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A wave expansion method for acoustic propagation in lined flow ducts

Ciarán J. O'Reilly

*KTH Royal Institute of Technology, Department of Aeronautical and Vehicle Engineering,
Teknikringen 8, SE-100 44 Stockholm, Sweden.*

Abstract

Acoustic liners are used extensively in engineering applications, particularly in aero-engines and automotive exhaust systems. In this paper, a flow impedance boundary conditions is introduced into the wave expansion method with the aim of providing an efficient methodology for computing the acoustic propagation through a lined duct with flow. For a potential flow, the boundary layer and the lined wall are included in the discretisation scheme by the Myers flow impedance boundary condition. The acoustic propagation through a flow-impedance tube is computed in order to validate the implementation of the impedance boundary condition in this scheme. The results show that this computationally light methodology provides generally good agreement with the experimental data.

Keywords:

Acoustic Liner, Frequency-domain propagation, Wave expansion method, Flow-impedance boundary condition.

1. Introduction

Acoustic liners are used to absorb sound in many engineering applications, but particularly in aero-engines [1, 2]. As aero-engines have become progressively larger in diameter, with the introduction of higher bypass-ratio flows, fan noise has become a prominent noise source. The accompanying reduction in jet noise has also unmasked combustion and turbine noise as significant noise sources. This has resulted in their already extensive, and ever increasing, use in aero-engines to treat fan, combustion and turbine noise.

Liners are typically composed of a single layer honeycomb structure with a perforated facing sheet and a rigid backing sheet, although more complex novel configurations are under investigation [3]. Quite often, it is the effect of the liner on sound propagating past them which is of primary interest, and liners are included in propagation predictions through an impedance model, rather than including individual cells, the perforations in the facing sheet, etc. Liners are therefore characterised by a complex impedance value, which is dependent on the grazing flow velocity and acoustic pressure spectrum at the liner surface. This impedance value is usually inversely determined from experimental data using an eduction technique with numerical or analytical propagation methods [4, 5, 6, 7]. For typical liner applications (i.e. with a honeycomb structure), the liner may reasonably be assumed to be locally reacting.

In this paper, a flow impedance boundary condition is implemented in a numerical propagation scheme known as the wave expansion method (WEM) [8, 9] with the intention of investigating acoustic propagation in lined ducts. The implementation of a flow impedance boundary condition in the WEM has not been examined in the past, but is of interest due to the inherent efficiency of the scheme. The WEM uses fundamental solutions of the wave operator and so accurate solutions to linearised propagation equations may be obtained with only two-to-three points-per-wavelength [10], which is significantly less than the six-to-ten points-per-wavelength of comparable traditional finite difference and finite element methods. The method is robust to meshing and could be implemented in a meshless manner (only points, no elements). These properties make it well suited for use in the optimisation of realistic complex lined engine nacelles.

This paper is structured as follows. The WEM discretisation scheme is presented in Section 2. The propagation equation and the flow-impedance boundary condition are implemented into the WEM in Section 3. The methodology

Email address: ciaran@kth.se (Ciarán J. O'Reilly)

is then used to compute the propagation through a flow impedance tube and to deduce impedance values in Section 4. The results are compared against the data from the grazing incidence tube (GIT) at the NASA Langley Research Center, published by Jones et al. [11] in Section 5. These data are the most extensive available for such a configuration and have been widely used to benchmark the use of flow impedance boundary conditions in numerical schemes [12, 13, 14]. These results are discussed further along with the improved efficiency of the methodology in Section 6, before conclusion are presented in Section 7.

2. Wave Expansion Discretisation Scheme

The wave expansion method (WEM) was first proposed by Caruthers et al. [8], and was more recently used by Ruiz and Rice [10] and Barrera Rolla and Rice [15] to investigate sound propagation in quiescent media. The discretisation scheme may be used for solving linearised time-harmonic propagation equations. The WEM is a physically-based numerical scheme, in that it uses fundamental solutions of the wave operator. As such, it has a very low dispersion error compared to those associated with other numerical schemes [8]. As a result, it is a highly efficient numerical procedure, requiring only two-to-three points-per-wavelength to obtain accurate solutions. This discretisation is perhaps optimal, as it is valid down to the Nyquist limit of two points-per-wavelength. The value of the unknown at each discrete point in the domain is related to the values at a selected set of neighbouring points by plane wave functions. The WEM represents a local interpolation formula to obtain the field values in the domain.

