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Peak Covariance Stability of Kalman Filtering with Markovian Packet Losses

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Abstract—In this paper, we consider state estimation using a Kalman filter of a linear time-invariant process over an unreliable network. The stability of Kalman filtering with random packet losses is studied, where the packet losses are modeled by the Gilbert-Elliott channel model and the stability is measured by the so-called peak covariance stability introduced in [1]. We give two sufficient conditions for the peak covariance stability: one combined with a numerical method provides an accurate criterion, and the other is in a simple form and easy to check; both of which are shown to be less conservative than existing works in practice. Numerical examples demonstrate the effectiveness of our result compared with relevant literature.

I. INTRODUCTION

Wireless communication has been commonly used nowadays in sensor networks and networked control systems for a wide spectrum of applications, such as environmental monitoring, health care, smart building, intelligent transportation and power grid, whereas channel fading ubiquitously exist in wireless propagation. Random packet dropouts, arising from channel fading and congestion, may deteriorate the performance of networked control systems. State estimation, offering the control unit with estimates of the system state based on the system outputs observed, is embedded in many networked control applications and is typically implemented recursively using the Kalman filter. In this work, we are concerned about the stability of Kalman filtering over random packet losses due to unreliability of the wireless propagation medium.

A burst of interest in the problem of Kalman filtering over lossy networks has been seen after the pioneering work [2], where *Sinopoli et al.* studied what the optimal state estimator is subject to an independent and identically distributed Bernoulli packet losses and how the packet loss process affects the stability of Kalman filtering. They proved that there exists a certain critical arrival probability for the packet losses, beyond which, $\mathbb{E}[P_k]$, the expected value of the error covariance matrix, is unbounded. An upper and lower bound for this critical value were given for general systems, and fortunately the lower bound is tight for some special cases, such as the observation matrix is invertible, and the system has a single unstable eigenvalue. *Mo and Sinopoli* [3], [4] and *Plarre and Bullo* [5] investigated the critical value under some less constrictive conditions for a wider class of systems. It is worthwhile to note that a surprising result of [3] revealed that, for the so-called non-degenerate systems, the lower bound for the critical

arrival probability is tight. Essentially, the probabilistic nature of P_k is fully captured by its probability distribution function. Motivated by this, *Shi et al.* [6] considered the same problem, but from a probabilistic perspective.

At the same time, to characterise the temporal correlation nature of realistic communication channels, the celebrated Gilbert-Elliott model [7], [8] that describes time-homogeneous Markovian packet losses was introduced. *Huang and Dey* [1] considered the similar problem with Markovian packet losses. To aid the analysis, they introduced the concept of peak covariance and focused on its stability. A sufficient condition for this stability was proposed. However, simulations showed conservativeness of their result for a general vector system. Their work was followed and improved by [9], [10]. Parallel to this, *You et al.* [11] proved that the stability at packet reception times is equivalent to the stability at sampling times (usual stability). For some certain classes of linear systems, necessary and sufficient conditions for the usual stability were derived.

In this paper, we will revisit the problem of peak covariance stability with Markovian packet losses, which has been studied by [1], [9], [10], [12], etc. Research along the pioneering work of [1] is still incomplete so far. We present this incompleteness mainly from three-folds:

- 1) It is shown by numerical examples that a variety of sufficient conditions for peak covariance stability in the existing literature only apply to relatively reliable channel models, where the failure rate must be small. This type of model is often invalid in reality. In this sense, the existing results are conservative.
- 2) From the point of view of computation, the existing results are also incomplete: 1. All aforementioned results have the difficulty in calculating a series of matrix norms; 2. Some results, such as Theorem 6 of [1] and Theorem 1 of [10], relied on some heuristic coefficients – how to appropriately choose them is still open.
- 3) The relationship between the peak covariance stability and usual stability was discussed in [1]. However, for general vector case, the explicit relationship is yet unknown.

This paper, as a more close look at this topic, attempts to (partially) solve these problems. Following the trajectory of previous works, this paper is a step forward. It gives two sufficient conditions for the peak covariance stability: one combined with a numerical method provides with an accurate criterion, the other is in a simple form and easy to check; both of which are shown to be less conservative than existing works in practice. The main part of this part culminates with

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numerical examples, which demonstrates effectiveness of our result compared with relevant literature.

The remainder of the paper is organized as follows. Section II presents the problem setup. Section III introduces the main results. Numerical examples are presented in Section IV. Conclusions and future work are provided in the end.

