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Precise Gravimetric-GPS Geoid Determination with Improved Topographic Corrections Applied over Sweden

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ABSTRACT

Stokes's formula published by George Gabriel Stokes in 1849 is still the most important formula in physical geodesy. It enables us to compute the geoidal height N from the gravity anomaly data Δg. But, the major drawback of the Stokes's formula is that it needs the coverage of the gravity over the whole Earth. To diminish this problem, we limit the integration area to a spherical cap with radius ψ_0 around the computation point and to reduce the truncation error committed, a modification of Stokes's integral is implemented. In the modified Stokes's formula, the long-to-medium-wavelength components of the geoid are determined from a global Earth Gravity Model (here, EGM96), whereas the short-wavelength contributions are obtained from terrestrial gravity information. A geoid model built up in this way has high relative accuracy and resolution, but its absolute accuracy is poor due to the long-wavelength errors in the geopotential model as the adopted reference.

The geoid serves as the most important reference surface for vertical datums. Conventional levelling heights, orthometric heights, are referred to the geoid surface. Hence, having an accurate geoid results in precise orthometric heights, which is very important to most of the geodetic and engineering projects. With the advent of satellite positioning, especially Global Positioning System (GPS), the geoid has become directly observable on land through a combination of GPS ellipsoidal height, h, and precise orthometric height H through levelling. The geoid derived with this method has high absolute and relative accuracy, but it is not enough dense to produce a national levelling reference surface.

This thesis is based on 11 papers, which are given in Part II. Part I begins with a brief introduction (Chapter 1). Chapter 2 deals with the concept of geoidal height computations using the modified Stokes's formula and the GPS-levelling derived geoid. Then, these two kinds of data are combined through the fitting process. They are also used for unification of vertical datums.

Chapter 3, which is a major contribution of this study, treats the problems encountered in compiling an accurate geoid model. To have a more accurate estimate of the geoid, we have reformulated the terrain corrections. Downward continuation of mean gravity anomalies is investigated. Kernel modification of Poisson's kernel, low frequency contribution, and truncation error are implemented in downward continuation. Atmospheric direct and indirect effects on geoid are reformulated and investigated precisely in the classical and truncated, as well as modified Stokes's formula. Although truncation error in Stokes's integral is minimized through modification process, its effect is derived using a low degree gravity presentation and added to the final geoid as a correction. One commits an error if the flattening of the reference ellipsoid is neglected. This error is called ellipsoidal correction and considered in the final estimation of geoid.

Chapter 4 is devoted to the numerical integration of terrestrial gravity anomalies with the modified Stokes's integral. An internal error propagation is evaluated to provide the statistical error properties of the gravimetric geoidal heights.

Numerical investigations, including all corrections and geoid determination over Sweden are discussed in Chapter 4. For the estimation of the geoid over Sweden, we have employed 4 different gravimetric geoid models: Vincent and Marsh (1974) model with the high degree reference gravity field but no kernel modifications, Wong and Gore (1969) and Molodenskii et al. (1962) models, which use a high degree reference gravity field and modification of Stokes's kernel and refined least squares spectral weighting proposed by Sjöberg (1991a). The refined least squares geoid estimator is used among the other models to present the final geoid over Sweden. It is constrained to the observed geoid values at 23 GPS stations. The mean and standard deviation of the geoid differences are computed to the 10.1 cm and 5.5 cm, which are the smallest values among the other estimators. The final geoid ranges from 17.22 m to 43.62 m with a mean value of 29.01 m. The standard deviations of computed geoid, through a simple error propagation of standard deviations of mean anomalies, are also computed. It ranges from 7.02 cm to 13.05 cm. The global root mean square error of the refined least squares model is the other estimation of the accuracy of the final geoid, which is computed to 25.3 cm. Thereafter, the gravimetric and GPS-levelling derived geoid are combined in a least-squares sense, benefitting from the high resolution and high relative accuracy of the gravimetric solution and the high absolute accuracy achieved by the GPS-levelling. The combined gravimetric-GPS geoid model improves (3.2 cm in mean value) the solution at the GPS-levelling stations, compared with gravimetric (only) geoid. Finally, we have combined gravimetric-GPS derived geoid to unify the Swedish and the Finnish height systems, indirectly. The difference between the Swedish and the Finnish height systems is computed to -16.1±2.3 cm.

Key words: geoidal height, Stokes's formula, modification, terrain correction, downward continuation, Earth gravity model

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Chapter 1

Introduction

The boundary value problem in physical geodesy can be solved by Stokes's well-known formula for the anomalous potential, and the geoidal height is the final output through Bruns's formula. As a fundamental reference surface of the Earth, the geoid has been served for heights in many countries for classical applications. Therefore, the accurate determination of geoid has always been in the centre of discussion for many of the geodesists.

With the event of the GPS, the geoid has become more important. Gravimetrically determined geoidal heights can be applied for orthometric height determination by GPS. This procedure replaces costly conventional levelling operations with quicker and cheaper GPS surveys, as long as the geoid has been computed to a high accuracy.

The objective of this thesis is to review the state of the art methodology for geoidal height determination over Sweden. To estimate the geoidal heights to a high accuracy, say decimetre level, the theory is reviewed to avoid some approximations used in the existing theories (see also Vaníček et al., 1996a). Special efforts are carried out for the corrections used in this study.

The thesis is based on 11 papers (submitted, accepted or published in refereed journals), which are included in Part II. Most of these papers have been investigated numerically in one or two test areas located in Sweden. A preface is written to the Part II, which contributes as Part I of this thesis. The papers are not summarized one by one in the first part. Instead, they are joined into one unit to compile an accurate geoid over Sweden. They have to be referred to for more details by readers. Part I consistes of 6 chapters. After the intoduction, Chapter 2 discusses the geoid determination by gravity and GPS-levelling data. Stokes's theory for original and higher degree and order reference field as well as kernel modification techniques are described. A combined adjustment of a gravimetrically determined geoid to the GPS-levelling stations is studied. Also, gravimetric-GPS geoid is used to unify vertical datums, indirectly.

Chapter 3, the most important part of this study, is devoted to the corrections which have to be implemented in the geoid determination process. Terrain and atmospheric corrections, ellipsoidal correction and truncation error as well as downward continuation of mean gravity anomalies are discussed. These corrections operate on both the satellite reference field and the terrestrial data.

In Chapter 4, the numerical integration technique used for the integration of Stokes's formula is discussed. An error estimate of the final geoid by propagating the estimated errors of the terrestrial mean gravity anomalies is also presented.

Chapter 5 deals with the numerical investigations. The final output is the geoidal heights over Sweden. Analysis of the gravity data, corrections mentioned in Chapter 3, different geoid estimators, an error analysis of the geoid and a fitting to the GPS-levelling stations are included in this chapter. Unification of the Swedish and the Finnish height systems, using modified Stokes's formula, is also investigated.

Finally, Chapter 5 comprises discussions, conclusions and some recommendations for future work.

Author's contribution

The Stokes's well known formula requires that masses exterior to the geoid are primarily removed (or at least reduced onto the geoid) and the gravity be referred to sea-level. The topographic and atmospheric masses violate the first requirements. The main contribution of this presentation is the improvement of the classical formulas for the terrain corrections, which suffers from the planar approximation and omission of some long-wavelength contributions. To satisfy the second requirements, the corrected gravity anomaly at the Earth's surface must be analytically continued downward to sea-level. The study of this effect is the second most important contribution of the author. The effect of atmosphere (direct and indirect) is presented as a correction applied to the geoid, directly, showing the bias of the IAG approach for the truncated Stokes's formula.

There are different estimators for geoid determination. The global EGM96 and GPS-levelling data are used to study the efficiency of some geoid models. Unification of vertical datums, indirectly, is a by-product of combined gravimetric-GPS geoid determination. Then, the Swedish and Finnish height systems are connected by the proposed model.

Chapter 2

Geoid determination

2.1 Stokes's theory for the original and higher degree reference field and kernel modifications

The gravimetric geoid determination has essentially been employed the original Stokes's formula. When working with the Stokes's kernel, one is supposed to evaluate a surface convolution integral over the whole Earth. This is, of course, an impractical requirement. Therefore, the area of integration is usually limited to a spherical cap around the computation point. This truncation causes an error to the computed geoid, which we will call here the truncation error. The truncation error can be reduced by introducing a modification of the Stokes's kernel. The lack of a global coverage of gravity data can be compensated by a combination of terrestrial gravity with Global Geopotential Models (GGM), i.e. the long-wavelength geoid contributions would be determined from a GGM and the short-wavelength informations from terrestrial gravity data.

In this study, three categories (four gravimetric geoid models) of combination of geopotential models with the Stokes's integral are used (see also Fan, 1989): i) using a high degree reference field, but no kernel modification, ii) combination of Stokes's kernel modification with high degree reference field, and iii) minimizing the global mean square error of the truncation error, the potential coefficients and terrestrial gravity anomalies in a least squares sense.

The original method to modify Stokes's formula was presented by Molodenskii et al. (1962). The main idea in this method is to reduce the truncation error committed by limiting the area of integration under Stokes's integral to a spherical cap around the computation point. The second model, Wong and Gore (1969), employs a residual field and a modified Stokes's kernel. This estimator corresponds to high degree reference gravity field and kernel modification method. Vincent and Marsh (1974) estimates geoid in a slightly different way, which is the third model used in this study. The principle in this method is to use a high degree reference gravity field in Stokes's formula, implying a localized gravity field, and then adding the long-wave contribution from geopotential coefficients. No kernel modification is employed in this model. Sjöberg's least squares modification of Stokes's formula reduces the impact of the errors stemming from truncation, erroneous terrestrial gravity data and potential harmonics in a least squares sense (Sjöberg, 1991a).

Assuming a cap of integration σ_0 with geocentric angle ψ_0 around the computation point, a general estimator that combines the Stokes's kernel modification and the high degree reference gravity field, can be written as (see e.g. Vaníček and Sjöberg, 1991):

$$\tilde{N} = \frac{c}{2\pi} \iint_{\sigma_0} S_M(\psi)(\Delta g - \Delta g_M) d\sigma + c \sum_{n=2}^{M} s_n^* \Delta g_n \ , \tag{2.1}$$

where
$S_M(\psi) = S(\psi) - \sum_{k=2}^{M} \frac{2k+1}{2} s_k P_k(\cos\psi)$,
$s_0, s_1, \cdots, s_M$ = selected parameters,
$s_2^*, \cdots, s_M^*$= selected parameters,
$S(\psi) = \sum_{k=2}^{\infty} \frac{2k+1}{k-1} P_k(\cos\psi)$,
$P_k(\cos\psi)$=k-th Legendres' polynomial,
Δg=observed gravity anomaly,
Δg_n= n-th Laplace harmonic of Δg determinaed from potential coefficients,
Δg_M=gravity anomaly computed from a GGM,
$c = \frac{R}{2\gamma}$,
R= mean Earth surface radius,
γ=mean normal gravity,
M=maximum degree of expansion of geopotential model.
Following Molodenskii et al. (1962), who made a modification to the spherical Stokes's kernel, Vaníček and Kleusberg (1987) made a modification to the spheroidal Stokes kernel. Then, the modification parameters, s_i, were determined from the system of linear equations (Vaníček and Kleusberg, 1987):

$$\sum_{i=2}^{M} (2i+1) e_{ki}(\psi_0) s_i = 2Q_k^M \quad \forall k \ . \tag{2.2}$$

Here

$$e_{ki}(\psi_0) = \int_{\psi=\psi_0}^{\pi} P_i(\cos\psi) P_k(\cos\psi) \sin\psi d\psi \ , \tag{2.3}$$

and the Vaníček and Kleusberg (or spheroidal Molodenskii) truncation coefficients are evaluated from

$$Q_k^M(\psi_0) = Q_k(\psi_0) - \sum_{j=2}^{M} \frac{(2j+1)}{(j-1)} e_{kj}(\psi_0) \ , \tag{2.4}$$

where

$$Q_k(\psi_0) = \int_{\psi=\psi_0}^{\pi} S(\psi) P_k(\cos\psi) \sin\psi d\psi \tag{2.5}$$

are the Molodenskii's truncation coefficients. This procedure to determine the modification coefficients, s_i, is here called Molodenskii et al. method. Furthermore, $s_n^* = s_n$ in this model and Molodenskii-modified spheroidal Stokes function $S_M^s(\psi)$ is used instead of $S_M(\psi)$ in Eq.

(2.1) (see Vaníček and Kleusberg, 1987). $S_M^s(\psi)$ is evaluated from

$$S_M^s(\psi) = S_{M+1}(\psi) - \sum_{k=2}^{M} \frac{2k+1}{2} s_k P_k(\cos\psi) \ , \tag{2.6}$$

where

$$S_{M+1}(\psi) = \sum_{k=M+1}^{\infty} \frac{2k+1}{k-1} P_k(\cos\psi) = S(\psi) - \sum_{k=2}^{M} \frac{2k+1}{k-1} P_k(\cos\psi) \ . \tag{2.7}$$

The modified Wong and Gore (1969) method employs a residual field and a modified Stokes's kernel. Moreover, $s_n = s_n^* = \frac{2}{n-1}$ in Eq. (2.1). It means that high degree reference gravity field and kernel modification are combined in this model.

Vincent and Marsh (1974) choices of the arbitrary parameters are $s_n = 0$ and $s_n^* = \frac{2}{n-1}$ in Eq. (2.1). This method corresponds to a high degree reference gravity field but no kernel modification.

