A Markov Process on Cyclic Words

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Doctoral Thesis
Stockholm, Sweden 2014

TRITA-MAT-A 2014:12
ISRN KTH/MAT/A–14/12–SE
ISBN 978-91-7595-357-1

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Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie doktorsexamen i matematik fredagen den 12 december 2014 kl 10.00 i sal E3, Kungl Tekniska högskolan, Lindstedtsvägen 26, Stockholm.

Tryck: Universitetsservice US AB

Abstract

The TASEP (totally asymmetric simple exclusion process) studied here is a Markov chain on cyclic words over the alphabet $\{1, 2, \ldots, n\}$ given by at each time step sorting an adjacent pair of letters chosen uniformly at random. For example, from the word 3124 one may go to 1324, 3124, 3124, 4123 by sorting the pair 31, 12, 24, or 43.

Two words have the same *t*ype if they are permutations of each other. If we restrict TASEP to words of some particular type $\mathbf{m}$ we get an ergodic Markov chain whose stationary distribution we denote by $\zeta_{\mathbf{m}}$. So $\zeta_{\mathbf{m}}(u)$ is the asymptotic proportion of time spent in the state u if the chain started in some word of type $\mathbf{m}$. The distribution ζ is the main object of study in this thesis. This distribution turns out to have several remarkable properties, and alternative characterizations. It has previously been studied both from physical, combinatorial, and probabilitistic viewpoints.

In the first chapter we give an extended summary of known results and results in this thesis concerning ζ. The new results are described (and proved) in detail in Papers I - IV.

The new results in Papers I and II include an explicit formula for the value of ζ at sorted words and a product formula for decomposable words. We also compute some correlation functions for ζ. In Paper III we study of a generalization of TASEP to Weyl groups. In Paper IV we study a certain scaling limit of ζ, finding several interesting patterns of which we prove some. We also study an inhomogenous version of TASEP, in which different particles get sorted at different rates, which generalizes the homogenous version in several aspects. In the first chapter we compute some correlation functions for ζ.

Sammanfattning

TASEP är en markovkedja på cykliska ord över alfabetet $\{1, 2, ...\}$, som ges av att i varje tidssteg sortera ett likformigt slumpmässigt valt par av närliggande positioner i ordet. Från ordet 1342 kan man exempelvis nå orden 2341, 1342, 1342, och 1324 beroende på om man väljer paret 21, 13, 34 eller 42.

Vi säger att två ord har samma *t*yp, om de är permutationer av varandra. TASEP begränsad till ord av en given typ **m** är ergodisk och har därför en väldefinierad stationärfördelning $\zeta_{\mathbf{m}}$. Så $\zeta_{\mathbf{m}}(u)$ för ett ord (tillstånd) u av typ **m** är den asymptotiska proportionen av tidssteg spenderade i u om kedjan startar på något ord av typ **m**. I denna avhandling fokuserar vi på frågor om $\zeta_{\mathbf{m}}$ (för allmänt **m**). Fördelningen ζ visar sig ha många oväntade egenskaper och alternativa karakteriseringar.

Avhandlingen börjar med en längre introduktion som beskriver redan kända resultat samt resultat i avhandlingen om $\zeta_{\mathbf{m}}$. De nya resultaten beskrivs utförligt i artikel I - IV.

Bland de nya resultaten finns en explicit formel för stationärfördelningen för sorterade ord (som t.ex. 12233344), en produktformel för ord som är direkta summor (dvs. alla bokstäver till vänster om en viss position är mindre än alla bokstäver till höger om positionen), och generaliseringar av några av egenskaper för TASEP till en generalisering av TASEP till allmänna Weylgrupper. Vi beräknar också korrelationsfunktioner för markovkedjan. I artikel IV studerar vi en viss skalningsgräns av ζ, och noterar många intressanta mönster (av vilka några bevisas). Vi undersöker också en inhomogen variant av kedjan, som generaliserar många av den homogena variantens egenskaper.

Contents

Part II: Scientific Papers

Acknowledgements

Several people contributed positively to this thesis.

First of all, I thank my advisor Svante Linusson, whose guidance and enthusiasm made this thesis possible.
Working with my coauthor and -advisor Jonas Sjöstrand has been fun and inspirational.
The Mathematics department at KTH and in particular the Combinatorics group has provided a stimulating environment.
I thank Alexander Engström and Bruno Benedetti for being great teachers.
I'm very glad to have shared office with my fellow graduate students, who I thank for all the distractions and discussions together.
Finally I thank my family and other friends for being there.

Chapter 1

Introduction

This first part of the thesis consists of an exposition of the results in the thesis and of relevant background material. In Section 1.12 we prove a new result not contained anywhere else. The thesis consists of this summary and the following papers.

- Paper A: *Stationary probability of the identity for TASEP on a ring* [1]

- Paper B: *A product formula for the TASEP on a ring* (joint with Jonas Sjöstrand) [4]

- Paper C: *TASEP in any Weyl group* [2]

- Paper D: *Continuous multi-line queues and TASEP* (joint with Svante Linusson) [3]

The summary is organized as follows. We first describe the homogenous TASEP in some detail. In Sections 1.1 - 1.5 we review the well-known multiline queues, the Matrix Ansatz and the concept of intertwining matrices. In Sections 1.6 - 1.9 we give some useful identities satisfied by the stationary distribution of the TASEP. In Section 1.10 we describe a generalization of the TASEP to general Weyl groups. The correlation functions for the homogenous TASEP are described in Section 1.11 and 1.12.

Then, in Section 1.13, we introduce an inhomogenous version and describe to what extent properties of the homogenous TASEP generalize to the inhomogenous case. Paper D studies a certain limit of the stationary distribution of the TASEP. This is described in Section 1.14. Finally in Section 1.15 we list the contributions of the thesis in a concise way.

1.1 TASEP on a ring

Given any word w of length n, let $\sigma_i(w)$ denote the word obtained by sorting the letters w_i and w_{i+1}, taking the indices modulo n. The totally asymmetric simple exclusion process (TASEP) on a ring is the Markov chain on words where we apply a random σ_i at each time step. In the *homogenous* case, which we consider first, we take the σ_i's with equal rate, say 1.

In general, we will assume that our words have letters from the ordered alphabet $\{1, 2, \ldots, r\}$ (for varying r), with m_i occurrences of letter i. We typically assume that $m_i > 0$ for all i. The vector $\mathbf{m} = (m_1, \ldots, m_r)$ is the *type* of the word. The set of words of type $\mathbf{m}$ is denoted $\Omega_{\mathbf{m}}$. Thus the length n of the words in Ω_n satisfies $n = m_1 + \ldots + m_r$. If $w_i = j$ we will say that a *particle* of *class* j occupies *site* i in w.

As an example, consider words of type $(1, 1, 1)$, i.e. permutations of $\{1, 2, 3\}$. The transition diagram is given in Figure 1.1. The $\Omega_{(1,1,1)} \times \Omega_{(1,1,1)}$ transition matrix (ordering rows and columns lexicographically), scaled to have all column sums equal to 3, is

$$M_{\mathbf{m}} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The stationary probabilities $\zeta = \zeta_{\mathbf{m}}$ (the asymptotic proportion of time spent in each state) in this case become $\zeta(123) = \zeta(231) = \zeta(312) = 2/9$ and $\zeta(132) = \zeta(321) = \zeta(213) = 1/9$. We may define $\zeta_{\mathbf{m}}$ as the unique solution to $M_{\mathbf{m}}\zeta_{\mathbf{m}} = n \cdot \zeta_{\mathbf{m}}$ whose components sum up to 1.

It will be convenient to rescale these to *amplitudes* $[u] = [u]_{\mathbf{m}}$ by letting $[u]_{\mathbf{m}} = \zeta_{\mathbf{m}}(u)/\min_u \zeta_{\mathbf{m}}(u)$. We will omit the index $\mathbf{m}$ when this unlikely to lead to confusion. Finally, the chain has a cyclic symmetry, so we can express the probabilities above more succinctly as $[123] = 2$, $[132] = 1$.

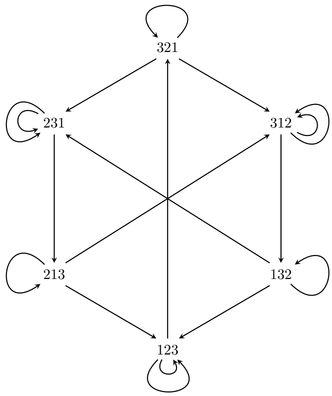

Figure 1.1: Transition diagram for TASEP on $\Omega_{(1,1,1)}$.

The following fact will be a consequence of Theorem 1.3.1. I don't know of a simpler derivation.

Proposition 1.1.1. *([7], [14]) Suppose* $u \in \Omega_{\mathbf{m}}$*. Then* $[u]_{\mathbf{m}} = Z_{\mathbf{m}}\zeta_{\mathbf{m}}(u)$*, where we define* $Z_{\mathbf{m}} = \binom{n}{m_1}\binom{n}{m_1+m_2}\cdots\binom{n}{m_1+\ldots+m_{r-1}}$*. Furthermore, the amplitudes are integers.*

The number $Z_{\mathbf{m}}$ is referred to as the *partition function.* Let us also make the following important observation.

Proposition 1.1.2. *(Projection Principle) Fix some class* i*. For any word* u*, we have* $\zeta_{\mathbf{m_1}}(u) = \sum_v \zeta_{\mathbf{m_2}}(v)$*, where the sum is taken over all* v *such that merging classes* i *and* $i+1$ *in* v *gives* u*.*

It is probably easier to explain what we mean by "merge" with an example rather than a formal definition: the result of merging classes 2 and 3 in 1523425 is 1422324.

