Networked Control with Stochastic Scheduling

Kun Liu 1, Emilia Fridman 2 and Karl Henrik Johansson 1

Abstract—This paper develops the time-delay approach to networked control systems with scheduling protocols, variable delays and variable sampling intervals. The scheduling of sensor communication is defined by a stochastic protocol. Two classes of protocols are considered. The first one is defined by an independent and identically-distributed stochastic process. The activation probability of each node for this protocol is a given constant, whereas it is assumed that collisions occur with a certain probability. The resulting closed-loop system is a stochastic impulsive system with delays both in the continuous dynamics and in the reset equations, where \( \eta_c \) is the computational delay in the controller node. Denote \( s_k + \eta_k \) by \( t_k \). Differently from [8], [9], we derive linear matrix inequalities (LMIs) conditions for the exponential mean-square stability of the closed-loop system. As in [3] and [4], differently from the hybrid and discrete-time approaches, we allow the transmission delays to be larger than the sampling intervals in the presence of scheduling protocols. The efficiency of the presented approach is illustrated by a batch reactor example.

The rest of this note is organized as follows. Section II presents the model of NCS and the hybrid delayed system model for the closed-loop system. In Section III below, the exponential mean-square stability of the closed-loop system under i.i.d stochastic protocol will be studied. The exponential mean-square stability of the closed-loop system under Markovian stochastic protocol will be presented in Section IV. In Section V, the efficiency and advantages of the presented approach are illustrated by a batch reactor example.

Keywords: networked control systems, Lyapunov functional, stochastic protocols, stochastic impulsive system.

I. INTRODUCTION

Networked control systems (NCSs) have received considerable attention in recent years (see e.g., [1], [2]). In many such systems, only one node is allowed to use the communication channel at a time. The communication is orchestrated by a scheduling rule called a protocol. The time-delay approach was recently developed for the stabilization of NCSs under the round-robin (RR) protocol [3] and under the try-once-discard (TOD) protocol [4]. The closed-loop system was modeled as a switched system with multiple and ordered time-varying delays under RR scheduling or as a hybrid system with time-varying delays in the dynamics and in the reset equations under the TOD scheduling. Differently from the existing results on NCSs in the presence of scheduling protocols (in the frameworks of hybrid and discrete-time systems), the transmission delay is allowed to be larger (than the sampling interval), but a crucial point is that data packet dropout is not allowed for large delays in either [3] or [4].

In the framework of hybrid systems, a stochastic protocol was introduced in [5] and analyzed for the input-output stability of NCSs in the presence of data packet dropouts or collisions. An i.i.d. (independent and identically-distributed) sequence of Bernoulli random variables is applied to describe the stochastic protocol. Communication delays, however, are not included in the analysis. The stability of NCSs under a stochastic protocol, where the activated node is modeled by a Markov chain, was studied in [6] by applying the discrete-time modeling framework. In [6], data packet dropouts can be regarded as prolongations of the sampling interval for small delays.

In the present note, to overcome the lack of stability analysis of NCS under scheduling protocols with large communication delays and data packet dropouts, we develop a time-delay approach considering multiple sensors under a stochastic scheduling protocol. The resulting closed-loop system is a stochastic impulsive system with delays both in the continuous dynamics and in the reset equations. We treat two classes of stochastic protocols. The first one is defined by an i.i.d. stochastic process. The activation probability of each node for this protocol is a given constant, whereas it is assumed that collisions occur with a certain probability. The second protocol is defined by a discrete-time Markov chain with a known transition probability matrix taking into account collisions.

By developing appropriate Lyapunov-Krasovskii techniques, we derive linear matrix inequalities (LMIs) conditions for the exponential mean-square stability of the closed-loop system. As in [3] and [4], differently from the hybrid and discrete-time approaches, we allow the transmission delays to be larger than the sampling intervals in the presence of scheduling protocols. The efficiency of the presented approach is illustrated by a batch reactor example.

II. SYSTEM MODEL

A. NCS model

Consider the system architecture in Figure 1 with plant

\[ \dot{x}(t) = Ax(t) + Bu(t), \]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control input and \( A, B \) are system matrices of appropriate dimensions. The initial condition is given by \( x(0) = x_0 \).