Notations: $\mathbb{N}$ is the set of positive integers. $k \in \mathbb{N}$ is time index. $\mathbb{S}_+^n$ is the set of n by n symmetric positive semi-definite matrices. For random variables, $\sigma(\cdot)$ is the σ -algebra generated by random variables. For a matrix $X \in \mathbb{R}^{n \times n}$, $\sigma(X)$ denotes the spectrum of X , i.e., $\sigma(X) = \{\lambda : \det(\lambda I - X) = 0\}$, and λ_X denotes the eigenvalue of X that has the largest magnitude. For any two finite set $\mathcal{A}$ and $\mathcal{B}$, we define $\mathcal{AB} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$. Moreover, $\|X\|$ means the 2-norm on $\mathbb{R}^n$ or the induced 2-norm on $\mathbb{R}^{n \times n}$.

II. PROBLEM SETUP

Consider a linear time-invariant (LTI) process:

$$x_{k+1} = Ax_k + w_k, \quad (1)$$

$$y_k = Cx_k + v_k, \quad (2)$$

where $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$, $x_k \in \mathbb{R}^n$ is the process state vector, $y_k \in \mathbb{R}^m$ is the observation vector, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero-mean Gaussian random vectors with $\mathbb{E}[w_k w_j'] = \delta_{kj} Q$ ($Q \geq 0$), $\mathbb{E}[v_k v_j'] = \delta_{kj} R$ ($R > 0$), $\mathbb{E}[w_k v_j'] = 0 \forall j, k$. The δ_{kj} is the Kronecker delta function with $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise. The initial state x_0 is a zero-mean Gaussian random vector that is uncorrelated with w_k and v_k and has covariance $P_0 \geq 0$. We assume that the pair (C, A) is detectable and $(A, \sqrt{Q})$ is stabilizable. It can be seen that, by applying a similarity transformation, the unstable and stable modes of the LTI system are decoupled. An open-loop prediction of the stable mode ever has a bounded estimation error covariance, therefore, this mode does not play any key in the problem considered below. Without loss of generality, we henceforth assume that all of the eigenvalues of A have magnitudes not less than 1. Then certainly A is invertible.

Fig. 1: Estimation over a fading channel

In this work, we consider an estimation scheme illustrated in Fig. 1, where the raw measurements of the sensor $\{y_k\}_{k \in \mathbb{N}}$ are transmitted to the estimator over an unreliable communication channel. Packets may be dropped randomly when they are transmitted via the channel. We assume that the channel is the Gilbert-Elliott channel, wherein the packet loss process is modelled by a time-homogeneous two-state Markov chain. Denote by $\gamma_k \in \{0, 1\}$ the arrival of y_k at time k : If $\gamma_k = 1$, it indicates that y_k arrives error-free at the estimator; otherwise $\gamma_k = 0$. Then $\{\gamma_k\}_{k \in \mathbb{N}}$ is the state of the Markov chain with

initial condition, without loss of generality, $\gamma_0 = 1$, which has a transition probability matrix as follows

$$\mathbf{P} = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}, \quad (3)$$

where $\mathbf{P}_{ij} = \mathbb{P}(\gamma_{k+1} = j | \gamma_k = i)$, and p and q are respectively called failure rate and recovery rate. To make sure of a meaningful channel model, both p and q are assumed to be strictly positive and less than 1, so that the packet loss process $\{\gamma_k\}_{k \in \mathbb{N}}$ is ergodic. Other effects introduced by, such as quantization and data rate, bit errors, and random delays, are not considered in this work.

Define $\mathcal{F}_k$ as the filtration generated by all the measurements received by the estimator up to time k , i.e., $\mathcal{F}_k \triangleq \sigma(\gamma_t y_t, \gamma_t; 1 \leq t \leq k)$ and $\mathcal{F} = \sigma(\cup_{k=1}^{\infty} \mathcal{F}_k)$. We will use a triple $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the common probability space for all random variables. For a set $\mathcal{A} \in \mathcal{F}$, $1_{\mathcal{A}}$ denotes the indicator function of $\mathcal{A}$, i.e., $1_{\mathcal{A}} = 1$ if $\omega \in \mathcal{A}$, otherwise 0. The estimator computes $\hat{x}_{k|k}$, the minimum mean-squared error estimate, and $\hat{x}_{k+1|k}$, the one-step prediction, according to $\hat{x}_{k|k} = \mathbb{E}[x_k | \mathcal{F}_k]$ and $\hat{x}_{k+1|k} = \mathbb{E}[x_{k+1} | \mathcal{F}_k]$. Let $P_{k|k}$ and $P_{k+1|k}$ be the corresponding estimation and prediction error covariance matrices, i.e., $P_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(\cdot)'] | \mathcal{F}_k]$ and $P_{k+1|k} = \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(\cdot)'] | \mathcal{F}_k]$, which are computed recursively via a modified Kalman filter [2]:

