Investigation of the Dynamics and Modeling of a Triangular Quadrotor Configuration

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Abstract

In this paper the dynamics of a new type of quadrotor configuration, called Y4, is investigated. The configuration is proposed to be more energy efficient than the traditional design where four rotors are placed around the center of the aircraft frame. This by taking advantage of the benefits of a larger centered rotor in order to produce more lift. A complete mathematical model of the configuration is derived and modeled in the software Simulink. Different control laws is then developed and derived in order to control the Y4, starting with a simple PID control and then moving to non linear control methods using Lyapunov theory. The dynamics of the Y4 is then investigated by simulating different maneuvers, starting with hover and then continuing with attitude, altitude and position maneuvers. The Y4 is shown to be controllable but is less responsive than a typical quadrotor. It show more dynamics of an ordinary helicopter. The conclusion is that the Y4 might have a place in aeronautics if one prioritize power efficiency or lift power and not maneuverability, but still needs the benefits of a VTOL aircraft. However, more research regarding power optimization, design and aerodynamics needs to be done before one can say exactly how much less power the Y4 consumes compared with the standard design.
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Nomenclature

\( \kappa \)  Cant angle for the boom rotors
\( \Omega \)  Angular velocity of the air frame
\( \omega_i \)  Angular velocity for boom rotor \( i \)
\( \omega_M \)  Angular velocity for main rotor
\( \phi \)  Roll angle
\( \psi \)  Yaw angle
\( \rho \)  Air density
\( \rho_{rod} \)  Density of the thin rods
\( \tau \)  Moment vector of the air frame
\( \theta \)  Pitch angle
\( A \)  Rotor area
\( B \)  Body fixed frame
\( D \)  Diameter of the rotor
\( E \)  Inertial frame
\( F \)  Force vector of the air frame
\( g \)  Gravitational constant
\( h \)  Length of the vertical rod
\( I \)  Moment of inertia matrix of the air frame
\( i \)  Current
\( I_{3\times3} \)  3 by 3 identity matrix
\( J_m \)  Motor inertia
CONTENTS

\( J_{pl} \)  Moment of inertia around the rotational axis for the main rotor
\( J_{ps} \)  Moment of inertia around the rotational axis for the boom rotors
\( k_e \)  Motor constant
\( k_{dl} \)  Drag constant for the main rotor
\( k_{tl} \)  Thrust constant for the main rotor
\( k_{ts} \)  Thrust constant for the boom rotors
\( L \)  Inductance
\( L \)  Length of the horizontal rods
\( M \)  Total mass of the Y4
\( m_{el} \)  Mass of the electrical components
\( m_h \)  Mass of the vertical rod
\( m_L \)  Mass of the horizontal rods
\( m_{ml} \)  Mass of large motor
\( m_{ms} \)  Mass of small motor
\( R \)  Rotation Matrix
\( R_{ml} \)  Inner motor resistance for large motor
\( R_{ms} \)  Inner motor resistance for small motor
\( V \)  Velocity vector of the air frame
\( v \)  Speed of the Y4
\( X = (x, y, z) \), position vector of the air frame relative the inertia system
\( X' = (x', y', z') \), Coordinates in the body fixed frame
Chapter 1

Introduction

1.1 Background and Objectives

The quadrotor platform has the last decade become very popular for use in research regarding unmanned aerial vehicles (UAV), this partly due to their simple mechanical design, interesting control challenges and real life applications. In comparison with other VTOL aircraft, quadrotors do not require any linkage to vary the rotor blade pitch since they have fixed propellers, which reduces maintenance, cost and simplifies design. Another advantage by using four small rotors instead of one large is that the total rotational energy of the rotors are spread out, which in return reduces damage both to the vehicle itself and the surroundings in the case of a collision.

Another large factor to why the quadrotor platform has become so popular both as a research platform and for real life applications is the progress in sensor technology. The cost and size of sensors and micro controllers that are needed in order to control the quadrotor has decreased significantly over the last years, making it possible to build small and cheap quadrotors. However, the drawback of cheap sensor is the inaccuracy in data, making it necessary to develop sophisticated and robust control laws.

The real life applications for the quadrotor spreads over various fields such as:

- Surveillance, where the quadrotor can patrol an area, looking for specific events or people.
- Assistance in rescue missions, where the quadrotor can be equipped with e.g. infrared cameras in order to find people.
- Film making, in order to achieve inexpensive aerial shots.
- Measuring and modeling of land areas.
- Transportation of small packages. Both the companies Amazon [1] and DHL [5] are currently developing deliverance services using quadrotors.
CHAPTER 1. INTRODUCTION

- 3D-mapping. 3D view ability have become standard in modern map systems.

In all these examples the quadrotor can replace or assist the old solution to the problem, while offering cheap and simple operation and maintenance. However, one of the things that prevents the quadrotor platform from reaching its full potential in applications is the flight time and lifting power. Most of today’s quadrotors have a flight time around 10-20 minutes, they can only lift about 1 kg and the charging time often exceeds several hours. The batteries are also very heavy and the progress in high density energy storage is slow. This motivates research regarding more energy efficient mechanical and system designs.

One design that has the potential to be more energy efficient is the so-called triangular quadrotor. The purpose of this design is to combine the benefits from a traditional helicopter with those of a quadrotor, keeping the simple mechanical design while increasing the lifting area. The authors of [6] have built a triangular quadrotor called Y4, see figure 1.1, and compared it with a standard design quadrotor test bed. Both aircraft were built as similar as possible in the sense of weight and size. They concluded by performing real life tests that the standard design quadrotor consumed 15 per cent more power than the Y4 design, meaning approximately a 15 percent increase in flight time for the Y4 over the standard design. The design of the Y4 was not optimized in any sophisticated way and the authors believe that the power reduction can be increased even further, up to 25 per cent. The authors used a PID controller in order to stabilize the Y4 under hover conditions, but no deeper investigations were done regarding control or flight dynamics.

The motivation for the work in this thesis was to investigate if the triangular quadrotor design could be controlled in an efficient way using more advanced control methods then a PID controller under different flight conditions such as hover, level flight and different flight maneuvers. The robustness against disturbances was also investigated. This was done by building a dynamical model of the Y4 and implement it in the software Simulink. Simulation of different flight conditions with different control methods were then investigated. If the triangular quadrotor could be controlled in a reasonable way while using less power compared to the typical quadrotor design, then it has the potential to steer the typical quadrotor design towards a more energy efficient standard.
CHAPTER 1. INTRODUCTION

Figure 1.1. The Y4 built by the authors in [6].
Chapter 2

The Triangular Quadrotor Configuration

This chapter will go through the mechanical design of the Y4, the advantages and disadvantages as well as some aerodynamics and other details that distinguish the Y4 from an ordinary quadrotor.

2.1 Mechanical Design

As the name implies the Y4 has four rotors, but in contrast to the traditional quadrotor design three of them are placed on booms in a triangle around the center and the fourth rotor is placed above the center. The centered rotor is also much larger than the boom rotors. The three boom rotors are tilted with a fixed angle, $\kappa$, relative to the boom axis, see figure 2.1. This is to produce active counter moment for the larger main rotor in the same way as a helicopter. But in contrast to a helicopter that uses a single vertical tail rotor, the boom rotors also contributes to the lift force. The reason for the larger main rotor is to get the benefits from a traditional helicopter since a larger rotor produces more lift, but at the same time retaining the mechanical simplicity of a quadrotor by using fixed pitched propellers. Compared with a traditional quadrotor of the same footprint diameter, the Y4 achieves a greater lifting area, see figure 2.2, and motivates the decreased power consumption. This due to that the rotors of a typical quadrotor cannot be placed too close to each other due to interactions of blade tip vortices. Placing the rotors too close to each other will then decrease performance. A rule of thumb is a spacing of $\sqrt{2}$ times the rotor radius between the center of one rotor to the blade tip of another. The lifting areas for the Y4 and standard design with a unity footprint diameter are

$$ \frac{A_{Y4}}{A_{standard}} = \frac{\frac{\pi}{4}}{\left(\frac{1}{3+\sqrt{2}}\right)^2 \pi} \approx 0.21 \pi \Rightarrow \frac{A_{Y4}}{A_{standard}} \approx 1.22 $$

(2.1)

