}(\mathrm{sSets}) \arrow[r, "- \otimes_h X"] & \mathrm{Ho}(\mathcal{M}) \end{tikzcd} .$$

This functor has the following important property.

Proposition 3.0.2 ([**7**, Proposition 9.4])**.** *Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$ any object. Then $- \otimes_h X$ has a right adjoint.*

This right adjoint turns out to be the defining universal property of the mapping space.

Definition 3.0.3. Given $X \in \mathcal{M}$, the right adjoint of the functor $- \otimes_h X \colon \mathrm{Ho}(\mathrm{sSets}) \to \mathrm{Ho}(\mathcal{M})$ is denoted

$$\mathrm{map}(X, -) : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathrm{sSets})$$

and its values are the *weak homotopy type of the mapping spaces.*

Given the universal property for the mapping space, the next step is to look for a construction that satisfies this universal property. For our purposes we need such a construction together with a strictly associative composition. Our goal is thus to construct a functor

$$\mathrm{map}_{\mathcal{M}}(-,-) \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{sSets}$$

together with:

- a composition map $\circ \colon \mathrm{map}_{\mathcal{M}}(X,Y) \times \mathrm{map}_{\mathcal{M}}(Y,Z) \to \mathrm{map}_{\mathcal{M}}(X,Z)$ for any triple of objects X, Y, Z in $\mathcal{M}$;
- a choice of a vertex $e_X \colon \Delta[0] \to \mathrm{map}_{\mathcal{M}}(X,X)$ for any object X in $\mathcal{M}$,

such that:

(1) the spaces $\mathrm{map}_{\mathcal{M}}(X, Y)$ together with the maps e_X and the compositions define an enrichment (see Definition 2.4.1) indexed by the objects of $\mathcal{M}$;

(2) the spaces $\mathrm{map}_{\mathcal{M}}(X, Y)$ are fibrant for all X and Y;

(3) the functor $\pi_0\mathrm{map}_{\mathcal{M}}(-,-) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{Sets}$ is naturally isomorphic to $\mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(-,-) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{Sets}$.

Given a simplicial model category (as defined in [**13**, Chapter II 3.1]) such a construction can always be found by composing the given mapping space functor with a functorial cofibrant-fibrant replacement. For a general model category one can use the Dwyer and Kan hammock construction from [**11**]. However for our purposes it much more advantageous to use the construction of Chahcólski and Scherer from [**7**] which we now recall. This construction is based on the notion of *bounded functors*.

3.1. Bounded functors

Definition 3.1.1. Let K be a simplicial set and $\mathcal{C}$ any category. We say that a functor $F\colon K \to \mathcal{C}$ is *bounded* if $F(s_i)$ is an isomorphism for any degeneracy morphism $s_i : \Delta[n+1] \to \Delta[n]$ in K. The full subcategory of $\mathrm{Fun}(K, \mathcal{C})$ of bounded functors is denoted $\mathrm{Fun}^b(K, \mathcal{C})$.

One of the reasons why we want to work with bounded functors is because of the following model structure.

Theorem 3.1.2 ([**6**, Theorem 21.1])**.** *Let $\mathcal{M}$ be a model category. The following describes a model structure on $\mathrm{Fun}^b(K, \mathcal{M})$:*

- *$\varphi : F \to G$ is a weak equivalence if $\varphi_\sigma : F(\sigma) \to G(\sigma)$ is a weak equivalence for any simplex σ in K,*
- *$\varphi : F \to G$ is a fibration if $\varphi_\sigma : F(\sigma) \to G(\sigma)$ is a fibration for any simplex σ in K,*
- *$\varphi : F \to G$ is an (acyclic) cofibration if, for any non-degenerate simplex $\sigma : \Delta[n] \to K$, the morphism:*

  $$\mathrm{colim}\,\big(F(\sigma) \leftarrow \mathrm{colim}_{\partial\Delta[n]}F \to \mathrm{colim}_{\partial\Delta[n]}G\big) \to G(\sigma)$$

  is an (acyclic) cofibration in $\mathcal{M}$.

Remark 3.1.3. One big advantage of bounded diagrams and the model structure in Theorem 3.1.2 is that the conditions on both cofibrations and fibrations are local, meaning that they only depend on the effect of natural transformations on individual non-degenerate simplicies and their boundaries. This is to be compared with the Reedy model structure for the category of all functors $\mathrm{Fun}(K, \mathcal{M})$ where fibrations and cofibrations instead depend on the entire structure of the domain K. The above model structure is thus considerably easier to work with.

Bounded functors are preserved by many fundamental operations induced by maps of simplicial sets. In particular if we let $f\colon L \to K$ be a morphism of simplicial sets and $F \in \mathrm{Fun}^b(K, \mathcal{M})$ then we get that $F \circ f \in \mathrm{Fun}^b(L, \mathcal{M})$ (see [**7**, Corollary 10.5]). Something that is not as obvious is that if $G\colon L \to \mathcal{M}$ is bounded, then so is its left Kan extension $f^k G\colon K \to \mathcal{M}$ (see [**7**, Theorem 33.1]). In this way we obtain a pair of adjoint functors:

$$\mathrm{Fun}^b(K, \mathcal{M}) \underset{f^k}{\overset{f^*}{\rightleftarrows}} \mathrm{Fun}^b(L, \mathcal{M})$$

These functors form a Quillen adjunction:

Proposition 3.1.4. [**6**, Theorem 11.2.3] *Let $f\colon L \to K$ be a map of simplicial sets. If $\varphi\colon F \to G$ is an (acyclic) cofibration in* $\mathrm{Fun}^b(L, \mathcal{M})$*, then so is its left Kan extension $f^k\varphi\colon f^kF \to f^kG$ in* $\mathrm{Fun}^b(K, \mathcal{M})$.

It is important to note that the cofibrancy condition on functors in $\mathrm{Fun}^b(K, \mathcal{M})$ is entirely dependent on the non-degenerate simplices of K. Given a morphism of simplicial set $f\colon K \to L$ we have seen that the left Kan extension f^k preserves this cofibrancy structure. The same cannot be said about its adjoint f^*. This is because in general f does not send non-degenerate simplices in K to non-degenerate simplices in L (see e.g. [**7**, Example 12.8]). The property of preserving non-degeneracy is called being *reduced.*

Definition 3.1.5. Let $f : K \to L$ be a morphism in sSets. We say that f is *reduced* if for any non-degenerate simplex $\sigma : \Delta[n] \to K$ the simplex $f(\sigma)$ is non-degenerate.

Applying the nerve to the simplex category of a simplicial set gives us what is called a barycentric subdivision. The barycentric subdivisions of maps between simplicial sets are the key examples of reduced maps that we are going to use.

Proposition 3.1.6 ([**6**, Example 12.10])**.** *Let $f : K \to L$ be a morphism in* sSets*. Then the induced morphism*

$$N(f) : N(K) \to N(L)$$

is reduced.

Because of this property we want to work with the system of categories $\mathrm{Fun}^b(N(-), \mathcal{M})$. They are in fact exactly the categories we need, to be able to construct the mapping space functor.

3.2. The mapping space functor

The reason why the categories $\mathrm{Fun}^b(N(K), \mathcal{C})$ are important for us is because they have a *simplicial structure* (see e.g. [**13**, Chapter II]). This means that there exists a construction for a mapping space $\mathrm{map}_{\mathrm{Fun}}(-,-)$ available for $\mathrm{Fun}^b(N(K), \mathcal{C})$. This is our stepping stone to the mapping space for any model category. Before getting there we first need some simplifying notation.

Definition 3.2.1. For any $n \in \mathbb{N}$ let $p : \Delta[n] \to \Delta[0]$ denote the unique map. Then we denote the effect of the functor $N(p)^* : \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}) \to \mathrm{Fun}^b(N(\Delta[n]), \mathcal{M})$ by

$$F[n] := N(p)^*(F).$$

In the same way for a natural transformation $f \colon F \to G$ in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ we denote by

$$f[n] \colon F[n] \to G[n]$$

the corresponding natural transformation in $\mathrm{Fun}^b(N(\Delta[n]), \mathcal{M})$.

We are now ready to define the mapping space for $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$.

Definition 3.2.2. Let F and G be functors in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Then the simplicial set $\mathrm{map}_{\mathrm{Fun}}(F, G)$ is defined degree-wise by

$$\mathrm{map}_{\mathrm{Fun}}(F, G)_n := \mathrm{Nat}(F[n], G[n])$$

for $n \in \mathbb{N}$ and by sending any morphism $\alpha \colon \Delta[n] \to \Delta[m]$ to the morphism $\mathrm{map}_{\mathrm{Fun}}(F, G)_\alpha \colon \mathrm{map}_{\mathrm{Fun}}(F, G)_m \to \mathrm{map}_{\mathrm{Fun}}(F, G)_n$ defined by

$$\mathrm{map}_{\mathrm{Fun}}(F, G)_\alpha := N(\alpha)^*$$

Further let $f \colon E \to F$ and $g \colon G \to H$ be morphisms in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Then the simplicial map $\mathrm{map}_{\mathrm{Fun}}(f, g) \colon \mathrm{map}_{\mathrm{Fun}}(F, G) \to \mathrm{map}_{\mathrm{Fun}}(E, H)$ is defined by

$$\mathrm{map}_{\mathrm{Fun}}(f, g)_n := \mathrm{Nat}(f[n], g[n])$$

for $n \in \mathbb{N}$.

The construction in Definition 3.2.2 give us a functor

$$\mathrm{map}_{\mathrm{Fun}}(-,-) \colon \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}) \times \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})^{\mathrm{op}} \to \mathrm{sSets}$$

which we call the *mapping space functor* for $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. This functor have additional structure that reinforces the idea of having a space of maps.