$$K_k = P_{k|k-1} C' [C P_{k|k-1} C' + R]^{-1}, \quad (4)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \gamma_k K_k (y_k - C \hat{x}_{k|k-1}), \quad (5)$$

$$P_{k|k} = (I - \gamma_k K_k C) P_{k|k-1}, \quad (6)$$

$$\hat{x}_{k+1|k} = A \hat{x}_{k|k}, \quad (7)$$

$$P_{k+1|k} = A P_{k|k} A' + Q. \quad (8)$$

It can be seen that $\hat{x}_{k|k}$ and $P_{k|k}$ now become random variables of $\{\gamma_t\}_{1 \leq t \leq k}$. In what follows, we are devoted to characterizing the impacts of $\{\gamma_k\}_{k \in \mathbb{N}}$ on $P_{k+1|k}$. To simplify notations in the sequel, let us use a convenient notation $P_{k+1} \triangleq P_{k+1|k}$, and define the functions g and $g^k: \mathbb{S}_+^n \rightarrow \mathbb{S}_+^n$ as follows:

$$g(X) \triangleq A X A' + Q - A X C' (C X C' + R)^{-1} C X A',$$

and $g^k(X) \triangleq \underbrace{g \circ g \circ \dots \circ g(X)}_{k \text{ times}}$, where $\circ$ denotes the function

composition. It is well known that, for a standard Kalman filter without packet losses, $\lim_{k \rightarrow \infty} P_k = \bar{P}$ with $\bar{P} > 0$ (see [13]).

III. MAIN RESULT

First we shall follow [1] and introduce the definition of peaking covariance stability of Kalman filtering. To this end, we define

$$\begin{aligned} \tau_1 &\triangleq \min\{k : k \geq 1, \gamma_k = 0\}, \\ \beta_1 &\triangleq \min\{k : k > \tau_1, \gamma_k = 1\}, \\ &\vdots \end{aligned} \quad (9)$$

$$\begin{aligned} \tau_j &\triangleq \min\{k : k > \beta_{j-1}, \gamma_k = 0\}, \\ \beta_j &\triangleq \min\{k : k > \tau_j, \gamma_k = 1\}. \end{aligned} \quad (10)$$

It is straightforward to verify that $\{\tau_j\}_{j \in \mathbb{N}}$ and $\{\beta_j\}_{j \in \mathbb{N}}$ are two sequences of stopping times because both $\{\tau_j \leq k\}$

and $\{\beta_j \leq k\}$ are $\mathcal{F}_k$ -measurable; see [14] for details. Due to strong Markov property and the ergodic property of the Markov chain defined by (3), it is shown in [1] that the sequences $\{\tau_j\}_{j \in \mathbb{N}}$ and $\{\beta_j\}_{j \in \mathbb{N}}$ have finite values $\mathbb{P}$ -almost surely. Then we can define the sojourn times at the state 1 and state 0 respectively by τ_j^* and $\beta_j^* \forall j \in \mathbb{N}$ as

$$\begin{aligned}\tau_j^* &\triangleq \tau_j - \beta_{j-1}, \\ \beta_j^* &\triangleq \beta_j - \tau_j,\end{aligned}$$

where we assume $\beta_0 = 1$. The following result given by [1] demonstrates that $\{\tau_k^*\}_{k \in \mathbb{N}}$ and $\{\beta_k^*\}_{k \in \mathbb{N}}$ are, respectively, i.i.d. and these two process are mutually independent.

