Summary

1 Introduction

In the past few decades, a new branch of structural mechanics — Computational Structural Mechanics — has emerged. The mission of conventional structural mechanics was mainly to establish mathematical descriptions for problems in structural mechanics. In a conventional way, the behavior of a deformable solid is described by differential equations, which are established by equilibrium of a typical infinitesimal from the solid. To know the behavior of a specific structure, with certain loads and boundary conditions, the obtained differential equations have to be solved. But properties of the equations are quite diverse, depending on the studied structure. In the most complicated case, there might be a set of coupled partial differential equations with non-constant coefficients. Only in very simple cases, the equations can be solved in a conventional analytical way. Even in mathematics, there is no a systematic way to analytically deal with such equations.

The development of the digital computer has provided a powerful tool and led to substantive changes in computational structural mechanics. It is since the occurrence of digital computer that various numerical techniques were rapidly developed. Among numerical techniques, the finite element method is the most mature. In the development of structural finite element methods, matrix algebra, Rayleigh-Ritz approximation and other techniques are very important stages, but the key step is the establishment of variational principles, which form the very basis of finite element methods. As an alternative way, most of the differential equations, which were originally derived from infinitesimal equilibrium in conventional structural mechanics, can be easily obtained from a properly formulated variational principle. But some generalized equilibrium equations are difficult, if not impossible, to derive by the conventional way. The establishment of various variational principles paved the way for structural finite element methods to develop rapidly and to form a theoretical system.

Today, structural finite element methods are widely applied in various branches of engineering such as aeronautic and astronautic, civil and mechanical engineering. But structural finite element method is not so perfected yet. Not to mention complicated problems concerned with geometrical [7-14,20] or material [2,3,23,28,29] nonlinearities and simulation of (quasi-)dynamic problems [3,4,25,29], even in the basic region with assumptions of static, elastic and small linear deformation, there are still some problems, e. g. various lockings, spurious zero energy modes (or rank deficiency), element coarse-mesh deficiency, cf. [19,22,27,30,32-34,36,37] among many others. Several questions can be asked, for instance about locking. Shear locking is a problem concerned with Mindlin type of beam, plate and shell elements in the region of thin structures. Membrane locking usually strikes in curved geometry, while volumetric locking is related to incompressibility of material. Indeed, we have some solutions to these problems. The question is: although these lockings are
quite different in phenomenon, is there any intrinsic connection between them? If there is, do we have a reasonable explanation for their origins? Do we have a unified remedy for them? The second area is related to isoparametric finite elements. An obvious advantage of isoparametric elements is the simplicity in selecting interpolation functions. The introduction of isoparametric description is believed a break-through in structural finite element method for providing a systematic and simple way in constructing element interpolations. The central idea of isoparametric finite element method is to use shape functions, which describe the geometric shape of the studied structure (element), to interpolate displacements. An implicit assumption in using isoparametric shape interpolations is that the deformation (or displacements) of a load-bearing structure is mainly, if not solely, determined by geometrical factors, the effects of other factors being neglected without giving any reason. As we know, the deformation of a load-bearing structure is related to geometry, material, loads and boundary conditions of the structure. It might be unreasonable to only take the geometrical factors into account while ignoring the others in constructing element interpolations. Some troubles might be introduced while adopting the strategy of isoparametric finite element method in constructing displacement interpolations. Third, about different elements in structural mechanics. We have many finite elements with different bases: displacement-based elements with full or various reduced integrations, incompatible displacement model, hybrid stress elements, assumed strain elements, elements based on mixed interpolation of tensorial components (MITC). Some of them are easy to understand and accept. Some of them are believed to be just numerical tricks. Beginners may be completely confused by the disorder. Although a lot of work has been done to straighten out the relations between the different methods or elements, the relationship diagram is still in pieces and incomplete. In a word, some further work need be done, even in the very basic region of structural finite element methods.

There are two tasks for this thesis. The first one is to find an unified way for explaining and eliminating various locking phenomena. If this can be done, the found way should also be useful in developing efficient finite elements. The other task is to obtain a more overall picture about relations between different finite elements, by a close investigation. To this end, some existing formulations need be extended.

2 Layout of this thesis

This thesis consists of this summary and the following nine papers


1All the papers have been published or submitted with the Ph.D. candidate's name put in the Chinese convention, that is, the family name, Luo, is placed first and followed by the given name, Yanhua, without a comma. From this thesis, the English convention is followed, but be aware that Yanhua are two Chinese characters and abbreviated as Y. H. or Y.-H.


The papers can be naturally grouped under three topics, which constitute the three parts of this thesis. The first part, including Papers A, B and C, focuses on the theme *Field Consistence Approach*. In Paper A and in Reference [19], the field consistence approach, a new way for constructing element interpolation functions, is put forward and applied to improve beam elements. In the context of 2-node 2-D beam elements, shear locking and membrane locking are unified explained with respect to their origins and completely eliminated by applying the field consistence approach. In Paper B, the approach is extended into developing 2-D elements by the aid of two methods for finding quasi-general solutions to a set of partial differential equations. The conventional 4-node plane stress element, based on the Hellinger-Reissner variational principle, is improved by applying the assumed stresses constructed from the field consistence approach. Relations between different variational principles are tentatively explored. In Paper C, based on the field consistence approach and starting from the Hu-Washizu variational principle, an alternative assumed strain method is formulated and applied in developing efficient Mindlin plate elements.