So the Y4 has approximately 22 percent more lifting area.
2.2 How to Maneuver

In order to maneuver the Y4 the angular velocity of the rotors are altered similarly to an ordinary quadrotor so that roll, pitch and yaw moment are produced. To produce roll moment the front boom rotor is kept at hover speed while one of the side rotor increases speed and the other decreases, producing the same balancing yaw moment while producing an asymmetric thrust in the vertical direction. To achieve pitch moment the front boom rotor increases speed while the other two decreases, this produces a negative pitch moment. A positive pitch moment can be achieved by doing the opposite. The main rotor is kept at hover speed under both roll and pitch maneuvers. The yaw moment is produced by either increasing the boom rotors speeds while decreasing the main rotors speed or vice versa. An illustration of these different situations are shown in figure 2.3. Note that the front rotor is aligned with the $x$-axis, this is to simplify later calculations.
CHAPTER 2. THE TRIANGULAR QUADROTOR CONFIGURATION

Figure 2.3. Illustration of the different rotor velocity combinations in order to achieve hover, pitch, roll and yaw moment. Yellow arrow implies hover reference rotor velocity, green arrow increased rotor velocity and red arrow decreased rotor velocity

2.3 Aerodynamics and Cant Angle

Since the main rotors contributes with the majority of the lift, the boom rotors are placed underneath the main rotor so that their shed wake do not interfere with main rotor, decreasing its lifting power. Also, the boom rotors should be placed around the main rotors vena contracta so that the insulation is maximized.

The cant angle both affect the maneuverability and the total lifting force. A smaller cant angle makes the contribution to the lift greater, but reduces the maneuverability in the yaw direction. So the cant angle must be picked in such way so the control bandwidth and hover power are balanced.

2.4 Disadvantages and Aerodynamics

The cost of having a larger main rotor is an increase in the moment of inertia of the rotor, giving it a slower rise time. This can mean less maneuverability since changing rotor speed is essential for stabilizing the aircraft. A larger main rotor introduces more gyroscopic effects which can make the aircraft less responsive to attitude changes. Also, in the ordinary quadrotor design the rotor gyroscopic effects balance each other since the rotational inertia sum up to zero. This is not the case for the Y4 and has to be considered when developing control laws.
Chapter 3

Dynamic Model

In this chapter the model of the Y4 platform used in simulations will be derived. The mechanical model, equation of motions, aerodynamic forces and rotor dynamics. This model was then implemented in Simulink to perform simulations of different flight situations using different control methods. Figure 3.1 shows an illustrative block diagram of the model implemented in Simulink.

Since the main objective of this thesis was to investigate the maneuverability and flight dynamics of the Y4, all the physical data and constants were taken from the configuration built by the authors in [6]. In table 3.1 all the parameters and their values are shown. The values of the motor resistances were found on the manufacturers web pages and the motor torque constants were found by performing simulations, knowing the increase in angular velocity per voltage.
Table 3.1. Data used for the dynamic model of the Y4

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total mass</td>
<td>$M$</td>
<td>0.953</td>
<td>kg</td>
</tr>
<tr>
<td>Mass of the electrics</td>
<td>$m_{el}$</td>
<td>0.5</td>
<td>kg</td>
</tr>
<tr>
<td>Mass of each small motor</td>
<td>$m_{ms}$</td>
<td>0.038</td>
<td>kg</td>
</tr>
<tr>
<td>Mass of the large motor</td>
<td>$m_{ml}$</td>
<td>0.187</td>
<td>kg</td>
</tr>
<tr>
<td>Mass of each small rotor</td>
<td>$m_{ps}$</td>
<td>0.012</td>
<td>kg</td>
</tr>
<tr>
<td>Mass of the large rotor</td>
<td>$m_{pl}$</td>
<td>0.040</td>
<td>kg</td>
</tr>
<tr>
<td>Radius of the large rotor</td>
<td>$R_l$</td>
<td>0.23</td>
<td>m</td>
</tr>
<tr>
<td>Radius of the small rotors</td>
<td>$R_s$</td>
<td>0.05</td>
<td>m</td>
</tr>
<tr>
<td>Length of the vertical boom</td>
<td>$h$</td>
<td>0.1</td>
<td>m</td>
</tr>
<tr>
<td>Length of the boom arms</td>
<td>$L$</td>
<td>0.23</td>
<td>m</td>
</tr>
<tr>
<td>Cant angle</td>
<td>$\kappa$</td>
<td>$\pi/6$</td>
<td>rad</td>
</tr>
<tr>
<td>Thrust coefficient of the large rotor</td>
<td>$C_{T_l}$</td>
<td>0.0230</td>
<td>N/A</td>
</tr>
<tr>
<td>Drag coefficient of the large rotor</td>
<td>$C_{Q_l}$</td>
<td>0.0037</td>
<td>N/A</td>
</tr>
<tr>
<td>Thrust coefficient of each small rotor</td>
<td>$C_{T_s}$</td>
<td>0.0302</td>
<td>N/A</td>
</tr>
<tr>
<td>Drag coefficient of each small rotor</td>
<td>$C_{Q_s}$</td>
<td>3.62 \times 10^{-4}</td>
<td>N/A</td>
</tr>
<tr>
<td>Inner resistance of the large motor</td>
<td>$R_{ml}$</td>
<td>50 \times 10^{-3}</td>
<td>Ohm</td>
</tr>
<tr>
<td>Inner resistance of the small motors</td>
<td>$R_{ms}$</td>
<td>59 \times 10^{-3}</td>
<td>Ohm</td>
</tr>
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<td>Increase in RPM per voltage</td>
<td>$kv_l$</td>
<td>380</td>
<td>rpm/V</td>
</tr>
<tr>
<td>Increase in RPM per voltage</td>
<td>$kv_s$</td>
<td>2600</td>
<td>rpm/V</td>
</tr>
</tbody>
</table>

3.1 Mechanical Model

In order to keep the model as simple as possible without sacrificing too much of the accuracy, the Y4 was modeled as a rigid body consisting of a set of point masses connected with thin rods with uniform density. The rotors were also modeled as rigid thin rods with uniform density. Figure 3.2 shows the complete mechanical model used in simulations. The values for the different masses and lengths are shown in table 3.1.
CHAPTER 3. DYNAMIC MODEL

Figure 3.2. The mechanical model used for representing the Y4 in simulations.

The masses of the thin rods were calculated as:

\[ m_h = \rho_{rod,h} h \]
\[ m_L = \rho_{rod,L} L \]  

(3.1)

where \( \rho_{rod} \) is the density of the rods, \( L \) the length of the booms and \( h \) the distance from the center of the air frame to the main rotor.

3.2 Moment of Inertia

The moments of inertia for the model in figure 3.2 can be calculated as:

\[ I_{xx} = \int (y^2 + z^2) \, dm \]
\[ = (m_{ml} + \frac{m_h}{3})h^2 + \frac{3}{2}(m_{ms} + \frac{m_L}{3})L^2 \]  

(3.2)

\[ I_{yy} = \int (x^2 + z^2) \, dm \]
\[ = (m_{ml} + \frac{m_h}{3})h^2 + \frac{3}{2}(m_{ms} + \frac{m_L}{3})L^2 \]
\[ = I_{xx} \]  

(3.3)

\[ I_{zz} = \int (x^2 + y^2) \, dm \]
\[ = (3m_{ms} + m_L)L^2 \]  

(3.4)
CHAPTER 3. DYNAMIC MODEL

The propeller masses are neglected since the mass of the motors are substantially greater. Due to symmetry the cross terms become zero and the full inertia matrix with values inserted is:

\[
I = \begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix} = \begin{bmatrix}
0.0036 & 0 & 0 \\
0 & 0.0036 & 0 \\
0 & 0 & 0.0069
\end{bmatrix} \quad (3.5)
\]

The moment of inertia of the propellers around their rotational axis are:

\[
\begin{align*}
J_{pl} &= \frac{m_{pl}R^2_l}{12} \\
J_{ps} &= \frac{m_{ps}R^2_s}{12}
\end{align*} \quad (3.6)
\]

