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Postprint

This is the accepted version of a paper presented at *2014 1st IEEE International Conference on Communications, ICC 2014, Sydney, NSW, Australia, 10 June 2014 through 14 June 2014*.

Citation for the original published paper:

Stathakis, E., Rasmussen, L. K., Skoglund, M. (2014)

Closed-form capacity formula for multi-antenna cognitive radio networks with asymmetric fading.

In: *2014 IEEE International Conference on Communications, ICC 2014* (pp. 2141-2146). IEEE Computer Society

<http://dx.doi.org/10.1109/ICC.2014.6883640>

N.B. When citing this work, cite the original published paper.

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Closed-form Capacity Formula for Multi-antenna Cognitive Radio Networks with Asymmetric Fading

Efthymios Stathakis, Lars K. Rasmussen and Mikael Skoglund

Abstract—A cognitive radio network with a multiple-input single-output secondary link and a multi-antenna primary receiver is considered. The secondary transmitter steers its transmission into the direction of its intended destination in order to maximize the received signal-to-noise ratio. Under this beam-forming strategy, the power allocation is optimized to achieve the ergodic capacity under the constraint that the long-term interference, caused at the primary receiver, will not exceed a predefined threshold. Assuming line-of-sight communication (Ricean fading) for the secondary link and Rayleigh fading for the secondary-to-primary link channel, we derive the exact closed-form expression for the ergodic capacity. Numerical results corroborate theoretical findings and illustrate the manifold impact of the system parameters into its performance.

I. INTRODUCTION

Cognitive radio (CR) networks [1] have been recognized as the new paradigm that will resolve the acknowledged spectrum inefficiency, in an effort to harvest all possible gains by reusing or moderately improving existing technological know-how through efficient spectrum utilization. Three main approaches have been suggested within the CR framework [2]; the interweave, where secondary (unlicensed) users (SU) opportunistically occupy licenced (primary) vacant bands; the underlay where both primary users (PUs) and SUs coexist, with priority on the primary Quality-of-Service (QoS), and the overlay where both user classes transmit simultaneously and actively cooperate in order to guarantee high performance.

Among the aforementioned alternatives, the underlay constitutes the most appealing; it reconciliates the concept of frequency reuse, hence is more spectrally efficient than the interweave counterpart, with reasonable operational assumptions and performance goals. Therefore it obviates the unrealistic requirements that are necessary in order to deliver the high QoS that the overlay approach promises. In particular, the secondary network has to ensure that it will not degrade the primary QoS below a predefined acceptable threshold. To achieve this the secondary transmitter utilizes available channel side information (CSI), which is exploited to properly design the transmission strategy and control the generated interference. This level of cooperation, i.e., information exchange, does not require significant changes in existing standards.

An information-theoretic study of the basic single-user CR model with spectrum sharing constraints was carried out in [3],

where the author characterized capacity for various scenarios. The Gaussian channel model with Rayleigh fading was considered in [4] where the authors provided an analytical result for the single-user link capacity, under ergodic interference constraints. The optimal CR transceiver architecture design, under various combinations of objectives and constraints, was studied in the work of [5], [6]. However, the solutions to the optimization problems considered therein, albeit analytical did not facilitate a detailed stochastic characterization of the design, except for special cases. A more general case, pertaining to the emerging femtocell architecture [7], corresponds to line-of-sight (LoS) communication [8] for the secondary link. Channel models, addressing such practical scenario, were considered in [9], [10], yet the associated analysis was carried out numerically [9] or asymptotically [10].

This theoretical gap suggests that a revisit of the basic setup, under a general channel model, is necessary prior to addressing more sophisticated designs. This work contributes a *closed-form* solution to the optimal design of the fundamental CR network with multiple antennas at the secondary transmitter SU_{Tx} and the primary receiver PU_{Rx} . For a linear beam-forming strategy, we evaluate analytically the ergodic capacity under an ergodic interference constraint, assuming LoS fading for the secondary link and non-LoS (NLoS) fading for the secondary-to-primary link. A key insight that facilitates the associated derivations follows the observation that for this class of channel models the $\mathcal{F}$ -distribution [11] is involved in the stochastic characterization. Numerical results assert the validity of theoretical derivations and offer insights into the impact of network parameters on performance.

The paper is organized as follows. The system model is described in Section II and the main results are presented in Section III. Numerical results are discussed in Section IV and, finally, Section V concludes the paper.

Notation: Bold lower (upper) case letters stand for vectors (matrices). We denote the N -length all-zero (all-one) vector as $\mathbf{0}_N$ ($\mathbf{1}_N$) and the $N \times N$ identity matrix as $\mathbf{I}_N$. The operator $\|\cdot\|$ stands for the Euclidean norm and $\{x\}^+$ is equivalent to $\max\{x, 0\}$.