In the second part of this thesis, which is composed of Papers D, E and F, several existing finite element techniques or formulations are extended. The 'optimal' reduced integration scheme in [26] and the MITC approach in [1], which were originally developed aiming at eliminating shear locking in the 4-node displacement-based Mindlin plate element, are generalized, in Papers D and E, to improve the performance of the 8-node displacement-based solid element. In Paper F, the conventional mixed
field variational principles, the Hellinger-Reissner and the Hu-Washizu principles, are hybridized with the principle of minimum potential energy. Two hybrid mixed field principles, the hybrid Hellinger-Reissner principle and the hybrid Hu-Washizu principle, are thereby obtained. The two hybrid principles have potential application in developing highly efficient elements. Furthermore, they provide a variational basis for some finite element techniques such as reduced integration, which were not developed directly from a variational principle.

In the third part, consisting of Papers G, H and I, the relations between different finite element techniques or formulations are closely investigated by the aid of a symbolic computational software, Maple [39]. The relations between different variational principles and the relations between uniform/selective reduced integrations and variational formulations are studied in Paper G with a 2-node Timoshenko beam element and a 4-node plane stress element. The relations between the MITC (Mixed Interpolation of Tensorial Components) approach, the 'optimal' reduced integration and the hybrid mixed formulations are explored in Paper H, with a 4-node Mindlin plate element and a 8-node solid element. Finally, in Paper I, the incompatible displacement model is linked to the relation diagram, based on an investigation around a 4-node plane stress element.

3 (Relaxed) Field Consistence Approach

The central idea of the field consistence approach is to construct element interpolation functions from (quasi-)general solutions to the Euler-Lagrangian equations corresponding to the adopted variational principle. All the factors which might contribute to the structural (element) deformation can be considered in the element interpolations. The establishment of finite element equations with this idea applied can be conceptually depicted with the diagram in Fig. 1, which can be more precisely described in words.

![Diagram of Field Consistence Approach](image-url)

**Figure 1:** Field consistence approach in establishing FE equations
(A) Having selected a variational principle, which could be the principle of minimum potential energy, the Hellinger-Reissner principle or the Hu-Washizu principle, or even the hybrid mixed field principles set up in Paper F, the expression of (generalized) potential energy in an element can be written;

(B) From the (generalized) potential energy, a set of Euler-Lagrangian equations is derived. Depending on the adopted structural theory, geometrical assumptions and the selected variational principle, the Euler-Lagrangian equations might be ordinary or partial. The most complicated situation is when the Euler-Lagrangian is a set of coupled partial differential equations with non-constant coefficients.

(C) These equations are solved. Only in very simple cases, e.g. the 2-node Timoshenko beam element in Paper A, the equations can be explicitly solved and real general solutions can be obtained. In most cases, such as the 2-D plane stress problem in Paper B, quasi-general solutions, which are essentially approximate analytical solutions, to the Euler-Lagrangian equations are obtained by using the undetermined coefficient or the semi-explicit method.

(D) Element interpolation functions are constructed from (quasi-) general solutions to the Euler-Lagrangian equations by the aid of element 'boundary conditions'.

(E) The element interpolations are used to derive the discretized form of the (generalized) potential energy. Element matrices take their analytical forms in this step;

(F) By applying variational operation, the standard finite element equation, characterized by element stiffness matrix, nodal displacement vector and nodal force vector in static structural mechanics, is established.

The procedure including steps (B), (C) and (D), enclosed in the dashed box in the figure, is the core of the field consistence approach. Element information such as material, geometry, loads and coupling relations between field functions is inherited by the Euler-Lagrangian equations. The field consistence approach inputs the above information at (B) and outputs element interpolations at (D). In constructing element interpolation functions, the field consistence approach provides a more rational way than a conventional one does. Several classes of conventional interpolations are listed in the dotted box at the upper-right corner in Fig. 1. In conventional ways, less or even no element information is taken into account in selecting element interpolations, which might be the source of some deficiencies such as shear locking and membrane locking.

In applying the field consistence approach, if the process in Fig. 1 is started from the principle of minimum potential energy, the Euler-Lagrangian equations are equilibrium relations expressed with displacements. There is less flexibility in dealing with these equations. But if the procedure begins with a mixed field principle, the Euler-Lagrangian equations are of different form. Beside the equilibrium equations, which are now expressed with stresses or internal forces rather than displacements, there are also other equations, e.g. stress-strain relations and/or strain-displacement
relations, depending on the used mixed field variational principle, [19]. The equilibrium equations expressed with stresses or internal forces are more general than those expressed with displacements in the sense that the former is still true even if stress-strain relations (e.g. a different material is used) or strain-displacement relations (say, a different strain definition or a different geometric assumption is adopted) are changed. In structural mechanics, the requirement of equilibrium is the most basic and the most strict, while the others are quite dependent on the adopted structural theory and geometric assumptions. A load-bearing structure could have a different deformed configuration and a different stress distribution if the material or the strain definition is substituted by another one, but the equilibrium must be upheld. Based on this reasoning, the field consistence approach can be relaxed if starting from a mixed field variational principle. In the relaxed field consistence approach, the equilibrium equations are satisfied point-wise within an element, while the stress-strain relation and/or strain-displacement relations are satisfied at element level. With the relaxed field consistence approach, the conventional hybrid stress model is improved in Paper B and an alternative assumed strain method is formulated in Paper C. The assumed stresses or assumed strains are constructed from the requirement of equilibrium.

The field consistence approach has two main applications. One application is to explain various locking phenomena. In Paper A, with beam elements investigated, it is concluded that shear locking and membrane locking have the same origin — coupling between concerned displacements in the Euler-Lagrangian equations and the adoption of low order interpolation functions. More specifically, when transverse displacements are coupled with section rotations in Euler-Lagrangian equations and low order interpolations are adopted, shear locking will occur. Similarly, when in-plane displacements are coupled with section rotations in Euler-Lagrangian equations and low order interpolations are used, membrane locking will happen. Although volumetric locking is not closely investigated in this thesis, it is quite possible that this phenomenon has a similar origin. It is obvious that the three displacement components describing a typical particle in a solid element are coupled in the equilibrium equations expressed with the displacements, and the adopted interpolations in a displacement-based isoparametric solid element, e.g. 8-node solid element, are too low and too simple without taking into account element material parameters such as Young's modulus and Poisson's ratio. This is one work to be done after the thesis.