Definition 3.2.3. Let $F, G, H \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Then we define the *composition map*,

$$\circ \colon \mathrm{map}_{\mathrm{Fun}}(F, G) \times \mathrm{map}_{\mathrm{Fun}}(G, H) \to \mathrm{map}_{\mathrm{Fun}}(F, H)$$

to be given degree-wise by the composition of natural transformations. Further for every $F \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ define the *unit map* to be the simplicial map $e_F \colon \Delta[0] \to \mathrm{map}_{\mathrm{Fun}}(F, F)$ given by

$$e_F(\Delta[0]) := \mathrm{id}_{F[0]} \in \mathrm{map}_{\mathrm{Fun}}(F, F)_0.$$

The composition maps and the unit maps in the definition above would not be very useful if they did not satisfy conditions for associativity.

Proposition 3.2.4 ([**3**, Proposition 11.2 (1)])**.** $\mathrm{map}_{\mathrm{Fun}}$ *is an enrichment indexed by the objects of* $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$*.*

This construction of the mapping space functor is not without its problems. One such problem is the compatibility with the model structure in Theorem 3.1.2. For example it is not true in general that $\pi_o\mathrm{map}(F, G)$ is in bijection with the set of morphisms $[F, G]$ in $\mathrm{Ho}(\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}))$ for arbitrary F and G (this follows from [**7**, Example 8.1]). For that to be true we need an additional assumption on the considered functors. One such assumption is that the functors are homotopically constant.

Definition 3.2.5. We say that a functor $F \in \mathrm{Fun}^b(K, \mathcal{M})$ is *homotopically constant* if it is weakly equivalent to some functor $cX \colon K \to \mathcal{M}$ with constant value $X \in \mathcal{M}$.

This definition can be combined with the following result.

Proposition 3.2.6 ([**7**, Corollary 8.4])**.** *Let φ be a weak equivalence between cofibrant objects and ψ be a weak equivalence between fibrant objects in* $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$*. Then* $\mathrm{map}_{\mathrm{Fun}}(\varphi, \psi)$ *is a weak equivalence in* sSets*.*

This prompts us to define the following subcategory.

Definition 3.2.7. We denote by $\mathrm{Cons}(N(K), \mathcal{M})$ the full subcategory in $\mathrm{Fun}^b(N(K), \mathcal{M})$ with objects being cofibrant, fibrant and homotopically constant.

Remark 3.2.8. Exactly how and why homotopically constant functors resolve the compatibility issue with the model structure is unfortunately outside the scope of this paper. The interested reader is instead encouraged to read Section 8 of the paper [**7**] for more information on the subject.

What one gains by looking only at the mapping spaces for the functors in $\mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ can be summarized in the following proposition.

Proposition 3.2.9 ([**7**, Corollary 8.6])**.** *Let F, G be two functors in* $\mathrm{Cons}(N(K), \mathcal{M})$*. Then* $\mathrm{map}_{\mathrm{Fun}}(F, G)$ *is fibrant. Moreover there is a bijection natural in F and G, beetween the set of connected components* $\pi_0\mathrm{map}_{\mathrm{Fun}}(F, G)$ *and the set of morphisms* $[F, G]$ *in* $\mathrm{Ho}(\mathrm{Fun}^b(N(K), \mathcal{M}))$*.*

We now have the tools we need to construct the mapping space functor for a general model category $\mathcal{M}$. We use the following simplifying notation.

Definition 3.2.10. Let $\mathcal{M}$ be a model category. Choose a functorial fibrant replacement, $R_{\mathcal{M}}$. Let P_{Fun} denote the functorial cofibrant replacement in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. We define the functor $Q\colon \mathcal{M} \to \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ by the composition

$$\begin{tikzcd} \mathcal{M} \arrow[r, "R_{\mathcal{M}}"] \arrow[rrr, bend right, "Q"'] & \mathcal{M} \arrow[r, "c"] & \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}) \arrow[r, "P_{\mathrm{Fun}}"] & \mathrm{Cons}(N(\Delta[0]), \mathcal{M}) \end{tikzcd}$$

where $c\colon \mathcal{M} \to \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ is the constant functor.

The mapping space for any given model category can now be defined as follows.

Definition 3.2.11. Let $\mathcal{M}$ be a model category and let $X, Y \in \mathcal{M}$. Define

$$\mathrm{map}_{\mathcal{M}}(X, Y) := \mathrm{map}_{\mathrm{Fun}}(QX, QY)$$

According to Proposition 3.2.6 the composition of $\mathrm{map}_{\mathcal{M}}(X, -)\colon \mathcal{M} \to$ sSets and the localization $\gamma_{\mathrm{sSets}}\colon \mathrm{sSets} \to \mathrm{Ho}(\mathrm{sSets})$ factors through the localization $\gamma_{\mathcal{M}}\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. The obtained functor $\mathrm{map}_{\mathcal{M}}(X, -)\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathrm{sSets})$ satisfies the following.

Proposition 3.2.12 ([**7**, Proposition 9.4])**.**

$$\mathrm{map}_{\mathcal{M}}(X, -)\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathrm{sSets})$$

is the right adjoint to $- \otimes_h X$.

Thus the values of $\mathrm{map}_{\mathcal{M}}(X, -)$ are the weak homotopy type of the mapping spaces from Definition 3.0.3. Proposition 3.2.9 leads to the following corollary.

Corollary 3.2.13 ([**7**, Proposition 9.2])**.** *Let X, Y be any objects in $\mathcal{M}$. Then $\mathrm{map}_{\mathcal{M}}(X, Y)$ is fibrant. Moreover there is a bijection, natural in X and Y, between the set of connected components $\pi_0\mathrm{map}_{\mathcal{M}}(X, Y)$ and the set of morphisms $\mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$ in $\mathrm{Ho}(\mathcal{M})$.*

Proposition 3.2.4, Proposition 3.2.12 and Corollary 3.2.13 tell us that we have acquired a construction of the mapping space for a model category $\mathcal{M}$ that satisfies the conditions we sought in the beginning of the chapter. The next chapter is devoted to show how we can replace this construction with something much simpler and less technical which is the main goal of this paper.

3.3. The space of weak equivalences

One of the tools we need is a subspace of the mapping space called the space of weak equivalences. We start with a preliminary definition.

Definition 3.3.1. Let $F, G \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Then we define the set

$$\mathrm{Natwe}(F, G) := \{f \in \mathrm{Nat}(F, G) \mid f \text{ is a weak equivalence}\}.$$

Note that for any $\alpha\colon \Delta[n] \to \Delta[m]$, the induced map

$$\mathrm{map}_{\mathrm{Fun}}(F, G)_\alpha\colon \mathrm{Nat}(F[m], G[m]) \to \mathrm{Nat}(F[n], G[n])$$

takes $\mathrm{Natwe}(F[m], G[m])$ to $\mathrm{Natwe}(F[n], G[n])$. We can thus use the functoriality of our constructions of mapping spaces to define spaces of weak equivalences.

Definition 3.3.2. Let F and G be any two functors in $\mathrm{Cons}(N(\Delta[0]), \mathcal{M})$. Then the simplicial set $\mathrm{we}(F, G)$ is defined as the subspace of $\mathrm{map}_{\mathrm{Fun}}(F, G)$ given degree-wise by

$$\mathrm{we}(F, G)_n := \mathrm{Natwe}(F[n], G[n]) \quad \subseteq \ \mathrm{map}_{\mathrm{Fun}}(F, G)_n$$

for $n \in \mathbb{N}$. The composition map $\circ\colon \mathrm{we}(F, G) \times \mathrm{we}(G, H) \to \mathrm{we}(F, H)$ and the map $e_F\colon \Delta[0] \to \mathrm{we}(F, F)$ for $F, G, H \in \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ are defined as in $\mathrm{map}_{\mathrm{Fun}}$.

Seeing how the composition and unit maps of the mapping space made $\mathrm{map}_{\mathrm{Fun}}$ into an enrichment it seems reasonable to assume that the space of weak equivalences is also an enrichment. In fact one can show even more.

Proposition 3.3.3. *Let* $H \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ *be a fibrant functor. Then* $\mathrm{we}(-, -)$ *is a small homotopy groupoid indexed by* $\mathrm{Cons}(N(\Delta[0]), \mathcal{M})$*. Furthermore* $\mathrm{map}_{\mathrm{Fun}}(-, H)$ *is a* we*-space indexed by* $\mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ *with the homotopy groupoid action being the composition.*

Proof. That $\mathrm{we}(-, -)$ is a homotopy groupoid follows from [**3**, Corollary 12.2 (1)&(2)] together with [**3**, Proposition 12.3].