II. SYSTEM MODEL

Consider the fundamental underlay CR network model, depicted in Fig. 1. In this setup, a node SU_{Tx} intends to transmit data to its designated destination SU_{Rx} , while refraining from inadvertently degrading the *long-term* primary QoS, measured through the aggregate interference temperature, at the PU_{Rx} node. The secondary link consists of a transmitter with an M -

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The research leading to these results has received funding from the European Research Council under the European Communitys Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement number 228044.

antenna array and a single antenna receiver, while the primary receiver node PU_{Rx} has an N -antenna array.

Let $\mathbf{h} = \sqrt{\frac{K}{K+1}}\mathbf{h}_{\text{LoS}} + \sqrt{\frac{1}{K+1}}\mathbf{h}_{\text{NLoS}}$ denote the secondary link channel with mean $\mathbf{h}_{\text{LoS}}$ that satisfies $\|\mathbf{h}_{\text{LoS}}\|^2 = M$, NLoS component $\mathbf{h}_{\text{NLoS}} \sim \mathcal{CN}(\mathbf{0}_M, \mathbf{I}_M)$ and Rician factor K . Moreover, let $\mathbf{G} = [\mathbf{g}_1 \dots \mathbf{g}_N]^H \in \mathcal{C}^{N \times M}$ represent the $\text{SU}_{\text{Tx}}\text{-PU}_{\text{Rx}}$ link channel matrix, whose elements are i.i.d. zero-mean Gaussian distributed with variance σ_g^2 . The secondary transmitter employs a linear beamforming strategy, i.e., it multiplies its data s with a unit-norm vector $\mathbf{w}$ such that the power P of the transmitted signal $\mathbf{x} \triangleq \mathbf{w}s$ satisfies $P = \mathbf{E}\{\|\mathbf{x}\|^2\} = \mathbf{E}\{|s|^2\}$. The node SU_{Tx} knows $\mathbf{h}$ perfectly, though side-information for the cross-link is limited to the instantaneous interference $I_i = \|\mathbf{G}\mathbf{w}\|^2$, inflicted on the PU_{Rx} node, which is measured at the primary receiver and fed back to the secondary transmitter. This information exchange allows for power control at the secondary link while at the same time the network overhead is kept low. To fully exploit the available CSI, the SU_{Tx} node designs the steering vector as $\mathbf{w} = \frac{\mathbf{h}}{\|\mathbf{h}\|}$, which corresponds to the maximum ratio transmission (MRT) strategy [12]. Based on this description, the received signal $\mathbf{y}_s$ at the secondary receiver is written as

$$\mathbf{y}_s = \|\mathbf{h}\|s + n, \quad (1)$$

where $n \sim \mathcal{CN}(0, \sigma^2)$ is the additive noise. Under the underlay CR paradigm, the secondary communication, modeled by Eq. (1), has to obey certain QoS criteria. In the absence of instantaneous cross-link channel CSI,¹ the primary QoS guarantee is realized by constraining the average interference $I_a \triangleq \mathbf{E}\{P\|\mathbf{G}\mathbf{w}\|^2\}$ to satisfy always $I_a \leq Q$.² The power allocation scheme is then selected to maximize the ergodic capacity under the particular interference power constraint (IPC). Noting that the secondary link SNR is $\frac{P\|\mathbf{h}\|^2}{\sigma^2}$, one can obtain the optimal power allocation policy P^* by solving the following optimization problem,

$$\begin{aligned} & \text{maximize} \quad \int \int \ln \left[1 + \frac{yP(y, z)}{\sigma^2} \right] f_{y,z}(y, z) dy dz \\ & \text{subject to} \quad \int \int zP(y, z) f_{y,z}(y, z) dy dz \leq Q, \end{aligned} \quad (\text{P1})$$

where the logarithm is base e and, for ease of exposition, we have defined the quantities $y \triangleq \|\mathbf{h}\|^2$ and $z \triangleq \|\mathbf{G}\mathbf{w}\|^2$. Problem (P1) is convex and can always be solved using a numerical solver, e.g., CVX [13]. However, our target is to characterize the solution analytically, since an analytical characterization can shed light on how the network parameters influence performance.

III. ANALYSIS OF THE SCHEME

A. An analytical approach to system capacity

The solution to (P1) can be retrieved, in closed-form, by solving the Karush-Kuhn-Tucker (KKT) conditions and it has

¹ If full CSI was available, then the beamformer design could be modified to take into account the extra side-information.

² It is also customary to impose a constraint on the instantaneous interference $I_i \triangleq P\|\mathbf{G}\mathbf{w}\|^2$, expressed as $I_i \leq Q$.