The other application of the field consistence approach is to develop efficient finite elements. With the field consistence approach, several beam elements are developed in Paper A. With a relaxed field consistence approach, several plane stress elements and several Mindlin plate elements are derived, respectively, in Papers B and C. The elements obtained from the (relaxed) field consistence approach are not only free of locking but also computationally much more efficient than their displacement-based counterparts.

Although only low order elements such as 2-node beam elements, 4-node plane stress or Mindlin plate elements are developed from the (relaxed) field consistence approach in this thesis, in principle, the approach can also be applied in developing high order elements. There is an increasing trend in developing high order $p$-version
finite elements [31, 35], but low order elements are preferred for several reasons. Low order elements are simple to derive, reliable in performance and easy to be fitted into an existing program. They might not be so inferior to high order elements with respect to computational efficiency and convergence rate, if considering that with the same number of D.O.F., a 9-node quadrilateral element is equivalent to four 4-node quadrilateral elements and if considering the extra work spent in the pre- and post-processing for high order elements; furthermore, deficiencies occurring in a high order element is much more difficult to handle and eliminate. Listed in Table 1 are some elements derived from the (relaxed) field consistency approach in this thesis.

Table 1: Efficient elements developed from (relaxed) field consistency approach

<table>
<thead>
<tr>
<th>Identifier</th>
<th>in Paper</th>
<th>brief description</th>
</tr>
</thead>
</table>
| B2D2       | A       | • Timoshenko beam theory  
• 2 nodes, 4 degrees of freedom (D.O.F.), straight beam  
• interpolations from the field consistency approach |
| B2DH       | A       | • Bernoulli beam theory with high order strain  
• shallow-arch assumption  
• 2 nodes, 6 D.O.F., straight beam  
• interpolations from the field consistency approach |
| B2D2C      | A       | • Marguerre's shallow-arch assumption and Timoshenko's shear deformation  
• 2 nodes, 6 D.O.F., curved beam  
• interpolations from the field consistency approach |
| PLN4-5     | B       | • plane stress theory  
• 4 node, 8 D.O.F., quadrilateral  
• bilinear, isoparametric interpolations for displacements; assumed stresses are constructed from the relaxed field consistency approach with the following form before satisfying equilibrium

\[
\begin{align*}
N_x &= a_1 + a_2 s \\
N_y &= a_3 + a_4 t \\
N_{xy} &= a_5 + a_6 t + a_7 s
\end{align*}
\]

continued on next page
<table>
<thead>
<tr>
<th>Identifier</th>
<th>in Paper</th>
<th>brief description</th>
</tr>
</thead>
</table>
| PLT3–2     | C        | ● Mindlin plate theory  
● 3 nodes, 9 D.O.F., triangle  
● isoparametric interpolations for displacements; assumed strains are constructed from the relaxed field consistency approach with the following form, before the corresponding internal forces satisfying equilibrium  
\[ \begin{align*} 
\kappa_x &= a_1 + a_2 s \\
\kappa_y &= a_3 + a_4 r \\
\kappa_{xy} &= a_5 
\end{align*} \] |
| PLT4–5     | C        | ● Mindlin plate theory  
● 4 nodes, 12 D.O.F., quadrilateral  
● bilinear isoparametric interpolations for displacements; assumed strains are constructed from the relaxed field consistency approach with the following form before the corresponding internal forces satisfying equilibrium  
\[ \begin{align*} 
\kappa_x &= a_1 + a_2 r + a_3 s \\
\kappa_y &= a_4 + a_5 r + a_6 s \\
\kappa_{xy} &= a_7 + a_8 r + a_9 s 
\end{align*} \] |

The field consistency approach has a vast potential application area. In principle, the approach can be applied to any area where finite element methods are used and there is a need to construct efficient interpolation functions. The key step in applying the approach is to derive Euler-Lagrangian equations. The most difficult step is to obtain general solutions to those equations. In most cases, it is very difficult or even impossible to find the real general solutions and quasi-general solutions from an approximate analytical method have to be used. It is very important to be aware that the original consistency between field functions may have been ruined in a set of quasi-general solutions. At this moment, there is no measure to examine and to guarantee that the consistency is not ruined with quasi-general solutions used.

4 Extension of Several Existing Formulations

The process of establishing a finite element equation \( \mathbf{K} \mathbf{u} = \mathbf{P} \) can be technically separated into several stages:

- (I) \( \text{selection of variational principle} \)
- (II) \( \text{element stiffness matrix} \ (\text{unintegrated}) \)
The three stages marked by (I), (II) and (III) can be called in short pre-variational principle, post-variational principle & pre-formation of element stiffness matrix and formation of element stiffness matrix. Although there are quite many finite element techniques or formulations, any of them is, without exception, a technique introduced in one of the three stages. Therefore, it might be more reasonable to group finite element techniques or formulations according to their time introduced in establishing the finite element equation. In Table 4, the commonly used techniques or formulations are grouped according to the above method.