Thus we only need to show that $\mathrm{map}_{\mathrm{Fun}}(-, H)$ is a we-space. Property (1) of Definition 2.4.4, associativity, follows from the fact that $\mathrm{we}(F, G)$ is a subspace of $\mathrm{map}(F, G)$ for any $F, G \in \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ and Proposition 3.2.4. Thus we only need to show that the diagram

$$\begin{tikzcd}
\mathrm{we}(F, G) \times \mathrm{map}_{\mathrm{Fun}}(G, H) \arrow[r, "\circ"] \arrow[d, "\mathrm{pr}"] & \mathrm{map}_{\mathrm{Fun}}(F, H) \arrow[d] \\
\mathrm{we}(F, G) \arrow[r] & \Delta[0]
\end{tikzcd}$$

is a homotopy pullback for any $F, G \in \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$. The proof for this is exactly the same as the one for [**3**, Proposition 12.3]. We assume that all the simplicial sets are non-empty since the empty case is easy. The restriction of the composition map $\circ$ to the homotopy fiber over $f \colon F[0] \to G[0] \in \mathrm{we}(F, G)_0$ is given by $\mathrm{map}_{\mathrm{Fun}}(f, \mathrm{id})$ which is a weak equivalence by Proposition 3.2.6 since f is a weak equivalence. By Lemma 2.3.9 we are done. □

If we, for any $\mathcal{S} \subset \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$, denote by $\mathrm{we}_{\mathcal{S}}$ the homotopy groupoid $\mathrm{we}(-,-)$ restricted $\mathcal{S}$ and similarly denote $\mathrm{map}_{\mathrm{Fun}}(-, H)_{\mathcal{S}}$ for the corresponding we-space restricted to $\mathcal{S}$ we get the following corollary.

Corollary 3.3.4. *Let $X, Y \in \mathcal{M}$ and $\mathcal{S} \subset \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ be any set such that $QX \in \mathcal{S}$. Then the homotopy fiber of the map*

$$\mathrm{diag}(\pi) \colon \mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-, QY)_{\mathcal{S}}) \to \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}})$$

over QX is $\mathrm{map}_{\mathcal{M}}(X, Y)$.

Proof. Follows from Corollary 2.4.13 and Proposition 3.3.3. □

Chapter 4

The Hammock Category

-Really?! I can give you some of the ham. It's home made.
-Home made ham?! Amazing!
Deadly Premonition

As we already mentioned Dwyer and Kan introduced in the 80's the notion of hammocks to explain what is now known as mapping spaces. The hammock construction, or *the hammock localization* as it was called, was at the time the perfect way to showcase mapping spaces. In this part of the paper we show that in most cases the full hammock construction is superfluous and one only needs to consider "hammocks" of at most length 2.

Definition 4.0.1. Let $\mathcal{M}$ be a model category and let $X, Y \in \mathcal{M}$ be any objects with Y being fibrant. We define $\operatorname{Ham}(X, Y)$ to be the category whose

- objects are triples (Z, f, g) where $Z \in \operatorname{ob} \mathcal{M}$, $f\colon Z \xrightarrow{\sim} X$ is a weak equivalence and $g\colon Z \to Y$ is a morphism. We depict such an object by a diagram

$$X \underset{\sim}{\xleftarrow{f}} Z \xrightarrow{g} Y .$$

- morphisms from (Z, f, g) to (Z', f', g') are morphisms $h\colon Z \to Z'$ in $\mathcal{M}$ such that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[dl, "f"', "\sim"] \arrow[dr, "g"] \arrow[d, "h"] & \\ X & & Y \\ & Z' \arrow[ul, "f'", "\sim"'] \arrow[ur, "g'"'] & \end{tikzcd} .$$

Sometimes we omit the subscript $\mathcal{M}$ and just write $\mathrm{Ham}(X, Y)$ when it is clear from the context which model category is being considered.

Remark 4.0.2. In Definition 4.0.1 we see that the hammock category is small. But if we for comparison ignore this fact and allow ourselves to take the nerve of the hammock category we get a “simplicial set” with k-simplices of the form

$$\begin{tikzcd} & Z_0 \arrow[dl, "\sim"'] \arrow[dr] \arrow[d] & \\ X & \vdots \arrow[d] & Y \\ & Z_k \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

Comparing this with the definition of the hammock localization (see [**11**, 3.1]) we see that our hammock category can be viewed as having objects that are “reduced hammocks of fixed length 2” as opposed to having any length. This is why we have chosen to call our category the hammock category.

There is a way to define the hammock categories as subcategories of more familiar categories as the following proposition tells us.

Proposition 4.0.3. $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ *is a full subcategory of the category* $(\mathcal{M} \downarrow X \times Y)_{\mathrm{we}}$.

Proof. Morphisms in $\mathcal{M} \downarrow X \times Y$ are of the form

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr] & & Z' \arrow[dl] \\ & X \times Y & \end{tikzcd}$$

which using the projections of the product translate into diagrams of the form

$$\begin{tikzcd} X & Z \arrow[l] \arrow[r] \arrow[d, "h"] & Y \\ & Z' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Thus we see that $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ is a subcategory of $\mathcal{M} \downarrow X \times Y$. Given any morphism

$$\begin{tikzcd} X & Z \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\alpha"] & Y \\ & Z' \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

in $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ we see by the 2-out-of-3 property of weak equivalences that the map $\alpha\colon Z \to Z'$ must be a weak equivalence in $\mathcal{M}$. Since this is true for any morphism in $\mathrm{Ham}(X, Y)$ we see that $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ is indeed a subcategory of $(\mathcal{M} \downarrow X \times Y)_{\mathrm{we}}$. Choosing the objects in $(\mathcal{M} \downarrow X \times Y)_{\mathrm{we}}$ whose projections onto X are weak equivalences gives us the full subcategory $\mathrm{Ham}_{\mathcal{M}}(X, Y)$. $\square$

Seeing as we want our hammock category to represent the mapping space we need to be able to define pre-composition and post-composition of maps.

Definition 4.0.4. Let $\beta\colon Y \to Y'$ be a morphism in $\mathcal{M}$ and $X \in \mathrm{ob}\,\mathcal{M}$. Then we define

$$\mathrm{Ham}(X, \beta)\colon \mathrm{Ham}(X, Y) \to \mathrm{Ham}(X, Y')$$

to be the functor that sends (Z, f, g) to $(Z, f, \beta \circ g)$ and sends a morphism $h\colon Z \to Z'$ in $\mathrm{Ham}(X, Y)$ to the same morphism in $\mathrm{Ham}(X, Y')$. Further for a weak equivalence $\alpha\colon X \to X'$ in $\mathcal{M}$ and $Y \in \mathrm{ob}\,\mathcal{M}$ we define

$$\mathrm{Ham}^{\alpha}\colon \mathrm{Ham}(X, Y) \to \mathrm{Ham}(X', Y)$$

to be the functor that sends (Z, f, g) to $(Z, \alpha \circ f, g)$ and sends a morphism $h\colon Z \to Z'$ in $\mathrm{Ham}(X, Y)$ to the same morphism in $\mathrm{Ham}(X', Y)$.

In the above definition we see that the functor Ham^{α} is covariant. To be able to define a functor $\mathrm{Ham}(\alpha, Y)$ we instead need a functor that acts in a contravariant way. For this to be true we need to be careful how we define $\mathrm{Ham}(\alpha, Y)$. Consider the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, equal] & Y \arrow[r, equal] & Y \\ Z \arrow[u, "g"] \arrow[r, hook, "\sim"] \arrow[dr] \arrow[dd, "f"', "\sim"] & W \arrow[u] \arrow[d, two heads] & P \arrow[l] \arrow[u, "g'"'] \arrow[dd, "f'"] \\ & X' \times Y \arrow[d, "\mathrm{pr}"] & \\ X' \arrow[r, equal] & X' & X \arrow[l, "\alpha"', "\sim"] \end{tikzcd}$$

where

$$\begin{tikzcd} Z \arrow[r, hook, "\sim"] \arrow[dr] & W \arrow[d, two heads] \\ & X' \times Y \end{tikzcd}$$

is a functorial factorisation in $\mathcal{M}$ and

$$\begin{array}{ccc} W & \longleftarrow & P \\ \downarrow & & \downarrow f' \\ X' & \overset{\alpha}{\underset{\sim}{\longleftarrow}} & X \end{array}$$

is a pullback diagram. The morphism $W \to X'$ is a weak equivalence by the 2-out-of-3 property. If Y was to be fibrant it would also be a fibration because the morphism $\mathrm{pr}\colon X' \times Y \to X'$ would then be a fibration. Since acylic fibrations are stable under base change (see e.g. [**12**, Proposition 3.14 (iv)]) this would also make the morphism $f'\colon P \to X$ into an acyclic fibration. Since all these steps in this construction passing from $X' \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ to $X' \overset{f'}{\leftarrow} P \overset{g'}{\rightarrow} Y$ are functorial, by demanding that Y is fibrant, we obtain a functor that we denote by $\mathrm{Ham}(\alpha, Y)$.

Definition 4.0.5. Let $\alpha\colon X \to X'$ be a weak equivalence in $\mathcal{M}$ and let $Y \in \mathcal{M}$ be fibrant. Then we define $\mathrm{Ham}(\alpha, Y)\colon \mathrm{Ham}(X', Y) \to \mathrm{Ham}(X, Y)$ to be the functor that assigns $X \overset{f'}{\underset{\sim}{\longleftarrow}} P \overset{g'}{\longrightarrow} Y$ to an object $X' \overset{f}{\underset{\sim}{\longleftarrow}} Z \overset{g}{\longrightarrow} Y$ by the above construction.

Since the functor $\mathrm{Ham}(\alpha, Y)$ is crucial to our construction and it is only defined as long as Y is fibrant we will from now on demand that Y is fibrant.

4.1. Basic properties of the hammock category

Since the hammock categories are supposed to represent mapping spaces Proposition 3.2.6 tells us that we want them to be homotopy invariant under weak equivalences. Fortunately when Y is fibrant the functor $\mathrm{Ham}(\alpha, Y)$ can be used to show that this is indeed the case.

Proposition 4.1.1. *Let $\alpha\colon X \to X'$ be any weak equivalence in $\mathcal{M}$ and let $\beta\colon Y \to Y'$ be a weak equivalence between fibrant objects in $\mathcal{M}$. Then* $\mathrm{Ham}(\alpha, Y)$ *and* $\mathrm{Ham}(X, \beta)$ *are both homotopy equivalences. Futher* Ham^{α} *is a homotopy inverse to* $\mathrm{Ham}(\alpha, Y)$.