3.3 Equations of Motions

The equation of motions was created with the same approach as in the papers [2,3,6,7]. Using Newton-Euler formalism the dynamics of a rigid body in the body-frame, under external forces and moments applied to the center of mass, can be expressed in matrix form as

\[
\begin{bmatrix}
F \\
\tau
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} m I_{3 \times 3} \begin{bmatrix}
\dot{V} \\
\dot{\Omega}
\end{bmatrix} + \begin{bmatrix}
\Omega \times mV \\
\Omega \times I \Omega
\end{bmatrix} \quad (3.7)
\]

where \( F \) is the force, \( V \) the velocity, \( \Omega \) the angular velocity and \( \tau \) the moment. Introduce an inertial frame, \( E \), with coordinates \( x, y, z \) and a body fixed frame, \( B \), with coordinates \( x', y', z' \) where the \( x' \)-axis is aligned with one of the arms of the Y4 as seen in figure 3.3. The transformation of a vector defined in the body frame to the inertial frame using Euler angles is given by the rotation matrix

\[
R = \begin{bmatrix}
c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\
c_\theta s_\psi & c_\theta s_\phi c_\psi + s_\phi s_\psi & c_\phi s_\phi c_\psi - s_\phi s_\psi \\
-s_\theta & -s_\phi c_\theta & c_\phi c_\theta
\end{bmatrix}
\quad (3.8)
\]

where \( c_x \) and \( s_x \) are shorthand for \( \cos(x) \) and \( \sin(x) \), respectively. Using (3.8) the equations in (3.7) can be rewritten with position and velocity expressed in the inertial frame as

\[
\begin{align*}
\dot{X} &= V \\
m\dot{V} &= RF - R\Omega \times mV \\
I\dot{\Omega} &= -\Omega \times I\Omega + \tau
\end{align*} \quad (3.9)
\]
The moment applied to the air frame by the rotors can be expressed in terms of the thrust produced by the rotors. By using figure 3.4 and the fact that the thrust is proportional to the square of the angular velocity of the rotors, the moment around the different axis can be expressed as

\[
\tau = \begin{bmatrix}
0 & \frac{\sqrt{3}}{2} L k_{ts} \cos(\kappa) & \frac{\sqrt{3}}{2} L k_{ts} \cos(\kappa) & 0 \\
-L k_{ts} \cos(\kappa) & \frac{1}{2} L k_{ts} \cos(\kappa) & \frac{1}{2} L k_{ts} \cos(\kappa) & 0 \\
L k_{ts} \sin(\kappa) & L k_{ts} \sin(\kappa) & L k_{ts} \sin(\kappa) & -k_{dl}
\end{bmatrix} \begin{bmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_3^2 \\
\omega_M^2
\end{bmatrix}
\] (3.10)

where \(k_{ts}\) is the thrust constant of the small rotor, \(k_{dl}\) is the drag constant for the main rotor and \(\omega_i\) is the angular velocity of rotor \(i\). The rows in (3.10) corresponds to moment in roll, pitch and yaw respectively.
CHAPTER 3. DYNAMIC MODEL

Figure 3.4. The Y4 seen from above with the vertical force components marked with red circles. The horizontal force components produced by the boom rotors are also marked out with red arrows. Using the angles and the boom length the moment produced around the different axis can be calculated.

The force along the $z'$ axis in the body frame can be expressed as

$$ F = k_{ts} \cos(\kappa)(\omega_1^2 + \omega_2^2 + \omega_3^2) + k_{tl} \omega_M^2 + MgR^{-1} \hat{z}' $$

(3.11)

where $g$ is the gravity constant. $\hat{z}'$ denotes the unit vector in the $z'$ direction in the body-fixed frame. Note that $R^{-1} \hat{z}' = \hat{z}$. Taking (3.11) and removing the gravitational term and combining it with (3.10) the forces and moments produced by the rotors can be expressed together in matrix form as

$$
\begin{bmatrix}
T \\
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix} =
\begin{bmatrix}
 k_{ts} \cos(\kappa) & k_{ts} \cos(\kappa) & k_{ts} \cos(\kappa) & k_{tl} \\
 0 & -\frac{\sqrt{3}}{2} Lk_{ts} \cos(\kappa) & \frac{\sqrt{3}}{2} Lk_{ts} \cos(\kappa) & 0 \\
 -Lk_{ts} \sin(\kappa) & \frac{3}{2} Lk_{ts} \cos(\kappa) & \frac{3}{2} Lk_{ts} \cos(\kappa) & 0 \\
 Lk_{ts} \sin(\kappa) & Lk_{ts} \sin(\kappa) & Lk_{ts} \sin(\kappa) & -k_{dl}
\end{bmatrix}
\begin{bmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_3^2 \\
\omega_M^2
\end{bmatrix}
$$

(3.12)

The thrust constants $k_{ts}$ and $k_{tl}$ for the small and large rotor, as well as the drag constants $k_{ds}$ and $k_{dll}$ were calculated using the equations

$$
\begin{cases}
 k_t = \frac{1}{4} C_T \rho A D^2 \omega^2 \\
 k_d = \frac{1}{8} C_Q \rho A D^3 \omega^2
\end{cases}
$$

(3.13)

Combining equations (3.5), (3.7), (3.8) and (3.9) the final equations of motion with position and velocity expressed in the inertial frame can be stated as
CHAPTER 3. DYNAMIC MODEL

\[
\begin{align*}
\ddot{x} &= \frac{1}{M}(a_1 \dot{c}_\psi c_\psi + a_2 (\dot{c}_\phi \dot{c}_\theta c_\psi - c_\phi \dot{s}_\psi) + (a_3 + T)(c_\phi \dot{s}_\theta c_\psi + s_\phi \dot{s}_\psi)) \\
\ddot{y} &= \frac{1}{M}(a_1 \dot{c}_\theta c_\psi + a_2 (\dot{c}_\phi \dot{s}_\theta s_\psi + c_\phi c_\psi) + (a_3 + T)(c_\phi \dot{s}_\theta s_\psi - s_\phi c_\psi)) \\
\ddot{z} &= -g + \frac{1}{M} \left( (-a_1 s_\theta + a_2 s_\phi c_\theta) + (a_3 + T)(c_\phi c_\theta) \right) \\
\ddot{\phi} &= \frac{\dot{\theta} \dot{\psi}}{I_{yy} - I_{zz}} - \frac{J_{pl}}{I_{yy}} \dot{\omega}_M + \frac{\tau_1}{I_{xx}} \\
\ddot{\theta} &= \frac{\dot{\phi} \dot{\psi}}{I_{zz} - I_{xx}} + \frac{J_{pl}}{I_{zz}} \dot{\omega}_M + \frac{\tau_2}{I_{xx}} \\
\ddot{\psi} &= \frac{\dot{\phi} \dot{\theta}}{I_{xx} - I_{yy}} + \frac{\tau_3}{I_{xx}} \quad (3.14)
\end{align*}
\]

where \( \phi, \theta \) and \( \psi \) are the roll, pitch and yaw angle and

\[
\begin{align*}
a_1 &= \Omega_z V_y - \Omega_y V_z \\
a_2 &= \Omega_x V_z - \Omega_z V_x \\
a_3 &= \Omega_y V_x - \Omega_x V_y \\
\end{align*}
\]  

(3.15)

3.4 Dynamics of the Rotors

The Y4 uses brushless DC-motors with well known equations which are

\[
\begin{align*}
L \frac{di}{dt} &= u - R_m i - k_e \omega \\
J_m \frac{d\omega}{dt} &= \tau_q - \tau_l \\
\end{align*}
\]  

(3.16)

where \( L \) is the inductance, \( i \) the current, \( u \) the voltage, \( R_m \) the motor resistance, \( k_e \) the motor constant, \( J_m \) the motor inertia, \( \tau_q \) the torque produced and \( \tau_l \) the motor load. Since small brushless DC motors has very low inductance the first equation in (3.16) can be approximated as:

\[
i = \frac{u}{R_m} - \frac{k_e \omega}{R_m} \\
\]  

(3.17)

Combining (3.17) with the second equation in (3.16) and using the fact that the torque is proportional to the current we have:

\[
J_m \frac{d\omega}{dt} = \frac{k_q}{R_m} u - \frac{k_q^2}{R_m} \omega - \tau_l \\
\]  

(3.18)

where \( k_q \) is the torque constant. Adding the rotor and using that the motor load is equal to the drag torque produced by the rotor gives the final rotor dynamics as

\[
\begin{align*}
\frac{d\omega}{dt} &= \frac{1}{J_{r}} - \frac{1}{J_{r}} \omega - \frac{d}{J_r} \omega^2 \\
\frac{1}{J_{r}} &= \frac{k_e}{R_m} \\
\end{align*}
\]  

(3.19)

The inertia of the propeller is much greater than that of the motor so \( J_m \) is replaced by \( J_p \) in (3.19).
Chapter 4

Control of The Y4

In this chapter the main contributions of this work will be derived and presented. Several control methods will be investigated and applied to the Y4 model built in Simulink, analyzing the performance. The focus will start on stabilizing the Y4 at hover conditions using a simple control approach and later add disturbances and different flight conditions that demand more advanced control laws.