Table 2: Finite element techniques grouped according to their introduced time

<table>
<thead>
<tr>
<th>(I) pre-variational principle</th>
<th>(II) post-variational principle &amp; pre-formation of element stiffness matrix</th>
<th>(III) formation of element stiffness matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. assumed (hybrid) stress elements</td>
<td>1. incompatible displacement model</td>
<td>1. uniform reduced integration</td>
</tr>
<tr>
<td>2. assumed strain elements</td>
<td>2. MITC approach</td>
<td>2. selective reduced integration</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3. 'optimal' reduced integration</td>
</tr>
</tbody>
</table>

Category (I), including assumed (hybrid) stress and assumed strain methods, is characterized by altering or modifying variational principles, with the principle of minimum potential energy seen as the most basic one. In Category (II), consisting of incompatible displacement model and the MITC approach, something is done after selecting the principle of minimum potential energy and before deriving the analytical expression of element stiffness matrix. In Category (III), including various reduced integrations, a measure is taken in integrating the element stiffness matrix, when the element stiffness matrix actually has taken its analytical form.

None of the methods is clearly superior to the others. From the view point of computational efficiency, the techniques in categories (II) and (III) are more efficient than those in category (I), because at least one matrix inversion is needed in the formation of element stiffness matrix if a technique in category (I) is used. Sometimes, techniques from different categories are combined to improve element performance, [5, 6].

It should be noted that spurious zero energy modes or hourglassing is not always the 'crime' of reduced integration. There are two proofs which can justify the statement. The first one is the 2-D 2-node displacement-based Timoshenko beam element in Reference [19], in which a third order isoparametric interpolation is used for the transversal displacement, a first order isoparametric interpolation is adopted for the section rotation and exact integration is used. The element, however, has one
spurious zero energy mode. The second proof is the alternative assumed strain method in Paper C. If the assumed strains are all taken as constants, the resulting element has one spurious zero energy mode. We can also find other proofs in Tables 5 and 7.

Table 3 shows the reduced integrations applicable in different finite elements.

<table>
<thead>
<tr>
<th>Element</th>
<th>uniform RI</th>
<th>selective RI</th>
<th>'optimal' RI</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-node Timoshenko beam element</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4-node plane stress/strain element</td>
<td>x</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4-node Mindlin plate element</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8-node solid element</td>
<td>x</td>
<td>x</td>
<td>0</td>
</tr>
</tbody>
</table>

In the table, RI stands for Reduced Integration. Symbol 'o' indicates that a reduced integration scheme works without any problem, while 'x' represents that an element with reduced integration shows spurious zero energy modes or hourglassing. Some interesting observations can be made from the table. Both uniform and selective reduced integrations work in a Timoshenko beam element and a Mindlin plate element. Only selective reduced integration can be applied in a plane stress/strain element. Neither uniform nor selective reduced integration works in a solid element. According to numerical experiments [19], uniform reduced integration is more effective in 'removing element extra constraints'. That means, for an element, if selective reduced integration can just right remove all extra constraints, then uniform reduced integration will produce one or more spurious zero energy modes. On the other hand, structural theories can be seen as degeneration of continuum mechanics after imposing certain geometric assumptions. If a geometric assumption is considered as a constraint, then a solid element has no extra constraint, a plane stress/strain element is constrained in the sense that deformations or stresses along thickness direction are not free. But this constraint is not so severe as those imposed in a Timoshenko beam element or in a Mindlin plate element. In Timoshenko beam theory and Mindlin plate theory, three constraints are imposed: zero normal stress along the thickness, straight normal line and the special assumption for shear strain. Among the imposed constraints, the assumption of transverse shear strain has very profound influence over element performances. It might not be so clear at this moment, but there seems be a connection between the practicability of a reduced integration scheme and the geometric assumption of corresponding structural theory.

The 'optimal' reduced integration and the MITC approach were originally developed for eliminating shear locking in the 4-node Mindlin plate element. In Papers D and E, the techniques are extended to a 8-node solid element for two goals: to improve
the performance of the 8-node solid element and to investigate their relations with other finite element formulations in a 8-node solid element.

In Paper F, two hybrid mixed principles, the hybrid Hellinger-Reissner principle and the hybrid Hu-Washizu principle, are formulated. The main idea of hybrid mixed principles is to combine the principle of minimum potential energy with one conventional mixed field variational principle. A basic assumption of hybrid mixed principles is that it is possible to write the strain energy in an element in two or more independent parts, which implicitly supposes that a single or a set of stresses is uncoupled from the rest of stresses. Owing to this, the hybrid mixed principle may have a limited application area, but the principles provide sound variational bases for some existing finite element techniques such as the reduced integration and the MITC approach.

5 Relations between Different Formulations

In the third part of this thesis, the relations between different finite element techniques or formulations are closely investigated by the aid of Maple [39], a symbolic computational software. The strategy for the investigation is to compare element stiffness matrices from different formulations. This might be also the most advisable way at this moment. An element, usually characterized by its stiffness matrix in statics, is a composite product of many ingredients: structural theory, variational principle, element configuration, interpolation function and integration technique. Longitudinally, operations involved in deriving element stiffness matrix are complex and of different mathematical natures at different stages, as divided in the last section. Transversally, different replacements at a stage are independent and can not be related by a mathematical function, e. g. plane stress/strain theory and Mindlin plate theory. It is very difficult, if not impossible, to establish a mathematical theorem for a relation between two finite element techniques or formulations, especially those introduced at different stages. The equivalence between two finite element techniques or formulations only means that they can yield identical element stiffness matrices under certain conditions.