The solution to a linear propagation equation of the form $L\phi = 0$ at each point in a computational lattice may be approximated by the superposition of the field generated by N hypothetical plane waves of strength γ_n and with unit propagation in direction $\mathbf{v}_n$. The solution at a discrete point $\mathbf{x}_0$ at the centre of a small computational stencil, as depicted in Fig. 1, is

$$\phi_0 = \sum_{n=1}^N \gamma_n \exp[-iq_n(\mathbf{v}_n \cdot \mathbf{x}_0)] = \mathbf{h}\boldsymbol{\gamma}, \quad (1)$$

where $\mathbf{h}$ is a $1 \times N$ vector containing fundamental plane wave solution $h_n = \exp[-iq_n(\mathbf{v}_n \cdot \mathbf{x}_0)]$, $\boldsymbol{\gamma}$ is a $N \times 1$ vector containing γ_n , and q_n is the wave number. Similarly, the solution at neighbouring points at $\mathbf{x}_m$ in the computational stencil (see Fig. 1 again), where $m = 1, 2, \dots, M$ and M is the number of neighbours, is given as

$$\boldsymbol{\phi}_m = \sum_{n=1}^N \gamma_n \exp[-iq_n(\mathbf{v}_n \cdot \mathbf{x}_m)] = \mathbf{H}\boldsymbol{\gamma}, \quad (2)$$

where $\mathbf{H}$ is a $M \times N$ matrix containing $H_{m,n} = \exp[-iq_n(\mathbf{v}_n \cdot \mathbf{x}_m)]$.

Within the computational stencil, the wave strengths $\boldsymbol{\gamma}$ must simultaneously satisfy the central and neighbouring points so in this way the central point is linked to its neighbours. This assumes that within the stencil it is possible to approximate any mean flow as locally uniform.

Equation 2 is therefore used as a constraint on Eqn. 1 by solving Eqn. 2 and substituting into Eqn. 1. By choosing the number of waves to be greater than the number of neighbours, $N > M$, Eqn. 2 is under-constrained and so there exists an infinite set of solutions for $\boldsymbol{\gamma}$. However, if $\mathbf{H}^+$ is the Moore-Penrose pseudo-inverse of $\mathbf{H}$ then

$$\boldsymbol{\gamma} = \mathbf{H}^+ \boldsymbol{\phi}_m, \quad (3)$$

is the minimum-norm out of all solutions. Upon substituting back into Eqn. 1 this leads to

$$\phi_0 - \mathbf{h}\mathbf{H}^+ \boldsymbol{\phi}_m = 0, \quad (4)$$

which may be written in vector form as

$$\left\{ \begin{array}{cc} 1 & -\mathbf{h}\mathbf{H}^+ \end{array} \right\} \left\{ \begin{array}{c} \phi_0 \\ \boldsymbol{\phi}_m \end{array} \right\} = 0. \quad (5)$$

The central point is thus linked to its neighbours in an optimal or least-squares way.

For the total computational lattice made up of overlapping stencils, a linear system of equations may be constructed, using Eqn. 5 to determine each row. This takes the form

$$\mathbf{K}\boldsymbol{\phi} = \mathbf{f}, \quad (6)$$

where $\mathbf{K}$ is the overall sparse unsymmetric stiffness matrix $ndof \times ndof$ in dimension, where $ndof$ is the number of degrees of freedom in the overall problem. The vector $\mathbf{f}$ is been added to the right-hand-side and may be used to add point sources or impose boundary values. This system of equations may be solved for $\boldsymbol{\phi}$, a vector containing the solution at each point in the overall computational lattice.

It is assumed in this present formulation that the values of the coefficients are constant within a computational stencil so that there are only small variations across overlapping stencils. The grid must therefore be sufficiently fine to resolve flow variations. However, this is also true of other numerical schemes. Alternatively, a higher-order stencil could be employed if there are large variations in the background flow as discussed by Caruthers et al. [8].

From experience it has been found that an approximately uniform distribution of plane wave vectors in all directions gives the most accurate results. In 2D this can be achieved easily, whilst in 3D a Buckminster Fuller geodesic dome gives an effective distribution of 60 uniformly spaced vectors.

Additionally, out-going and in-coming waves may both be included for q_+ and q_- , although one set is sufficient. This is achieved by augmenting the number of columns in $\mathbf{h}$ and $\mathbf{H}$ such that $\mathbf{h}$ becomes $\{\mathbf{h}_{q_+} \mathbf{h}_{q_-}\}$ and similarly for $\mathbf{H}$. For example, if a distribution of 20 vectors is used then $N = 40$.

It is worth noting that although the computational stencil depicted in Fig. 1 shows a central point and its eight neighbours on a regular lattice, there is nothing in the present formulation to restrict the method to this type of grid. Indeed all that is required for the WEM are the grid points and information about neighbouring points. In practice, it is often easiest to use the mesh element connectivity generated by (regular or irregular) meshing routines to determine the point neighbours. There may, however, be circumstances when it is more efficient or reliable to determine the neighbours directly from a list of points (or use a combination of surface meshes with distributed points within the domain). In this regard the WEM may be regarded mesh-less or loosely mesh dependent.

3. A WEM-Based Methodology for Computing the Propagation in a Lined Flow Duct

In order to compute the linear acoustic propagation in a lined duct, it is necessary to decide how the lined wall or impedance boundary are to be introduced into the computation and what assumption can be made about the background flow. In the absence of a mean flow, the impedance relates the Fourier transformed acoustic pressure and normal acoustic particle velocity at the surface. This is also the case when mean flow is present where the velocity goes to zero at the wall (no-slip condition). However, in this case, the effect of the viscous boundary layer over the liner must be included in the propagation. This may be done by solving the linearised Navier-Stokes equations. The linearised Navier-Stokes equations may also support vorticity and entropy waves but these are not of primary interest here. A pragmatic alternative frequently employed has been to include the boundary layer in the impedance model and to assume that the flow is potential [2]. This is computational much lighter and is attractive particularly for inverse searches of liner impedance values from experimental data or large-scale design optimisations.