The NCS has \( N \) distributed sensors, a controller and an actuator connected via two wireless networks. Their measurements are given by \( y_i(t) = C_i \tilde{x}(t), i = 1, \ldots, N \). Let \( C = [C_1^T \cdots C_N^T]^T \), \( y(t) = [y_1(t) \cdots y_N(t)]^T \in \mathbb{R}^n \). We denote by \( s_k \) the unbounded and monotonously increasing sequence of sampling instants \( 0 = s_0 < s_1 < \cdots < s_k < \cdots, k \in \mathcal{Z}_{>0}, \lim_{k \to \infty} s_k = \infty, s_{k+1} - s_k \leq \tau \) MATI, whereMATI denotes the maximum allowable transmission interval. At each sampling instant \( s_k \), at most one of the outputs \( y_i(s_k) \in \mathbb{R}^n \), \( \sum_{i=1}^{N} y_i(s_k) \) is transmitted over the network.

We suppose that the transmission of the information (between the sensor and the actuator) is subject to a variable delay \( \eta_p = \eta_{p_i} + \eta_{c_i} + \eta_{c_{i+1}}, \) where \( \eta_{p_i} \) and \( \eta_{c_i} \) are the network-induced delays (from the sensor to the controller and from the controller to the actuator, respectively), and where \( \eta_{c_{i+1}} \) is the computational delay in the controller node. Denote \( s_k + \eta_k \) by \( t_k \). Differently from [8], [9], we...
do not restrict the network delays to be small with \( y_k < s_{k+1} - s_k \). Following [3], [4], [10], we allow the delay to be large provided that packet ordering is maintained. Assume that the network-induced delay \( y_k \) and the time span between the instant \( t_{k+1} \) and the current sampling instant \( s_k \) are bounded:

\[
t_{k+1} - t_k + y_k \leq \tau_M, \quad 0 \leq \eta_m \leq y_k \leq \text{MAD}, \quad k \in \mathbb{Z}_>0.
\]

where MAD denotes the maximum allowable delay. Here \( \eta_m \) and MAD are known bounds and \( \tau_M = \text{MAT + MAD} \). The inequality \( \eta_m > \tau_M / 2 \) implies the case of large delay. For the given example in Section V, we show that our method is applicable also for \( \eta_m > \tau_M / 2 \).

**Remark 1** Differently from [10], where subscript \( k \) in \( t_k \) corresponds to the measurements that are not lost, in our paper \( k \) corresponds to the sampling time. This is because we consider the probability of collisions or data packet dropouts (see further details below). Therefore, \( t_k \) is the actual or the fictitious (when collisions occur or the data packet is lost) updating time instant of the zero-order hold (ZOH) device.

**Remark 2** We follow a commonly used assumption on the boundedness of the network-induced delays, e.g., [8], [11]. Another possibility is the Markov chain model of the network-induced delays, e.g., [12].

**B. The impulsive model**

At each sampling instant \( s_k \), at most one of the system nodes \( i \in \{1, \ldots, N\} \) is active. In some cases, collisions may occur when nodes access the network [5]. If this happens, then packet with sensor data is dropped. At the sampling instant \( s_k \), let \( \sigma_k \in \mathcal{I} = \{0, 1, \ldots, N\} \) denote the active output node, which will be chosen according to the stochastic protocol. Here \( \sigma_k = 0 \) means that either collisions occur when nodes access the network or the data packet is lost during the transmission over the network from the sensor to the controller. We suppose data loss is not possible during the transmission from the controller to the actuator.

Denote by \( \hat{y}(s_k) = [\hat{y}_1(s_k) \ldots \hat{y}_N(s_k)]^T \in \mathbb{R}^{N_y} \) the most recently received output information on the controller side. We consider the error between the system output \( y(s_k) \) and the last available information \( \hat{y}(s_k-1) \):

\[
e(t) = \text{col}\{e_1(t), \ldots, e_N(t)\} = \hat{y}(s_{k-1}) - y(s_k), \quad \text{for } t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_>0.
\]

We suppose that the controller and the actuator are event-driven (in the sense that the controller and the ZOH device update their outputs as soon as they receive a new sample).

**Static output feedback control:** Assume that there exists a matrix \( K = [K_1 \cdots K_N], K_i \in \mathbb{R}^{n_y \times n_x} \) such that \( A + BK_i \) is Hurwitz. Then the static output feedback controller has the form

\[
u(t) = K_s y(s_k) + \sum_{i=1,i \neq \sigma_k} \gamma_i(s_k-1), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_>0.
\]

where \( K_s y(s_k) \) is the error between the system output \( y(s_k) \) and the controller output \( \nu(s_k) \). To avoid the control input being unbounded, we assume that the controller is active. In some cases, collisions may occur when nodes access the network [5]. If this happens, then packet with sensor data is dropped.