Lemma 3.1 (Lemma 2 in [1]) The following statements on $\{\tau_j^*\}_{j \in \mathbb{N}}$ and $\{\beta_j^*\}_{j \in \mathbb{N}}$ hold:

- 1) $\{\tau_j^*\}_{j \in \mathbb{N}}$ are i.i.d random sequence, and $\tau_j^* - 1$ is geometrically distributed, i.e., $\mathbb{P}(\tau_j^* - 1 = k) = (1-p)^k p$, $k \in \mathbb{N}$;
- 2) $\{\beta_j^*\}_{j \in \mathbb{N}}$ are i.i.d random sequence, and $\beta_j^* - 1$ is geometrically distributed, i.e., $\mathbb{P}(\beta_j^* - 1 = k) = (1-q)^k q$, $k \in \mathbb{N}$;
- 3) $\{\tau_j^*, \beta_j^*\}_{j \in \mathbb{N}}$ are independent of each other.

Let us denote the prediction error covariance matrix at the stopping time β_j by P_{β_j} . Now we introduce the concept of peak covariance stability, which is originated from [1], as follows: *The Kalman filter with packet losses is said to be peak covariance stable if $\sup_{j \geq 1} \mathbb{E} \|P_{\beta_j}\| < \infty$.* To analyze the peak covariance stability, we also need to introduce the so-called observability index $I_o \in \{1, \dots, n\}$ as in [1], which is defined as smallest integer such that $[C', A'C', \dots, (A^{I_o-1})'C']'$ is full-ranked.

Lemma 3.2 (Lemma 5 in [1]) Define

$$\mathbb{S}_0^n = \{P : 0 \geq P \geq \tilde{P}A' + Q, \text{ for some } \tilde{P} \geq 0\}.$$

Then there exists a constant $L > 0$ such that

- 1) for any $X \in \mathbb{S}_0^n$, $g^k(X) \geq LI$ for all $k \geq I_o$;
- 2) for any $X \in \mathbb{S}_0^n$, $g^{k+1}(X) \geq LI$ for all $k \geq I_o$, where I is $n \times n$ identity matrix.

Before proceeding to the main result, let us first provide with some properties related to discrete-time algebraic Riccati Equation (DARE). Consider the following DARE

$$P = APA' + Q - APC'(CPC' + R)^{-1}CPA'. \quad (11)$$

Define the operator $\phi(K, X) = (A + KC)X(A + KC)' + KKK' + Q$, where K is a matrix with compatible dimension, and sequentially denote

$$\phi^k(K, P) = \phi(K, \phi^{k-1}(K, P)), \quad \forall k \geq 2.$$

Then (11) can be also written as

$$P = \phi(\bar{K}, P), \quad (12)$$

with $\bar{K} = -APC'(CPC' + R)^{-1}$. It is well known that (11) has a unique positive definite solution $\bar{P}$ when $(A, \sqrt{Q})$ is controllable and (C, A) is observable as assumed in this work. See [15]. In addition, $A + \bar{K}C$ is stable, i.e., all the eigenvalues

of A lie inside the open unit circle. This result is summarized as the following lemma. The proof, provided in [15], is omitted here.

Lemma 3.3 Consider the DARE given in (11). If $(A, \sqrt{Q})$ is controllable and (C, A) is observable, then it has a unique positive definite solution $\bar{P}$ and $A + \bar{K}C$ is stable.

Theorem 3.4 If the following two conditions hold:

- 1) $|\lambda_A|^2(1-q) < 1$,
- 2) $\exists K \in \mathbb{R}^{n \times m}$ such that $|\lambda_{H(K)}| < 1$, where

$$H(K) = qp \left[(A \otimes A)^{-1} - (1-q)I \right]^{-1} \sum_{i=1}^{I_o-1} [(A + KC) \otimes (A + KC)]^i (1-p)^{i-1}, \quad (13)$$

and $\otimes$ represents the Kronecker product, then $\sup_{j \geq 1} \mathbb{E} \|P_{\beta_j}\| < \infty$.

Proof. See Appendix. $\square$

As the condition in Theorem 3.4 is difficult to verify directly, in the following corollaries we will give another sufficient condition, which is, despite being conservative, easy to check.

