

CONTINUOUS MULTI-LINE QUEUES AND THE TASEP

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ABSTRACT. In this paper, we study a distribution Υ of labeled particles $\{1, 2, \dots, n\}$ on a continuous ring. It arises in three different ways, all related to the multi-type TASEP on a ring. We prove formulas for the probability density function for some (cyclic) permutations and give conjectures for a larger class. We give a complete conjecture for the probability of two particles i, j being next to each other on the cycle, for which we prove some cases.

We end with observations of the similarities between the TASEP on a ring and the well-studied Razumov-Stroganov process.

1. INTRODUCTION

In this paper, we study a distribution Υ of labeled particles on a continuous ring. We call this distribution continuous TASEP on a ring, and it arises in three different ways:

- (1) As the limit of the stationary distribution of totally asymmetric exclusion process (TASEP) on a ring.
- (2) As the projection to the last row of a random continuous multiline queue.
- (3) As the stationary distribution of the so-called (continuous) process of the last row.

Exact definitions are given in Section 2. The equivalence of these three descriptions already in the discrete case follows from the seminal work conducted by Ferrari and Martin [12]. The first two explicitly and the third implicitly, as described in Section 2.3. The limit of the TASEP considered here keeps the number of jumping particles constant and let the number of vacant positions tends to infinity.

In previous works of several authors e.g. [2, 8, 14, 15, 12], the finite version of this TASEP has been proved to have many remarkable properties and to be connected to other mathematical objects: shape of random n -core partitions, random walks in an affine Weyl group, and multiline queues. In the present paper we study properties of the limit distribution Υ and find some remarkable properties of it. For simplicity, most of our results are focused on the case when the n particles are labelled by $\{1, \dots, n\}$, so we have a cyclic permutation.

If we condition on the permutation π of the distribution we can in some special cases give an exact description of the density function g_π of how the particles are located on the circle, see Section 3. For the reverse permutation $w_0 = n \dots 321$ the density function is the Vandermonde determinant. For a number of other permutations the density function is, mostly conjecturally, a sum of derivatives of the Vandermonde determinant. In the cases we can

prove, we first prove an exact formula for the discrete case with a given number of empty sites and then take the limit. An interesting observation is that the the density functions in several cases satisfy the Laplace equation.

The probability for the particles to form a given permutation seems in general difficult to compute and we have no general conjecture. It can be
 SL computed for w_0 , see Theorem 3.9. In Section 4 we study the probability of two given labels being next to each other. There is a tantalising pattern for this correlation that we formulate as a general conjecture. We prove the conjecture in a few special cases. In Section 5 we study the probability that k particles adjacent to each other form a descending sequence. This is also proven to be the Vandermonde determinant. The proof is in the finite case and it proves Theorem 8.1 announced in [9]. We don't know if these two different probabilities both given by the Vandermonde determinant are just coincidences or if there is a deeper connection. See beginning of Section 5 for a discussion. In Section 6 we end with some speculations about the similar nature of this model and the famous Razumov-Stroganov model for linking patterns in a disk.

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2. BACKGROUND AND DEFINITIONS

2.1. Multi-type TASEP on a ring. Consider a vector $\mathbf{m} = (m_1, \dots, m_n)$ of positive integers and a ring with $N \geq \sum m_i$ sites labelled $\{0, \dots, N-1$. A state of the \mathbf{m} -TASEP chain is a placement of m_i particles labelled i on the ring $1 \leq i \leq n$ such that no two particles are on the same site. A transition in this chain is one particle being chosen uniformly at random and the particle tries to jump left. If the site to the left is vacant, it jumps there. If the site to the left has a particle with a larger label they trade places. If the site to the left has a particle with a smaller or equal label no jump will happen. So we can think of vacancies as particles labelled $n+1$, but the notation becomes simpler by not doing so.

Exclusion processes has been studied extensively in general and this cyclic \mathbf{m} -TASEP has been considered by several authors [3, 2, 9, 10, 11, 12, 14, 17]. Both Matrix Ansatz solutions and more combinatorial solutions have been suggested. Let $\Upsilon_{\mathbf{m}}(N)$ be the stationary distribution of the TASEP, then we define Υ as the limit when \mathbf{m} is fixed and N tends to infinity while scaling the ring to have length 1. Note that we define a limit of stationary distributions and not the limit of the TASEP itself. We leave that as an interesting challenge.

2.2. Multiline queues. We will make extensive use of *multiline queues* (MLQs), originally defined by Ferrari and Martin [12]. We will distinguish between *discrete* MLQs and *continuous* MLQs.

A *discrete MLQ* of type $\mathbf{m} = (m_1, \dots, m_n)$ is an $(n+1) \times N$ array, with $m_1 + \dots + m_i$ boxes in row i for $1 \leq i \leq n$. Given such an array, there is a labelling procedure which assigns a label to each box. See Figure 2.2 for an example. We label the boxes row by row from top to bottom. Suppose we

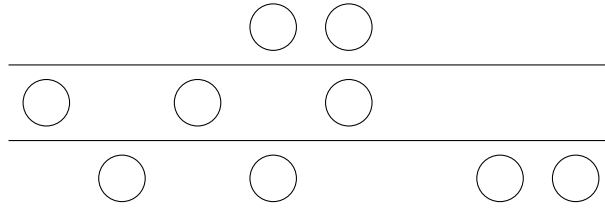


FIGURE 1. A discrete multiline queue

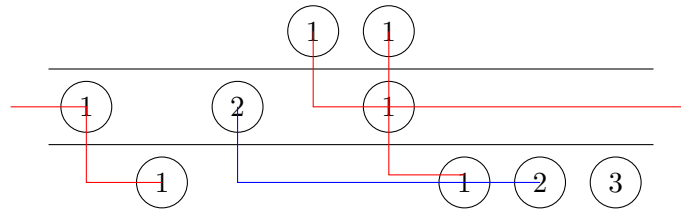


FIGURE 2. The labelling of the multiline queue in Figure 2.2

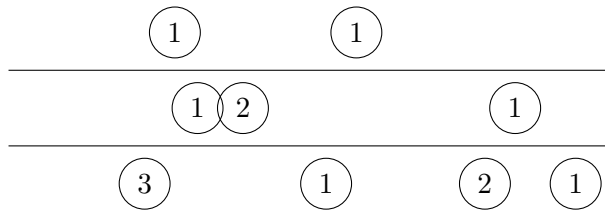


FIGURE 3. A continuous multiline queue

have just labeled row i . Pick any order of the boxes in row i such that boxes with smaller label come before boxes with larger label. Now go through the boxes in this order. When considering a box labeled k , find the first unlabelled box in row $(i + 1)$, going weakly to the right (cyclically) from the column of the box, and label that box k . When this is done, some boxes (in total m_{i+1}) remain unlabelled in row $i + 1$. Label these $i + 1$. Thus all boxes in the first row are labelled 1. With a k -bully path we will mean the path of a label k from its starting position on row k directly down and then along the $k + 1$ st row to its box with label k , and so forth all the way down to the bottom row. If two k -bully-paths are arriving at the same label k it is not well defined which one turns downwards and which one that continues on the row, but it will not matter for our purposes.

A *continuous MLQ* of type \mathbf{m} is a sequence of n 'continuous' rows with $m_1 + \dots + m_i$ boxes in row i . In this case we consider the location of the boxes to be numbers in the continuous interval $[0, 1)$. We label the boxes using the same labelling procedure as for discrete MLQs (we disregard the set of measure zero where two or more of the positions of the boxes coincide). Clearly, continuous MLQs are limits of discrete MLQs.

The motivation of these definitions is the following theorem.

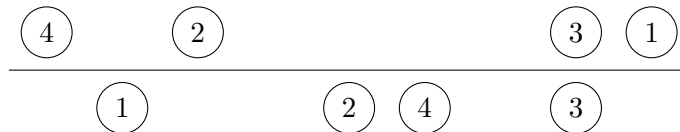


FIGURE 4. The output of the process of the last row when the top row together with the circles in the second row are given.