The finite element techniques or formulations involved in relationship investigation are: reduced integration (including uniform, selective and 'optimal' versions), MITC approach, incompatible displacement model, mixed field formulations and hybrid mixed formulations. Relations between these formulations are investigated, respectively, around a 2-node Timoshenko beam element, a 4-node plane stress/strain element, a 4-node Mindlin plate element and an 8-node solid element, depending on the applicability of the formulations in these elements. In the investigation, the shape of 2-D and 3-D elements was successively changed from rectangular to parallelogram and to arbitrary quadrilateral (brick), in order to show the influence of element shape over a relationship. It was found that Poisson's ratio also has effects on some relations. The investigation results from Papers G, H and I are summed up to obtain an overall picture about the relations. For clarity, elements involved in an investigation are briefly described in a table with unified identifiers. Closely
following the table is a figure describing relations between the listed elements. In the figures, symbol ‘\(\equiv\)' represents an equivalence relationship which is not affected by the mentioned conditions. ‘\(=\)' indicates that an equivalence relation is conditionally true with symbols under or over it describing the conditions satisfied. ‘\([\square]\)' and ‘\(\square\)' stands for, respectively, a rectangular or a parallelogram 4-node quadrilateral element. ‘\(\nu = 0\)' means that Poisson’s ratio is zero.

5.1 Equivalences in a 2-node Timoshenko beam element

In a Timoshenko beam element, the applicable finite techniques or formulations are: uniform/selective reduced integration, the (hybrid) Hellinger-Reissner formulations and the (hybrid) Hu-Washizu formulations. Beam elements from different formulations and their identifiers are listed in Table 4.

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>investigated in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B_{2h})</td>
<td></td>
<td>(h_i = \frac{1}{2}(1 + r_ir),) (\quad (r_i = \pm 1, \quad i = 1, 2))</td>
</tr>
<tr>
<td>(B_{2_{u1}})</td>
<td>(G)</td>
<td>one Gaussian point for integrating bending and shear stiffness matrices</td>
</tr>
<tr>
<td>(B_{2_{r1}})</td>
<td>(G)</td>
<td>two Gaussian points for integrating bending stiffness matrix and one Gaussian point for integrating shear stiffness matrix</td>
</tr>
<tr>
<td>(B_{2hx})</td>
<td>(G)</td>
<td>the Hellinger-Reissner formulation with assumed stresses (\quad M = a_1, \quad Q = a_2)</td>
</tr>
<tr>
<td>(B_{2hw})</td>
<td>(G)</td>
<td>the Hu-Washizu formulation with assumed strains (\quad \kappa = b_1, \quad \gamma = b_2)</td>
</tr>
</tbody>
</table>

Table 4: Beam elements involved in relationship investigation

continued on next page
Investigation of equivalences in a 2-node Timoshenko beam element is conducted in Paper G. The equivalence relations are described in Fig. 2. From the diagram one can see that the six beam elements, all elements in Table 4 except $B_{2n}$, have identical stiffness matrices. But it is too early to conclude that all the corresponding finite element formulations are equivalent. An equivalence relationship in a Timoshenko beam element may in general not exist in, e.g. plane stress/strain elements, as the structural theories of these elements have different geometrical assumptions.

### 5.2 Equivalences in a 4-node plane stress/strain element

The techniques or formulations applicable to a 4-node plane stress/strain element are: uniform/selective reduced integration, incompatible displacement model, the (hybrid) Hellinger-Reissner formulations and the (hybrid) Hu-Washizu formulations. Listed in Table 5 are plane stress/strain elements from the above formulations.

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<table>
<thead>
<tr>
<th>unified identifier</th>
<th>investigated in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{2hr}$</td>
<td>G</td>
<td>• the hybrid Hellinger-Reissner formulation with assumed shear stress $Q = a_1$.</td>
</tr>
<tr>
<td>$B_{2hv}$</td>
<td>G</td>
<td>• the hybrid Hu-Washizu formulation with assumed shear strain $\gamma = b_1$.</td>
</tr>
</tbody>
</table>

**Figure 2**: Equivalence in a 2-node Timoshenko beam element
Table 5: Plane stress/strain elements involved in relationship investigation

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier</th>
<th>in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>QUAD4f1</td>
<td>QUAD4</td>
<td>G</td>
<td>• plane stress/strain theory</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• 4 nodes, 8 degrees of freedom</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• bilinear isoparametric interpolations</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$h_i = \frac{1}{4}(1 + r_i r)(1 + s_i s)$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$r_i, s_i = \pm 1, \ i = 1, 2, 3, 4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• full (2 × 2 Gaussian points) integration</td>
</tr>
<tr>
<td>QUAD4ux1</td>
<td>QUAD4+U.R.I.</td>
<td>G</td>
<td>• a (1 × 1) scheme for integrating both normal- and shear-related stiffness matrices</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
<tr>
<td>QUAD4ex1</td>
<td>QUAD4+S.R.I.</td>
<td>G</td>
<td>• a (2 × 2) scheme for integrating normal-related stiffness matrix and a (1 × 1) scheme for integrating shear-related stiffness matrices</td>
</tr>
<tr>
<td>QUAD40.hr</td>
<td>HRpln4</td>
<td>G</td>
<td>• the Hellinger-Reissner formulation with assumed stress resultants $N_x = a_1$, $N_y = a_2$, $N_{xy} = a_3$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
<tr>
<td>QUAD40_hv</td>
<td>HWpln4</td>
<td>G</td>
<td>• the Hu-Washizu formulation with assumed strains $\varepsilon_x = b_1$, $\varepsilon_y = b_2$, $\gamma_{xy} = b_3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
<tr>
<td>QUAD41_hr</td>
<td>QUAD4_hr</td>
<td>l</td>
<td>• the Hellinger-Reissner formulation with assumed stress resultants $N_x = a_1 + a_2 s$, $N_y = a_3 + a_4 r$, $N_{xy} = a_5$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• the Hu-Washizu formulation with assumed strains,</td>
</tr>
<tr>
<td>QUAD41_hv</td>
<td>QUAD4_hv</td>
<td>l</td>
<td>$\begin{bmatrix} \varepsilon_x \ \varepsilon_y \ \gamma_{xy} \end{bmatrix} = d \begin{bmatrix} N_x \ N_y \ N_{xy} \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$N_x$, $N_y$, $N_{xy}$ as in the above row, $d$ is the flexibility matrix, cf. Paper l</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>QUAD4_{hr}^0</td>
<td>HHRpln_4</td>
<td>G</td>
</tr>
<tr>
<td>QUAD4_{hr}^1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>QUAD4_{hv}^0</td>
<td>HHWpln_4</td>
<td>G</td>
</tr>
<tr>
<td>QUAD4_{hv}^1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>QUAD4_{inc}</td>
<td>QUAD4_{inc}</td>
<td>l</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It should be noted that elements QUAD4_{hr}^0 and QUAD4_{hr}^1 (QUAD4_{hv}^0 and QUAD4_{hv}^1) have different assumed stresses (or strains). The relations between different finite element techniques or formulations in a 4-node plane stress/strain element are investigated in Papers G and l. Fig. 3 shows relations between the plane stress/strain elements. Compared with the relation diagram from Timoshenko beam elements