Corollary 3.5 $\sup_{j \geq 1} \mathbb{E} \|P_{\beta_j}\| < \infty$ holds if the following condition is satisfied:

$$pq|\lambda_A|^2 \sum_{i=1}^{I_o-1} |\lambda_A|^{2i} (1-p)^{i-1} < 1 - |\lambda_A|^2(1-q). \quad (14)$$

Proof. Define a sequence of polynomials at matrix $A \otimes A$ as $\{p_n(A \otimes A)\}_{n \in \mathbb{N}}$, where

$$p_n(A \otimes A) = \sum_{i=1}^n (A \otimes A)^i (1-q)^{i-1} q \sum_{j=1}^{I_o-1} (A \otimes A)^j (1-p)^{j-1} p.$$

In light of Lemma A.3 and A.5, the spectrum of $p_n(A \otimes A)$ is given by $\sigma(p_n(A \otimes A)) = \{p_n(\lambda_i \lambda_j) : \lambda_i, \lambda_j \in \sigma(A)\}$. Since A is a real matrix, its complex eigenvalues, if any, always occur in conjugate pairs. Therefore, $|\lambda_A|^2$ must be an eigenvalue of $A \otimes A$, and the spectral radius of $p_n(A \otimes A)$ can be computed as

$$|\lambda_{p_n(A \otimes A)}| = \sum_{i=1}^n |\lambda_A|^{2i} (1-q)^{i-1} q \sum_{j=1}^{I_o-1} |\lambda_A|^{2j} (1-p)^{j-1} p.$$

It is seen that the sequence $\{|\lambda_{p_n(A \otimes A)}|\}_{n \in \mathbb{N}}$ is monotonically increasing. When $|\lambda_A|^2(1-q) < 1$, we have

$$\lim_{n \rightarrow \infty} p_n(A \otimes A) = H(0) \quad (15)$$

and

$$\lim_{n \rightarrow \infty} |\lambda_{p_n(A \otimes A)}| = \frac{q|\lambda_A|^2}{1 - |\lambda_A|^2(1-q)} \sum_{j=1}^{I_o-1} |\lambda_A|^{2j} (1-p)^{j-1} p. \quad (16)$$

As $|\lambda_X|$ is continuous with respect to X , (15) and (16) altogether lead to

$$|\lambda_{H(0)}| = \frac{q|\lambda_A|^2}{1 - |\lambda_A|^2(1-q)} \sum_{j=1}^{I_o-1} |\lambda_A|^{2j} (1-p)^{j-1} p.$$

Letting $K = 0$, then the sufficient condition provided in Theorem 3.4 becomes: 1) $|\lambda_A|^2(1 - q) < 1$, 2) $|\lambda_{H(0)}| < 1$; whereby the conclusion follows. $\square$

Remark 3.6 In Theorem 6 of [1] and Theorem 1 of [10], $d_j^{(1)}$ is chosen as $\|A + KC\|^2$ or $\|A\|^2$ for practical application. Theorem 3.4 and Corollary 3.5 establish direct connections between λ_A (or $\lambda_{H(k)}$), p , q , the most essentials of system dynamic and channel characteristics on one hand, and the peaking covariance stability on the other hand; while other relative literature failed to do that. These two results includes the results proposed by [1], [9], [10], [12] once being aware of the subadditivity property of matrix norm, and that spectral radius is the infimum of all possible matrix norms, and therefore are less conservative.

On the other hand, we consider the lower bound of P_{β_k} and have $P_{\beta_k} \geq \sum_{i=0}^{\beta_k^*} A^i Q (A^i)'$. Taking randomness of β_k^* into account, this eventually gives rise to a necessary condition for peak covariance stability: $\sup_{j \geq 1} \mathbb{E} \|P_{\beta_j}\| < \infty$ holds only if $|\lambda_A|^2(1 - q) < 1$. The result has been present in [9], where the detailed proof was provided.

Remark 3.7 Since the left side of (14) is nonnegative, the criterion given in corollary 3.5 imposes the positivity of $1 - |\lambda_A|^2(1 - q)$. The left side of (14) is strictly positive when $I_o \geq 2$. And it vanishes when $I_o = 1$, $|\lambda_A|^2(1 - q) < 1$ thereby being a necessary and sufficient condition for peak covariance stability in this case.

IV. EXAMPLES

To compare with the work done in [1], [9], [12], we will examine the same vector example considered in therein. The parameters are specified as follows:

$$A = \begin{bmatrix} 1.3 & 0.3 \\ 0 & 1.2 \end{bmatrix}, \quad C = [1, 1],$$

$Q = I_{2 \times 2}$ and $R = 1$. As illustrated in [1], it is easily checked that $I_o = 2$ and the eigenvalues of A are $\sigma(A) = \{1.2, 1.3\}$, and that $\lambda_A = 1.3$.

