On Spectral Inequalities in Quantum Mechanics and Conformal Field Theory

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On Spectral Inequalities in Quantum Mechanics and Conformal Field Theory

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Master’s Thesis in Mathematics (30 ECTS credits)
Master Programme in Mathematics (120 credits)
Royal Institute of Technology year 2015
Supervisor at KTH was Ari Laptev
Examiner was Ari Laptev

TRITA-MAT-E 2015:17
ISRN-KTH/MAT/E--15/17--SE

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Abstract

Following Exner et al. (Commun. Math. Phys. 26 (2014), no. 2, 531–541), we prove new Lieb-Thirring inequalities for a general class of self-adjoint, second order differential operators with matrix-valued potentials, acting in one space-dimension. This class contains, but is not restricted to, the magnetic and non-magnetic Schrödinger operators. We consider the three cases of functions defined on all reals, all positive reals, and an interval, respectively, and acquire three different kinds of bounds.

We also investigate the spectral properties of a family of operators from conformal field theory, by proving an asymptotic phase-space bound on the eigenvalue counting function and establishing a number of spectral inequalities. These bound the Riesz-means of eigenvalues for these operators, together with each individual eigenvalue, and are applied to a few physically interesting examples.

Keywords: Schrödinger operators, Lieb-Thirring inequalities, commutation method, conformal field theory, Birman-Schwinger principle.

Även undersöker vi spektralegenskaperna för en klass operatorer från konform fältteori, genom att asymptotiskt begränsa antalet egenvärden med ett fasrymdsuttryck, samt genom att bevisa ett antal spekralolikheter. Dessa begränsar Riesz-medelvärdena för operatorerna, samt varje enskilt egenvärde, och tillämpas på ett par fysikaliskt intressanta exempel.

**Nyckelord:** Schrödingeroperatorer, Lieb-Thirring-olikheter, kommutationsmetoden, konform fältteori, Birman-Schwing- principen.
I am indebted, and would like to express my sincere gratitude, to my supervisor Ari Laptev, for introducing me to the use of spectral theory in mathematical physics. By suggesting more interesting problems than those that could be approached in this thesis, and by continually providing support and encouragement along the way, he has been a great inspiration. For all of your time spent, all theorems and approaches you have taught me, and all of our chats: thank you!

Since this thesis marks the end of my studies at KTH, I would also like to thank all of the people I have gotten to know. As usual, listing all the names would not be possible on a single page, so is therefore left as an exercise to the reader. A last expression of gratitude is due to the many professors who have fueled my interest in finding things out, and then especially to those who responded to my many questions “why is X?”, with a smile and “because Y”.

Oscar Mickelin
Stockholm, May 2015
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1.1 Motivation

Many of the tools in functional analysis have been created in order to answer questions from physics. Harmonic analysis, variational calculus and the theory of partial differential equations all originate from, and continue to contribute to, physical applications. In particular, purely mathematical work concerning the Schrödinger operator, which can be written as

$$-\Delta + V(x),$$

(1.1)

has led to advances in perturbation theory, operator theory and spectral theory, on the mathematical side of things. On the physical side, the theoretical work has also found immediate applications, as well as opened up new means of interpretation and methodology. There are therefore both mathematical and physical reasons to rigorously study questions arising in physics, as will be done in this thesis.

One question which has received significant attention during the latter half of the previous century concerns the stability of matter. The mechanisms underlying stability properties of both isolated atoms and bulk matter have been understood in a number of settings, but are neither physically nor mathematically obvious. The main tool used in proving that fermionic bulk mass is stable, is an inequality due to Lieb and Thirring [Lie76], which has since been found useful in both physics and mathematics. More results have been discovered after suitably generalizing this inequality, and the first part of this thesis proves a number of new generalized Lieb-Thirring inequalities.

Another area of active mathematical and physical research is conformal field theory, which describes quantum field theories invariant under conformal transformations [Blu09]. Examples of these include string theory and the Ising model, where a number of mathematically interesting operators are defined. The second part of this thesis starts with a detailed example concerning the spectrum of such an operator, and concludes by proving a number of spectral bounds for a family of operators.
These bounds are obtained by adapting arguments and techniques from classical spectral theory to this new setting, and can therefore also be seen as something of an introduction to many classical results.

1.2 Organization

The remainder of this text is organized as follows. Notation and definitions are first listed in Chapter 2, together with a few useful theorems to be used in the remainder of the text. Chapter 3 then introduces the stability of matter problem and motivates the questions investigated in the first part of the thesis. A summary of the results obtained is then provided in Section 3.3 with full details available in [Mic15]. In Chapter 4, we lastly investigate a family of operators from conformal field theory. Motivated by the first part of the thesis, we begin with a detailed study of an example in Section 4.2, after which we prove a number of spectral inequalities in Section 4.3.
2.1 Notation, conventions and definitions

We first fix the notation and conventions to be used in the text. All concepts in this section are standard and can be found in e.g. [Ree72, Ree78, Fri82, Maz85]. In what follows, the reader will be assumed familiar with basic quantum mechanics and spectral theory at the level of e.g. [Ree72, Tak08].

Sets and symbols

We let $\mathbb{R}^+ := [0, \infty)$ as a shorthand. The letter $\Omega$ will usually refer to a subset of $\mathbb{R}^n$. For $R$ a ring, we take $R^{p \times q}$ as the ring of all $p \times q$-matrices with entries in $R$. When $R$ denotes functions with values in $\mathbb{R}$ or $\mathbb{C}$, integrability and differentiability assumptions of elements in $R^{p \times q}$ will typically be understood element-wise. The restriction of an operator $T$ to a space $X$ will be denoted by $T|_X$, and the domain of $T$ by $D(T)$. We will write $|A|$ for the Lebesgue-measure of a set $A$, and $\chi_A$ for the characteristic function of $A$.

For two normed spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, we will say that there is a continuous inclusion from $X$ into $Y$ if there is a positive constant $C$ with

$$\|\psi\|_Y \leq C \|\psi\|_X,$$

for all $\psi \in X$. We will denote this as $X \hookrightarrow Y$.

Function spaces

Take $\Omega \subseteq \mathbb{R}^n$ and $(X, \|\cdot\|_X)$ as a normed space. For some $p$ with $1 \leq p \leq \infty$, we let $L^p(\Omega; X)$ denote

$$L^p(\Omega; X) := \left\{ \psi : \Omega \rightarrow X : \|\psi\|_{L^p(\Omega; X)} := \left( \int_\Omega \|\psi(x)\|_X^p \,dx \right)^{\frac{1}{p}} < \infty \right\}.$$
CHAPTER 2. PRELIMINARIES

When \( X = \mathbb{R} \) with the usual norm, we will also simply write \( L^p(\Omega) \). We will most often use \( X = \mathbb{C}^m \) for some integer \( m \) and the natural norm. For the case \( X = \mathbb{C}^{p \times q} \), the norm \( \| \cdot \|_X \) used in Equation (2.2) is the maximum norm of the entries (i.e. not the operator norm) to be able to denote element-wise integrability assumptions in a convenient manner.

We denote the set of \( k \) times differentiable functions with values in \( \mathbb{R} \) or \( \mathbb{C} \) on \( \Omega \) by \( C^k(\Omega) \), the set of smooth functions by \( C^\infty(\Omega) \), and the set of smooth functions with compact support by \( C^\infty_0(\Omega) \).

A (not necessarily continuous) function \( f_i' : \mathbb{R}^n \to \mathbb{R} \) is a weak partial derivative of a function \( f : \mathbb{R}^n \to \mathbb{R} \) if
\[
\int_{\mathbb{R}^n} f_i'(x)\phi(x)dx = -\int_{\mathbb{R}^n} f(x)\frac{\partial \phi(x)}{\partial x_i}dx,
\]
for all \( \phi \in C^\infty_0(\mathbb{R}^n) \). Iterating this definition, we will use multi-index notation to denote higher derivatives. Let \( |\alpha| = k \); the order \( \alpha \) weak derivative of \( f \) will then be denoted by \( \nabla^\alpha f \). Using this, we let \( W^{k,p}(\Omega) \) denote the Sobolev space of functions on the open set \( \Omega \subseteq \mathbb{R}^n \), by defining
\[
W^{k,p}(\Omega) := \{ \psi \in L^p(\Omega) : \nabla^\alpha f \in L^p(\Omega), \text{ for all } |\alpha| \leq k \}.
\]
We recall that \( W^{k,p}(\Omega) \) is a Banach-space with the norm
\[
\| \psi \|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \| \nabla^\alpha \psi \|_{L^p(\Omega)}.
\]
The convenient notation \( H^k(\Omega) := W^{k,2}(\Omega) \) will also be used in the following, and \( H^k(\Omega) \) is a Hilbert-space with the inner product
\[
\langle \psi, \phi \rangle_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \langle \nabla^\alpha \psi, \nabla^\alpha \phi \rangle_{L^2(\Omega)}.
\]

We will also refer to the Schwartz-space \( \mathcal{S}(\mathbb{R}^n) \), consisting of smooth functions with derivatives decreasing faster than any polynomial. In symbols, this can be defined as
\[
\mathcal{S}(\mathbb{R}^n) := \left\{ \psi \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x|^\alpha \cdot \left| \nabla^\beta \psi(x) \right| \leq \infty, \text{ for all } \alpha, |\beta| \geq 0 \right\}.
\]
Lastly, we will talk about Hölder spaces, or Hölder continuous functions, which consist of functions with strong regularity properties. Let \( l \in \mathbb{N} \) be a positive integer, and \( \alpha \in [0,1] \). We denote by \( C^{l,\alpha} \) the space of functions \( \psi \) for which
\[
|\psi|_{C^{l,\alpha}} := \sup_{x \neq y, \frac{|\beta|}{|\beta|} = l} \frac{|\nabla^\beta \psi(x) - \nabla^\beta \psi(y)|}{|x-y|^\alpha}
\]
is finite. We endow this space with the norm
\[
\|\psi\|_{C^{l,\alpha}} := \sup_{x \in \mathbb{R}^n, |\beta| \leq l} |\nabla^\beta \psi(x)| + |\psi|_{C^{l,\alpha}},
\]
which turns $C^{l,\alpha}$ into a Banach-space.

**Operator spaces**

We next define two useful spaces of operators. Let $(X, \langle \cdot, \cdot \rangle_X)$ be a separable Hilbert-space with basis $\{e_n\}_{n=1}^\infty$. If $T$ is a bounded, linear operator on $X$, we define the trace of $T$ by
\[
\text{Tr}(T) := \sum_{n=1}^\infty \langle Te_n, e_n \rangle_X.
\]
This sum is independent of the choice of the basis $\{e_n\}_{n=1}^\infty$; if it is finite, $T$ is said to be of *trace-class*. We denote all such $T$ by
\[
\mathcal{J}_1(X) := \{T : X \to X : \text{Tr}(T) < \infty\},
\]
with the associated and actually well-defined norm
\[
\|T\|_{\mathcal{J}_1(X)} := \text{Tr}(|T|).
\]

We use this, in turn, to define the class of *Hilbert-Schmidt operators*. Take
\[
\mathcal{J}_2(X) := \{T : X \to X : \text{Tr}(T^*T) < \infty\},
\]
and denote by $\|T\|_{\mathcal{J}_2(X)}$ the natural norm
\[
\|T\|_{\mathcal{J}_2(X)} := \sqrt{\text{Tr}(T^*T)}.
\]

These two classes of functions are related, as seen from the following result.

**Theorem 1** (Reed72). The following properties hold.

1. $S$ is in $\mathcal{J}_1(X)$ if and only if $S = T_1T_2$, for $T_1, T_2 \in \mathcal{J}_2(X)$.

2. $\|S\|_X \leq \|S\|_{\mathcal{J}_2(X)} \leq \|S\|_{\mathcal{J}_1(X)}$.

3. Every $S$ in $\mathcal{J}_1(X)$ or $\mathcal{J}_2(X)$ is compact.

Out of $\mathcal{J}_1(X), \mathcal{J}_2(X)$, the latter will be of more use in the following, since its norm can be explicitly computed in most situations of interest. We have the following result.
Theorem 2 ([Ree72]). An operator $T$ is Hilbert-Schmidt in $L^2(\mathbb{R})$ if and only if there exists some function $K(x, y)$ with

$$ (T\psi)(x) = \int_{\mathbb{R}} K(x, y)\psi(y)dy, \quad (2.15) $$

and

$$ \int_{\mathbb{R}^2} |K(x, y)|^2 dx\,dy < \infty. \quad (2.16) $$

In this case, we have

$$ \|T\|_{\mathcal{B}(X)}^2 = \int_{\mathbb{R}^2} |K(x, y)|^2 dx\,dy. \quad (2.17) $$

2.2 Useful theorems

This section lists some theorems in forms which will be useful at some point in the text. They are stated mostly without proof, and in less general form than is possible to achieve, for the sake of accessibility. This is intended to provide a quick reference for the reader, in a form streamlined for the exposition. Full proofs, statements, and longer discussions of the results can be found where indicated.

Theorem 3 (Sobolev embedding theorem; [Bre83, Maz85, Str08]). Let $\Omega \subseteq \mathbb{R}^n$ be an open ball of arbitrary radius. Depending on the relationship of the parameters $k, p, n$, we then have the following inclusions.

1. If $kp < n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $1 \leq q \leq \frac{np}{n - kp}$.
2. If $kp = n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$.
3. If $kp > n$, and $k - \frac{n}{p} \not\in \mathbb{N}$, then $W^{k,p}(\Omega) \hookrightarrow C^{[k - \frac{n}{p} - \frac{\alpha}{p}]}(\Omega)$.
4. If $kp > n$, and $k - \frac{n}{p} \in \mathbb{N}$, then $W^{k,p}(\Omega) \hookrightarrow C^{[k - \frac{n}{p} + \alpha]}(\Omega)$, for any $0 \leq \alpha \leq 1$.

Corollary 1 (Adapted from [Str08]). For $k > \frac{n}{2} + 2$, we have the (not necessarily continuous) inclusion

$$ H^k(\mathbb{R}^n) \subseteq C^2(\mathbb{R}^n). \quad (2.18) $$

Proof. Since any element of $H^k(\mathbb{R}^n)$ is square-integrable, it will be enough to show that

$$ H^k(\Omega) \hookrightarrow C^2(\Omega), \quad (2.19) $$

where $\Omega$ is an open ball of arbitrary radius. This follows from the third case in Theorem 3 above.

Theorem 4 (Young’s inequality; [Lie01c]). Take $f, g, h : \mathbb{R}^n \to \mathbb{R}$, and let $p, q, r \geq 1$ be real numbers with $p^{-1} + q^{-1} + r^{-1} = 2$. We then have the inequality

$$ \int_{\mathbb{R}^n} f(x)g(x - y)h(y)dx\,dy \leq C_{p,q,r,n} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}. \quad (2.20) $$
2.2. USEFUL THEOREMS

Here, the constant $C_{p,q,r:n}$ can be written as

$$C_{p,q,r:n} = (C_p C_q C_r)^n,$$  \hspace{1cm} (2.21)

where

$$C_p^2 = \frac{p^{1/p}}{p^{1/p'}},$$  \hspace{1cm} (2.22)

and $p, p'$ are Hölder conjugate through $p^{-1} + p'^{-1} = 1$. $C_q$ and $C_r$ are defined analogously. The constant $C_{p,q,r:n}$ is also sharp.

**Theorem 5** (Schur’s lemma; [Gra09]). Consider the integral operator

$$(T_K\psi)(x) := \int_{\mathbb{R}} K(x,y)\psi(y)dy.$$  \hspace{1cm} (2.23)

A sufficient condition for $T_K$ to be a bounded, linear operator on $L^2(\mathbb{R})$ is that $K$ satisfies the two conditions

$$\int_{\mathbb{R}} |K(x,y)|\,dx \leq C \text{ a.e. in } y,$$  \hspace{1cm} (2.24)

$$\int_{\mathbb{R}} |K(x,y)|\,dy \leq C \text{ a.e. in } x,$$  \hspace{1cm} (2.25)

for some constant $C$. In this case, it also holds that

$$\|T_K\|^2 \leq \left( \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |K(x,y)|\,dy \right) \cdot \left( \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |K(x,y)|\,dx \right).$$  \hspace{1cm} (2.26)
This chapter deals with the first problem in the thesis, namely Lieb-Thirring inequalities, and their relation to the stability of matter. Serving as motivation for the problems in the thesis, the exposition in Section 3.1 largely follows those of [Lie01c, Lie09, Sei10], where more details can be found. We then define a problem statement in Section 3.2, provide a solution in Section 3.3, and consider a few examples in Section 3.4.

3.1 The stability of matter problem

Stability of the first kind

One of the chronologically first important results in quantum mechanics concerns why isolated atoms are stable and their electrons do not collapse into their nuclei. This empirical fact is termed *stability of the first kind* and will now be explained.

By several orders of magnitude, the largest force acting on an electron bound by an atom with atomic number $Z$, is the Coulomb interaction. For the sake of simplicity, we will therefore work in the non-relativistic limit, neglect other forces, e.g. gravity and magnetic forces, and write our Hamiltonian as

$$H := -\frac{1}{2}\Delta - \frac{Z\alpha}{|x|},$$

after choosing our units appropriately. Here, $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine-structure constant, and $x \in \mathbb{R}^3$. The total energy of a wave-function $\psi \in H^1(\mathbb{R}^3)$ can then be written as

$$E(\psi) := \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \psi(x)|^2 - \frac{Z\alpha}{|x|} |\psi(x)|^2 \, dx.$$  \hspace{1cm} (3.2)

which gives the ground-state energy

$$E_0 := \inf_{\psi \in D(H), \|\psi\|_{L^2(\mathbb{R}^3)} = 1} E(\psi).$$  \hspace{1cm} (3.3)
Both $E(\psi)$ and $E_0$ are defined analogously when the Coulomb potential is replaced by a more general potential $V$.