![Diagram](image)

**Figure 3:** Equivalence in a 4-node plane stress/strain element

in Fig. 2, the join of incompatible displacement model (QUAD4_{inc}) and two mixed field elements (QUAD4_{hr}^1 and QUAD4_{hv}^1) with first order assumed stresses or strains make the relationship diagram much more complicated, but the equivalence relations found in beam elements, that is urihvrh and srihrhvh, are still true in plane stress/strain elements. It should be mentioned that although elements
QUAD4$_{ur}$, QUAD4$_{hr}$ and QUAD4$_{hv}$ suffer from hourglassing, they do have identical element stiffness matrices.

### 5.3 Equivalences in a 4-node Mindlin plate element

All the mentioned finite element techniques or formulations are applicable to a 4-node Mindlin plate element. The 4-node Mindlin plate elements involved in the investigation are listed and described in Table 6.

Three chains of equivalence, i.e. three boxes marked by (a), (b) and (c) in Fig. 4, exist in a 4-node Mindlin plate element. From boxes (a) and (b), one can see that the equivalence between uniform/selective reduced integration and (hybrid) mixed formulations, which is unconditionally true in a Timoshenko beam element and in a 4-node plane stress/strain element, becomes a conditional relation which can be true only if a 4-node Mindlin element is a parallelogram. It can be observed from box (c) that the equivalence between the 'optimal' reduced integration and hybrid mixed formulations with linear assumed stresses/strains is very similar to those, indicated in boxes (a) and (b), between uniform/selective reduced integrations and (hybrid) mixed field formulations with constant assumed (shear) stresses/strains, cf. Fig. 4.

Table 6: Mindlin plate elements involved in relationship investigation

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier</th>
<th>in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4$_{r1}$</td>
<td>Q4</td>
<td>G</td>
<td>Mindlin plate theory</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4 nodes, 12 degrees of freedom</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>bilinear isoparametric interpolation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$h_i = \frac{1}{4}(1 + r_i r)(1 + s_i s)$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$r_i, s_i = \pm 1, i = 1, 2, 3, 4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>full (2 x 2) integration</td>
</tr>
<tr>
<td>Q4$_{ur1}$</td>
<td>Q$_4$+U.R.I.</td>
<td>G</td>
<td>uniform reduced integration with a (1 x 1) scheme for integrating both bending and shear stiffness matrices</td>
</tr>
<tr>
<td>Q4$_{sr1}$</td>
<td>Q$_4$+S.R.I.</td>
<td>G</td>
<td>selective reduced integration with a (2 x 2) scheme for integrating bending stiffness matrix and a (1 x 1) scheme for integrating shear stiffness matrices</td>
</tr>
</tbody>
</table>

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Summary

5. Relations between Different Formulations

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier in Paper</th>
<th>brief description</th>
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</thead>
<tbody>
<tr>
<td>Q4_{ori}</td>
<td>Q4_{ori} H</td>
<td>'optimal' reduced integration with a $(2 \times 2)$ scheme for integrating bending stiffness matrix, a $(1 \times 2)$ scheme for integrating shear stiffness matrices related to shear strain $\gamma_2$ and a $(2 \times 1)$ scheme for integrating shear stiffness matrices related to shear strain $\gamma_y$.</td>
</tr>
<tr>
<td>Q4_{mitc}</td>
<td>Q4_{mitc} H</td>
<td>the mixed interpolation of tensorial components (MITC) sampling points substitution of shear strains.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\bar{\gamma}<em>{zz} = \frac{1}{2} (1 + s) \gamma</em>{zz}^A + \frac{1}{2} (1 - s) \gamma_{zz}^B$ $\bar{\gamma}<em>{yz} = \frac{1}{2} (1 + r) \gamma</em>{yz}^C + \frac{1}{2} (1 - r) \gamma_{yz}^D$</td>
</tr>
<tr>
<td>Q4_{hr}</td>
<td>HRplt H G</td>
<td>the Hellinger-Reissner formulation with assumed stress resultants $M_x = a_1$, $M_y = a_2$, $M_{xy} = a_3$, $Q_x = a_4$, $Q_y = a_5$.</td>
</tr>
<tr>
<td>Q4_{hw}</td>
<td>HWplt H G</td>
<td>the Hu-Washizu formulation with assumed strains $\kappa_x = b_1$, $\kappa_y = b_2$, $\kappa_{xy} = b_3$, $\gamma_x = b_4$, $\gamma_y = b_5$.</td>
</tr>
<tr>
<td>Q4_{0 hr}</td>
<td>HHRplt H G</td>
<td>the hybrid Hellinger-Reissner formulation with assumed shear stress resultants $Q_x = a_1$, $Q_y = a_2$.</td>
</tr>
<tr>
<td>Q4_{0 hw}</td>
<td>HHWplt H G</td>
<td>the hybrid Hu-Washizu formulation with assumed shear strains $\gamma_x = b_1$, $\gamma_y = b_2$.</td>
</tr>
</tbody>
</table>