First let us compare the sufficient condition we provide with in Corollary 3.5 with the counterpart provided in [1]. It is seen that both of them require $|\lambda_A|^2(1 - q) < 1$. To make the comparison persuasive, we also take $q = 0.65$ as what was done in [1]. If no other specifications are stated, we will use $q = 0.65$ by default in the rest of this section. As for the failure rate p , [1] requires $p < 0.04$ to guarantee peak covariance stability; while Corollary 3.5 requires

$$p < \frac{1 - |\lambda_A|^2(1 - q)}{|\lambda_A|^4 q},$$

which generates a less conservative condition, that is $p < 0.22$. In [1], all numerical simulations were implemented with parameters $(p, q) = (0.03, 0.65)$. Note that, for the associated channel with $(p, q) = (0.04, 0.65)$, $\mathbb{P}(\gamma_k = 0) = 0.0580$ when

¹In [1] and [10], $d_j^{(0)}$ and $d_j^{(1)}$ are defined as two positive constants satisfy the inequality $g^j(X) \leq d_j^{(1)} \|X\| + d_j^{(0)}$, $1 \leq j \leq \max\{1, I_o - 1\}$ for any $X \in \mathbb{S}_0^n$.

the packet loss process enters stationary distribution, which means that the allowed long term packet loss rate is at most 5.80%. However, by choosing a larger p , Corollary 3.5 permits $\mathbb{P}(\gamma_k = 0) = 0.2529$ at stationary distribution at most, i.e., as high as 25.39% the long term packet dropout rate is allowed. Similarly, the example in [9] allows $p = 0.1191$ at most.

Moreover, it is rather convenient to check the condition in Corollary 3.5 even with manual calculation; while, to check the conditions in [1] and [9], it involves a considerable amount of complex numerical calculation.

In what follows, we still consider the above vector example and devote ourselves to improving the foregoing result attained by Corollary 3.5 with the aid of Theorem 3.4. Despite of heavy computation, Theorem 3.4 may establish an accurate criterion for the peak covariance stability than Corollary 3.5 does and than [1], [9] do. To this end, we adopt a numerical method to search for an K^* over a prescribed subset $\mathcal{R}_K$ of $\mathbb{R}^{n \times m}$ so that

$$K^* = \arg \max_K \{p : |\lambda_{H(K)}| < 1 \text{ and } K \in \mathcal{R}_K\}.$$

Going back to the example considered herein, we denote $K = [k_1, k_2]' \in \mathbb{R}^2$ and search for (k_1, k_2) over $\mathcal{R}_K = \{-10, -9.95, \dots, 10\} \times \{-10, -9.95, \dots, 10\}$ with a constant step size 0.05. Then we obtain $K^* = [0.45, -1.95]'$, which allows $p = 1^2$. It should be pointed that at least for the parameters specified in this example, the criterion of [1] only covers the Gilbert-Elliott models with failure rate as low as 4.5%. The criterion provided by Theorem 3.4 combined with a numerical procedure is much relaxed: p is allowed to take value 1^2 for this examples. Fig. 2 and Fig. 3 illustrate sample paths of $\|P_k\|$ and γ_k with $(p, q) = (0.5, 0.65)$ and $(p, q) = (0.99, 0.65)$ respectively, where they show that even an extremely high value of p may not effect the peak covariance stability with the system parameters specified in this section.

V. CONCLUSION

In this paper, we study the peak covariance stability Kalman filtering over Markovian packet losses. We propose two sufficient conditions for the peak covariance stability: one combined with a numerical method provides with an accurate criterion, and the other is in a simple form and easy to check; both of which are shown to be less conservative than existing works. Numerical examples demonstrate the effectiveness of our result compared with relevant literature. Future work includes establishing the connection between the peak covariance stability and the usual stability for general systems.

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² Unfortunately, when $p = 1$ the Markov Chain defined in (3) will lose ergodicity property. At expense of a little decrease in failure rate by means of setting $p = 1 - \epsilon$ for an arbitrary small positive ϵ , the ergodicity requirement is met.