Let’s first rewrite the equations in (3.14) on the form

\[ \dot{x} = f(x, u) \]  

where \( x \) is the state vector of the system with states defined by

\[
\begin{align*}
  x_1 &= x \\
  x_2 &= \dot{x}_1 \\
  x_3 &= y \\
  x_4 &= \dot{x}_3 \\
  x_5 &= z \\
  x_6 &= \dot{x}_5 \\
  x_7 &= \phi \\
  x_8 &= \dot{x}_7 \\
  x_9 &= \theta \\
  x_{10} &= \dot{x}_{10} \\
  x_{11} &= \psi \\
  x_{12} &= \dot{x}_{11}
\end{align*}
\]  

(4.2)

and \( u \) is the input vector with inputs

\[
\begin{align*}
  u_1 &= T \\
  u_2 &= \tau_1 \\
  u_3 &= \tau_2 \\
  u_4 &= \tau_3
\end{align*}
\]  

(4.3)
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\( f(x, u) \) can then be written as

\[
\begin{bmatrix}
  x_2 \\
  x_4 \\
  x_6 \\
  x_8 \\
  x_{10} \\
  x_{12}
\end{bmatrix} =
\frac{1}{m}(a_1 c_{x9} c_{x11} + a_2(s_{x7} s_{x9} c_{x11} - c_{x7} s_{x11}) + (a_3 + u_1)(c_{x7} s_{x9} c_{x11} + s_{x7} s_{x11}))
\]

\[
\frac{1}{m}(a_1 c_{x9} s_{x11} + a_2(s_{x7} s_{x9} s_{x11} + c_{x7} c_{x11}) + (a_3 + u_1)(c_{x7} s_{x9} s_{x11} - s_{x7} c_{x11}))
\]

\[
-g + \frac{1}{m}((-a_1 s_{x9} + a_2 s_{x7} c_{x9}) + (a_3 + u_1)(c_{x7} c_{x9}))
\]

\[
\frac{x_{10}}{I_{zz} - I_{xx}} + \frac{x_{12}}{I_{xx} - I_{yy}} (I_{yy} - I_{xx}) - \frac{x_9 x_{11}}{I_{xx}} - \frac{x_7 x_{11}}{I_{xx}} + \frac{u_1}{I_{xx}}
\]

\[
\frac{x_9 x_{11}}{I_{yy}} - \frac{x_7 x_{11}}{I_{yy}} + \frac{u_2}{I_{yy}}
\]

\[
\frac{x_7 x_{11}}{I_{yy}} - \frac{x_9 x_{11}}{I_{yy}} + \frac{u_3}{I_{yy}}
\]

(4.4)

where \( a_1, a_2 \) and \( a_3 \) are the same as in (3.15). (4.1) and (4.4) then holds the complete system dynamics and inputs in state space form.

4.1 Open Loop

In order to validate the model several simulations were done with no feedback. A simulation was done with an initial angular velocity of 90 deg/s around the roll axis. The result can be seen in figures 4.1 and 4.2. The Y4 clearly has non-desirable dynamics. Strong oscillations can be seen as well as the characteristic circulating motion. This is expected since the equations of motion is similar to the ones of an ordinary quadrotor where this behavior is well known [4]. The circulating motion origins from the gyroscopic terms in the equations for the \( \phi \) and \( \theta \) angles in (3.14)

\[
\begin{align*}
\dot{\theta} &= \frac{J_{pl} \omega_M}{I_{yy}} \\
\dot{\phi} &= \frac{J_{pl} \omega_M}{I_{xx}}
\end{align*}
\]

(4.5)

A positive angular velocity around the roll angle results in a positive contribution to the angular acceleration in the pitch and a positive angular velocity around the pitch angle results in a negative angular acceleration in the roll. The circulating oscillations decays into a limit cycle after some seconds. One can also note that the pitch angle does not fall back to zero, but instead oscillates around approx 2.5 degrees.
Figure 4.1. Plot of the roll and pitch angle when starting with an initial angular velocity of 90 deg/s around the roll axis
Figure 4.2. Plot of the roll and pitch angle when starting with an initial angular velocity of 90 deg/s around the roll axis. Here the circulating motion due to the gyroscopic effects can be seen.

### 4.2 PID Controller

In order to replicate the result in [6] where a simple PID controller was implemented to achieve stable hover, the first controller applied in the Simulink model was a PID controller. A PID controller consists of proportional, integral and derivative components. Measuring the error between the desired and current state, the proportional part of the controller is the ratio between the error and the output. The integral part is the integral of the error and the derivative part is the rate of change of the error. So a PID controller can be formulated as

\[
output = K_p e + K_i \int_0^t e \, dt + K_d \frac{de}{dt}
\]  

(4.6)

where \( e \) is the error and \( K_p, K_i \) and \( K_d \) are control parameters needed to be tuned in order to achieve the desired system response. However, its more common to
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express the controller in the Laplace domain, then it transforms to

\[ output = K_p e + \frac{1}{s} K_i e + sK_d e \]  

(4.7)

4.2.1 Altitude Control

For controlling the altitude a PID controller was added to the z direction and the control parameters was tuned by performing several simulations. Figure 4.3 shows the step response for a step in altitude when the control parameters was set to: \( K_p = 8, \ K_I = 0 \) and \( K_d = 7 \). The rise time is quite slow, but there is no overshoot and zero settling time. Note that since the integral part is zero this is just a PD controller.

![Figure 4.3. Step response in altitude when using a PD controller](image)

Figure 4.3. Step response in altitude when using a PD controller
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4.2.2 Attitude Control

Since the gyroscopic effects of the Y4 are very large due to the large main rotor it is natural to add a correction for this in the control. Looking at the equations in (3.14) the correction should be

\[ \frac{J_{pl}}{I_{xx}} \omega_M \frac{d\phi}{dt} \]

(4.8)

for the roll angle and

\[ -\frac{J_{pl}}{I_{yy}} \omega_M \frac{d\theta}{dt} \]

(4.9)

for the pitch angle. Since the gyroscopic effects when turning around the z axis are small no correction is needed. The PID controllers with the gyroscopic correction for the Euler angles in matrix form are

\[
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix} = (K_p + K_i \frac{1}{s} + K_d s) \begin{bmatrix}
\phi_{\text{error}} \\
\theta_{\text{error}} \\
\psi_{\text{error}}
\end{bmatrix} + \omega_M J_M I^{-1} \begin{bmatrix}
0 & -s & 0 \\
0 & s & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix}
\]

(4.10)

By using (4.10) and (3.18) the inputs can be converted to voltage and sent into the motors. The PID controllers was implemented on the orientation angles and their control parameters was tuned by performing several simulations. Figure 4.4 shows a simulation where the Y4 started with a tilt of 45 degrees in each direction and the goal was to bring back the angles to zero. The controller managed to do this relatively quickly and stabilize the Y4. However, these simulations were done without any noise on the feedback data and this is of course not reasonable. To account for this Gaussian noise with mean value 0 and variance 10 deg/s were added to the angular velocities in the feedback loop. The simulations were performed again and figure 4.5 shows that the controller is unable to stabilize the aircraft and bring back the angles to zero when noise is present. In order to handle noise and model uncertainties more sophisticated control methods need to be considered.
Figure 4.4. Plot showing the response when the initial angles were 45 degrees. The goal was to bring back the angles to zero.
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Figure 4.5. Plot showing the response when using a PID controller and noise is present. The controller does not manage to bring back roll, pitch and yaw angles to zero.

4.3 Lyapunov

One of the most common approaches when it comes to non-linear control methods is to use Lyapunov theory. It gives a simple framework with specific mathematical criterion in order to achieve stabilizing control laws. The basic principle is to find a function, usually called $V(x)$, that depends on the states of the system that should be stabilized. If this function fulfills specific mathematical requirements or if it is possible to choose the inputs in such a way so that it does, one is guaranteed that the origin is stable to some degree. The function $V(x)$ is then called a Lyapunov function. The main motivation behind this theory is that if the total energy of a system dissipates the system must be stable and a Lyapunov function shows that. Note that the function is only called a Lyapunov function if it fulfills the necessary requirements, otherwise it is only a candidate. The theorem for asymptotic stability is
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Theorem 4.1. Let $V(x)$ be a Lyapunov function candidate. $V : \Omega \rightarrow \mathbb{R}$ and $V \in C^1$. Let $\dot{x} = f(x)$, $f(0) = 0$ and $0 \in \Omega \in \mathbb{R}^n$. If $V(x)$ satisfies

1. $V(0) = 0$
2. $V(0) > 0 \quad \forall x \in \Omega, \quad x \neq 0$
3. $\dot{V}(x) < 0 \quad \forall x \in \Omega, \quad x \neq 0$

Then $x = 0$ is asymptotically stable.