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e(t) = \text{col}\{e_1(t), \ldots, e_N(t)\} = \hat{y}(s_{k-1}) - y(s_k), \quad \text{for } t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_>0.
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We suppose that the controller and the actuator are event-driven (in the sense that the controller and the ZOH device update their outputs as soon as they receive a new sample).
1) i.i.d scheduling: The choice of \( \sigma_k \) is assumed to be i.i.d with the probabilities given by
\[
\text{Prob}\{\sigma_k = i\} = \beta_i, \quad i \in I,
\]
where \( \beta_i, i = 0, \ldots, N \) are non-negative scalars and \( \sum_{i=0}^{N} \beta_i = 1 \).
Here \( \beta_j, j = 1, \ldots, N \) are the probabilities of the measurement \( y_j(s_k) \) to be transmitted at \( s_k \), whereas \( \beta_0 \) is the probability of collision.

2) Markovian scheduling: The protocol determines \( s_k \) through a Markov Chain. The conditional probability that node \( j \in I \) gets access to the network at time \( s_k \), given the values of \( s_{k-1} \in I \), is defined by
\[
\text{Prob}\{\sigma_k = j|\sigma_{k-1} = i\} = \pi_{ij},
\]
where \( 0 \leq \pi_{ij} \leq 1 \) for all \( i, j \in I, \sum_{j=1}^{N} \pi_{ij} = 1 \) for all \( i \in I \) and \( \sigma_0 \in I \) is assumed to be given. The transition probability matrix is denoted by \( \Pi = \{\pi_{ij}\} \in \mathbb{R}^{(N+1) \times (N+1)}. \)

Remark 3 The i.i.d scheduling is a special case of the Markovian scheduling. For instance, assume that there are \( N = 2 \) sensor nodes and collisions do not occur; the Markovian scheduling with \( \Pi = \left[ \begin{array}{cc} 1 & -p \\ p & 1 - p \end{array} \right] \), \( 0 \leq p \leq 1 \), is an i.i.d scheduling with \( \beta_1 = p, \beta_2 = 1 - p. \)

Definition 1 The hybrid system (5)-(6) is said to be exponentially mean-square stable with respect to \( x \) if there exist constants \( b > 0, \alpha > 0 \) such that the following bound holds
\[
\mathbb{E}\{|x(t)|^2\} \leq be^{-2(\alpha-t)}\mathbb{E}\{|x(0)|^2\} e^{2b(t-t_0)} \quad t \geq t_0.
\]
for the solutions of the stochastic impulsive system (5)-(6) initialized with \( e(t_0) \in \mathbb{R}^n \) and \( x(t) = \phi(t) \), \( t \in [\tau_0, \tau_1). \) The hybrid system (5)-(6) is exponentially mean-square stable if additionally the following bound is valid
\[
\mathbb{E}\{|e(t)|^2\} \leq be^{-2(\alpha-t)}\mathbb{E}\{|e(0)|^2\} e^{2b(t-t_0)} \quad t \geq t_0.
\]

III. NCSS UNDER I.I.D STOCHASTIC SCHEDULING PROTOCOL
A. Stochastic impulsive time-delay model with Bernoulli distributed parameters
Following [14], we introduce the indicator functions
\[
\pi_{(\sigma_k=i)} = \begin{cases} 1, & \sigma_k = i, \\ 0, & \sigma_k \neq i, \end{cases} \quad i \in I, \quad k \in \mathbb{Z}_{\geq 0}.
\]
Thus, from (7) it follows that
\[
\mathbb{E}\{\pi_{(\sigma_k=i)}\} = \mathbb{E}\{\pi_{(\sigma_k=i)}\} = \text{Prob}\{\sigma_k = i\} = \beta_i,
\]
\[
\mathbb{E}\{\pi_{(\sigma_k=i)} - \beta_i\pi_{(\sigma_k=i)} - \beta_i\} = \left\{\begin{array}{ll} -\beta_i\beta_j, & i \neq j, \\ \beta_i(1-\beta_i), & i = j. \end{array}\right.
\]

Therefore, the stochastic impulsive system model (5)-(7) can be rewritten as
\[
\dot{x}(t) = Ax(t) + A_1 x(t-h) + \sum_{i=1}^{N} (1-\pi_{(\sigma_k=i)})B_ie_i(t), \quad e(t) = 0, \quad t \in [t_k, t_{k+1})
\]
with the delayed reset system
\[
x(t_{k+1}) = x(t_{k+1}), \\
\dot{e}(t_{k+1}) = (1-\pi_{(\sigma_k=i)})e_i(t_{k+1}) + C_i[x(t_{k+1}) - x(t_{k+1})], \quad i = 1, \ldots, N.
\]

Remark 4 Applying the Bernoulli-distributed stochastic variables \( \pi_{(\sigma_k=i)} \), \( i = 0, \ldots, N \), the closed-loop system (10)-(11) is presented as an impulsive time-delay system with stochastic parameters in the system matrices. Note that the Bernoulli distribution has previously been applied to NCS with probabilistic measurements missing [15], stochastic sampling intervals [16], time-delay system with stochastic interval delays [14], output tracking control under unreliable communication [17] and fuzzy control for nonlinear NCSS [18].