Fig. 2: A sample path of $\|P_k\|$ and γ_k with $p = 0.50$, $q = 0.65$

Fig. 3: A sample path of $\|P_k\|$ and γ_k with $p = 0.99$, $q = 0.65$

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APPENDIX A

Lemma A.1 (Lemma A.1 [6]) For any matrices $X \geq Y \geq 0$ in $\mathbb{S}_n^+$, the following inequalities hold

$$h(X) \geq h(Y), \quad (17)$$

$$g(X) \geq g(Y), \quad (18)$$

Lemma A.2 Consider the operator $\phi(K, X)$. For any $X \geq 0$ and K , the following statement always holds

$$g(X) = \min_K \phi(K, X) \leq \phi(K, X).$$

Proof. The result is straightforward when setting $\lambda = 1$ in Lemma 1 in [2]. $\square$

The following lemma is easily checked by using the knowledge of linear algebra.

Lemma A.3 Let A and B be square matrices. Then $\sigma(A \otimes B) = \sigma(A)\sigma(B)$.

Lemma A.4 The series of matrices $\sum_{i=1}^{\infty} (A \otimes A)^i (1-q)^{i-1} q$ and $\sum_{i=1}^{\infty} \sum_{j=0}^{i-1} (A \otimes A)^j (1-q)^{i-1} q$ converge if and only if $|\lambda_A|^2 (1-q) < 1$.

Proof. First observe that

$$\sum_{i=1}^{\infty} \sum_{j=0}^{i-1} (A \otimes A)^j (1-q)^{i-1} q = \sum_{i=0}^{\infty} (A \otimes A)^i (1-q)^i.$$

The geometric series generated by $(A \otimes A)(1-q)$ converges if and only if $|\lambda_{A \otimes A}|(1-q) < 1$. Therefore the conclusion follows from the fact that $|\lambda_{A \otimes A}| = \max\{|\lambda_i \lambda_j| : \lambda_i, \lambda_j \in \sigma(A)\} = |\lambda_A|^2$. $\square$

Lemma A.5 (Theorem 1.1.6 in [16]) Let $p(\cdot)$ be a given polynomial. If λ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ while x is an associated eigenvector, then $p(\lambda)$ is an eigenvalue of the matrix $p(A)$ and x is associated an eigenvector of $p(A)$.

APPENDIX B: PROOF OF THEOREM 3.4

Proof. Fix an $k \geq 0$. First note that, for any $j \in [\tau_{k+1}, \beta_{k+1} - 1]$, $\gamma_j = 0$. Hence we have

$$\begin{aligned} P_{\beta_{k+1}} &= \sum_{i=1}^{\infty} 1_{\{\beta_{k+1}^* = i\}} A^i P_{\tau_{k+1}} (A^i)' \\ &\quad + \sum_{i=1}^{\infty} 1_{\{\beta_{k+1}^* = i\}} \underbrace{\sum_{j=0}^{i-1} A^j Q (A^j)'}_{\triangleq V_i} \end{aligned} \quad (19)$$

Now let us consider the random interval $[\beta_k, \tau_{k+1} - 1]$ where τ_{k+1}^* packets are successfully received. We will analyze the relationship between $P_{\tau_{k+1}}$ and P_{β_k} in two separated cases, which are $\tau_{k+1}^* \leq I_o - 1$ and $\tau_{k+1}^* \geq I_o$. Computation yields the following result

$$\begin{aligned}
P_{\tau_{k+1}} &= \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} g^i(P_{\beta_k}) + \sum_{j=I_o}^{\infty} 1_{\{\tau_{k+1}^*=j\}} g^j(P_{\beta_k}) \\
&\leq \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} g^i(P_{\beta_k}) + LI \sum_{j=I_o}^{\infty} 1_{\{\tau_{k+1}^*=j\}} \\
&\leq \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} \phi^i(A + KC, P_{\beta_k}) + LI \sum_{j=I_o}^{\infty} 1_{\{\tau_{k+1}^*=j\}} \\
&= \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} (A + KC)^i P_{\beta_k} (A' + C'K')^i \\
&\quad + LI \sum_{j=I_o}^{\infty} 1_{\{\tau_{k+1}^*=j\}} \\
&\quad + \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} \sum_{j=0}^{i-1} (A + KC)^j (Q + KRK') (A' + C'K')^j \\
&\triangleq \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} (A + KC)^i P_{\beta_k} (A' + C'K')^i \\
&\quad + LI \sum_{j=I_o}^{\infty} 1_{\{\tau_{k+1}^*=j\}} + \sum_{i=1}^{I_o-1} 1_{\{\tau_{k+1}^*=i\}} U_i \quad (20)
\end{aligned}$$

where the first inequality is from Lemma 3.2 and the second one follows from Lemma A.2. By substituting (20) into (19), it yields