If we also demand that $V(x)$ is radial unbounded we achieve global stability.

Theorem 4.2. Let $V(x)$ be a Lyapunov function candidate. $V : \Omega \rightarrow \mathbb{R}$ and $V \in C^1$. Let $\dot{x} = f(x)$, $f(0) = 0$ and $0 \in \Omega \in \mathbb{R}^n$. If $V(x)$ satisfies

1. $V(0) = 0$
2. $V(0) > 0 \quad \forall x \in \Omega, \quad x \neq 0$
3. $\dot{V}(x) < 0 \quad \forall x \in \Omega, \quad x \neq 0$
4. $V(x) \rightarrow \infty \quad \text{as} \quad ||x|| \rightarrow \infty$

Then $x = 0$ is globally asymptotically stable.

For proof the reader is referred to literature, i.e [8].

4.3.1 Attitude Control

The orientation dynamics are encapsulated in the six last equations in (4.4) and in order to stabilize the aircraft we want to stabilize roll, pitch and yaw with corresponds to the states $x_7, x_9$ and $x_{11}$. A first guess of a simple Lyapunov function candidate would be

$$V(x) = \frac{1}{2}(x_7^2 + x_8^2 + x_9^2 + x_{10}^2 + x_{11}^2 + x_{12}^2)$$

(4.11)

Clearly $V(x)$ satisfies the first two and the last requirement in theorem 4.2. Consider the time derivative of $V(x)$ which becomes

$$\dot{V}(x) = x_7x_8 + x_8\dot{x}_8 + x_9x_{10} + x_{10}\dot{x}_{10} + x_{11}x_{12} + x_{12}\dot{x}_{12}$$

$$= x_7x_8 + x_8(x_{10}\dot{x}_{12}c_1 - d_1x_{11} + \frac{u_2}{I_{xx}}) + x_9x_{10} +$$

$$+ x_{10}(x_8x_{12}c_2 + d_2x_8 + \frac{u_3}{I_{yy}}) + x_{11}x_{12} + x_{12}(x_8x_{10}c_3 + \frac{u_4}{I_{zz}})$$

(4.12)

$$= \{I_{xx} = I_{yy}\}$$

$$= x_7x_8 + x_8\frac{u_2}{I_{xx}} + x_9x_{10} + x_{10}\frac{u_3}{I_{yy}} + x_{11}x_{12} + x_{12}\frac{u_4}{I_{zz}}$$
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By choosing

\[
\begin{align*}
    u_2 &= -I_{xx}(x_7 + k_1 x_8) \\
    u_3 &= -I_{yy}(x_9 + k_2 x_{10}) \\
    u_4 &= -I_{zz}(x_{11} + k_3 x_{12})
\end{align*}
\] (4.13)

the time derivative becomes

\[
\dot{V}(x) = -k_1 x_8^2 - k_2 x_{10}^2 - k_3 x_{12}^2
\] (4.14)

which is semi-negative definite since \(\dot{V}(x) = 0\) even if \(x_7, x_9\) or \(x_{11}\) is \(\neq 0\). Thus, \(V(x)\) can not guarantee global asymptotic stability. However, one can use LaSalle’s theorem in order to pursue global stability.

**Theorem 4.3.** Let \(\Omega \subset D\) be a compact set that is positively invariant with respect to \(\dot{x} = f(x)\). Let \(V : D \rightarrow \mathbb{R}\) be a continuously differentiable function such that \(\dot{V}(x) \leq 0\) in \(\Omega\). Let \(E\) be the set of all points in \(\Omega\) where \(\dot{V}(x) = 0\). Let \(M\) be the largest invariant set in \(E\). Then every solution starting in \(\Omega\) approaches \(M\) as \(t \rightarrow \infty\).

Again, for proof see for example [8]. If one can show that the set \(E\) only contains the origin and \(V(X)\) is continuously differentiable, positive definite, radial unbounded and \(\dot{V}(x) = 0\) \(\forall x \in \mathbb{R}^n\), then the origin is globally asymptotically stable.

**Corollary 4.1.** Let \(x = 0\) be an equilibrium point for \(\dot{x} = f(x)\). Let \(V : \mathbb{R}^n \rightarrow \mathbb{R}\) be a continuously differentiable, radially unbounded, positive definite such that \(\dot{V}(x) = 0\) \(\forall x \in \mathbb{R}^n\). Let \(S = \{x \in \mathbb{R}^n | \dot{V}(x) = 0\}\) and suppose that no solution to \(\dot{x} = f(x)\) can stay identically in \(S\), other than the trivial solution \(x(t) \equiv 0\). Then the origin is globally asymptotically stable.

To find \(S\) in our case note that

\[
\dot{V}(x) = 0 \implies x_8 = x_{10} = x_{12} = 0
\]

Hence, \(S = \{x \in \mathbb{R}^6 | x_8 = x_{10} = x_{12} = 0\}\). If a solution, \(x(t)\), should belong identically to \(S\) this means that

\[
x_8 = x_{10} = x_{12} = 0 \implies \dot{x}_8 = \dot{x}_{10} = \dot{x}_{12} = 0 \implies \dot{x}_7 = \dot{x}_9 = \dot{x}_{11} = 0
\]

Using this fact and inserting the inputs from (4.13) in the six last equations in (4.4) they are reduced to

\[
\begin{align*}
    \dot{x}_7 &= 0 \\
    \dot{x}_8 &= -x_7 = 0 \\
    \dot{x}_9 &= 0 \\
    \dot{x}_{10} &= -x_9 = 0 \\
    \dot{x}_{11} &= 0 \\
    \dot{x}_{12} &= -x_{11} = 0
\end{align*}
\] (4.15)
Which shows that the only solution, \( x(t) \), that can belong identically to \( S \) is \( x(t) \equiv 0 \) and thereby global asymptotic stability is ensured.

The control laws in (4.13) was implemented in Simulink and simulations regarding attitude control were done since this is essential for stable flight. In figure 4.6 the initial roll and pitch angle are 10 degrees. One can see that the angles are brought back to zero, but it takes a large amount of time. The control parameters were set to \( k_1 = k_2 = k_3 = 10 \). Different values were tried for the parameters, but the performance could not reach a satisfying level. The reason for the very large settling time is the fact that the controller does not take the large gyroscopic effects into account, so the response when trying to move the aircraft around the roll and pitch axis becomes very slow. This shows that just because a Lyapunov function is found does not mean that the corresponding controller is satisfying or even suitable for the application.

![Figure 4.6](image.png)

**Figure 4.6.** The response when using the control inputs from equation (4.13), the system is stable but the response is very slow.
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4.4 Backstepping

In order to develop a more advance non-linear control law one can use backstepping. The backstepping technique uses Lyaponov theory to recursively calculate an origin stabilizing controller. The main idea of backstepping is to let certain state variables act as virtual inputs in order to achieve the desired system behavior. The backstepping technique has been used before on traditional quadrotors with good results [4].