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### 5. Relations between Different Formulations

<table>
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<tr>
<th>unified identifier</th>
<th>original identifier</th>
<th>in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q₄¹ₜₜr</td>
<td>Q₄ₜₜr</td>
<td>H</td>
<td>the hybrid Hellinger-Reissner formulation with assumed shear stress resultants</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
Q_x &= \frac{1}{2}a_1(1 - s) + \frac{1}{2}a_2(1 + s), \\
Q_y &= \frac{1}{2}a_3(1 - r) + \frac{1}{2}a_4(1 + r)
\end{align*}
\]

| Q₄¹ₜₜv | Q₄ₜₜv | H | the hybrid Hu-Washizu formulation with assumed shear strains |

\[
\begin{align*}
\gamma_z &= \frac{1}{2}b_1(1 - s) + \frac{1}{2}b_2(1 + s), \\
\gamma_y &= \frac{1}{2}b_3(1 - r) + \frac{1}{2}b_4(1 + r)
\end{align*}
\]

---

**Figure 4:** Equivalence in a 4-node Mindlin plate element

---

Although a Mindlin plate element can also be developed from the incompatible displacement model [5, 38], it is difficult to relate it with other elements. The reason for this might be the special geometric assumption adopted in Mindlin plate theory. On the other hand, even in a 4-node plane stress/strain element, the incompatible displacement model is weakly connected to other formulations, cf. Fig. 3 (b).
5.4 Equivalences in a 8-node solid element

The mentioned finite element techniques or formulations are all applicable to an 8-node solid element. Described in Table 7 are the 8-node solid elements involved in the relationship study.

Table 7: Solid elements involved in relationship investigation

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier</th>
<th>Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>H8xi</td>
<td></td>
<td></td>
<td>● continuum mechanics</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>● 8 nodes, 24 degrees of freedom</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>● trilinear isoparametric interpolations</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( h_i = \frac{1}{8} (1 + r_i) (1 + s_i) (1 + t_i) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>((r_i, s_i, t_i = \pm 1, (i = 1, 2, \cdots, 8)))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>● full ((2 \times 2 \times 2)) integration</td>
</tr>
<tr>
<td>H8auri</td>
<td></td>
<td></td>
<td>● uniform reduced integration with a ((1 \times 1 \times 1)) scheme for integrating all stiffness matrices</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>● hourglassing</td>
</tr>
<tr>
<td>H8auri</td>
<td></td>
<td></td>
<td>● selective reduced integration with a ((2 \times 2 \times 2)) scheme for integrating normal-related stiffness matrix and a ((1 \times 1 \times 1)) scheme for shear-related matrix</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>● hourglassing</td>
</tr>
<tr>
<td>H8auri H</td>
<td>H8auri H</td>
<td></td>
<td>● 'optimal' reduced integration with a ((2 \times 2 \times 2)) scheme for integrating normal-related stiffness matrix, a ((2 \times 1 \times 1)) scheme for matrix related to shear strain (\gamma_{xy}), a ((1 \times 2 \times 1)) scheme for matrix related to shear strain (\gamma_{xx}) and a ((1 \times 1 \times 2)) scheme for matrix related to shear strain (\gamma_{yy})</td>
</tr>
</tbody>
</table>

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5. Relations between Different Formulations

Summary

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>H6mitc</td>
<td>H6mitc</td>
<td>• the mixed interpolation of tensorial components (MITC)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• sampling points and calculated shear strains</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Point identifier</td>
</tr>
<tr>
<td>A</td>
<td>(1, 0, 0)</td>
<td>$\gamma_{yz}^A$</td>
</tr>
<tr>
<td>B</td>
<td>(-1, 0, 0)</td>
<td>$\gamma_{yz}^B$</td>
</tr>
<tr>
<td>C</td>
<td>(0, 1, 0)</td>
<td>$\gamma_{yz}^C$</td>
</tr>
<tr>
<td>D</td>
<td>(0, -1, 0)</td>
<td>$\gamma_{yz}^D$</td>
</tr>
<tr>
<td>E</td>
<td>(0, 0, 1)</td>
<td>$\gamma_{xy}^{E}$</td>
</tr>
<tr>
<td>F</td>
<td>(0, 0, -1)</td>
<td>$\gamma_{xy}^{F}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\tilde{\gamma}<em>{yz} = \frac{1}{2}(1 + r)\gamma</em>{yz}^A + \frac{1}{2}(1 - r)\gamma_{yz}^B$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\tilde{\gamma}<em>{xx} = \frac{1}{2}(1 + s)\gamma</em>{xx}^C + \frac{1}{2}(1 - s)\gamma_{xx}^D$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\tilde{\gamma}<em>{xy} = \frac{1}{2}(1 + t)\gamma</em>{xy}^{E} + \frac{1}{2}(1 - t)\gamma_{xy}^{F}$</td>
</tr>
<tr>
<td>H6hhr</td>
<td></td>
<td>• the Hellinger-Reissner formulation with assumed stresses $\sigma_x = a_1$, $\sigma_y = a_2$, $\sigma_z = a_3$, $\tau_{yz} = a_4$, $\tau_{xx} = a_5$, $\tau_{xy} = a_6$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
<tr>
<td>H6hw</td>
<td></td>
<td>• the Hu-Washizu formulation with assumed strains $\varepsilon_x = b_1$, $\varepsilon_y = b_2$, $\varepsilon_z = b_3$, $\gamma_{yz} = b_4$, $\gamma_{xx} = b_5$, $\gamma_{xy} = b_6$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
<tr>
<td>H6^0_hhr</td>
<td></td>
<td>• the hybrid Hellinger-Reissner formulation with assumed shear stresses $\tau_{yz} = a_1$, $\tau_{xx} = a_2$, $\tau_{xy} = a_3$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
<tr>
<td>H6^0_hhw</td>
<td></td>
<td>• the hybrid Hu-Washizu formulation with assumed shear strains $\gamma_{yz} = b_1$, $\gamma_{xx} = b_2$, $\gamma_{xy} = b_3$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• hourglassing</td>
</tr>
</tbody>
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5. Relations between Different Formulations