These three estimators use higher degree reference field in Stokes's integral, through the subtraction of the long-wavelength contribution of gravity anomalies (computed from an EGM) from the terrestrial gravity anomalies. This subtraction is a time consuming work which has to be done in each computation point, especially for large values of M. Molodenskii et al. (1962) used the original Pizzetti reference field and Jekeli (1981) and Sjöberg (1984) also emphasized this point. The least squares estimator below uses a model gravity field of a degree and order 2 as a reference field.

Sjöberg (1991a) proposed a refined least squares modification of Stokes's formula, which reduces the truncation error, erroneous terrestrial gravity data and potential harmonics in a least squares sense. It is provided by the formula

$$\tilde{N}_1 = \frac{c}{2\pi} \iint_{\sigma_0} S_N(\psi) \Delta g d\sigma + c \sum_{n=2}^{M} (Q_{Mn} + s_n') \Delta g_n \ , \tag{2.8}$$

where

$$S_N(\psi) = S(\psi) - \sum_{n=2}^{N} \frac{2n+1}{2} s_n' P_n(\cos\psi) \ , \tag{2.9}$$

with $N \geq M$, i.e we allow for more parameters s_n' than there are degrees of potential coefficients available. This means that the solution is unbiased through degree M and the solution contains no truncation error to degree and order M (Sjöberg, 1991a). Q_{Mn} in Eq. (2.8) can be written as:

$$Q_{Mn} = Q_n - \sum_{k=2}^{M} \frac{2k+1}{2} s_k' e_{nk} \ . \tag{2.10}$$

The expected mean square error of the refined least squares estimator is minimized in a least squares sense, resulting to the arbitrary parameters s'_n given by the following system of linear symmetric equations

$$\sum_{r=2}^{N} a_{kr} s'_r = h_k \qquad k = 2, 3, \cdots, N \ , \tag{2.11}$$

where

$$\begin{aligned} a_{kr} &= (\sigma_k^2 + dc_k^*)\delta_{kr} - \frac{2r+1}{2}(\sigma_k^2 + dc_n^*)e_{kr} - \frac{2k+1}{2}(\sigma_r^2 + dc_r^*)e_{rk} \\ &\quad + \frac{2k+1}{2}\frac{2r+1}{2}\sum_{n=2}^{\infty} e_{nk}e_{nr}(\sigma_n^2 + dc_n^*) \end{aligned} \tag{2.12}$$

and

$$h_k = \frac{2\sigma_k^2}{k-1} - Q_k(\sigma_k^2 + dc_n^*) + \frac{2k+1}{2}\sum_{n=2}^{\infty}(Q_n e_{nk}(\sigma_n^2 + dc_n^*) - \frac{2}{n-1}e_{nk}\sigma_n^2) \ , \tag{2.13}$$

where

$$dc_n^* = \begin{cases} dc_n & 2 \le n \le M \\ c_n & n > M \end{cases} \tag{2.14}$$

and σ_n^2 is the n-th anomaly error degree variance, c_n is gravity anomaly degree variance, dc_n is expected mean square error of Δg_n and δ_{kr} is Kronecker's delta symbol. c_n can be computed from a global EGM as:

$$c_n = \frac{(GM)^2}{a^4}(n-1)^2 \sum_{m=0}^{n} (C_{nm}^2 + S_{nm}^2) \ , \tag{2.15}$$

where GM is the product of the universal gravitational constant G and the mass of the Earth, M, a is the equatorial radius of the reference ellipsoid, and C_{nm} and S_{nm} are potential coefficients. The error anomaly degree variance due to erroneous potential coefficients is computed from:

$$dc_n = \frac{(GM)^2}{a^4}(n-1)^2 \sum_{m=0}^{n} (\delta_{C_{nm}}^2 + \delta_{S_{nm}}^2) \ , \tag{2.16}$$

where $\delta_{C_{nm}}$ and $\delta_{S_{nm}}$ are the standard deviations of potential coefficients taken from an EGM. The error anomaly degree variances for the terrestrial gravity anomalies (σ_n^2) can be estimated from a knowledge of an error degree covariance function. This covariance function is, for example, given by (Sjöberg, 1986):

$$C(\psi) = c_1\Big[\frac{1-\mu}{\sqrt{1 - 2\mu\cos\psi + \mu^2}} - (1-\mu) - (1-\mu)\mu\cos\psi\Big] \ , \tag{2.17}$$

where σ_n^2 can be expressed from

$$\sigma_n^2 = c_1(1-\mu)\mu^n \ . \tag{2.18}$$

The parametrers c_1 and μ can be determined from a knowledge of the error variance C_0 (the value of the covariance function $C(\psi)$ for $\psi = 0$) and the correlation length ζ (the value of the argument for which $C(\psi)$ has decreased to half of its value at $\psi = 0$) (Moritz, 1980). $C(0) = 10$ $mGal^2$ and correlation length 0.1° are used in this study (Sjöberg, 1986).

2.2 GPS-levelling geoid

The geoidal height N can be directly computed, on land, through space techniques with combination of the ellipsoidal height h, computed from GPS (for example), and orthometric height H, computed from precise levelling, by the following well-known formula

$$N = H - h \,. \tag{2.19}$$

It has to be mentioned that if the normal height system is used instead of the orthometric heights H, then a geoidal height computed by the Eq. (2.19) is the quasi-geoid rather than the geoid. This is the case in this study, over Sweden, where the RH70 normal height system is used on the GPS stations. To correct for this separation between the orthometric height, H, and normal height, H^N, formula (Sjöberg, 1995a)

$$H_P - H_P^N = -\frac{H_P \Delta g^B}{\gamma} + \frac{H_P^2}{2\gamma}\left(\frac{\partial \Delta g^F}{\partial H}\right)_P \tag{2.20}$$

is used, where Δg^B and Δg^F are the Bouguer and free-air anomalies and (Heiskanen and Moritz, 1967)

$$\left(\frac{\partial \Delta g^F}{\partial H}\right)_P = \frac{R^2}{2\pi}\iint_\sigma \frac{\Delta g^F - \Delta g_p^F}{\ell_0^3} d\sigma - \frac{2}{R}\Delta g_P^F \,, \tag{2.21}$$

where ℓ_0 is the spatial distance between the computation point P and the running point and σ is the unit sphere.

The other correction which affects the geoidal height determination by Eq. (2.19) is the postglacial rebound of the crust and mantle in the Fennoscandian area. The GPS-levelling stations used in this study are taken from the National Land Survey of Sweden. They are the Swedish Permanent GPS Network (SWEPOS). The zero-point epoch of these GPS stations is agreed to the EUREF-89 which refers to epoch 1989.0. The Swedish height system RH70 refers to epoch 1970.0. Therefore, the orthometric heights, H, should be reduced to the 1989.0. The absolute rate of the land uplift, referred to the ellipsoid, is determined by (see e.g. Sjöberg and Fan, 1986; Ekman, 1992; Pan and Sjöberg, 1998; Nahavandchi and Sjöberg, 1998a [Paper k]):

$$\dot{h} = \dot{H}_a + \dot{H}_e + \dot{N} = \dot{H}_e + 1.07\dot{H}_a \,, \tag{2.22}$$

where $\dot{H}_e$ is the eustatic rise of sea-level (1mm/yr) and $\dot{H}_a$ is the apparent land uplift relative to the mean sea-level. $\dot{H}_a$ is taken from Mäkinen et al. (1986). See also Sjöberg et al. (1988).

A fitting between the gravimetric and GPS-levelling geoid can be conducted by a simplified 4-parameter transformation, represented by the geographical coordinates, as (see e.g. Forsberg and Madsen, 1990):

$$N^G - N^{GPS} = \Delta N = \cos\phi\cos\lambda\Delta X + \cos\phi\sin\lambda\Delta Y + \sin\phi\Delta Z + kR \tag{2.23}$$

where R is the mean radius of the Earth, ϕ and λ are geographical coordinates, ΔX, ΔY, ΔZ are the three translations and k is the scale factor. This 4-parameter regression is a very appropriate formula, but, unfortunately, some long-wavelength geoid errors may be absorbed by the parameters.

2.2.1 Combined adjustment of gravimetric-GPS geoid

Gravimetric geoidal height can have a very high resolution and relative accuracy, but its absolute accuracy can be poor due e.g. to the long-wavelength errors coming from global geopotential models. Rapp (1992) reported an average accuracy of about 36 cm due to the long-wavelength error in global geopotential model on land. There are also some short-wavelength errors in gravimetric geoid stemming e.g. from the lack of the gravity data. On the other hand, the geoid computed through GPS-levelling can have a high absolute and relative accuracy, especially when those stations are tied to VLBI or SLR stations. The disadvantage of the GPS-levelling geoid is its poor resolution, which prevents us to use them to define a national levelling surface.

Hence, a combined adjustment of gravimetric geoid to the GPS-levelling stations is here conducted. It allows us to benefit from the advantages of the respective methods. In this case, the long- and short-wavelength (local deformations) errors can be carefully approximated by the global and local transformation models, respectively.

For the gravimetric geoid points, the difference of the geoidal heights is taken as the observation to establish a relative observation equation. This corrects local deformation, which is due to the shortage and errors of gravity or DTM data and the bias due to the differences between data sources which have different reference systems. A local transformation model is used, here, to take into consideration these deformations. We have divided the computation area (central part of Sweden, which has shortage of gravity data) to two pieces, and in each piece a local transformation model is established. The local deformation in the differences of the geoidal heights will be mostly cancelled out, which makes the realtive observations to be relatively accurate.

On the GPS stations, the GPS-levelling geoidal height itself is taken as the observation to establish the absolute observation equation. This corrects the long-wavelength error in the gravimetric solution, which is due to the errors in the geopotential model. A global transformation model is used to take into consideration the long-wavelength deformations.

The unknowns in the adjustment process are the global and local transformation parameters, the geoidal heights in the common points, in the border of the two pieces, and the geoidal heights in the uncommon points.

Observation equations

Denoting i, j, k as the points on the geoid, the relative observation equation for the gravimetric geoid solution is (see Jiang and Duquenne, 1996):

$$V_{ij} = \bar{N}_j - \bar{N}_i - \Delta N_{ij}^G + \Delta T(Z) + \Delta t(z) \ . \tag{2.24}$$

Similarly the absolute observation equation for the GPS-levelling stations is:

$$V_k = N_k^G - N_k^{GPS} + T(Z) \ , \tag{2.25}$$

where
N^{GPS}=the geoidal height computed in GPS-levelling stations,
N^G= the gravimetric geoidal height,

$\bar{N}$= the adjusted geoidal height,
V= residuals,
ΔN^G= the difference of geoidal height between two points,
$T(Z)$ and $\Delta T(Z)$ = global transformation models with Z as the set of transformation parameters,
$\Delta t(z)$= the local transformation model with z as the set of transformation parameters.

The global transformation model, used in the adjustment process, is the model in Eq. (2.23). The relative form of this equation, between two points i, j, can be expressed as:

$$\begin{aligned} \Delta T_{ij} &= (\cos\phi_j \cos\lambda_j - \cos\phi_i \cos\lambda_i)\Delta X + (\cos\phi_j \sin\lambda_j - \cos\phi_i \sin\lambda_i)\Delta Y \\ &+ (\sin\phi_j - \sin\phi_i)\Delta Z \ . \end{aligned} \tag{2.26}$$

The local transformation model in relative form, between two points i, j, is approximated from:

$$\Delta t_{ij} = a(\phi_j - \phi_i) + b\cos\phi_0(\lambda_j - \lambda_i) \ , \tag{2.27}$$

where ϕ_0 is the mean latitude in area, a and b are the local transformation coefficients. A relative form of Eq. (2.27) for two points i, j, in the border of the two pieces 1, 2, is written as:

$$\Delta t_{ij}^{12} = a^1(\phi_j - \phi_0^1) + b^1 \cos\phi_0^1(\lambda_j - \lambda_0^1) - a^2(\phi_i - \phi_0^2) - b^2 \cos\phi_0^2(\lambda_i - \lambda_0^2) \ , \tag{2.28}$$

where λ_0 is the mean value of the longitude in the area.

To combine the gravimetric geoid in two pieces in a least squares sense, some constraints are needed (see also Jiang and Duquenne, 1996):

a) the gravimetric geoid in each piece has to be constrained to the GPS points in the piece itself.

b) The common points between the two pieces must have the same adjusted value.

For each GPS-levelling station, there are two kinds of geoid (GPS-levelling and gravimetric) and subsequently two kinds of observation equations (relative and absolute). Relative observation equation in GPS-levelling stations are established between a GPS station and its nearest 4 corner knots of the gravity geoid grid. We have implemented the combined adjustment as an adjustment in two groups. Let L_1 and L_2 be respectively two groups of the gravimetric and GPS-levelling geoid observations, respectively. Then, one can write:

$$\begin{aligned} V_1 &= A\Delta X_a + B_1 \Delta X_b - \Delta L_1 \\ V_2 &= B_2 \Delta X_b + C\Delta X_c - \Delta L_2 \ , \end{aligned} \tag{2.29}$$

where A, B_1, B_2, C are matrices of coefficients; ΔX_b are the common unknowns among the two observation groups such as global transformation parameters and the geoid at the common points; $\Delta X_a, \Delta X_c$ are the geoidal heights at the uncommon points and the local tansformation parameters; $\Delta L_1, \Delta L_2$ include the observations L_1, L_2, approximate coordinates and constants. These two group observations are considered uncorrelated. This system of the observation equations can be solved by the least squares method for two groups of observations such as the one described in Bjerhammar (1973), Vaníček and Krakiwsky (1986) or the method explained in Jiang and Duquenne (1996).