B. Exponential mean-square stability of stochastic impulsive delayed system
Our objective of this section is to derive LMI conditions for exponential mean-square stability of the stochastic impulsive system (10)-(11). Consider Lyapunov-Krasovskii functional (LKF)
\[
V(t) = V(t, x_t, \dot{x}_t) + \sum_{i=1}^{N} e_i^T(t)Q_i e_i(t),
\]
\[
V(t, x_t, \dot{x}_t) = \bar{V}(t, x_t, \dot{x}_t) + \sum_{i=1}^{N} e_i^T(t)Q_i e_i(t),
\]
\[
\bar{V}(t, x_t, \dot{x}_t) = \bar{V}(t, x_t, \dot{x}_t) + \sum_{i=1}^{N} e_i^T(t)Q_i e_i(t),
\]
\[
\sum_{i=1}^{N} E\{|\sqrt{Q_i e_i(t)}|\|2\} \leq \tilde{c} e^{-2\alpha(t-t_0)}E\{V(e(t_0))\}, \quad t \geq t_0.
\]

Lemma 1 If there exist positive constant \( \alpha, 0 < Q_i \in \mathbb{R}^{n_i \times n_i}, \quad 0 < U_i \in \mathbb{R}^{n_i \times n_i}, \quad 0 < G_i \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, \ldots, N, \) and \( V_i(t) \) of (12) such that along (10) for \( t \in [t_k, t_{k+1}) \)
\[
\mathbb{E}\{(V_i(t) + 2\alpha V_i(t) - \sum_{i=1}^{N} e_i^T(t)U_i e_i(t))\} \leq 0,
\]
\[
\mathbb{E}\{(V_i(t) + 2\alpha V_i(t) - \sum_{i=1}^{N} e_i^T(t)U_i e_i(t))\} \leq 0,
\]

The following lemma gives sufficient conditions for exponential stability of (10)-(11) in the mean-square sense:

Lemma 2 If there exist positive constant \( \alpha, 0 < Q_i \in \mathbb{R}^{n_i \times n_i}, \quad 0 < U_i \in \mathbb{R}^{n_i \times n_i}, \quad 0 < G_i \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, \ldots, N, \) and \( V_i(t) \) of (12) such that along (10) for \( t \in [t_k, t_{k+1}) \)
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\]
\[
\mathbb{E}\{(V_i(t) + 2\alpha V_i(t) - \sum_{i=1}^{N} e_i^T(t)U_i e_i(t))\} \leq 0,
\]

Moreover, the following bounds hold for the solutions of (10)-(11) with the initial condition \( x(t_0), e(t_0) \):
\[
\mathbb{E}\{V_i(t_{k+1})\} \leq e^{-2\alpha(t-t_0)}\mathbb{E}\{V_i(t_k)\}, \quad t \geq t_0,
\]
\[
\mathbb{E}\{V_i(t_{k+1})\} \leq e^{-2\alpha(t-t_0)}\mathbb{E}\{V_i(t_k)\}, \quad t \geq t_0,
\]
\[
\sum_{i=1}^{N} E\{|\sqrt{Q_i e_i(t)}|\|2\} \leq \tilde{c} e^{-2\alpha(t-t_0)}E\{V(e(t_0))\}, \quad t \geq t_0.
\]
where $\tilde{c} = e^{2\alpha (\tau_M - \eta_m)}$, implying exponential mean-square stability of (10)-(11).