Physically, electrons described by the wave-function $\psi$ will tend towards lower, more energetically favorable energy levels, and the difference in energy when changing levels is emitted in the form of light. Stability of the first kind is then equivalent to the emission of at most a finite energy, i.e. to imposing that $E_0 > -\infty$. This is the case for the Coulomb potential, and even for more general types of potentials. Indeed, one can use the Sobolev embedding theorem to show that if $V \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ denotes a potential function, then the Hamiltonian

$$H_V := -\frac{1}{2}\Delta + V(x)$$

induces stability of the first kind. In fact, there exist strictly positive constants $C$ and $D$ such that

$$CE(\psi) \geq \|\nabla \psi\|_{L^2(\mathbb{R}^3)} - D\|\psi\|_{L^2(\mathbb{R}^3)} \geq -D\|\psi\|_{L^2(\mathbb{R}^3)}.$$  \hspace{1cm} (3.5)

See e.g. [Lie01c] for a derivation.

**Stability of the second kind**

Another more recently considered stability problem arises when modelling bulk matter. We will study a system consisting of $N$ electrons orbiting $M$ nuclei, with atomic number $Z_k$, $k = 1, \ldots, M$, respectively. The nuclei will be assumed fixed at the positions $R_k$, and the positions of the electrons will be denoted by $x_i$, $i = 1, \ldots, N$. Neglecting magnetic and gravitational forces once again, we will study the Hamiltonian

$$H := -\frac{1}{2}\Delta + aV(x, R),$$

where $x = (x_1, \ldots, x_N)$, $R = (R_1, \ldots, R_M)$ and the multi-body Coulomb potential $V$ is given by

$$V(x, R) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|} - \sum_{i=1}^{N} \sum_{k=1}^{M} \frac{Z_k}{|x_i - R_k|} + \sum_{1 \leq k < l \leq M} \frac{Z_k Z_l}{|R_k - R_l|}.$$ \hspace{1cm} (3.7)

The first term here is the electron-electron repulsion, the second the electron-nucleus attraction, and the third the nucleus-nucleus repulsion. We can now define the ground state energy as

$$E_0 := \inf_{\psi \in D(H), \|\psi\|_{L^2(\mathbb{R}^3N)} = 1} E(\psi).$$

Removing the two positive terms in $V$, our potential is bounded below by sums of potentials of the form in the case of isolated atoms. Since all of these induce stability of the first kind, it follows that also $V$ does. The bulk matter does however introduce the following additional stability problem. In the thermodynamic limit of
large $N$ and $M$, the total energy of the modelled system equals $k \cdot (N + M)$, where $k = k(Z_1, \ldots, Z_M)$ is a constant. Relaxing this equality to an inequality, it follows that, at least in this limit, we certainly have the bound

$$E_0 \geq -k \cdot (N + M).$$  \hspace{1cm} (3.9)

A system for which Equation (3.9) holds for all $N$ and $M$, and not just asymptotically, is said to be stable of the second kind [Fis66, Dys67, Lie09]. The first point of this definition is that systems that do not have this extensive stability property would behave differently from physical intuition. If, for instance, $E_0$ would be proportional to a higher power of the particle numbers, then mixing two sets of particles would release a large amount of energy spontaneously. The second point of the definition is that not all matter is stable of the second kind. In fact, bosonic bulk matter asymptotically satisfies

$$E_0 = -C \min(N, M)^{5/3},$$  \hspace{1cm} (3.10)

for some constant $C$ [Lie09]. This can be expected intuitively: the ground-state energy in Equation (3.10) means that it is energetically favorable for bosons to clump together, and more so than fermions. This is in line with an intuitive interpretation of the exclusion principle.

Stability of the second kind is therefore intrinsically fermionic in nature, and, in particular, does not follow from stability of the first kind. Understanding the mechanisms giving rise to stability of the second kind is therefore of physical interest, and we will see below that the proof is also mathematically interesting. We remark in passing that also matter with a relativistic Hamiltonian, even with magnetic fields present, can be shown to be stable. This is however only true if the fine-structure constant $\alpha$ is smaller than some critical value $\alpha_c$; fortunately, the numerical value of $\alpha$ does happen to be smaller than $\alpha_c$. See e.g. [Lie88, Lie97a, Lie97b, Fra07] and the references therein for more information.

To conclude this section and to motivate the theme of the thesis, we now proceed to prove Equation (3.9). We will need the following three theorems, stated without proof. The last of these are the so-called Lieb-Thirring inequalities, which will be central in the remainder of this chapter.

**Theorem 6** (Lie09). $E_0$ decreases if all atomic numbers $Z_j$ for $j = 1, \ldots, M$ are replaced by some $Z \geq \max_j Z_j$.

**Theorem 7** (Baxter’s inequality; Lie09). Let $V(x, R)$ be a Coulomb potential with $N$ electrons and $M$ nuclei of equal atomic number $Z$, i.e.

$$V(x, R) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|} - \sum_{i=1}^{N} \sum_{k=1}^{M} \frac{Z}{|x_i - R_k|} + \sum_{1 \leq k < l \leq M} \frac{Z^2}{|R_k - R_l|},$$  \hspace{1cm} (3.11)

and take $D_j = \frac{1}{2} \min_{i \neq j} |R_i - R_j|$. Then

$$V(x, R) \geq -(2Z + 1) \sum_{i=1}^{N} \frac{1}{\min_j |x_i - R_j|} + \frac{Z^2}{8} \sum_{j=1}^{M} \frac{1}{D_j},$$  \hspace{1cm} (3.12)
Theorem 8 (Lieb-Thirring inequalities; [Hun02, Lie76, Wei96, Cwi77, Lap97, Lie01b, Roz76]). Study $L^2(\mathbb{R}^d)$ and take $\gamma \in \mathbb{R}_+$, with

$$
\begin{cases}
\gamma \geq \frac{1}{2}, & d = 1, \\
\gamma > 0, & d = 2, \\
\gamma \geq 0, & d \geq 3.
\end{cases}
$$

(3.13)

Assume $V_\gamma^{+d/2} \in L^1(\mathbb{R}^d)$ and let $\{-E_k\}_{k=1}^\infty$ denote the non-positive eigenvalues of $H = -\Delta + V(x)$.

(3.14)

There then exist constants $L_{\gamma,d}$ such that

$$
\sum_{k=1}^\infty |E_k|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+d/2} \, dx.
$$

(3.15)

Using these results, we can prove stability of the second kind for the Coulomb potential, following [Sol06]. We study electrons, but might just as well allow for any fermions and therefore let our Hamiltonian act on anti-symmetric wavefunctions with $q$ possible spin-states. By Theorem 6, we can assume that all nuclei have the same atomic number $Z$. Baxter’s inequality now implies

$$
-\frac{1}{2} \Delta + \alpha V \geq \sum_{i=1}^N \left( -\frac{1}{2} \Delta_i - \frac{(2Z + 1)\alpha}{\min_j |x_i - R_j|} \right),
$$

(3.16)

where $\Delta_i$ is the Laplacian acting on the coordinates of the $i$th fermion. Fixing a $\lambda \in \mathbb{R}_+$, we will perform a trick and bound $E_0$ by minimizing

$$
\langle H \psi, \psi \rangle_{L^2(\mathbb{R}^3N)} \geq \sum_{i=1}^N \left( -\frac{1}{2} \Delta_i - \frac{(2Z + 1)\alpha}{\min_j |x_i - R_j|} \right) \psi, \psi \rangle_{L^2(\mathbb{R}^3N)}
$$

(3.17)

$$
= -\lambda N + \sum_{i=1}^N \left( -\frac{1}{2} \Delta_i + \lambda - \frac{(2Z + 1)\alpha}{\min_j |x_i - R_j|} \right) \psi, \psi \rangle_{L^2(\mathbb{R}^3N)}.
$$

(3.18)

Since we are dealing with fermions, at most $q$ particles can occupy the same state, by the exclusion principle. Using the min-max principle, we can then bound the right hand side of Equation (3.18) from below using the sum of the $N$ lowest eigenvalues of the isolated Hamiltonian

$$
-\frac{1}{2} \Delta + \lambda - \frac{(2Z + 1)\alpha}{\min_j |x - R_j|}
$$

(3.19)

together with this spin constraint. Overcounting grossly, we instead sum over all the negative eigenvalues $-\lambda_n \leq 0$ of the isolated Hamiltonian in Equation (3.19), and do this $q$ times to obtain

$$
\langle H \psi, \psi \rangle_{L^2(\mathbb{R}^3N)} \geq -\lambda N - q \sum_{n=0}^\infty |\lambda_n|.
$$

(3.20)
3.2. Problem statement

Using the Lieb-Thirring inequality with $\gamma = 1, d = 3$ for the operator in Equation (3.19), we can write

\[
-\sum_{n=0}^{\infty} |\lambda_n| \geq -\frac{1}{2} 2^{\frac{5}{2}} L_{1,3} \int_{\mathbb{R}^3} \left( \lambda - \frac{(2Z + 1)\alpha}{\min_j |x - R_j|} \right)^{\frac{3}{2}} dx
\]

\[
\geq -2^{\frac{3}{2}} L_{1,3} M \int_{\mathbb{R}^3} \left( \lambda - \frac{(2Z + 1)\alpha}{|x|} \right)^{\frac{3}{2}} dx
\]

\[
= -2^{\frac{3}{2}} L_{1,3} \frac{5\pi^2}{4} (2Z + 1)\alpha^3 \sqrt{\lambda} M.
\]

The factor $2^{\frac{3}{2}}$ in Equation (3.22) is inherited from the factor $\frac{1}{2}$ in front of the Laplacian in Equation (3.6), as compared to Equation (3.14). Inserting this into Equation (3.20) yields

\[
E_0 \geq -\lambda N - q 2^{\frac{3}{2}} L_{1,3} \frac{5\pi^2}{4} (2Z + 1)\alpha^3 \sqrt{\lambda} M,
\]

which implies stability of the second kind using any positive $\lambda$. In order to achieve a tighter bound, we minimize the magnitude of this expression with respect to $\lambda$, which also allows us to compare our bound to empirical results. This gives

\[
E_0 \geq -\frac{3}{2} (5\pi^2 L_{1,3} q)^{\frac{3}{2}} (2Z + 1)^2 \alpha^2 M^{\frac{4}{3}} N^{\frac{1}{3}}
\]

\[
\geq -\frac{3}{2} (5\pi^2 L_{1,3} q)^{\frac{3}{2}} (2Z + 1)^2 \alpha^2 (M + N).
\]

As an example, we can mention neutral hydrogen, which has $Z = 1, N = M$, and $q = 2$. Inserting numerical values into Equation (3.26) yields the bound $E_0 \geq -30.52 \cdot M$ Rydberg [Lie09], which has an order of magnitude consistent with empirical results. A more refined analysis can decrease the magnitude of the constant by roughly four times; see e.g. [Lie09] for details.

3.2 Problem statement

The main tools of the last section were the Lieb-Thirring inequalities, which have found versatile applications in other parts of physics and mathematics, e.g. in analysis of Navier-Stokes equations [Con88, Lie02], and studies of ionization of atoms [Lie01a, Nam12]. They constitute interesting tools on their own, and therefore warrant even further study. This part of the thesis therefore treats one type of generalization of Theorem 8 by first passing to matrix-valued potentials and by also including a first order derivative term in the Hamiltonian. More specifically, we will study

\[
\mathcal{L} := -\frac{d^2}{dx^2} \otimes I + P(x) \frac{d}{dx} \otimes I + \tilde{Q}(x),
\]

(3.27)
where $P$ and $\tilde{Q}$ are given $n \times n$-matrix functions. The case where $P = 0$ has been treated in [Lap00, Ben00, Exn14] and we seek to generalize their results. This gives the following problem.

**Problem 1.** Under which conditions on $P$, $\tilde{Q}$, and the domain of $\mathcal{L}$ will an analogue of Theorem 8 hold?

### 3.3 Summary of results

This section provides one answer to the open-ended Problem 1, solving it for all self-adjoint $\mathcal{L}$ bounded from below. We will study three different settings of the problem, where the operator $\mathcal{L}$ acts on functions defined on all of $\mathbb{R}$, on $\mathbb{R}^+$, and on $[0,1]$, respectively. The last two cases will be endowed with Robin boundary conditions at their respective endpoints. In detail, let $\mathcal{S}, \mathcal{S}_0, \mathcal{S}_1$ be Hermitian $n \times n$-matrices. We define three different domains of $\mathcal{L}$ by

\begin{align}
\mathcal{D}(\mathcal{L}) : &= L^2(\mathbb{R}; \mathbb{C}^n), \\
\mathcal{D}_+(\mathcal{L}) : &= \{ \psi \in L^2(\mathbb{R}^+; \mathbb{C}^n) : \psi'(0) - \mathcal{S} \psi(0) = 0, \quad P(0) = 0 \}, \\
\mathcal{D}_{[0,1]}(\mathcal{L}) : &= \{ \psi \in L^2([0,1]; \mathbb{C}^n) : \\
&\quad \psi'(0) - \mathcal{S}_0 \psi(0) = 0 = \psi'(1) - \mathcal{S}_1 \psi(1), \quad P(0) = P(1) = 0, \}
\end{align}

respectively. Here, the conditions that $P$ vanishes at the end-points of the respective intervals are necessary for $\mathcal{L}$ to be symmetric. In fact, one can show that $\mathcal{L}$ is symmetric if and only if this is the case, together with

\begin{align}
P' &= -P, \\
\tilde{Q}' &= \tilde{Q} - P'.
\end{align}

Assuming this to be true, we can then rewrite $\mathcal{L}$ on the form

\begin{equation}
\mathcal{L} := -\frac{d^2}{dx^2} \otimes \mathbb{1} + P(x) \frac{d}{dx} \otimes \mathbb{1} + \frac{P'(x)}{2} - \frac{P(x)^2}{4} + Q(x),
\end{equation}

where

\begin{equation}
Q = \tilde{Q} - \frac{P'}{2} + \frac{P^2}{4}
\end{equation}

is an effective potential (and, importantly, a Hermitian matrix function). This will simplify the notation in the following results, where we establish Lieb-Thirring inequalities for all settings in Equations (3.28)-(3.30). A longer discussion, together with the proofs, can be found in [Mic15], where we also give a simple criterion for $\mathcal{L}$ to be bounded from below. The following theorems are the main results in the first part of the thesis.
3.3. SUMMARY OF RESULTS

Theorem 9. [Mic15] Let $\mathcal{L}$ act on $\mathcal{D}(\mathcal{L})$ in Equation (3.28). Let also $P(x)$ be anti-Hermitian, termwise weakly differentiable and $Q(x)$ Hermitian. If $\mathcal{L}$ is bounded from below and $\text{Tr}(Q^2) \in L^1(\mathbb{R})$, then

$$
\sum_{i=1}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}} \text{Tr}(Q^2) dx,
$$

(3.35)

where $\kappa_i$ denotes the multiplicity of the negative eigenvalue $-\lambda_i$. This bound is also sharp, in the sense that there are $P, Q$ such that the resulting inequality is false if the constant is replaced by a smaller number.

Theorem 10. [Mic15] Let $P(x)$ be anti-Hermitian, termwise weakly differentiable with $P(0) = 0$ and $Q(x)$ Hermitian. If $\mathcal{L}$ acts on $\mathcal{D}_+(\mathcal{L})$ in Equation (3.29), is bounded from below and if $\text{Tr}(Q^2) \in L^1(\mathbb{R}_+)$, then

$$
\frac{3}{4} \lambda_1 \text{Tr} \mathcal{S} + \frac{1}{2} (2 \kappa_1 - n) \lambda_1^{3/2} + \sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}_+} \text{Tr}(Q^2) dx + \frac{1}{4} \text{Tr} \mathcal{S}^3.
$$

(3.36)

Theorem 11. [Mic15] Let $P(x)$ be anti-Hermitian, termwise weakly differentiable with $P(0) = P(1) = 0$ and $Q(x)$ Hermitian. If $\mathcal{L}$ acts on $\mathcal{D}_{[0,1]}(\mathcal{L})$ in Equation (3.30), is bounded from below and if $\text{Tr}(Q^2) \in L^1([0,1])$, then

$$
\frac{3}{4} \lambda_1 \text{Tr} (\mathcal{S}_0 - \mathcal{S}_1) + \sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{0}^{1} \text{Tr}(Q^2) dx + \frac{1}{4} \text{Tr} (\mathcal{S}_0^3 - \mathcal{S}_1^3).
$$

(3.37)

Corollary 2. [Mic15] Let $\mathcal{L}$ act on $\mathcal{D}(\mathcal{L})$ in Equation (3.28) and take $\gamma \geq 3/2$. Let $P(x)$ be anti-Hermitian, termwise weakly differentiable and $Q(x)$ Hermitian. If $\mathcal{L}$ is bounded from below and $\text{Tr}(Q^{\gamma+1/2}) \in L^1(\mathbb{R})$, then

$$
\sum_{i=1}^{\infty} \kappa_i \lambda_i^\gamma \leq L_{\gamma,1} \int_{\mathbb{R}} \text{Tr}(Q^{\gamma+1/2}) dx.
$$

(3.38)

These bounds are also sharp.