As far as the weight problem of the relative and absolute observations is concerened, the variances of the observations, gravimetric geoid differences and GPS-levelling geoid, are considered constant. They are also supposed to have the same accuracy as far as random error is concerned. Thereafter, the priori variance factor is considered unit. Hence, the weights of the geoid difference observations are equal to one and through the law of error propagation, the weights of GPS-levelling observations become equal to two. This might not be resonable for this study and a more sophisticated procedure has to be investigated.

2.2.2 Unification of vertical datums

Neighbouring vertical datums can be directly compared at one or more common points on the border between the datums. This direct method of the unification needs an information of gravity and levelling. But, if it is not possible to connect the vertical datums directly (e.g. if they are separated by an ocean), then a rigorous mathematical model may be useful. This model, called indirect approach, is derived based on a method by Rummel and Teunissen (1988). It connects vertical datums by means of a combination of GPS and gravimetric geoid models at the tide gauge sites using modified Stokes's formula. The following observation equation for the unknowns N_0 and C_{P_jO} $(j = 1, 2, \cdots, n)$ can be established (see, Pan and Sjöberg, 1998 and Nahavandchi and Sjöberg, 1998a [Paper K]):

$$y_i = N_0 + \frac{C_{P_iO}}{\gamma} + \frac{R}{4\pi\gamma} \sum_{j=1}^{n} \int\int_{\Delta\sigma_j} S_l^s(\psi) \frac{2}{R} C_{P_jO} d\sigma, \tag{2.30}$$

where C_{P_jO} is the potential differences between point O and P and the observations y_i are given by

$$y_i = h - H^i - \frac{R}{4\pi\gamma} \sum_{j=1}^{n} \int\int_{\Delta\sigma_j} S_l^s(\psi) \Delta g^j \, d\sigma - \frac{R}{2\gamma} \sum_{n=2}^{l} s_n \Delta\hat{g}_n. \tag{2.31}$$

The constant N_0 is

$$N_0 = -\frac{\Delta W_0}{\gamma} + \frac{G\delta M}{R\gamma} , \tag{2.32}$$

where G is the gravitational constant, ΔW_0 is the difference between the constant gravity potential at the geoid and constant normal potential at the reference ellipsoid, δM is the difference between the mass of the Earth and the mass of the ellipsoid, h is the ellipsoidal height and H is the orthometric height . The upper index (i) of N and H expresses the fact that P lies in the datum zone with datum point P_i and refers to the i-th regional vertical datum, $j = 1, 2, .., n$ being the number of regional vertical datums, Δg^j is the observed gravity anomaly reduced to the level surface passing through P_i and subtracted from it the gravity anomalies computed from a global geopotential model, $\Delta\hat{g}_n$ is the n-th Laplace harmonic of Δg derived from potential coefficients, γ is the normal gravity on the reference ellipsoid, R is the mean radius of the Earth, ψ is the spherical distance between the computation and running points, $\Delta\sigma_j$ is the solid angle of datum zone j and s_n are selected parameters. $S_l^s(\psi)$ is the modified spheroidal Stokes's function defined before, where the reference spheroid is given by spherical harmonic function up to degree and order l.

Chapter 3

Corrections

This chapter is considered the most important part of this study, and most efforts are put on it. The application of Stokes's formula for the computation of the geoid requires that the disturbing potential is harmonic outside the geoid. This is implied by removing the external masses or reducing them inside the geoid (direct effect). The masses are then restored after applying Stokes's integral (indirect effect). The formulas derived for terrain corrections are based on a constant topographic density. These formula can also be generalized to a laterally variable density simply by putting it under the surface integrals on the direct and indirect topographic effects. Geoid determination by Stokes's well known formula also requests that gravity anomalies Δg must refer to the geoid. For satisfying this second condition, as the gravity anomalies Δg are available on the surface of the Earth, a reduction of them from the Earth's surface to the geoid is in order. This reduction is called downward continuation.

The corrections mentioned above with the combined idea of Stokes-Helmert integration are realized by the formula (see e.g. Heiskanen and Moritz, 1967, p. 324):

$$N = \frac{R}{4\pi\gamma} \iint_\sigma S(\psi) \Delta g^{H^*} d\sigma + \delta N_I \ , \tag{3.1}$$

where Δg^{H^*} is the gravity anomaly corrected for terrain correction and reduced to sea-level and δN_I is the indirect geoid effect. In this study, we have decided to use the Helmert's second condensation method to replace the external masses by a condensed layer placed on the geoid. For more details see e.g. Wichiencharoen (1982), Vaníček and Martinec (1994) and Nahavandchi and Sjöberg (1998b). The notation Δg^H at the ground level can be expressed via

$$\Delta g^H = \Delta g + \delta \Delta g_{dir} \ , \tag{3.2}$$

where Δg is the surface free-air anomaly and $\delta \Delta g_{dir}$ is the direct effect determined at the Earth's surface. The notation Δg^{H^*} is the analytically downward continued Δg^H from the surface to sea-level. This process can be done e.g. by a Taylor expansion. It should be mentioned that Eq. (3.1) can also be rewritten as (see e.g. Sjöberg, 1998a):

$$N = \frac{R}{4\pi\gamma} \iint_\sigma S(\psi)(\Delta g + \delta \Delta g^*_{dir} + \delta \Delta g_{dc}) d\sigma + \delta N_I \ , \tag{3.3}$$

where $\delta\Delta g^*_{dir}$ is the direct gravity anomaly downward continued to sea-level and $\delta\Delta g_{dc}$ is the correction due to the downward continuation of free-air anomaly Δg. However, Eq. (3.1) is preferred to Eq. (3.3) as far as numerical point of view is concerned.

The effect of the atmosphere on the geoid determination and the truncation error as well as ellipsoidal corrections are also studied in this chapter.

3.1 Terrain corrections

3.1.1 Direct terrain correction in Stokes's formula

The correction due to the removing the masses above geoid (or shifting them inside the geoid) is, here, simply called direct effect. The Helmert's second condensation method is used which preserves the mass but changes the potential of the topography.

The classical integral formula for direct effect determination at point P, on the surface of the Earth, can be approximated from (see Vaníček et al., 1986; Vaníček and Kleusberg, 1987):

$$\delta A(H_P)^{classic} = \frac{\mu R^2}{2} \int\int_\sigma \frac{H^2 - H_P^2}{\ell_0^3} d\sigma \ , \tag{3.4}$$

where
$\mu = G\rho_0$, G being the universal gravitational constant, and ρ_0 is the density of topography, assumed to be constant,
H, H_P=orthometric heights of the running and computation points, respectively,
$\ell_0 = R\sqrt{2(1-\cos\psi)} = 2R\sin\frac{\psi}{2}$,
R = mean Earth surface radius,
σ = the unit sphere,
ψ = geocentric angle between computation point P and running point on the sphere.
This formula can only be used for the far zone integration area, where $\ell_0 \gg H$, and the effect of the near zone and the Bouguer shell (which cannot be derived from a planar model) are completely missing (see Martinec and Vaníček, 1994a). It should also be mentioned that the accuracy of power series used in integration is limited to the second-order terms in the height H.

Sjöberg (1994), (1995) and (1997) developed the direct effect, $\delta A(H_P)$, in spherical harmonics to power H^2, and Nahavandchi and Sjöberg (1998b) [Paper A] extended this approach to power H^3. The result can be summarized as:

$$\begin{aligned} \delta A(H_P) \ \doteq \ & \frac{\pi\mu}{2R}\left[5H_P^2 + 3\bar{H}_p^2 + 2\sum_{n,m} n(H^2)_{nm}Y_{nm}(P)\right] + \frac{\pi\mu}{2R^2}\left[\frac{28}{3}H_P^3 + \frac{9}{2}\bar{H}_P^2 H_P - \frac{1}{2}\bar{H}_P^3\right. \\ & \left. + H_P\sum_{n,m} n(2n+9)(H^2)_{nm}Y_{nm}(P) - \frac{1}{3}\sum_{n,m} n(2n+7)(H^3)_{nm}Y_{nm}(P)\right] \ , \end{aligned} \tag{3.5}$$

where Y_{nm} are fully normalized spherical harmonics obeying

$$\frac{1}{4\pi}\iint_\sigma Y_{nm}Y_{n'm'}d\sigma = \begin{cases} 1 & \text{if } \ n = n' \ \text{ and } \ m = m' \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

and

$$(H^v)_{nm} = \frac{1}{4\pi}\iint_\sigma H_P^v Y_{nm} d\sigma \ ; \ v = 1,2,3,\cdots \ , \tag{3.7}$$

$$H_P^v = \sum_{n,m}(H^v)_{nm} Y_{nm} \ , \tag{3.8}$$

$$\bar{H}_P^v = \sum_{n,m}\frac{1}{2n+1}(H^v)_{nm} Y_{nm}(P) \ . \tag{3.9}$$

This harmonic presentation of the direct terrain effect is simple for the computations. It is also free from the problems encountered in integral formulas, e.g. the singularity and topographic limitations. But, the harmonic expansion series of H^2 and H^3 will include the long-wavelengths. To incorporate all significant contributions from both short- and long- wavelengths, it needs an expansion of spherical harmonics presentation of H^2 and H^3 to very high degrees, which is practically difficult and it ruins the simplicity of this method. Nahavandchi and Sjöberg (1998b) [Paper A] showed that the dominant part of power series in formula (3.5) was the second power of elevation H. The contribution from the harmonic expansion series H^3 on geoid was within 9 cm in Himalayas region. Nahavandchi (1998a)[Paper E] also showed that the contributions from H^4 and H^5 can safely be neglected for geoid terrain correction of satellite derived geopotential model, to a maximum degree of expansion of M=360. Sjöberg (1996) showed that the expansion of these series to infinity would converge.

On the other hand, the integral classic formulas are not practical for numerical computations, as they require a global integration to include the long-wavelength information. Then, a middle compromise may be of the form (Nahavandchi, 1998b [Paper C]):

$$\Delta\delta A(H_P) \ = \ \delta A(H_P)^{classic} - \delta A(H_P)^{new} = -\frac{4\pi\mu}{R}H_p^2 - \frac{3\mu}{8}\iint_\sigma \frac{H^2 - H_P^2}{\ell_0} d\sigma \tag{3.10}$$

or

$$\begin{aligned}\delta A(H_P)^{new} \ &= \ \delta A(H_P)^{classic} - \Delta\delta A(H_P) \\ &= \ -\frac{4\pi\mu}{R}H_p^2 - \frac{3\mu}{8}\iint_\sigma \frac{H^2 - H_P^2}{\ell_0} d\sigma + \frac{\mu R^2}{2}\iint_\sigma \frac{H_P^2 - H^2}{\ell_0^3} d\sigma \ ,\end{aligned} \tag{3.11}$$

where $\Delta\delta A(H_P)$ in spectral form is estimated to:

$$\Delta\delta A(H_P) = -\frac{5\pi\mu}{2R}H_p^2 - \frac{3\pi\mu}{2R}\bar{H}_P^2 \ . \tag{3.12}$$

In Eq. (3.11), the Bouguer shell effect (first term on right hand side) and some long-wavelength contributions (second term on right hand side) are present. But, the problem with this formula is the third term, which is singular for the near zone integration area and can only be used for the far zone contributions (following Eq. (3.4) where $\ell_0 \gg H$). It has to be modified in some way to include both far and near zones effects (see below).

Formula (3.12) shows that there are some long-wavelength differences of power H^2 between the classical and new formulas. The most likely explanation to this difference is that the classical method suffers from the planar approximation. Hence $\Delta\delta A(H_P)$ above can be regarded as a correction to this method.

Sjöberg (1998a) also derived a new formula for direct terrain effect to the second power of elevation H including both far and near zone effects. This formula has also included the terms to the fourth power of elevation H_P and is derived as:

$$\delta A(H_P)^S = -\frac{4\pi\mu}{R}H_p^2 - \frac{3\mu}{8}\iint_\sigma \frac{H^2 - H_P^2}{\ell_0}d\sigma + \frac{\mu R^2}{2}\iint_\sigma \frac{H_P^2 - H^2}{\ell^3}\Big(1 - \frac{3H_P^2}{\ell^2}\Big)d\sigma \ , \quad (3.13)$$

where $\ell = \sqrt{r_P^2 + r^2 - 2r_P r \cos\psi}$, and $r_P = R + H_P$. As it can be seen from Eq. (3.13), the first two terms are the same with Eq. (3.11). The third term uses ℓ instead of ℓ_0 and also an additional term $-\frac{\mu R^2}{2}\iint_\sigma \frac{H_P^2 - H^2}{\ell^3}\frac{3H_P^2}{\ell^2}d\sigma$ is present, which is insignificant. Therefore, Eq. (3.11) is modified to

$$\delta A(H_P)^{new} = -\frac{5\pi\mu}{2R}H_p^2 - \frac{3\pi\mu}{2R}\bar{H}_P^2 + \frac{\mu R^2}{2}\iint_\sigma \frac{H_P^2 - H^2}{\ell^3}\Big(1 - \frac{3H_P^2}{\ell^2}\Big)d\sigma \ , \quad (3.14)$$

considering all the significant contributions.