Proof. We start by showing that $\mathrm{Ham}(\alpha, Y)$ has homotopy inverse Ham^{α}. We have that $\mathrm{Ham}(\alpha, Y) \circ \mathrm{Ham}^{\alpha}(\ X \overset{f}{\underset{\sim}{\leftarrow}} Z \overset{g}{\rightarrow} Y\ ) = \ X \overset{f'}{\underset{\sim}{\leftarrow}} P \overset{g'}{\rightarrow} Y$, where

$X \overset{f'}{\underset{\sim}{\leftarrow}} P \overset{g'}{\to} Y$ fits into the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, equal] & Y \arrow[r, equal] & Y \\ Z \arrow[u, "g"] \arrow[r, hook, "\sim"] \arrow[d, "f"', "\sim"] & W \arrow[u] \arrow[d, two heads, "\sim"] & P \arrow[l] \arrow[u, "g'"'] \arrow[d, "\sim"', "f'"] \\ X \arrow[r, "\alpha", "\sim"'] & X' & X \arrow[l, "\alpha"', "\sim"] \end{tikzcd} .$$

By the universality of the pullback we have a map $Z \xrightarrow[\sim]{p} P$ which gives us the following commuting diagram.

$$\begin{tikzcd} Y \arrow[r, equal] & Y \\ Z \arrow[u, "g"] \arrow[r, "p", "\sim"'] \arrow[d, "f"', "\sim"] & P \arrow[u, "g'"'] \arrow[d, "\sim"', "f'"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

These maps form a natural transformation

$$\mathrm{id}_{\mathrm{Ham}(X,Y)} \to \mathrm{Ham}(\alpha, Y) \circ \mathrm{Ham}^{\alpha}.$$

Thus $\mathrm{Ham}(\alpha, Y) \circ \mathrm{Ham}^{\alpha}$ is homotopic to $\mathrm{id}_{\mathrm{Ham}(X,Y)}$. Further $\mathrm{Ham}^{\alpha} \circ \mathrm{Ham}(\alpha, Y)(\, X' \overset{f}{\underset{\sim}{\leftarrow}} Z \overset{g}{\to} Y \,) = \, X' \overset{f'}{\underset{\sim}{\leftarrow}} P \overset{g'}{\to} Y$, where $X' \overset{f'}{\underset{\sim}{\leftarrow}} P \overset{g'}{\to} Y$ fits into the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, equal] & Y \arrow[r, equal] & Y \\ Z \arrow[u, "g"] \arrow[r, hook, "\sim"] \arrow[d, "f"', "\sim"] & W \arrow[u] \arrow[d, two heads, "\sim"] & P \arrow[l] \arrow[u, "g'"'] \arrow[d, "\sim"', "f'"] \\ X' \arrow[r, equal] & X' \arrow[r, equal] & X' \end{tikzcd} .$$

Here the map $f'\colon P \overset{\sim}{\to} X'$ factors as $P \underset{\sim}{\to} X \overset{\alpha}{\underset{\sim}{\to}} X'$. These maps form natural transformations

$$\mathrm{id}_{\mathrm{Ham}(X',Y)} \to h_1 \leftarrow \mathrm{Ham}^{\alpha} \circ \mathrm{Ham}(\alpha, Y)$$

and thus we have that $\mathrm{Ham}^{\alpha} \circ \mathrm{Ham}(\alpha, Y)$ is homotopic to $\mathrm{id}_{\mathrm{Ham}(X',Y)}$. This shows that $\mathrm{Ham}(\alpha, Y)$ is a homotopy equivalence with homotopy inverse Ham^{α}.

Let RX be the fibrant replacement of X. Then by what we have just proven $\mathrm{Ham}(RX, Y)$ is homotopically equivalent to $\mathrm{Ham}(X, Y)$ so we can assume, without loss of generality, that X is fibrant. Because of the symmetry of the problem, we then can use the same proof as for $\mathrm{Ham}(\alpha, Y)$ to show that $\mathrm{Ham}(X, \beta)$ also is a homotopy equivalence. □

In the last part of the above proof we argued why we always could assume that X is fibrant when working with $\mathrm{Ham}(X, Y)$. We restate this fact in a corollary.

Corollary 4.1.2. *Let X be any object of $\mathcal{M}$, let RX be its fibrant replacement and let QX be its cofibrant replacement. Further let Y be a fibrant object of $\mathcal{M}$ and QY its cofibrant replacement. Then $\mathrm{Ham}(X, Y)$, $\mathrm{Ham}(RX, Y)$, $\mathrm{Ham}(QX, Y)$ and $\mathrm{Ham}(X, QY)$ are all homotopically equivalent.*

Thus we assume in the remainder of this paper that, as long as Y is fibrant, then Y is also cofibrant and X is fibrant and cofibrant.

4.2. Essentially smallness of the Hammock category

The hammock categories were defined with the explicit purpose of representing mapping spaces. Since we have seen in Definition 3.0.3 that mapping spaces are universally characterized by having a certain weak homotopy type we need to be able to talk about the weak homotopy types of hammock categories to be able to connect the two concepts. The first step in doing this is proving that the hammock categories are in fact essentially small. This is the aim of this section.

By Theorem 3.1.2 we have that the category $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ is a model category, thus it makes sense to talk about the hammock category in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Given two objects $H, G \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ we denote the hammock category from H to G by $\mathrm{Ham}_{\Delta[0]}(H, G)$. For $X, Y \in \mathcal{M}$ we denote by $cX, cY \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ the constant functors with values X and Y respectively.

Proposition 4.2.1. *Let $Y \in \mathcal{M}$ be fibrant, $X \in \mathcal{M}$ any object. Then the categories $\mathrm{Ham}_{\Delta[0]}(cX, cY)$ and $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ are homotopically equivalent.*

Proof. We first define the functor $\varphi : \mathrm{Ham}_{\Delta[0]}(cX, cY) \to \mathrm{Ham}_{\mathcal{M}}(X, Y)$ by the following assignment on morphisms

$$\begin{tikzcd} cX & F \arrow[l, "\sim"'] \arrow[r] \arrow[d] & cY \\ & G \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} X & \mathrm{colim}\ QF \arrow[l, "\sim"'] \arrow[r] \arrow[d] & Y \\ & \mathrm{colim}\ QG \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

where Q is the functorial cofibrant replacement in $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Next define the functor $\psi : \mathrm{Ham}_{\mathcal{M}}(X, Y) \to \mathrm{Ham}_{\Delta[0]}(cX, cY)$ by the following assignment on morphisms

$$
\begin{tikzcd}
X & Z \arrow[l, "\sim"'] \arrow[r] \arrow[d] & Y \\
 & Z' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
cX & cZ \arrow[l, "\sim"'] \arrow[r] \arrow[d] & cY \\
 & cZ' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd} .
$$

We want to prove that these two functors are each others homotopic inverses. We have that $\varphi \circ \psi(\ X \overset{\sim}{\leftarrow} Z \rightarrow Y \ ) = \ X \overset{\sim}{\longleftarrow} \operatorname{colim} Q(cZ) \rightarrow Y$, where $X \overset{\sim}{\longleftarrow} \operatorname{colim} Q(cZ) \rightarrow Y$ fits into the following diagram

$$
\begin{tikzcd}
\varphi \circ \psi : & X \arrow[d, equal] & \operatorname{colim} Q(cZ) \arrow[l, "\sim"'] \arrow[r] \arrow[d, two heads, "\sim"] & Y \arrow[d, equal] \\
\mathrm{id}_{\mathrm{Ham}_{\mathcal{M}}(X,Y)} : & X & Z \arrow[l, "\sim"'] \arrow[r] & Y
\end{tikzcd} .
$$

These maps form a natural transformation

$$\varphi \circ \psi \rightarrow \mathrm{id}_{\mathrm{Ham}(X,Y)}.$$

Thus $\varphi \circ \psi$ is homotopic to $\mathrm{id}_{\mathrm{Ham}_{\mathcal{M}}(X,Y)}$. Further $\psi \circ \varphi(\ cX \overset{\sim}{\leftarrow} F \rightarrow cY \ ) = \ cX \overset{\sim}{\longleftarrow} c(\operatorname{colim} QF) \rightarrow cY$, where $cX \overset{\sim}{\longleftarrow} c(\operatorname{colim} QF) \rightarrow cY$ fits into the following diagram

$$
\begin{tikzcd}
\psi \circ \varphi : & cX \arrow[d, equal] & c(\operatorname{colim} QF) \arrow[l, "\sim"'] \arrow[r] & cY \arrow[d, equal] \\
 & cX \arrow[d, equal] & QF \arrow[l, "\sim"'] \arrow[r] \arrow[u, tail, "\sim"] \arrow[d, two heads, "\sim"] & cY \arrow[d, equal] \\
\mathrm{id}_{\mathrm{Ham}_{\Delta[0]}(X,Y)} : & cX & F \arrow[l, "\sim"'] \arrow[r] & cY
\end{tikzcd} .
$$

These maps form natural transformations

$$\psi \circ \varphi \leftarrow h_1 \rightarrow \mathrm{id}_{\mathrm{Ham}_{\Delta[0]}(X,Y)}$$

and thus $\psi \circ \varphi$ is homotopic to $\mathrm{id}_{\mathrm{Ham}_{\Delta[0]}(X,Y)}$. It follows that the categories $\mathrm{Ham}_{\Delta[0]}(cX, cY)$ and $\mathrm{Ham}_{\mathcal{M}}(X,Y)$ are homotopically equivalent. $\square$

We denote by $\mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$ the full subcategory of $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ of cofibrant, fibrant functors that are weakly equivalent to the constant functor cX. Then we have the following corollary.