Fig. 1. The basic cognitive radio network setting.

already been shown to be given by $P^* = \left[\frac{1}{z\mu} - \frac{\sigma^2}{y} \right]^+$ [4]. The center of gravity in the subsequent analysis will be the stochastic behavior of the ratio $\frac{y}{z}$ and, to proceed with the characterization of the optimal solution to (P1), we need to determine its distribution function.

We can see that $\tilde{z} \triangleq \frac{z}{\sigma_g^2/2} = \frac{\mathbf{w}^H \mathbf{G}^H \mathbf{G} \mathbf{w}}{\sigma_g^2/2} = \sum_{k=1}^N \frac{|\mathbf{g}_k^H \mathbf{w}|^2}{\sigma_g^2/2} \sim \chi_2^{2N}(0)$ and also observe that $\tilde{y} \triangleq \frac{y}{\sigma_h^2/2} \sim \chi_2^{2M}(\lambda)$, where $\lambda = \frac{\|\mathbf{h}_{\text{LoS}}\|^2}{\sigma_h^2/2} = 2MK$. Upon noticing that $\tilde{y}$ and $\tilde{z}$ are independent variables and further letting $c \triangleq \frac{M\sigma_h^2}{N\sigma_g^2}$, it becomes insightful to express $\frac{y}{z}$ as $\frac{y}{z} = \frac{2M}{2N} \frac{\sigma_h^2/2}{\sigma_g^2/2} \frac{\tilde{y}}{\tilde{z}} = c\gamma$, where $\gamma \triangleq \frac{\tilde{y}/(2M)}{\tilde{z}/(2N)}$ because in this form we can recognize that $\gamma \sim \mathcal{F}_{2M, 2N}(\lambda, 0)$ [11]. To the extent of the authors' knowledge, this observation has not been made before in the relevant literature. We can exploit this insight to directly evaluate the expectations involving γ , since its distribution is known to be given in closed-form as

$$f_\gamma(x) = e^{-\frac{\lambda}{2}} \sum_{n=0}^{\infty} \frac{(\lambda/2)^n (M/N)^{M_n}}{B(M_n, N) n!} \left(\frac{1}{1 + \frac{M}{N}x} \right)^{M_n+N} x^{M_n-1},$$

where, for brevity, we have defined $M_n \triangleq M + n$. Since the inequality constraint is active at the optimum, the value for μ is obtained by solving $E \left[\frac{1}{\mu} - \frac{\sigma^2}{y/z} \right]^+ = Q$ which is equivalent to $E \left[\frac{1}{\gamma_0} - \frac{1}{\gamma} \right]^+ = \tilde{Q}$, where $\gamma_0 \triangleq \frac{\mu\sigma^2}{c}$ and $\tilde{Q} \triangleq \frac{cQ}{\sigma^2}$. The solution is described in the following Proposition.

Proposition 1. Define the scalar quantity $\tilde{\gamma}_0 \triangleq \frac{M}{N}\gamma_0$ and further let $\mathbf{I}_{\{\Omega\}}$ be the indicator function of the event $\Omega \triangleq \{(M, n) = (1, 0)\}$. Then, the optimal γ_0 (equivalently μ) that solves (P1) is found as the solution to the following equation

$$\begin{aligned} \tilde{Q} = & \frac{1}{\gamma_0} [1 - F_\gamma(\gamma_0)] - e^{-\frac{\lambda}{2}} \sum_{n=0}^{\infty} \frac{(\lambda/2)^n (M/N)}{B(M_n, N) n!} \\ & \left\{ \mathbf{I}_{\{\Omega\}} \left[\ln \left(\frac{1 + \tilde{\gamma}_0}{\tilde{\gamma}_0} \right) - \sum_{j=2}^{N+1} \frac{(1 + \tilde{\gamma}_0)^{-(j-1)}}{(j-1)} \right] \right. \\ & \left. + \mathbf{I}_{\{\Omega^c\}} \mathcal{B} \left(\frac{1}{1 + \tilde{\gamma}_0}; N+1, M_n-1 \right) \right\}. \end{aligned} \quad (2)$$

Proof: Given in Appendix B. ■

Albeit its complicated form, Eq. (2) involves elementary functions that can be easily evaluated. In order to recover the root γ_0 of this equation, one can use high resolution search, or bisection, within an interval $\mathcal{S}_{\gamma_0} \triangleq [\gamma_{\min}, \gamma_{\max}]$ assuming that $\gamma_0 \in \mathcal{S}_{\gamma_0}$. Then, a simple iterative process for obtaining the

root of this equation is to set $\gamma_{\min} = (k-1)\gamma$ and $\gamma_{\max} = k\gamma$ for some positive γ and k being the iteration index. The index is initialized with $k = 1$ and keeps updating in an incremental fashion, as $k \rightarrow k + 1$, until the *unique* root to Eq. (2) is retrieved. Note that uniqueness is guaranteed by the strict monotonicity of the right-hand-side (RHS) of Eq. (2).