As the viscous boundary layers of high-Reynolds-number flows are typically much thinner than a characteristic wavelength, Ingard [16] and later Myers [17] derived an impedance wall model for sound at a lined wall in a mean flow with slip at the wall. This is the limit of a vanishing boundary layer and assumes that the acoustic pressure and particle velocity are continuous across the thin boundary layer. This boundary condition is known as the Myers boundary condition (or alternatively as the Ingard-Myers boundary condition). As the boundary layer has been effectively removed from the propagation, the mean flow may be assumed to be inviscid and potential. The perturbed field can be written in terms of an acoustic velocity potential and the propagation equations may be reduced down to a convected Helmholtz equation.

The Myers boundary condition has been widely used to achieve numerical predictions of lined duct propagation [2, 18], which are considered to be sufficiently accurate for engineering applications. However, there have been a number of studies in recent years, which have identified problems with Myers condition resulting from the modelling assumptions. There are doubts as to whether the Myers model deals correctly with hydrodynamic modes in the boundary layer [19, 20, 21]. Since the boundary condition is the limit of an infinitely thin boundary layer, or the

boundary layer being collapsed into a vortex sheet, when applied with slipping mean flow it may exhibit an instability comparable to the Kelvin-Helmholtz free-shear-layer instability. Furthermore, the model has been shown to be mathematically ill-posed problem [22]. These two issues are problematic for time-domain simulations, where errors generate perturbations at every frequency, and may result in difficulties with numerical convergence. For frequency-domain computations, however, they are of minor practical importance [23]. It has also been shown that the Myers model fails to result in a single impedance value for wave propagation downstream and upstream in a lined duct when the acoustic boundary layer is not much smaller than the mean flow boundary layer thickness, i.e. at low frequencies [24]. This finding does not negate the use of the model altogether but rather it limits its applicability. Caution should be exercised when applying the Myers boundary condition to cases which are not consistent with those used to evaluate the impedance.

In this paper, the acoustic propagation in a potential flow is solved with Myers boundary condition implemented for the lined wall impedance condition. This choice could be changed later. The questions at hand here is how to implement this type of boundary condition in the WEM and how does this computationally light approach compare with available experimental data and alternative computationally more demanding approaches?

3.1. Acoustic Propagation in Inhomogeneous Irrotational Steady Flows

Wave propagation in an inhomogeneous irrotational homentropic steady flow may be described in terms of an acoustic velocity potential, ϕ , and is governed by [25]

$$\frac{1}{\bar{\rho}} \nabla \cdot (\bar{\rho} \nabla \phi) - (i\omega + \bar{\mathbf{u}} \cdot \nabla) \left(\frac{1}{\bar{c}^2} (i\omega + \bar{\mathbf{u}} \cdot \nabla) \phi \right) = 0, \quad (7)$$

where $\bar{\mathbf{u}}$, $\bar{\rho}$, $\bar{c}$, are the local mean velocity vector, density, and speed of sound respectively¹. ω is the angular frequency and $i \equiv \sqrt{-1}$. ∇ is a vector differential operator, which depends on the chosen dimensions and coordinate system. The acoustic pressure and velocity are related to the potential by $p = -i\omega \bar{\rho} \phi - \bar{\rho} \bar{\mathbf{u}} \cdot \nabla \phi$ and $\mathbf{u} = \nabla \phi$.

Eqn. 7 may be expanded into the form

$$\nabla^2 \phi - (\mathbf{M} \cdot \nabla)^2 \phi - \mathbf{B} \cdot \nabla \phi + K^2 \phi = 0, \quad (8)$$

where

$$\mathbf{B} = 2ik\mathbf{M} + \left(\bar{\mathbf{u}} \cdot \nabla \frac{1}{\bar{c}^2} \right) \bar{\mathbf{u}} + \frac{1}{\bar{c}^2} \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} - \frac{1}{\bar{\rho}} \nabla \bar{\rho}, \quad (9)$$

$$K^2 = k^2 - i\omega \bar{\mathbf{u}} \cdot \nabla \frac{1}{\bar{c}^2}, \quad (10)$$

and where $\mathbf{M} = \bar{\mathbf{u}}/\bar{c}$ is the local Mach vector and $k = \omega/\bar{c}$.