**Proof:** Since $\int_{t_k}^{t} e^{-2\alpha (t-s)} ds \leq \tau_M - \eta_m$, $t \in [t_k, t_{k+1})$, and $L^2[0, V(t)] = e^{2\alpha [2V(t) + CV(v(t))]}$, $\alpha > 0$, then (15) implies

$$E[V(t)] \leq e^{-2\alpha (t-t_k)} E[V(t_k)]$$

$$+ \sum_{i=1}^{N} E[e^T(t_k) u_i(t_k) e_i(t)] t \in [t_k, t_{k+1}).$$

(20)

Because (16) yields $U_i \leq \beta_i Q_i < Q_i$, $i = 1, \ldots, N$, we have

$$E[V(t, x_t, x_{t_1})] \leq e^{-2\alpha (t-t_k)} E[V(t_k)] t \in [t_k, t_{k+1}).$$

(21)

Note that

$$E[V(t_{k+1})] = E[\tilde{V}(t_{k+1}) + V_G(t_{k+1}) + \sum_{i=1}^{N} e^T(t_{k+1}) Q_i e_i(t_{k+1})]$$

and

$$E[e^T(t_{k+1}) Q_i e_i(t_{k+1})]$$

$$= E[[\sqrt{Q_i}][1 - \pi_{m, \eta_m}]) v_i(t_{k+1})$$

$$+ C_i x(t_k - \eta_m) - C_i x(t_k - \eta_m)]^2$$

$$\leq E[(1 - \beta_i) e^T(t_{k+1}) Q_i e_i(t_{k+1})$$

$$+ 2(1 - \beta_i) e^T(t_{k+1}) Q_i C_i x(t_k - \eta_m - x(t_k - \eta_m))$$

$$+ |\sqrt{Q_i} C_i x(t_k - \eta_m - x(t_k - \eta_m))]^2$$

$$i = 1, \ldots, N.$$}

Taking (13) and (16) into account, we obtain

$$\Theta = E[\sum_{i} ||\sqrt{Q_i} e_i(t_k + 1)||^2 - ||\sqrt{Q_i} e_i(t_k)||^2]$$

$$\leq \sum_{i} E[||\sqrt{Q_i} e_i(t_k + 1)||^2 - ||\sqrt{Q_i} e_i(t_k)||^2]$$

$$- e^{-2\alpha (t_k - t_{k+1})} \sum_{i} E[||\sqrt{Q_i} C_i x(t_k - \eta_m - x(t_k - \eta_m))]^2$$

$$\leq \sum_{i} E(\omega_i(t_k) \Phi_i(t_k))^2$$

$$\leq \omega \sum_{i} \Phi_i(t_k)^2$$

where $\phi_i(t_k) = \text{col}(e_i(t_k), C_i x(t_k - \eta_m - x(t_k - \eta_m))]$ and $\Omega_i$ is given by (16).

Therefore, the inequalities (17) and (20) with $t = t_{k+1}$ imply

$$E[V(t_{k+1})] \leq e^{-2\alpha (t_{k+1} - t_k)} E[V(t_k)]$$

$$\leq e^{-2\alpha (t_{k+1} - t_k)} E[V(t_{k+1})]$$

(22)

The latter inequality, with $k + 1$ replaced by $k$ and (21) give (18). The inequality (18) implies exponential mean-square stability of (10)-(11) with respect to $x$ because

$$\lambda_{\min}(P) E[x(t)^2] \leq E[V(t, x_t, x_{t_1})]$$

$$E[V(t, x_t, x_{t_1})] \leq E[||e(t)||^2]$$

for some scalar $\nu > 0$. Moreover, the inequality (22) with $k + 1$ replaced by $k$ implies (19) (since for $t \in [t_k, t_{k+1})$, $e^{-2\alpha (t_{k+1} - t_k)} e^{-2\alpha (t_{k+1} - t_{k+1})} \leq e^{-2\alpha (t_{k+1} - t_{k+1})}$.)

By using Lemma 1 and the standard arguments for the delay-dependent analysis, we derive LMI conditions for the exponential mean-square stability of (10)-(11):

**Theorem 1** Given $0 \leq \eta_m < \tau_M$, $\alpha > 0$, $\beta_0 \geq 0$, $\beta_i \geq 0$, $\sum_{i=1}^{N} \beta_i = 1$ and $K_i$, $i = 1, \ldots, N$. Suppose there exists $n \times n$ matrices $P > 0$, $S_j > 0$, $R_i > 0$, $j = 0, 1, S_{12}$ and $\tau_i$ of $n \times n$ matrices $Q_i > 0$, $U_i > 0$, $G_i > 0$, $i = 1, \ldots, N$, such that (16) and

$$\Phi = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \geq 0,$$

$$\Sigma + \Xi H \Xi^T + \sum_{i=1}^{N} \beta_i \Xi \Xi H \Xi < 0$$

(24)

are feasible, where

$$H = \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m) \sum_{i=1}^{N} C_i^T G_i C_i,$$