Theorem 2.1 (Ferrari-Martin). *Let X be \mathbf{m} -TASEP distributed word. Then*

$$\mathbb{P}[X = u] = \frac{n_u}{\prod_{i=1}^r \binom{n}{m_1 + \dots + m_i}},$$

where n_u is the number of \mathbf{m} -MLQs whose bottom row is labelled u .

From this theorem it follows that Υ is the distribution on the bottom row for a uniformly chosen continuous MLQ.

2.3. The process of the last row. Theorem 2.1 can easily be extended to the case where $m_n = 0$, as we now explain. Consider a $(m_1, \dots, m_{n-1}, 0)$ -MLQ. By the theorem, the labelling of row $n - 1$ has TASEP distribution. Using the proof of [12], it is easy to show that row n also has TASEP distribution, *of the same type*. It follows that the probabilistic map from the $(n - 1)$ st row to the n th row represents another Markov chain with the same distribution! Here is an example of how we will use this fact. Consider Figure 2.3. In the top row we have sampled a word from the TASEP distribution, and on the second row we have selected 4 positions for the boxes uniformly at random. Then we use the same labelling procedure as previously. The claim, then, is that the bottom row also has TASEP distribution. We have been deliberately vague as to whether we are talking about discrete or continuous MLQs here – of course, both interpretations are valid¹.

3. DISCRETE AND CONTINUOUS DENSITY FUNCTIONS

In this section we will assume that $\mathbf{m} = (1, \dots, 1)$, so the particles form a permutation. Our main interest lies in the following quantities.

- (a) For a permutation π , the number $G_\pi(b_1, \dots, b_n; N)$ of discrete MLQs of length N such that the letters in the bottom row are π_1, \dots, π_n , at positions $b_1 < \dots < b_n$. For fixed N , summing $G_\pi(b_1, \dots, b_n; N)$ over all permutations π and all increasing sequences $b_1 < \dots < b_n$, we get $Z_N = \binom{N}{1} \binom{N}{2} \dots \binom{N}{n}$, the total number of discrete MLQs.
- (b) We get the corresponding continuous probability density function for $0 < q_1 < \dots < q_n < 1$ as a limit,

$$g_\pi(q_1, \dots, q_n) = \lim_{N \rightarrow \infty} \frac{1}{Z_N} G_\pi(\lfloor N \cdot q_1 \rfloor, \dots, \lfloor N \cdot q_n \rfloor; N).$$

The number g_π can also be obtained by a finite computation by looking at the $\binom{n}{2}!$ permutations the boxes in the MLQ can form

¹Strictly speaking, we proved it for discrete MLQs, and it is trivial to deduce from this that the continuous version is also valid.

(when ordered from the left to right on a *single* line), or by taking the part of highest degree in G_π appropriately normalized.

- (c) The probability $p_\pi = \int_{0 < q_1 < \dots < q_n < 1} g_\pi(q_1, \dots, q_n) dq_1 \dots dq_n$ that the letters at the bottom of a random continuous MLQ form the permutation π .

3.1. Probabilities of the discrete chain. We are not able to give an exact formula for G_π except for a few special permutations and conjecturally for some more.

As an example one can by studying MLQs with two rows see that $G_{12}(b_1, b_2; N) = N - b_2 + b_1$ and $G_{21}(b_1, b_2; N) = b_2 - b_1$.

More generally, we have the following formula for the reverse permutation $w_0 = n(n-1)\dots 1$.

Proposition 3.1. *For any $N \geq n \geq 2$ we have*

$$G_{w_0}(b_1, \dots, b_n; N) = \det \left\{ \binom{b_i + j - 1}{j - 1} \right\}_{1 \leq i, j \leq n} = \prod_{1 \leq k < l \leq n} (b_l - b_k) \prod_{d=1}^{n-1} \frac{1}{d!}.$$

Proof. Suppose we have a discrete MLQ whose bottom row n is labelled by the reverse permutation, with a particle labeled $n+1-i$ at position b_i , for $1 \leq i \leq n$, where $b_1 < \dots < b_n$. It is a direct consequence of the construction of MLQs that the position of the boxes on row $n-1$, $b'_1 < \dots < b'_{n-1}$ must be such that $b_i < b'_1 \leq b_{i+1}$ for $1 \leq i < n$ and they must also correspond to the reverse permutation (of length $n-1$). It follows by induction that each row is labelled by a reverse permutation.

Thus the bully paths will never touch each other and we may use the Lindström-Gessel-Viennot lemma, see e.g. [22, Chapter 2.7]. We may say that each bully path starts at the beginning of the row, that is on positions $(r, 0)$ for $1 \leq r \leq n$ (matrix notation).

The number of (non-cyclic) paths using only right and down steps from $(r, 0)$ to (n, b_i) is $\binom{b_i + n - r}{n - r}$. Setting $j = n + 1 - r$ gives the determinant in the proposition. The factor $\prod_{d=1}^{n-1} \frac{1}{d!}$ can be taken out and using column operations we reduce to the standard form of the Vandermonde determinant. \square

Next we study the permutations $s_k w_0 = n \dots (k+2)k(k+1)(k-1) \dots 21$, where the numbers k and $k+1$ have switched places in w_0 . For $1 \leq k \leq n$, let A_k be the matrix with entries $\binom{b_i + j - 1}{j - 1}$ in rows $1 \leq i \leq n - k$ and entries $\binom{b_i + j - 2}{j - 2}$ in rows $n - k < i \leq n$. We conjecture the following.

Conjecture 3.2. *For $N \geq n > k \geq 1$ and $0 \leq b_1 < \dots < b_n \leq N - 1$ the number of MLQs with bottom row $s_k w_0$ is*

$$G_{s_k w_0}(b_1, \dots, b_n; N) = \binom{N}{k} \det A_k - G_{w_0}.$$

We can prove this conjecture for $k = 1, 2$.

Theorem 3.3. *For $k = 1, 2$, Conjecture 3.2 is true, that is,*

$$G_{s_k w_0} = \binom{N}{k} \det A_k - G_{w_0}.$$

Proof. We will first consider the case $k = 1$. We distinguish between two different types of MLQs that can result in the permutation $s_1 w_0$, depending on whether the bully path for the first class particle wrapping from rightmost position to the leftmost.

The first type is an MLQ \mathbf{q} where there is no bully path wrapping. Then the bully paths for the first and second class particles will touch at some point (r, c) and after that first class particles path will be below or on the path of the second class particle. We could also describe this as the permutations on the first $r - 1$ rows of \mathbf{q} is the reverse permutation and after that the 1 and 2 has switched places.

To count MLQs of the first type we define an injection into sets of certain non-intersecting paths $\mathcal{P} = \{L_n, \dots, L_1\}$. The path L_1 is formed by concatenating the bully path of the first class particle of \mathbf{q} from $(1, 0)$ to (r, c) with the bully path of second class particle from (r, c) to (n, b_n) and then lifting the resulting path one step upwards. So, L_1 is a path from $(0, 0)$ to $(n - 1, b_n)$ passing $(r - 1, c)$. The path L_2 is formed by concatenating the second class particles bully path from $(2, 0)$ to (r, c) with the first class particles path from (r, c) to (n, b_{n-1}) . For $3 \leq i \leq n$, L_i is the bully path for the i th class particle. Thus \mathcal{P} is a set of non-intersecting lattice paths with starting positions $\mathcal{S}_1 = \{(n, 0), \dots, (3, 0), (2, 0), (0, 0)\}$ and ending positions $\mathcal{M}_1 = \{(n, b_1), \dots, (n, b_{n-2}), (n, b_{n-1}), (n - 1, b_n)\}$. All such sets of paths are counted by the determinant of the matrix

$$D_1 = \left[\begin{array}{ccc|c} \ddots & \vdots & \ddots & \vdots \\ \cdots & \binom{b_i+j-1}{j-1} & \cdots & \binom{b_i+n}{n} \\ \ddots & \vdots & \ddots & \vdots \\ \hline \cdots & \binom{b_n+j-2}{j-2} & \cdots & \binom{b_n+n-1}{n-1} \end{array} \right].$$

Sets of paths \mathcal{P} where the vertical distance between L_1 and L_2 is always two or more do not come from an MLQ \mathbf{q} of the first type. To subtract this overcount we need to count the number of non-intersecting paths from $\mathcal{S} = \{(n, 0), \dots, (2, 0), (1, 0)\}$ to $\mathcal{M} = \{(n, b_1), \dots, (n, b_{n-1}), (n, b_n)\}$, which is precisely G_{w_0} by Proposition 3.1. Thus the number of MLQs of the first type is $\det(D_1) - G_{w_0}$.