Once we have recovered the optimal γ_0 , using some method, we can plug it in the objective of (P1) in order to determine the corresponding capacity. The closed-form expression for the ergodic capacity is presented in the following result.

Proposition 2. *Let $M_n, \tilde{\gamma}_0$ be defined as in Proposition 1 and let $F_\gamma(x)$ be the distribution function of an $\mathcal{F}$ -distributed random variable γ . Then, for any given γ_0 , the objective function of Problem (P1) is given in closed-form by Eq. (3) (see top of next page).*

Proof: Given in Appendix C. ■

It is important to note here that it is the particular description of the LoS channel model, which is based on the Gaussian distribution, that facilitated the analytical evaluation of all the related quantities. Although the Ricean distribution is, in fact, more appropriate for LoS communication modeling, it is not directly amenable to theoretical analysis, as suggested in [9]. Note here that, there is no fundamental distinction between the Ricean model and the Gaussian distribution with non-zero mean for LoS channel modeling, since each model can always be translated into the other with proper transformation of the involved parameters. Therefore, we can judiciously argue that a non-zero mean Gaussian model is more insightful and tractable from a mathematical viewpoint while someone can always convert it into the equivalent Ricean model in order to draw conclusions concerning the practical aspects.

B. A note on implementation

When solving Eq. (2) or evaluating Eq. (3) one has, in practice, to truncate the infinite series and consider only a finite number T of terms in the implementation. In this case truncation errors will inevitably appear thus it is necessary to quantify the impact of the chosen value of T . The following proposition asserts that the truncation error will decrease rapidly with T , when T is chosen to be larger than $\frac{e\lambda}{2}$.

Proposition 3. *Assume that the infinite summations, involved in Eqs. (2),(3) are truncated to the first T terms. Then the truncation error is bounded and decays exponentially with T , given that $T > \frac{e\lambda}{2}$.*

Proof: Given in Appendix D. ■

Recalling from the previous subsection that $\frac{\lambda}{2} = KM$ we can claim that for practical CR systems, with a moderate number of antennas (up to four) and a Ricean factor up to 20dB, the condition $T > \frac{e\lambda}{2}$ can be easily satisfied. The most salient implication of the above proposition is that if we consider a sufficiently high number of terms in the truncated series then we will approximate the exact solution with very high accuracy. This certificate provides an appealing guarantee which is especially important for numerical implementation; one does not need to resort to heuristic selection criteria, e.g.,

Fig. 2. Capacity of a CR network with $Q = 1$ and $(M, N) = (2, 1)$, for different values of the Ricean factor K .

choose T such that the (relative) distance between the resulting series for T and $T + 1$ drops below a predefined threshold. On the contrary, for any $T > MKe$, one can rely in Proposition 3 to quantify the truncation error (in order-of-magnitude).

IV. NUMERICAL RESULTS

In this section we investigate numerically some scenarios to quantify how the system parameters influence performance. In all cases the channel mean vector is fixed to $\mathbf{h}_{\text{LoS}} = (\sqrt{2}/2)(1+i)\mathbf{1}_M$ such that $\|\mathbf{h}_{\text{LoS}}\|^2 = M$. Numerical optimization was performed with CVX [13] assuming a channel block length of $2.5 \cdot 10^3$ states.³ Theoretical results were obtained using MATHEMATICA [14], where the infinite series were truncated to $T = 10^3$ terms to get the guarantees stemming from Proposition 3. We also define the threshold-to-noise ratio $\text{TNR} \triangleq \frac{Q}{\sigma^2}$ that is used for presentation of the data.

Fig. 2 illustrates the impact of TNR, for various values of K , in the capacity of the secondary link. We can see that with increasing K the capacity of the secondary link increases, since the effect of fading in the $\text{SU}_{\text{Tx}}\text{-SU}_{\text{Rx}}$ link becomes weaker. We also observe that the improvement is more salient between $K = 0\text{dB}$ and $K = 10\text{dB}$. For $K < 0\text{dB}$, where the fading effect is prominent the system experiences small rate increments, with increasing K , and a similar statement holds for the region $K > 0\text{dB}$, where the fixed component dominates. Finally, there is good agreement between numerical results and theoretical predictions, supporting our analysis.