Corollary 4.2.2. *Let $Y \in \mathcal{M}$ be fibrant and $X \in \mathcal{M}$ any object. Further let $H \in \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$. Then the categories $\mathrm{Ham}_{\Delta[0]}(H, cY)$ and $\mathrm{Ham}_{\mathcal{M}}(X,Y)$ are homotopically equivalent.*

Proof. This follows from Proposition 4.2.1 combined with Proposition 4.1.1. $\square$

The above proposition tells us that to prove essential smallness of $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ it is enough to prove essential smallness for $\mathrm{Ham}_{\Delta[0]}(cX, cY)$. We do this by looking at an even smaller category which we call Ham_{QX}.

Definition 4.2.3. Let $H, G \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Define $\mathrm{Ham}_{QX}(H, G)$ to be the full subcategory of $\mathrm{Ham}_{\Delta[0]}(H, G)$ with objects of the form

$$H \xleftarrow{\sim} QX \longrightarrow G$$

where $QX \in \mathrm{Cons}(N(\Delta[0]), \mathcal{M})$ is defined as in Definition 3.2.10.

The category $\mathrm{Ham}_{QX}(H, G)$ is not only essentially small but also small.

Lemma 4.2.4. *Let $H, G \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$. Then $\mathrm{Ham}_{QX}(H, G)$ is a small category.*

Proof. Denote by $\mathrm{Natwe}(QX, H)$ the set of all weak equivalences $QX \xrightarrow{\sim} H$. Then we have that

$$\mathrm{ob}\ \mathrm{Ham}_{QX}(H, G) = \coprod_{\substack{f \in \mathrm{Natwe}(QX,H) \\ g \in \mathrm{Nat}(QX,G)}} H \xleftarrow{f} QX \xrightarrow{g} G$$

Since $\mathrm{Natwe}(QX, H)$ and $\mathrm{Nat}(QX, G)$ are both sets we must have that $\mathrm{ob}\ \mathrm{Ham}_{QX}(H, G)$ is a set making $\mathrm{Ham}_{QX}(H, G)$ a small category. $\square$

If a category is small then this means that the components of the category must also be a set. This fact makes the following proposition useful.

Proposition 4.2.5. *The inclusion map* $\mathrm{Ham}_{QX}(cX, cY) \hookrightarrow \mathrm{Ham}_{\Delta[0]}(cX, cY)$ *induces a surjection*

$$\pi_0 \mathrm{Ham}_{QX}(cX, cY) \twoheadrightarrow \pi_0 \mathrm{Ham}_{\Delta[0]}(cX, cY)$$

Proof. Given an object $cX \xleftarrow{\sim} F \longrightarrow cY$ in $\mathrm{Ham}_{\Delta[0]}(cX, cY)$ we get a morphism $F \to cX \times cY$. We can factor this morphism

$$\begin{tikzcd} & & W \arrow[dr, two heads] & \\ 0 \arrow[r, hook] & F \arrow[ur, hook, "\sim"] \arrow[rr] & & cX \times cY \end{tikzcd}$$

which then gives us the object $cX \overset{\sim}{\twoheadleftarrow} W \twoheadrightarrow cY$ which also lies in $\mathrm{Ham}_{\Delta[0]}(cX, cY)$. This allows us to look at the following commuting diagram

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & W \arrow[d, two heads, "\sim"] \\ QX \arrow[r, two heads, "\sim"] \arrow[ur, dotted, "\exists \sim"] & cX \end{tikzcd}$$

where the lifting in the diagram exists because we are working in a model category (see e.g. [**12**, Proposition 3.13]). We can then create a sequence of morphisms in $\mathrm{Ham}_{\Delta[0]}(cX, cY)$

$$\begin{tikzcd}
cX \arrow[d, equal] & F \arrow[l, "\sim"'] \arrow[r] \arrow[d, hook, "\sim"'] & cY \arrow[d, equal] \\
cX \arrow[d, equal] & W \arrow[l, two heads, "\sim"'] \arrow[r, two heads] & cY \arrow[d, equal] \\
cX & QX \arrow[l, two heads, "\sim"'] \arrow[r] \arrow[u, tail, two heads, "\sim"] & cY
\end{tikzcd} .$$

These objects must therefore all lie in the same component so $\pi_0\mathrm{Ham}_{QX}(cX, cY)$ surjects onto $\pi_0\mathrm{Ham}_{\Delta[0]}(cX, cY)$. □

Remembering the importance of components when showing essential smallness from Lemma 2.2.19 we now have all the tools we need to prove the essential smallness of $\mathrm{Ham}_{\mathcal{M}}(X, Y)$.

Theorem 4.2.6. $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ *is essentially small.*

Proof. Lemma 4.2.4 tells us that $\pi_0\mathrm{Ham}_{QX}(cX, cY)$ is a set. By Proposition 4.2.5 we then get that $\pi_0\mathrm{Ham}_{\Delta[0]}(cX, cY)$ is also a set. Since $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ is homotopically equivalent to $\mathrm{Ham}_{\Delta[0]}(cX, cY)$ we get that $\pi_0\mathrm{Ham}_{\mathcal{M}}(X, Y)$ must also be a set. The result then follows from Proposition 4.0.3 together with Lemma 2.2.19. □

4.3. The Hammock category as a fiber

What is left to show is that the weak homotopy type of the hammock category is the mapping space. To accomplish this we are going to create a sequence of maps. We show that the homotopy fiber of these maps have the same weak homotopy type. We also identify these fibers with hammock categories and mapping spaces respectively. Recall from Definition 3.2.1 that given a functor $F \in \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ and $n \in \mathbb{N}$ the effect of the functor $N(p)^*\colon \mathrm{Fun}^b(N(\Delta[0]), \mathcal{M}) \to \mathrm{Fun}^b(N(\Delta[n]), \mathcal{M})$ on F is denoted by $F[n]$.

Definition 4.3.1. Let $Y \in \mathcal{M}$ be fibrant, let $\mathcal{S} \subset \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$ be any class and let $n \in \mathbb{N}$. Define $(D_{\mathcal{S}})_n$ to be the category whose

- objects are pairs $(F[n], f)$ where $F \in \mathcal{S}$ and $f\colon F[n] \to cY$ is a natural transformation.

- morphisms from $(F[n], f)$ to $(G[n], g)$ are morphisms $h\colon F[n] \to G[n]$ in $\mathrm{Cons}(N(\Delta[n]), X_{\mathrm{we}})$ such that the following diagram commutes
$$\begin{tikzcd} F[n] \arrow[r, "f"] \arrow[d, "h"] & cY \\ G[n] \arrow[ur, "g"] & \end{tikzcd} .$$

Further given any $n \in \mathbb{N}$ we define $(C_{\mathcal{S}})_n$ to be the category whose

- objects are functors $F[n]$ where $F \in \mathcal{S}$.
- morphisms from $F[n]$ to $G[n]$ are morphisms $h\colon F[n] \to G[n]$ in $\mathrm{Cons}(N(\Delta[n]), X_{\mathrm{we}})$.

For any morphism $\alpha\colon \Delta[m] \to \Delta[n]$ we define the functors $(D_{\mathcal{S}})_\alpha\colon (D_{\mathcal{S}})_n \to (D_{\mathcal{S}})_m$ and $(C_{\mathcal{S}})_\alpha\colon (C_{\mathcal{S}})_n \to (C_{\mathcal{S}})_m$ by
$$(D_{\mathcal{S}})_\alpha := N(\alpha)^* \quad \text{and} \quad (C_{\mathcal{S}})_\alpha := N(\alpha)^*.$$
Lastly by forgetting the natural transformations onto cY we obtain a functor $\mu_n\colon (D_{\mathcal{S}})_n \to (C_{\mathcal{S}})_n$ for every $n \in \mathbb{N}$. If $\mathcal{S} = \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$ we write D and C instead for $D_{\mathcal{S}}$ and $C_{\mathcal{S}}$.

The above definition makes $D_{\mathcal{S}}$ and $C_{\mathcal{S}}$ into a system of categories indexed by Δ. Taking the respective Grothendick constructions we get an induced map
$$\mu\colon \mathbf{Gr}_\Delta D_{\mathcal{S}} \to \mathbf{Gr}_\Delta C_{\mathcal{S}}.$$
Our strategy is to find different models for the weak homotopy type of this map. The homotopy fibers for these models are going to be our connection between the hammock categories and the mapping spaces. One of these models is given by the following proposition.

Proposition 4.3.2. *Let $\mathcal{S} \subset \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$ be a set. Then the diagram*
$$\mathbf{Gr}_\Delta D \xrightarrow{\mu} \mathbf{Gr}_\Delta C$$
is essentially small with weak homotopy type
$$\mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-, cY)_{\mathcal{S}}) \xrightarrow{\mathrm{diag}\ N(\pi)} \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}})$$

Using Corollary 3.3.4 this proposition allows us to conclude the following.

Corollary 4.3.3. *Let $Y \in \mathcal{M}$ be fibrant. Then* $\mathrm{hofib}_{QX}(N(\mu))$ *is weakly equivalent to* $\mathrm{map}_{\mathcal{M}}(X, Y)$.