$$\Sigma = F_1^T F_2 \Xi + \Xi F_2^T F_1 + \Xi F_2 H \Xi F_2^T - \Xi F_2 R_0 \Xi F_2^T$$

$$F_1 = [I_{n} 0_{n \times (n + m)}] F_2 = [I_{n} - I_{n} 0_{n \times (2n + m)}],$$

$$F = \left[ \begin{array}{c} 0_{n \times n} - I_{n} 0_{n \times n} I_{n} - I_{n} 0_{n \times n} \\ 0_{n \times n} 0_{n \times n} 0_{n \times n} \end{array} \right],$$

$$= [A_{n \times n} A_{1, n \times n} (1 - \beta_1) B_1 \cdots (1 - \beta_N) B_N],$$

$$A_{n \times n}, B_0, B_2, \ldots, B_1, B_0, \ldots, B_N$$

$$E_i = [0_{n \times 2n - B_1}, 0_{n \times 2n - B_2}, \ldots, 0_{n \times 2n - B_{N+1}}$$

Then the solutions of (10)-(11) satisfy the bounds (18) and (19). Hence, the closed-loop system (10)-(11) with initial condition $x(t_0)$, $e(t_0)$ is exponentially mean-square stable. If the aforementioned matrix inequalities are feasible with $\alpha = 0$, then the bounds (18) and (19) hold also for a sufficiently small $\alpha_0 > 0$.

**IV. NCSS UNDER MARKOVIAN STOCHASTIC CONTINGENCY PROTOCOL**

In this section, we will derive LMI conditions for exponential mean-square stability of the Markovian jump impulsive system (5), (6), (8) with respect to $x$. Note that the differential equation for $x$ given by (5) depends on $e_j(t) = e_j(t_k)$, $t \in [t_k, t_{k+1})$ with $j \neq \pi_k, \pi_k \in \mathcal{I}$ (0) only. Consider LKF:

$$V(t) = V(t, x_t, x_{t_1}) + \sum_{j=1, j \neq \pi_k}^{N} e_j^T(t) Q_j e_j(t),$$

$$\pi_k \in \mathcal{I},$$

$$V(t, x_t, x_{t_1}) = \bar{V}(t, x_t, x_{t_1}) + V_Q$$

$$V_Q = (\tau_M - \eta_m) \int_{t_0}^{t} e^{-2\alpha (t-s)} ds,$$

$$t \in [t_k, t_{k+1}), k \in \mathbb{Z} > 0, Q > 0, Q_1 = 0, j = 1, \ldots, N,$$

$$\bar{V}(t, x_t, x_{t_1})$$

is given by (12). The following statement holds:

**Lemma 2** If there exist positive constant $\alpha$, matrices $0 < Q \in \mathbb{R}^{n \times n}$, $0 < Q_j \in \mathbb{R}^{n \times n}$, $0 < U_j \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, N$, and $V(t)$ of (26) such that for any $i \in \mathcal{I}$ along (5)

$$E[V(t)] + 2a V(t) - \frac{1}{\tau_m - \eta_m} \sum_{i=1}^{N} e_j^T(t) Q_j e_j(t) \leq 0, t \in [t_k, t_{k+1})$$

(27)

with

$$\Phi_i = \begin{bmatrix} \Phi_{i1} & \Phi_{i2} \\ \Phi_{i1} & \Phi_{i2} \end{bmatrix} \leq 0,$$

(28)

holds, where

$$\Phi_{i1} = \sum_{j=1}^{N} \sum_{j \neq j}^{N} \pi_{j} C_j Q_j C_j^T Q_j - e^{-2\alpha (t-t_0)} Q_j,$$

$$\Phi_{i2} = \sum_{j=1}^{N} \pi_{j} C_j Q_j C_j^T Q_j,$$

$$\sum_{j \neq j}^{N} \pi_{j} C_j Q_j C_j^T Q_j,$$

$$\Phi_{i2} = \text{diag} \left[ \pi_{j} C_j Q_j C_j^T Q_j - \sum_{j \neq j}^{N} \pi_{j} C_j Q_j C_j^T Q_j \right]$$

Then $V(t)$ satisfies

$$E[V(t_{k+1}) - V(t_{k})] \leq \sum_{j=1, j \neq \pi_k}^{N} e_j^T(t_{k}) (Q_j - U_j) e_j(t_{k}) \leq 0, i \in \mathcal{I}.$$
Therefore,
\[ E\{V(t,x_{i1})\} \leq e^{-2\alpha (t-t_0)} \sum_{i=1}^{N} \{V_i(t_0)\}, \quad t \in [t_0,t_k+1]. \]