In Fig. 3 we can see how the number of antennas in the network influences performance. Following intuition, by increasing the number of secondary transmit antennas the capacity increases, too, since the system has more degrees

³The authors acknowledge that larger block size might be needed to ensure "good" statistical averaging, yet computational limitations were imposed by the software. We note, though, that the smoothness of the resulting curves advocates that $\mathcal{O}(10^3)$ states are sufficient, for the setups investigated herein.

$$C = \sum_{n=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} (\lambda/2)^n}{B(M_n, N)n!} \left(\sum_{k=1}^{M_n-1} \frac{\Gamma(M_n)\Gamma(M_n+N-k)}{\Gamma(M_n-k+1)\Gamma(M_n+N)} \left[\frac{\tilde{\gamma}_0^{M_n-k} \ln \tilde{\gamma}_0}{(1+\tilde{\gamma}_0)^{M_n+N-k}} + \mathcal{B}\left(\frac{1}{1+\tilde{\gamma}_0}; N, M_n-k\right) \right] \right. \\ \left. + \frac{\Gamma(M_n)\Gamma(N)}{\Gamma(M_n+N)} \left\{ \frac{\ln \tilde{\gamma}_0}{(1+\tilde{\gamma}_0)^N} + \left[\ln\left(\frac{1+\tilde{\gamma}_0}{\tilde{\gamma}_0}\right) - \sum_{j=2}^N \frac{(1+\tilde{\gamma}_0)^{-(j-1)}}{(j-1)} \right] \right\} \right) - \ln(\tilde{\gamma}_0)[1 - F_{\gamma}(\gamma_0)]. \quad (3)$$

Fig. 3. Capacity of a CR network with $Q = 1$ and Ricean factor $K = 10$ dB, for various combinations of (M, N) .

Fig. 4. Capacity of a CR network with $Q = 1$ and $(M, N) = (2, 1)$ for different TNR values.

of freedom and the transmitter can steer its transmission into the desired direction more efficiently. On the contrary, an increase in the number of primary receive antennas has a negative effect on capacity because the average interference will increase, thereby tightening the IPC. It is worth noting here that, according to Fig. 3, the negative impact of adding one extra antenna at the primary receiver is higher than the positive return of adding one extra transmit antenna.

In Fig. 4 we can see how the capacity evolves with respect to the Ricean factor. We observe that with increasing K the secondary system exhibits positive shifts in capacity, with those increments varying in magnitude depending on the operating TNR. While system performance is essentially the same for any value of the Ricean factor K , when TNR = 0dB, it is seen that the CR network experiences some rate improvement with respect to K when TNR is 10dB or 20dB. The reason is that with increasing K the MRT beamforming direction contains less randomness which is reflected in the concentration around the mean of the distribution, the latter having a proportional connection to the average received power. Therefore, with increasing TNR the system can benefit from a higher K .

V. CONCLUSIONS

In this work we considered a single-user cognitive radio network in the presence of a licensed primary receiver with minimal information exchange between the two network entities. Based on the available CSI, a linear beamforming strategy was adopted at the SU_{Tx} node and the power was thereafter

optimized to achieve the ergodic capacity. A crucial element in our analysis was the entanglement of the $\mathcal{F}$ -distribution in the stochastic modeling of the relevant quantities, which assisted the theoretical derivations.

Our mathematical treatment has shed light onto the manifold impact of the network parameters into the secondary link capacity, under the particular primary QoS metric (IPC). Numerical simulations supported the validity of our theoretical claims. Our insights can be easily applied to analyze other similar scenarios, e.g., the setup considered in this work but under an instantaneous IPC. Another relevant CR network model corresponds to the scenario with Rayleigh fading for the SU_{Tx} - SU_{Rx} link and Ricean fading for the SU_{Tx} - PU_{Rx} cross-link.

APPENDIX

A. Some useful integrals

In this section we present some basic integrals that appear in this paper. We treat them separately in this appendix only to discretize the various technical levels, in our later derivations. The first result is the following

$$I_1 = \int_a^{\infty} \frac{t^{N-1}}{(1+t)^{L+N}} dt = \int_a^{\infty} \left(\frac{1}{1+t} \right)^{L+1} \left(\frac{t}{1+t} \right)^{N-1} dt \\ \stackrel{(a)}{=} \int_0^{\frac{1}{1+a}} z^{L-1} (1-z)^{N-1} dz = \mathcal{B}\left(\frac{1}{1+a}; L, N\right),$$

where (a) follows by the change of variables $z \triangleq \frac{1}{1+t}$ and $\mathcal{B}(x; L, N)$ denotes the *incomplete Beta function* [15] which,

for integer arguments can be expanded as $\mathcal{B}(x; L, N) = \mathcal{B}(L, N) \sum_{m=L}^{L+N-1} \binom{L+N-1}{m} x^m (1-x)^{L+N-1-m}$. The next integral under consideration is