The second type of MLQs \mathbf{q} that project to the permutation $s_1 w_0$ has the bully path of the first class particle wrapping on row two before finding a box to label 1, see Figure 5. The bully path of the first class particle will overlap with the bully path of the second class particle in the beginning of row two and possibly more later. Apart from that, no paths are touching. We now describe a bijection from such MLQs to non-intersecting lattice paths $\mathcal{P} = \{L_n, \dots, L_1, L_0\}$ with starting positions $\mathcal{S}_2 = \{(n, 0), \dots, (2, 0), (1, 0), (0, 0)\}$ and ending positions $\mathcal{M}_2 = \{(n, b_1), \dots, (n, b_{n-2}), (n, b_{n-1}), (n - 1, b_n), (1, N - 1)\}$. We define L_0 as a translation one step upwards of the bully path for the first class particle from $(1, 0)$ until it comes to $(2, N - 1)$. The path L_1 is a translation one step upwards of the bully path of the second class particle. L_2 is the part of the bully path of the first class particle starting at $(2, 0)$. For $3 \leq i \leq n$, L_i is the bully path for the i th class particle.

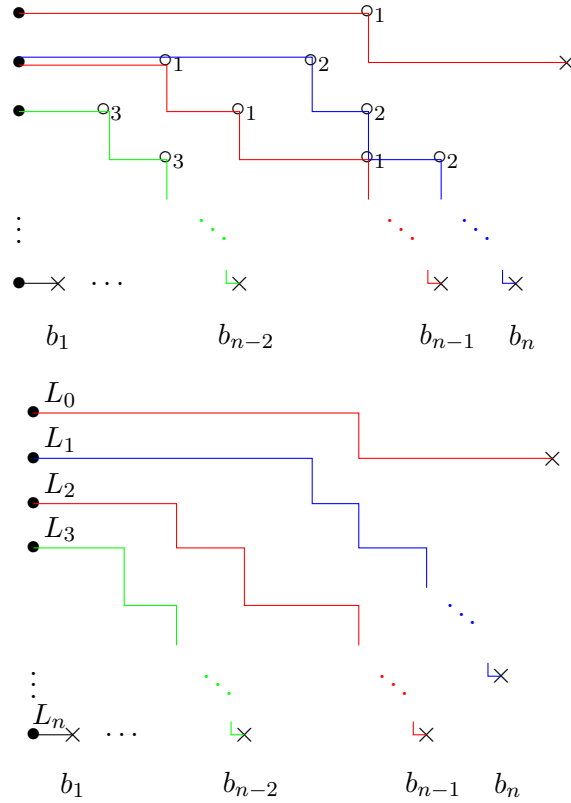


FIGURE 5. On top a schematic image of an MLQ projecting to $s_1 w_0$ of type 2. Below the corresponding set of $n + 1$ non-intersecting lattice paths as in the proof of Theorem 3.3.

By the Lindstöm-Gessel-Viennot Lemma all such sets of non-intersecting paths are counted by the determinant of the $(n + 1) \times (n + 1)$ -matrix

$$D_2 = \begin{bmatrix} 1 & \cdots & \binom{b_1+j-1}{j-1} & \cdots & \binom{b_1+n}{n} \\ \vdots & & \vdots & & \vdots \\ 1 & \cdots & \binom{b_{n-1}+j-1}{j-1} & \cdots & \binom{b_{n-1}+n}{n} \\ 0 & \cdots & \binom{b_n+j-2}{j-2} & \cdots & \binom{b_n+n-1}{n-1} \\ 0 & \cdots & 0 & 1 & \binom{N}{1} \end{bmatrix}.$$

Hence the total number of MLQs is

$$G_{s_1 w_0}(b_1, \dots, b_n; N) = \det D_2 + \det D_1 - G_{w_0}.$$

Expanding D_2 along the bottom row gives $\det D_2 = N \cdot \det A_1 - \det D_1$ and the statement for $k = 1$ follows.

The case $k = 2$ is very similar but more complicated. This time there are three different types of MLQs: no bully paths are wrapping, the second class path is wrapping on row 3, and the third type is when the second class path is wrapping on row 3 and first class path is wrapping on row 2. These are all the cases since if the 1 is wrapping then the 2 must also be wrapping for

the 1 to end up last in the permutation. For each type we give an injection to a set of tuples of non-intersecting paths, which can be counted using the Lindström-Gessel-Viennot Lemma.

Type 1: Assume the bully paths of the second and third class particle intersect for the first time at (r, c) . No other bully paths touch. We map such MLQs injectively to $\mathcal{P} = \{L_n, \dots, L_1\}$, with starting positions $\mathcal{S}_1 = \{(n, 0), \dots, (3, 0), (1, 0), (0, 0)\}$ and ending positions $\mathcal{M}_1 = \{(n, b_1), \dots, (n, b_{n-2}), (n-1, b_{n-1}), (n-1, b_n)\}$. Let L_1 be the bully path of the first class particle translated one step upwards. Let L_2 be the translation one step upwards of the concatenation of the second bully path from $(2, 0)$ to (r, c) and the third bully path from (r, c) to (n, b_{n-1}) . Let L_3 be the concatenation for the remaining pieces and L_i is the i th bully path for $4 \leq i \leq n$. The total number of such n -tuples of non-intersecting paths is the determinant of the following $n \times n$ -matrix

$$E_1 = \left[\begin{array}{ccc|cc} \ddots & \vdots & \ddots & \vdots & \vdots \\ \dots & \binom{b_i+j-1}{j-1} & \dots & \binom{b_i+n-1}{n-1} & \binom{b_i+n}{n} \\ \ddots & \vdots & \ddots & \vdots & \vdots \\ \hline \dots & \binom{b_n+j-2}{j-2} & \dots & \binom{b_{n-1}+n-2}{n-2} & \binom{b_{n-1}+n-1}{n-1} \\ \dots & \binom{b_n+j-2}{j-2} & \dots & \binom{b_n+n-2}{n-2} & \binom{b_n+n-1}{n-1} \end{array} \right].$$

As in the type 1 case above we must subtract those where the vertical distance between L_2 and L_3 is at least 2 all the time, which again is G_{w_0} .

Type 2: Let (r, c) be the point where the bully paths for the first and second class particle intersect the first time. The wrapping second bully path and the third bully path will overlap in the beginning of row 3. We map such MLQs injectively to $\mathcal{P}_2 = \{L_n, \dots, L_1, L_0\}$, with starting positions $\mathcal{S}_2 = \{(n, 0), \dots, (2, 0), (1, 0), (-1, 0)\}$ and ending positions $\mathcal{M}_2 = \{(n, b_1), \dots, (n, b_{n-2}), (n-1, b_{n-1}), (n-1, b_n), (1, N-1)\}$.