At a discrete point $\mathbf{x}_0$ in the small sub-domain covering any stencil of the entire computational lattice the coefficients of Eqn. 8 may be considered to be constant, and a fundamental plane wave solution may be determined, via solving for the roots of its characteristic equation, as

$$\phi_0 = \exp[-iq_{\pm}(\mathbf{v} \cdot \mathbf{x}_0)], \quad (11)$$

where $\phi_0 = \phi(\mathbf{x}_0)$, $\mathbf{v}$ is an arbitrary unit vector in the direction of the plane wave propagation, $q_{\pm}$ is the wave number for out-going and in-coming waves [8]

$$q_{\pm} = \frac{i\mathbf{B}_0 \cdot \mathbf{v} \pm \sqrt{4K_0^2(1 - (\mathbf{M}_0 \cdot \mathbf{v})^2) - (\mathbf{B}_0 \cdot \mathbf{v})^2}}{2(1 - (\mathbf{M}_0 \cdot \mathbf{v})^2)}, \quad (12)$$

and $\mathbf{B}_0 = \mathbf{B}(\mathbf{x}_0)$, $K_0^2 = K^2(\mathbf{x}_0)$ and $\mathbf{M}_0 = \mathbf{M}(\mathbf{x}_0)$ are constant coefficients. These wave may be used in the WEM discretisation described in the previous section.

¹ An over-line is used throughout to denote mean flow, as opposed to fluctuating, quantities. Bold characters denote vector or matrix quantities.

3.2. Implementation of a Flow Impedance Boundary Condition in the WEM

In the absence of flow the complex impedance on the boundary is simply $Z = p/\mathbf{n} \cdot \mathbf{u}$, where p and $\mathbf{u}$ are the acoustic pressure and velocity. $Z \equiv Z_R + iZ_X$, where Z_R and Z_X are the resistance and reactance of the liner. For a steady mean flow that slips at the boundary, this boundary condition must be modified [16, 17]. If the fluid velocity at the boundary is $\bar{\mathbf{u}} + \mathbf{u} \exp(i\omega t)$, then a flow impedance boundary condition may be given by

$$\mathbf{n} \cdot \mathbf{u} - \frac{p}{Z} - \bar{\mathbf{u}} \cdot \nabla \left(\frac{p}{i\omega Z} \right) + \frac{p}{i\omega Z} \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \bar{\mathbf{u}}) = 0. \quad (13)$$

This is known as the Myers boundary condition [17].

In order to impose this boundary condition in the discretisation scheme outlined above, it must firstly be expressed in terms of the acoustic velocity potential. The acoustic velocity and pressure are related to the potential by $\mathbf{u} = \nabla \phi$ and $p = -i\omega \bar{\rho} \phi - \bar{\rho} \bar{\mathbf{u}} \cdot \nabla \phi$. Inserting these relations into the Myers equation and expanding using the vector dot product identity² leads to

$$\begin{aligned} \mathbf{n} \cdot \nabla \phi + \frac{i\omega \bar{\rho}}{Z} \phi + \frac{2\bar{\rho}}{Z} \bar{\mathbf{u}} \cdot \nabla \phi + \frac{\bar{\rho}}{i\omega Z} \bar{\mathbf{u}}^2 \nabla^2 \phi + \frac{\bar{\rho}}{i\omega Z} \bar{\mathbf{u}} \cdot \nabla \phi \nabla \cdot \bar{\mathbf{u}} \\ - \frac{\bar{\rho}}{Z} \phi \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \bar{\mathbf{u}}) - \frac{\bar{\rho}}{i\omega Z} \bar{\mathbf{u}} \cdot \nabla \phi \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \bar{\mathbf{u}}) = 0. \end{aligned} \quad (14)$$

As mentioned previously, Eqn. 2 is used as a constraint on Eqn. 1 in the discretisation scheme. Therefore, this boundary condition may be imposed by augmenting Eqn. 2 at each boundary point with an additional constraint row [10]. As the potential at each point is the superposition of plane wave functions, its derivatives in Eqn. 14 may be evaluated explicitly in the WEM so that $\mathbf{H}$ is augmented with a row containing

$$\begin{aligned} \left(-iq_n \mathbf{n} \cdot \mathbf{v}_n + \frac{i\omega \bar{\rho}}{Z} - \frac{2i\bar{\rho}}{Z} q_n \bar{\mathbf{u}} \cdot \mathbf{v}_n - \frac{\bar{\rho}}{i\omega Z} q_n^2 \bar{\mathbf{u}}^2 v_n^2 - \frac{\bar{\rho}}{\omega Z} q_n \bar{\mathbf{u}} \cdot \mathbf{v}_n \nabla \cdot \bar{\mathbf{u}} \right. \\ \left. - \frac{\bar{\rho}}{Z} \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \bar{\mathbf{u}}) + \frac{\bar{\rho}}{\omega Z} q_n \bar{\mathbf{u}} \cdot \mathbf{v}_n \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \bar{\mathbf{u}}) \right) h_n. \end{aligned} \quad (15)$$

in each column for each vector direction n . $\mathbf{H}$ becomes a $(M+1) \times N$ matrix denoted as $\mathbf{H}_{aug}$.