3.1.2 Direct terrain correction for potential coefficients

In determining the geoidal undulations from an EGM, we must expect a bias of the external harmonic series when applied at the geoid within the topographic masses. The bias can be estimated by removing the terrain masses, which implies a direct effect on the geopotential. Helmert second condensation method is used for reducing the masses (see also Vaníček et al., 1995). The direct effect on geopotential to the third power of elevation, H, is estimated directly on geoid to (Nahavandchi and Sjöberg, 1998b [Paper A]):

$$\begin{aligned}\delta N_{\rm dir}^M \doteq & -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{n+2}{2n+1}(H^2)_{nm}Y_{nm}(P) \\ & -\frac{2\pi\mu}{R\gamma}\sum_{n=0}^{M}\sum_{m=-n}^{n}\frac{(n+2)(n+1)}{3(2n+1)}(H^3)_{nm}Y_{nm}(P) \ , \end{aligned} \quad (3.15)$$

with the same notations as previously defined.

3.2 Primary indirect effect

The effect of restoration of reduced masses is the indirect effect. The classical formula for determining the indirect effect on the geoid for Helmert's second condensation method is (Wichiencharoen, 1982):

$$N_I(P) = \frac{-\pi\mu H^2}{\gamma} - \frac{\mu R^2}{6\gamma}\iint_\sigma \frac{H^3 - H_P^3}{\ell_0^3}d\sigma \ , \quad (3.16)$$

with the same notations as before.

Sjöberg (1994), (1995) and (1997) developed the indirect effect in spherical harmonics to power H^2, and Nahavandchi and Sjöberg (1998b) [Paper A] extended this approach to power H^3. The result can be summarized as:

$$\delta N_{\mathrm{I}}(P) = -\frac{2\pi\mu}{\gamma}\sum_{n=0}^{\infty}\frac{n-1}{2n+1}H_n^2(P) + \frac{2\pi\mu}{3R\gamma}\sum_{n=0}^{\infty}\frac{n(n-1)}{2n+1}H_n^3(P) \ , \tag{3.17}$$

where

$$H_n^{\nu}(P) = \frac{2n+1}{4\pi}\iint_{\sigma} H^{\nu}P_n(cos\psi)d\sigma; \ \nu = 2,3, \tag{3.18}$$

and $P_n(cos\psi)$ is Legendre's polynomial of order n. The classical formula is not practical for computation, as it requires an integration over the whole Earth to include long-wavelength contributions. It also suffers from planar approximation (see Martinec and Vaníček, 1994b; Sjöberg and Nahavandchi, 1998a). On the other hand, the spherical presentation of indirect effect needs a very high maximum degree of expansion, to consider all short- and long- wavelengths information. Therefore, a compromise between these two method is derived as (Sjöberg and Nahavandchi, 1998a [Paper D]):

$$\begin{aligned}\delta N_{\mathrm{I}}(P) \quad &= \quad N_{\mathrm{I}}^{classic} - N_{\mathrm{I}}^{new} = -\frac{3\pi\mu}{\gamma}H_p^2 - \frac{3R\mu}{4\gamma} \\ &\times \iint_{\sigma}\frac{H^2 - H_P^2}{\ell_0}d\sigma - \frac{\mu}{8\gamma}\iint_{\sigma}\frac{H^3 - H_P^3}{\ell_0}d\sigma \ ,\end{aligned} \tag{3.19}$$

or

$$N_{\mathrm{I}} = N_{\mathrm{I}}^{classic} - \delta N_{\mathrm{I}} \ , \tag{3.20}$$

where δN_{I} in spectral form is estimated to:

$$\delta N_{\mathrm{I}}(P) = -\frac{3\pi\mu}{\gamma}\bar{H}_p^2 + \frac{\pi\mu}{2R\gamma}(H_P^3 - \bar{H}_P^3) \ . \tag{3.21}$$

Formula (3.21) shows that there are some long-wavelength differences of power H^2 and H^3 between the classical and new formulas. The most likely explanation to this difference is that the classical method suffers from the planar approximation. Hence, $-\delta N_{\mathrm{I}}$ above can be regarded as a correction to this method.

3.3 Secondary indirect effect

The secondary indirect effect is a free-air correction of gravity from geoid to cogeoid, i.e. $2\gamma\delta N_{dir}/R \doteq -2\gamma\delta\zeta_I/R$, where ζ is height anomaly. This yields the following correction, directly on geoid, to the third power of elevation H (Nahavandci and Sjöberg, 1998b [Paper A]):

$$\begin{aligned}\delta N_{\mathrm{I2}} \quad \doteq \quad &\frac{4\pi\mu}{\gamma}\sum_{n,m}\frac{n+2}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P) \\ &-\frac{\pi\mu}{R\gamma}H_P\sum_{n,m}\frac{4n^2+2n+3}{(2n+1)(n-1)}(H^2)_{nm}Y_{nm}(P) \\ &+\frac{2\pi\mu}{3R\gamma}\sum_{n,m}\frac{2n^2-8n-3}{(2n+1)(n-1)}(H^3)_{nm}Y_{nm}(P).\end{aligned} \tag{3.22}$$

According to our experience the secondary indirect terrain effect is at least 2 orders of magnitude smaller than the direct terrain effect and, therefore, is expressed with a spherical harmonic presentation.

3.4 Atmospheric effect

The effect of the mass of the atmosphere must also be removed prior to the application of Stokes's formula. This corresponds to the direct atmospheric effect. After the application of Stokes's formula the effect of restoring the atmosphere should be applied. Sjöberg (1993) emphasized that there could be additional significant direct and indirect atmospheric effects stemming from a more detailed treatment of the Earth's topography than is made in the classical IAG approach. Sjöberg and Nahavandchi (1998b) [Paper I] derived the total of direct and indirect effects, called the total atmospheric effect, to the so-called modified Stokes's formula, implying the combination with potential coefficients. This total atmospheric effect is derived as (Sjöberg and Nahavandchi, 1998b [Paper I]) :

$$\begin{aligned} N^a_{\text{total}} &= c_1 \sum_{n=0}^{1} (s_n + Q_{Mn}) H_n(P) - c_1 \sum_{n=2}^{M} \Big(\frac{2}{n-1} - Q_{Mn} - s_n \Big) H_n(P) \\ &\quad - c_1 \sum_{n=M+1}^{\infty} \Big(\frac{2}{n-1} - \frac{n+2}{2n+1} Q_{Mn} \Big) H_n(P) \ , \end{aligned} \tag{3.23}$$

where $c_1 = \frac{2\pi R \rho^0 G}{\gamma}$. ρ^0 is the density of the atmosphere at the radius of sea level.

3.5 Downward continuation of terrain corrected free-air anomaly

To obtain the boundary values in the Stokes's well known formula, the gravity anomalies Δg at the Earth's surface have to be reduced onto the geoid. This reduction is the downward continuation. The main problem with the downward continuation is the masses between the surface level and the geoid and irregularity of the density distribution, which causes that the disturbing potential is non-harmonic outside the geoid. Vaníček et al. (1996b) examined downward continuation of Helmert anomaly and found out that the determination of this effect is a well posed problem in $5' \times 5'$ cells. Nahavandchi (1998c) [Paper H] investigated four models of downward continuation of free-air anomalies: the Poisson's integral, a simple method, the Pellinen approximation, and a method based on the topographic-isostatic compensation potential (proposed by Sjöberg (1998b)). Among them, the first model is used in this study.

Prior to the downward continuation, the gravity anomalies Δg are corrected for the direct terrain correction. The application of the direct effect makes the gravity anomalies smoother and a better suited for downward continuation.

After Bjerhammar (1962), assume a fictious field of gravity anomalies Δg^* on the geoid, which generate on the surface the gravity anomalies Δg. These two anomalies can be related by the Poisson's formula (excluding the spherical harmonics of degrees zero and one) (Kellogg, 1929; MacMillan, 1930):

$$\Delta g = \frac{R}{4\pi} \iint_\sigma \Delta g^* K(r, \psi, R) d\sigma \ , \tag{3.24}$$

where $K(r, \psi, R)$ is the Poisson's kernel described by

$$K(r, \psi, R) = \sum_{n=0}^{\infty} (2n+1) \Big(\frac{R}{r}\Big)^{n+1} P_n(\cos\psi) = R \frac{r^2 - R^2}{\ell^3} \ , \tag{3.25}$$

where $r = R + H_P$. In this equation, spherical approximation is used; R is the mean radius of the Earth, H is the topographic height of the surface point P, σ is the unit sphere and ℓ and ψ are the spatial and spherical distances between the surface point P and the surface element of the terrestrial sphere $R = r$. The gravity anomalies Δg at level surface are known, and the gravity anomalies Δg^* at sea-level are desired. In this sense, Eq. (3.24) can be solved in different ways; for example by a linear approximation. We have used an iterative process to solve the integral (3.24) (see, Heiskanen and Moritz, 1967; Bjerhammar, 1969).

The Poisson's kernel which dominates the behaviour of the Poisson's integral tappers off rapidly with the growing of the distances from computation point. Therefore, it can only be integrated over a small spherical cap ψ_0 instead of the whole Earth. We reduce the truncation error using the Molodenskii's truncation modification technique (Molodenskii et al., 1962). A spherical cap with radius equal to 1° asures us that the contribution from the rest of the world is small (see Vaníček et al., 1996; Nahavandchi, 1998c [Paper H]). Hence, the truncation error is computed from a global gravity model. To minimize the contribution from the rest of the world, the long-wavelength contribution is also subtracted from the terrain corrected gravity anomalies. This long-wavelength part is downward continued, separately, and added together with the correction due to the truncation error to the short-wavelength contributions (downward continued with the Poisson's integral).

3.6 Truncation error in Stokes's formula

Because of the gravity coverage of the Earth's surface and the huge computation time, the integration area in Stokes's integral is splitted to a resonably small cap σ_0 (with radius ψ_0) around the computation point and the rest of the world. The short-wavelength part of the geoid can then be written as:

$$\frac{R}{4\pi\gamma} \iint_\sigma S_M(\psi) \Delta g d\sigma = \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_M(\psi) \Delta g d\sigma + \frac{R}{4\pi\gamma} \iint_{\sigma - \sigma_0} S_M(\psi) \Delta g d\sigma \ . \tag{3.26}$$

As the contribution from the rest of the world is sufficiently small (considering the use of the modified Stokes's formula), it can be evaluated from a global gravity model. Writing the modified Stokes's formula as

$$S_M(\psi) = S(\psi) - \sum_{k=2}^{M} \frac{2k+1}{2} s_k P_k(cos\psi) \tag{3.27}$$

and the gravity anomaly (derived from a GGM) as

$$\Delta g = \gamma \sum_{j=2}^{M} (j-1) \sum_{m=-j}^{j} U_{jm} Y_{jm}(P) \ , \tag{3.28}$$

one obtains the correction due to the truncation error in the Stokes's integral as:

$$T_g(P) = \frac{R}{2\pi} \sum_{j=2}^{M} (j-1) Q_{Mj}(\psi_0) \sum_{m=-j}^{j} U_{jm} Y_{jm}(P) \ , \tag{3.29}$$

where U_{jm} are potential coefficients and Q_{Mj} is defined before. This correction is already considered in the refined least squares estimator. However, this correction is applied in the other models.

3.7 Ellipsoidal correction

3.7.1 Ellipsoidal correction for terrestrial gravity data

As the quantities of the anomalous gravity field are relatively small, one usually neglect the terms of the order e^2, the flattening of the reference ellipsoid, in computations of the formulas used in physical geodesy. Therefore, these expressions hold only for a spherical approximation. The ellipsoidal correction on the terrestrial gravity anomalies, ϵ_s, is evaluated by the formulas given in Moritz (1980) as:

$$\epsilon_s = e^2 \Delta g^1 \ , \tag{3.30}$$

where

$$\Delta g^1 = \frac{1}{R} \sum_{n=2}^{\infty} \sum_{m=0}^{n} (G_{nm} \cos m\lambda + H_{nm} \sin m\lambda) P_{nm}(\sin\phi) \ . \tag{3.31}$$

$$G_{nm} = k_{nm} A_{n-2,m} + \lambda_{nm} A_{nm} + \mu_{nm} A_{n+2,m} \ , \tag{3.32}$$

$$H_{nm} = k_{nm} B_{n-2,m} + \lambda_{nm} B_{nm} + \mu_{nm} B_{n+2,m} \ , \tag{3.33}$$

where

$$k_{nm} = -\frac{3(n-3)(n-m-1)(n-m)}{2(2n-3)(2n-1)} \ , \tag{3.34}$$

$$\lambda_{nm} = \frac{n^3 - 3m^2 n - 9n^2 - 6m^2 - 10n + 9}{3(2n+3)(2n-1)} \ , \tag{3.35}$$

$$\mu_{nm} = -\frac{3n+5)(n+m+2)(n+m+1)}{2(2n+5)(2n+3)} \ . \tag{3.36}$$

A_{nm} and B_{nm} are the coefficients of the disturbing potential T.

As the ellipsoidal correction is relatively small, thus it can be estimated by using the truncated spherical harmonic coefficients from a GGM. It has to be mentioned that the summation in Eq. (3.31) starts from n=2. The zero and first degree terms used in the Eqs. (3.32) and (3.33) can be evaluated from, e.g.:

$$\sum_{n=2}^{\infty} G_{nm} A_{nm} P_{n-2,m} = \sum_{n=0}^{\infty} G_{n+2,m} A_{n+2,m} P_{nm} \ , \tag{3.37}$$

$$\sum_{n=2}^{\infty} H_{nm} A_{nm} P_{n-2,m} = \sum_{n=0}^{\infty} H_{n+2,m} A_{n+2,m} P_{nm} \ . \tag{3.38}$$

3.7.2 Ellipsoidal correction for geopotential coefficients

The Earth's gravitational potential is usually decomposed into a set of spherical harmonic components. Recognizing that the Earth's surface resembles more an ellipsoid, the decomposition of the gravitational potential can be performed more accurately using ellipsoidal harmonics. The Earth gravity model in this study is EGM96. The difference between an ellipsoidal and a spherical spectrum of gravimetric data is already accounted in this model. The corrections due to the spherical approximation effects in the fundamental boundary condition, relating the gravity anomaly to the disturbing potential, is described by Rapp and Pavlis (1990). The ellipticity of the boundary surface (the ellipsoid) is also accounted for. This is implied either by precise calculation of the geocentric distance to the centre of the cell in question, residing on the surface of the ellipsoid, or by the use of the exact ellipsoidal-to-spherical harmonic transformation developed by Jekeli (1988).