L_0 is a translation two steps upwards of the concatenation of the first bully path to (r, c) and the bully path of the second class particle from (r, c) to $(3, N-1)$ before it is wrapping. L_1 is a translation one step upwards of the concatenation of the second bully path to (r, c) and the first bully path from (r, c) to (n, b_n) . Let L_2 be a translation one step upwards of the bully path of the third class particle and let L_3 be the bully path of the second class particle after it has cycled, that is, from $(3, 0)$ to (n, b_{n-2}) . To count the number of MLQs of type 2 we have to subtract of the sets \mathcal{P}_2 where the vertical distance between L_0 and L_1 is 2 or more at all times, which can be counted by lowering the start and endpoints of L_0 . This means that the number of MLQs is counted by the difference $\det(E_2) - \det(E'_2)$, where E_2, E'_2 are the following $(n+1) \times (n+1)$ -matrices

$$E_2 = \left[\begin{array}{ccc|c} \ddots & \vdots & \ddots & \vdots \\ \dots & \binom{b_i+j-1}{j-1} & \dots & \binom{b_i+n+1}{n+1} \\ \ddots & \vdots & \ddots & \vdots \\ \dots & \binom{b_n+j-2}{j-2} & \dots & \binom{b_{n-1}+n}{n} \\ \dots & \binom{b_n+j-2}{j-2} & \dots & \binom{b_n+n}{n} \\ 0 & \dots & 0 & 1 \\ & & & \binom{N+1}{2} \end{array} \right],$$

$$E_2' = \left[\begin{array}{cccc|c} \ddots & \vdots & \ddots & \vdots & \vdots \\ \dots & \binom{b_i+j-1}{j-1} & \dots & \binom{b_i+n-1}{n-1} & \binom{b_i+n}{n} \\ \ddots & \vdots & \ddots & \vdots & \vdots \\ \dots & \binom{b_n+j-2}{j-2} & \dots & \binom{b_{n-1}+n-2}{n-2} & \binom{b_{n-1}+n-1}{n-1} \\ \dots & \binom{b_n+j-2}{j-2} & \dots & \binom{b_n+n-2}{n-2} & \binom{b_n+n-1}{n-1} \\ 0 & \dots & 0 & 1 & \binom{N}{1} \\ & & & & \binom{N+1}{2} \end{array} \right].$$

Type 3: The MLQs where both the first and the second bully path wraps around can similarly be bijectively mapped to $n+2$ -tuples of non-intersecting paths with starting positions $\mathcal{S}_3 = \{(n, 0), \dots, (1, 0), (0, 0), (-1, 0)\}$ and ending positions $\mathcal{M}_3 = \{(n, b_1), \dots, (n, b_{n-2}), (n-1, b_{n-1}), (n-1, b_n), (1, N-1), (0, N-1)\}$. These are counted by the determinant of the matrix,

$$E_3 = \left[\begin{array}{ccc|cc} \ddots & \vdots & \ddots & & \\ \dots & \binom{b_i+j-1}{j-1} & \dots & & \\ \ddots & \vdots & \ddots & & \\ \dots & \binom{b_n+j-2}{j-2} & \dots & & \\ \dots & \binom{b_n+j-2}{j-2} & \dots & & \\ 0 & \dots & 0 & 1 & \binom{N}{1} & \binom{N+1}{2} \\ 0 & \dots & 0 & 0 & 1 & \binom{N}{1} \end{array} \right].$$

Expanding the matrices E_1, E_2, E_2', E_3 along the bottom rows most terms cancel and give the claimed result. \square

We will also state the following more general conjecture for commuting simple reflections. For a partition $\mathbf{k} = (k_1 \geq \dots \geq k_r \geq 1)$, let \mathbf{k}' denote the conjugate partition. For a subset $S \subseteq [r]$, let $\mathbf{k}(S)$ be the partition consisting of the parts $k_i, i \in S$. With $\mathbf{k}(S)'$ we denote the conjugate of $\mathbf{k}(S)$. Let A_S be the matrix with entries $\binom{b_i+j-1-k_{n+1-i}(S)'}{j-1-k_{n+1-i}(S)'}$, where $k_i(S)' = 0$ if $i > k_{\min} S$.

Conjecture 3.4. *For $N \geq n$ and \mathbf{k} such that $n > k_1 > k_2 + 1 > k_3 + 2 > \dots > k_r + r - 1 > r - 1$ and $0 \leq b_1 < \dots < b_n \leq N - 1$, the number of MLQs is*

$$G_{s_{k_1} \dots s_{k_r} w_0}(b_1, \dots, b_n; N) = \sum_{S \subseteq [r]} (-1)^{|S|} \prod_{i \in S} \binom{N}{k_i} \det A_S.$$

Note that A_\emptyset is the Vandermonde matrix so it specializes to Conjecture 3.2 for $r = 1$.

3.2. Probability density functions for the continuous chain. We will now use the above results on the discrete chain to understand the probability density function g_π for the continuous distribution Υ when the particles form a certain permutation π .

By setting $q_i = b_i/N$ in the formula in definition (b) and letting $N \rightarrow \infty$ we can from Proposition 3.1 deduce the following.

Corollary 3.5. *For any $n \geq 2$ and $0 \leq q_1 < \dots < q_n < 1$ we have*

$$g_{w_0}(q_1, \dots, q_n) = n! \prod_{1 \leq k < l \leq n} (q_l - q_k)$$

Similarly, Conjecture 3.2 translates to the following.

Conjecture 3.6. *For any $n > k \geq 1$ and any $0 \leq q_1 < \dots < q_n < 1$ we have*

$$g_{s_k w_0} = \left(\frac{1}{k!} \frac{\partial^k}{\partial q_{n-k+1} \dots \partial q_n} - 1 \right) g_{w_0}.$$

Corollary 3.7. *Conjecture 3.6 is true for $k = 1, 2$.*

Example.

$$g_{4321} = \prod_{1 \leq i < j \leq 4} (q_j - q_i), \quad g_{4312} = \left(\frac{\partial}{\partial q_4} - 1 \right) g_{4321}$$

$$g_{4231} = \left(\frac{1}{2} \frac{\partial^2}{\partial q_3 \partial q_4} - 1 \right) g_{4321}, \quad g_{3421} = \left(\frac{1}{6} \frac{\partial^3}{\partial q_2 \partial q_3 \partial q_4} - 1 \right) g_{4321}.$$

The polynomials $g_w(\dots)$ satisfy some interesting relations whose general form we have not been able to pin down exactly. For example, for $n \leq 4$, all the $g_w(x_1, \dots, x_n)$ satisfy Laplace's equation

$$\frac{\partial^2 g_w}{\partial x_1^2} + \frac{\partial^2 g_w}{\partial x_2^2} + \dots + \frac{\partial^2 g_w}{\partial x_n^2} = 0.$$

It's a classical fact [6] that any such *harmonic polynomial* can be expressed as a linear combination of partial derivatives of a Vandermonde determinant (the converse, that any such combination is harmonic, is immediate). Since g_{w_0} is a Vandermonde determinant, it follows that there is *some* linear combination of its partial derivatives whose value is g_w for each w . In each case there seems to be a particularly simple one, for example:

$$g_{132} = \left(1 + \frac{\partial}{\partial q_1} + \frac{1}{2} \frac{\partial^2}{\partial q_1^2} \right) g_{321}$$

$$g_{1432} = \left(-1 - \frac{\partial}{\partial q_1} - \frac{1}{2} \frac{\partial^2}{\partial q_1^2} - \frac{1}{6} \frac{\partial^3}{\partial q_1^3} \right) g_{4321}$$

$$g_{4132} = \left(1 - \frac{\partial}{\partial q_3} - \frac{\partial}{\partial q_4} + \frac{1}{2} \frac{\partial^2}{\partial q_3 \partial q_4} \right) g_{4321}$$

$$g_{4213} = \left(1 - \frac{\partial}{\partial q_4} + \frac{1}{2} \frac{\partial^2}{\partial q_4^2} \right) g_{4321}$$

For $n = 5$, we have calculated that 15 of the 24 polynomials g_w satisfy Laplace's equation (cyclicity modded out). So there cannot be any linear recursion using differentiation operators between the polynomials g_w in general. Perhaps there is some other set of operators which coincide with differentiation for small n and does extend to larger n ?

The reader may note that the part of maximal degree in g_u appears to be $\pm g_{w_0}$, where $w_0 = 4321$, and we choose $+$ if and only if $\ell(w_0) - \ell(u)$ is even.