Note that
\[ E\{V_i(t_{k+1})\} = E\{V_i(t_{k+1}) + V_j(t_{k+1}) + \sum_{j=1}^{N} \sum_{j \neq i} e_j^T(t_{k+1}) Q_j e_j(t_{k+1})\} \]

and
\[ \sum_{j=1}^{N} \|e_j(t_{k+1})\|^2_{Q_j} \leq \|e(t_{k+1})\|^2_{Q}. \]

Taking (28) into account, we obtain
\[ E\{V_{i}(t_{k+1}|\sigma_k) = \delta = e_i^T(t_{k+1}) Q e_i(t_{k+1})\}. \]

Theorem 2
\[ \sum_{i=1}^{N} \sum_{j \neq i} e_j^T(t_{k+1}) Q_j e_j(t_{k+1}) \leq e^{-2\alpha (t_{k+1} - t_0)} \{V_i(t_{k+0})\} \leq e^{-2\alpha (t_{k+1} - t_0)} \{V_i(t_{k})\}. \]

The latter inequality, with \( k+1 \) replaced by \( k \) and (30) (18), which implies exponential mean-square stability of (5), (6), (8) with respect to \( x \).

Remark 5

Differently from Lemma 1, in Lemma 2 the inequality
\[ E\{V_{i}(t_{k+1})\} \leq e^{-2\alpha (t_{k+1} - t_0)} \{V_i(t_{k})\} \]

does not give a bound on \( e_{i,\sigma} \) since \( V_i(t_{k+1}) \) does not depend on \( e_{i,\sigma} \). That is why Lemma 2 guarantees only mean-square stability with respect to \( x \).

By using the above lemma and the arguments of Theorem 1, we arrive at the following result:

**Theorem 2**

Given \( 0 \leq \eta_m < \tau_M, \alpha > 0, 0 \leq \eta_i < 1, \sum_{i=1}^{N} \eta_i = 1, i, j \in I, l = 1, \ldots, N. \) Suppose there exist \( n \times n \) matrices \( P > Q, R_i > 0, \eta_j > 0, j = 0, 1, \ldots, N. \) The matrix inequalities (28), (29) and \( \sum_{i=1}^{N} \sum_{j \neq i} e_j^T(t_{k+1}) Q_j e_j(t_{k+1}) \leq e^{-2\alpha (t_{k+1} - t_0)} \{V_i(t_{k})\} \) are feasible, where the notation \( \Xi \) is given by (28), and where
\[ \dot{\tilde{X}} = A - \alpha \tilde{X} + \sum_{i=1}^{N} \sum_{j \neq i} e_j^T(t_{k+1}) Q_j e_j(t_{k+1}), \]
\[ \dot{\Xi} = A - \alpha \Xi + \sum_{i=1}^{N} \sum_{j \neq i} e_j^T(t_{k+1}) Q_j e_j(t_{k+1}), \]
\[ \dot{\tilde{X}} = A - \alpha \tilde{X} + \sum_{i=1}^{N} \sum_{j \neq i} e_j^T(t_{k+1}) Q_j e_j(t_{k+1}). \]

Then the solutions of (5), (6), (8) satisfy the bound (18), implying exponential mean-square stability with respect to \( x \). If the aforementioned matrix inequalities are feasible with \( \alpha = 0 \), then the solution bound holds also for a sufficiently small \( \alpha > 0 \).

Remark 6

Note that Theorem 1 under i.i.d. scheduling protocol guarantees exponential mean-square stability with respect to the

### Table I

**Complexity of stability conditions under different protocols**

| Method             | Decision variables | Number and order of LMI
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(RR)</td>
<td>8.5m^2 + 2.5n</td>
<td>two of 6n × 6n, 2n of 3n × 3n</td>
</tr>
<tr>
<td>(TOD/RR)</td>
<td>3.5n^2 + 3n</td>
<td>two of 5.5n × 5.5n, one of 2n × 2n</td>
</tr>
<tr>
<td>Theorem 1 (i.i.d.)</td>
<td>4.25m^2 + 4n</td>
<td>one of 8n × 8n, two of 2n × 2n</td>
</tr>
<tr>
<td>Theorem 2 (Markovian)</td>
<td>4.5m^2 + 4n</td>
<td>two of 5.5n × 5.5n, one of 2n × 2n, two of 3.5n × 1.5n</td>
</tr>
</tbody>
</table>

Note: Table I lists the complexity of stability conditions under different protocols. The results show that the (RR) and (TOD/RR) methods provide lower complexity compared to the i.i.d. scheduling protocol and the Markovian approach, which requires more complex calculations.