Corollary 3. [Mic15] Take $\gamma \geq 3/2$ and assume that $\text{Tr} \mathcal{S}^3 \leq 0$. Let $P(x)$ be anti-Hermitian, termwise weakly differentiable with $P(0) = 0$ and $Q(x)$ Hermitian. If $\mathcal{L}$ acts on $\mathcal{D}_+(\mathcal{L})$ in Equation (3.29), is bounded from below and if $\text{Tr}(Q^{\gamma+1/2}) \in L^1(\mathbb{R}_+)$, then

$$
\frac{3}{4} \frac{B(\gamma - 3/2, 2)}{B(\gamma - 3/2, 5/2)} \lambda_1^{\gamma-1/2} \text{Tr} \mathcal{S} + \frac{1}{2} (2 \kappa_1 - n) \lambda_1^\gamma + \sum_{i=2}^{\infty} \kappa_i \lambda_i^\gamma \leq L_{\gamma,1} \int_{0}^{\infty} \text{Tr} \left( Q^{\gamma+1/2} \right) dx,
$$

(3.39)

where $B(p,q)$ denotes the Beta function

$$
B(p,q) = \int_{0}^{1} (1 - t)^{q-1} t^{p-1} dt.
$$

(3.41)
CHAPTER 3. LIEB-THIRRING INEQUALITIES

Corollary 4. [Mic15] Take $\gamma \geq 3/2$ and assume that $\text{Tr} \left( \mathfrak{G}_0^3 - \mathfrak{G}_1^3 \right) \leq 0$. Let $P(x)$ be anti-Hermitian, termwise weakly differentiable with $P(0) = P(1) = 0$ and $Q(x)$ Hermitian. If $L$ acts on $\mathbb{D}_{[0,1]}(L)$ in Equation (3.30), is bounded from below and if $\text{Tr}(Q_{\gamma + \frac{3}{2}}) \in L^1([0,1])$, then

$$3 \frac{B(\gamma - 3/2, 2)}{4 B(\gamma - 3/2, 5/2)} \lambda_{\gamma - 1/2} \text{Tr} (\mathfrak{G}_0 - \mathfrak{G}_1) + \sum_{i=2}^{\infty} \kappa_i \lambda_i^\gamma \leq L_{\gamma, 1} \int_0^1 \text{Tr} \left( Q^{\gamma + 1/2} \right) dx. \quad (3.42)$$

3.4 Examples, remarks and comments

We now illustrate the results in the preceding section by a few examples.

**Example 1.** Let $n = 1$, $P = Q = \mathfrak{G}_0 = 0$ and let us study the case of functions defined on the interval $[0,1]$. All eigenfunctions $\psi$ with eigenvalue $-\lambda \leq 0$ to the operator in Equation (3.33) then satisfy

$$\psi'(1) = \sqrt{\lambda} \frac{\sinh(\sqrt{\lambda})}{\cosh(\lambda)} \psi(1) = \mathfrak{G}_1 \psi(1). \quad (3.43)$$

The function $\lambda \mapsto \sqrt{\lambda} \frac{\sinh(\sqrt{\lambda})}{\cosh(\lambda)}$ is a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, so there is a unique eigenfunction for any $\mathfrak{G}_1 \geq 0$. The bound in Theorem [17] then becomes

$$\frac{3}{4} \lambda \text{Tr} (\mathfrak{G}_0 - \mathfrak{G}_1) = -\frac{3}{4} \lambda \sqrt{\lambda} \frac{\sinh(\sqrt{\lambda})}{\cosh(\lambda)} \leq \frac{1}{4} \text{Tr} (\mathfrak{G}_0^3 - \mathfrak{G}_1^3) = -\frac{1}{4} \left( \sqrt{\lambda} \frac{\sinh(\sqrt{\lambda})}{\cosh(\lambda)} \right)^3, \quad (3.44)$$

i.e.

$$\lambda^{3/2} \frac{\sinh(\sqrt{\lambda})}{\cosh(\lambda)} \left( 3 - \left[ \frac{\sinh(\sqrt{\lambda})}{\cosh(\lambda)} \right]^2 \right) \geq 0, \quad (3.45)$$

which is clearly true.

**Example 2.** By letting $\mathfrak{G}_1 \to 0$ in the example above, the inequality becomes an equality.

**Example 3.** One might hope to use Theorem [9] to exclude the existence of negative eigenvalues. Note that $Q_{\gamma + 1/2}^{-} = \left( \widetilde{Q} - \frac{P'}{2} + \frac{P_2}{4} \right)^{\gamma + 1/2} = 0$ for some $\gamma \geq 3/2$ would imply that our operator $L$ in Equation (3.33) has no negative eigenvalues. However, this is true precisely when $\left( \widetilde{Q} - \frac{P'}{2} + \frac{P_2}{4} \right)^{-} = 0$. To see this, note that under our
assumptions on $P$ and $\tilde{Q}$, the matrix $Q_-$ is Hermitian. We can then diagonalize $Q_-(x) = U(x)^* D(x) U(x)$, where

$$D(x) = \begin{bmatrix}
\mu_1(x) & 0 & 0 & \cdots & 0 \\
0 & \mu_2(x) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \mu_{n-1}(x) & 0 \\
0 & 0 & \cdots & 0 & \mu_n(x)
\end{bmatrix}, \quad (3.47)$$

has diagonal equal to the negative eigenvalues of $Q(x)$, and $U(x)$ is unitary. If $Q_+^{-1/2} = U(x)^* D(x)^{+1/2} U(x)$ is zero, then

$$D(x)^{+1/2} = \begin{bmatrix}
\mu_1(x)^{+1/2} & 0 & 0 & \cdots & 0 \\
0 & \mu_2(x)^{+1/2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \mu_{n-1}(x)^{+1/2} & 0 \\
0 & 0 & \cdots & 0 & \mu_n(x)^{+1/2}
\end{bmatrix}, \quad (3.48)$$

must be zero. This implies $D(x) = 0$, so $Q_- = U(x)^* D(x) U(x) = 0$, and the claim is proven.

**Example 4.** The main parts of the proofs of the theorems in Section 3.3 consist in making the substitution $\psi = \Psi \tilde{\psi}$ for some appropriate matrix function $\Psi$. This can physically be interpreted as a generalized gauge transformation. Together with the conditions on $P, \tilde{Q}$ in Equations (3.31) and (3.32), this allows us to reformulate the spectral problem into one that is well-known. This substitution can also be of interest when studying not necessarily self-adjoint operators of the form

$$H_1 := -\frac{d^2}{dx^2} + q(x), \quad (3.49)$$

acting on functions defined on $\mathbb{R}$. Here, $q$ is a complex-valued scalar function, and this situation was studied e.g. in [Abr01]. There, the authors showed that all eigenvalues $\lambda \not\in \mathbb{R}$ to $H_1$ satisfy

$$|\lambda| \leq \|q\|_{L^2(\mathbb{R})} / 4, \quad (3.50)$$

provided that $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Note that these eigenvalues will in general be complex-valued. However, by adding to this operator an appropriate term of the form $p(x) \frac{d}{dx}$, we can ensure that the resulting operator

$$H_2 := -\frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x) \quad (3.51)$$

only has real eigenvalues. The physical interpretation of this would be that an added generalized magnetic field lifts the non-self-adjointness. This procedure will work for certain conditions on $q$; in fact, we have the following result.
CHAPTER 3. LIEB-THIRRING INEQUALITIES

Proposition 1. Take \( q(x) = h(x) + ik(x) \). If there is a positive \( s \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \cap C^2(\mathbb{R}) \) s.t.

\[
I_{s,k}(x) := s(x) \int_0^x \frac{k(y)}{s(y)} \, dy \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}),
\]

(3.52)

\[
\frac{s'(x)}{s(x)} \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}),
\]

(3.53)

\[
\frac{s''(x)}{s(x)} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),
\]

(3.54)

then there is a \( p : \mathbb{R} \to \mathbb{C} \) s.t.

\[
H_2 = -\frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x)
\]

(3.55)

only has real eigenvalues \( \lambda \). The negative eigenvalues of \( H_2 \) all satisfy

\[
4 |\lambda| \leq \left\| q + \frac{p^2}{4} - \frac{p'}{2} \right\|^2_{L^1(\mathbb{R})}.
\]

(3.56)

Proof. Take

\[
f(x) := \frac{s'(x)}{s(x)}.
\]

(3.57)

Then \( \exp(\int_0^x f(y) \, dy) = s(x) \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \), so take

\[
g(x) := \exp \left( \int_0^x f(y) \, dy \right) \left[ 1 + 2 \int_0^x k(y) \exp \left( -\int_0^y f(t) \, dt \right) \, dy \right]
\]

(3.58)

\[
= s(x) + I_{s,k}.
\]

(3.59)

By assumption, \( g \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \), and differentiating \( g \) results in \( g'(x) - f(x)g(x) = 2k(x) \). If we take \( p := f + ig \), then

\[
0 = k + \frac{f g}{2} - \frac{g'}{2} = \text{Im} \left( q + \frac{p^2}{4} - \frac{p'}{2} \right).
\]

(3.60)

Applying the generalized gauge transformation to \( H_2 \) in this scalar case shows that it has the same eigenvalues as the operator

\[
-\frac{d^2}{dx^2} + q(x) + \frac{p(x)^2}{4} - \frac{p'(x)}{2}.
\]

(3.61)

The details of this calculation can be found in [Mic15]. Equation (3.60) now shows that the gauge-transformed operator has real potential part. It is therefore symmetric and we have

\[
\|g'\|_{L^1(\mathbb{R})} \leq \|fg\|_{L^1(\mathbb{R})} + 2\|k\|_{L^1(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} + 2\|k\|_{L^1(\mathbb{R})},
\]

(3.62)

\[
\|g'\|_{L^2(\mathbb{R})} \leq \|fg\|_{L^2(\mathbb{R})} + 2\|k\|_{L^2(\mathbb{R})} \leq \|f\|_{L^4(\mathbb{R})} \|g\|_{L^4(\mathbb{R})}^2 + 2\|k\|_{L^2(\mathbb{R})}.
\]

(3.63)
Together with the assumptions in Equation (3.52), this guarantees that the potential of the gauge-transformed operator is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Applying the result from \cite{Abr01} cited above, we conclude the desired bounds on the eigenvalues.

An example of a function satisfying the last two assumptions in Equations (3.53) and (3.54) is e.g. $s(x)$ decaying as $\frac{1}{|x|^{\beta}}$ for $\beta \geq 1$, when $|x|$ is large. The first assumption is satisfied e.g. when $k(x) = s(x)$, since then

$$
\int_{\mathbb{R}} |I_{s,k}|^2 \, dx = \int_{\mathbb{R}} |s(x)|^2 \left| \int_0^{|x|} \frac{k(y)}{s(y)} \, dy \right|^2 \, dx \leq \|s\|_{L^2(\mathbb{R})}^2 : \|k/s\|_{L^1(\mathbb{R})}^2 ,
$$

(3.64)

and

$$
\int_{\mathbb{R}} |I_{s,k}|^4 \, dx = \int_{\mathbb{R}} |s(x)|^4 \left| \int_0^{|x|} \frac{k(y)}{s(y)} \, dy \right|^4 \, dx \leq \|s\|_{L^4(\mathbb{R})}^4 : \|k/s\|_{L^1(\mathbb{R})}^4 .
$$

(3.65)

**Example 5.** The following is a formal example of how the bounds in Corollary 4 relate to those in Corollary 2. It is included for the sake of physical intuition and is not intended to be mathematically rigorous. In a special case, we will see that Corollary 2 can be seen as a consequence of Corollary 4, albeit with less tight control in the resulting bounds.

We now deal with trapped particles, i.e. quantum mechanical systems confined to the bounded interval $I := [0,1]$ of the real line, when subjected to a potential $V$. This can physically be modelled by forming a potential

$$V_a(x) = \begin{cases} V(x), & x \in I, \\ a \otimes 1, & x \notin I, \end{cases}
$$

(3.66)

for some positive $a$. Solving the associated Schrödinger equation shows that its solutions have components of the form $\psi(x) = C e^{\pm \sqrt{a-x}}$ outside $I$, for all eigenvalues $\lambda \leq a$. Formally letting $a \to \infty$, this means that all $L^2(\mathbb{R})$-solutions to the Schrödinger equation with the formal potential

$$V_\infty(x) = \begin{cases} V(x), & x \in I, \\ \infty \otimes 1, & x \notin I, \end{cases}
$$

(3.67)

should be interpreted as being zero on $\mathbb{R} \setminus I$. Assuming that the Lieb-Thirring inequalities are continuous under this limiting procedure, we obtain

$$
\sum_{i=1}^{\infty} \kappa_i \lambda_i^\gamma \leq L_{\gamma,1} \int_{\mathbb{R}} \text{Tr}((V_\infty)^{\gamma+\frac{1}{2}}) \, dx = L_{\gamma,1} \int_{I} \text{Tr}(V_\infty^{\gamma+\frac{1}{2}}) \, dx.
$$

(3.68)

Another way of modelling this physical scenario is contained in our setting of the interval. If we put $\mathcal{G}_0 = \mathcal{G}_1$ to be invertible, we fulfill the conditions of Corollary 4. Here, the associated solutions to the Schrödinger equation have components which satisfy $\psi(0) = \mathcal{G}_0^{-1} \psi'(0)$ and $u(1) = \mathcal{G}_1^{-1} u'(1)$. If we put $\mathcal{G}_0 = r \otimes 1 = \mathcal{G}_1$ and formally let $r \to \infty$, we obtain the boundary conditions $\psi(0) = 0 = \psi'(1)$, i.e. the
CHAPTER 3. LIEB-THIRRING INEQUALITIES

same as in the case above. Again, assuming that our Lieb-Thirring inequalities are continuous in this limit, we obtain

\[ \sum_{i=2}^{\infty} \kappa_i \lambda_i^\gamma \leq L_{\gamma,1} \int_0^1 \text{Tr} \left( V_{\gamma}^{\gamma+1/2} \right) \, dx. \]  

This bound is of the same form as in Equation (3.68), but with the ground state excluded.

As a simple numerical example of this formal setting, we consider the case of a square potential well, i.e. we put \( P = 0 \) and \( V(x) = -M \) for \( x \in [0, 1] \). Here, we consider the formal boundary values \( \psi(0) = 0 = \psi(1) \). Solving the Schrödinger equation results in a discrete spectrum of negative eigenvalues of the form \( \lambda_n = M - n^2 \pi^2 \), for \( n \in \{1, 2, \ldots, \lfloor \sqrt{M/\pi^2} \rfloor \} \). In Figure 3.1(a), we plot the quotient

\[ Q := L_{\gamma,1} \int_0^1 \text{Tr} \left( V_{\gamma}^{\gamma+1/2} \right) \, dx \cdot \left( \sum_{i=2}^{\infty} \kappa_i \lambda_i^\gamma \right)^{-1}, \]

as a function of the parameters \( M \geq 0, \gamma \geq 3/2 \). The quotient is always greater than one and Figure 3.1(b) shows the parameter interval in which it asymptotically tends to one.
4.1 Introduction and problem statement

This chapter investigates spectral questions arising in conformal field theory, which describes quantum field theories invariant under conformal transformations [Blu09]. We will study a family of operators from two-dimensional conformal field theory, which model phenomena in topological string theory and in quantum Liouville theory. The physical backgrounds are technically involved, and will not be elaborated upon. Interested readers are instead referred to e.g. [Fad86, Gra14], and the remainder of the thesis will treat only spectral-theoretical notions.

We will be interested in qualitative and quantitative spectral properties, together with concrete examples. The first of these items includes e.g. criteria for discreteness of spectra, and the second consists in bounds on the number and the moments of eigenvalues of the operators. Because of the immediate physical applications, concrete examples also give valuable information and will be discussed below.

The remainder of this chapter is organized as follows. We first define the family of operators under consideration and introduce some auxiliary notation. Section 4.2 then considers a detailed example of a generalization of an operator treated in [Fad13]. We lastly turn to a more general class of operators in Section 4.3, and study the asymptotic behaviour of the distribution of eigenvalues as well as establish a number of spectral bounds.

Definition of operators

Let \( \omega' \) be an imaginary number. We study functions in \( L^2(\mathbb{R}) \) and define an operator \( U \) with inverse \( U^{-1} \) by

\[
(U\psi) (x) := \psi(x + 2\omega'),
\]
\[
(U^{-1}\psi) (x) := \psi(x - 2\omega').
\]
With these definitions, the free operator
\[ H_0 := U + U^{-1} \] (4.3)
can be seen as a kind of discretized Laplacian, since
\[ \left( H_0 - \frac{2}{(2\omega')^2} \psi \right) (x) \to \psi''(x), \] (4.4)
as \( \omega' \to 0 \), for \( \psi \) analytic at \( x \). Including some potential function \( V : \mathbb{R} \to \mathbb{R} \), we will therefore consider the discretized Schrödinger operator defined by
\[ H := H_0 + V. \] (4.5)

In the applications to conformal field theory treated here, it is common to follow [Zam96] and adopt the parametrization \( \omega' = \frac{ib}{\pi} \), for some \( b \in \mathbb{R}_+ := [0, \infty) \), and to introduce the auxiliary constant \( \omega = \frac{i}{2\pi} \). We will use this in the following, together with the resulting normalization condition \( \omega \omega' = -\frac{1}{4} \). We also define \( \omega'' = \omega + \omega' \).