Chapter 4

Stokes's integration

The short-wavelength component of the geoidal height is given by the Stokes's integral as:

$$N_s = \frac{R}{4\pi\gamma} \iint_{\sigma_0} S_M(\psi)\Delta g d\sigma \ . \tag{4.1}$$

For numerical integration, Eq. (4.1) is expressed in discrete form:

$$N_s = \frac{R}{4\pi\gamma} \sum_i S_M(\psi)_i \Delta g_i \Delta A_i \ , \tag{4.2}$$

where the subscript i refers to the i-th cell of dimension $\Delta\phi$ and $\Delta\lambda$ along meridian and parallel, respectively. The spherical surface area $\Delta A_i = \Delta\phi\Delta\lambda\cos\phi_i$. The values $S_M(\psi)_i$ and Δg_i are determined at the centre of the cell. However, a singularity occurs when the gravity data point coincides with the computation point (i.e. $\psi = 0$). In this case, the contribution of innermost compartment can be evaluated separately, using (see Heiskanen and Moritz, 1967; Strang van Hees, 1990):

$$N_i^s = \sqrt{\frac{R^2 \Delta A_i}{\pi\gamma^2}} \Delta g_i \ . \tag{4.3}$$

The innermost zone can be considered the area which covers the immediate $6' \times 10'$ ($6' \times 20'$ for high latitudes) of the point of interest. An additional consideration arises for the cells close to the computation point. The values of the Stokes's kernel changes very rapidly for small values of ψ. Therefore, its value at the centre of a compartment is not a real value of the whole compartment. Hence, a weighting factor can be applied as (see Strang van Hees, 1990):

$$W(\psi_c) = \frac{\psi_c}{\Delta\psi} \ln\Big(\frac{2\psi_c + \Delta\psi}{2\psi_c - \Delta\psi}\Big) \ , \tag{4.4}$$

where $\Delta\psi$ is the spherical distance across the compartment and ψ_c is the spherical distance to the cenre of the cell.

We have used $6' \times 10'$ mean gravity anomalies over the whole integration cap σ_0. However, the other numerical integration techniques of the Stokes's integral are not investigated. Vaníček et al. (1986) and Vaníček et al. (1990) have treated the Stokes's integration in a different way, dividing the integration cap to the innermost, inner and outer zones. Vaníček et al. (1996a) modified this technique as: the treatment of the innermost zone is changed to consider the mean gravity anomalies instead of point gravity anomalies and a tear reparation, which is caused by discontinuity in the border line between the inner and innermost zone integration, is implemented.

To provide the statistical error properties of the gravimetric geoidal heights, by propagating the estimated errors of the mean terrestrial gravity anomaly data, $\sigma^2_{\Delta g}$, an internal error propagation might be useful. Assuming that the variances of the mean gravity anomaly data are known and uncorrelated, by the law of error propagation through Eq. (4.1), the effect of the terrestrial data error on geoidal height, $\sigma^2_{N_{\Delta g}}$, can be evaluated from

$$\sigma^2_{N_{\Delta g}} = \Big(\frac{R\Delta\phi\Delta\lambda}{4\pi\gamma}\Big)^2 \sum_i (S_M(\psi)\cos\phi)^2_i \sigma^2_{\Delta g}(\phi,\lambda)_i \; . \tag{4.5}$$

Chapter 5

Numerical investigations

The area of study is limited by latitudes 54° N and 70° N and longitudes 10° E and 25° E. This area includes the whole Sweden. We intend to determine the geoidal heights in this area. Priliminary to this computation all corrections mentioned in the Chapter 3 have been evaluated. Since the picture is worth a thousand words, we decided to illustrate the numerical results in a graphical form.

The height coefficients $(H)_{nm}$, $(H^2)_{nm}$ and $(H^3)_{nm}$ are determined from Eqs. (3.7) and (3.8). For this, a $30' \times 30'$ Digital Terrain Model (DTM) is generated using the GETECH $5' \times 5'$ DTM (GETECH, 1995a). This $30' \times 30'$ DTM is averaged using area weighting. Since the interest is in continental elevation coefficients, the heights below sea level are all set to zero. The spherical harmonic coefficients are computed to degree and order 360. The parameter $\mu = G\rho_0$ is computed using $G = 6.673 \times 10^{-11} m^3 \ kg^{-1} \ s^{-2}$ and $\rho_0 = 2670 \ kg/m^3$. R=6371 km and γ = 981 Gal are also used in computations. In integral formulas the $2.5' \times 2.5'$ DTM (GETECH, 1995b) is used. Global EGM96 (Lemoine, et al., 1997) to degree and order 360 is used in computations.

The gravity data over Scandinavia are received from the National Survey and Cadastre of Denmark (KMS). The Bjerhammar's deterministic method (Bjerhammar, 1973) has been used to compute mean free-air gravity anomalies over Scandinavia. As the free-air gravity anomalies are highly correlated with topography, then they are not appropriate for prediction procedure on land. We have used the height information for correcting the height bias. On the other hand, the Bouguer anomaly field is smoother and the least correlated with topography. Over water, the free-air and Bouguer anomalies are assumed to have the same value. Therefore, the Bouguer anomaly on land and free-air anomaly on sea are used for prediction. The final computed mean free-air gravity anomalies in $6' \times 10'$ cells over Scandinavia are shown in Figure 5.1 (see, Nahavandchi, 1998d [Paper J]). It ranges from -125.19 mGal to 193.18 mGal with a mean value of -0.29 mGal and standard deviation (SD) of 25.04 mGal. The original point free-air gravity anomaly on sea and Bouguer anomaly on land ranges from -172.91 mGal to 134.84 mGal with a mean value of -15.63 mGal and SD of 30.05 mGal.

The direct terrain effect on gravity in Stokes's formula is plotted in Figure 5.2. Equation (3.14) is used to compute this effect. It ranges from -35.38 mGal to 59.89 mGal with a mean value of 10.42 mGal. This effect contains both short- and long-wavelength contributions. It is

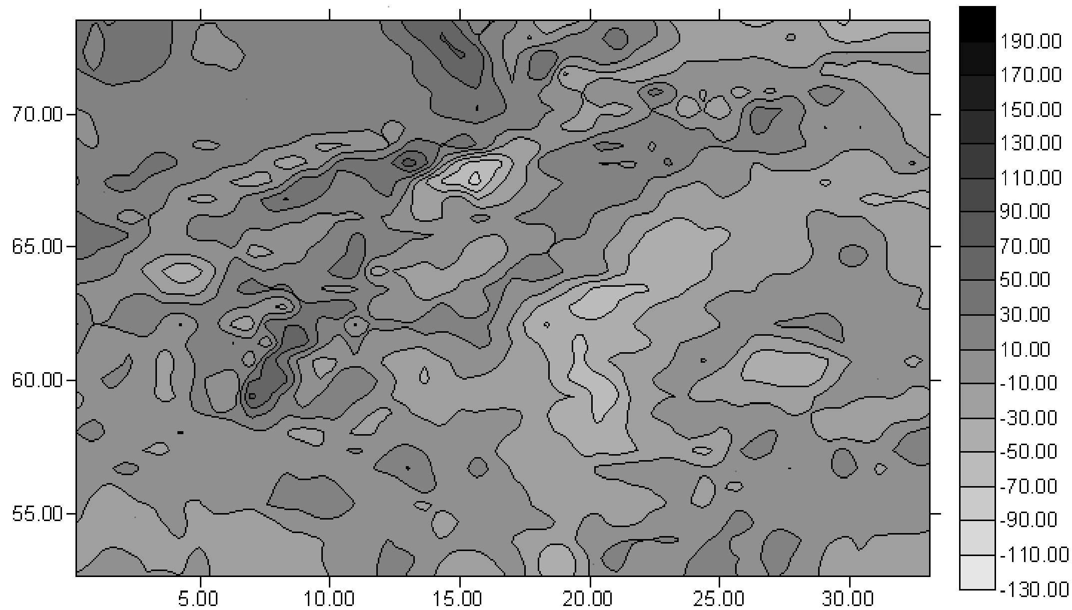

Figure 5.1: The mean free-air gravity anomalies over Scandinavia. Contour interval is 20 mGal.

also correlated with topography. The application of the direct terrain effect to the mean free-air anomalies has reduced the original values of (-125.19 mGal to 193.18 mGal) to (-105.26 mGal to 171.53 mGal), a reduction of 42 mGal of the span.

The direct terrain effect on geopotential (Eq. 3.15), to degree 360, has directly been computed on geoid and illustrated in Figure 5.3. This effect is small compared to the effect in Stokes's integral and always negative. It ranges from -13.88 cm to -6.89 cm with a mean value of -7.92 cm.

Figure 5.4 shows the secondary indirect terrain effect. It is computed from Eq. (3.22) to degree 360. This effect has directly been computed on geoid using a spherical harmonics presentation of the heights. It is 2 orders of magnitude smaller than the direct terrain effect in Stokes's formula. It ranges from -1.7 cm to 3.1 cm with a mean value of 1.1 cm. This effect is relatively small and contributes very little to the final geoid, but it affects systematically and has to be considered when an accurate geoid is desired.

The ellipsoidal correction is computed from Eq. (3.30)-(3.38) using the global EGM96. It is shown in Figure 5.5. It is even smaller than the secondary indirect effect. It ranges from 0.007 mGal to 0.055 mGal. We have also computed its contribution to geoid, which has a maximum value of 0.07 cm.

The application of direct terrain correction to the free-air gravity anomaly makes it a better choice for downward continuation. The downward continuation correction from surface level to the geoid is plotted in Figure 5.6. Equation (3.24) is used for the computation of this effect. It ranges from -42.59 mGal to 81.97 mGal with a mean value of 0.13 mGal. Figure 5.6 shows that this contribution has very short-wavelength character and has mostly large values. It has to be mentioned that this illustration is not appropriate to show this effect. The reason is its very

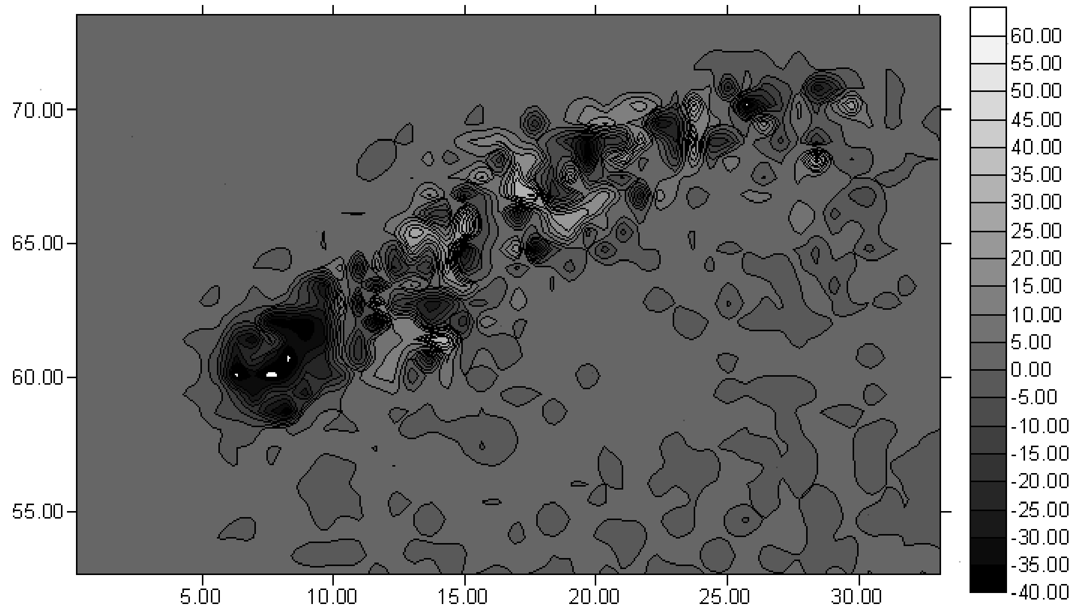

Figure 5.2: The direct terrain effect. Contour interval is 5 mGal.