The implication of Conjecture 3.4 to the continuous distributions is the following.

Conjecture 3.8. *For \mathbf{k} such that $n > k_1 > k_2 + 1 > k_3 + 2 > \dots > k_r + r - 1 > r - 1$ and any $0 \leq q_1 < \dots < q_n < 1$,*

$$g_{s_{k_1} \dots s_{k_r} w_0} = \left(\frac{1}{k_r!} \frac{\partial^{k_r}}{\partial q_{n-k_r+1} \dots \partial q_n} - 1 \right) g_{s_{k_1} \dots s_{k_{r-1}} w_0}.$$

An example, where the conjecture is true, is $n = 4, k_1 = 3, k_2 = 1$,

$$g_{3412} = \left(1 - \frac{1}{6} \frac{\partial^3}{\partial q_2 \partial q_3 \partial q_4} - \frac{\partial}{\partial q_4} + \frac{1}{6} \frac{\partial^4}{\partial q_2 \partial q_3 \partial^2 q_4} \right) g_{4321}.$$

3.3. Probability of given permutation. One obvious question to ask about the distribution Υ is the probability p_π that the particles form a certain permutation π . We can compute the exact probability of the reverse permutation.

Theorem 3.9. *The probability that the particles form the reverse permutation is*

$$p_{w_0} = \frac{1}{\prod_{k=1}^{n-1} \binom{2k+1}{k+1}}.$$

Proof. If the labelling of the boxes on the last row of an MLQ is w_0 then also the word on the row above have to be labelled by the reverse permutation, as noted in the proof of Proposition 3.1, and this holds inductively for all rows. All the relative positions are however not fixed since rows two or more apart could be on either side of each other. For example for $n = 3$ the box on the top row could be to the left or right of the middle box in the last row. Thus giving 2 different possible relative positions of all boxes. Such triangular arrays of boxes have been studied previously in other contexts and are called e.g. Gelfand-Tsetlin patterns, see [20]. The enumeration of all such patterns seems to have been done first in [24] where it is proven that the number of such patterns is

$$\frac{\binom{k+1}{2}! \prod_{i=1}^{n-1} i!}{\prod_{i=1}^{n-1} (2i+1)!}.$$

Think of the $\binom{n+1}{2}$ boxes in the MLQ as chosen in the interval $[0, 1)$, and then selecting which boxes ends up on which line. Then there are a total $\binom{\binom{n+1}{2}}{1, 2, 3, \dots, n}$ ways to do this. Dividing the number of Gelfand Tsetlin patterns with the total numbers gives the formula claimed. \square

We have computed p_π for all permutations for $n \leq 6$ and we can unfortunately not see any obvious general pattern. For instance, (i) p_{w_0} is

$i \setminus j$	1	2	3	4	5	6
1	0	1/2	1/6	2/15	6/55	1/11
2	1/14	0	25/42	2/15	6/55	1/11
3	5/42	1/21	0	19/30	6/55	1/11
4	16/105	17/210	1/30	0	106/165	1/11
5	68/385	81/770	19/330	4/165	0	7/11
6	37/77	41/154	34/231	5/66	1/33	0

TABLE 1. Table over $c_{i,j}(n)$, for $n = 6$.

the smallest of the probabilities and p_{id} is the largest, but p_π/p_{w_0} is not an integer in general, and (ii) the chain is not symmetric: $p_{w_0\pi w_0} \neq p_\pi$ in general. SL

4. CORRELATIONS

Even though the probability p_π of a given permutation π to appear at stationarity seems difficult to describe in general, the correlation of two adjacent elements seems to exhibit very interesting patterns in the continuous TASEP. Let $c_{i,j}(n) = \mathbb{P}(w_a = i, w_{a+1} = j)$ for some a . See Table 1 for the values of $c_{i,j}(6)$.

The most obvious observation is that the columns in the upper right part seem to be constant. To be more precise:

Conjecture 4.1. *For every $i + 1 < j$ we have $c_{i,j}(n) = n/\binom{n+j}{2}$.*

From this it would also follow that $c_{n-1,n} = (n+1)/(2n-1)$ and $c_{1,2} = 4/(n+2)$.

It seems the denominator always is a product of small primes. The data for $n \leq 6$ suggest in fact a conjecture covering all the $c_{i,j}$ s. Our main conjecture for the correlations in the continuous TASEP on a ring is the following.

Conjecture 4.2. *For $n \geq 2$, we have the following two-point correlations at stationarity*

$$c_{i,j}(n) = \begin{cases} \frac{n}{\binom{n+j}{2}}, & \text{if } i + 1 < j \leq n, \\ \frac{n}{\binom{n+j}{2}} + \frac{ni}{\binom{n+i}{2}}, & \text{if } i + 1 = j \leq n, \\ \frac{n}{\binom{n+j}{2}} - \frac{n}{\binom{n+i}{2}}, & \text{if } j < i < n, \\ \frac{n(j+1)}{\binom{n+j}{2}} - \frac{n(j-1)}{\binom{n+j-1}{2}} - \frac{n}{\binom{2n}{2}}, & \text{if } j < i = n. \end{cases}$$

So, for any $j < n$ the most likely situation is that it is (directly) to the right of $j - 1$. If j is very close to n then this will happen more than half the time. For small j ($1 < j < n/6$) the second most likely thing is that j is to the right of n . The most unlikely of all events is that n is to the right of $n - 1$.

Remark 4.3. *According to the conjecture, n is always a factor for the probabilities. It could be tempting to divide with n and say that we are interested*

in the case when $w_1 = i, w_2 = j$. This is however not true. The spacing between the particles is not uniform and hence it is not uniform which particle is first in a given $[0, 1)$ interval.

We can prove a few of the cases of this conjecture.

Proposition 4.4. *The following two-point correlations hold for any $n \geq 3$.*

$$\begin{aligned} (1) \quad c_{2,1} &= \frac{n}{\binom{n+1}{2}} - \frac{n}{\binom{n+2}{2}} = \frac{4}{(n+1)(n+2)} \\ (2) \quad c_{1,2} &= \frac{n}{\binom{n+2}{2}} + \frac{n}{\binom{n+1}{2}} = \frac{4}{(n+2)} \\ (3) \quad c_{n,n-1} &= \frac{n^2}{\binom{2n-1}{2}} - \frac{n(n-2)}{\binom{2n-2}{2}} - \frac{n}{\binom{2n}{2}} = \frac{3}{(2n-1)(2n-3)} \end{aligned}$$

Proof. For the first two statements we will use the process of the last row. Assume that the particles of classes 2 and 1 are positioned at q_2 and q_1 respectively before the process of the last row. By rotation we may assume that $q_2 < q_1$. The only way to obtain a 2 followed by a 1 after the process of the last row is to have exactly one particle in the interval $[q_2, q_1]$. That is easy to see since the class 1 particle will land at the first available position and after that the class 2 particle will do the same. Let $y = 1 - (q_1 - q_2)$. We know by Corollary 3.5 that $g_{21}(q_1, q_2) = 2(q_2 - q_1) = 2(1 - y)$. If we think of this as the limit of the stationary distribution of TASEP it is clear that the particles of classes higher than 2 will not influence the relative position of 2 and 1. The probability that there is exactly one of the n particles in the interval $[q_2, q_1]$ is $\binom{n}{1}(1 - y)y^{n-1}$. We thus obtain

$$c_{2,1} = \mathbb{P}(2 \text{ followed by } 1) = \int_0^1 2n(1 - y)^2 y^{n-1} dy = \frac{4}{(n+1)(n+2)}.$$

The computation of $c_{1,2}$ is similar. This time there are three possibilities to get a 1 followed by a 2; either there are no particles in the interval $[q_2, q_1]$ or all the particles are in this interval or there are all particles but one in the interval. Summing these three integrals gives the desired formula. Note that the method used above could in principle (but it quickly gets more complicated) be extended to $c_{i,j}$, for $i, j \leq x$, if we know all the density functions g_π for all permutations of length at most x . The reader is invited to try e.g. $c_{3,1}$.