This in particular means that one way to compute $\zeta_{\mathbf{m}}$ for some type of length n is to compute the distribution $\zeta_{(1,1\ldots,1)}$ for *permutations* of that length and then take a sum.

Proposition 1.1.3. *(Duality) Suppose* $w \in \Omega_{\mathbf{m}}$ *and let* $w' \in \Omega_{\mathbf{m}'}$ *be the word obtained from* w *by reversing values and positions;* $w'_i = r+1-w_{n+1-i}$ *(so that* $\mathbf{m}'$ *is* $\mathbf{m}$ *reversed). Then* $\zeta_{\mathbf{m}}(w) = \zeta_{\mathbf{m}'}(w')$*.*

The main object of this thesis is to study the probabilities ζ. In the course of doing so, we will introduce more general models which reduce to the case already described.

1.2 Small cases

How does one go about studying a process like the TASEP? A priori, there's no obvious reason why there should be anything to study. However, looking at small examples, one finds many remarkable patterns (of which Proposition 1.1.1 is one example).

We already considered the chain on $\Omega_{(1,1,1)}$. If we do the same thing for permutations of length 4, then we get $[1234] = 9, [1243] = [1324] = [1342] = 3, [1423] = 5, [1432] = 1$. The only non-self-dual pair of words in this case is 1342 and 1243. They have the same amplitude, as they should, by Proposition 1.1.2.

If we look at the case $r = 2$, $\mathbf{m} = (m_1, m_2)$, the stationary distribution is not very interesting; every state has the same probability, and there are $\binom{n}{m_1}$ states in total. To see that the stationary distribution is uniform, note that for each state, the number of in-arrows equals the number of out-arrows.

Nevertheless, this observation has an interesting consequence, using Proposition 1.1.2: in the general case (with an arbitrary number of classes), for any k, the set of positions occupied by particles of class k or lower is a uniformly distributed subset of $[n]$ of size $m_1 + \ldots + m_k$. However, varying k, these subsets are of course strongly dependent (for example they are nested).

The case $\mathbf{m} = (m_1, m_2, m_3)$ was studied in [7]. This case is much more interesting, even if it does not have all the complexities of the general case. Namely, for the 3-class system there are especially simple recursion relations for the amplitudes. To describe them, let u be any word with letters from $\{1, 2, 3\}$ containing at least one letter 2. Then the following recursions hold,

$$\begin{aligned} [31u] &= [3u] + [1u], \\ [32u] &= [2u], \\ [21u] &= [2u]. \end{aligned} \tag{1.2.1}$$

Finally we define $[u] = 1$ for any word u containing only 2's. Using these recursions and cyclic invariance ($[u_1 \ldots u_n] = [u_2 u_3 \ldots u_n u_1]$) it is easy to compute, for example, that $[231323131] = 10$, as the reader is invited to check!

To prove the correctness of these two recursions one needs to check two things. First, we need to check that they are consistent with each other (if we think of the amplitudes as variables and the recursions as equations then there are many more equations than variables when considering words of length at most some given bound). Second, we need to plug the recursions into the equilibrium equation to show that it is consistent with the recursions. In any particular example this is of course a simple but tedious calculation. For the general case one needs some bookkeeping device, such as the so-called Matrix Ansatz. A proof along these lines (for a slightly modified TASEP) can be found in Paper C [1]. We will come back to the Matrix Ansatz in Section 1.5.

A useful and non-trivial consequence of the recursions above is that for words $u^1, \ldots, u^m$ with letters from $\{1, 3\}$, we have $[2u^1 2u^2 \ldots 2u^m] = [2u^1] \cdot [2u^2] \cdot \ldots \cdot [2u^m]$. So to compute amplitudes it suffices to be able to compute amplitudes of words of the form $[2u]$ for words u with letters from $\{1, 3\}$ (i.e. excluding 2) with m_1 1's and m_3 3's. These numbers have a nice combinatorial interpretation, first pointed out to me by Henrik Eriksson [12]. In fact, for such a word $2u$, its amplitude $[2u]$ counts the number of subpartitions of the partition λ_u, obtained as follows. To draw the diagram of λ_u, draw a polygonal line starting at $(m_1, 0)$ (in matrix notation) and going to $(0, m_3)$, taking a step $(0, 1)$ for each letter 3 in u and a step $(-1, 0)$ for each letter 1 in u, when reading u from left to right. It is easy to see that this combinatorial construction satisfies the recursions above.

The interpretation of amplitudes as subdiagram counts does not appear to generalize. Instead, the next section describes a combinatorial interpretation for the amplitudes of arbitrary words.

1.3 Multiline queues

The most useful device for computing amplitudes are the *multiline queues*, which were discovered by Ferrari and Martin [14], building on a simpler variant of Angel's [7].

A *queue* is given by a subset of $[n]$ of some size k. The size k is referred to as the *capacity* of the queue. Let $\mathbf{m} = (m_1, \ldots, m_r)$ be a type and $\mathbf{m}' = (m_1, \ldots, m_{r-2}, m_{r-1}+m_r)$. We think of a queue q of capacity $m_1+\ldots+m_{r-1}$ as a function on words u of type $\mathbf{m}'$ to words of type $\mathbf{m}$ as follows. Create

[1] The method is not original with this paper.

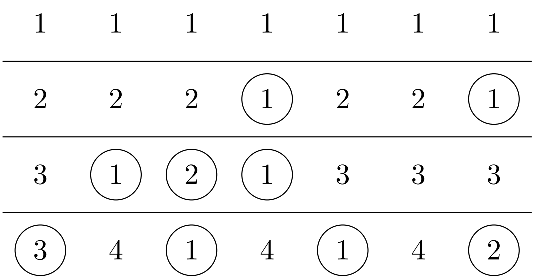

Figure 1.2: A multiline queue. The output of this queue is 3414142. There are 161 multiline queues with this output, which means that [3414142] = 161.

a $2 \times n$ array, writing u in the top row. Place empty circles in the second row at the sites in the subset associated to q. Write r in the non-circled spots. Now go through the letters that are strictly less than $r-1$ in the first row in any order such that small letters come before larger ones. When considering a letter l, find the first empty circle in the second row weakly cyclically to the right and fill it with l. This fills all but m_i empty circles. Fill these remaining circles with $r-1$. We then define the output $q(u)$ of q with respect to u as the word obtained by reading the labels in the second row from left to right.

Now, a *multiline queue* of type $\mathbf{m} = (m_1, \ldots, m_r)$ is simply a sequence of $r-1$ queues, where the ith queue has capacity $m_1 + \ldots + m_i$. The output $b(\mathbf{q})$ of a multiline queue $\mathbf{q}$ is the result of composing the individual queues as functions, in order, with the first taking the all-ones word of length n as input. Clearly, the number of multiline queues of type $\mathbf{m}$ is $Z_{\mathbf{m}}$. See Figure 1.3 for an example of a $(2,1,1,3)$-multline queue.

We can now state the main result in [14].

Theorem 1.3.1. *([14]) For any word u of type $\mathbf{m}$, the amplitude $[u]$ equals the number of multiline queues $\mathbf{q}$ of type $\mathbf{m}$ such that $b(\mathbf{q}) = u$.*

There are two independent proofs of this theorem (in [14] and [5]). The original one by Ferrari and Martin goes by constructing a Markov chain on the set of multiline queues of type $\mathbf{m}$ and showing that (i) that chain has a uniform stationary distribution (ii) the map b projects (probabilistically) the chain to the TASEP. We will say more about the other proof in Section

1.4.

It follows easily (see [14]) from Theorem 1.3.1 that $\min_{u\in\Omega_{\mathbf{m}}}\zeta(u)$ is attained for those words $u \in \Omega_{\mathbf{m}}$ satisfying $u_i \geq u_{i+1} - 1$ for all i (mod n). This includes in particular the reverse sorted word $\overleftarrow{u}$, for which we have $[\overleftarrow{u}] = 1$.

1.4 Intertwining matrices

There is a stronger, more illuminating, version of Theorem 1.3.1. Let $\mathbf{m}$, $\mathbf{m}'$ be as before, and let $U = U_{\mathbf{m}',\mathbf{m}}$ be the $\Omega_{\mathbf{m}} \times \Omega_{\mathbf{m}'}$ -matrix whose (u, v) entry is the number of queues q such that $v = q(u)$ (this number is clearly 0 or 1).

Let Ψ be the vector of amplitudes for $\mathbf{m}$-words and Ψ' that for $\mathbf{m}'$-words. Then the claim of the Theorem 1.3.1 is equivalent with the following equation between vectors,

$$\Psi = U\Psi'. \tag{1.4.1}$$

A stronger statement is that the following matrix equation holds.

Theorem 1.4.1. *([5]) In the notation above, we have*

$$UM_{\mathbf{m}'} = M_{\mathbf{m}}U. \tag{1.4.2}$$

Indeed, since Ψ' satisfies $M_{\mathbf{m}'}\Psi' = n\Psi'$, we see that if (1.4.2) holds, then $U\Psi'$ satisifies $M_{\mathbf{m}}(U\Psi') = n \cdot U\Psi'$. Therefore (since $M_{\mathbf{m}}$ is ergodic), $U\Psi'$ is parallel to Ψ and by looking more carefully at U it is easy to see that in fact $U\Psi' = \Psi$.

More general than [14], where only equation (1.4.1) is proved, in [5] (where U is defined in a very different way) equation (1.4.2) is proved. However the proof of [5] is less combinatorial than that of [14]. It is proved in [13] and [6] that the definition of U given here and the definition in [5] are indeed equivalent. The way we have defined U, equation (1.4.2) is a purely combinatorial statement: for each word $v \in \Omega_{\mathbf{m}'}$ and $u \in \Omega_{\mathbf{m}}$, the number of ways to obtain u from v by first going through a queue of capacity of $m_1 + \ldots + m_{r-1}$ and then taking one step in the TASEP on $\Omega_{\mathbf{m}}$ should equal the number of ways to obtain u from v by first taking a step in the TASEP on $\Omega_{\mathbf{m}'}$ and then going through a queue of capacity of $m_1 + \ldots + m_{r-1}$. To my knowledge there is no proof of Theorem 1.3.1 along these lines in the literature.