Proof of 4.3.2. From Thomason's theorem and Lemma 2.3.6 we have that the nerve of
$$\mathbf{Gr}_\Delta D_{\mathcal{S}} \xrightarrow{\mu} \mathbf{Gr}_\Delta C_{\mathcal{S}}$$

is weakly equivalent to

$$\mathrm{diag}\ N(D_{\mathcal{S}}) \xrightarrow{\mathrm{diag}\ N(\mu)} \mathrm{diag}\ N(C_{\mathcal{S}})\ .$$

This map can be shown to be equal to the map

$$\mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-, cY)_{\mathcal{S}}) \xrightarrow{\mathrm{diag}\ N(\pi)} \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}})\ .$$

For example we have by definition that diag $N(D_{\mathcal{S}})$ is a simplicial set with n-simplices being commuting diagrams of the form

$$\begin{tikzcd} F_0[n] \arrow[r, "\sim"] \arrow[dr] & \cdots \arrow[r, "\sim"] \arrow[d] & F_n[n] \arrow[dl] \\ & Y & \end{tikzcd}$$

for $F_0, \ldots, F_n \in \mathcal{S}$. Thus we get

$$\mathrm{diag}\ N(D_{\mathcal{S}})_n = \coprod_{F_0,\ldots,F_n \in \mathcal{S}} \left(\prod_{k=0}^{n-1} \mathrm{we}(F_k, F_{k+1})_n \right) \times \mathrm{map}_{\mathrm{Fun}}(F_n, cY).$$

This last expression can be identified with $\mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-, cY)_{\mathcal{S}})_n$. Using the same argument one also gets that

$$\mathrm{diag}\ N(C_{\mathcal{S}})_n = \coprod_{F_0,\ldots,F_n \in \mathcal{S}} \left(\prod_{k=0}^{n-1} \mathrm{we}(F_k, F_{k+1})_n \right) = \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}})_n.$$

One can also check that the simplicial maps are identical and also that diag $N(\mu)$ equals diag $N(\pi)$. Thus we get that

$$\mathbf{Gr}_\Delta D_{\mathcal{S}} \xrightarrow{\mu} \mathbf{Gr}_\Delta C_{\mathcal{S}}$$

is weakly equivalent to

$$\mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-, cY)_{\mathcal{S}}) \xrightarrow{\mathrm{diag}\ N(\pi)} \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}})$$

What remains is to show that $\mathbf{Gr}_\Delta D_{\mathcal{S}} \xrightarrow{\mu} \mathbf{Gr}_\Delta C_{\mathcal{S}}$ is a core to $\mathbf{Gr}_\Delta D \xrightarrow{\mu} \mathbf{Gr}_\Delta C$. Let $I \subset \mathbf{Gr}_\Delta D$ and $J \subset \mathbf{Gr}_\Delta C$ be two small subcategories such that $\mathbf{Gr}_\Delta D_{\mathcal{S}} \subset I$, $\mathbf{Gr}_\Delta C_{\mathcal{S}} \subset I$ and μ maps I to J. Let $\mathcal{S}'$ be the set of all objects F in $\mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$ such that either $F[n] \to cY \in I$ for some $n \in \mathbb{N}$ or $F[m] \in J$ for some $m \in \mathbb{N}$. Then we can create a commuting diagram of inclusions

$$\begin{tikzcd} \mathbf{Gr}_\Delta D_{\mathcal{S}} \arrow[r, hook] \arrow[d, "\mu"'] & I \arrow[r, hook] \arrow[d, "\mu"'] & \mathbf{Gr}_\Delta D_{\mathcal{S}'} \arrow[d, "\mu"'] \\ \mathbf{Gr}_\Delta C_{\mathcal{S}} \arrow[r, hook] & J \arrow[r, hook] & \mathbf{Gr}_\Delta C_{\mathcal{S}'} \end{tikzcd} .$$

We have identified the weak homotopy type of the left and right vertical maps in the above diagram with

$$\mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-,cY)_{\mathcal{S}}) \xrightarrow{\mathrm{diag}\ N(\pi)} \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}})$$

$$\mathrm{diag}(\mathcal{E}\mathrm{map}_{\mathrm{Fun}}(-,cY)_{\mathcal{S}'}) \xrightarrow{\mathrm{diag}\ N(\pi)} \mathrm{diag}(\mathcal{B}\mathrm{we}_{\mathcal{S}'})\ .$$

We can therefore use Corollary 2.4.15 to conclude that the composition of the horizontal maps are weak equivalences. □

Another model for the map μ is given by μ_0.

Proposition 4.3.4. *In the following diagram*

$$\begin{tikzcd} D_0 \arrow[r, hook, "i"] \arrow[d, "\mu_0"] & \mathbf{Gr}_\Delta D \arrow[d, "\mu"] \\ C_0 \arrow[r, hook, "i"] & \mathbf{Gr}_\Delta C \end{tikzcd}$$

the horizontal inclusion functors are homotopy equivalences.

Proof. To show that $i\colon D_0 \hookrightarrow \mathbf{Gr}_\Delta D$ is a homotopy equivalence we need to find a functor $\varphi\colon \mathbf{Gr}_\Delta D \to D_0$ such that $\varphi \circ i$ is homotopic to id_{D_0} and $i \circ \varphi$ is homotopic to $\mathrm{id}_{\mathbf{Gr}_\Delta D}$.
Given an object $F[n] \to Y$ let $N(p)^k F[n] \in \mathrm{Fun}^b(N(\Delta[0]), X_{\mathrm{we}})$ be its left Kan extenstion, where $p\colon \Delta[n] \to \Delta[0]$ is the unique map. By the universal property of the colimit we get that $F[n] \to Y$ factors as

$$F[n] \xrightarrow{\sim} c(\mathrm{colim} F[n]) \longrightarrow cY$$

where the morphism $F[n] \xrightarrow{\sim} c(\mathrm{colim} F[n])$ is a weak equivalence because of [**3**, Proposition 10.2]. The left Kan extension then fits into the adjoint diagram

$$N(p)^k F[n] \longrightarrow c(\mathrm{colim} F[n]) \longrightarrow cY$$

since $\mathrm{colim} N(p)^k F[n] = \mathrm{colim} F[n]$. Using the model structure of $\mathrm{Fun}^b(N(\Delta[0]), \mathcal{M})$ we can factor the morphism $N(p)^k F[n] \to c(\mathrm{colim} F[n])$ twice in a functorial way to get a commutative diagram

(4.3.1)
$$\begin{tikzcd} & W \arrow[r, hook, "\sim"] \arrow[dr, two heads, "\sim"] & \overline{W} \arrow[drr, two heads] & & \\ N(p)^k F[n] \arrow[ur, hook] \arrow[rr] & & c(\mathrm{colim} F[n]) \arrow[rr, two heads] & & cY \end{tikzcd} .$$

$N(p)^k F[n]$ is cofibrant by Proposition 3.1.4 so W is also cofibrant and thus $\overline{W}$ is cofibrant. Further since $c(\mathrm{colim} F[n])$ is weakly equivalent to cX so is

W and thus also $\overline{W}$. Finally since cY is fibrant so is $\overline{W}$. The functor $\overline{W}$ is then an object in $\mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$. Thus the natural transformation

$$\overline{W} \twoheadrightarrow cY$$

is an object in D_0. All the steps in the above construction are functorial (Kan extension, colimit, morphism factorization). Thus we obtain a functor $\varphi\colon \mathbf{Gr}_\Delta D \to D_0$ that assigns $\overline{W} \to Y$ to an object $F[n] \to Y$. We want to prove that the functors $\varphi \circ i$ and $i \circ \varphi$ are homotopic to id_{D_0} and $\mathrm{id}_{\mathbf{Gr}_\Delta D}$ respectively.
The case of $\varphi \circ i$ is straightforward since in this case $p = \mathrm{id}_{\Delta[0]}$ so $N(p)^k F[0] = F[0] = F$ for any $F \in \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$. Thus we get a commuting diagram

$$\begin{tikzcd}
\mathrm{id}_{D_0}\colon & F \arrow[r] \arrow[d, "\sim"] & cY \arrow[d, equal] \\
\varphi \circ i\colon & \overline{W} \arrow[r] & cY
\end{tikzcd}$$

which gives us the natural transformation needed to show that $\varphi \circ i$ is homotopic to id_{D_0}.
The case of $i \circ \varphi$ is not as straightforward. Given an element $F[n] \to Y$ in $\mathbf{Gr}_\Delta D$ we can fit it into the commuting diagram in (4.3.1). Using the adjoint and the adjointness property of the left Kan extension of this we get the diagram

$$\begin{tikzcd}
& W[n] \arrow[r, "\sim"] \arrow[dr, "\sim"'] & \overline{W}[n] \arrow[dr, two heads] & \\
F[n] \arrow[ur] \arrow[rr, "\sim"] & & c(\mathrm{colim} F[n]) \arrow[r, two heads] & cY
\end{tikzcd}$$

where the weak equivalences come from having objectwise weak equivalences. But by the two-out-of-three property we then get that the map $F[n] \to W[n]$ is a weak equivalence. Thus the composition $F[n] \to \overline{W}[n]$ is a weak equivalence so we get a diagram

$$\begin{tikzcd}
\mathrm{id}_{\mathbf{Gr}_\Delta D}\colon & F[n] \arrow[r] \arrow[d, "\sim"] & cY \arrow[d, equal] \\
\imath \circ \varphi\colon & \overline{W}[n] \arrow[r] & cY
\end{tikzcd}$$

which gives us the natural transformation needed to show that $i \circ \varphi$ is homotopic to $\mathrm{id}_{\mathbf{Gr}_\Delta D}$.
That C_0 is homotopically equivalent to $\mathbf{Gr}_\Delta C$ follows as a corollary by choosing $Y = *$. □

It follows that $\mu_0\colon D_0 \to \mathcal{C}_0$ is essentially small because of Proposition 4.3.2. It remains to identify the homotopy fiber of μ_0.