4.4.1 Attitude Control

Start by considering the roll angle dynamics which are

\[
\begin{aligned}
    \dot{x}_7 &= x_8 \\
    \dot{x}_8 &= x_{10}x_{12}\left(\frac{I_{yy} - I_{zz}}{I_{xx}}\right) - \frac{J_{pl}}{I_{xx}}x_{10}\omega_M + \frac{u_1}{I_{xx}}
\end{aligned}
\]  

(4.16)

then consider the tracking error

\[ z_1 = x_{7d} - x_7 \]  

(4.17)

where \( x_{7d} \) is the desired roll angle. A possible Lyaponov function candidate is then

\[ V(z_1) = \frac{1}{2}z_1^2 \]  

(4.18)

since it is positive definite. The derivative becomes

\[ \dot{V}(z_1) = z_1(\dot{x}_{7d} - x_8) \]  

(4.19)

In order to stabilize \( z_1 \) we need the derivative to be at least semi negative-definite and we achieve this by introducing a virtual input in the form of state \( x_8 \) as

\[ x_8 = \dot{x}_{7d} - \lambda_1 z_1 \]  

(4.20)

where \( \lambda_1 \) is a positive constant. The derivative of \( V(z_1) \) then becomes

\[ \dot{V}(z_1) = -\lambda_1 z_1^2 \]  

(4.21)

which is semi negative-definite and \( V(z_1) \) then fulfills theorem 4.1. The next step is to introduce a variable change by letting

\[ z_2 = x_8 - \dot{x}_{7d} - \lambda_1 z_1 \]  

(4.22)

then consider the augmented Lyaponov function candidate

\[ V(z_1, z_2) = \frac{1}{2}(z_1^2 + z_2^2) \]  

(4.23)

The time derivative is
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\[ \dot{V}(z_1, z_2) = z_1 \dot{z}_1 + z_2 \dot{z}_2 \]
\[ = z_1(\dot{x}_{7d} - x_8) + z_2(\dot{x}_8 - \dot{x}_{7d} - \lambda_1 \dot{z}_1) \]
\[ = z_1(-\lambda_1 z_1 - z_2) + z_2(\dot{x}_8 - \dot{x}_{7d} - \lambda_1 (\lambda_1 z_1 - z_2)) \]
\[ = z_2 \dot{z}_8 + z_2 \lambda_1 (\lambda_1 z_1 + z_2) - z_1 \dot{z}_2 - \lambda_2 \dot{z}_2^2 \]
\[ = z_2(b_1 x_{10} x_{12} - b_2 x_{10} + \frac{b_1}{I_{xx}}) + z_2 \lambda_1 (\lambda_1 z_1 + z_2) - z_1 \dot{z}_2 - \lambda_1 \dot{z}_1^2 \]  

where \( b_1 = \frac{(I_{yy} - I_{xx})}{I_{xx}} \) and \( b_2 = \frac{J_{xx}}{I_{xx}} \omega_M \). By choosing \( u_1 \) equal to

\[ u_1 = I_{xx}(b_2 x_{10} - b_1 x_{10} x_{12} - \lambda_1 (\lambda_1 z_1 + z_2) + z_1 - \lambda_2 \dot{z}_1) \]
\[ = I_{xx}(b_2 x_{10} - b_1 x_{10} x_{12} - (x_8 - \dot{x}_{7d})(\lambda_1 + \lambda_2) + (1 + \lambda_1 \lambda_2)(x_{7d} - x_7)) \]  

the time derivative of \( V(z_1, z_2) \) becomes

\[ \dot{V}(z_1, z_2) = -\lambda_1 z_1^2 - \lambda_2 \dot{z}_2^2 \]  

which again is semi negative-definite and by using LaSalle’s theorem \( V(z_1, z_2) \) fulfills theorem 4.2. \( \lambda_2 \) is introduced in order to stabilize \( z_2 \). With the same approach \( u_2 \) and \( u_3 \) can be extracted as

\[ u_2 = I_{yy}(-b_4 x_8 - b_3 x_8 x_{12} - (\lambda_3 + \lambda_4)(x_{10} - \dot{x}_{9d}) + (1 + \lambda_3 \lambda_4)(x_{9d} - x_9)) \]  

\[ u_3 = I_{zz}(-b_5 x_8 x_{10} - (\lambda_5 + \lambda_6)(x_{12} - \dot{x}_{11d}) + (1 + \lambda_5 \lambda_6)(x_{11d} - x_{11})) \]  

where

\[
\begin{align*}
    b_3 &= \frac{(I_{xx} - I_{zz})}{I_{yy}} \\
    b_4 &= \frac{J_{xx}}{I_{yy}} \omega_M \\
    b_5 &= \frac{(I_{yy} - I_{xx})}{I_{yy}}
\end{align*}
\]  

The control laws take all the gyroscopic effects in account as well as the error and the change of the error. In order to tune the six control parameters, \[ [\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6] \] the response optimization tool in Simulink was used. Simulations was then performed in the same way as in the previous section where the goal was to bring back the orientation angles to zero from a 45 degree initial condition on each angle. The result is shown in figure 4.7. The angles are quickly brought back to zero. Especially the roll angle is brought back mush faster than when using a PID controller and thus yielding a better result. Noise was then added to the angular velocities in the same way as in section 4.2. Gaussian noise with mean value zero and variance 10 deg/s. As seen in figure 4.8 the controller manages to keep to aircraft stable despite the noise.
Figure 4.7. Simulation using backstepping controller. The goal was to bring back the orientation angles to zero from a 45 degree initial condition in roll, pitch and yaw.
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Figure 4.8. Simulation using backstepping controller. Gaussian noise with mean value 0 deg/s and variance 10 deg/s was added to the angular velocities. The controller manages to stabilize the aircraft despite the noise.

4.5 Integral Backstepping

To combine the good disturbance rejection provided by the backstepping controller with the robustness against model uncertainties achieved by integral action a so called integral backstepping controller can be considered. The derivation is similar to the Backstepping controller and the control law for the roll angle will be derived first.

4.5.1 Attitude Control

Again, first consider the roll angle and its error.

\[ e_1 = x_{7d} - x_7 \]

The error has its own dynamics
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\[ \frac{de_1}{dt} = \dot{x}_7d - x_8 \quad (4.30) \]

If we again let \( x_8 \) be our virtual control input and set for it a desirable behavior and introduce an integral term as

\[ x_{8d} = k_1 e_1 + \dot{x}_7d + \lambda_1 \Gamma_1 \quad (4.31) \]

where \( k_1 \) and \( \lambda_1 \) are positive constants and \( \Gamma_1 \) is integral of the roll error, \( \Gamma_1 = \int_0^t e_1 d\tau \). If one can achieve \( x_{8d} = x_8 \) the dynamics of \( e_1 \) would become

\[ \frac{de_1}{dt} = -k_1 e_1 - \lambda_1 \Gamma_1 \quad (4.32) \]

which is stable. However, we only can set a desirable behavior for the roll angular velocity and it has its own dynamics and error. Let \( e_2 = x_{8d} - x_8 \) be the error between the desired- and actual roll angular velocity, then the dynamics can be calculated as

\[ \frac{de_2}{dt} = \dot{x}_{8d} - \dot{x}_8 = k_1 (\ddot{x}_7d - x_8) + \ddot{x}_7d + \lambda_1 e_1 - \dot{x}_8 \quad (4.33) \]

Using (4.4) and replacing \( \dot{x}_8 \) results in

\[ \frac{de_2}{dt} = k_1 (\ddot{x}_7d - x_8) + \ddot{x}_7d + \lambda_1 e_1 - b_1 x_{10} x_{12} + b_2 x_{10} - \frac{u_2}{I_{xx}} \quad (4.34) \]

where our actual control input has appeared. By using (4.30) and (4.31) we can rewrite the dynamics of \( e_1 \) and \( e_2 \) as

\[ \frac{de_1}{dt} = e_2 - k_1 e_1 - \lambda_1 \Gamma_1 \quad (4.35) \]

\[ \frac{de_2}{dt} = k_1 (e_2 - k_1 e_1 - \lambda_1 \Gamma_1) + \ddot{x}_7d + \lambda_1 e_1 - b_1 x_{10} x_{12} + b_2 x_{10} - \frac{u_2}{I_{xx}} \quad (4.36) \]

From (4.35) we see that we need \( e_2 \) to be stable and converge to zero in order for \( e_1 \) to converge. So a desirable behavior for \( e_2 \) would be

\[ \frac{de_2}{dt} = -k_2 e_2 - c_1 \quad (4.37) \]

where \( c_2 \) is a positive constant. To obtain this the input \( u_2 \) can be chosen as

\[ u_2 = I_{xx} ((1 - k_1^2 + \lambda_1) e_1 + (k_1 + k_2) e_2 - c_1 \lambda_1 \Gamma_1 + \ddot{x}_7d - b_1 x_{10} x_{12} + b_2 x_{10}) \quad (4.38) \]

Similarly, the control laws for the pitch and yaw can be obtained as

\[
\begin{cases}
  u_3 = I_{yy} ((1 - k_3^2 + \lambda_2) e_3 + (k_3 + k_4) e_4 + \ddot{x}_{9d} - b_3 x_{8x} x_{12} - b_4 x_8) \\
  u_4 = I_{zz} ((1 - k_5^2 + \lambda_3) e_5 + (k_5 + k_6) e_6 + \ddot{x}_{11d} - b_5 x_8 x_{10})
\end{cases}
\quad (4.39)
\]
4.5.2 Altitude Control