Taking the pseudo-inverse of $\mathbf{H}_{aug}$ gives

$$\boldsymbol{\gamma} = [\mathbf{H}_{aug}^+] \begin{Bmatrix} \phi_m \\ 0 \end{Bmatrix}, \quad (16)$$

where $\mathbf{H}_{aug}^+$ is a $N \times (M+1)$ matrix. As before this is then inserted into Eqn. 1. However, Eqn. 4 is now partitioned on boundary points, and so becomes

$$\phi_0 - \mathbf{h} \mathbf{H}_{aug,L}^+ \phi_m - \mathbf{h} \mathbf{H}_{aug,R}^+ 0 = 0, \quad (17)$$

where $\mathbf{H}_{aug,L}^+$ is the first M columns of $\mathbf{H}_{aug}^+$ and $\mathbf{H}_{aug,R}^+$ is the remaining column. As the right-hand-side of Eqn. 14 is zero, the last term here does not contribute. However, it is worth noting that other boundary conditions with non-zero right-hand-sides may be imposed in this way with last term being entered in the forcing vector $\mathbf{f}$. The first two terms are used to assemble the overall stiffness matrix exactly as for non-boundary points. This implementation may be used to impose both soft- and hard-wall boundary conditions, with the impedance Z set to infinity in the latter case. It is also used to implement the exit impedance at the end of the duct.

4. Validation of the Propagation Methodology

The application of the WEM with a flow impedance boundary condition to acoustic liner problems is validated by comparing results obtained using this methodology with experimental data from the grazing incidence tube (GIT) at the NASA Langley Research Center. This data, presented by Jones et al. in Ref. [11], was collected specifically for the purpose of validating acoustic propagation codes and is the most comprehensive available. As such, it has been widely used to benchmark the use of flow impedance boundary conditions in numerical schemes [12, 26, 14, 13].

² $\bar{\mathbf{u}} \cdot \nabla (\bar{\mathbf{u}} \cdot \nabla \phi) = \bar{\mathbf{u}}^2 \nabla^2 \phi + \bar{\mathbf{u}} \cdot \nabla \phi \nabla \cdot \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot (\bar{\mathbf{u}} \times (\nabla \times \nabla \phi)) + \bar{\mathbf{u}} \cdot (\nabla \phi \times (\nabla \times \bar{\mathbf{u}}))$ but for acoustic propagation in a potential flow $\nabla \times \nabla \phi = 0$ and $\nabla \times \bar{\mathbf{u}} = 0$, so the last two terms disappear.

4.1. Validation Cases

The GIT test section, illustrated in Fig. 2, was 0.8128 m long and had a square cross-section with an internal width and height of 0.0508 m. It contained an axially-centred liner with a length of 0.4064 m, which formed the centre of the upper wall. The remaining walls were rigid. Acoustic waves in the plane wave range were generated on the upstream side and a near anechoic termination is downstream. The tube exit impedances from Ref. [11] are given in Tab. 1. Two flow conditions are considered here – firstly, with no mean flow ($M_x = 0.0$) and, secondly, with a mean flow of average axial Mach number $M_x = 0.335$. Acoustic pressure measurements were made at thirty-one points along the lower surface of the GIT (opposite the liner).

4.2. Liner Impedance Eduction Procedure

Normalised impedance values for the liner have been educed from the pressure data by Jones et al. [11] using a 2D finite element method (FEM)-based procedure. In addition to using these Jones et al. educed liner impedance values, impedance values have also been educed here to best-fit the Jones et al. complex pressure data using the WEM-based methodology. This is achieved, in the same way as used by Jones et al. for consistency, by iteratively minimising the objective function

$$F(Z) = \sum_{i=1}^{N_p} (p_{WEM}(x_i) - p_{Data}(x_i)) (p_{WEM}^*(x_i) - p_{Data}^*(x_i)), \quad (18)$$

where p is the complex acoustic pressure along the lower surface of the tube, N_p is the total number of sampled points and $*$ denotes the complex conjugate. These WEM educed values are presented alongside the Jones et al. educed values in Tabs. 2 and 3 for the two flow cases, and are discussed further in the following sections.

4.3. Scattering Matrix Components

The transmission and reflection coefficients are also computed as a further test of the agreement between the experimental data and the WEM predictions for the upstream source. The acoustic pressure values on either side of the liner are decomposed in upstream and downstream propagating plane wave

$$p = p_+ \exp(-ik_+x) + p_- \exp(ik_-x), \quad (19)$$

where $k_{\pm} = k/(1 \pm M_x)$ and M_x is the mean axial Mach number. For a number of sample points $1, 2, \dots, N_s$ with $N_s \geq 2$ an system can be formed [27]

$$\begin{bmatrix} \exp(-ik_+x_1) & \exp(ik_-x_1) \\ \exp(-ik_+x_2) & \exp(ik_-x_2) \\ \vdots & \vdots \\ \exp(-ik_+x_{N_s}) & \exp(ik_-x_{N_s}) \end{bmatrix} \begin{bmatrix} p_+ \\ p_- \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N_s} \end{bmatrix}, \quad (20)$$

which upon solving via a pseudo-inverse gives the values of p_+ and p_- . The transmission and reflection coefficients are defined respectively as

$$T = \frac{p_+^d}{p_+^u}, \quad R = \frac{p_-^u}{p_+^u}, \quad (21)$$

where the superscript u denotes the value on the upstream side of the liner and d the downstream side of the liner.