Even without knowing the particular structure of U (other than not being of too low rank – clearly, $U = 0$ is a solution), equation (1.4.2) is a very strong statement about the relation between $M_{\mathbf{m}}$ and $M_{\mathbf{m}'}$. If we think of U as unknown then there are as many equations as variables in (1.4.2). Furthermore, an eigenvector v of $M_{\mathbf{m}'}$ gives an eigenvector Uv of $M_{\mathbf{m}'}$ with the same eigenvalue whenever $Uv \neq 0$. In particular, $M_{\mathbf{m}}$ and $M_{\mathbf{m}'}$ have many shared eigenvalues.

However, we should already have suspected this! There is a more obvious $\Omega_{\mathbf{m}'} \times \Omega_{\mathbf{m}}$-matrix $D = D_{\mathbf{m},\mathbf{m}'}$ defined by $D(v, u) = 1$ if v is obtained from u by replacing all occurrences of r by $r - 1$, and 0 otherwise. Then

$$M_{\mathbf{m}'}D = DM_{\mathbf{m}}. \tag{1.4.3}$$

Equation (1.4.3) is a more general version of the Projection Principle in the case of merging classes $r - 1$ and r, and it tells us that indeed, $M_{\mathbf{m}'}$ and $M_{\mathbf{m}}$ have many shared eigenvalues. So, heuristically, the existence of the (obvious) map D should make us optimistic about finding a (more complicated) map U. This is a key observation. It would probably be valuable to make the correspondence between D and U more rigorous.

Merging other classes

Of course, we can merge any pair of adjacent classes (not necessarily the highest pair $r-1$ and r) and obtain a relation like (1.4.3). In [5], queues are generalized to construct a corresponding U operator. To describe these it will be convenient to briefly leave our convention to consider only standardized words aside. Thus suppose $\mathbf{m}$ is a type and that $m'_i = m_i$ for $i \neq k, k+1$ and that $m'_{k+1} = 0$, $m'_k = m_k + m_{k+1}$. Similar to queues, a *generalized queue* q is determined by a subset S of sites. If $|S| = m_1 + \ldots + m_k$ we think of q as a function from $\Omega_{\mathbf{m}'}$ to $\Omega_{\mathbf{m}}$. Let $u \in \Omega_{\mathbf{m}'}$. We construct $q(u)$ as follows. Write u in the top row of a $2 \times n$ array. Go through the letters in u which are between 1 and $k-1$ inclusive in any order and put them in the sites of S as before. This leaves m_k empty circles. Fill these with k. Now, go through the letters in u which are between $k+2$ and r inclusive in *decreasing* order. When considering a letter l, put it in the first empty *non-circled* site, going weakly cyclically *left* from l. This leaves m_{k+1} empty non-circled sites. Fill these with $k+1$. In [5] it is proven that the corresponding map $U_{\mathbf{m}',\mathbf{m}}$ (with the (u, v) entry counting the number of generalized queues with top row v and bottom row u) satisfies $M_{\mathbf{m}}U_{\mathbf{m}',\mathbf{m}} = U_{\mathbf{m}',\mathbf{m}}M_{\mathbf{m}'}$. Thus, if u is picked

1	3	5	2	6	3	3	5	1	2	3
5	1	3	4	6	2	4	5	1	2	4

Figure 1.3: A generalized queue, mapping $13526335123 \in \Omega_{(2,2,4,0,2,1)}$ to $51346245124 \in \Omega_{(2,2,\mathbf{2},\mathbf{2},2,1)}$.

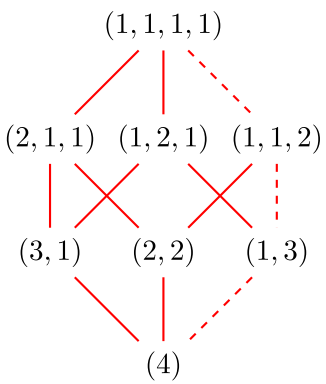

Figure 1.4: A node labelled $\mathbf{m}$ represents TASEP on $\Omega_{\mathbf{m}}$. For each edge going up from $\mathbf{m}'$ to $\mathbf{m}$ we have operators $D_{\mathbf{m},\mathbf{m}'}$ (given by Proposition 1.1.2) and $U_{\mathbf{m}',\mathbf{m}}$ given by (generalized) queues satisfying intertwining relations. Dashed edges correspond to ordinary queues.

from $\zeta_{\mathbf{m}'}$ and if q is the queue given by a uniformly random $m_1 + \ldots + m_k$-subset of $[n]$, then $q(u)$ is distributed according to $\zeta_{\mathbf{m}}$.

An example of a generalized queue is given in Figure 1.4. The generalized queues provide us with a system of pairs of intertwining matrices, which we have illustrated in Figure 1.4 for $n = 4$.

1.5 The Matrix Ansatz

As mentioned earlier, the proof of [5] is quite indirect compared to how Theorem 1.3.1 is described here. It is based on the *Matrix Ansatz*, of which we will give a brief account here. Like multiline queues, the Matrix Ansatz is a heavily used tool for proving statements about the TASEP.

Let $\mathbb{R}^\infty = \mathrm{span}(e_0, e_1, \ldots)$ be an countably infinite dimensional vector space with basis $e_0, e_1, \ldots$, and define linear maps δ, ε and A by letting $\delta e_i = e_{i-1}$ for $i > 0$, $\delta e_0 = 0$, $\varepsilon e_i = e_{i+1}$ for $i \geq 0$, $Ae_i = 0$ for $i > 0$ and $Ae_0 = e_0$. Further let $\mathbf{1}$ denote the identity map on this space, and let $D = \mathbf{1} + \delta$, $E = \mathbf{1} + \varepsilon$. These maps satisfy
$DE = D + E$,
$DA = A$,
$AE = A$.

Furthermore, the trace of any positive power of A equals 1. Thus, from the recursion relations (1.2.1), we see that for any word $u = u_1 u_2 \ldots u_n$ with letters in $\{1, 2, 3\}$ with at least one letter 2, we have

$$\mathrm{tr}(X^{(3)}_{u_1} \ldots X^{(3)}_{u_n}) = [u], \tag{1.5.1}$$

if we let $X^{(3)}_1 = E$, $X^{(3)}_2 = A$, and $X^{(3)}_3 = D$. This argument goes back to [11], where the Matrix Ansatz was first introduced (for a slightly different process).

Let $\mathbf{E}$ be the set of linear maps on $\mathbb{R}^\infty$, and let $\mathbf{R} = \bigotimes^\bullet \mathbf{E}$ be the corresponding algebra of tensors.

There are two notions of multiplication on $\mathbf{R}$. The first one, $\cdot$, is defined on each graded piece $\mathbf{R}_d$, and is induced by composition of maps. For example $(1 \otimes D \otimes E) \cdot (E \otimes E \otimes A) = E \otimes DE \otimes EA = E \otimes D \otimes EA + E \otimes E \otimes EA$ in $\mathbf{R}_3$. The second one is simply the tensor product: $(1 \otimes D) \otimes E = 1 \otimes D \otimes E$ gives the product of an element in $\mathbf{R}_2$ and an element in $\mathbf{R}_1$, resulting in an element in $\mathbf{R}_3$.

The method (1.5.1) for computing amplitudes has been extended to arbitrary words in [13].

For each r, we define $X^{(r)}_i \in \mathbf{R}$ for $1 \leq i \leq r$ by letting

$$X^{(r)}_i = \sum_{j=1}^{j-1} a^{(r)}_{ij} \otimes X^{(r-1)}_j,$$

for a certain explicitly defined matrix $a^{(r)}_{ij}$ with entries in $\mathbf{R}$.

The result of [13] is then that for any word $u_1 \ldots u_n$ with $1 \leq u_i \leq r$, we have[2]

$$[u_1 \ldots u_n] = \mathrm{tr}\left(X^{(r)}_{u_1} \cdot \ldots \cdot X^{(r)}_{u_n}\right).$$

[2] In [13], smaller particles jump to the right rather than to the left so if we were more explicit we would have to reverse u to get conventions right.

1.6 The jump operator

If w is a word and j_0 is a position in w, we define a new word $w^{j_0\to}$ as follows. Let r be the largest class in w. First define a sequence $j_1, j_2, \dots$ by letting j_{k+1} be the first position to the right (cyclically) of j_k such that $w_{j_k} < r$ and $w_{j_{k+1}} > w_{j_k}$. If there is no such j_{k+1}, then we stop and consider the thus obtained sequence $j_0, \dots, j_k$. Now $w^{j_0\to}$ is obtained from w by putting the letter w_{j_t} at j_{t+1} instead, for $t = k-1, k-2, \dots, 0$. The letter w_{j_0} is replaced by a new letter r (the letter $w_{j_k} = r-1$ disappears).

This is simpler than it sounds. For example,

$$\underset{\cdot}{1}\underset{\cdot}{3}21532\underset{\cdot}{4}5415^{1\to} = \mathbf{51}21532\mathbf{3}5415.$$

The following recursion is the key lemma in Paper B.

Lemma 1.6.1. *([4]) Suppose u is a word of type $(m_1, \dots, m_{r-2}, m_{r-1} - 1, m_r + 1)$ such that $u_1 = r$.*

$$[u] = \sum_{v \in \Omega_{\mathbf{m}} : v^{1\to} = u} [v]$$

This is proved in Paper B. Here's a proof synopsis: look at a multiline queue whose bottom row starts with r and ask: what happens if we add a box to this position? The answer is that if the bottom row was u it will become $u^{1\to}$.