### V. Example: Batch Reactor

We illustrate the efficiency of the given conditions on a benchmark example of a batch reactor under the dynamic output feedback (see e.g., [8], [9], [19]), where \( N = 2 \) and

\[ A = \begin{bmatrix} 1.380 & -0.208 & 6.715 & -5.676 \\ -0.581 & -4.2902 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} \]

\[ B = \begin{bmatrix} 5.679 & 0 & 0 & 0 \\ 0 & 1.136 & -3.146 & 0 \\ 0 & 1.136 & 0 & 0 \end{bmatrix} \]

\[ C = \begin{bmatrix} \frac{C_1}{C_2} \\ \frac{C_1}{C_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \]

Assume that \( \beta_0 = 0, \sigma_0 = \pi_0 = 0, i = 0, 1, 2, \) which means that collisions do not occur. Let \( \beta_i = 0.6 \) and the transition matrix of Markov chain \( \sigma_k \in [1,2] \) as \( \Pi = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \). For the values of \( \eta_m \), given in Table I, by applying Theorems 1 and 2 with \( \beta_0 = 0 \), we obtain the maximum values of \( \tau_M = 2\alpha + 1 \) that preserve mean-square stability of the impulsive system (5)-(6) (see Table II). From Table II it is seen that for small transmission delays, our method essentially improves the results of [9], but is more conservative than the results obtained via the discrete-time approach. However, the latter approach becomes complicated for uncertain systems. Polytopic uncertainties in the system model can be easily included in our analysis [3], [4]. When \( \eta_m > 3\alpha/2 \) (\( \eta_m = 0.03, 0.04 \)), note that our method is still applicable.

Choosing \( \eta_m = 0.02 \), by Theorem 1 with \( \alpha = 0 \), we obtain the corresponding maximum values of \( \tau_M \) shown in Figure 2(a) for

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TABLE II
ESTIMATED MAXIMUM VALUES OF $\tau_M = M_\text{ATI} + M_\text{AD}$ FOR DIFFERENT $\eta_m$

<table>
<thead>
<tr>
<th>$\tau_M \setminus \eta_m$</th>
<th>0</th>
<th>0.004</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>[9] (MAD = 0.004, TOD)</td>
<td>0.0108</td>
<td>0.0133</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>[9] (MAD = 0.004, RR)</td>
<td>0.0088</td>
<td>0.0088</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>[8] (MAD = 0.03, TOD)</td>
<td>0.069</td>
<td>0.069</td>
<td>0.069</td>
<td>0.069</td>
<td>-</td>
</tr>
<tr>
<td>[8] (MAD = 0.03, RR)</td>
<td>0.068</td>
<td>0.068</td>
<td>0.068</td>
<td>0.068</td>
<td>0.068</td>
</tr>
<tr>
<td>[4] (TOD/RR)</td>
<td>0.035</td>
<td>0.031</td>
<td>0.037</td>
<td>0.053</td>
<td>0.059</td>
</tr>
<tr>
<td>[3] (RR)</td>
<td>0.042</td>
<td>0.044</td>
<td>0.053</td>
<td>0.058</td>
<td>0.063</td>
</tr>
<tr>
<td>Theorem 1 ($\beta_1 = 0.6$)</td>
<td>0.022</td>
<td>0.025</td>
<td>0.039</td>
<td>0.048</td>
<td>0.056</td>
</tr>
<tr>
<td>Theorem 2 ($\Pi_1$)</td>
<td>0.035</td>
<td>0.038</td>
<td>0.049</td>
<td>0.055</td>
<td>0.061</td>
</tr>
</tbody>
</table>

Fig. 2. (a) Estimated maximum values of $\tau_M(\beta_1)$ by Theorem 1 with $\alpha = 0$; (b) Estimated maximum values of $\tau_M(\pi_{11})$ by Theorem 2 with $\alpha = 0$.

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REFERENCES


VI. CONCLUSIONS

In this note, a time-delay approach has been developed for the stabilization of NCSs under stochastic protocol. Two types of stochastic protocols, which are defined by the i.i.d and Markovian processes are proposed. By developing appropriate Lyapunov methods, the exponential mean-square stability conditions for the delayed stochastic impulsive system were derived in terms of LMIs. Future work will involve the optimization of $\beta_i$, $i = 0, 1, \ldots, N$ and $\Pi$ to obtain less conservative results, the implementation aspects of the stochastic protocol in a real wireless network and the consideration of more general NCS models, including stochastic communication delays and scheduling protocols for the actuator nodes.