**Domains and notation**

We define the domain of \( U \) to be those \( \psi \in L^2(\mathbb{R}) \) which can be continued analytically into the set \( \{ x + iy : x, y \in \mathbb{R}, 0 < y < 2|\omega'| \} \), with \( \psi(x + iy) \in L^2(\mathbb{R}) \) for \( 0 \leq y < 2|\omega'| \). We also require that the limit
\[ \psi(x + 2\omega' - i\epsilon) := \lim_{\epsilon \searrow 0} \psi(x + 2\omega' - i\epsilon) \] (4.6)
extists in \( L^2(\mathbb{R}) \). Likewise, the domain of \( U^{-1} \) is those \( \psi \in L^2(\mathbb{R}) \) with analytic continuation into the strip \( \{ x + iy : x, y \in \mathbb{R}, -2|\omega'| < y < 0 \} \), wherein \( \psi(x + iy) \in L^2(\mathbb{R}) \). Again, we require that
\[ \psi(x - 2\omega' + i\epsilon) := \lim_{\epsilon \searrow 0} \psi(x - 2\omega' + i\epsilon) \] (4.7)
extists in \( L^2(\mathbb{R}) \). The domain of multiplication by \( V \) is naturally defined to be
\[ D(V) := \{ \psi \in L^2(\mathbb{R}) : V(x)\psi(x) \in L^2(\mathbb{R}) \}. \] (4.8)

It can be verified by computation that \( U \) and \( V \) are self-adjoint on their respective domains. This can also be seen by using the functional calculus form of the spectral theorem, since
\[ U = \exp \left[ 2\omega' i \left( -\frac{i}{dx} \right) \right], \] (4.9)
shows that \( U \) is a real function of the self-adjoint operator \( -i \frac{d}{dx} \).

We will in the following use the Fourier-transform to obtain spectral information, and will adopt the convention of the Fourier-transform given by
\[ (\mathcal{F}\psi)(p) := \int_{\mathbb{R}} e^{-2\pi ipx} \psi(x) dx, \] (4.10)
\[ (\mathcal{F}^{-1}\hat{\psi})(x) := \int_{\mathbb{R}} e^{2\pi ipx} \hat{\psi}(p) dp. \] (4.11)
Remark 1. For later use, we rewrite the domain $D(U) \cap D(U^{-1})$. By the Paley-Wiener theorem [Ree75], this set consists precisely of those functions $\psi \in L^2(\mathbb{R})$ with $\cosh\left(\frac{\pi ip}{\omega}\right) \hat{\psi}(p) \in L^2(\mathbb{R})$. Any $\psi \in D(U) \cap D(U^{-1})$ then satisfies

$$\int_{\mathbb{R}} \cosh^2\left(\frac{\pi ip}{\omega}\right) |\hat{\psi}(p)|^2 \, dp \geq C_k \int_{\mathbb{R}} |p|^{2k} |\hat{\psi}(p)|^2 \, dp,$$

(4.12)

for all positive integers $k$, and some $C_k > 0$ which are independent of $\psi$. It then follows that $D(U) \cap D(U^{-1}) \subseteq H^k(\mathbb{R})$, for all $k$. By Corollary 1, we then in particular have $\psi \in C^\infty(\mathbb{R})$, so $\psi \in L^p(\mathbb{R})$, for $2 \leq p \leq \infty$.

4.2 Example: spectral properties of $H^\mu$

Retaining the notation from Section 4.1, we will consider a potential given by

$$V(x) = e^{\frac{\pi i x}{\omega}}.$$

(4.13)

The associated discretized Schrödinger operator will be denoted by

$$H^0\psi(x) := \psi(x + 2\omega') + \psi(x - 2\omega') + e^{\frac{\pi i x}{\omega}} \psi(x),$$

(4.14)

on the domain specified in Section 4.1. Originally defined in [Fad86], this operator has applications in representations of quantum groups [Pen01], and in quantum Teichmüller theory [Kas01a, Kas01b], where related difference equations can be reduced to this form. It is therefore of interest to study the spectral properties of this operator. In [Kas01a, Kas01b], it was shown to have absolutely continuous spectrum $[2, \infty)$, and in [Fad15], explicit formulas were given for its resolvent, Jost solutions, and eigenfunction expansion.

In this section, we consider a generalization of the operator in Equation (4.14), by including a first-derivative term representing a magnetic field. This is in analogue with the procedure in Chapter 3 and leads to an interesting mathematical structure. More specifically, we define a new operator $H^\mu$ by

$$H^\mu\psi(x) := \psi(x + 2\omega') + \psi(x - 2\omega') + 2i\mu \omega' \left[\psi(x + 2\omega') - \psi(x - 2\omega')\right] + e^{\frac{\pi i x}{\omega}} \psi(x).$$

(4.15)

Here, the term in brackets is a discretization of a first-derivative term, which physically models a constant magnetic field with strength $\mu$. We will also refer to the operator

$$H^0_\mu\psi(x) := \psi(x + 2\omega') + \psi(x - 2\omega') + 2i\mu \omega' \left[\psi(x + 2\omega') - \psi(x - 2\omega')\right]$$

(4.16)

as the free operator. Our focus will be to derive detailed spectral properties of $H^\mu$ in Equation (4.15) as a function of the coupling constant $\mu$, by following the techniques in [Fad15]. We will show that the spectra for the free operators are
qualitatively distinct for the three cases $2|\mu\omega'| < 1$ (weak magnetic field), $2|\mu\omega'| = 1$ (critical magnetic field), and $2|\mu\omega'| > 1$ (strong magnetic field), respectively. This corresponds to an interplay of the first derivative term and the second derivative term in Equation (4.16), with $H_0^\mu$ assuming the spectrum of the dominating term. We will find these spectra, and compute explicit formulas for the generalized eigenfunctions for the non-free operator, in all these three cases. Where successful, formulas for Jost solutions and resolvents will be calculated, and we will also derive appropriate eigenfunction expansion theorems.

**Origin of operators**

Our operators are constituted by instances of the *Weyl-operators* $U(u), V(v)$, which act on $L^2(\mathbb{R})$ for $u, v \in \mathbb{C}$. These are defined by

\[
[U(u)\psi](x) = \psi(x - u), \quad (4.17)
\]

\[
[V(v)\psi](x) = e^{-ivx}\psi(x), \quad (4.18)
\]

and satisfy the commutation relation

\[
U(u)V(v) = e^{iuv}V(v)U(u). \quad (4.19)
\]

By the Stone-von Neumann theorem [Ree72, Tak08], this commutation relation characterizes the operators $U$ and $V$ up to unitary equivalence and has been historically important in formulating the foundations of quantum mechanics [Ros04]. In our case, we define two operators as

\[
[U\psi](x) = \psi(x + 2\omega'), \quad (4.20)
\]

\[
[V\psi](x) = e^{\pi ix\omega}\psi(x), \quad (4.21)
\]

and note that we can then write

\[
H^\mu = U + U^{-1} + 2\mu i\omega'[U - U^{-1}] + V. \quad (4.22)
\]

**Remark 2.** Note that for analytic $\psi$, we have

\[
\lim_{\omega' \to 0} \frac{H^\mu - V - 2}{(2\omega')^2} \psi(x) = -\psi''(x) + i\mu \psi'(x), \quad (4.23)
\]

so $H^\mu$ is indeed a discretized version of the Schrödinger operator in the presence of a constant magnetic field.

We note for use with the algebra arising below that our specific choice of $V$ gives

\[
\mathcal{F}U\mathcal{F}^{-1} = V^{-1}, \quad \mathcal{F}V\mathcal{F}^{-1} = U^{-1}, \quad (4.24)
\]

and remark that our operator can also be written as

\[
H^\mu = (1 - b\mu)U + (1 + b\mu)U^{-1} + V. \quad (4.25)
\]
The modular quantum dilogarithm

A special function, termed the modular quantum dilogarithm, was shown to be linked to the spectral properties of \( H^0 \) in [Rui97, Kas01a, Kas01b, Der14, Fad15]. This function will be important also in this note, and is defined by

\[
\gamma(z) = \exp \left( -\frac{1}{4} \int_{\mathbb{R}} \frac{e^{itz}}{\sin(\omega t) \sin(\omega' t)} \, dt \right).
\]

(4.26)

Here, \( z \in \mathbb{C} \) has \( \text{Im}(z) \leq |\omega + \omega'| \) and the contour of integration avoids the singularity at \( t = 0 \) by an indentation into the upper complex plane. \( \gamma(z) \) has a number of important algebraic properties that will be used in the following and are listed below; they are by now practically standard to use in connection with the spectral properties of operators like \( H^0 \). Proofs and additional properties can be found in e.g. [Kas01a, Fad01, Vol05], and the formulations below are taken from [Fad15].

**Lemma 1.** (Fad15) The quantum dilogarithm has the following properties.

1. \( \gamma(z) \) can be continued meromorphically to all of \( \mathbb{C} \). It has poles at \( -(2m + 1)\omega - (2n + 1)\omega' \), for \( m, n \in \mathbb{N} \). If we take

\[
\beta = \frac{\pi}{12} \left( \tau + \frac{1}{\tau} \right),
\]

(4.27)

then we also have

\[
\gamma(z - \omega'') = \frac{e^{i(z - \beta)}}{2\pi z} + O(1),
\]

(4.28)

as \( z \to 0 \).

2. \( \gamma(z) \) satisfies

\[
\gamma(z + \omega') = \left( 1 + e^{-\frac{\pi i}{\omega'}} \right) \gamma(z - \omega'),
\]

(4.29)

\[
\gamma(z + \omega) = \left( 1 + e^{-\frac{\pi i}{\omega}} \right) \gamma(z - \omega),
\]

(4.30)

The main use of the quantum dilogarithm in this context comes from the fact that it can be used to describe eigenfunctions to \( H^0 \) in the momentum-basis. By applying the Fourier-transform to the eigenvalue equation of Equation (4.14), we obtain

\[
\left( \mathcal{F} H^0 \mathcal{F}^{-1} \hat{\psi} \right)(p) = \hat{\psi}(p + 2\omega') + 2 \cosh \left( \frac{\pi i p}{\omega} \right) \hat{\psi}(p) = \lambda \hat{\psi}(p).
\]

(4.31)

This was shown in [Kas01a, Kas01b] to have a generalized solution for \( \lambda \in [2, \infty) \), by using the parametrization \( \lambda = 2 \cosh \left( \frac{\pi k}{\omega} \right) \). Real \( k \) then correspond to \( \lambda \in \sigma(H^0) \) and \( k \) with \( 0 < \text{Im}(k) \leq |\omega| \) correspond to \( \lambda \in \rho(H^0) \). Equation (4.31) is explicitly solved by

\[
\varphi_0(p, k) = c(k)e^{-\pi \omega(p - \omega'')^2} \gamma(p - k - \omega'') \gamma(p + k - \omega'').
\]

(4.32)
Here, \( c(k) = e^{-i \beta - i \pi k^2} \) is a phase normalization constant. We will use and modify this formula in what follows below.

We next investigate the spectrum of \( H^\mu \) and divide into three cases. The analysis of each section is similar and therefore structured in the same way.

**Weak magnetic field:** \( |\mu| < 1 \)

**Spectrum of \( H^\mu \)**

**Proposition 2.** \( H^\mu \) is unitarily equivalent to \( \sqrt{1 - b^2 \mu^2} H^0 \).

**Proof.** Let \( a, b \) be real numbers and define

\[
T_{a,b} : D(H^0) \to D(H^\mu)
\]

\[
\psi(x) \mapsto e^{i a x} \psi(x + b).
\]

(4.33)

(4.34)

Integrating by parts, we see that \( T_{a,b} \) is unitary. If we now choose

\[
a_* = \frac{1}{4i \omega'} \ln \left( \frac{1 + b \mu}{1 - b \mu} \right) \in \mathbb{R},
\]

(4.35)

\[
b_* = -\frac{\omega}{2\pi i} \ln \left( 1 - b^2 \mu^2 \right) \in \mathbb{R},
\]

(4.36)

then

\[
H^\mu T_{a_*,b_*} \psi(x) = (1 - b \mu) U T_{a_*,b_*} \psi(x) + (1 + b \mu) U^{-1} T_{a_*,b_*} \psi(x)
\]

\[
+ V T_{a_*,b_*} \psi(x)
\]

(4.37)

\[
= e^{i a_* x} (1 - b \mu) e^{2i \omega' a_*} \psi(x + b + 2\omega') + e^{i a_* x} \psi(x + b)
\]

(4.38)

\[
+ e^{i a_* x} (1 + b \mu) e^{-2i \omega' a_*} \psi(x + b - 2\omega')
\]

(4.39)

Note that this in particular implies that

\[
\sigma(H^\mu) = \sqrt{1 - b^2 \mu^2} \sigma(H^0) = [2 \sqrt{1 - b^2 \mu^2}, \infty),
\]

(4.40)

with multiplicity two. Next, in order to obtain explicit formulas for the remaining spectral properties of interest, we apply this to the results in [Fad15]. We emphasize that all calculations below reduce to those of [Fad15] if we put \( \mu = 0 \).

**Resolvent and eigenfunctions of the free operator**

Applying the Fourier-transform to the eigenvalue equation corresponding to Equation (4.16), we obtain

\[
\left( \mathcal{F} H_0^\mu \mathcal{F}^{-1} \psi \right)(p) = f(p) \hat{\psi}(p),
\]

(4.41)
4.2. EXAMPLE: SPECTRAL PROPERTIES OF $H^\mu$

for $f(p) := 2 \left[ \cosh \left( \frac{\pi i p}{\omega} \right) + b \mu \sinh \left( \frac{\pi i p}{\omega} \right) \right]$. We can also rewrite

$$f(p) = A \cosh \left( \frac{\pi i p}{\omega} + \theta \right) = A \cosh \left( \frac{\pi i \tilde{p}}{\omega} \right),$$  \hspace{1cm} (4.42)

where we introduced the auxiliary parameters

$$A = 2 \sqrt{1 - b^2 \mu^2},$$  \hspace{1cm} (4.43)

$$\cosh (\theta) = \frac{1}{\sqrt{1 - b^2 \mu^2}},$$  \hspace{1cm} (4.44)

$$\sinh (\theta) = \frac{b \mu}{\sqrt{1 - b^2 \mu^2}},$$  \hspace{1cm} (4.45)

$$\tilde{p} = p + \frac{\omega \theta}{\pi i}. \hspace{1cm} (4.46)$$

The resolvent at $\lambda \in \mathbb{C}$ is therefore a bounded operator in the momentum representation precisely when $\lambda \not\in \text{Ran}(f) = [2 \sqrt{1 - b^2 \mu^2}, \infty)$. We can then form

$$R_0(x; \lambda) = \int_\mathbb{R} \frac{e^{2\pi i px}}{f(p) - \lambda} dp,$$  \hspace{1cm} (4.47)

which gives the resolvent to be

$$(R(\lambda)\psi)(x) = \int_\mathbb{R} R_0(x - y; \lambda)\psi(y) dy,$$  \hspace{1cm} (4.48)

after applying the inverse Fourier-transform. We now parametrize the spectrum by writing $\lambda = f(k) = A \cosh \left( \frac{\pi i k}{\omega} \right)$, for $\tilde{k} = k + \frac{\omega \theta}{\pi i}$. Here, $k \in \mathbb{R}$ corresponds to $\lambda \in \sigma(H^\mu)$, which shows that the spectrum has multiplicity two. A $\lambda$ in the resolvent-set corresponds to a $k$ with $0 < \text{Im}(k) \leq |\omega|$. With this parametrization, we note that the integrand in Equation (4.47) has singularities at $p = k + 2n \omega$ and $p = -k - 2n \omega$, for $n, m$ integers. We can then compute the integral in Equation (4.47) using the residue theorem, and obtain

$$R_0(x; \lambda) = \frac{2\omega}{A \sinh \left( \frac{\pi i k}{\omega} \right)} \left[ \frac{e^{-2\pi i (k + \frac{2n \omega}{\pi i}) x}}{1 - e^{-4\pi i \omega x}} + \frac{e^{2\pi i \tilde{k} x}}{1 - e^{4\pi i \omega x}} \right],$$  \hspace{1cm} (4.49)

$$= \frac{2\omega e^{i \omega x}}{A \sinh \left( \frac{\pi i k}{\omega} \right)} \left[ \frac{e^{-2\pi i \tilde{k} x}}{1 - e^{-4\pi i \omega x}} + \frac{e^{2\pi i \tilde{k} x}}{1 - e^{4\pi i \omega x}} \right].$$  \hspace{1cm} (4.50)

If we put $\mu = 0$, this result is consistent with [Fad15]. With our parametrization of $\lambda$, we can also find the generalized eigenfunctions directly. Using Equation (4.16), we see that $f_+(x, k) := e^{2\pi i \tilde{k} x}$, $f_-(x, k) := e^{-2\pi i (k + \frac{2\omega}{\pi i}) x}$ both satisfy $H^\mu_0 f_\pm(x, k) = \lambda f_\pm(x, k)$. These are therefore analogous to the Jost solutions for Schrödinger operators.
Eigenfunctions of $H^\mu$

We retain the notation of Section 1.2 and study the eigenvalue problem in the momentum-basis. Fourier-transforming Equation (4.15), we obtain the associated eigenvalue equation

$$\hat{\psi}(p + 2\omega') + A \cosh \left( \frac{\pi ip}{\omega} \right) \hat{\psi}(p) = \lambda \hat{\psi}(p). \tag{4.51}$$

Parametrizing $\lambda = A \cosh \left( \frac{\pi i k}{\omega} \right)$ as in the last section, we compare this equation to Equation (4.31). An explicit solution to Equation (4.51) can then be written down as

$$\varphi_\mu(p,k) = e^{2\omega b_x} c_\mu(k) e^{2\pi ip_x} e^{-i\pi(\bar{p} - \omega'')}^2 \gamma(\bar{p} + \bar{k} - \omega'') \gamma(\bar{p} - \bar{k} - \omega''). \tag{4.52}$$

In this equation, $c_\mu(k) = e^{-i\beta - ik^2}$ and $b_x = -\frac{\omega}{i\pi} \ln \left( 1 - \beta^2 \mu^2 \right)$, so that $e^{4\pi i\omega/b_x} = A/2$. That this is indeed a solution to Equation (4.51) follows from a straightforward calculation using property two in Lemma 1. Since the function $k \mapsto A \cosh \left( \frac{\pi i k}{\omega} \right)$ for $0 \leq \text{Im}(k) \leq |\omega|$ is a $2 - 1$ mapping of the complex plane, it follows again that the spectrum above has multiplicity two.