A (very) special case of Lemma 1.6.1 appears in Paper A. We state it explicitly for later reference.

Proposition 1.6.2. *([1]) Suppose the word u is of the form $u = r(r-1)u'$, where all letters in u' are at most r. Then*

$$[r(r-1)u'] = [(r-1)(r-1)u'].$$

Since the proof of Lemma 1.6.1 focuses on the last queue of a multi-line queue rather than the entire queue, it seems possible to generalize the jump operator to one modifying only classes in some range $\{i, i+1, \dots, j\}$ using the generalized queues described above. Since we will not have use for such relations we omit the details.

1.7 Process of the last row

Recall from Theorem 1.3.1 that if u is a random word picked from the distribution $\zeta_{\mathbf{m}'}$, and S is a random subset of $[n]$ of size $n-m_r$, then $q(u)$ is a random word picked from $\zeta_{\mathbf{m}}$, where q is the queue associated to S. This is true even in the somewhat degenerate case when $m_{r-1}=0$ (both proofs of Theorem 1.3.1 that we have mentioned generalize to this setting). In this case we may rename all letters r in v to $r-1$ and thus obtain again a word $\tau_S(u)$ in $\Omega_{\mathbf{m}'}$, which will be distributed according to $\zeta_{\mathbf{m}'}$. The conclusion is that the chain on $\Omega_{\mathbf{m}'}$ obtained by at each step applying τ_S where S is a uniformly random $(n-m_r)$-subset of $[n]$, is again distributed according to $\zeta_{\mathbf{m}'}$.

1.8 Sorted words

Lam conjectured an interesting formula for the amplitude of the sorted permutation of length n. The formula is extended to arbitrary words in Paper A.

Theorem 1.8.1. *([1]) Let $\overrightarrow{u}$ be the sorted word of type $\mathbf{m}$. Then*

$$[\overrightarrow{u}] = \prod_{i=1}^{r} \binom{n-m_i}{m_1+\ldots+m_{i-1}}.$$

For permutations, this takes the form $[12\ldots n] = \prod_{i=1}^{n}\binom{n-1}{i}$, which happens to equal the partition function $Z_{(1,\ldots,1)}$ for permutations of size one smaller, $n-1$. Considering the general formula, this appears to be merely a coincidence.

Lam also conjectured that among all permutations of any length n, the amplitude of the sorted word (and its cyclic shifts) should be the largest. It seems reasonable this should hold for arbitrary type.

Conjecture 1.8.2. *([15]) Let $u \in \Omega_{\mathbf{m}}$ be any word, and $\overrightarrow{u}$ be the sorted word. Then*

$$[u] \leq [\overrightarrow{u}].$$

(Equivalently, $\zeta(u) \leq \zeta(\overrightarrow{u})$.)

In fact, for $r \geq 3$, it seems that the maximum is uniquely attained for the sorted permutation and its cyclic shifts. We will come back to Conjecture

1.8.2. Although it might not be a very important conjecture in itself, I think of it as a good guide for understanding ζ better.

1.9 The k-TASEP

We have already defined the sorting operators σ_i in Section 1.1. A natural extension of this definition is to define an operator σ_S, for any proper subset $S \subset [n]$, as the product of all σ_i for $i \in S$, where we take σ_{i-1} before σ_i when both $i-1$ and i belong to S. Clearly, σ_i and σ_j commute when i and j are further apart, so this is well-defined.

Fix any $k < n$. The k-TASEP is the chain on all words of length n where at each step we apply a random σ_S, picking a uniformly random subset $S \subset [n]$ of size k. The 1-TASEP is clearly the TASEP we have already considered.

The following surprising theorem was proved in [18].[3]

Theorem 1.9.1. *([18]) The k-TASEP has the same stationary distribution for all $k = 1, 2, \ldots, (n-1)$.*

Note that for small k, the transition matrix of the k-TASEP is similar but not identical to the kth power of the transition matrix of the 1-TASEP. Of course, the kth power of the transition matrix has the same stationary distribution for all k, but this fact is not used in the two known proofs of this theorem ([18] and [2]).

Recall the map τ_S of Section 1.7. Here's a mystery:

Proposition 1.9.2. *Suppose u is a word of type $\mathbf{m}$ where $m_r = 1$ (i.e. there is only one particle of class r). Then, $\tau_S(u)$ and $\sigma_S(u)$ differ only by a rotation by one step, for every subset S of $[n]$ of size $n-1$.*

Proof. When $|S| = n-1$, τ_S is similar to the jump operator $w^{j\to}$ (except the jump operator changes the type of the word whereas τ_S does not) where $\{j\}$ is the complement of S in $[n]$. It is easy to see that the 'jumping' happening in τ_S is simulated by the 'pushing' in σ_S. □

[3] It is phrased slightly differently in that paper. Let A_k be the transition matrix of the k-TASEP. In [18] it is shown that the unique positive eigenvector with sum 1 of $\sum_{k=0}^{n-1} \binom{n}{k} p^k (1-p)^{n-k} A_k$ is independent of $p \in (0,1)$. This is clearly equivalent to the statement here.

This seems to be a coincidence, since for smaller $|S|$, τ_S and σ_S are essentially different functions. Nevertheless we have showed that the operators τ_S for $|S| = n-1$ are essentially just σ_S, that is, built up from much simpler operators σ_i which already generate the same stationary distribution as τ_S.

It would be very interesting to see if this could be extended to the type-changing operators $w^{j\to}$ of Proposition 1.1.2 (which of course give a Markov chain with stationary distribution ζ). That is, could these operators be expressed as compositions of some simpler (type-changing) operators $\tilde{\sigma}_i$ that generate the stationary distribution ζ?

One amusing consequence of Theorem 1.9.1 is the following. Let A_2 be the transition matrix of the 2-TASEP, and $\tilde{A}_2$ that of the 'dual' process, where we sort the operators the other way (taking σ_i before σ_{i-1}). It follows from Theorem 1.9.1 and Proposition 1.1.3 that $\tilde{A}_2$ also has stationary distribution ζ. Of course, A_2 and $\tilde{A}_2$ have significantly many more non-zero transitions than A_1. Now consider the matrix $A_2 - \tilde{A}_2$. It also has ζ as an eigenvector[4], and it has few non-zero transitions. Yet it is significantly different from A_1. The fact that $A_2 - \tilde{A}_2$ has negative rates makes it difficult to interpret it as a transition matrix.

1.10 Weyl groups

In this section, we will assume some background knowledge of Weyl groups.

Fix some affine Weyl group $\hat{W}$ with root system Φ, and let W the corresponding finite Weyl group. We assume that Φ lies in $\mathbb{R}^d$, which is equipped with the standard inner product. The arrangement corresponding to $\hat{W}$, whose hyperplanes are $\mathcal{H}_{\varphi,k} = \{x \in \mathbb{R}^d : (x, \alpha) \in k\}$ for integers k and positive roots φ, divides space into regions called *alcoves*. See Figure 1.5. We choose some simple system $\Delta = \{\alpha_i\}_{i\in I}$ of roots and denote the simple reflections by s_i, $i \in I$. Further let ϑ be the longest root with respect to this choice.

Lam [15] defined a random walk $(X_0, X_1, \ldots)$ on the set of alcoves (or, equivalently, on $\hat{W}$) as follows. Let X_0 be the fundamental alcove. For $i \geq 1$, let X_i be a uniformly chosen neighbour of X_{i-1} which is farther away from X_0 than X_{i-1} (with respect to the length function $\ell = \ell_{\hat{W}}$ of $\hat{W}$). Another way of saying this is that at each step, the walk crosses an adjacent hyperplane that has not already been crossed. Let $v(X_i)$ be the unit vector

[4] I don't know if it has non-trivial multiplicity.

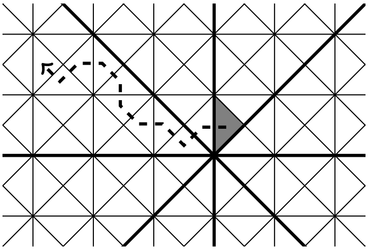

Figure 1.5: The affine Coxeter arrangement $\tilde{B}_2$. The fundamental alcove is shaded and the Weyl chamber are delineated by heavier lines. The dashed path shows one possible evolution of $(X_0, X_1, \ldots)$. This walk is stuck in the upper of the two leftmost chambers.

pointing from the center of X_0 to the center of X_i. This means that any outcome of the process X corresponds to an infinite reduced word of $\hat{W}$.

Next, we define a random walk $(Y_0, Y_1, \ldots)$ on the finite group W. For $w \in W$, there is a transition $w \to ws_i$ with rate 1 for each $i \in I$ such that $\ell(ws_i) < \ell(w)$. Finally there is a transition $w \to r_\vartheta(w)$ with rate 1 whenever $\ell(r_\vartheta w) > \ell(w)$. Here r_ϑ denotes reflection in the longest root ϑ.

The connection between X and Y is given by the following theorem.

Theorem 1.10.1. *([15]) There exists vector Ψ such that the limit of $v(X_i)$ as $i \to \infty$ almost surely equals one of the images $w\Psi, w \in W$ of Ψ under the finite Weyl group W. Furthermore, the probability of equalling $w\Psi$ is $\zeta(w)$, where $\zeta(w)$ is the stationary distribution of Y.*

We have been intentionally vague about the labelling of the alcoves and chambers by $\hat{W}$ and W. For details, see [15].