$$\begin{aligned} I_2 &= \int_a^\infty \frac{\ln t}{(1+t)^{N+1}} dt = \frac{\ln a}{N(1+a)^N} + \frac{1}{N} \int_a^\infty \frac{1}{t(1+t)^N} dt \\ &\stackrel{(a)}{=} \frac{\ln a}{N(1+a)^N} + \frac{1}{N} \int_a^\infty \left[\frac{1}{t} - \sum_{j=1}^N \frac{1}{(1+t)^j} \right] dt \\ &= \frac{\ln a}{N(1+a)^N} + \frac{1}{N} \left[\ln \left(\frac{1+a}{a} \right) - \sum_{j=2}^N \frac{1}{(j-1)(1+a)^{j-1}} \right], \end{aligned}$$

where (a) follows after the observation that $\frac{1}{t} \frac{1}{(1+t)^N} = \frac{1}{t} \frac{1}{(1+t)^{N-1}} - \frac{1}{(1+t)^N}$. This relation is applied recursively in order to yield the partial fraction decomposition $\frac{1}{t} \frac{1}{(1+t)^N} = \frac{1}{t} - \sum_{j=1}^N \frac{1}{(1+t)^j}$. These results will be exploited to simplify the evaluation of various quantities of interest.

B. Proof of proposition 1

Evaluation of the left-hand-side (LHS) of the equation $E\left[\frac{1}{\gamma_0} - \frac{1}{\gamma}\right]^+ = \tilde{Q}$ involves the solution of the integral

$$\begin{aligned} \tilde{Q} &= \int_{\gamma_0}^\infty \left(\frac{1}{\gamma_0} - \frac{1}{z} \right) f_\gamma(z) dz = \frac{\Pr(\gamma \geq \gamma_0)}{\gamma_0} - \int_{\gamma_0}^\infty \frac{f_\gamma(z)}{z} dz \\ &= \frac{1}{\gamma_0} [1 - F_\gamma(\gamma_0)] - e^{-\frac{\lambda}{2}} \sum_{n=0}^\infty \frac{(\lambda/2)^n (M/N)}{B(M_n, N) n!} \\ &\quad \underbrace{\left(\frac{M}{N} \right)^{M_n-1} \int_{\gamma_0}^\infty \frac{1}{z} \left(\frac{1}{1 + \frac{M}{N} z} \right)^{M_n+N} z^{M_n-1} dz}_{I_0}. \quad (\text{B1}) \end{aligned}$$

In order to obtain the closed-form solution to the above integral we first need to evaluate the inner integral I_0 . If we set $t \triangleq \frac{M}{N} z$ and $\tilde{\gamma}_0 \triangleq \frac{M}{N} \gamma_0$ then I_0 is equivalent to

$$I_0 = \int_{\tilde{\gamma}_0}^\infty \left(\frac{1}{1+t} \right)^{M_n+N} t^{M_n-2} dt, \quad (\text{B2})$$

Note here that we have to consider two separate cases for the exponent of the term t^{M_n-2} in order to capture all possible resulting formulations. Thus we have

- $M_n - 2 = -1$: This event occurs only for $\Omega \triangleq \{(M, n) = (1, 0)\}$. In this case the integral becomes

$$I_0' = \int_{\tilde{\gamma}_0}^\infty \frac{t^{-1}}{(1+t)^{N+1}} dt = \ln \left(\frac{1+\tilde{\gamma}_0}{\tilde{\gamma}_0} \right) - \sum_{j=2}^{N+1} \frac{(1+\tilde{\gamma}_0)^{-(j-1)}}{(j-1)},$$

where the second equality follows easily by exploiting the partial fraction expansion used to solve I_2 .

- $M_n - 2 \geq 0$: This case essentially captures all remaining admissible combinations of (M, n) . In this case we solve the following integral

$$I_0'' = \int_{\tilde{\gamma}_0}^\infty \frac{t^{M_n-2}}{(1+t)^{M_n+N}} dt = \mathcal{B} \left(\frac{1}{1+\tilde{\gamma}_0}; N+1, M_n-1 \right),$$

where the second equality follows by making the proper substitutions into the integral I_1 . Therefore, if we define the

indicator function of the event Ω as $\mathbf{I}_{\{\Omega\}}$ and assemble the above results then we obtain the following solution

$$\begin{aligned} I_0 &= \mathbf{I}_{\{\Omega\}} \left[\ln \left(\frac{1+\tilde{\gamma}_0}{\tilde{\gamma}_0} \right) - \sum_{j=2}^{N+1} \frac{1}{(j-1)(1+\tilde{\gamma}_0)^{j-1}} \right] \\ &\quad + \mathbf{I}_{\{\Omega^c\}} \mathcal{B} \left(\frac{1}{1+\tilde{\gamma}_0}; N+1, M_n-1 \right). \quad (\text{B3}) \end{aligned}$$

Plugging (B3) into (B1) yields Eq. (2).