Proposition 4.3.5. *Let $Y \in \mathcal{M}$ be fibrant. Then* $\mathrm{hofib}_{QX}(N(\mu_0))$ *is weakly equivalent to the nerve of* $\mathrm{Ham}_{\mathcal{M}}(X, Y)$.

Proof. We start by noting that we have an identification

$$\mu_0 \downarrow H = \mathrm{Ham}_{\Delta[0]}(H, cY).$$

This follows since an object in $\mu_0 \downarrow H$ is given by a diagram

$$H \xleftarrow{\sim} F \longrightarrow cY$$

for $F \in \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$ and similarly a morphism in $\mu_0 \downarrow H$ is given by a diagram

$$\begin{tikzcd} H & F \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & cY \\ & G \arrow[ul, "\sim"] \arrow[ur] & \end{tikzcd}$$

for $F, G \in \mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$. Conversely if $H \xleftarrow{\sim} F \to cY \ \in \mathrm{Ham}_{\Delta[0]}(H, cY)$ then F must be in $\mathrm{Cons}(N(\Delta[0]), X_{\mathrm{we}})$. Given any morphism $\alpha\colon H \to H'$ in C_0 we can further make the identification

$$\mu_0 \downarrow \alpha = \mathrm{Ham}^{\alpha}_{\Delta[0]}\colon \mathrm{Ham}_{\Delta[0]}(H, cY) \to \mathrm{Ham}_{\Delta[0]}(H', cY).$$

It then follows that μ_0 is a strong fibration from Proposition 4.1.1. Thus we can apply Quillens theorem and get that the following is a homotopy pullback

$$\begin{tikzcd} \mu_0 \downarrow QX \arrow[r, "\text{forget}"] \arrow[d] & D_0 \arrow[d, "\mu_0"] \\ C_0 \downarrow QX \arrow[r, "QX"] & C_0 \end{tikzcd} .$$

Since $C_0 \downarrow QX$ is contractible this tells us that if we take the nerve of the cores we get that $\mathrm{hofib}_{QX}(N(\mu_0)) \simeq N(\mu_0 \downarrow QX)$. But $\mu_0 \downarrow QX = \mathrm{Ham}_{\Delta[0]}(QX, cY)$ so by Corollary 4.2.2 we have that $\mu_0 \downarrow QX$ is homotopically equivalent to $\mathrm{Ham}_{\mathcal{M}}(X, Y)$ and so $\mathrm{hofib}_{QX}(N(\mu_0))$ is weakly equivalent to the nerve of $\mathrm{Ham}_{\mathcal{M}}(X, Y)$. $\square$

We are finally ready to prove our main theorem.

Theorem 4.3.6. *The category* $\mathrm{Ham}(X, Y)$ *is essentially small and has weak homotopy type* $\mathrm{map}(X, Y)$.

Proof. By Proposition 4.3.4 we have a commuting diagram

$$\begin{tikzcd} D_0 \arrow[r, hook, "i"] \arrow[d, "\mu_0"] & \mathbf{Gr}_{\Delta} D \arrow[d, "\mu"] \\ C_0 \arrow[r, hook, "i"] & \mathbf{Gr}_{\Delta} C \end{tikzcd}$$

where the horizontal maps are homotopy equivalences. Thus the homotopy fibers of μ_0 and μ are weakly equivalent. The theorem then follows from Corollary 4.3.3 and Proposition 4.3.5. □

4.4. Re-proving Retakh's theorem

The fact that the hammock category has the weak homotopy type of the mapping space allows us to re-prove classical results without getting as technical as otherwise would be needed. Another advantage is that we are equipped with the notion of essential smallness which allows us to make sense out of these classical statements which otherwise would be difficult to interpret (see the discussion at Section 2.1). One of those results is Retakh's theorem from [**23**], which we re-prove in this section. In the following we assume that R is a unital and associative ring. The category of R-modules is denoted by Mod_R. Further by a chain complex we mean a non-negatively graded chain complex with the differential lowering the degree. We denote the category of chain complexes over R by Ch_R. Before stating Retakh's theorem we need some preliminary definitions.

Definition 4.4.1. Let R be a ring and M and N left R-modules. Then an exact sequence of left R-modules

$$E : 0 \to N \to E_{n-1} \to \cdots \to E_0 \to M \to 0$$

with $n \geq 1$, is called an *n-fold extension of M by N*. We denote by $\mathscr{E}xt^n_R(M, N)$ the category whose

- objects are n-fold extensions of M by N ($n \geq 1$)
- morphisms from E to F are commutative diagrams of R-modules of the form

$$\begin{tikzcd} N \arrow[r] \arrow[d, equal] & E_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_0 \arrow[r] \arrow[d] & M \arrow[d, equal] \\ N \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & M \end{tikzcd} .$$

Definition 4.4.2. Let $\mathcal{C}$ be a essentially small category, I be a core and $v \in N(I)$ be a fixed base point for $N(I)$. Then the *i-th homotopy group of* $\mathcal{C}$, denoted $\pi_i\mathcal{C}$, is defined as

$$\pi_i\mathcal{C} := [S^i, N(I)]_{\mathrm{sSets}_*}.$$

Our version of Retakh's theorem can be stated as follows.

Theorem 4.4.11 ([**23**, Theorem 1])**.** *Let M and N be R-modules. Then $\mathscr{E}xt^n_R(M, N)$ is essentially small and for any choice of a base point*

$$\pi_i(\mathscr{E}xt^n_R(M, N)) \simeq \mathrm{Ext}^{n-i}_R(M, N).$$

Here Ext^k_R *is the usual* Ext *functor from homological algebra (see e.g.* [**18**, pp. 84-85]*).*

Remark 4.4.3. For a more modern proof of a special case of Retakh's theorem one can read [**26**]. Unfortunately this paper suffers from the same problem as [**23**], since it never address the problem of the largeness of the category $\mathscr{E}xt^n_R(M,N)$. We are instead left with the following comment in [**26**, p. 158].

> The cautious reader will have noticed that the category $\mathscr{E}xt^n_R(M,N)$ is not small, nor even equivalent to a small category. There are various ways to avoid this problem, depending on the underlying framework for set theory. We will ignore this point and treat $\mathscr{E}xt^n_R(M,N)$ as if it were a small category.

Proving Retakh's theorem involves working in the model category of chain complexes over R, which we also denote by Ch_R. This model structure is given by the following theorem.

Theorem 4.4.4 ([**12**, Theorem 7.2])**.** *The following describes a model structure on* Ch_R*:*

(i) $f : X \to Y$ *is a* weak equivalence *if the map* f *induces isomorphisms* $H_k(X) \to H_k(Y)$ *for* $k \geq 0$*,*

(ii) $f : X \to Y$ *is a* cofibration *if for each* $k \geq 0$ *the map* $f_k : M_k \to N_k$ *is a monomorphism with a projective* R*-module as its cokernel,*

(iii) $f : X \to Y$ *is a* fibration *if for each* $k > 0$ *the map* $f_k : M_k \to N_k$ *is an epimorphism.*

Suspensions in Ch_R are particularly easy to construct.

Definition 4.4.5. Let N be a R-module. Then we denote by $\Sigma^n N$ the chain complex

$$\cdots \to 0 \to N \to 0 \to \cdots \to 0 \to 0$$

where the module N appears in the n-th degree. The chain complex $\Sigma^n N$ is called the *n-th suspension of N.*
We simply write N for the chain complex $\Sigma^0 N$, where it is understood from the context if we are talking about the R-module or the corresponding chain complex.

Remark 4.4.6. Given left R-modules M and N, by definition an object in $\mathrm{Ham}(M, \Sigma^n N)$ is given by chain maps $M \xleftarrow{\sim} X \to \Sigma^n N$. Thus such an object consists of: a (left) resolution, $X \xrightarrow{\sim} M$, and an R-module homomorphism $\alpha : X_n \to N$ such that the composition $\alpha \circ d : X_{n+1} \to N$ is the zero map. We therefore write $(X, \alpha\colon X_n \to N)$, or simply (X, α), for an object of $\operatorname{Ham}(M, \Sigma^n N)$. Further since the category $\mathscr{E}xt^n_R(M, N)$ is in a natural way a subcategory to the category of resolutions of M we get that $\mathscr{E}xt^n_R(M, N)$ can be viewed as subcategory of $\operatorname{Ham}(M, \Sigma^n N)$ by identifying $E \in \mathscr{E}xt^n_R(M, N)$ with an object in $\operatorname{Ham}(M, \Sigma^n N)$ given by $(E, \mathrm{id}\colon N \to N)$.

Our most important tool in proving Retakh's theorem is the following proposition where we show that $\mathscr{E}xt^n_R(M, N)$ is essentially small.