Type A

For W of type A, it's easy to see that Y is nothing but the TASEP on permutations of length n. Now we can see Figure 1.1 in new light: we are simply going down randomly in the weak order of $\mathbb{S}_n$, occasionally going up with r_ϑ (for type A, r_ϑ corresponds to the transposition $(1 \quad n)$ in the standard permutation representation in which s_i corresponds to $(i \quad i+1)$). In this case, Lam conjectured explicit coordinates for the vector Ψ in Theorem 1.10.1. This conjecture was proved in [8], as we will come back to in Section 1.11.

Other types

For W of type other than A, the chain Y does not appear to have the same nice properties as it does for type A. For example, the stationary distribution can not be scaled to integers with minimum value 1. Lam suggested the following weighted variant $\tilde{Y}$, which indeed seems to have this property for types B, C, and D.

Express the longest root ϑ with respect to Δ as $\vartheta = \sum_{i \in I} a_i \alpha_i$. Then define $\tilde{Y}$ like Y, making the following modification. In the definition of Y, let the transitions $w \to ws_i$ occur with rate a_i and $w \to r_\vartheta(w)$ with rate 1 (as before).

The connection to reduced words is now lost, since Theorem 1.10.1 is a statement about Y and not about $\tilde{Y}$. Nevertheless $\tilde{Y}$ appears to be a nice generalization of the TASEP. In Paper C, this chain is studied in particular for type C.

As we have seen in Section 1.4, to study the type A chain, it was useful to consider the extended chain on arbitrary words, which does not have an obvious counterpart in the type A Weyl group consisting of permutations. The right generalization (generalizing to types other than A) turns out to be to think of words as left cosets of parabolic subgroups (i.e. subgroups of W generated by a subset of simple generators); for example, the word 1332 corresponds to the left coset $\{\mathrm{id}, (34)\} \cdot 1342$. In Paper C we prove that the discussion in Section 1.4 generalizes to $\tilde{Y}$ for any Coxeter type. This does not use the particular choice of weights for $\tilde{Y}$. Then we restrict attention to type C, where the corresponding rates a_i find a combinatorial interpretation. It seems that the analysis in type B and D are similar, but type C is simplest so we describe only that case. Now we can look at Figure 1.4 again: we have simply drawn the poset of parabolic subgroup of the type A Weyl group under inclusion. In Paper C we show that all the D operators in the corresponding diagram generalize to any Weyl group. Moreover we construct some corresponding U operators for groups of type C. They are similar to the queues in type A, yet interestingly different.

Note that for the type A chain, the product of the partition functions corresponding to maximal parabolic (proper) subgroups equals the partition function for the trivial group. In Paper C we conjecture that this holds for the type C chain and give formulas for these numbers. Though we have not examined the type B and D cases in detail, they seem to be simple variations of the type C case.

The weights

For the root systems A, B, C, and D, the coefficients a_i are all 1 or 2, and, at least in case C (see Paper C), they are shown to have a direct combinatorial meaning.

For F_4, for example, the coefficients are not so simple (they are given by $(a_1 \ldots, a_4) = (2, 2, 4, 2)$) and the stationary measure ζ of $\tilde{Y}$ on this group does not seem to have the desired integrality property.[5]

1.11 Correlations and independences

In [8], the vector Ψ of Theorem 1.10.1 is computed for Weyl groups of type A. This is done by computing the 2-*point correlation functions* of the TASEP. The relation is roughly that each jump $\ldots ji \ldots \to \ldots ij \ldots$ in the chain Y of Section 1.10 corresponds to movement of X in a direction which depends only on i and j. We make the following general definition, which contains the 2-point correlation functions as the case $k = 2$.

Definition 1.11.1. *Consider the homogenous TASEP on permutations of length n, i.e. of type $\mathbf{m} = (1^n)$. For a given word u, denote by E_u the probability that a TASEP distributed word starts with the word u.*

Of course, if we know say all E_u's for words of length k then it is easy to compute the probability that a TASEP distributed word of *any* type starts with any given word of length k, by using Proposition 1.1.2.

In [8], the case of words of length at most 3 is studied in detail. It is shown that for words $u = ij$ of length 2,

$$E_{ij} = \begin{cases} \frac{1}{n^2} & i+1 < j \\ \frac{1}{n^2} + \frac{i(n-i)}{n^2(n-1)} & i+1 = j \\ \frac{2(i-j)}{n^2(n-1)} & i > j \end{cases}.$$

Further, formulas are found[6] for E_{ijk} for all i, j, k. The proofs go by close study of multiline queues; clearly, computing E_u can be phrased as counting the number of multiline queues satisfying certain requirements.

[5]If my computation is correct!

[6]The formulas for the cases $u = ijk$ with $i < j < k$ appears as conjectures in [8]. However they follow from the special case of Conjecture 1.11.3 proved in the Section 1.12 together with the fact that the conjectured formulas together with the proved formulas for E_{ijk} add up to E_{ij} when one fixes i, j and lets k run over all possible values.

It turns out that for a fixed ordering of $i, i-1, j, j-1, k, k-1$, E_{ijk} is of the form

$$\frac{poly(i,j,k,n)}{n^3(n-1)^2(n-2)},$$

where $poly(i,j,k,n)$ is an integer-valued polynomial in i,j,k,n. This appears to be the case in general, so let us state it more explicitly.

Conjecture 1.11.2. *For any $u = u_1 \dots u_k$, E_u equals some integer t_u divided by $n^k(n-1)^{k-1} \dots (n-k+1)$. Moreover t_u is a piecewise polynomial function in $u_1, \dots, u_k$ and n.*

The following conjecture appears in [8].

Conjecture 1.11.3. *([8]) Suppose u, v are words satisfying* $\max(u)+1 < \min(v)$. *Then $E_{uv} = E_u E_v$.*

It follows from the main result of Paper B that this holds for words u, v with $|u|+|v| = n$, with the weaker restriction $\max(u) < \min(v)$ (clearly, we cannot have $\max(u)+1 < \min(v)$ if uv is a permutation of $[n]$ and $|u|+|v| = n$). The result actually precedes the conjecture. Let us state it explicitly for later reference.[7]

Theorem 1.11.4. *([4]) Suppose u, v are words such that each letter in v is larger than each letter in u. Then*

$$[uv] = [\tilde{u}v] \cdot [u\tilde{v}],$$

where $\tilde{u}v$ denotes the word obtained from uv by merging the classes represented in u to one class, and $u\tilde{v}$ that obtained by merging all classes represented in v to one class.

As an example, for $u = 121$ and $v = 3534$, the theorem asserts that $[1213534] = 70 = 14 \cdot 5 = [1112423] \cdot [1213333]$. Using Proposition 1.1.1, it is easy to see that Theorem 1.11.4 implies the formula in Conjecture 1.11.3 for $|u|+|v| = n$, even with the weaker restriction $\max(u) < \min(v)$.

In Section 1.12, we will prove Conjecture 1.11.3 in the case of words v of length 1. This contains the most interesting consequence of the conjecture, namely that for a 'strongly increasing' word u, satisfying $u_1 < u_2 - 1 <$

[7] The version here is slightly more explicit than that in Paper B; using Proposition 1.6.2 it is easy to see that they are equivalent.

$u_3 - 2 < \cdots < u_k - (k-1)$, we have $E_u = \frac{1}{n^k}$. This was already known for $k = 2$ (and, trivially, for $k = 1$).

In general, the formulas for the E_u become less and less tractable as the length of u increases. There are, however, two general cases that appear to have elegant answers. First, that of increasing words $u_1 < u_2 < \cdots < u_k$. Here the pieces of the piecewise polynomial function t_u of Conjecture 1.11.2 appear to be indexed by subsets $S \subseteq [k-1]$ by letting $i \in S$ if $u_i = u_{i+1} - 1$. Assuming Conjecture 1.11.3, it suffices to describe the case $S = [k-1]$, that is, to describe the function $E_{i(i+1)(i+2)\ldots(i+k-1)}$. The first two interesting cases are given (see [8]) by

$$E_{ij} = \frac{1}{n^2} + \frac{i(n-i)}{n^2(n-1)},$$

and

$$E_{ijk} = \frac{1}{n^3} + \frac{i(n-i)}{n^3(n-1)} + \frac{j(n-j)}{n^3(n-1)} + \frac{i(n-i)j(n-j)}{n^2(n-1)^2(n-2)},$$

where we let $j = i + 1$ and $k = j + 1$ [8].

Second, when u is a decreasing word, E_u is given by a Vandermonde determinant. This is proved in Paper D.

Theorem 1.11.5. *([3]) For a decreasing word $u = u_1 \ldots u_k$, we have*

$$E_u = \frac{k! \prod_{i<j}(u_i - u_j)}{n^k(n-1)^{k-1}\ldots(n-k+1)}.$$

1.12 Proof of Conjecture 1.11.3 for $|v| = 1$

It will be convenient to have special notation for certain sums of amplitudes. If u, v, w are arbitrary words, we define $[u\,\underline{w}\,v]$ to be the sum of $[uw'v]$ for all permutations w' of w. Thus this sum does not change if permute w itself (of course we could specify that w should be written in increasing order, but it will be convenient not to do so). So, for example, $[4\,\underline{131}\,28] = [411328] + [413128] + [431128]$. We also sometimes insert delimiters $|$ into words for readability.

First, we rephrase Conjecture 1.11.3 in terms of amplitudes. It is easy to prove that the following conjecture is stronger than Conjecture 1.11.3, using Proposition 1.1.2.