The computation of $c_{n,n-1}$ is more involved. We use the continuous multiline queues. As discussed in Section 3 it is only the relative positions of the $\binom{n+1}{2}$ chosen positions in the multiline queue that are important. By rotation we may assume that it is the leftmost box on the bottom line that is given class n . We number the relative position from the right and let $a_{m,j}$ be the j th box from the right on row m , so $a_{m,m} > a_{m,m-1} > \dots > a_{m,1}$. The key is to note first that for the leftmost box on row n to get class n there must be no box "queuing" at the far left which means we must have the inequalities $a_{n-1,j} > a_{n,j}$ for each $1 \leq j \leq n-1$ and $a_{n,n} > a_{n-1,n-1}$. Secondly, for the second box from the left on row n to get class $n-1$ the leftmost box on row $n-1$ must have class $n-1$ and be the only box on that row to the left of $a_{n,n-1}$. We may summarize the inequalities as follows.

$$\begin{array}{ccccccccc}
a_{n,n} & > & a_{n-1,n-1} & > & a_{n-2,n-2} & > & a_{n-2,n-3} & > & \cdots & > & a_{n-2,1} \\
& & \downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
& & a_{n,n-1} & > & a_{n-1,n-2} & > & a_{n-1,2} & > & \cdots & > & a_{n-1,1} \\
& & & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
& & & & a_{n,n-2} & > & a_{n,2} & > & \cdots & > & a_{n,1}
\end{array}$$

The position of the boxes on the first $n - 3$ rows will not matter and we may resort to studying the relative position of the $3n - 3$ boxes on the three bottom rows. From the inequalities it follows directly that $a_{n,n} = 3n - 3$ and $a_{n-1,n-1} = 3n - 4$. If $a_{n,n-1} = 2n + i - 1$ then $a_{n-2,j} = 2n + j - 3$ for all $i < j \leq n - 2$. The remaining entries form a standard Young tableau (transpose the figure of the inequalities above) with columns of length $n - 2, n - 2, i$. Let $SYT_{n-2,n-2,i}$ denote the number of standard Young tableaux of this type. We refer to [23] for basic facts about standard Young tableaux including the hook-content formula from which one may deduce that $SYT_{n-2,n-2,i} = \frac{(2n-4+i)!(n-i)(n-i-1)}{i!n!(n-1)!}$. The rotation gives a factor of $3n - 3$ and we get

$$\begin{aligned}
c_{n,n-1} &= \frac{(3n-3) \sum_{i=0}^{n-2} SYT_{n-2,n-2,i}}{\binom{3n-3}{n,n-1,n-2}} = \frac{(n-2)! \sum_{i=0}^{n-2} \frac{(2n-4+i)!(n-i)(n-i-1)}{i!}}{(3n-4)!} \\
&= \sum_{i=0}^{n-2} \frac{\binom{n-2}{i}}{\binom{3n-4}{n-i}} = \frac{3}{(2n-1)(2n-3)},
\end{aligned}$$

as desired. The last equality was obtained using MAPLE. \square

5. CORRELATION FUNCTION FOR INITIAL DECREASING SEQUENCE

In this section we will prove a formula for the probability that a word sampled from the discrete TASEP starts with a given decreasing word. By a remarkable coincidence it is almost the same formula as in Proposition 3.1. Fix the length N of the ring and let $\mathbf{m} = \underbrace{(1, \dots, 1)}_N$. Suppose u is picked

from the stationary distribution of the \mathbf{m} -TASEP. We now ask, what is the probability that u starts with some fixed word? In general the answer appears to be complicated (see [9] for words of length at most 3). However in the case of a word of the type $x_n x_{n-1} \dots x_2$ where $x_n > x_{n-1} > \dots > x_2$, we show that there is a simple answer to this question.

Theorem 5.1. *Suppose u is picked from the stationary distribution of the \mathbf{m} -TASEP. Fix $N \geq x_n > x_{n-1} > \dots > x_2 \geq 1$. Then, the probability $f_\pi(x_n, \dots, x_2)$ (π for 'permutation') that for some word v , $u = x_n x_{n-1} \dots x_2 v$, is*

$$\frac{1}{\prod_{i=1}^{n-1} \binom{N}{i}} \det \left(\binom{x_{i+1}}{j-1} \right)_{i,j=1}^{n-1}.$$

The formula in Theorem 5.1 has been previously been noted by Kirone Mallick [7]. One way to think about this theorem is to think of the state of the TASEP as a permutation matrix of size N with 1s in positions $(i, \pi(i))$. Theorem 5.1 below gives the probability that the first $n - 1$ rows have the 1s in positions $(1, x_n), \dots, (n - 1, x_2)$. By cyclic invariance the same is true for

any $n - 1$ consecutive rows. Proposition 3.1 on the other hand states that the probability is exactly the same that the first columns of the state matrix have 1s in positions $(x_n, n - 1), \dots, (x_2, 1)$. This is because the probability of the position of the smallest labels does not change if we change all labels n, \dots, N to just n . This interesting equality does not carry over to other patterns in general.

Proof. For a vector (m_1, \dots, m_n) of positive integers with sum N , write $N_i = (M_{i-1}, M_i]$ where $M_0 = 0$ and $M_i = \sum_{j \leq i} m_j$, $1 \leq i \leq n$ so that $[N] = \{1, 2, \dots, N\}$ is the disjoint union of the N_i 's.

Now, write $f_w(m_n, m_{n-1}, \dots, m_1)$ (w for 'word') for the probability that a TASEP distributed word of type \mathbf{m} starts with the word $n(n-1) \dots 32$.

Clearly,

$$(1) \quad f_w(m_n, \dots, m_1) = \sum_{x_n \in N_n} \cdots \sum_{x_2 \in N_2} f_\pi(x_n, \dots, x_2).$$

Moreover, for a fixed x_n, \dots, x_2 , the value of $f_\pi(x_n, \dots, x_2)$ can be computed from the values of all $f_w(m_n, \dots, m_1)$ where $m_1 + \dots + m_n = N$ (by applying Möbius inversion to equation (1)). Thus, to prove the theorem, it is sufficient to show that equation (1) is satisfied (for all \mathbf{m}) when substituting the stated formula for f_π . We will now make a series of manipulations to the right hand side of (1), after substituting. This expression is

$$\frac{1}{\prod_{i=1}^{n-1} \binom{N}{i}} \sum_{x_n \in N_n} \cdots \sum_{x_2 \in N_2} \det \left(\binom{x_{i+1}}{j-1} \right)_{i,j=1}^{n-1}.$$

Now, note that x_{i+1} only occurs in row i and use the multilinearity of the determinant to move each sum inside its respective row. We get

$$\begin{aligned} & \frac{1}{\prod_{i=1}^{n-1} \binom{N}{i}} \det \left(\sum_{x_{i+1} \in N_{i+1}} \binom{x_{i+1}}{j-1} \right)_{i,j=1}^{n-1} = \\ & \frac{1}{\prod_{i=1}^{n-1} \binom{N}{i}} \det \left(\binom{M_{i+1} + 1}{j} - \binom{M_i + 1}{j} \right)_{i,j=1}^{n-1}. \end{aligned}$$

We will use the following determinantal identity twice. It is easy to prove using row operations.

Lemma 5.2. *Suppose a_{ij} are the entries of an $n \times n$ matrix such that $a_{ij} = 1$ for $j = 1$. Then*

$$\det(a_{ij})_{i,j=1}^n = \det(a_{(i-1)(j+1)} - a_{1(j+1)})_{i,j=1}^{n-1}.$$

Some further manipulation of the determinant:

$$\begin{aligned} & \det \left(\binom{M_{i+1} + 1}{j} - \binom{M_i + 1}{j} \right)_{i,j=1}^{n-1} = \\ & \det \left(\binom{M_{i+1} + 1}{j} - \binom{M_1 + 1}{j} \right)_{i,j=1}^{n-1} = \det \left(\binom{M_i + 1}{j-1} \right)_{i,j=1}^n. \end{aligned}$$

The first equality follows from row operations. The second follows from Lemma 5.2 (note that the first column in the matrix on the right is constant).