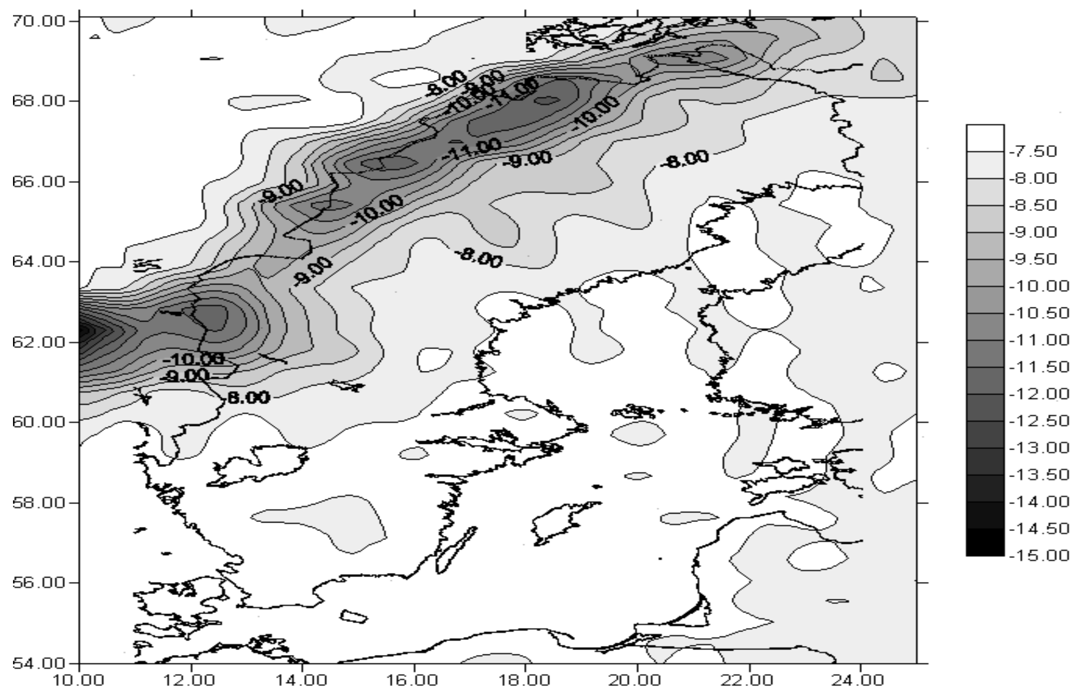

Figure 5.3: The direct terrain effect on geopotential. Contour interval is 0.5 cm.

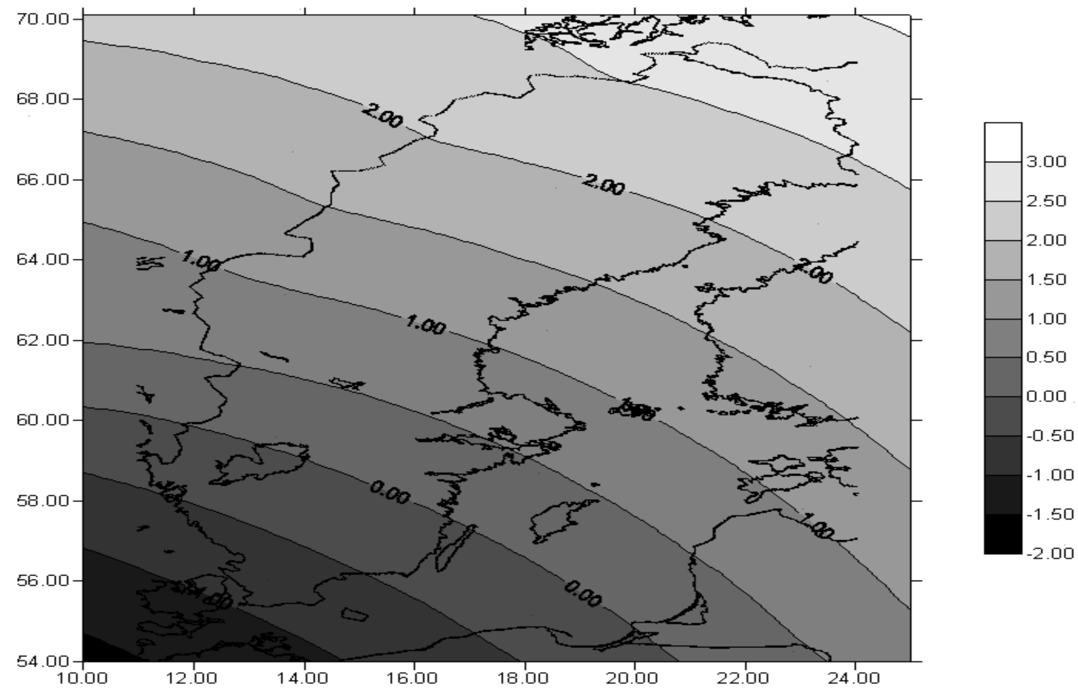

Figure 5.4: The secondary direct terrain effect on the geoid. Contour interval is 0.5 cm.

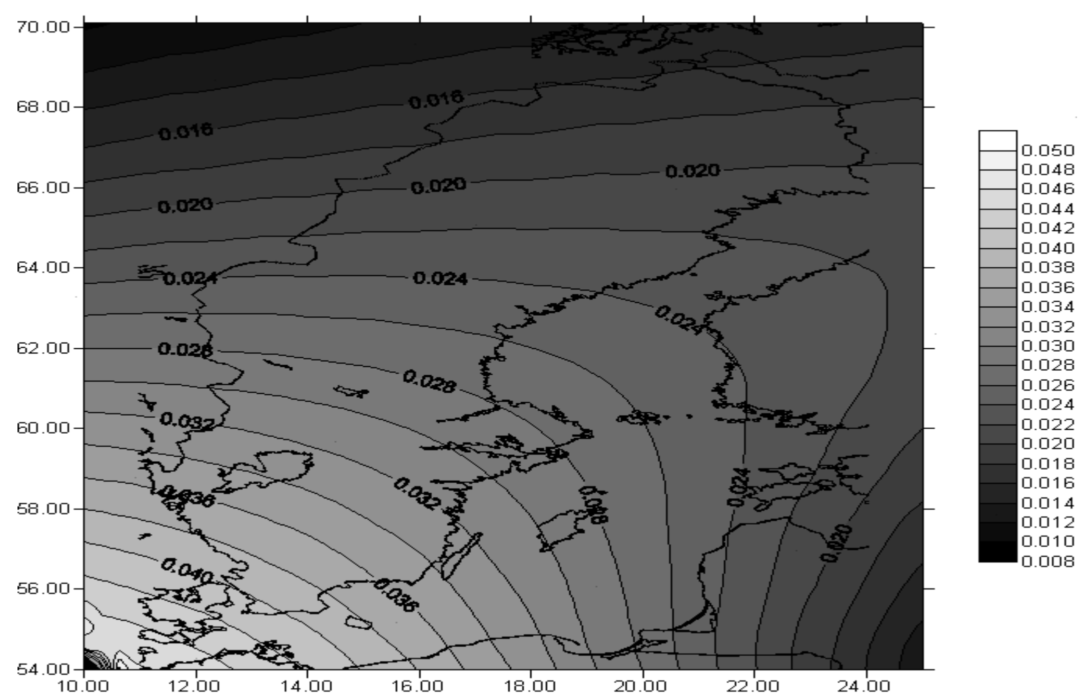

Figure 5.5: The ellipsoidal correction. Contour interval is 0.002 mGal.

high frequency nature. This effect on the geoid gives always positive values.

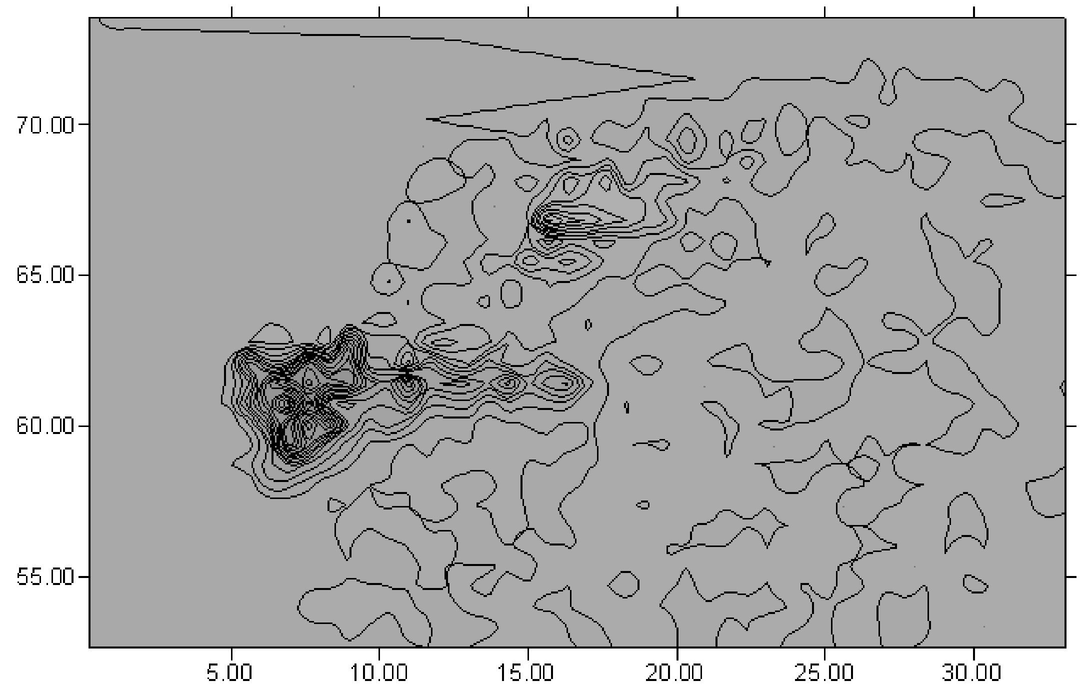

Figure 5.6: The difference between gravity anomalies on the Earth's surface and on the geoid. Contour interval is 6 mGal.

The primary indirect terrain effect, computed from Eqs. (3.20) and (3.21), is illustrated in Figure 5.7. It ranges from -11.26 cm to 13.46 cm with a mean value of 4.81 cm. It contributes from both short- and long-wavelengths to the geoid.

The total atmospheric effect on the geoid, including the direct and indirect atmospheric effects, is depicted in Figure 5.8. Equation (3.23) is used for the computation of this effect. However, the zero- and first-degree harmonic contributions to the total atmospheric geoid effect are not included in the solution. It ranges from -0.05 cm to 1.56 cm with a mean value of 0.95 cm. The total atmospheric effect has been evaluated on the geoid and can directly be added to the final geoid.

Figure 5.9 shows the correction due to the truncation error, for an integration cap 6°, computed from Eq. (3.29) using the global EGM96. Its effect ranges from 3.66 cm to 7.30 cm with a mean value of 5.38 cm. It has a quite significant contribution and has to be added to the final geoid, determined from a combination of terrestrial and satellite gravity data. This correction is already considered in the refined least squares model and, therefore, is not applied to the geoid estimated by this model.

Nahavandchi (1998e, f) [Papers G and F] showed the power of refined least squares estimator over the other gravimetric geoid models. The mean and standard deviation of the differences between geoid derived from the refined least squares estimator and 23 SWEPOS GPS stations are found to be 10.1 cm and 5.5 cm, respectively. They are computed to 13.1 cm and 7.1 cm for the Molodenskii et al. model, 18.2 cm and 11.2 cm for Wong and Gore estimator and 23.1 cm

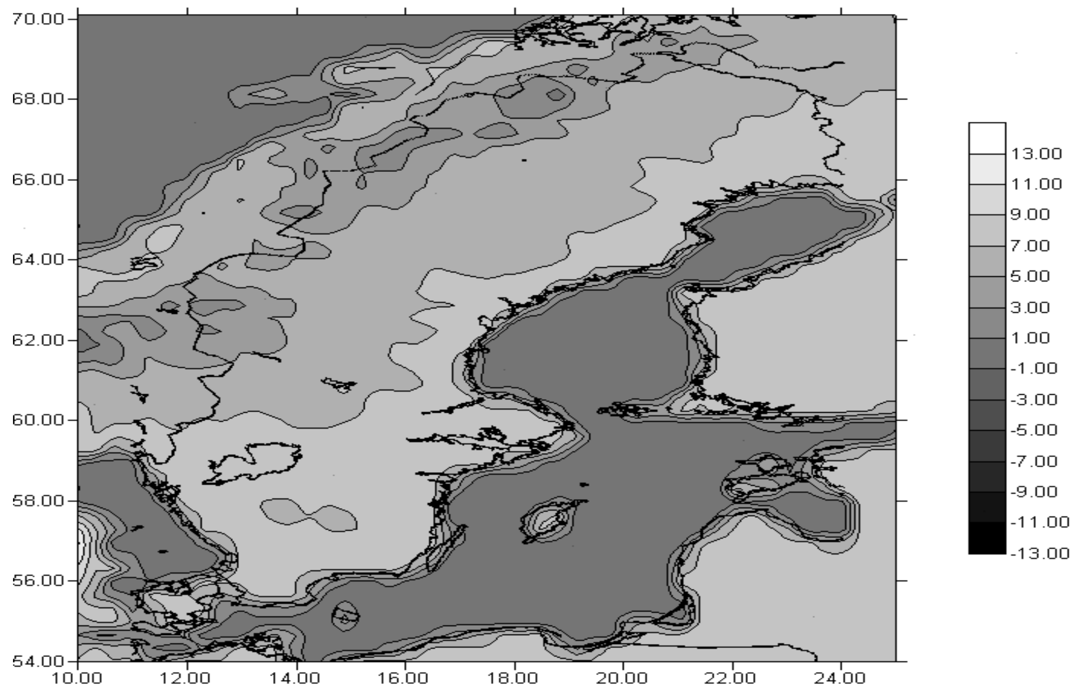

Figure 5.7: The primary indirect terrain effect. Contour interval is 2 cm.