Conjecture 1.12.1. *Let w be any word, and u, v such that there is exactly one class represented in w which is greater than all classes represented in u and smaller than all classes represented in v. Further, assume that* $\min(u) - 1 \leq \min(w)$ *and* $\max(w) \leq \max(v) + 1$. *Then*

$$[v|\underbracket{w}|u] = [v|\underbracket{\tilde{wu}}] \cdot [\underbracket{\tilde{vw}}|u],$$

where $\tilde{wu}$ denotes the result of merging all classes strictly greater than all classes represented in v into one class, and all classes strictly smaller than all classes represented in v to one class. (Similarly, $\tilde{vw}$ is wv merged as much as possible without merging classes between the smallest and largest class represented in u.)

Proof when $|v| = 1$

When v has length 1, it consists of a single letter, which we assume to be 2. Then w (up to permutation) can be written as a concatentation $s1^a2^b3^c$ for some word s all of whose letters are strictly smaller than 1, and integers $a > 0$, $b \geq 0$, $c \geq 0$. Allowing for words whose smallest letter is not necessarily 1 will simplify our notation. So we want to prove that

$$[2|\underbracket{s1^a2^b3^c}|u] = [2|\underbracket{1^{|s|+|u|+a}2^b3^c}] \cdot [\underbracket{s1^{a+b+c+1}}|u].$$

To do this, we will successively apply recursions to the left hand side. When this is done, we will obtain a numerical factor (coming from the coefficients in the recursions) times $[\underbracket{s1^{a+b+c+1}}|u]$. We then compute both the numerical factor and $[2|\underbracket{1^{|s|+|u|+a}2^b3^c}]$ and observe that they are equal.

First we modify the word so that we have a letter 3 at the leftmost site. This will then allow us to use Lemma 1.6.1. For integers x, we will use the notation $x^+ = x + 1$ and $x^- = x - 1$.

Note that

$$[2|\underbracket{s1^a2^b3^c}|u] + [3|\underbracket{s1^a2^{b^+}3^{c^-}}|u] = \binom{n}{c}[2|\underbracket{s1^a2^{b+c}}|u].$$

Here we have used Proposition 1.1.2 to merge classes 2 and 3, and Proposition 1.1.1. The factor $\binom{n}{c}$ is $Z_{\mathbf{m}}/Z_{\mathbf{m}'}$, where $\mathbf{m}$ is the type of the words on the left and $\mathbf{m}'$ is the type of the word on the right.

Similarly, by merging classes 1 and 2,

$$[3|\underbracket{s1^a2^{b^+}3^{c^-}}|u] = \binom{n}{b+c+1}[2|\underbracket{s1^{a+b+1}2^{c^-}}|u].$$

It remains to investigate expressions on the form $[2|\underbracket{s1^\alpha 2^\beta}|u]$ with $\alpha > 0$ and $\beta \geq 0$.

Lemma 1.12.2. *For any $\alpha > 0, \beta \geq 0$ such that $2s1^\alpha 2^\beta u$ has length n,*

$$[2|\underbracket{s1^\alpha 2^\beta}|u] = \binom{n-1}{\beta}[\underbracket{s1^{\alpha+\beta+1}}|u].$$

Proof. We first prove the formula

$$\binom{n}{\beta}[\underbracket{s1^{\alpha+\beta+1}}|u] = [\underbracket{s1^{\alpha^+}2^\beta}|u] = [2|\underbracket{s1^\alpha 2^\beta}|u] + [2|\underbracket{s1^{\alpha^+}2^{\beta^-}}|u].$$

The first equality is clear, so we focus on the second one. By applying Lemma 1.6.1 to each term in the sum denoted by $[2|\underbracket{s1^\alpha 2^\beta}|u]$ we get the sum over $[w|u]$ for all words w obtained by putting the letters in $s1^{\alpha^+}$ anywhere, and the letters in 2^β anywhere except for the first position. If we add to this sum the sum $[2|\underbracket{s1^{\alpha^+}2^{\beta^-}}|u]$ we get simply $[\underbracket{s1^{\alpha^+}2^{\beta^-}}|u]$, which is the second equality.

Now, let $f(\alpha, \beta) = [2|\underbracket{s1^\alpha 2^\beta}|u]$ and $A = [\underbracket{s1^{\alpha+\beta+1}}|u]$. Again, from Lemma 1.6.1 we see that $f(\alpha+\beta, 0) = A$. Thus $f(\alpha, \beta) = \binom{n}{\beta}A - f(\alpha+1, \beta-1) = \binom{n}{\beta}A - \binom{n}{\beta-1}A + f(\alpha+2, \beta-2) = \cdots = \left(\binom{n}{\beta} - \binom{n}{\beta-1} + \cdots \pm \binom{n}{0}\right) A = \binom{n-1}{\beta}A$, as claimed.

□

Therefore,

$$[2|\underbracket{s1^a2^b3^c}|u] = \binom{n}{c}[2|\underbracket{s1^a2^{b+c}}|u] - \binom{n}{b+c+1}[2|\underbracket{s1^{a+b+1}2^{c^-}}|u] =$$

$$\left(\binom{n}{c}\binom{n-1}{b+c} - \binom{n}{b+c+1}\binom{n-1}{c-1}\right)[\underbracket{s1^{a+b+c+1}}|u].$$

There we have the numerical factor! We now compute $[2|\underbracket{1^{|s|+|u|+a}2^b3^c}]$. But this is easy, since it's simply $\binom{n}{b+c+1}\binom{n}{c}$ times the probability that a

$\zeta_{(a,b+1,c)}$ distributed word starts with 2 (which is $\frac{b+1}{n}$). So we need to show that

$$\binom{n}{c}\binom{n-1}{b+c} - \binom{n}{b+c+1}\binom{n-1}{c-1} = \frac{b+1}{n}\binom{n}{b+c+1}\binom{n}{c},$$

which is an easy exercise.

Remark 1.12.3. *In the proof above, we repeatedly applied Lemma 1.6.1 (and Propostions 1.1.1 and 1.1.2) to* $[v|\underbracket{w}|u]$ *to eventually arrive at some multiple of* $[\underbracket{v\tilde{w}}|u]$. *In general, for* $|v| > 1$, *it seems that the same approach does indeed yield a multiple of* $[\underbracket{v\tilde{w}}|u]$, *though the computation becomes progressively more involved. One could also apply the dual version of Lemma 1.6.1 to arrive at a multiple of* $[v|\underbracket{\tilde{w}u}]$, *or mix these two approaches. Of course, a proof not relying on this kind of computation would be more desirable.*

1.13 Inhomogenous versions

So far we have given the transitions $w = \dots ji \dots \to \sigma_k(w) = \dots ij \dots$ $(i < j)$ the same rate 1. An interesting weighted variant was suggested in [16], where instead the transition above occurs at rate x_i, where $x_1, \dots, x_n$ are indeterminates.[8] We will refer to this as the *(singly) inhomogenous* chain.

Knutson and Lam also suggested giving weight $x_i + y_j$ to the general transition above. We will refer to this as the *doubly inhomogenous* chain.

In this section, we will go through results already mentioned in previous sections and explain to what extent they generalize to the these two inhomogenous cases. Since the doubly inhomogenous case contains the singly inhomogenous case, we omit the singly inhomogenous version of a statement if there is a doubly inhomogenous version.

Placing this section last in the summary helps the exposition, but for several properties of the TASEP, the homogenous case was found only after specializing the inhomogenous case.

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[8] In that paper, attention is restricted to permutations w, and the moves in the chain are described in terms of w^{-1}. The version for arbitrary words appears in [9].

The singly inhomogenous case

We must first generalize the definition of amplitudes. The most commonly used one, which is the one we'll use here[9], is to scale the stationary distribution so that the amplitudes are polynomials in the x_i's without a common factor.

In [8], a monomial is assigned to each multiline queue such that $[u]$ conjecturally is the sum of the weights of all multiline queues whose bottom row is labelled u. The conjecture is proved in [6] and [17]. The first proof generalizes the proof of Theorem 1.3.1 by [5] and the second proof generalizes the original proof in [14]. We summarize this in the following theorem.

Theorem 1.13.1. *([6], [17]) There is a function w mapping multi-line queues of type $\mathbf{m}$ to monomials in $x_1, \ldots, x_{r-1}$ such that for words $u \in \Omega_{\mathbf{m}}$, the singly inhomogenous amplitude $[u]$ equals the sum of $w(\mathbf{q})$ over all multi-line queues $\mathbf{q}$ whose bottom row is u.*

We will not give the definition of w in this summary. It can be found in Paper B, where we use it to prove an inhomogenous generalization of Lemma 1.6.1. Similarly, we prove an inhomogenous version of Theorem 1.11.4 there.

The generalized Matrix Ansatz proof of Theorem 1.13.1 in [6] proves a generalized version of the intertwining relation (1.4.2) mentioned in Section 1.4. It is interesting to note that the matrix D corresponding to merging the two highest classes of Section 1.4 still satisfies $DM_{\mathbf{m}} = M_{\mathbf{m}'}D$ with respect to the inhomogenous transition matrices. Now, however, merging the two highest classes becomes essential; the matrix D corresponding to merging any other pair of adjacent classes will *not* satisfy this intertwining property. It also appears difficult to generalize the generalized queues of Section 1.4 to the inhomogenous case (as should be expected, by the discussion in Section 1.4). The reason that the highest pair of classes is special is that the variable x_{r-1} only occurs in transitions involving classes $r-1$ and r (and these transitions become loops when we merge the classes).

Here are the amplitudes for words in $\Omega_{(1,1,1,1)}$ in the singly inhomogenous case.

$[1234] = (x_1^2 + x_1x_2 + x_2^2)(x_1x_2 + x_1x_3 + x_2x_3)$
$[1243] = x_1x_2(x_1^2 + x_1x_2 + x_2^2)$
$[1324] = x_1^2x_2(x_1 + x_2 + x_3)$

[9] Except in Paper B!