Let $F_w(m_n, m_{n-1}, \dots, m_1) = \prod_{i=1}^n \binom{N}{M_i} \cdot f_w(m_n, m_{n-1}, \dots, m_1)$ be the number of multi-line queues whose bottom row starts $n(n-1) \dots 32$ and has type \mathbf{m} .

To prove equation (1), we should prove that

$$F_w(m_n, m_{n-1}, \dots, m_1) = \prod_{i=1}^n \frac{\binom{N}{M_i}}{\binom{N}{i-1}} \det \left(\binom{M_i+1}{j-1} \right)_{i,j=1}^n$$

Move the product in the numerator into the rows of the matrix and the product in the denominator into the columns. Simplify the resulting expression $\frac{\binom{N}{M_i} \binom{M_i+1}{j-1}}{\binom{N}{j-1}} = \frac{M_i+1}{N+2-j} \binom{N+2-j}{M_i+2-j}$ and move the prefactors out through the rows and columns again. We get

$$\begin{aligned} & \prod_{i=1}^n \frac{M_i+1}{N+2-i} \det \left(\binom{N+2-j}{M_i+2-j} \right)_{i,j=1}^n = \\ & \prod_{i=1}^{n-1} \frac{M_i+1}{N+1-i} \det \left(\binom{N+2-j}{M_i+2-j} \right)_{i,j=1}^n. \end{aligned}$$

This matrix has a row with all ones, allowing us to convert it back to a $(n-1) \times (n-1)$ -matrix again, using (the transposed version of) Lemma 5.2. This (and $\binom{N+2-j}{M_i+2-j} - \binom{N+1-j}{M_i+1-j} = \binom{N+1-j}{M_i+2-j}$) yields

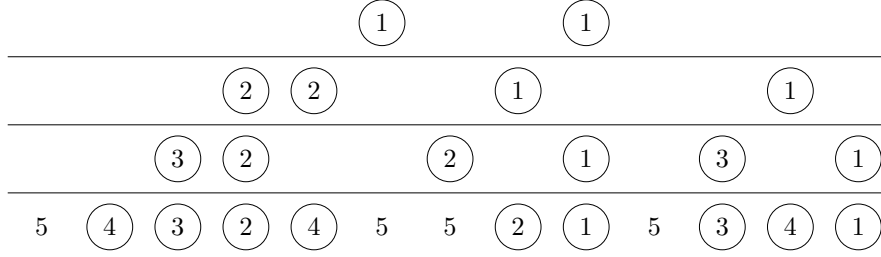
$$(2) \quad \prod_{i=1}^{n-1} \frac{M_i+1}{N+1-i} \det \left(\binom{N+1-j}{M_i+2-j} \right)_{i,j=1}^{n-1}.$$

It remains to show that the last expression equals $F_w(m_n, \dots, m_1)$. Consider a multiline queue counted by $F_w(m_n, \dots, m_1)$. It has $n-1$ rows, indexed $1, \dots, n-1$ (we use matrix notation; (i, j) refers to row i and column j). In row $1, 2, \dots, (n-1)$, there are $m_1, m_1+m_2, \dots, N-m_n$ particles. It is easy to see that the sites in the 'triangle' $(n-1, 1), \dots, (n-1, n-1)$; $(n-2, 2), \dots, (n-2, n-1)$; \dots ; $(2, n-1)$ are filled with boxes, and that a box in such a site (i, j) is labeled $n+1-j$. Denote by $z_{j,i}$ the distance from the right end of the multiline queue of the j th box from the right in the $(n-i)$ th row. That is, if the j th particle from the right in the $(n-i)$ th row is at (i, r) , we let $z_{j,i} = N-r$. Since the word starts with the descending sequence, no bully paths may wrap around to the beginning. Thus the numbers $z_{j,i}$ must form a semi-standard Young tableau (SSYT) of shape λ where the conjugate partition is $\lambda'_i = M_{n-i} - (n-i-1)$ for $1 \leq i \leq n-1$. Moreover this is a bijection from the MLQs counted by $F_w(m_n, \dots, m_1)$ to SSYT of shape λ with entries in $[t]$, where $t = N - n + 1$.

Example.

Here, $N = 13$, $n = 5$, $\mathbf{m} = (2, 2, 2, 3)$, $(M_1, \dots, M_5) = (2, 4, 6, 9, 13)$, $t = 9$, $\lambda' = (6, 4, 3, 2)$.

The multiline queue



corresponds to the tableau with rows 1125, 2368, 359, 57, 6, 9.

Now we are in a position to finish our argument.

By the definition of the Schur function, the number of SSYT of shape λ with entries in $[t]$ is $s_\lambda(1^t)$. Now, recall the hook-content formula and the Jacobi-Trudi identity, see e.g. [23].

Lemma 5.3. *The number of SSYT of shape λ and entries in $[t]$ equals*

$$\prod_{r \in \lambda} \frac{t + c_\lambda(r)}{h_\lambda(r)} = s_\lambda(1^t) = \det \left(\binom{t}{\lambda'_i - i + j} \right) = \det \left(\binom{t + j - 1}{\lambda'_i - i + j} \right),$$

where for a box $r = (i, j)$ in the Ferrers diagram of λ , we let $c_\lambda(r) = j - i$ and $h_\lambda(r) = \lambda_i + \lambda'_j - i - j + 1$.

The first two equalities are well-known, and the last is easily obtained by column operations.

So, by the lemma,

$$F_w(m_n, \dots, m_1) = \prod_{r \in \lambda} \frac{N - n + 1 + c_\lambda(r)}{h_\lambda(r)}.$$

Now let $s = t + 1 = N + 1 - n$ and $\mu'_i = \lambda'_i + 1$ for $1 \leq i \leq n - 1$. It is easy to see that our determinant in (2) equals (after reversing the numbering of rows and columns)

$$\prod_{i=1}^{n-1} \frac{M_i + 1}{N + 1 - i} \det \left(\binom{s + j - 1}{\mu'_i - i + j} \right)_{i,j=1}^{n-1},$$

which by the lemma (temporarily letting μ and s play the roles of λ and t) equals

$$\prod_{i=1}^{n-1} \frac{M_i + 1}{N + 1 - i} \prod_{r \in \mu} \frac{N - n + 2 + c_\mu(r)}{h_\mu(r)}$$

To prove that $F_w(m_n, \dots, m_1)$ equals the expression (2) it thus remains to show that

$$\prod_{i=1}^{n-1} \frac{M_i + 1}{N + 1 - i} \prod_{r \in \mu} \frac{N - n + 2 + c_\mu(r)}{h_\mu(r)} = \prod_{r \in \lambda} \frac{N - n + 1 + c_\lambda(r)}{h_\lambda(r)},$$

or, in terms of λ' ,

$$\prod_{r \in \lambda} \frac{N - n + 1 + c_\lambda(c)}{h_\lambda(c)} = \prod_{i=1}^{n-1} \frac{\lambda'_i + n - i}{N + 1 - i} \prod_{r \in \mu} \frac{N - n + 2 + c_\mu(c)}{h_\mu(c)}.$$

This is easily checked (think of μ as the result of adding a row on top of λ).

6. RAZUMOV-STROGANOV

During this work, we were struck several times by the similarity between our chain and the Markov chain of Razumov and Stroganov [21].

A *linking pattern* on $[2n]$ is a fixed point free non-crossing involution of $[2n]$. We draw linking patterns as diagrams as in Figure 6. Let Ω_n be the set linking patterns on $[2n]$. For any $i \in \{1, 2, \dots, 2n\}$ and any pattern L , let $e_i L$ be the pattern obtained from L by joining i with $i+1$, and $L(i)$ with $L(i+1)$. Here we take indices modulo $2n$. We can describe the image $e_i L$ of a generator e_i acting on L by adding a tile below L – see Figure 7. The RS chain on Ω_n is given by at each time step applying e_i where i is chosen uniformly at random from $\{1, 2, \dots, 2n\}$.