C. Proof of proposition 2

The ergodic capacity is found by averaging over all possible channel states as follows

$$\begin{aligned} C &= E \left\{ \ln \left(1 + \left[\frac{c\gamma}{\mu\sigma^2} - 1 \right]^+ \right) \right\} = \int_{\gamma_0}^\infty \ln \left(\frac{z}{\gamma_0} \right) f_\gamma(z) dz \\ &= \int_{\gamma_0}^\infty \ln \left(\frac{M}{N} z \right) f_\gamma(z) dz - \ln(\tilde{\gamma}_0) [1 - F_\gamma(\gamma_0)]. \quad (\text{C1}) \end{aligned}$$

The capacity is evaluated by exploiting the results from Appendix A to resolve an integral of the form

$$\begin{aligned} I_c &= \left(\frac{M}{N} \right)^{M_n} \int_{\gamma_0}^\infty \ln \left(\frac{M}{N} z \right) \left[\frac{1}{1 + (\frac{M}{N} z)} \right]^{M_n+N} z^{M_n-1} dz \\ &= \int_{\tilde{\gamma}_0}^\infty \left(\frac{1}{1+t} \right)^{M_n+N} t^{M_n-1} \ln t dt, \end{aligned}$$

solved as shown at the top of the next page, where (a) follows by the result $\int_{\alpha}^\infty \frac{t^{M_n-2}}{(1+t)^{M_n+N-1}} dt = \mathcal{B}(\frac{1}{1+\alpha}, N, M_n-1)$ (see I_1); (b) follows by mathematical induction and (c) follows from the integral I_2 , in Appendix A. Plugging this expression back in (C1) yields Eq. (3) for the ergodic capacity.

D. Proof of proposition 3

To investigate the trend of the truncation error we consider the series in Eqs. (2),(3), but with the summation index now starting at $n = T$. The following lemma will be useful.

Lemma 1. If $g(T)$ is a function bounded from above, by some finite positive constant K , and $S_T \triangleq e^{-\theta} \sum_{n=T}^\infty \frac{\theta^n}{n!}$ then $f(T) \triangleq |g(T)| \times S_T$ decays exponentially with T , for $T > e\theta$.

Proof: Since $g(T)$ is bounded from above then the decay rate of $f(T)$ depends solely on S_T . Therefore we need only investigate the behavior of S_T .

$$\begin{aligned} \sum_{k=T}^\infty \frac{e^{-\theta} \theta^k}{k!} &< \sum_{k=T}^\infty \frac{e^{-\theta} \theta^T}{T!} \left(\frac{\theta}{T+1} \right)^{k-T} \\ &< \frac{e^{-\theta} \theta^T e^T}{\sqrt{2\pi T T^T} e^{\frac{1}{12T+1}}} \frac{1}{1 - \frac{\theta}{T+1}} \\ &< \left(\frac{e^{-\theta}}{\sqrt{2\pi}} \right) \left(\frac{\theta e}{T} \right)^T \frac{(T+1)}{\sqrt{T}(T+1-\theta)} < C\alpha^T, \end{aligned}$$

where the second inequality follows by invoking $\sum_{k=0}^\infty x^k = \frac{1}{1-x}$, $|x| < 1$ and Stirling's approximation of $T!$ [16]. To obtain the last inequality we define $C \triangleq e^{-\theta} \sqrt{\frac{2}{\pi}}$, $\alpha \triangleq \frac{\theta e}{T}$, which is smaller than one, and exploit the fact that the remaining