$$\begin{aligned}
P_{\beta_{k+1}} &= \sum_{i=1}^{\infty} 1_{\{\beta_{k+1}^*=i\}} A^i \left[\sum_{j=1}^{I_o-1} 1_{\{\tau_{k+1}^*=j\}} (A + KC)^j P_{\beta_k} \right. \\
&\quad \left. (A' + C'K')^j \right] (A^i)' \\
&\quad + \sum_{i=1}^{\infty} 1_{\{\beta_{k+1}^*=i\}} A^i \left(\sum_{j=1}^{I_o-1} 1_{\{\tau_{k+1}^*=j\}} U_j \right) (A^i)' \\
&\quad + \sum_{i=1}^{\infty} 1_{\{\beta_{k+1}^*=i\}} V_j \\
&\triangleq \sum_{i=1}^{\infty} 1_{\{\beta_{k+1}^*=i\}} A^i \left[\sum_{j=1}^{I_o-1} 1_{\{\tau_{k+1}^*=j\}} (A + KC)^j P_{\beta_k} \right. \\
&\quad \left. (A' + C'K')^j \right] (A^i)' + W. \quad (21)
\end{aligned}$$

In what follows, we will introduce the so-called vectorization operator. Let $X = [x_1 \ x_2 \ \dots \ x_n] \in \mathbb{R}^{m \times n}$. Then we define

$$\text{vec}(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{mn}.$$

Notice that $\text{vec}(AXB) = (B' \otimes A) \text{vec}(X)$. For Kronecker product, we have $(A_1 A_2) \otimes (B_1 B_2) = (A_1 \otimes B_1) (A_2 \otimes B_2)$. Take expectations and vectorization operators over both sides of (21). Using the property of Kronecker product and Lemma 3.1, we obtain

$$\begin{aligned}
\mathbb{E}[\text{vec}(P_{\beta_{k+1}})] &= \mathbb{E}[\text{vec}(W)] + \sum_{i=1}^{\infty} (A \otimes A)^i (1 - q)^{i-1} q \\
&\quad \cdot \sum_{j=1}^{I_o-1} [(A + KC) \otimes (A + KC)]^j p (1 - p)^{j-1} \text{vec}(P_{\beta_k}) \quad (22)
\end{aligned}$$

In the above equation $\mathbb{E}[\text{vec}(W)]$ can be written as

$$\begin{aligned}
\mathbb{E}[\text{vec}(W)] &= \sum_{i=1}^{\infty} (A \otimes A)^i (1 - q)^{i-1} q \sum_{j=1}^{I_o-1} (1 - p)^{j-1} p \\
&\quad \cdot \sum_{l=0}^{j-1} [(A + KC) \otimes (A + KC)]^l \text{vec}(Q + KRK') \\
&\quad + \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} (A \otimes A)^j (1 - q)^{i-1} q \text{vec}(Q). \quad (23)
\end{aligned}$$

In Lemma A.4, we show that both of the two terms in (23) converges if $|\lambda_A|^2(1 - q) < 1$.

We have shown that $\mathbb{E}[\text{vec}(P_{\beta_k})]$ for $k \in \mathbb{N}$ evolves following (22) and that $\mathbb{E}[\text{vec}(W)]$ in (22) is bounded if $|\lambda_A|^2(1 - q) < 1$. We conclude that if $|\lambda_A|^2(1 - q) < 1$ and there exists a matrix K such that $|\lambda_{H(K)}|$ defined in (13) is less than 1, then the spectral radius of $\sum_{i=1}^{\infty} (A \otimes A)^i (1 - q)^{i-1} q \sum_{j=1}^{I_o-1} [(A + KC) \otimes (A + KC)]^j (1 - p)^{j-1} p$ is less than 1. Hence all the above observations lead to $\sup_{j \geq 1} \mathbb{E}[\text{vec}_i(P_{\beta_j})] < \infty \ \forall 1 \leq i \leq n^2$, where $\text{vec}_i(X)$ represents the i th element of $\text{vec}(X)$. In addition, there holds

$$\mathbb{E}\|P_{\beta_k}\| \leq \mathbb{E}[\text{tr}(P_{\beta_k})] = [e'_1, \dots, e'_n] \mathbb{E}[\text{vec}(P_{\beta_k})],$$

where e_i denotes the vector with a 1 in the i th coordinate and 0's elsewhere, so the desired result follows. $\square$