Returning to the coordinate-representation, we define

$$\varphi_\mu(x,k) = \int_\mathbb{R} e^{2\pi i px} \varphi_\mu(p,k) dp \tag{4.53}$$

$$= e^{2\omega b_x} \int_\mathbb{R} e^{2\pi i px} c_\mu(k) e^{2\pi ip_x} e^{-i\pi(\bar{p} - \omega'')}^2 \gamma(\bar{p} + \bar{k} - \omega'') \gamma(\bar{p} - \bar{k} - \omega'') dp \tag{4.54}$$

$$= e^{2\omega b_x} \int_\mathbb{R} e^{2\pi i p(x + b_x)} c_0(\tilde{k}) e^{-i\pi(\bar{p} - \omega'')}^2 \gamma(\bar{p} + \bar{k} - \omega'') \gamma(\bar{p} - \bar{k} - \omega'') dp \tag{4.55}$$

$$= e^{2\omega b_x} e^{-2\omega \theta(x + b_x)} \int_\mathbb{R} e^{2\pi i p(x + b_x)} \varphi_0(\bar{p}, \tilde{k}) dp \tag{4.56}$$

$$= e^{2\omega b_x} e^{-2\omega \theta(x + b_x)} \varphi_0(x + b_x, \tilde{k}) = e^{ia_x x} \varphi_0(x + b_x, \tilde{k}). \tag{4.57}$$

The last equality used the fact that $\theta = -\frac{\omega a_x}{2\omega}$, as can be seen from the definitions above. $\varphi_\mu$ then satisfies $H^\mu \varphi_\mu(x,k) = \lambda \varphi_\mu(x,k)$.

Jost solutions

The Jost solutions $f_{\pm}^{(\mu)}(x,k)$ of $H^\mu$ can be obtained from the corresponding Jost solutions $f_{\pm}^{(0)}(x,k)$ of $H^0$, using Proposition 2.

**Proposition 3.** The functions

$$f_{\pm}^{(\mu)} := e^{ia_x x + 2\pi i \tilde{k} b_x} f_{\pm}^{(0)} \left( x + b_x, \pm \tilde{k} \right) \tag{4.58}$$
4.2. EXAMPLE: SPECTRAL PROPERTIES OF $H^\mu$

satisfy

$$H^\mu f_{\pm}^{(\mu)} = A \cosh \left( \frac{\pi \tilde{k}}{\omega} \right) f_{\pm}^{(\mu)}, \quad (4.59)$$

and

$$f_+^{(\mu)} = e^{2\pi ikx} + o(1),$$
$$f_-^{(\mu)} = e^{-2\pi i(k + \frac{2\theta}{\pi})x} + o(1), \quad (4.60)$$

as $x \to -\infty$.

**Proof.** We only perform the straightforward calculations for $f_+^{(\mu)}$. The case $f_-^{(\mu)}$ is similar. We first calculate

$$H^\mu f_+^{(\mu)} = T_{a^*, b^*} \frac{A}{2} H^0 T_{a^*, b^*}^{-1} f_+^{(\mu)} = A e^{ia_x - 2\pi i \tilde{k} b^*} 2 \cosh \left( \frac{\pi \tilde{k}}{\omega} \right) f_+^{(0)} (x + b, \tilde{k}) = \lambda f_+^{(\mu)}. \quad (4.62)$$

Next, it was shown in [Fad15] that $f_+^{(0)} (x, \tilde{k}) = e^{2\pi i \tilde{k} x} + o(1)$ as $x \to -\infty$, so

$$f_+^{(\mu)} (x, k) = e^{2\pi i \tilde{k} x + ia_x} + o(1) = e^{2\pi i k x} + o(1), \quad (4.63)$$

as $x \to -\infty$. This concludes the proof.

$f_\pm^{(\mu)}$ therefore behave asymptotically like the generalized eigenvectors of $H_0^\mu$ as $x \to -\infty$, and so play the role of Jost solutions.

**Resolvent of $H^\mu$**

By Proposition 2, we can express the resolvent of $H^\mu$ as

$$R_\mu(\lambda) := (H^\mu - \lambda)^{-1} = T_{a^*, b^*} \left( \frac{A}{2} H^0 - \lambda \right)^{-1} T_{a^*, b^*},$$

$$= \frac{2}{A} T_{a^*, b^*} \left( H^0 - \frac{2\lambda}{A} \right)^{-1} T_{a^*, b^*}^{-1}. \quad (4.65)$$

To calculate this, we will use the results from [Fad15] showing that

$$\left( (H^0 - \lambda)^{-1} \psi \right) (x) = \int_{\mathbb{R}} R_0(x - y; k) \psi (y) dy, \quad (4.66)$$

where $\lambda = 2 \cosh \left( \frac{\pi k}{\omega} \right)$ and

$$R_0(x; k) = \frac{\omega}{\sinh \left( \frac{\pi k}{\omega} \right)} \left[ \frac{f_+^{(0)} (x, k) \varphi_0 (y, k) + f_-^{(0)} (y, k) \varphi_0 (x, k)}{1 - e^{\frac{\pi k}{\omega}}} + \frac{f_+^{(0)} (y, k) \varphi_0 (x, k) - f_-^{(0)} (x, k) \varphi_0 (y, k)}{1 - e^{-\frac{\pi k}{\omega}}} \right]. \quad (4.67)$$
Using this in Equation (4.66), we obtain

\[
(R_\mu(\lambda)\psi) (x) = \frac{2}{A} e^{ia_{+}x} \int_{\mathbb{R}} R_0(x + b_{+} - y; \tilde{k}) T_{-b_{+}}^{-1} \psi(y) dy
\]

\[= \frac{2}{A} \int_{\mathbb{R}} e^{ia_{+}(x-y+b_{+})} R_0(x + b_{+} - y; \tilde{k}) \psi(y - b_{+}) dy\]

\[= \int_{\mathbb{R}} \frac{2}{A} e^{ia_{+}(x-y)} R_0(x - y; \tilde{k}) \psi(y) dy,
\]

i.e. \( R_\mu(\lambda) \) is the integral operator with kernel

\[
\frac{2\omega e^{ia_{+}(x-y)}}{A \sinh \left( \frac{\pi k}{\omega} \right)} \left[ \frac{f_{(0)}(x, \tilde{k}) \varphi_{0}(y, \tilde{k})}{1 - e^{\frac{\pi ik}{\omega}}} + \frac{f_{(0)}(y, \tilde{k}) \varphi_{0}(x, \tilde{k})}{1 - e^{-\frac{\pi ik}{\omega}}} \right]. \tag{4.72}
\]

**Remark 3.** This is a bounded operator on \( L^2(\mathbb{R}) \), since it is equal to the bounded operator \( R_0(x; \lambda) \), up to a phase term and a shift of the \( k \)-argument.

**Remark 4.** Note that \(|a_{+}| \to \infty\) as \(|b_{+}| \to 1\). The solutions to Equation (4.51) are therefore not continuous across the phase transition at \(|b_{+}| = 1\). We will see below that this is accompanied by an associated change in the spectrum of \( H^\mu \).

**Eigenfunction expansion**

In the case \( \mu = 0 \), it was shown in [Fred15] that

\[
\mathcal{U} : L^2(\mathbb{R}) \to L^2([0, \infty), \rho(k) dk)
\]

\[\psi \mapsto \int_{\mathbb{R}} \psi(x) \varphi_{0}(x, k) dx,\tag{4.74}
\]

where \( \rho(k) = 4 \sinh \left( \frac{\pi k}{\omega} \right) \sinh \left( \frac{\pi k}{\omega} \right) \) is an isometric bijection. Moreover, the operator \( \mathcal{U} H^0 \mathcal{U}^{-1} \) acts by multiplication by \( 2 \cosh \left( \frac{\pi k}{\omega} \right) \). In our case, Proposition 2 gives \( H^\mu = \frac{A}{2} T_{a_{+}, b_{+}} H^0 T_{a_{+}, b_{+}}^{-1} \), i.e.

\[\left( \mathcal{U} T_{a_{+}, b_{+}}^{-1} \right) H^\mu \left( \mathcal{U} T_{a_{+}, b_{+}}^{-1} \right)^{-1} = \frac{A}{2} \mathcal{U} H^0 \mathcal{U}^{-1}
\]

acts by multiplication with \( A \cosh \left( \frac{\pi k}{\omega} \right) \). Since \( T_{a_{+}, b_{+}} \) is unitary, \( H^\mu \) is unitarily equivalent to multiplication with \( A \cosh \left( \frac{\pi k}{\omega} \right) \) and so in particular has spectrum \([A, \infty)\). The eigenfunction expansion is therefore given by the operator

\[\left( \mathcal{U} T_{a_{+}, b_{+}}^{-1} \psi \right) (k) = \int_{\mathbb{R}} \left( T_{a_{+}, b_{+}}^{-1} \psi(x) \right) \varphi_{0}(x, k) dx = \int_{\mathbb{R}} \psi(x) \overline{\varphi_{\mu}(x, k)} dx. \tag{4.76}
\]

**Critical magnetic field:** \(|b_{+}| = 1\)

We consider the case \( b_{+} = -1 \). The case \( b_{+} = 1 \) can then be obtained by the substitution \( b \mapsto -b \), and is therefore omitted.
4.2. EXAMPLE: SPECTRAL PROPERTIES OF $H^\mu$

Resolvent and eigenfunctions of the free operator

We parametrize $\lambda = 2e^{\frac{\pi ik}{\omega}}$ and apply the Fourier-transform to Equation (4.16). This gives the eigenvalue equation

$$2e^{\frac{\pi ik}{\omega}} \hat{\psi} = 2e^{\frac{\pi ik}{\omega}} \hat{\psi}. \quad (4.77)$$

The spectrum of $H^\mu_0$ then consists of all $\lambda$ in the range of $2e^{\frac{\pi ik}{\omega}}$, i.e. $\sigma(H^\mu_0) = \mathbb{R}_+$, and it has multiplicity one. A $\lambda$ in the resolvent-set corresponds to a $k$ with $0 < \text{Im}(k) \leq |\omega|$. Applying the inverse Fourier-transform, we obtain the resolvent

$$(R(\lambda)\psi) (x) = \int_\mathbb{R} R_0(x - y; \lambda) \psi(y) dy, \quad (4.78)$$

where

$$R_0(x; \lambda) = \frac{1}{2} \int_\mathbb{R} \frac{e^{2\pi ipx}}{e^{\frac{\pi ik}{\omega}} - e^{\frac{\pi ik}{\omega}}} dp. \quad (4.79)$$

This integral can be evaluated using the residue theorem, with the result

$$R_0(x; \lambda) = \frac{\omega}{e^{\frac{\pi ik}{\omega}}} \frac{e^{2\pi ikx}}{1 - e^{4\omega\pi ix}}. \quad (4.80)$$

Note lastly that the generalized eigenfunction corresponding to a $k \in \mathbb{R}$ is simply $f(x) = e^{2\pi ikx}$.

Eigenfunctions of $H^\mu$

If we parametrize $\lambda = e^{\frac{\pi ik}{\omega}}$, the eigenvalue equation becomes

$$2\psi(x + 2\omega') + e^{\frac{\pi ik}{\omega}} \psi(x) = e^{\frac{\pi ik}{\omega}} \psi(x), \quad (4.81)$$

which can be solved explicitly. Take $c = \frac{1}{b} \ln(2)$; using property two in Lemma 1, a straightforward calculation shows that

$$\varphi_\mu(x, k) = e^{2\pi ikx+c\gamma}(-x + k - \omega'') \quad (4.82)$$

satisfies $H^\mu \varphi_\mu(x, k) = e^{\frac{\pi ik}{\omega}} \varphi_\mu(x, k)$.

Strong magnetic field: $|b_\mu| > 1$

Resolvent and eigenfunctions of the free operator

In this parameter regime, we Fourier-transform Equation (4.15) to obtain the eigenvalue equation

$$\left( \mathcal{F}H^\mu_0 \mathcal{F}^{-1} \hat{\psi} \right)(p) = f(p) \hat{\psi}(p). \quad (4.83)$$
Here, \( f(p) = A \sinh \left( \frac{\pi p}{\omega} \right) \), where we again introduced help variables through

\[
A = 2\sqrt{b^2 \mu^2 - 1},
\]

\[
\cosh(\theta) = \frac{b\mu}{\sqrt{b^2 \mu^2 - 1}},
\]

\[
\sinh(\theta) = \frac{1}{\sqrt{b^2 \mu^2 - 1}},
\]

\[
\tilde{p} = p + \frac{\omega\theta}{\pi i}.
\]

Precisely as in the cases above, we have the resolvent

\[
(R(\lambda)\psi)(x) = \int_{\mathbb{R}} R_0(x - y; \lambda)\psi(y)dy,
\]

for \( \lambda \in \mathbb{C} \) and

\[
R_0(x; \lambda) = \int_{\mathbb{R}} \frac{e^{2\pi ipx}}{f(p) - \lambda} dp,
\]

after applying the inverse Fourier-transform. \( R(\lambda) \) will be a bounded operator on \( L^2(\mathbb{R}) \) in the momentum representation precisely when \( \lambda \not\in \text{Ran}(f) = \mathbb{R} \). We therefore parametrize the spectrum by writing

\[
\lambda = f(k) = A \sinh \left( \frac{\pi k}{\omega} \right), \quad \tilde{k} = k + \frac{\omega\theta}{\pi i}.
\]

Again, \( k \in \mathbb{R} \) corresponds to \( \lambda \in \sigma(H_0^\mu) \), so the spectrum has multiplicity one. A \( \lambda \) in the resolvent-set corresponds to a \( k \) with \( 0 < \text{Im}(k) \leq |\omega| \). With this parametrization, the integrand in Equation (4.89) has singularities at

\[
p = k + 2n\omega, \quad p = -k - 2n\omega + (2m + 1)\omega,
\]

for \( m, n \) integers. We can then compute the integral in Equation (4.89) using the residue theorem, and obtain

\[
R_0(x; \lambda) = \frac{2\omega}{A \cosh \left( \frac{\pi k}{\omega} \right)} \frac{1}{1 - e^{4\pi i kx}} \left[ e^{2\pi ikx} - e^{-2\pi ikx + \frac{2\pi k}{\omega} + 2\pi i\omega x} \right].
\]

Note that \( R_0(x; \lambda) \) has a limit as \( x \to 0 \) and that \( |R_0(x; \lambda)| \leq e^{-2\pi |\text{Im}(k)|x-y} \), so this does indeed define a bounded operator on \( L^2(\mathbb{R}) \) by Schur’s lemma.