Consider the error in the $z$-direction

$$e_7 = x_{5d} - x_5$$

Its dynamics are

$$\dot{e}_7 = \dot{x}_{5d} - x_6 \quad (4.40)$$

If we treat $x_6$ as our virtual input and set for it a desirable behavior as

$$x_{6d} = c_7e_7 + \lambda_4 \Gamma_4 \quad (4.41)$$

$x_6$ has its own error and dynamics as

$$\left\{ \begin{array}{l}
    e_8 = x_{6d} - x_6 \\
    \dot{e}_8 = c_7\dot{e}_7 + \lambda_4 e_7 - \dot{x}_6
\end{array} \right. \quad (4.42)$$

Inserting the equation for $\dot{x}_6$ from (4.4) results in

$$\dot{e}_8 = c_7\dot{e}_7 + \lambda_4 e_7 + g - \frac{1}{M} \left((-a_1s_{x_9} + a_2s_{x_7}c_{x_9}) + (a_3 + u_1)(c_{x_7}c_{x_9})\right) \quad (4.43)$$

Using (4.40) and (4.41), (4.43) can be rewritten as

$$\dot{e}_8 = c_7\dot{e}_7 + \lambda_4 e_7 + g - \frac{1}{M} \left((-a_1s_{x_9} + a_2s_{x_7}c_{x_9}) + (a_3 + u_1)(c_{x_7}c_{x_9})\right) \quad (4.44)$$

In order to stabilize $e_8$, $u_1$ can be chosen as

$$u_1 = \frac{M}{c_{x_7}c_{x_9}} \left(-\frac{a_3 c_{x_7}c_{x_9}}{M} - \frac{1}{M} (-a_1s_{x_9} + a_2s_{x_7}c_{x_9}) + g + \dot{x}_{5d} + (1-c_7^2 + \lambda_4)e_7 + (c_7 + c_8)e_8 - c_7\lambda_4 \Gamma_4\right) \quad (4.45)$$

This results in the dynamics for $e_8$ as

$$\dot{e}_8 = -c_8e_8 - e_7 \quad (4.46)$$

**Stability**

To ensure stability of the controller Lyapunov theory is used as in the previous sections. Consider the Lyapunov function candidate

$$V = \frac{1}{2}(\lambda_1 \Gamma_1^2 + ... + \lambda_4 \Gamma_4^2 + e_1^2 + ... + e_8^2) \quad (4.47)$$

It is positive definite and the derivative becomes
\[ \dot{V} = -c_1 e_1^2 - c_2 e_2^2 + \ldots - c_8 e_8^2 \] (4.48)

which is semi-negative definite. Using theorem 4.1 and 4.2 and LaSalle’s theorem one can assure global asymptotic stability. The controller was tuned using the response optimization tool in Simulink and the values of the control parameters are shown in table 4.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
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</tr>
<tr>
<td>( c_2 )</td>
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</tr>
<tr>
<td>( \lambda_1 )</td>
<td>1.2</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>62</td>
</tr>
<tr>
<td>( c_4 )</td>
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</tr>
<tr>
<td>( \lambda_2 )</td>
<td>0.0</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>15</td>
</tr>
<tr>
<td>( c_6 )</td>
<td>13</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>9.4</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>6.9</td>
</tr>
<tr>
<td>( c_8 )</td>
<td>10</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>2.6</td>
</tr>
</tbody>
</table>

### 4.5.3 Position Control

In order for the Y4 to move around in the \( x \)-\( y \) plane it has to alter its \( \phi \) and \( \theta \) angle. When \( \phi \) and \( \theta \) are nonzero the vertical force in the body frame transforms into one vertical and one horizontal component in the Inertial frame, resulting in a net force perpendicular to the \( z \)-axis. This vector component needs to be oriented so that it points towards the desired \((x, y)\) coordinates in order to reach the desired point. So the input to the position controller presumable could be the angles required in order to reach the desired position in a reasonable time.

The derivation of an integral backstepping controller for the position is done the same way as for attitude and altitude. Consider the error in \( x \) and \( y \) position and their dynamics

\[
\begin{align*}
e_9 &= x_{1d} - x_1 \\
e_{11} &= x_{3d} - x_3
\end{align*}
\Rightarrow
\begin{align*}
\dot{e}_9 &= \dot{x}_{1d} - x_2 \\
\dot{e}_{11} &= \dot{x}_{3d} - x_4
\end{align*}
\] (4.49)

Set a desirable behavior for \( x_2 \) and \( x_4 \) as

\[
\begin{align*}
x_{2d} &= c_9 e_9 + \dot{x}_{1d} + \lambda_5 \Gamma_5 \\
x_{4d} &= c_{11} e_{11} + \dot{x}_{3d} + \lambda_6 \Gamma_6
\end{align*}
\] (4.50)
CHAPTER 4. CONTROL OF THE Y4

The error in \( x \) and \( y \) velocity components and their dynamics then are

\[
\begin{align*}
\dot{e}_{10} &= x_{2d} - x_2 \\
\dot{e}_{12} &= x_{4d} - x_4
\end{align*}
\]

Inserting the equations of motion for the \( x \) and \( y \) coordinates from 3.14 result in

\[
\begin{align*}
\dot{e}_{10} &= c_9 \dot{e}_9 + \ddot{x}_{1d} + \lambda_5 e_9 - \frac{1}{M} (a_1 x_9 c_{x11} + a_2 (s_{x7} s_{x9} c_{x11} - c_{x7} s_{x11})) + (a_3 + u_1) (c_{x7} s_{x9} c_{x11} + s_{x7} s_{x11})) \\
\dot{e}_{12} &= c_{11} \dot{e}_{11} + \ddot{x}_{3d} + \lambda_6 e_{11} - \frac{1}{M} (a_1 x_9 s_{x11} + a_2 (s_{x7} s_{x9} s_{x11} + c_{x7} c_{x11})) + (a_3 + u_1) (c_{x7} s_{x9} s_{x11} - s_{x7} c_{x11})
\end{align*}
\]  

(4.52)

In order to try and simplify this expression, small angle approximation can be applied. Then \( \cos(x) \approx 1, \sin(x) \approx x \) and \( \sin^2(x) \approx 0 \) and (4.52) then reduces to

\[
\begin{align*}
\dot{e}_{10} &= c_9 \dot{e}_9 + \ddot{x}_{1d} + \lambda_5 e_9 - \frac{1}{M} (a_1 - a_2 x_{11} + (a_3 + u_1) u_9) \\
\dot{e}_{12} &= c_{11} \dot{e}_{11} + \ddot{x}_{3d} + \lambda_6 e_{11} - \frac{1}{M} (a_1 x_{11} + a_2 + (a_3 + u_1) u_{x_7})
\end{align*}
\]  

(4.53)

Note that our inputs are the desired angles \( u_{x7} \) and \( u_{x9} \). To achieve stability for the errors one can choose

\[
\begin{align*}
u_{x7} &= \frac{1}{(a_3 + u_1)} (a_2 x_{11} - a_1 + M (c_9 \dot{e}_9 + \ddot{x}_{1d} + \lambda_5 e_9 + c_{10} \dot{e}_{10} + e_9)) \\
u_{x9} &= \frac{1}{(a_3 + u_1)} (a_1 x_{11} + a_2 + M (c_{11} \dot{e}_{11} + \ddot{x}_{3d} + \lambda_6 e_{11} + c_{12} \dot{e}_{12} + e_{11}))
\end{align*}
\]  

(4.54)

Using 4.49 and 4.50, 4.54 can be rewritten as

\[
\begin{align*}
u_{x7} &= \frac{1}{(a_3 + u_1)} (a_2 x_{11} - a_1 + M ((1 - c_9^2 + \lambda_5) e_9 + (c_9 + c_{10}) \dot{e}_{10} + \ddot{x}_{1d}) \\
u_{x9} &= \frac{1}{(a_3 + u_1)} (a_1 x_{11} + a_2 + M ((1 - c_{11}^2 + \lambda_6) e_{11} + (c_{11} + c_{12}) \dot{e}_{12} + \ddot{x}_{3d})
\end{align*}
\]  

(4.55)

Note also that the control laws in (4.55) depend on control input \( u_1 \). The control laws can be proven to be global asymptotically stable using the same arguments as in previous section.