Proposition 4.4.7. *$\mathscr{E}xt^n_R(M, N)$ and $\operatorname{Ham}(M, \Sigma^n N)$ are homotopically equivalent. In particular $\mathscr{E}xt^n_R(M, N)$ is essentially small.*

Proof. We need to find functors $\varphi\colon \operatorname{Ham}(M, \Sigma^n N) \to \mathscr{E}xt^n_R(M, N)$ and $\psi\colon \mathscr{E}xt^n_R(M, N) \to \operatorname{Ham}(M, \Sigma^n N)$ such that $\varphi \circ \psi \simeq \mathrm{id}$ and $\psi \circ \varphi \simeq \mathrm{id}$. Given any object (X, α) in $\operatorname{Ham}(M, \Sigma^n N)$ we can fit it, in a functorial way, into the following diagram of R-modules

(4.4.1)

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, "\alpha"] \arrow[dr, phantom, "\mathrm{I}"] & X_{n-1} \arrow[r] \arrow[d, "\hat{\alpha}"] & X_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_0 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & N \arrow[r] \arrow[rr, bend right, "0"'] & \hat{X}_{n-1} \arrow[r] & X_{n-2} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the square I is a pushout diagram. Note that the lower row of the above diagram is an n-fold extension of M by N which we denote by $\hat{X}$. Denote the induced functor that assigns $\hat{X}$ to the object (X, α) in $\operatorname{Ham}(M, \Sigma^n N)$ by $\varphi\colon \operatorname{Ham}(M, \Sigma^n N) \to \mathscr{E}xt^n_R(M, N)$. Since the category $\mathscr{E}xt^n_R(M, N)$ is a subcategory of $\operatorname{Ham}(M, \Sigma^n N)$ (see Remark 4.4.6) we can denote the inclusion functor $\mathscr{E}xt^n_R(M, N) \hookrightarrow \operatorname{Ham}(M, \Sigma^n N)$ by $\psi\colon \mathscr{E}xt^n_R(M, N) \to \operatorname{Ham}(M, \Sigma^n N)$. The next step is to show that $\varphi \circ \psi \simeq \mathrm{id}$ and $\psi \circ \varphi \simeq \mathrm{id}$.

Since the pushout of

$$E_{n-1} \leftarrow N \xrightarrow{\mathrm{id}} N$$

is just E_{n-1}, we have that $\varphi \circ \psi = \mathrm{id}$. It remains to show that $\psi \circ \varphi \simeq \mathrm{id}$. We do this by constructing a natural transformation $\Phi : \mathrm{id} \to (\psi \circ \varphi)$. We have that $(\psi \circ \varphi)(X, \alpha) = (\hat{X}, \mathrm{id}\colon N \to N)$, where $\hat{X}$ is defined as above. We define our family of morphisms

$$\begin{tikzcd}
M \arrow[d, equal] & X \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\Phi^{X,\alpha}"] & \Sigma^n N \arrow[d, equal] \\
M & \hat{X} \arrow[l, "\sim"'] \arrow[r] & \Sigma^n N
\end{tikzcd}$$

by having $\Phi_i^{X,\alpha} = \mathrm{id}$ for $i \leq n-2$, $\Phi_n^{X,\alpha} = \alpha$ and let $\Phi_{n-1}^{X,\alpha} = \hat{\alpha}$ be defined as in the diagram (4.4.1). By the diagram (4.4.1) these morphisms form the desired natural transformation and thus we can conclude that $\mathscr{E}xt_R^n(M,N)$ is homotopically equivalent to $\mathrm{Ham}(M, \Sigma^n N)$. By Theorem 4.2.6 $\mathrm{Ham}(M, \Sigma^n N)$ is essentially small and thus it follows from Proposition 2.2.7 that $\mathscr{E}xt_R^n(M,N)$ is also essentially small. □

With Proposition 4.4.7 we are just two lemmas away from proving Retakh's theorem. The lemmas are the following standard facts about the category of chain complexes.

Lemma 4.4.8 ([**12**, Proposition 7.3])**.** *Let M and N be two R-modules and $m, n \in \mathbb{N}$. Then there exists a natural isomorphism*

$$[\Sigma^m M, \Sigma^n N]_{\mathrm{Ch}_R} \simeq \mathrm{Ext}_R^{n-m}(M,N).$$

To state the second lemma we need to recall some definitions.

Definition 4.4.9. Let K be a simplicial set. The *free chain complex of K over R*, also denoted $R(K)$, is defined as being the chain complex with $R(K)_n$ being the free module generated by K_n and with differential given by $\partial_n := \Sigma_{i=0}^n d_i$, where $d_i \colon R(K)_n \to R(K)_{n-1}$ are induced by the simplicial face maps. Further we denote by $- \otimes - \colon \mathrm{Ch}_R \times \mathrm{Ch}_R \to \mathrm{Ch}_R$ the standard tensor product of Ch_R.

Lemma 4.4.10. *Let I be a small category and $X \in \mathrm{Ch}_R$ any object. Then $\mathrm{hocolim}_I cX$ is weakly equivalent to $R(N(I)) \otimes X$.*

Before we prove Lemma 4.4.10 we use these lemmas to prove Retakh's theorem.

Theorem 4.4.11. *For any choice of a base point we have*

$$\pi_i \mathscr{E}xt_R^n(M,N) \simeq \mathrm{Ext}_R^{n-i}(M,N).$$

Proof. By Proposition 4.4.7 and Theorem 4.3.6 it follows that $\mathscr{E}xt_R^n(M,N)$ is essentially small and has a core which is weakly equivalent to $\mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N)$. Thus we get that for a fixed choice of a base point

$$\pi_i \mathscr{E}xt_R^n(M,N) = \pi_i \mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N) = [S^i, \mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N)]_{\mathrm{sSets}_*}.$$

By the Dold-Kan correspondence $\mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N)$ has the homotopy type of a simplicial abelian group. Therefore

$$[S^i, \mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N)]_{\mathrm{sSets}_*} = [S^i, \mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N)]_{\mathrm{sSets}}$$

(see e.g. [**13**, III Corollary 2.5]). Using the adjointness from Proposition 3.0.2 we have that

$$[S^i, \mathrm{map}_{\mathrm{Ch}_R}(M, \Sigma^n N)]_{\mathrm{sSets}} \simeq [\mathrm{hocolim}_{S^i} cM, \Sigma^n N]_{\mathrm{Ch}_R}.$$

Lemma 4.4.10 gives us

$$[\mathrm{hocolim}_{S^i} cM, \Sigma^n N]_{\mathrm{Ch}_R} \simeq [S^i \otimes M, \Sigma^n N]_{\mathrm{Ch}_R} = [\Sigma^i M, \Sigma^n N]_{\mathrm{Ch}_R}$$

where we used the fact that $S^i \otimes X = \Sigma^i X$ for any $X \in \mathrm{Ch}_R$. Finally applying Lemma 4.4.8 we get

$$\pi_i \mathscr{E}xt^n_R(M,N) \simeq [\Sigma^i M, \Sigma^n N]_{\mathrm{Ch}_R} \simeq \mathrm{Ext}^{n-i}_R(M,N).$$

□

All that is left is to prove Lemma 4.4.10. For that we need to get a model for the homotopy colimit in Ch_R.

Definition 4.4.12. Let $F : I \to \mathrm{Ch}_R$ be any functor. We define the *simplicial replacement* of F to be the simplicial object repl F in Ch_R given degree-wise by

$$\mathrm{repl}\ F_n = \bigoplus_{i_0 \to \cdots \to i_n \in N(I)} F(i_0) = \bigoplus_{\sigma \in N(I)_n} F_\sigma$$

where we used the notation $F_\sigma := F(i_0)$ for a simplex $\sigma = i_0 \to \cdots \to i_n$. The face maps of repl F are defined as

$$d_k = \begin{cases} F(i_0 \to i_1), & \text{if } k = 0 \\ \mathrm{id}_{F(i_0)}, & \text{otherwise} \end{cases}$$

and the degeneracy maps are defined as $s_k = \mathrm{id}_{F(i_0)},\ \forall k$.

Given a simpicial object $S = \{S_n, d_i, s_j\}$ in Ch_R we get a double complex with one differential given by $d_{S_n} : S_{n,m} \to S_{n,m-1}$ and the other given by $\partial : S_{n,m} \to S_{n-1,m}$ defined as $\partial = \sum_{i=0}^n (-1)^i d_i$. We use this to define the following functor.

Definition 4.4.13. Given an simpicial object $S = \{S_n, d_i, s_j\}$ we define the *total complex* of S, denoted $\mathrm{Tot}(S)$, degree-wise by

$$\mathrm{Tot}(S)_p := \bigoplus_{n+m=p} S_{n,m} \quad , \qquad d_p = \bigoplus_{n+m=p} d_{S_n} + (-1)^m \partial.$$

These definitions allow us to find the following model for the homotopy colimit in Ch_R.

Proposition 4.4.14 ([**25**, Proposition 5.1.9] & [**24**, Theorem 4.1])**.** *Let I be a small category and $F : I \to \mathrm{Ch}_R$ any functor. Then* $\mathrm{Tot}(\mathrm{repl}\ F)$ *is a model for* $\mathrm{hocolim}_I F$*.*

We are now ready to prove Lemma 4.4.10.

Proof. *(of Lemma 4.4.10)*
We note that since $cX_\sigma = X$ for any $\sigma \in N(I)_n$ we have

$$\begin{aligned}\mathrm{Tot}(\mathrm{repl}\ cX)_p &= \bigoplus_{n+m=p} \bigoplus_{\sigma \in N(I)_n} X_m \\ &= \bigoplus_{n+m=p} (\oplus_{\sigma \in N(I)_n} R) \otimes X_m \\ &= \bigoplus_{n+m=p} R(N(I))_n \otimes X_m\end{aligned}$$

for any $p \in \mathbb{N}$. Since our differential is

$$d_p = \bigoplus_{n+m=p} d_{S_n} + (-1)^m \partial$$

we have that

$$\mathrm{Tot}(\mathrm{repl}\ cX) = R(N(I)) \otimes X.$$

Then by Proposition 4.4.14 we can conclude that $\mathrm{hocolim}_I cX$ is weakly equivalent to $R(N(I)) \otimes X$. □

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