With our parametrization of \( \lambda \), we can also find the generalized eigenfunctions directly. Using Equation (4.16), we see that

\[
f_+(x, k) := e^{2\pi i kx}, \quad f_-(x, k) := e^{-2\pi i(k + \frac{2\pi \omega}{\pi i})x + 2\pi i\omega x}
\]

both satisfy \( H_0^\mu f_\pm(x, k) = \lambda f_\pm(x, k) \). We can then rewrite Equation (4.90) as

\[
R_0(x - y; \lambda) = \frac{2\omega}{(1 - b\mu) C(f_-, f_+)(x)} \frac{f_+(x)f_-(y) - f_+(y)f_-(x)}{1 - e^{4\pi i \omega(x-y)}}.
\]

Here, \( C(f, g) \) denotes the Casorati-determinant

\[
C(f, g) := f(x + 2\omega')g(x) - f(x)g(x + 2\omega'),
\]

which is a discretized analogue of the Wronskian for differential equations.
4.2. EXAMPLE: SPECTRAL PROPERTIES OF $H^\mu$

Remark 5. For functions $f, g$ satisfying the eigenvalue equation associated to Equation (4.16), we have
\[
C(f, g)(x) = f(x + 2\omega')g(x) - f(x)g(x + 2\omega')
\]
\[
= -f(x - 2\omega')g(x)\frac{1 + b\mu}{1 - b\mu} + f(x)g(x - 2\omega')\frac{1 + b\mu}{1 - b\mu}
\]
\[
= \frac{1 + b\mu}{1 - b\mu} C(f, g)(x - 2\omega').
\]

We will use this to directly see that
\[
(H^\mu - \lambda) R(\lambda) \psi = \psi.
\]

We will only need to show that $R_0(x; \lambda)$ satisfies
\[
(1 - b\mu)R_0(x + 2\omega' - y - i0; \lambda) + (1 + b\mu)R_0(x - 2\omega' - y + i0; \lambda)
\]
\[
- \lambda R_0(x - y; \lambda) = \delta(x - y),
\]
in a distributional sense. This can be seen from Equation (4.91); since $f_\pm$ are eigenfunctions of $H^\mu$, both sides in Equation (4.97) are zero for $x \neq y$. For $x = y$, the left hand side has singular part
\[
-\frac{1}{2\pi i(1 - b\mu)C(f_-, f_+)(x)} \left[ \frac{f_+(x + 2\omega')f_-(y) - f_+(y)f_-(x + 2\omega')}{{x - y - i0}} \right.
\]
\[
- \frac{f_+(x - 2\omega')f_-(y) - f_+(y)f_-(x - 2\omega')}{{x - y + i0}} \right]
\]
\[
= \frac{1}{2\pi i} \left[ \frac{1}{{x - y - i0}} - \frac{1}{{x - y + i0}} \right]
\]
\[
= \delta(x - y).
\]

Here, the last step is precisely the so-called Sokhotski-Plemelj formula, which proves Equation (4.96).

Eigenfunctions of $H^\mu$

We again apply the Fourier-transform to the eigenvalue equation for the operator in Equation (4.15), and obtain
\[
\hat{\psi}(p + 2\omega') + A \sinh \left( \frac{\pi i\tilde{p}}{\omega} \right) \hat{\psi}(p) = \lambda \hat{\psi}(p).
\]

Using the same parametrization and notation as before, we can again use property two in Lemma 1 to find the generalized eigenfunctions. A straightforward calculation shows that these can be written as
\[
\hat{\varphi}_\mu(p, k) = c_\mu(k) e^{ipB} e^{-i\pi(p - \omega'')^2} \gamma(p + \tilde{k} - \omega') \gamma(p - \tilde{k} - \omega''),
\]
where \( B = \frac{1}{2} \ln \left( \frac{A}{2} \right) \), and \( c_\mu(k) = e^{-i\beta - i\pi k^2} \), as before. Note that the first \( \gamma \)-function has argument shifted by \(+\omega\) as compared to the operator for the weak magnetic field. We also define

\[
\varphi_\mu(x, k) = \int_\mathbb{R} e^{2\pi ipx} \hat{\varphi}_\mu(p, k) dp,
\]

which then satisfies \( H^\alpha \varphi_\mu(x, k) = A \sinh \left( \frac{\pi k}{\omega} \right) \varphi_\mu(x, k) \).

### 4.3 Eigenvalue estimates for general potentials

This last section returns to the case of a general potential function for the discretized Schrödinger operator in Equation (4.5). In order to gain a better understanding of the operator \( H \), we would like to obtain more quantitative information on the magnitude of its eigenvalues.

A first problem is to understand when \( H \) has discrete spectrum. When this is the case, we will denote its eigenvalues by \( \{\lambda_n\}_{n=1}^\infty \). The remainder of the chapter will then study two quantities; the first of these is the eigenvalue counting function \( N(\Lambda, H) \), defined by

\[
N(\Lambda, H) := |\{f_n \in D(H) : Hf_n = \lambda_n f_n, \text{ with } \lambda_n \leq \Lambda\}|,
\]

where \(|·|\) is the counting measure. The second quantity is the Riesz-mean, or eigenvalue-moment, which is defined as

\[
\sum_{n=1}^\infty |\lambda_n|^\gamma,
\]

for some \( \gamma \in \mathbb{R}_+ \). Both \( N(\Lambda, H) \) and the eigenvalue-moments are physical quantities, as seen e.g. in the Lieb-Thirring inequalities of Chapter 3. Our goal is to estimate these quantities in terms of the potential \( V \), which offers general spectral information without necessitating an explicit computation of the eigenvalues.

#### Discreteness of spectrum

For the continuous Schrödinger operator

\[
-\Delta + V(x),
\]

acting on functions defined on \( \mathbb{R}^d \), a classical and sufficient condition for the corresponding spectrum to be discrete is that \( V(x) \to +\infty \) when \( |x| \to \infty \), as shown in [Fri34]. A similar result was proven in [Sim09], originally for the continuous Schrödinger operator, but also generalized to a wider class of operators. This section applies this condition also to the operator

\[
H := H_0 + V = U + U^{-1} + V,
\]
4.3. EIGENVALUE ESTIMATES FOR GENERAL POTENTIALS

where \( V(x) \) is real-valued and essentially bounded from below. We could show that our operator is of the kind permitted in [Sim09] by redefining it in terms of quadratic forms, but instead opt for adapting the proof used therein, for the sake of completeness. To do so, we will need the following.

**Lemma 2.** The operator \( H_0 \) generates a positive contraction semigroup \( e^{-tH_0} \), given by

\[
(e^{-tH_0}\psi)(x) = \int_\mathbb{R} \hat{g}_t(x-y)\psi(y)dy,
\]

for \( g_t(p) = \exp \left( -2t \cosh \left( \frac{\pi p}{\omega} \right) \right) \).

**Proof.** Since \( H_0 \) is positive and self-adjoint, the first statement follows from the Hille-Yosida theorem [Ree75], and it only remains to show positivity and the representation as an integral operator. Note that \( e^{-tH_0}\psi(x) = \psi(x,t) \), where the latter is given by

\[
\frac{\partial}{\partial t} \psi(x,t) = -H_0\psi(x,t),
\]

\[
\psi(x,0) = \psi(x).
\]

Applying the Fourier-transform to the \( x \)-coordinate, this is equivalent to

\[
\frac{\partial}{\partial t} \hat{\psi}(p,t) = -2 \cosh \left( \frac{\pi p}{\omega} \right) \hat{\psi}(p,t),
\]

\[
\hat{\psi}(p,0) = \hat{\psi}(p),
\]

with the solution \( \hat{\psi}(p,t) = g_t(p)\hat{\psi}(p) \). Since \( g_t(p) > 0 \), it follows that \( e^{-tH_0} \) is positive in the momentum-representation, so then also in the coordinate-representation. Taking the inverse Fourier-transform results in

\[
e^{-tH_0}\psi(x) = \int_\mathbb{R} \hat{g}_t(x-y)\psi(y)dy.
\]

\[\square\]

**Remark 6.** We note for later use that \( g_t(p) \) is in the Schwartz space, meaning that \( \hat{g}_t \) is as well, so in particular \( \hat{g}_t \in L^p(\mathbb{R}) \) for all \( 1 \leq p \leq \infty \).

We can now adapt the proof from [Sim09] to prove the main result of this section.

**Theorem 12** (adapted from [Sim09]). Assume that \( V(x) \) is essentially bounded from below, and that \( S_m := \{ x : V(x) \leq m \} \) has finite Lebesgue-measure for all \( m \in \mathbb{R} \). Then \( H \) has discrete spectrum.

**Proof.** One can show that \( H \) having only discrete spectrum is equivalent to \( e^{-tH} \) being compact, for some positive \( t \) [Ree78]. Moreover, it is true that \( e^{-A}e^{-B} \) being compact for some non-negative, self-adjoint operators \( A, B \) implies that \( e^{-(A+B)} \) is
compact \cite{Sim99}. In our case, we therefore only need to prove that $T := e^{-H_0}e^{-V}$ is compact, and proceed just as in \cite{Sim99}. We decompose

$$T = T_m + R_m,$$

(4.114)

with $T_m := T\chi_{S_m}$, $R_m = T\chi_{S_m}^c$. This gives

$$\|R_m\| \leq \|e^{-H_0}\| e^{-m} \to 0,$$

(4.115)

as $m \to \infty$. It follows that $T_m \to T$ in the norm sense, so we need only show that each $T_m$ is compact to conclude that also $T$ is. In fact, $T_m$ is even Hilbert-Schmidt, since

$$(T_m\psi)(x) = \int_{S_m} \hat{g}_1(x-y)e^{-V(y)}\psi(y)dy$$

(4.116)

is an integral operator, with

$$\int_R \int_{S_m} \hat{g}_1^2(x-y)e^{-2V(y)}dydx \leq \|\hat{g}_1\|_{L^2(\mathbb{R})}^2 \cdot \exp \left(-2 \inf_{x \in \mathbb{R}} V(x) \right) |S_m| < \infty.$$  

(4.117)

As this last expression is finite, $T_m$ is compact, and so also $T$, which concludes the proof, by the remarks above.

From this, we can deduce the following standard result, which concludes the considerations about the discreteness of the spectrum under the conditions in Theorem 12.

**Corollary 5.** If $H$ has discrete spectrum, then there is a complete set in $L^2(\mathbb{R})$ consisting of orthonormal eigenfunctions to $H$ with eigenvalues $\lambda_n$ that satisfy $\lambda_n \to \infty$ as $n \to \infty$.

**Proof.** We first show that eigenfunctions $\{f_n\}$ to $H$ form a complete set. To see this, form $X$ as the space spanned by the eigenfunctions and study $X^\perp$. Now, $HX^\perp \subseteq X^\perp$, since for any $x \in X^\perp \cap D(H)$, we have

$$(Hx, f_n)_{L^2(\mathbb{R})} = (x, \lambda_n f_n)_{L^2(\mathbb{R})} = 0,$$

(4.118)

by the symmetry of $H$. Since $X^\perp$ is a closed subset of a Hilbert space, it is also complete, and therefore a Hilbert space. This means that $H|_{X^\perp}$ is a symmetric operator on a Hilbert space, with empty essential spectrum (inherited from $H$) and no eigenvalues (by definition of $X^\perp$). $\sigma(H|_{X^\perp})$ is therefore empty, which is impossible unless $X^\perp = \{0\}$.

Next, $L^2(\mathbb{R})$ has infinite dimension, so there are infinitely many $f_n$. Their corresponding eigenvalues cannot have a finite accumulation point, since $\sigma_{ess}(H) = \emptyset$, which shows that the set $\{\lambda_n\}$ cannot be bounded. This concludes the proof. \(\square\)
4.3. EIGENVALUE ESTIMATES FOR GENERAL POTENTIALS

Asymptotic behaviour of eigenvalues

This section studies $N(\Lambda, H)$, when $\Lambda$ tends to infinity. The main result is an asymptotic phase-space expression for $N(\Lambda, H)$, which has a classical analogue in quantum mechanics and also makes an argument in [Gra14] precise. Results of this kind are inspired by Weyl [Wey11], who showed that $N(\Lambda, -\Delta)$ for the operator $-\Delta$ with Dirichlet boundary conditions in some bounded $\Omega \subseteq \mathbb{R}^d$, satisfies

$$\lim_{\Lambda \to \infty} \Lambda^{-d/2} N(\Lambda, -\Delta) = (2\pi)^{-d} \omega_d |\Omega|. \quad (4.119)$$

Here, $\omega_d$ denotes the volume of the unit ball in $\mathbb{R}^d$.

Our approach is standard and follows [Sim79, Lap97], adapted to the setting of the operator $H$. The main tools are Abelian-Tauberian theorems, together with the Golden-Thompson formula and the method of coherent states, which allow us to prove the following.

**Theorem 13.** Let $\alpha, \beta \geq 0$. Assume that $V(x)$ is continuous, bounded from below, and that there exists some $K \in \mathbb{R}_+$ s.t.

$$\int_{\mathbb{R}} V(x)e^{-K|x-y|^2}dx \quad (4.120)$$

is finite, and determines a continuous function in $y$, for all $y \in \mathbb{R}$. If $H$ also has discrete spectrum, and

$$\lim_{\Lambda \to \infty} \Lambda^{-\alpha} (\ln \Lambda)^{-\beta} \left\{ (p,x) \in \mathbb{R}^2 : 2 \cosh \left( \frac{\pi i p}{\omega} \right) + V(x) \leq \Lambda \right\} = C \quad (4.121)$$

holds for some $C \in \mathbb{R}$, then

$$\lim_{\Lambda \to \infty} \Lambda^{-\alpha} (\ln \Lambda)^{-\beta} N(\Lambda, H) = C. \quad (4.122)$$

**Remark 7.** Note that the assumption that $V(x)$ is bounded from below is implicitly assumed in Equation (4.121), since the phase space volume would otherwise be infinite for any finite $\Lambda$. It can therefore not be relaxed. The assumption in Equation (4.120) is also needed in our proof; without it, we can, however, still conclude the weaker statement that

$$\Lambda^{-\alpha} (\ln \Lambda)^{-\beta} N(\Lambda, H) \leq C(1 + o(1)), \quad (4.123)$$

for all $\Lambda$ large enough.

**Remark 8.** This result can be compared to the analogue for the continuous Schrödinger equation, where the term $2 \cosh \left( \frac{\pi i p}{\omega} \right)$ is replaced by the kinetic energy term $p^2$. We therefore interpret the hyperbolic term as analogous to kinetic energy. The physical interpretation of the theorem is then that the number of eigenstates of $H$ asymptotically approaches the volume of the phase-space with total energy less than a prescribed level.
The proof of this fact relies on the following lemma, which is adapted and slightly generalized from [Sim79].

**Lemma 3** (Abelian-Tauberian theorem; slightly generalized from [Sim79]). Take \( \alpha, \beta \geq 0 \), and \( C \in \mathbb{R} \). If \( \mu \) is a non-negative measure on \( \mathbb{R}_+ \), then

\[
\lim_{\Lambda \to \infty} \Lambda^{-\alpha} (\ln \Lambda)^{-\beta} \mu(0, \Lambda) = \frac{C}{\Gamma(\alpha + 1)} 
\]

is equivalent to

\[
\lim_{t \to 0} t^\alpha (\ln t^{-1})^{-\beta} \int_{\mathbb{R}_+} e^{-tx} d\mu(x) = C, 
\]

where we denote the Gamma-function by

\[
\Gamma(t) = \int_{\mathbb{R}_+} x^{t-1} e^{-x} dx. 
\]

**Proof.** Assume first that Equation (4.124) holds. Integrating by parts, we obtain

\[
t^\alpha (\ln t^{-1})^{-\beta} \int_{\mathbb{R}_+} e^{-tx} d\mu(x) = t^{\alpha+1} (\ln t^{-1})^{-\beta} \int_{\mathbb{R}_+} e^{-tx} \mu(0, x) dx 
\]

\[
= t^{\alpha+1} (\ln t^{-1})^{-\beta} \int_{\mathbb{R}_+} e^{-tx} x^\alpha (\ln x)^\beta \left( x^{-\alpha} (\ln x)^{-\beta} \mu(0, x) \right) dx. 
\]

If we now set \( y = tx \), then the last term equals

\[
\int_{\mathbb{R}_+} y^\alpha e^{-y} \left( \frac{\ln y}{\ln t^{-1}} \right)^\beta R(y, t) dy, 
\]

for \( R(y, t) = \left( \frac{y}{t} \right)^{-\alpha} (\ln \frac{y}{t})^{\beta} \mu(0, \frac{y}{t}) \). By hypothesis, \( R(y, t) \to \frac{C}{\Gamma(\alpha + 1)} \), and \( \left( \frac{\ln \frac{y}{t}}{\ln t^{-1}} \right)^\beta \)
tends to 1 pointwise in \( y \), as \( t \to 0 \). This shows Equation (4.125).

Conversely, assume that Equation (4.125) holds. Letting \( t = \Lambda^{-1} \), we have

\[
\lim_{\Lambda \to \infty} \Lambda^{-\alpha} (\ln \Lambda)^{-\beta} \mu(0, \Lambda) = \lim_{t \to 0} t^\alpha (\ln t^{-1})^{-\beta} \mu(0, t^{-1}) = \lim_{t \to 0} \mu_t(0, 1), 
\]

where we define \( \mu_t(M) := t^\alpha (\ln t^{-1})^{-\beta} \mu(tM) \), for \( M \subseteq \mathbb{R} \) a Borel-set. Setting \( d\nu(x) = x^{\alpha-1} dx \), we see that Equation (4.124) is equivalent to

\[
\lim_{t \to 0} \mu_t[0, 1] = \frac{C}{\Gamma(\alpha)} \nu[0, 1]. 
\]

This will follow after showing that \( \mu_t \to \frac{C}{\Gamma(\alpha)} \nu \) weakly, i.e. that

\[
\lim_{t \to 0} \int_{\mathbb{R}_+} f(x) d\mu_t(x) = \frac{C}{\Gamma(\alpha)} \int_{\mathbb{R}_+} f(x) d\nu(x), 
\]
provided either limit exists. Define a second measure $\mu_2$ by

$$
\mu_2(M) := \left\{ (p, x) \in \mathbb{R}^2 : 2 \cosh \left( \frac{\pi i p}{\omega} \right) + V(x) \in M \right\},
$$

(4.141)

for $M \subseteq \mathbb{R}$ a Borel-set, we can use Lemma 3 again to conclude

$$
\lim_{t \to 0} t^n (\ln t^{-1})^{-\beta} I(t) = \Gamma(\alpha + 1) \lim_{\Lambda \to \infty} \Lambda^{-\alpha} (\ln \Lambda)^{-\beta} A(\Lambda),
$$

(4.142)

since the second term exists by assumption. It will therefore be enough to show that

$$
\lim_{t \to 0} t^n (\ln t^{-1})^{-\beta} \text{Tr} \left( e^{-tH} \right) = \lim_{t \to 0} t^n (\ln t^{-1})^{-\beta} I(t),
$$

(4.143)

which we will now do.