5. GIT Validation Results

The propagation of acoustic plane waves through the GIT has been computed using the WEM with the newly implemented Myers boundary condition. The computed results presented have been made on a uniform lattice of 65×9 points. Grid convergence of the solution was checked with finer grids with no appreciable difference observed. The number of nodal neighbours, M , is taken to be eight for internal grid points, i.e. those immediately surrounding the point in question as illustrated in Fig. 1. Ten uniformly spaced wave vectors were used for both q_+ and q_- so $N = 20$.

5.1. No Flow Case

In the absence of a mean flow, Eqn. 7 effectively reduces to the Helmholtz equation and the impedance condition to $\mathbf{n} \cdot \mathbf{u} = p/Z$. Figure 3 shows a contour plot of the solution at 3000 Hz. The effectiveness of the liner in reducing the amplitude of the acoustic wave is clearly visible. The impedance boundary condition described in Sec. 3.2 has been used in this solution to impose the hard-wall, soft-wall, and exit impedance boundary conditions. At the inlet of the tube (the left-hand boundary) a Dirichlet condition is used to introduce plane waves. Dirichlet type boundary conditions may be implemented in the WEM by simply constraining the appropriate entries in the overall stiffness matrix $\mathbf{K}$ [10], i.e. setting the appropriate row to zeros with a one on the diagonal and entering the desired solution values into $\mathbf{f}$.

The SPL along the wall opposite the liner is shown in Fig. 4. It can be seen that there is very good agreement across all frequencies, with only small disagreements at 500 Hz and 2000 Hz, where the WEM educed impedance values provide slightly better agreement. This is to be expected, of course, as these values have been educed using the WEM to best-fit the data. The phase across the GIT is plotted in Fig. 5. The agreement here between the computational values and the data is excellent throughout.

Importantly, there is little difference in the impedance values obtained using the WEM- and FEM-based eduction procedures. Figure 6 shows a plot of the impedance values from Tab. 2. This shows that there is a robustness to these impedance values, in that they are variations of the same global converged value rather than quite different values that are perhaps locally optimal. There is a constancy in the characterisation of the liner and so in the performance of the two methods. Figure 7 shows the transmission and reflection coefficients, and once again the agreement is good.

5.2. Grazing Flow Case

The WEM methodology has also been used to compute the propagation of acoustic plane waves through the GIT configuration when a grazing flow of $M_x = 0.335$ is present. This uniform Mach number flow is the average of the measured profile across the tube. For this flow, the mean flow velocity divergence is zero, so the impedance condition in Eqns. 14 and 15 is simplified as the last three terms are equivalent to zero. Also the other gradient terms in the propagation operator (in $\mathbf{B}$ and K) are all zero so the wave number $q = \pm k/(1 \pm \mathbf{M}_0 \cdot \mathbf{v})$.

A contour plot of the solution at 3000 Hz with the uniform flow is shown in Fig. 8. The boundary condition types used are the same as in the no-flow case. The longer wavelength in this case due the mean flow is visible. The SPL along the wall opposite the liner is shown in Fig. 9. The SPL reduction exiting the liner is captured quite well at all frequencies, with the exception of at 1000 Hz. The behaviour of the SPL does not show as good of agreement across the liner as in the no flow case³ and there is some small deviations between the two, particularly at lower frequencies. This will be discussed in more detailed in the next section.

A comparison of the phase values is presented in Fig. 10. Here again the agreement between the experimental data and the numerical values is excellent, with the exception of at 1000 Hz. Again, the WEM educed impedance values provide an improved fit. The educed impedance values are illustrated in Fig. 11. The same consistency noted for the no-flow case is evident here and the linear behaviour of this particular liner is of note.

6. Discussion

In the previous section, it has been demonstrated the WEM provides a reasonably robust method for computing the propagation in a flow impedance tube. When used to educe the impedance of a liner, the values obtained are consistent with those found using a FEM to solve the propagation equation. Despite the limitations of the Myers boundary condition, the methodology has captured the reduction in SPL across the liner using impedance values which are consistent with those educed using more computationally demanding procedures in Refs. [11, 26, 14].

In order to further justify the use of the present methodology, two issues need to be discussed in greater detail – the small departures between the experimental and numerical results that are visible in the grazing flow validation case in the previous section, and the efficiency of the methodology compared to alternatives methods.

³Note that the y-axis range used is different for the different frequency plots so variation are more obvious on plots with a narrower SPL range.