The controller was tuned using the response optimization tool in Simulink and the values for the control parameters is shown in table 4.2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<td>( c_9 )</td>
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<tr>
<td>( c_{10} )</td>
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</tr>
<tr>
<td>( \lambda_5 )</td>
<td>0.5</td>
</tr>
<tr>
<td>( c_{11} )</td>
<td>1.6</td>
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<tr>
<td>( c_{12} )</td>
<td>0.029</td>
</tr>
<tr>
<td>( \lambda_6 )</td>
<td>0.016</td>
</tr>
</tbody>
</table>
Chapter 5

Simulations

In this chapter different simulations regarding the maneuverability of the Y4 will be presented. The control used will be the integral backstepping controller derived in the previous chapter.

5.1 Maneuvers

5.1.1 Hover and Attitude Control

It is crucial for a multirotor to be able to hover, despite noise and disturbance. Several simulations regarding this were done with the integral backstepping controller in order to investigate how well the attitude of the Y4 can be controlled. Figure 5.1 shows a simulation when the goal was to pitch, roll and yaw the Y4 to 15 degrees and back to zero, this without noise. In figure 5.2 Gaussian noise with variance 10 deg/s was added to the angular velocities. The controller manages to maneuver and keep the Y4 stable, despite the noise.
Figure 5.1. The Y4 moves to 15 degree in roll, pitch and yaw and then back to zero using the integral backstepping controller.
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Figure 5.2. The backstepping controller manages to maneuver the Y4 to 15 degree in roll, pitch and yaw despite added noise.

Figure 5.3 shows a simulation when the roll angle was excited to 45 degrees while keeping the yaw and altitude at zero. It can be seen that the yaw angle cannot be held at zero, this is due to the lack of bandwidth in the boom rotors. Figure 5.4 shows the rotor velocities under this maneuver and it can be seen that the right back motor is coming to a stand still. The controller tries to compensate, but since the motor cannot go into reverse the controller fails. Figure 5.5 shows the angular velocities of the Y4 and it seems like a critical angular velocity for the roll and pitch angle are about 0.14 rad/s. If the aircraft rotates faster than this the angles and altitude are not fully controllable and the aircraft cannot be flown safely. In figure 5.6 two simulations are shown, one when the controller was tuned to be more gentle and one when the controller was more aggressive. Both controllers manage to keep the aircraft stable, but the angles are not fully controllable for the more aggressive controller. Again, it can be seen that about 0.14 rad/s seems to be a critical value for the angular velocities and this limits the Y4 attitude maneuverability. Starting from the angles equal to zero the Y4 can reach about 30 degrees in roll, pitch and yaw in 5 seconds. In these simulations no noise was added since the goal was to investigate the dynamics of the aircraft.
Figure 5.3. The Y4 move to 45 degree in roll and pitch. It can be seen that the yaw angle cannot be held at zero.
Figure 5.4. Rotor velocities when the Y4 tries to achieve 45 degrees in roll and pitch.
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Figure 5.5. Angular velocities when the Y4 tries to achieve 45 degrees in roll and pitch.

Figure 5.6. Simulation when the roll and pitch angle are excited to 45 degrees. In the left picture a more aggressively tuned controller is used and in the right a more gentle controller is used. Both controllers keeps the Y4 stable but the angles can not be controlled fully if the angular velocity exceeds approximately 0.14 rad/s.
5.1.2 Simultaneous Vertical and Horizontal Translation

In figure 5.7 the Y4 is moved in a square in space with side 1 m while increasing its altitude 1 m when traveling each side, the yaw angle is kept at zero. And in figure 5.9 the Y4 performs the opposite maneuver, moving in a square but with opposite direction and decreasing its altitude. This shows that the Y4 can alter its altitude while moving in the horizontal plane. It can also be seen that the Y4 is much more responsive in the vertical direction than the horizontal. This due to the large main rotor, which contributes with the majority of the lift, and also since there are no substantial gyroscopic effects in the vertical direction. Figure 5.8 shows the rotor velocities when performing the maneuver and it can be seen that the controller is far from being saturated by the boom rotors.

Figure 5.7. The Y4 moves in a square in the horizontal plane while increasing the altitude.
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Figure 5.8. Rotor velocities while moving in a square in the horizontal plane while increasing the altitude.

Figure 5.9. The Y4 moves in a square in the horizontal plane while decreasing its altitude.
5.1.3 Yaw Performance

To get a sense of the yaw performance and dynamics one simulation were done when the Y4 was told to rotate 180 degrees and back and then rotate 180 degrees and back in the opposite direction. The simulation is shown in figure 5.10 and in figure 5.11 the motor velocities during the maneuver are shown. It can be seen that control bandwidth is asymmetrical and the controller is closer to being saturated when moving in the negative direction around the yaw axis. This since the motor velocities depend on the direction of rotation hence, to achieve position rotation the main rotor needs to slow down while the boom rotors increases their velocity and vice versa. Since the dynamics of the large rotor are completely different from the smaller boom rotors the result is expected.

Figure 5.10. The yaw angle is excited to 180 degrees and -180 degrees.
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Figure 5.11. Motor velocities when the yaw angle is excited to 180 degrees and then to -180 degrees. The controller is closer to being saturated when the Y4 is moving around the $\hat{z}$ axis in the negative direction.

5.1.4 Maximum Attitude Angles and Turn Radius

In order to investigate the maximum roll and pitch angle the Y4 can obtain while keeping the altitude constant simulations were performed when the pitch and roll angle were increased gradually until the controller became saturated and could not retain the altitude. The simulations showed that the maximum angle in roll or pitch while keeping the altitude constant were about 50 degrees. This can be used to give an indication of the turn radius of the Y4. If the Y4 was at level flight with velocity 5 m/s and a banking angle, i.e. roll angle, of 45 degrees the turn radius becomes

$$r = \frac{v^2}{g \tan \theta} \approx 2.55 \text{ m}$$  \hspace{1cm} (5.1)

The velocity is chosen low so the aerodynamic effects can be neglected and the assumption that the Y4 can keep constant velocity can be made. 45 degree bank angle was used so the aircraft would have control bandwidth left to be able to move back to level flight. No deeper investigation could be made about the turn radius since the mathematical model used did not take aerodynamics into account, so the result is just meant to be an indication of the turn radius of the Y4 at low speeds.
Chapter 6

Conclusion and Future Work

The main conclusion of this work is that simulations show that the Y4 can be controlled in a reasonable way, but compared with a standard quadrotor it is less maneuverable and less responsive. The attitude control is quite slow due to the lack of control bandwidth in the boom rotors and the large gyroscopic effect from the main rotor. In order to keep the Y4 at constant altitude while tilting in the roll and pitch direction one can not exceed approximately 0.14 rad/s in angular velocity. This can narrow the applicable area of usage for the Y4. Also the control bandwidth around the yaw axis is asymmetrical, hence the maximum rotation speed depends in the direction of rotation. The Y4 show more dynamics of an ordinary helicopter, which is expected because of the larger main rotor.

The Y4 configuration may be an alternative if the demand is power efficiency or lift power and not high maneuverability. For example, if one need to be able to surveil an area for a long time or transport goods, the ordinary choice might be an unmanned airplane since they often have a far longer range than a quadrotor. But if one also wants or need the benefits of being able to land vertically and hover the Y4 might be an alternative. In order to perform tasks like transportation and surveillance, long operation time or large lift power or both are often more important than high performance when it comes to maneuverability. This makes the Y4 more suitable for these kinds of tasks than an ordinary quadrotor. But for tasks where the aircraft needs to be able to respond extremely fast, the ordinary quadrotor might be preferable. For example in military applications where the aircraft might have to avoid different threats and change speed and direction fast. The Y4 might not be able to replace the ordinary quadrotor configuration, but instead it might be seen as an alternative, so it comes down to choose the right configuration for the right task.

Future work would be to test the Y4 configuration with different motors and rotors in order to see if the handling could be improved without increasing power consumption. The authors in [6] clearly states that no deep investigation about the chosen hardware were done. This probably means that the power consumption can be decreased even more if the design and hardware were optimized. The fact
that the authors found indication that the Y4 consumes less power than the standard configuration without any optimization, and that the work in this thesis show that the Y4 can be controller motivates deeper research regarding optimization of hardware and design.

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Bibliography