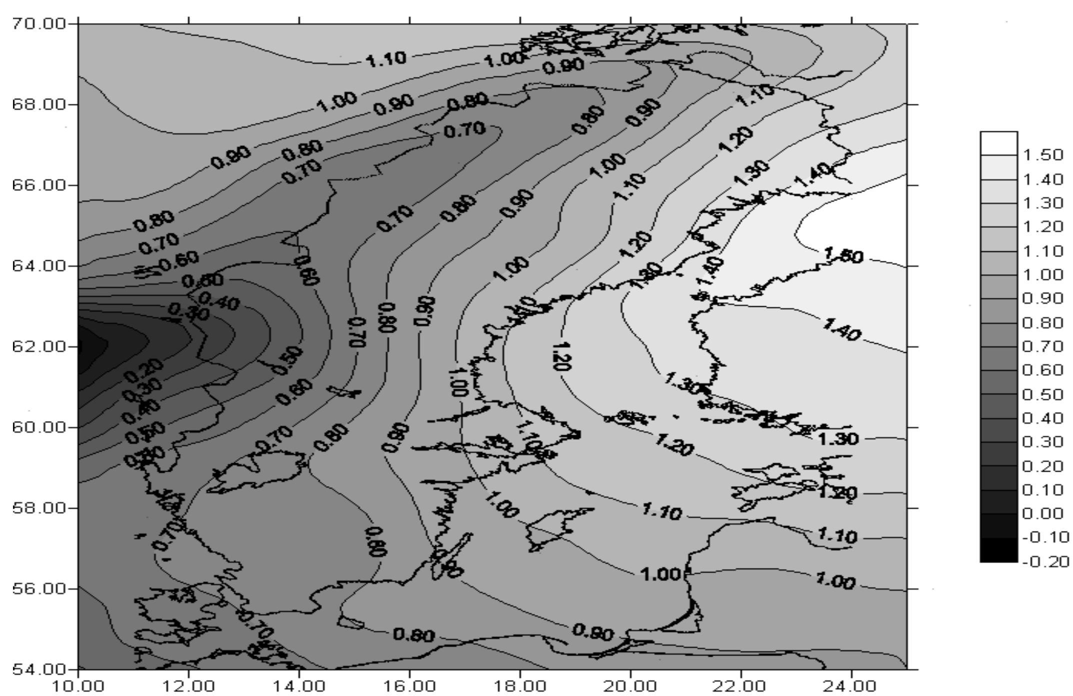

Figure 5.8: The total atmospheric effect on the geoid. Contour interval is 0.1 cm.

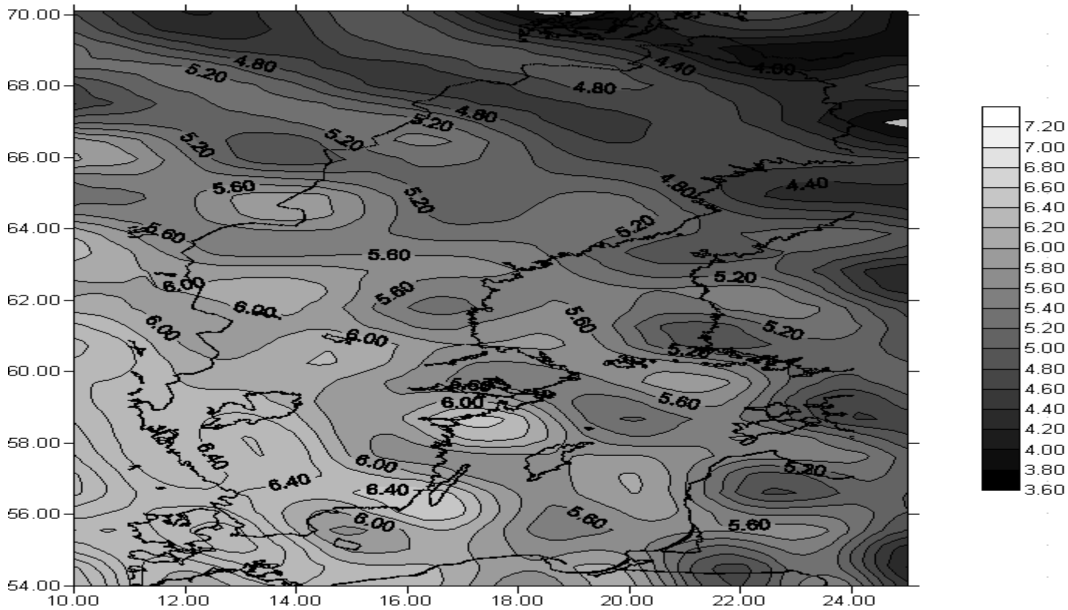

Figure 5.9: The correction due to the truncation of Stokes's integral to a spherical cap with radius $\psi_0 = 6^\circ$. Contour interval is 0.2 cm.

and 16.1 cm for Vincent and Marsh model (see Nahavandchi, 1998f). Therefore, this estimator is used for the final gravimetric geoid determination. Nahavandchi (1998e) also used EGM96 to compare different geoid estimator changing the maximum degree of expansion (M) of the reference field. This analysis resulted the best solution with large values of M. To get further insight into this analysis, we use the observed geoid values at 23 SWEPOS GPS stations to compare the gravimetric geoidal heights determined with different values of M (=20, 180, 360). The refined least squares model is used to estimate the gravimetric geoid. All corrections are applied in this investigation. $N = M + 40$ is used in this model. Table 5.1 shows the statistics of difference between gravimetric and GPS-levelling geoid. The results of Table 5.1 justify our

Table 5.1: The statistics of differences between gravimetric and GPS-levelling geoid with different values of M. Units in m.

	M=360	M=180	M=20
Min	0.044	0.081	0.095
Max	0.260	0.281	0.361
Mean	0.101	0.122	0.181
SD	0.055	0.071	0.119

belief that a reference field of an order as high as possible, in this study, gives the best results at GPS stations. Standard deviation and mean values for M=360 is computed to 5.5 cm and 10.1 cm, respectively, while it is found to be 11.9 cm and 18.1 cm for M=20.

Some studies propose for using a reference field constructed using only satellite orbit analysis (see e.g. Vaníček and Kleusberg, 1987 and Vaníček et al. 1996a). They argue that a reference field with higher degree and order than 20 by 20 is constructed using the same terrestrial gravity data that one used in geoid determination referred to this reference surface. Also, the other disadvantage of a combined field is its spatial inhomogeneity. However, some other researches have been carried out using a combined reference field of an order as high as possible (see e.g Forsberg, 1990; Forsberg, et al., 1996; Fan, 1993; Zhao, 1989: Despotakis, 1987 etc.) The use of a satellite only model versus a combined model (to degree and order e.g. 20 by 20) is not here investigated. Finally, M=360 and N=400 is used in the refined least squares estimator. Figure 5.10 shows the plot of the geoidal heights determined with this model. It has been computed in $6' \times 10'$ grids. The integration cap is selected equal to $6°$ (see also Nahavandchi, 1998f). The Geodetic Reference System (GRS80) normal field and its reference ellipsoid is used in this study. Then, the final gravimetric geoid is referred to the GRS80 ellipsoid. In this plot, all corrections have been carried out, resulting in the final geoid. It ranges from 17.22 metres to 43.62 metres with a mean value of 29.01 metres.

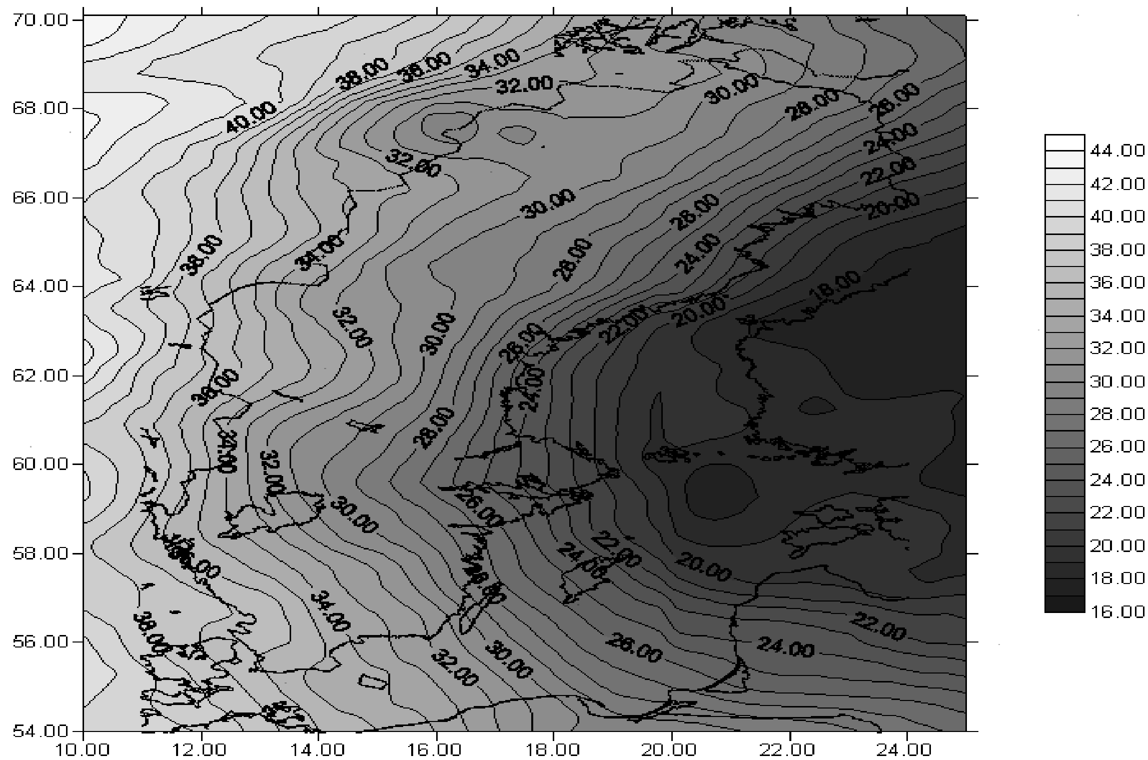

Figure 5.10: The final gravimetric geoid determined by the refined least squares estimator. Contour interval is 1 m.

We have also computed the standard deviations of this solution with a simple error propagation of standard deviations of mean anomalies (see Eq. 4.5). The statistics of these geoid undulation errors are given in Table 5.2. These standard deviations are highly correlated specially in short distances. It ranges from 7.02 cm to 13.05 cm with a root mean square value of

Table 5.2: The statistics of geoidal height errors through propagating the estimated errors of mean gravity anomalies. Units in cm.

Min	Max	Mean	SD	rms
7.02	13.05	9.12	0.90	9.17

9.17 cm. This represents the accuracy estimate of the gravimetric geoid obtained from the internal error propagation. It has to be mentioned that the errors in all the employed corrections are considered much smaller than the error in mean free-air gravity anomalies and neglected in this computation. Also, the errors due to the low frequency reference field are not applied here. This method to estimate the geoidal height errors may not be accurate but it, however, represents valuable information on the expected relative accuracies. It also indicates locations where the gravity anomalies have poor quality and quantity.

The accuracy of the geoidal height estimations can also be investigated in form of the global mean square error of the estimators. The global mean square error of the refined least-squares model is derived as (Sjöberg, 1991a):

$$\delta\bar{N}^2 = \left(\frac{R}{2\gamma}\right)^2 \sum_{n=2}^{n_{max}} \left[\left(\frac{2}{n-1} - s_n^* - Q_{Nn}\right)^2 \sigma_n^2 + (Q_{Nn} + s_n^*)^2 dc_n^*\right] , \tag{5.1}$$

where

$$s_n^* = \begin{cases} s_n' & 2 \le n \le N \\ 0 & n > N, \end{cases} \tag{5.2}$$

For n_{max}=1000, maximum degree of modification N=400, maximum degree of expansion M=360 and truncation radius of $\psi_0 = 6°$, the global root mean square error is computed to 25.3 cm. This result includes the erorrs from truncation, erroneous terrestrial gravity data and potential coefficients. It is comparable with the estimated standard deviation in Table 5.2, which only considers the errors due to the terrestrial gravity data.

However, it has to be noted that the most reliable way to estimate the accuracy of the gravimetric geoid is the comparison of its result with externally derived geoidal height data such GPS-levelling derived geoidal heights. But, it only reflects the geoid estimates in approximately flat areas. This is investigated in Nahavandchi (1998f).

The geoidal undulations are also computed by the other three estimators, Molodenskii et al., Wong and Gore and Vincent and Marsh models, respectively. The statistics of differences between these three estimators and the refined least squares model are shown in Table 5.3. It shows the best agreement of the least squares estimator with Molodenskii et al. model. The mean of differences is computed to 2.41 cm with a standard deviation of 1.40 cm. It should also be mentioned that all corrections are already applied to these estimators.

To investigate how each of the direct and indirect effects as well as atmospheric correction improve the agreement gravimetric geoid to the GPS points, a specific analysis with each of these corrections are carried out. Table 5.4 shows the statistics of the differences between the

Table 5.3: The statistics of differences between 3 geoid models with the refined least squares estimator. Units in cm.

	Molodenskii et al.	Wong and Gore	Vincent and Marsh
Min	-2.21	-4.13	-10.45
Max	6.12	10.41	20.15
Mean	2.41	4.08	15.11
SD	1.40	2.32	9.59

gravimetric and GPS geoid at 23 GPS stations, including and excluding all corrections. The results of Table 5.4 show an improvement of 18.6 cm in mean value of differences and the standard deviation improves about 50% from 10.5 cm to 5.5 cm.