6.1. Differences between the Experimental and Numerical Values

The agreement between the experimental and numerical values in the grazing flow validation case is poorest at 1000 Hz. This is visible in both the SPL and phase plots in Fig. 9 and 10. A relatively small difference in the impedance value results in quite a large change in the acoustic field. Even with the improvement of the WEM educed impedance value, there remains a difference of about 6 dB in the exit SPL compared to the experimental data – much larger than at any other frequency. The disagreement in the phase is also the largest at this frequency. This disagreement may be attributed to the presence of a hydrodynamic instability in the experimental test at this frequency. This instability has been reported previously in Refs. [26, 14]. As it is not included in the present methodology, the disagreement with the data here is to be expected. Interestingly, it can be noted that the WEM educed impedance value is still comparable to a value which has been shown to give good agreement in a propagation methods that include this instability – Burak et al. for example used an impedance value of $Z = 0.48 + 0.03i$ (compared to the WEM educed value of $Z = 0.43 + 0.07i$) in an LES simulation and achieved excellent agreement with the same data. This strengthens the argument that the present methodology is robust, at least for impedance education.

At 500 Hz, the numerical values underestimate the strength of the standing-wave-like formation across the liner that is visible in the experimental data. Note, the peaks and troughs are present in the numerical values at the same axial locations that appear in the experiments, but that the range of variation is much lower, and so the lines appear to be almost smooth on the axis presented here. Note that this agreement does not occur if the impedance condition is taken to be the no flow impedance boundary condition $\mathbf{n} \cdot \mathbf{u} - p/Z = 0$. The phase values at 500 Hz do not fully capture the shape of the experimental curves either although the difference is not so obvious. The same trend is also visible at 1500 Hz but to a much lesser extent. Both the length of the domain and the computational mesh have been investigated and excluded as possible explanations for this behaviour. Also, the exit impedance has been investigated, but this was seen to effect mostly the SPL downstream of the liner, and not across the liner itself. This same trend is also visible in the results presented in Ref. [26] for example (which also uses the Myers boundary condition), where it was not specifically discussed.

Some insight may be gained from looking at the transmission and reflection coefficients shown in Fig. 12 (the WEM results shown here were computed with the WEM educed impedance values). Although the transmission coefficient displays the same agreement as has been discussed for the total pressure values, the reflection coefficient shows considerable divergence particularly at lower frequency. The higher reflection coefficient in the experimental data explains the standing wave across the liner noted in Fig. 9. Three influences may explain for this difference. Firstly, that a hydrodynamic instability is influencing the result. As already discussed, this is most likely the case at 1000 Hz. However, at other frequencies, this seems unlikely as the results presented in the literature do not indicate that this is the case. Secondly, it may be that the implementation of the numerical boundary condition is done in a way that causes less reflection the end of the liner than in the experimental setup. This explanation must be counterbalanced with the fact the no flow case does capture a similar formation. Thirdly, and perhaps most likely, this disagreement at lower frequencies is consistent with the type of failure of the Myers boundary condition discussed in Ref. [24], where the model impedance values were found to be dependent on the direction of propagation. Therefore, waves reflected back upstream see a different impedance.

6.2. WEM Efficiency

The advantage of using the present methodology is that it requires less computational resources to achieve these positive results, compared to alternative numerical schemes and/or flow-impedance assumptions, which makes it interesting for many practical applications. To demonstrate this, the accuracy of the WEM is compared to a finite element method (FEM) solution (computed using the commercial solver COMSOL Multiphysics) with linear shape functions. Using the same domain as the GIT, this time with hard walls, and varying the frequency, the error relative to the exact reference solution was determined and is shown in Fig. 13. Included in this plot is the WEM error for different numbers of wave vectors, i.e. varying N in the discretisation. As can be seen, the relative error is considerably lower for the WEM for a given number of points-per-wavelength. The WEM requires roughly half the number of points-per-wavelength for the same level of accuracy. This translates into a significant saving in computational resources as the size of the overall problem to be solved – the size of the overall stiff matrix $ndof$ to be inverted – will be $1/2^D$ the size, where D is the number of spatial dimensions.

As this discretisation method works by finding an optimal solution to the local wave vector amplitudes via a Moore-Penrose pseudo-inverse (computed here via a singular value decomposition), it is important that the size of the

matrix $\mathbf{H}$ is kept to a minimum so as to limit the assembly requirements for the overall stiffness matrix. For the present 2D computation, the number of neighbours on the regular grid is $M = 8$, so the method requires that $N > 8$ wave vectors be used for internal points and $N > 9$ for boundary points, which have an additional row. Figure 13 shows that the accuracy of the solution with $N = 10$ used throughout the domain is almost identical to that with a larger number of wave vectors used. The size of the pseudo-inverse should therefore not be adversely large.

7. Conclusions

In this paper, a methodology is presented which couples the WEM with the Myers flow-impedance boundary condition in order to examine acoustic propagation through a lined duct with flow. The Myers boundary condition has been implemented for the first time in the WEM by explicitly evaluating the derivatives of the base plane wave functions and augmenting the local stiffness values accordingly. The computed reduction in SPL across the NASA GIT configuration agrees well with the measured data in terms of both SPL and phase across the liner. Deviations in this agreement have been accounted for. The method has been used to deduce impedance values, which are consistent with those determined using alternative, often more demanding, approaches. Analysis of the reflection coefficient shows that the Myers boundary condition fails to fully model the wave behaviour across the liner at low frequencies. Aside from this, the discretisation scheme itself has been shown to be considerably more efficient than a comparable FEM discretisation. The WEM with the Myers boundary condition offers a highly efficient numerical methodology for computing acoustic propagation in lined ducts with a grazing flow.