$$\begin{aligned}
I_c &= \frac{1}{M_n + N - 1} \left\{ - \frac{t^{M_n-1} \ln t}{(1+t)^{M_n+N-1}} \Big|_{\tilde{\gamma}_0}^{\infty} + \int_{\tilde{\gamma}_0}^{\infty} \frac{(M_n-1)t^{M_n-2} \ln t + t^{M_n-2}}{(1+t)^{M_n+N-1}} dt \right\} \\
&\stackrel{(a)}{=} \frac{1}{M_n + N - 1} \left\{ \frac{\tilde{\gamma}_0^{M_n-1} \ln \tilde{\gamma}_0}{(1+\tilde{\gamma}_0)^{M_n+N-1}} + \mathcal{B}\left(\frac{1}{1+\tilde{\gamma}_0}; N, M_n-1\right) + \int_{\tilde{\gamma}_0}^{\infty} \frac{(M_n-1)t^{M_n-2} \ln t}{(1+t)^{M_n+N-1}} dt \right\} = \dots \\
&\stackrel{(b)}{=} \sum_{k=1}^{M_n-1} \frac{\Gamma(M_n)\Gamma(M_n+N-k)}{\Gamma(M_n-k+1)\Gamma(M_n+N)} \left[\frac{\tilde{\gamma}_0^{M_n-k} \ln \tilde{\gamma}_0}{(1+\tilde{\gamma}_0)^{M_n+N-k}} + \mathcal{B}\left(\frac{1}{1+\tilde{\gamma}_0}; N, M_n-k\right) \right] + \frac{\Gamma(M_n)\Gamma(N+1)}{\Gamma(M_n+N)} \int_{\tilde{\gamma}_0}^{\infty} \frac{\ln t}{(1+t)^{N+1}} dt \\
&\stackrel{(c)}{=} \sum_{k=1}^{M_n-1} \frac{\Gamma(M_n)\Gamma(M_n+N-k)}{\Gamma(M_n-k+1)\Gamma(M_n+N)} \left[\frac{\tilde{\gamma}_0^{M_n-k} \ln \tilde{\gamma}_0}{(1+\tilde{\gamma}_0)^{M_n+N-k}} + \mathcal{B}\left(\frac{1}{1+\tilde{\gamma}_0}; N, M_n-k\right) \right] + \frac{\Gamma(M_n)\Gamma(N)}{\Gamma(M_n+N)} \\
&\quad \left\{ \frac{\ln \tilde{\gamma}_0}{(1+\tilde{\gamma}_0)^N} + \left[\ln\left(\frac{1+\tilde{\gamma}_0}{\tilde{\gamma}_0}\right) - \sum_{j=2}^N \frac{1}{(j-1)(1+\tilde{\gamma}_0)^{j-1}} \right] \right\} \quad (\{I_{ck}\}_{k=1}^3 \triangleq \text{the first two summations and the last term})
\end{aligned}$$

quantity is less than two. Thus, the series S_T belongs to the family $\mathcal{O}(\alpha^T)$ which has an exponential decay with T . ■

Remark: A similar conclusion can be reached by recognizing that S_T is the tail probability of a Poisson random variable [17] with parameter θ . In that case, we can utilize the properties of the Poisson distribution to establish similar results.

Let us now consider each existing summation separately, starting from the series that appears in Eq. (2) and pertains to the event $I_{\{\Omega^c\}}$.⁴ After some manipulations one can show that

$$\frac{\mathcal{B}\left(\frac{1}{1+\tilde{\gamma}_0}; N+1, M_n-1\right)}{NB(M_n, N)} = \frac{\mathcal{I}\left(\frac{1}{1+\tilde{\gamma}_0}; N+1, M_n-1\right)}{M_n-1} < \frac{1}{T},$$

where $\mathcal{I}(x; a, b)$ is the *regularized incomplete Beta function* [15], which is smaller or equal to one, and the last inequality holds because $n \geq T$ and $M \geq 1$. Plugging this bound into the series under consideration and invoking lemma 1, with $g(T) = T^{-1}$ and $\theta = \frac{\lambda}{2}$, concludes the proof for this series.

In a similar way we characterize the behavior of the truncation error arising in Eq. (3), where the components $\{I_{ck}\}_{k=1}^3$ are involved. These terms correspond to the three separate quantities in I_c (last equality at the top of the page). Due to space limitations we do not detail the exact algebraic manipulations but instead provide a brief sketch.

The last term I_{c3} is already a constant thus when it is multiplied with S_T it readily yields the sought result. The summation part involved in first term I_{c1} (the term containing $\ln \tilde{\gamma}_0$) can be shown to be bounded by

$$I_{c1}^B = B(M_n, N) \left[1 - \frac{1}{(1+\tilde{\gamma}_0)^N} \right],$$

where the term $B(M_n, N)$ will be cancelled by its counterpart in the denominator of the outer summation. Bounding the remaining term, I_{c2} is slightly more involved because we need to manipulate both the inner and the outer (involving S_T) summation. The main idea is to interchange the order of summation and reshape the resulting formulation such that it is expressed in terms of S_T multiplied by a quantity that is independent of the summation index n . Application, thereafter, of lemma 1 completes the proof.

It is worth pointing out that the bounds are not tight, in general, and this analysis is conservative because the focus

has been on establishing an exponential, at least, trend of the truncation error reduction. We conjecture that the actual error demonstrates even sharper diminishing behavior, with looser conditions on T . Thus approximation of the original series will be even more accurate than what Proposition 3 predicts.

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⁴It is easy to see that the term pertaining to $I_{\{\Omega\}}$ is never truncated.