### Proof of Theorem 13

Let $\{\lambda_n\}_{n=0}^{\infty}$ denote the eigenvalues of $H$. Note that this set is bounded from below, since $H_0$ is positive and $V(x)$ is bounded from below. Since Equation (4.121) is invariant with respect to equal translations of $V$ and $\Lambda$, we can without loss of generality assume that $\lambda_n \geq 0$, for all $n$. Define

$$
A(\Lambda) := \left\{ (p, x) \in \mathbb{R}^2 : 2 \cosh \left( \frac{\pi i p}{\omega} \right) + V(x) \leq \Lambda \right\},
$$

(4.138)

$$
I(t) := \int_{\mathbb{R}^2} \exp \left( -t \left[ 2 \cosh \left( \frac{\pi i p}{\omega} \right) + V(x) \right] \right) \, d\nu, \quad \text{if } t > 0,
$$

(4.139)
Claim 1. \( \text{Tr} (e^{-tH}) \leq I(t) \).


\[
\text{Tr} (e^{-tH}) \leq \text{Tr} \left( e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \right),
\]

and we will find an expression for the right-hand side. Using Lemma 2, we can write

\[
e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \psi(x) = \int_{\mathbb{R}} e^{-\frac{t}{2}V(x)} \hat{g}_t(x - y) e^{-\frac{t}{2}V(y)} \psi(y) dy
\]

where \( K_t(x, y) := e^{-\frac{t}{2}V(x)} \hat{g}_t(x - y) e^{-\frac{t}{2}V(y)} \). Now, by [Bri88], we can compute

\[
\text{Tr} \left( e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \right) = \int_{\mathbb{R}} K_t(x, x) dx,
\]

provided that \( K_t(x, y) \in L^1_{\text{loc}}(\mathbb{R}^2) \), and that \( e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \) is positive as well as Hilbert-Schmidt. This first statement follows from \( \hat{g}_t \) being in the Schwartz space and that \( V \) was assumed continuous; the second statement is true, since

\[
\langle e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \psi, \psi \rangle_{L^2(\mathbb{R})} = \langle e^{-tH_0} \varphi, \varphi \rangle_{L^2(\mathbb{R})},
\]

for \( \varphi(x) = e^{-\frac{t}{2}V(x)} \psi(x) \). Since \( e^{-tH_0} \) was seen to be positive in Lemma 2, the expression in Equation (4.148) is positive, so \( e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \) is a positive operator. For the third statement, note that it is true if \( e^{-tV(x)} \) is in \( L^1(\mathbb{R}) \); if this is not the case, then \( I(t) = \infty \), so the bound in Claim 1 is void. We can therefore return to Equation (4.147), and compute

\[
\text{Tr} \left( e^{-\frac{t}{2}V} e^{-tH_0} e^{-\frac{t}{2}V} \right) = \int_{\mathbb{R}} K_t(x, x) dx = \int_{\mathbb{R}} \hat{g}_t(0) e^{-tV(x)} dx = I(t).
\]

Inserting this into Equation (4.144) concludes the proof of the claim.

Claim 2. \( \lim_{t \to 0} t^\alpha \left( \ln t^{-1} \right)^{-\beta} \text{Tr} (e^{-tH}) \geq \lim_{t \to 0} t^\alpha \left( \ln t^{-1} \right)^{-\beta} I(t) \).

Proof of Claim 2. We will use the method of coherent states, and therefore define

\[
F_{k,y}(x) = e^{2\pi i x k} G(x - y),
\]

for \( G(x) = \sqrt{\frac{K}{2\pi}} e^{-K|y|^2}/2 \). Here, the constant is chosen such that \( \|G\|_{L^2(\mathbb{R})} = \|F_{k,y}(\cdot)\|_{L^2(\mathbb{R})} = 1 \). Let \( \{\psi_n\}_{n=0}^\infty \) be the complete, orthogonal set of eigenfunctions with eigenvalues \( \{\lambda_n\}_{n=0}^\infty \) guaranteed by Corollary 3, and denote by \( E_\nu \) the
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associated spectral measure. We have

$$\text{Tr} \left( e^{-tH} \right) = \sum_n e^{-t\lambda_n} \int_\mathbb{R} |\psi_n(x)|^2 \, dx$$

(4.151)

$$= \sum_n e^{-t\lambda_n} \int_{\mathbb{R}^4} F_{k,y}(p) \psi_n(p) \overline{F_{k,y}(q)} \, dp dq dy$$

(4.152)

$$= \int_{\mathbb{R}^2} \int_0^\infty e^{-t\nu} \langle dE_{\nu} F_{k,y}(\cdot), F_{k,y}(\cdot) \rangle \, dk dy.$$  (4.153)

Note that we have $\int_0^\infty \langle dE_{\nu} F_{k,y}(\cdot), F_{k,y}(\cdot) \rangle = \langle F_{k,y}(\cdot), F_{k,y}(\cdot) \rangle_{L^2(\mathbb{R})} = 1$, so we can apply Jensen’s inequality to the last term above. We obtain

$$\text{Tr} \left( e^{-tH} \right) \geq \int_{\mathbb{R}^2} \exp \left( -t \left[ \int_0^\infty \nu \langle dE_{\nu} F_{k,y}(\cdot), F_{k,y}(\cdot) \rangle \right] \right) \, dk dy.$$  (4.154)

Here

$$\int_0^\infty \nu \langle dE_{\nu} F_{k,y}(\cdot), F_{k,y}(\cdot) \rangle = \langle F_{k,y}(\cdot), HF_{k,y}(\cdot) \rangle_{L^2(\mathbb{R})}$$

(4.155)

$$= \exp \left( -\frac{\pi ik}{\omega} \right) \int_\mathbb{R} e^{K\tau^2/2} e^{-2K\omega'(x-y)} G(x-y)^2 \, dx$$

$$+ \exp \left( \frac{\pi ik}{\omega} \right) \int_\mathbb{R} e^{K\tau^2/2} e^{2K\omega'(x-y)} G(x-y)^2 \, dx$$

(4.156)

$$+ \int_\mathbb{R} V(x) G(x-y)^2 \, dx$$

$$\leq 2 \cosh \left( \frac{\pi ik}{\omega} \right) e^{K\tau^2/2} + \int_\mathbb{R} V(x) G(x-y)^2 \, dx.$$  (4.157)

Let now $-a = \min_{y \in \mathbb{R}} V(y)$, which is finite by assumption. This implies that $e^{K\tau^2/2} (V(y) + a) \geq 0$. We add this to the exponent in the right side of Equation (4.154) and obtain

$$\text{Tr} \left( e^{-tH} \right) \geq \int_{\mathbb{R}^2} \exp \left( -t \left[ 2 \cosh \left( \frac{\pi ik}{\omega} \right) e^{K\tau^2/2} + e^{K\tau^2/2} (V(y) + a) \right] \right) \, dk dy$$

(4.158)

$$+ \int_\mathbb{R} V(x) G(x-y)^2 \, dx$$

(4.159)

$$\geq \int_\Omega \exp \left( -t \left[ 2 \cosh \left( \frac{\pi ik}{\omega} \right) e^{K\tau^2/2} + e^{K\tau^2/2} (V(y) + a) \right] \right) \, dk dy,$$  (4.160)

(4.161)

for $\Omega \subseteq \mathbb{R}^2$ an arbitrary compact set. By the continuity assumption, the function

$$y \mapsto \int_\mathbb{R} V(x) G(x-y)^2 \, dx$$  (4.162)
assumes a maximum magnitude value on $\Omega$ projected onto the $y$-coordinate, so we obtain

$$
\lim_{t \to 0} \text{Tr} \left( e^{-tH} \right) \geq \lim_{t \to 0} \int_{\Omega} \exp \left( -te^{Kt^2/2} \left[ 2 \cosh \left( \frac{\pi ik}{\omega} \right) + V(y) \right] \right) \, dk \, dy
$$

(4.163)

$$
= \lim_{t \to 0} \int_{\Omega} \exp \left( -t \left[ 2 \cosh \left( \frac{\pi ik}{\omega} \right) + V(y) \right] \right) \, dk \, dy.
$$

(4.164)

Inserting the term $t^\alpha (\ln t - 1) - \beta$ ($\geq 0$, for $t$ small enough) on each side of the inequality, the right hand side converges as $\Omega \to \mathbb{R}^2$. We can conclude that

$$
\lim_{t \to 0} t^\alpha (\ln t - 1)^{-\beta} \text{Tr}(e^{-tH}) \geq \lim_{t \to 0} t^\alpha (\ln t - 1)^{-\beta} I(t),
$$

(4.165)

which finishes the proof of the claim.

We have therefore shown that

$$
\lim_{t \to 0} t^\alpha (\ln t - 1)^{-\beta} \text{Tr}(e^{-tH}) = \lim_{t \to 0} t^\alpha (\ln t - 1)^{-\beta} I(t).
$$

(4.166)

By the remarks at the beginning of the proof, the theorem is now established.

Example 6. The potential $V(x) = 2 \cosh \left( \frac{\pi x}{\omega} \right)$ occurs in a model of conformal field theory discussed in e.g. [Gr14]. We can compute the asymptotics of $N(\Lambda, H)$. Denoting

$$
E(\Lambda) = \left\{ (p, x) \in \mathbb{R}^2 : 2 \cosh \left( \frac{\pi ip}{\omega} \right) + 2 \cosh \left( \frac{\pi ix}{\omega} \right) \leq \Lambda \right\},
$$

(4.167)

we note that the maximal $x, p \in E(\Lambda)$ are $x, p = \frac{\omega}{\pi i} \text{acosh} \left( \frac{\Lambda}{4} \right)$. This means that $|E(\Lambda)| \leq 4 \left( \frac{\omega}{\pi} \text{acosh} \left( \frac{\Lambda}{2} \right) \right)^2$. Moreover, by symmetry, the set

$$
E'(\Lambda) = \left\{ (p, x) \in \mathbb{R}^2 : |x|, |p| \leq \frac{\omega}{\pi i} \text{acosh} \left( \frac{\Lambda}{4} \right) \right\}
$$

(4.168)

is contained in $E(\Lambda)$, so

$$
4 \left( \frac{\omega}{\pi i} \text{acosh} \left( \frac{\Lambda}{4} \right) \right)^2 = |E'(\Lambda)| \leq |E(\Lambda)| \leq 4 \left( \frac{\omega}{\pi i} \text{acosh} \left( \frac{\Lambda}{2} \right) \right)^2.
$$

(4.169)

For $\Lambda$ large, this reduces to

$$
4 \left( \frac{\omega}{\pi i} \ln \left( \frac{\Lambda}{2} \right) \right)^2 \leq |E(\Lambda)| \leq 4 \left( \frac{\omega}{\pi i} \ln \left( \Lambda \right) \right)^2,
$$

(4.170)

and Theorem \[13\] results in the asymptotics

$$
\lim_{\Lambda \to \infty} \ln (\Lambda)^{-2} N(\Lambda, H) = \frac{1}{b^2 \pi^2}.
$$

(4.171)
Spectral bounds

The bound in the last section concerns only the asymptotic behaviour for large eigenvalues, and it is also of interest to obtain general bounds, valid regardless of the size of the eigenvalue. This section derives various spectral bounds, which are not necessarily sharp, but still of interest. This section does not assume that $V \to \infty$ as $|x| \to \infty$, so in particular does not assume that we are studying a discrete spectrum.

Non-self adjoint operators

In this section, we prove a general bound for eigenvalues of the discretized Schrödinger equation, under certain integrability assumptions on the potential. Let
\[ V \in L^1(\mathbb{R}) \]  
and study the operator
\[ H = H_0 + V, \]
acting on $L^2(\mathbb{R})$. Note that we allow for $V$ to be complex-valued, meaning that the operator in Equation (4.173) in general is not, and will not be assumed to be, symmetric. Instead, we follow the techniques of [Abr01] to obtain a general upper bound on the magnitude of the eigenvalues of $H$. The main result of this section is the following.

**Theorem 14.** Let $V$ be as in Equation (4.172). Any eigenvalue $\lambda$ of $H$ with $\lambda \not\in \sigma(H_0) = [2, \infty)$ then satisfies
\[ \|V\|^2_{L^1(\mathbb{R})} \geq \frac{|\lambda^2 - 1|}{\frac{1}{4\pi} + \frac{\pi}{8} \ln \left( \frac{\lambda}{2} + \sqrt{\frac{\lambda}{2} + 1} \right)^2}, \]

where the square roots are principal.

**Remark 9.** The right hand side of Equation (4.174) is plotted in Figure 4.1, for complex and real $\lambda \in \mathbb{C} \setminus [2, \infty)$, respectively. Note that the bound is void for $\lambda = -2$. It also implies that eigenvalues to the operator
\[ H_\alpha := H_0 + \alpha V, \quad \alpha \in \mathbb{R}, \]

can only cluster around the point $\lambda = -2$ or lie on $[2, \infty)$, as $\alpha \to 0$.

**Proof.** Any eigenvalue $\lambda$ with eigenfunction $\psi \in L^2(\mathbb{R})$ satisfies
\[ (H_0 - \lambda) \psi = -V \psi, \]
i.e.
\[ \varphi = -\frac{V}{|V|^{1/2}} (H_0 - \lambda)^{-1} |V|^{1/2} \varphi, \]
(a) The righthand side of Equation (4.174) plotted for complex $\lambda$.

(b) The righthand side of Equation (4.174) plotted for real $\lambda$.

Figure 4.1: Eigenvalue-bounding functions in Theorem 14.

for $\varphi := \frac{V}{|V|^{1/2}} \psi$. We note that $\varphi \in L^2(\mathbb{R})$, since

$$\int_{\mathbb{R}} \left| \frac{V}{|V|^{1/2}} \psi \right|^2 \, dx = \int_{\mathbb{R}} |V| |\psi|^2 \, dx \leq \|V\|_{L^1(\mathbb{R})} \|\psi\|_{L^\infty(\mathbb{R})}^2,$$  (4.178)

which is finite, as $\psi \in L^\infty(\mathbb{R})$ by Remark 1.

Because of this, we will study

$$T := -\frac{V}{|V|^{1/2}} (H_0 - \lambda)^{-1} |V|^{1/2},$$  (4.179)

as an operator with domain $L^2(\mathbb{R})$. In the following, we will show that $T$ also maps continuously into $L^2(\mathbb{R})$. In fact, $T$ is even Hilbert-Schmidt on $L^2(\mathbb{R})$, for $\lambda \in \rho(H_0)$.