$[1342] = x_1^2(x_1x_2 + x_1x_3 + x_2x_3)$
$[1423] = x_1(x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_1^2x_3 + x_1x_2x_3)$
$[1432] = x_1^3x_2$

The partition function $Z_{(1,1,1,1)} = [1234] + [1243] + \ldots + [4321]$ is given by $4(3x_1^2 + 2x_1x_2 + x_2^2)(2x_1x_2 + x_1x_3 + x_2x_3)$.

A general formula for $Z_{\mathbf{m}}$ (generalizing Proposition 1.1.1) is given in Papers A and B. To state it, let $h_k(y_1, \ldots, y_m)$ be the complete symmetric polynomial of degree k in m variables.

Theorem 1.13.2. *([1], [4]) The sum of the amplitudes of all words in $\Omega_{\mathbf{m}}$ is*

$$n\prod_{i=1}^{r-1}(x_1\ldots x_i)^{n-m_1-\ldots-m_i}h_{n-m_1-\ldots-m_i}(\underbrace{\frac{1}{x_1},\ldots,\frac{1}{x_1}}_{m_1},\ldots,\underbrace{\frac{1}{x_{i-1}},\ldots,\frac{1}{x_{i-1}}}_{m_{i-1}},\underbrace{\frac{1}{x_i},\ldots,\frac{1}{x_i}}_{m_i+1}).$$

The product formula for sorted words (see Theorem 1.8.1) generalizes as follows.

Theorem 1.13.3. *([1]) Let $\overrightarrow{u}$ be the sorted word in $\Omega_{\mathbf{m}}$. Then amplitude $[\overrightarrow{u}]$ is given by*

$$\prod_{i=1}^{r-1}(x_1\ldots x_i)^{n-m_1-\ldots-m_i}h_{n-m_1-\ldots-m_i}(\underbrace{\frac{1}{x_1},\ldots,\frac{1}{x_1}}_{m_1},\ldots,\underbrace{\frac{1}{x_{i-1}},\ldots,\frac{1}{x_{i-1}}}_{m_{i-1}},\frac{1}{x_i}).$$

Theorem 1.13.3 was conjectured[10] in [16] and proved in Paper A.

Theorem 1.9.1 on the k-TASEP is extended [11] to the inhomogenous case in Paper C. The proof method in Paper C is different from that in [18]. The proof in [18] builds on the ideas of the original proof [14] of Theorem 1.3.1. However, it does not appear to extend to the inhomogenous case.

Instead, in Paper C, we prove that the (inhomogenous) transition matrix of the k-TASEP commutes with that of the k'-TASEP for any k, k'. This proves in particular that they have the same stationary distribution.

A natural question is whether the correlations studied in Section 1.11 generalize. To take a prototypical case, we have $E_{135} = \frac{1}{5^3}$ for $n = 5$ in the homogenous case. In the inhomogenous case we have instead to consider

[10]The conjecture is formulated in terms of Schubert polynomials, but by well-known properties of these it is easy to show that that formula is equivalent to the one given here.

[11]Here the rate of a transition $w \to \sigma_S w$ is given by $\prod_{i\in S} x_{w_i}$.

$$\frac{[13524]+[13542]}{Z_5} = \frac{x_1^4 x_2 x_3 (2x_1^2 x_2^2 + \ldots)}{5(4x_1^3+\ldots)(3x_1^2x_2^2+\ldots)(2x_1x_2x_3+\ldots)},$$

where the part in the parentheses in the numerator is a certain *irreducible* polynomial in $\mathbf{x}$ which evaluates to 20 when letting $\mathbf{x}=\mathbf{1}$ and the denominator is given by Theorem 1.13.2 (and evaluates to 2500). It seems difficult to find a way to express this in a way analogous to $\frac{1}{5^3}$.

The doubly inhomogenous case

This weighting seems to retain many of the properties of the singly inhomgenous case, but there is no known generalization of multiline queues to this case, or of the Matrix Ansatz. The transition matrix $M_{(1,1,1)}$ for $x_3 = y_1 = 0$ in this case is

$$\begin{pmatrix} x_2+y_2 & x_2+y_3 & x_1+y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1+y_3 & 0 \\ 0 & 0 & 0 & x_1+y_3 & 0 & 0 \\ 0 & x_1+y_2 & 0 & x_2+y_2 & 0 & x_2+y_3 \\ 0 & 0 & x_2+y_3 & 0 & x_2+y_2 & x_1+y_2 \\ x_1+y_3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(Note that x_3 and y_1 only occur on the diagonal, so there is no loss in restricting to the case $x_3 = y_1 = 0$.) If we rescale the stationary distribution to get polynomials without a common factor we get $[123] = x_1 + x_2 + y_2 + y_3, [132] = x_1 + y_3$.

In general, for permutations of length n, the scaling that achieves polynomial amplitudes without common factors appears to be the one which assigns $[\overleftarrow{u}] = \prod_{i+1<j}(x_i + y_j)^{j-i-1}$ for the reverse permutation $[\overleftarrow{u}]$. This way we get the amplitudes for the singly inhomogenous case from the doubly inhomogenous case simply by letting $\mathbf{y} = 0$.

There appears to be no product formula generalizing $Z_{\mathbf{m}}$ of Proposition 1.13.2 in the doubly inhomogenous case; the sum of all the amplitudes of all permutations is an irreducible polynomial for each length $3, 4, 5$.

However, in the case of three types, the recursions (1.2.1) do generalize, and allow us to compute the amplitudes in this case.

$$
\begin{aligned}
[31u] &= (x_1 + y_2)[3u] + (x_2 + y3)[1u], \\
[32u] &= (x_1 + y_3)[2u], \\
[21u] &= (x_1 + y_3)[2u].
\end{aligned}
\tag{1.13.1}
$$

These recursion relations can be proved in the same way as the original recursions (1.2.1). A proof can be found in the excellent survey paper [10].

The amplitude of the sorted word (a polynomial in the x_i's and y_j's) appears to factor in the same number of factors as in the singly inhomogenous case (see Theorem 1.13.3). It would be interesting to identify these factors explicitly.

Duality is elegant in the doubly inhomogenous case.

Proposition 1.13.4. *As functions of* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$, *we have*

$$[w](\mathbf{x}; \mathbf{y}) = [w'](\mathbf{y}'; \mathbf{x}'),$$

for any w *when* w' *is the word gotten from* w *by reversing values and positions, and* $\mathbf{x}'$, $\mathbf{y}'$ *from reversing* $\mathbf{x}$ *and* $\mathbf{y}$.

Since, as a function of $\mathbf{x}$ and $\mathbf{y}$, the transition matrix satisfies $M_{\mathbf{m}}(\mathbf{x} + a, \mathbf{y}) = M_{\mathbf{m}}(\mathbf{x}, \mathbf{y} + a)$, the corresponding invariance $\zeta_{\mathbf{m}}(\mathbf{x} + a, \mathbf{y}) = \zeta_{\mathbf{m}}(\mathbf{x}, \mathbf{y} + a)$ holds for the stationary distribution $\zeta_{\mathbf{m}}$ (and consequently also the amplitudes). Here, $\mathbf{x} + a$ means $(x_1 + a, \ldots, x_n + a)$.

Proposition 1.13.5. *Let* $p(\mathbf{x}, \mathbf{y}) \in \mathbb{Q}[\mathbf{x}, \mathbf{y}]$ *be a polynomial. The following are equivalent. (i) For each* a, $p(\mathbf{x} + a; \mathbf{y}) = p(\mathbf{x}; \mathbf{y} + a)$ *(ii)* p *is a polynomial in* $\mathbb{Q}[x_i + y_j]_{1 \le i, j \le n}$.

Proof. We give a proof sketch. Fix some integer $d \ge 0$. The vector space of polynomials of degree at most d described in (ii) is clearly a vector subspace of those described in (i) of degree at most d. It is easy to show that the dimension of these two spaces is the same. Since d was general, this proves the proposition. □

So all the amplitudes can be written as polynomials in $(x_i + y_j)$. Surprisingly, it seems there is always a positive expression. This in particular means (assuming the conjectural normalization of the amplitudes above) that the total number of $(x_i + y_j)$-monomials occurring in a doubly inhomogenous amplitude is the same as the number of x_i-monomials occurring in the corresponding singly inhomogenous amplitude. This gives some hope for the existence of a generation rule of these monomials in the doubly inhomogenous case which is not much more complicated than that in the singly inhomogenous case (given by Theorem 1.13.1).

So, to find such a rule, one should wonder what is the right way to express the polynomials. Here are the polynomials for type $(1,1,1,1)$.

$[1234] = \Big((x_1+y_2)(x_1+y_4)+(x_1+y_2)(x_2+y_3)+(x_2+y_3)(x_2+y_4)\Big)\Big((x_1+y_4)(x_3+y_4)+(x_2+y_3)(x_3+y_4)+(x_1+y_3)(x_2+y_3)\Big)$

$[1243] = (x_1+y_4)(x_2+y_4)\Big((x_1+y_2)(x_1+y_4)+(x_1+y_2)(x_2+y_3)+(x_2+y_3)(x_2+y_4)\Big)$

$[1324] = (x_1+y_4)(x_2+y_4)(x_1+y_3)\Big((x_1+y_2)+(x_2+y_3)+(x_3+y_4)\Big)$

$[1342] = (x_1+y_4)(x_1+y_3)\Big((x_1+y_4)(x_3+y_4)+(x_2+y_3)(x_3+y_4)+(x_1+y_3)(x_2+y_3)\Big)$

$[1423] = (x_1+y_4)\Big((x_1+y_2)(x_1+y_3)(x_2+y_3)+(x_1+y_2)(x_1+y_3)(x_3+y_4)+(x_1+y_3)(x_2+y_3)(x_2+y_4)+(x_1+y_2)(x_2+y_4)(x_3+y_4)+(x_2+y_3)(x_2+y_4)(x_3+y_4)\Big)$

$[1432] = (x_1+y_4)^2(x_1+y_3)(x_2+y_4)$

Above, we have first of all factored the polynomials as much as possible, then tried to express them in terms of (x_i+y_j) for $i<j$. In the singly inhomogenous case, there is one monomial M (equal to the amplitude of the reverse sorted word) appearing in each amplitude. If we want to generalize this property, we run into trouble; the monomial M should be generalized by $[1432]$ for $\mathbf{m}=(1,1,1,1)$, but $[1243]-[1432]$ has a factor (x_2+y_2). So the expressions cannot have both of these desired properties.