The generators e_i satisfy the following relations.

- (A) $e_i e_{i+1} e_i = e_i = e_i e_{i-1} e_i$
- (B) $e_i^2 = e_i$
- (C) $e_i e_j = e_j e_i$ when i, j are distinct and non-adjacent modulo $2n$

We list some similarities between the TASEP and the RS chain.

TASEP

- the states are permutations
- stationarity measure (conjecturally) is largest at identity, smallest at reverse identity. [14], [1], [12] (both TASEP and continuous TASEP)
- sum of entries of previous stationary measure equals largest component of next [12]
- defined in terms of the NilCoxeter algebra [14]
- (continuous TASEP) recursions that are rooted at the reverse identity.
- (continuous TASEP) measure at reverse identity is Vandermonde
- the k -TASEP has the same stationary distribution for all k [18]

Razumov-Stroganov

- the states are linking patterns
- stationarity measure is largest at least nested linking patterns, smallest at most nested patterns.
- sum of entries of previous stationary measure equals largest component of next. [21]
- defined in terms of the Temperley-Lieb algebra. [21]
- recursions that are rooted at the most nested pattern [13].
- measure at most nested pattern is Vandermonde [13].
- the k -RS has the same stationary distribution for all k . [13]

We now explain the last point in more detail. The k -TASEP is a generalisation of the TASEP where in each time step, a k -subset of positions is chosen, and then ringing a TASEP bell at each position with the rule that for any pair of neighbouring positions, the position to the left is activated before the one to the right. Remarkably, the stationary measure of this chain is independent of k . It makes perfect sense to define a " k -Razumov-Stroganov" chain in the same way; for a k -subset S of $[2n]$, define e_S as the product of all e_i for $i \in S$, taking e_i before e_{i+1} if both i and $i+1$ belong to S . For example, for $n = 4$, $e_{\{1,4,7,8\}} = e_4 e_1 e_8 e_7 = e_1 e_8 e_7 e_4$.

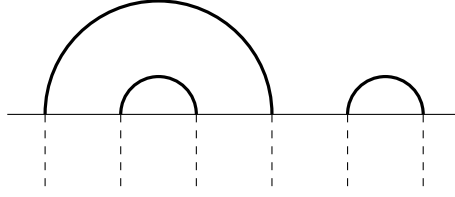


FIGURE 6. A linking pattern L on $[2n]$ for $n = 3$ with $L(1) = 4$, $L(2) = 3$, $L(5) = 6$.

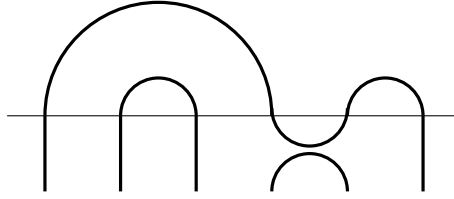


FIGURE 7. The linking pattern $L' = e_4L$, where L is from Figure 6. We have $L'(1) = 6$, $L'(2) = 3$, $L'(4) = 5$.

Let $M_n^{(k)}$ be the transition matrix of the k -RS chain on Ω_n . The following theorem is an analog of the k -TASEP from [18]. The theorem is not new, it is a consequence of the results in [13]. We will give a bijective proof for completeness.

Theorem 6.1 (DiFrancesco, Zinn-Justin). *The stationary distribution of $M_n^{(k)}$ is the same for all $k = 1, 2, \dots, 2n - 1$.*

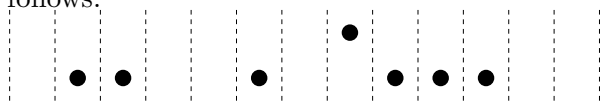
Proof. We prove that

$$M_n^{(k)} M_n^{(1)} = M_n^{(1)} M_n^{(k)}$$

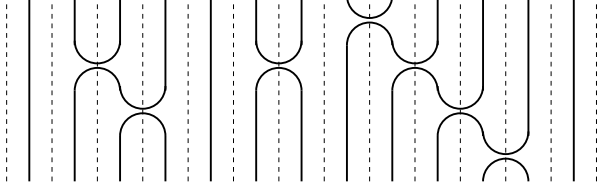
for each k . This is sufficient, since if ζ is the stationary distribution of $M_n^{(1)}$, then $M_n^{(k)} \zeta = M_n^{(k)} M_n^{(1)} \zeta = M_n^{(1)} M_n^{(k)} \zeta$ so that $M_n^{(k)} \zeta$ is some multiple of ζ (since $M_n^{(1)}$ is ergodic). The multiple cannot be the zero multiple, since the entries of $M_n^{(k)}$ and ζ are nonnegative.

So we should prove that for any pair of linking patterns L, L' , the number of pairs (i, S) satisfying $L' = e_i e_S L$ equals the number of pairs (S, i) satisfying $L' = e_S e_i L$. We will prove a stronger statement, namely for each pair (i, S) we will produce a pair (S', i') in a bijective way, such that $e_i e_S = e_{S'} e_{i'}$ as functions on Ω_n .

To do this, we visualise pairs (i, S) as $2 \times n$ diagrams filled with dots; for example, for $n = 13$ and $k = 7$, the pair $(8, \{2, 3, 6, 9, 10, 11\})$ will be drawn as follows.



This diagram corresponds to the following tile (which we think of as a map on Ω_{2n} , like in Figure 7).



These diagrams should be thought of as cyclic in the horizontal direction. We will not mark the 'boundary' (between positions 1 and $2n$) in what follows.

So for each (i, S) we should produce a pair (S', i') such that $e_i e_S = e_{S'} e_{i'}$. The definition of our map splits up in four cases.

- (1) $i \in S$: Now choose $r \geq 0$ maximal so that $i, i+1, \dots, i+r \in S$. Add i to S' , and let $i' = i+r$. For $k \in S$ not between i and $i+r$ inclusive, add k to S' .
 - (a) $i-1 \in S$: For k between $i-1$ and $i+r-1$ inclusive (all of which belong to S by assumption), add $k+2$ to S' . We need to verify that $e_{i+r} \dots e_i e_{i-1} \cdot e_i = e_{i+r} \cdot e_{i+r+1} \dots e_{i+2} e_{i+1} e_i$. Using the rule (A) $e_i e_{i-1} e_i = e_i$ on the left and $e_{i+r} e_{i+r+1} e_{i+r} = e_{i+r}$ on the right hand side they become the same expression.
 - (b) $i-1 \notin S$: For k between i and $i+r-1$ inclusive, add $k+1$ to S' . Now we need to check that $e_{i+r} \dots e_{i+1} e_i \cdot e_i = e_{i+r} \cdot e_{i+r} \dots e_{i+1} e_i$. This is clear since by rule (B) both expressions are equal to $e_{i+r} \dots e_{i+1} e_i$.
- (2) $i \notin S$
 - (a) $i-1 \in S$: Let $i' = i-1$, and $S' = S - \{i-1\} + \{i\}$. That $e_S e_i = e_{S'} e_{i'}$ follows easily from the fact that e_i and e_j commute when i and j are far apart, rule (C).
 - (b) $i-1 \notin S$:
 - (i) $i+1 \in S$: Choose $r \geq 1$ maximal such that $i+1, i+2, \dots, i+r$ all belong to S . Add $i, i+1, \dots, i+r-1$ to S' , and let $i' = i+r$. That $e_S e_i = e_{S'} e_{i'}$ again follows easily from relation (C).
 - (ii) $i+1 \notin S$: Let $i' = i$ and $S' = S$.

In each of the four cases above the image of the map is the dual of the domain in the sense that $i-1$ is replaced with $i'+1$. That is, in case 1a) we see that the image of the map is all pairs (S', i') such that $i' = i+r \in S'$ and $i'+1 \in S'$. Note that the cyclic symmetry mod $2n$ is essential here. The other three cases are easily checked and the map has thus a well-defined inverse and is hence a bijection. \square

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