We now introduce the parametrization $\lambda = 2 \cosh \left( \frac{2\pi ik}{\omega} \right)$, where $k \in \mathbb{R}$ corresponds to a $\lambda \in \sigma(H_0)$ and $0 < \Im(k) \leq |\omega|$ to a $\lambda \in \rho(H_0)$. In the latter case, the resolvent of $H_0$ was calculated in [Fad15] and in Section 4.2 to be an integral operator with kernel $R_0(x - y; \lambda)$ for

$$R_0(x; \lambda) := \frac{\omega}{\sinh \left( \frac{2\pi k}{\omega} \right)} \left( \frac{e^{-2\pi ikx}}{1 - e^{-4\pi i\omega x}} + \frac{e^{2\pi ikx}}{1 - e^{4\pi i\omega x}} \right).$$  (4.180)

We can then write

$$(T \varphi)(x) = \int_{\mathbb{R}} -\frac{V}{|V|^{1/2}}(x)R_0(x - y; \lambda)|V|^{1/2}(y)\varphi(y) \, dy,$$  (4.181)

which will be used to derive the desired inequality. To proceed, we will bound $R_0(x; \lambda)$ uniformly in $x$. 
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Lemma 4. Write \( \lambda = 2 \cosh \left( \frac{\pi k}{\omega} \right) \), for \( 0 < \text{Im}(k) \leq |\omega| \). We then have the bound

\[
|R_0(x; \lambda)|^2 \leq \left( 1 + \left| k \right|^2 / |\omega|^2 \right) \left( |\omega|^2 / |\sinh \left( \frac{\pi k}{\omega} \right)| \right)^2.
\] (4.182)

Proof of Lemma 4. Note that \( R_0(x; \lambda) \) is symmetric in \( x \), so we can without loss of generality assume \( x \geq 0 \). A straightforward calculation shows that

\[
\left| \frac{\sinh \left( \frac{\pi k}{\omega} \right)}{|\omega|^2} \right| |R_0(x; \lambda)|^2 = \frac{\cosh^2 \left( \frac{\pi}{b} - 2\text{Im}(k)x \right) - \cos^2 (2\pi \text{Re}(k)x)}{\sinh^2 \left( \frac{\pi}{b} \right)}
\]

\[
\leq \frac{\cosh^2 \left( \frac{\pi}{b} \right) - \cos^2 (2\pi \text{Re}(k)x)}{\sinh^2 \left( \frac{\pi}{b} \right)} = 1 + \frac{1 - \cos^2 (2\pi \text{Re}(k)x)}{\sinh^2 \left( \frac{\pi}{b} \right)}
\]

\[
\leq 1 + \frac{1 - \cos^2 (2\pi \text{Re}(k)x)}{\left( \frac{\pi}{b} \right)^2} =: f(x, k),
\] (4.186)

where the first inequality used that \( k \) has positive imaginary part and the last that \( x \leq \sinh (x) \), which is obtained by differentiation four times. \( f(x, k) \) is smooth and tends to zero as \( x \to \infty \). The end-point \( x = 0 \) has zero slope, and any zeros \( x_n \) of

\[
\frac{df(x, k)}{dx} = 2 \frac{\pi^2 x \pi \text{Re}(k) \sin (4\pi \text{Re}(k)x) x - \left( 1 - \cos^2 (2\pi \text{Re}(k)x) \right)}{\left( \frac{\pi}{b} \right)^4}
\]

satisfy

\[
f(x_n) = 1 + \frac{\pi \text{Re}(k) \sin (4\pi \text{Re}(k)x_n) x_n}{\left( \frac{\pi}{b} \right)^2} = 1 + 4b^2 \text{Re}(k)^2 \sin (4\pi \text{Re}(k)x_n) / 4\pi \text{Re}(k)x_n
\]

\[
\leq 1 + \left| \frac{k}{\omega} \right|^2.
\] (4.189)

Inserting this into Equation (4.186) concludes the proof of the lemma. \( \square \)

Returning to Equation (4.181), we fix a \( \lambda \in \rho(H_0) \) and note that \( T \) is then Hilbert-Schmidt with respect to the Lebesgue measure, since

\[
\|T\|_{L_2(X)}^2 = \int_{\mathbb{R}^2} \left| V \right|^2 (x) R_0(x-y; \lambda) \left| V \right|^{1/2} (y) \right|^2 \, dx \, dy
\]

\[
\leq \left( 1 + \left| k \right|^2 / |\omega|^2 \right) \left( |\omega|^2 / |\sinh \left( \frac{\pi k}{\omega} \right)| \right)^2 \|V\|_{L_2(\mathbb{R})}^2,
\] (4.191)
which is finite, by our assumptions on $V$. Now, this implies that
\[
\left(1 + \frac{|k|^2}{|\omega|^2}\right) \frac{|\omega|^2}{\sinh ^2 \left( \frac{\pi |k|}{2 |\omega|} \right)} \|V\|^2_{L^1(\mathbb{R})} \geq \|T\|^2_{L^2(\mathbb{R})} \geq \|T\|^2_{L^2(\mathbb{R})} \geq \frac{1}{16\tau} \int_{\mathbb{R}^2} \frac{(V(x) - \Lambda)_-(V(y) - \Lambda)_-}{\cosh ^2 \left( \frac{\pi |x-y|}{2 \delta} \right)} \, dx \, dy.
\]
where we used Equation (4.177) in the last step. Rewriting this using
\[
k = \frac{\omega}{\pi i} \ln \left( \frac{\lambda}{2} + \sqrt{\frac{\lambda}{2} + 1} \sqrt{\frac{\lambda}{2} - 1} \right)
\]
yields the desired bound.

Self-adjoint operators

This section treats some general techniques for the discretized Schrödinger equation with a real-valued potential $V$. We use the Birman-Schwenk principle to derive a general bound on the number of eigenvalues less than some pre-designated $\Lambda \in \mathbb{R}$, as a function of the potential $V$. For potentials with $V \to \infty$ as $|x| \to \infty$, this in particular implies that the discrete spectrum has no accumulation points, so the essential spectrum equals the continuous one. We then bound also higher moments of eigenvalues.

Let now as above
\[
H := U + U^{-1} + V = H_0 + V,
\]
for some real-valued potential $V$. The eigenvalues of $H$ less than $\Lambda$, which we denote by $N(\Lambda, H)$, are the negative eigenvalues of
\[
H' := U + U^{-1} + (V - \Lambda).
\]
By the min-max principle, the eigenvalues of $H'$ are all more positive than the eigenvalues of
\[
H_- := U + U^{-1} + (V - \Lambda)_-,
\]
meaning that we have the triplet
\[
N(\Lambda, H) = N(0, H') \leq N(0, H_-).
\]
We next proceed to bound this last term by using a Birman-Schwenk argument from [Bir61, Sch61]. This gives the following two main results.

**Proposition 4.** Let $V(x)$ be essentially bounded from below, and take $\Lambda \in \mathbb{R}$. The eigenvalue counting function then satisfies
\[
N(\Lambda, H) \leq \frac{1}{16\tau} \int_{\mathbb{R}^2} \frac{(V(x) - \Lambda)_-(V(y) - \Lambda)_-}{\cosh ^2 \left( \frac{\pi |x-y|}{2 \delta} \right)} \, dx \, dy.
\]
Using Young’s inequality for the last integral, we also obtain a uniform bound

\[
N(\Lambda, H) \leq \frac{C_{p,q,r}}{16\tau} \|\cosh(\frac{\pi x}{2b})\|_{L^r(\mathbb{R})} \cdot \| (V - \Lambda)^{-1} \|_{L^p(\mathbb{R})},
\]

(4.200)

which we will use in a variation of an Aizenman-Lieb argument \[Aiz78\] to obtain bounds on moments of eigenvalues to \(H\). This results in the following corollary.

**Corollary 6.** Let \(V(x)\) be essentially bounded from below, and let \(\{\lambda_n\}\) be eigenvalues to \(H\). If \(\gamma \in \mathbb{R}^+\), then

\[
\sum_{n=1}^{\infty} (\lambda_n - \Lambda)^\gamma \leq \frac{\gamma \Gamma(\gamma) \Gamma(3) D(b)}{16 \tau \Gamma(\gamma + 3)} \int_{\mathbb{R}} (V(x) - \Lambda)^{\gamma + 2} \, dx,
\]

(4.201)

where

\[
D(b) := \min_{p \in [1,2]} D(p,b) \cdot |\{x : 0 < \Lambda - V(x)\}|^{\frac{2}{p} - 1}
\]

(4.202)

\[
= \min_{p \in [1,2]} \frac{p}{2} \left( \frac{p'p}{2(2-p)} \right)^{\frac{2}{p'}} \left( \int_{\mathbb{R}} \frac{dx}{\cosh(\frac{\pi x}{2b})} \right)^{\frac{2(p-1)}{p}}
\]

(4.203)

\[
\cdot |\{x : 0 < \Lambda - V(x)\}|^{\frac{2}{p} - 1}
\]

(4.204)

is a constant, which can be bounded by \(D(b) \leq \min(|\{x : 0 < \Lambda - V(x)\}|, \frac{4b}{\pi})\).

**Remark 10.** We can numerically compute the constants \(D(p,b)\) and \(D(b)\) in Corollary 6. The result is shown in Figure 4.2.

Before turning to the proofs of these claims, we first list some of their consequences. Firstly, since the left hand side of Equation (4.199) is integer-valued, Proposition 4 implies that potentials sufficiently small in \(L^2(\mathbb{R})\) have no negative eigenvalues. This contrasts with the continuous Schrödinger equation in one dimension, where any potential will have at least one negative eigenvalue, independent of its size. Proposition 4 also implies parts of the results in Theorem 12, as in the following corollary.

**Corollary 7.** If \(V(x) \to +\infty\) as \(x \to \pm \infty\), then \(N(\Lambda, H)\) is finite, so the discrete spectrum of \(H\) has finite multiplicity and no accumulation points. In particular, the essential spectrum equals the continuous one.

**Proof of Corollary 7.** Under these assumptions, the integral in the right hand side of Equation (4.199) is finite for every \(\Lambda\), so the conclusion follows. \(\square\)

**Remark 11.** Note that the right hand side in Equation (4.199) tends to \(+\infty\) as \(\omega'\) tends to zero in the limit of a continuous Schrödinger equation. The bound of the Proposition then becomes void, reflecting the fact that the resolvent changes form in the limit \(\omega' \to 0\).
Example 7. We take $V(x) = 2\cosh\left(\frac{\pi i k}{\omega}\right)$, and parametrize $\Lambda = \cosh\left(\frac{\pi i k}{\omega}\right)$. Using Corollary 6 with $\gamma = 0$, we obtain

$$N(\Lambda, H) \leq \frac{D(b)}{16\pi} \int_{\mathbb{R}} (V - \Lambda)^2 \, dx$$

$$= \begin{cases} 
2k \cosh^2\left(\frac{\pi i k}{\omega}\right) + k - \frac{3\omega \sinh(2\pi i k)}{2\pi i k}, & \Lambda \geq 2, \\
0, & \Lambda < 2.
\end{cases}$$

(4.205)

For $\Lambda \to \infty$, this implies that the bound on $N(\Lambda, H)$ behaves as $\Lambda^2 \ln(\Lambda)$, which is to be compared to the actual behaviour $\ln(\Lambda)^2$, obtained in Example 6.

Proof of Proposition 4. Put $P = (V - \Lambda)_-$ and take

$$K_\lambda := P^{1/2}(H_0 + \lambda)^{-1} P^{1/2} = B^* B,$$

(4.207)

for $B = P^{1/2}(H_0 + \lambda)^{-1/2}$. Since $V$ was assumed to be essentially bounded from below, $P$ is in $L^\infty(\mathbb{R})$. Moreover, for $\lambda \notin -\sigma(H_0) = (-\infty, -2], (H_0 + \lambda)^{-1}$ in Equation (4.180) satisfies

$$|R_0(x - y; \lambda)| \leq C e^{-2\pi \text{Im}(k)x},$$

(4.208)

which is in $L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$. Schur’s lemma then implies that $(H_0 + \lambda)^{-1}$ is a bounded operator from $D(H)$ to $L^2(\mathbb{R})$, meaning that also $K_\lambda$ is, since $P \in L^\infty(\mathbb{R})$. $K_\lambda$ is then a bounded, self-adjoint operator on $L^2(\mathbb{R})$.

We next show the duality result that eigenfunctions with eigenvalue $\lambda$ to $H$ correspond to eigenfunctions with eigenvalue $1$ to $K_\lambda$. If we have a $\psi \in D(H)$ with $H\psi = -\lambda \psi$, for some $\lambda \leq \Lambda$, then

$$(H_0 - P)\psi = -\lambda \psi,$$

(4.209)
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i.e.

\[ P^{1/2} \psi = P^{1/2}(H_0 + \lambda)^{-1} P^{1/2} P^{1/2} \psi, \]

or \( K_\lambda \varphi = \varphi \), for \( \varphi = P^{1/2} \psi \in L^2(\mathbb{R}) \).

This shows that the eigenvalue \( \lambda \) to \( H \) corresponds to the eigenvalue 1 of \( K_\lambda \) acting on \( L^2(\mathbb{R}) \). Next, we claim that the eigenvalues of \( K_\lambda \) are decreasing functions of \( \lambda \). To see this, note that in the momentum representation, we have

\[
(F(H_0 + \lambda)^{-1} F^{-1})(p) = \frac{\psi(p)}{2 \cosh \left( \frac{\pi p}{\omega} \right)} \leq \frac{\psi(p)}{2 \cosh \left( \frac{\pi p}{\omega} \right) + \lambda'}
\]

for \( \lambda' \leq \lambda \). It follows that

\[
\langle \psi, K_\lambda \psi \rangle_{L^2(\mathbb{R})} = \langle P^{1/2} \psi, (H_0 + \lambda)^{-1} P^{1/2} \psi \rangle_{L^2(\mathbb{R})} \leq \langle \varphi, (H_0 + \lambda)^{-1} \varphi \rangle_{L^2(\mathbb{R})},
\]

for \( \varphi = P^{1/2} \psi \). Since the Fourier-transform is unitary, we obtain

\[
\langle \psi, K_\lambda \psi \rangle_{L^2(\mathbb{R})} = \langle \varphi, (H_0 + \lambda)^{-1} \varphi \rangle_{L^2(\mathbb{R})} \leq \langle \hat{\varphi}, F(H_0 + \lambda)^{-1} \hat{\varphi} \rangle_{L^2(\mathbb{R})} \leq \langle \psi, K_\lambda \psi \rangle_{L^2(\mathbb{R})}.
\]

By the min-max principle, the eigenvalues of \( K_\lambda \) must then all decrease with \( \lambda \). It follows that \( N(0, H_-) \) is not greater than the number of eigenvalues of \( K_0 \) that are greater than or equal to one. If we denote these by \( \lambda_j \geq 1 \), then certainly

\[
N(0, H_-) \leq \sum_{j \geq 1} \lambda_j^2 \leq \| K_0 \|_{L^2(\mathbb{R})}^2.
\]

Since the operator \( K_0 = P^{1/2} H_0^{-1} P^{1/2} \) is an integral operator, we can calculate the last norm to obtain

\[
N(0, H_-) \leq \int_{\mathbb{R}^2} P(x) P(y) |R_0(x - y; 0)|^2 \, dx \, dy.
\]

Equation (4.183) from the calculations in Lemma 4 give

\[
|R_0(x - y; \lambda)|^2 = \frac{|\omega|^2 \left[ \cosh^2 \left( \frac{\pi}{\omega} \right) - 2 \text{Im}(k) x \right] - \cos^2 \left( 2 \pi \text{Re}(k) x \right)}{\sinh^2 \left( \frac{\pi k}{\omega} \right) \sinh^2 \left( \frac{\pi k}{\omega} \right)},
\]

where we use the paremetrization \( \lambda = 2 \cosh \left( \frac{\pi k}{\omega} \right) \), for \( 0 \leq \text{Im}(k) \leq \omega \). \( \lambda = 0 \) then corresponds to \( k = \frac{\omega}{2} \), so

\[
|R_0(x - y; 0)|^2 = \frac{1}{16 \pi \cosh^2 \left( \frac{\pi k}{\omega} \right)}.
\]

Inserting this into Equation (4.218) concludes the proof. \( \square \)
Proof of Corollary 4. By the min-max principle, we can without loss of generality assume that \( \Lambda - V(x) \geq 0 \). We also introduce an auxiliary number \( \mu \in \mathbb{R}_+ \) and note that the number of eigenvalues of \( H - \Lambda \) less than \(-\mu\) equals the number of negative eigenvalues of \( H - \Lambda + \mu \). Aiming to set \( \mu = 0 \), we sum over eigenvalues less than \(-\mu\) and write

\[
\sum_{\Lambda - \lambda_n \leq -\mu} (\lambda_n - \Lambda)_+^\gamma \leq - \int_{\mathbb{R}_+} \mu^\gamma dN(H - \Lambda, -\mu) \quad (4.221)
\]

\[
= \gamma \int_{\mathbb{R}_+} \mu^{\gamma-1} N(H - \Lambda, -\mu) d\mu, \quad (4.222)
\]

after integrating by parts. We now use the bound from Equation (4.200), with \( 1 \leq p = q \leq 2, \ r = \frac{p}{2(p - 1)} \). This results in the upper bound

\[
\sum_{\Lambda - \lambda_n \leq -\mu} (\lambda_n - \Lambda)_+^\gamma \leq \frac{\gamma D(p, b)}{16\tau} \int_{\mathbb{R}_+} \mu^{\gamma-1} \left( \int_{\mathbb{R}} (V(x) - \Lambda + \mu)^p dx \right)^{\frac{2}{p}} d\mu, \quad (4.223)
\]

where

\[
D(p, b) = C_{p, p, r} \left\| \cosh \left( \frac{\pi x}{2b} \right) \right\|_{L^r(\mathbb{R})} \quad (4.224)
\]

is a constant. We put \( E(\mu) = |\{x : V(x) < \Lambda - \mu\}| \) and use Jensen’s inequality to obtain the bound

\[
\sum_{\Lambda - \lambda_n \leq -\mu} (\lambda_n - \Lambda)_+^\gamma \leq \frac{\gamma D(p, b)}{16\tau} \int_{\mathbb{R}_+} \mu^{\gamma-1} E(\mu)^{\frac{2}{p} - 1} \int_{\mathbb{R}} (V(x) - \Lambda + \mu)^2 dx d\mu. \quad (4.225)
\]

Changing from \( \mu \) to the variable \( y = \frac{\mu}{\Lambda - V} \), we can rewrite this bound as

\[
\frac{\gamma D(p, b)}{16\tau} \int_0^1 \int_{\mathbb{R}} y^{\gamma-1} E \left[ (V - \Lambda)y \right]^{\frac{2}{p} - 1} (L - V)^\gamma [(V(x) - \Lambda)(1 - y)]^2 dx dy \quad (4.226)
\]

\[
\leq E(0)^{\frac{2}{p} - 1} \left( \frac{\gamma D(p, b)}{16\tau} \int_0^1 y^{\gamma-1} (1 - y)^{\frac{2}{p}} dy \right) \int_{\mathbb{R}} (L - V(x))^{\gamma+2} dx \quad (4.227)
\]

\[
= \frac{\gamma \Gamma(\gamma+3)}{16\tau \Gamma(\gamma + 3)} D(p, b) E(0)^{\frac{2}{p} - 1} \int_{\mathbb{R}} (L - V)^{\gamma+2} dx. \quad (4.228)
\]

It therefore only remains to give a bound on \( D(b) := \min_{p \in [1,2]} D(p, b) E(0)^{\frac{2}{p} - 1} \). Taking \( p = 1, p = 2 \), we can explicitly compute \( D(1, b) = E(0), D(2, b) = \frac{4b}{\pi} \), respectively, which finishes the proof. \( \square \)


Paper A.

*Lieb–Thirring inequalities for generalized magnetic fields*