Table 5.4: The statistics of differences between gravimetric and GPS geoid, with and without the corrections, in metre.

	without corrections	with all corrections
Min	0.201	0.044
Max	0.402	0.260
Mean	0.287	0.101
SD	0.105	0.055

Thereafter, we make specific investigations (compare with GPS results) for the gravimetric geoid on each of the direct and indirect as well as atmospheric effects. Table 5.5 shows the results of this investigation. It indicates that the direct terrain correction of gravity anomalies reduces the differences between the gravimetric and GPS geoid, significantly. An improvment of about 17.5 cm is computed for the mean value of differences. The indirect terrain correction and atmospheric effect also improve the gravimetric geoid on GPS stations. An improvement of 6.0 cm and 0.8 cm is found for the mean value of differences, respectively.

The results of Table 5.5 also show that the standard deviation of differences decreases from 0.105 m to 0.071 m, 0.073 m and 0.098 m after applying the corrections, respectively.

We have also computed a gravimetric-GPS geoid from the combined least squares adjustment of the gravimetric geoid to the GPS-levelling stations (see Eq. 2.29). 9 SWEPOS GPS stations, used in this investigation, are located in the central part of the Sweden. The gravimetric geoid is the refined least squares model, which is in $6' \times 10'$ cells. To see how well the combined geoid fits the GPS points, 3 of the GPS stations are excluded from the computations. Finally, the combined gravimetric-GPS geoid model is compared on these three stations. We have also compared the gravimetric (only) geoid (refined least squares) on these 3 GPS points. For a geoid grid of size $N' \times M'$ and L' GPS stations, there will be altogether $2 \times (N'-1) \times (M'-1) + 4 \times L' + L'$ observation equations.

Table 5.5: The statistics of differences between gravimetric and GPS geoid on direct, indirect and atmospheric corrections in metre.

	Direct effect + Downward continuation correction	Indirect effect	atmospheric effect
Min	0.021	0.173	0.195
Max	0.285	0.350	0.390
Mean	0.112	0.227	0.279
SD	0.071	0.073	0.098

Table 5.6 shows the local transformation parameters. It shows that the local deformation exists and can be determined by the combined adjustment. But, as the number of GPS stations (6 in this investigation) is not enough, the errors in a^1, a^2 coefficients are greater than the values itself. The more GPS stations, the better results are expected (see, also Jiang and Duquenne).

Table 5.6: Local transformation parameters of constraining gravimetric geoid to GPS stations. Units in metre.

a^1	b^1	a^2	b^2
-0.035±0.052	-0.0715±0.052	-0.0429±0.058	-0.051± 0.031

The results of comparison of gravimetric-GPS and gravimetric (only) solutions at 3 GPS-levelling stations are shown in Table 5.7. We have considered the observed GPS values at GPS-levelling stations as the reference for the comparison. The results of Table 5.7 show that the gravimetric-GPS geoid agrees better at 3 GPS stations compared to the gravimetric (only) solution. An improvment of 3.2 cm (in mean value) is resulted. The standard deviation also decreases from 3.1 cm to 0.3 cm. Hence, it can be concluded that providing more GPS stations over the whole area of computation, the gravimetric geoid might be improved more significantly.

Table 5.7: The statistics of differences of the gravimetric (only) and gravimetric-GPS geoid at 3 GPS sites in metre.

	Gravimetric-GPS geoid	Gravimetric geoid
Min	0.128	0.131
Max	0.133	0.193
Mean	0.131	0.163
SD	0.003	0.031

As a by-product of the combination of gravity and GPS data, gravimetric and GPS geoid models are used to unify the Swedish and the Finnish height systems. This unification has already been carried out using NKG89 (Forsberg, 1990) and NKG96 (Forsberg et al., 1996) geoid models (see, Pan and Sjöberg, 1998 and Nahavandchi and Sjöberg, 1998a [Paper K]). Here, the same procedure (see, Eqs. 2.30 and 2.31) is implemented, but using the geoid computed by the refined least squares estimator in this study. The difference between the Swedish and the Finnish height systems is then computed to -16.1 $\pm$2.3 cm. This difference was computed to -19.3$\pm$6.5 cm with NKG89 geoid model (Pan and Sjöberg, 1998) and -12.1$\pm$2.7 cm with NKG96 geoid model (Nahavandchi and Sjöberg, 1998a [Paper K]). The difference between the Swedish and the Finnish height systems was computed, directly, by gravity and levelling observations to -19.2 cm (Sjöberg, 1991b) and 16.2 cm (Ekman, 1992).

Chapter 6

Discussions and recommendations

The geoid has been served as an important tool by geo-related scientists for many years. For example, one of the main concerns of geophysics has been the study of the inaccessible interior of the Earth. This can be done only through physical data which one of them is the geoid. The geoid has served as the reference surface for orthometric heights which is very important for geodetic and engineering operations. With the advent of satellite techniques, it has been possible to determine the geoid directly from satellite observation combining with the levelling data. On the other hand, since satellite observations, say GPS, furnishes the ellipsoidal heights (which are not directly usable for the surveying and engineering operations), then the reduction of these quantities to the more usable orthometric heights needs the information of an accurate geoid. Understanding this important role of the geoid, determination of an accurate geoid has been in the centre of the discussions by many geodesists during the last decades. Therefore, the author undertook the task of determination of an accurate geoid over Sweden. To do this, we had to reformulate some theoretical parts of the existing theories for geoid determination and also to use the new ideas and developments in the recent years.

Part II of this thesis consists some papers which have already been submitted, accepted and published in the refereed journals. As numerical investigations in theses papers are limited to some test areas within Sweden, therfore, all theoretical aspects are reviewed for geoid determination over Sweden, started with Chapter 1 as introduction and ended with Chapter 6 as discussions and recommedations in the Part I.

Chapter 2 is devoted to the original Stokes's theory and reformulation of it for higher than second-degree reference feield and modifications of Stokes's kernel. This theory combines the long-wavelength geoid, computed from a set of potential coefficients, with the short-wavelength geoid, obtained by the modified Stokes's integration in a limited spherical cap, yielding the final geoid. Various authors have discussed different methods to minimize the truncation error and different ways to compensate the lack of the terrestrial gravity data. In this chapter, four estimators are selected to determine the geoid. Vincent and Marsh (1974) estimator employs the high degree reference gravity field with the low degree satellite derived gravity anomalies, subtracted from the terrestrail gravity anomalies. No kernel modification is used in this model. Wong and Gore (1969) and Molodenskii's et al. (1962) estimators use the Stokes's kernel modification and high degree reference field. The refined least squares model, proposed by Sjöberg

(1991a), combines the terrestrial and satellite derived gravity anomalies in a least-squares sense to minimize the global mean square error of the truncation error and the global mean square error of the potential coefficients as well as terrestrial gravity anomalies

In Chapter 3, we have reviewed corrections needed to be applied to the gravimetrically computed geoid in combination with geopotential coefficients. Pertaining to the question of the geoid determination is also the question of direct and indirect terrain corrections. Sections 3.1-3.3 devoted to these effects. Classical integral formulas for the terrain corrections suffer from the planar approximations and some long-wavelength contributions are also missing. Spherical harmonic presentation of terrain correction is simple but practically needs to be expanded to a very high degree to consider short-wavelength information. Therefore, we have reformulated whole story to compromise between these two ideas considering both short- and long-wavelength contributions. The direct effect on geopotential is considered, separately, using spherical harmonic presentation, knowing the smallness of this effect compared with terrestrial data. Terrain corrections are very significant and have to be considered in both geopotential model and Stokes's formula.

The atmospheric geoid effect is presented in Section 3.4. It has been derived as the total terrain correction, including direct and indirect atmospheric effects. It has been presented by spherical harmonics as a correction to the modified Stokes's formula. Its effect is significant for accurate geoid determination and has always to be considered.

The downward continuation of gravity anomalies is the other correction which has been tackled in Section 3.5. It is applied to the terrain corrected gravity anomalies at the surface level to be reduced to the geoid. As the terrain correction is related to the points on the surface level in this study, this effect has been considered here. On computing this effect, the Poisson's kernel is modified. The correction due to the truncation error (due to the limiting of integration area) is computed using global EGM96 and added to the final solution. The low frequency contributions, computed from EGM96, are subtracted in the Poisson's kernel and downward continued, separately. It has then been added to the downward continued short-wavelength part coming from the iterative process. The results show that this effect is very important agumented by the height difference between the surface level and the geoid. The effect of downward continuation on geoid is everywhere positive.

Truncating the Stokes's integral to a spherical cap of specific radius causes the truncation error which has not been neglected in our study. It has been discussed in the Section 3.6. The global EGM96 is used to evaluate the truncation error. Its effect ranges to a few centimetres, which is significant in precise geoid determination. This effect is already considered in the refined least squares estimator.

Section 3.7 is devoted to the ellipsoidal correction. This effect which emanates from the neglecting of the terms of the order e^2, is very small and its contribution to the resulting geoid is within a few millimetres.

Chapter 4 reviews the numerical integration technique, applied to the Stokes's integral, in this study. The standard deviations associated with the short-wavelength contribution of geoidal heights, emanating from the estimated errors of mean terrestrial gravity data, are evaluated in this chapter.

Chapter 5 is devoted to the numerical investigations. The area of computation includes the whole of Sweden. Four different gravimetric geoid estimators, mentioned before, have been selected, and the results are compared with each other and on SWEPOS GPS stations. The results of the refined least squares estimator agree better with the GPS-levelling derived geoid than the other methods. It has the smallest differences in mean value (10.1 cm) and standard deviation (5.5 cm). The global root mean square error of this estimator is computed to 25.3 cm.

A least-squares procedure is used to constrain the gravimetric geoid to GPS-levelling stations. In this case, the long-wavelength errors can be carefully approximated by a global transformation model. A local transformation model is also used to eliminate the shortage and errors of gravity and DTM data as well as the bias due to the differences between data sources. The results of the gravimetric-GPS geoid shows an improvement of 3.2 cm compared to the gravimetric (only) geoid. This improvement is examined at the observed geoid values at 3 GPS stations.

The GPS and gravimetric geoid models are also used to unify the Swedish and the Finnish height systems, indirectly. The results are comparable with the previous computations with NKG89 and NKG96 models as well as with direct gravity and levelling observations. The standard error is also somewhat smaller.

Some recommendations for future work are in order:

a) If a precise geoid, say centimetre, is desired, the topographical density correction, discussed in Martinec (1993) must be considered in the final computations, especially in mountainous area.

b) A denser DTM, especially in mountainous area in the border of Norway and Sweden, is needed to evaluate terrain corrections more precisely. According to our experience, the use of $2.5' \times 2.5'$ DTM in this study may cause several centimetres errors in the direct terrain effects and thus in the resulting geoid.

c) An investigation of different methods to determine the direct terrain correction is also suggested. See Eqs. (3.1) and (3.3).

d) The atmospheric geoid effects in the Stokes's formula is determined from a spherical harmonic presentation of the heights over the world. The author believes that, considering the significant effect of this correction to the final geoid, especially in mountainous area, same procedure as the topographic crrection, i.e. a compromise between short- and long-wavelength contributions has to be carried out.

e) As it is shown, different Stokes's kernel modifications result in different geoidal heights with a maximum difference of about 20 cm in this study. Therefore, the author believes that Stokes's kernel modification techniques have to be investigated. For example, the models for the the error anomaly degree variances of the terrestrial gravity anomalies and the potential coefficients, especially the first one, has still to be studied in the refined least squares geoid estimator. The use of a satellite only model versus a combined model for the reference field has still a room for investigation.

f) Increasing GPS-levelling stations will improve the combined gravimeric-GPS geoidal height computation, compared to the gravimetric (only) solution. The combined model eliminates both long-wavelength error and local deformations (e.g. due to the shortage of gravity or DTM data). A more careful study is needed for the large number of observation equations.

g) In the internal error estimation of the final geoid, a simple error propagation of standard deviations of mean anomalies is used and the uncertainty in the reference field and the errors in all the employed corrections are disregarded. However, the errors in the applied corrections may require further investigations. Also, the contribution of the low frequency part of the geoid model has to be included.

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Part II

Papers

A. Terrain corrections to power H^3 in gravimetric geoid determination (Published in Journal of Geodesy 72:124-135)

B. Terrain correction computations by spherical harmonics and integral formulas (Presented in 21th General Assembly of the European Geophysical Society. Accepted for publication in Journal of Physics and Chemistry of the Earth)

C. The direct terrain correction in gravimetric geoid determination by the Stokes-Helmert method (Submitted to Journal of Geodesy)

D. On the indirect effect in the Stokes-Helmert method of geoid determination (Submitted to Journal of Geodesy)

E. Geoid terrain correction to power H^5 of satellite derived geopotential models. (Accepted for the publication in Bollettino di Geodesia e Scienze Affini)

F. Precise geoid determination on SWEPOS GPS stations: comparison of some estimators to modify Stokes's formula. (Submitted to Journal of Geodesy)

G. A comparison of some models of gravimetric geoid determination by the modification of Stokes's formula (Submitted to Journal of Geodesy)

H. On some methods of downward continuation of mean free-air gravity anomaly (Submitted to Journal of Geodesy)

I. The atmospheric geoid effects in Stokes's formula (Submitted to Geophysical Journal International)

J. Computation of mean free-air gravity anomalies by Bjerhammar's deterministic and collocation methods over Scandinavia (Accepted for publication in Survey Review)

K. Unification of vertical datums by GPS and gravimetric geoid models using modified Stokes formula (Published in Journal of Marine Geodesy. Vol. 21, No. 4, 1998)