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Figures

Figure 1: Schematic of the wave expansion discretisation stencil with a central node, its neighbours and hypothetical plane waves.

Figure 2: Schematic of the GIT test section from Ref. [11]. Acoustic measurements were made on the lower surface, opposite the liner.

Figure 3: Normalised contours of the acoustic pressure at 3000 Hz and with $M_x = 0.0$ mean flow computed with the WEM. The boundary conditions are illustrated with plane-wave input (cyan), hard-wall impedance (black), soft-wall impedance $Z = 0.72 - 0.27i$ (grey) and exit impedance $Z = 1 + 0.13i$ (green).

Figure 4: Sound pressure levels (SPL) on the surface opposite the liner for $M_x = 0.0$ with experimental results from Ref. [11] (+), and WEM computed levels using WEM reduced impedance values (—) and impedance values from Ref. [11] (---).

Figure 5: Acoustic pressure phase on the surface opposite the liner for $M_x = 0.0$ with experimental results from Ref. [11] (+), and WEM computed levels using WEM educed impedance values (—) and impedance values from Ref. [11] (---).

Figure 6: Educed liner impedance values with $M_x = 0.0$ using the WEM-based procedure (—) and the FEM-based procedure from Ref. [11] (---).

Figure 7: Transmission and reflection coefficients with $M_x = 0.0$ from the WEM (—) and from the data from Ref. [11] (+).

Figure 8: Normalised contours of the acoustic pressure at 3000 Hz and with a uniform $M_x = 0.335$ mean flow computed with the WEM. The boundary conditions are illustrated with plane-wave input (cyan), hard-wall impedance (black), soft-wall impedance $Z = 0.66 - 0.25i$ (grey) and exit impedance $Z = 0.95 + 0.16i$ (green).

Figure 9: Sound pressure levels (SPL) on the surface opposite the liner for $M_x = 0.335$ with experimental results from Ref. [11] (+), and WEM computed levels using WEM reduced impedance values (—) and impedance values from Ref. [11] (---).

Figure 10: Acoustic pressure phase on the surface opposite the liner for $M_x = 0.335$ with experimental results from Ref. [11] (+), and WEM computed levels using WEM educed impedance values (—) and impedance values from Ref. [11] (---).

Figure 11: Educated liner impedance values with $M_x = 0.335$ using the WEM-based procedure (—) and the FEM-based procedure from Ref. [11] (---).

Figure 12: Transmission and reflection coefficients with $M_x = 0.335$ from the WEM (—) and from the data from Ref. [11] (+).

Figure 13: Relative error for propagation in the GIT domain with hard walls using FEM (—) and WEM with $N = 10$ ($\cdots$), $N = 20$ ($---$) and $N = 40$ ($- \cdot -$) wave vectors.

Tables

Table 1: Normalised exit impedances from Jones et al. [11].

f [Hz]	$M_x = 0.0$		$M_x = 0.335$	
	$Z_R/(\rho_\infty c_\infty)$	$Z_X/(\rho_\infty c_\infty)$	$Z_R/(\rho_\infty c_\infty)$	$Z_X/(\rho_\infty c_\infty)$
500	0.99	0.06	0.85	-0.13
1000	1.02	0.06	1.09	0.28
1500	1.03	0.08	0.86	-0.17
2000	0.98	0.13	1.04	0.24
2500	1.02	0.11	1.00	-0.20
3000	1.00	0.13	0.95	0.16

Table 2: Normalised GIT liner impedances at $M_x = 0.0$.

f [Hz]	Jones et al. educed		WEM educed	
	$Z_R/(\rho_\infty c_\infty)$	$Z_X/(\rho_\infty c_\infty)$	$Z_R/(\rho_\infty c_\infty)$	$Z_X/(\rho_\infty c_\infty)$
500	0.51	-1.68	0.57	-1.61
1000	0.46	0.00	0.47	0.00
1500	1.02	1.30	1.08	1.30
2000	4.05	0.62	4.68	0.77
2500	1.54	-1.60	1.55	-1.63
3000	0.70	-0.29	0.72	-0.27

Table 3: Normalised GIT liner impedances at $M_x = 0.335$.

f [Hz]	Jones et al. educed		WEM educed	
	$Z_R/(\rho_\infty c_\infty)$	$Z_X/(\rho_\infty c_\infty)$	$Z_R/(\rho_\infty c_\infty)$	$Z_X/(\rho_\infty c_\infty)$
500	0.61	-0.59	0.43	-0.72
1000	0.17	0.14	0.43	0.07
1500	1.18	1.27	1.15	1.43
2000	4.40	-1.64	5.08	-2.13
2500	0.93	-1.43	1.00	-1.39
3000	0.73	-0.24	0.66	-0.25