What happens to Lam's conjecture 1.8.2 in the doubly inhomogenous case? Based on evidence for $n \leq 5$ for general $\mathbf{x}$, $\mathbf{y}$ and for $n \leq 10$ for $\mathbf{y}=0$, I conjecture the following.

Conjecture 1.13.6. *Fix a word u of type $\mathbf{m}$. Let $\overrightarrow{u}$ and $\overleftarrow{u}$ be u in sorted respectively reverse sorted order Then $[\overrightarrow{u}]-[u]$ and $[u]-[\overleftarrow{u}]$ have only positive coefficients.*

Setting all the x_i's to 1 and the y_j's to 0, this recovers Lam's conjecture 1.8.2. For $\mathbf{y}=0$, it is easy to show using Theorem 1.13.1 that $[u]-[\overleftarrow{u}]$

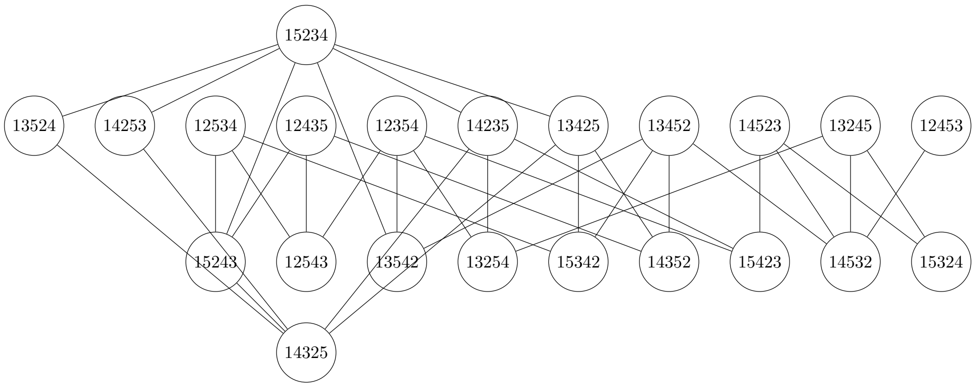

Figure 1.6: An arrow going up from a node labelled u to a node labelled v means that $[v]-[u]$ is a polynomial with positive coefficients. The reverse permutation 15432 and the sorted permutation 12345 have been omitted (they should be smallest respectively largest elements in this poset). So whatever the values of x_i and y_j, the state v is more likely than state u in the doubly inhomogenous TASEP.

have positive coefficients. If one is looking for an injective map on multiline queues to prove Conjecture 1.8.2, then it would be reasonable to look for one that preserves the weight of the queue in the singly inhomogenous case. In fact, there seem to be more domination relations between the amplitudes than those mentioned in Conjecture 1.13.6. See figure 1.13

The formula in Paper B can conjecturally be generalized to the doubly inhomogenous case.

Conjecture 1.13.7. *Let u, v be any words, u' and v' permutations of u and v respectively. Then*

$$[uv][u'v'] = [u'v][uv'],$$

where the amplitudes are taken with respect to the doubly inhomogenous chain.

Theorem 1.9.1 does not seem to extend to the doubly inhomogenous case. Indeed, if it did, then, by duality, the mirrored operator $\tilde{\sigma_S}$ which sorts the product the other way (by taking σ_{i-1} before σ_i) would give the same stationary distribution in the inhomogenous case for each k. But this is not true, as can be seen already for words of type $(1, 1, 1)$.

We finish with some remarks on possible ways to generalize Theorem 1.13.1 to the doubly inhomogenous case. A first step would be to be find an up-operator corresponding to the down-operator merging the two *lowest* classes in the singly inhomogenous case. (We would be able to do this if we were able to unmerge the highest pair of classes with an up-operator in the doubly inhomogenous case.)

Let $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mathbf{m'} = (m_1 + m_2, m_3, \ldots, m_r)$. Consider the (singly inhomogenous) transition matrices $M_{\mathbf{m}}$, $M_{\mathbf{m}|x_1=x_2}$, $M_{\mathbf{m}'}$. It is easy to see that the *same* matrix U going from $M_{\mathbf{m}'}$ to $M_{\mathbf{m}}$ in the homogenous case $\mathbf{x} = 1$ again satisfies $M_{\mathbf{m}|x_1=x_2}U = UM_{\mathbf{m}'}$.

So the difficult part is to find a matrix U satisfying

$$M_{\mathbf{m}}U = UM_{\mathbf{m}|x_1=x_2}. \tag{1.13.2}$$

We *do* have a down-operator D, which however is linear only over $\mathbb{Q}$ (and not $\mathbb{Q}[x_1, \ldots, x_{r-1}]$ as those considered previously have been). It is given by setting x_2 equal to x_1. It is then clear that

$$M_{\mathbf{m}}D = DM_{\mathbf{m}|x_1=x_2}.$$

So, by the discussion in Section 1.4 we should expect a map U as in (1.13.2) to exist. It will however be a linear map on $\mathbb{Q}[x_1, \ldots, x_{r-1}](\Omega_{\mathbf{m}}$ over $\mathbb{Q}$ rather than on $\mathbb{Q}[x_1, \ldots, x_{r-1}](\Omega_{\mathbf{m}}$ over $\mathbb{Q}[x_1, \ldots, x_{r-1}]$. It is easy to find such maps at least for $\mathbf{m} = (1, 1, 1)$.

Very inhomogenous case

One could consider giving completely independent weights x_{ij} to the transitions $\ldots ij \ldots \to \ldots ji \ldots$. However, as noted already by Lam, in this case, the amplitudes do not have positive coefficients, and hence might be very difficult to describe combinatorially. In the case $r = 3$, this case is equivalent to the doubly inhomogenous case, for which we have described a positive rule (1.13.1).

1.14 Paper D

In paper D we study the TASEP on $\Omega_{\mathbf{m}}$ with m_r going to infinity and the other m_i (and r) staying fixed. More precisely, fix a permutation π of $[n]$ and numbers $1 \leq b_1 < b_2 < \cdots < b_n \leq N$, and consider the number

$G_\pi(b_1, \ldots, b_n; N)$ of multiline queues of type $(1, 1, \ldots, 1, N - n)$ whose bottom row has the letter π_i at position b_i for $i = 1, \ldots, n$ (thus the remaining letters in the bottom row are all $n+1$). It turns out that G_π is a polynomial in the b_i's for each π. The most striking result on the G_π's is the following.

Theorem 1.14.1. *Let $\pi_0 = n(n-1)\ldots 21$ be the reverse permutation on n letters. Then*

$$G_{\pi_0}(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} \frac{1}{i!} \prod_{i<j} (x_j - x_i).$$

Curiously, this expression is very similar to the formula in Theorem 1.11.5 (which is also proven in Paper D). We show that for some π close to π_0, the polynomial G_π can be expressed as a certain operator applied to G_{π_0}.

In the limit, we get a process which can be described as follows. Choose a *continuous* multi-line queue of some type, where the circles in each row now are chosen uniformly at random from the real interval $[0, 1)$. Label the boxes as before. Then the labels in the bottom row, excluding those of highest class, are well-defined (except on a set of measure 0).

We can take a limit of the polynomial G_π to obtain the probability density $g_\pi(p_1, \ldots, p_n)$ that the ith box in the bottom row occurs at position $p_i \in [0, 1)$ and is labeled π_i.

We find some recursion relations among the g_π's for fixed n. We conjecture there are more relations, inspired by a similar system of recursion relations for a related-in-spirit Markov chain, the *Razumov-Stroganov (RS) chain.*

1.15 Contributions in this thesis

Paper A

In this paper, we prove Lam's conjectured formula for $\zeta_{(1,\ldots,1)}(12\ldots n)$ and introduce the simple version Proposition 1.6.2 of Lemma 1.6.1 to do so. The same method also proves Lam and Williams' conjectured inhomogenous generalization (stated in Theorem 1.13.3). Theorem 1.13.2 is claimed in Paper A, and proved more carefully in Paper B.

Paper B

In this paper we prove the (previously unknown) product formula 1.11.4 and the jumping lemma (Lemma 1.6.1), which is the essential ingredient in the proof.

Paper C

We study Lam's chain $\tilde{Y}$ in type C. In particular, the 'strange' weights $1, 2, \ldots, 2, 1$ for type C are given a combinatorial interpretation. We conjecture an explicit form of the partition function for type C, and describe the stationary distribution of some projections of the type C chain. We also prove an inhomogenous version of Theorem 1.9.1.

Paper D

We introduce the polynomials G_π in Section 1.14, and prove some relations among these. We also show that conjecturally, the independences in 1.11 extend to some extent to the limit TASEP. Theorem 1.11.5 is proved.

This summary

Section 1.12 contains a proof of [8, Conjecture 8.3], which implies Conjectures 8.2 and 7.4 in the same paper.

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Paper A

Stationary probability of the identity for the TASEP on a ring

Paper B

A product formula for the TASEP on a ring
(joint with Jonas Sjöstrand)

Paper C

TASEP in any Weyl group

Paper D

Continuous multiline queues and TASEP
(joint with Svante Linusson)