<table>
<thead>
<tr>
<th>unified identifier</th>
<th>original identifier</th>
<th>in Paper</th>
<th>brief description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^1_{hhr}$</td>
<td>$H^8_{hhr}$</td>
<td>$H$</td>
<td>- the hybrid Hellinger-Reissner formulation with assumed shear stresses</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\tau_{yz} = \frac{1}{2} a_1 (1 - r) + \frac{1}{2} a_2 (1 + r)$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\tau_{zx} = \frac{1}{2} a_3 (1 - s) + \frac{1}{2} a_4 (1 + s)$,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\tau_{xy} = \frac{1}{2} a_5 (1 - t) + \frac{1}{2} a_6 (1 + t)$,</td>
</tr>
</tbody>
</table>

| $H^1_{hhw}$         | $H^8_{hhw}$         | $H$      | - the hybrid Hu-Washizu formulation with assumed shear strains  |
|                     |                     |          | $\gamma_{yz} = \frac{1}{2} b_1 (1 - r) + \frac{1}{2} b_2 (1 + r)$, |
|                     |                     |          | $\gamma_{xz} = \frac{1}{2} b_3 (1 - s) + \frac{1}{2} b_4 (1 + s)$, |
|                     |                     |          | $\gamma_{xy} = \frac{1}{2} b_5 (1 - t) + \frac{1}{2} b_6 (1 + t)$, |

The investigation is conducted in Paper $H$. Fig. 5 shows the obtained relation diagram. Also three chains of equivalence exist in a 8-node solid element, from which

![Equivalence Diagram](image)

(a) $H^8_{uri}$ - $H^8_{hr}$ - $H^8_{hv}$  
(b) $H^8_{sri}$ - $H^8^0_{hhr}$ - $H^8^0_{hhw}$  
(c) $H^8_{hhr}$ $\Leftarrow$ $H^8_{mitc}$  
$H^8_{hhw}$ $\equiv$ $H^8_{sri}$

**Figure 5:** Equivalence in a 8-node solid element

one can pick up three unconditional equivalence relations: $uri\equiv hr\equiv hv$, $sri\equiv hhr\equiv hhw$ and $ori\equiv hhr\equiv hhw$. Like the situation in a 4-node plane stress/strain element, elements $H^8_{uri}$, $H^8_{hr}$ and $H^8_{hv}$ (also $H^8_{sri}$, $H^8^0_{hhr}$ and $H^8^0_{hhw}$) suffer from hourglassing, but they do have identical element stiffness matrices, which can be easily verified with the Maple code given in Paper $H$. Although the incompatible displacement
model is applicable to a 8-node solid element, it is difficult to find a connection between it and other formulations, as pointed out by Pian [24]: "unlike the 4-node plane problem, there exists no equivalent hybrid stress formulation to the Wilson's three-dimensional element".

From the investigation conducted, the following conclusions or observations can be stated about the relations between different finite element techniques or formulations:

- Equivalence between two elements does not mean that the two finite element techniques or formulations behind them are completely equivalent. In a different class of element, the equivalence may be untrue. It is for this that establishing a general mathematical theorem for a finite element relation is very difficult, if not impossible.

- the equivalence between the (hybrid) Hellinger-Reissner formulation and the (hybrid) Hu-Washizu formulation is a quite general relation. The only condition for this equivalence is that the assumed stresses in the former formulation and the assumed strains in the later formulation satisfy the stress-strain relations, cf. Paper G. The reason for the generality might be that both of them are pre-variational formulations.

- compared with the finite element techniques introduced between post-variational principle and pre-formation of element stiffness matrix, such as the MIOTC approach and the incompatible displacement model, reduced integrations (including uniform, selective and 'optimal' versions), which are introduced in forming element stiffness matrix, are more 'closely' related to a variational principle. An equivalence relationship between a reduced integration and a variational principle is nearly not affected by element shape, except the irregularity in a 4-node Mindlin plate element.

- most equivalence relations are influenced by element geometry. This is true especially for those describing a relation between two finite element techniques introduced at different stages in establishing the finite element equation.

- the assumption of shear strain in Timoshenko beam theory or Mindlin plate theory is demonstrated to have a very profound influence over element behavior. An equivalence relation existing in a 4-node plane stress/strain element or in a 8-node solid element is ruined in a 4-node Mindlin plate element, quite possibly because of this assumption. The 2-node Timoshenko beam element is different from a 4-node Mindlin plate element in some aspects, as they have different dimensions.

6 The Role of Symbolic Computational Software

It must be pointed out that symbolic computational techniques played an important role in the research reported in this thesis. As is well known, finite element formulas are characterized by matrix (vector) operations and large expressions for
the introduction of interpolation functions and nodal parameters. There are many factors which can affect the performance of a finite element. These factors might be the adopted structural theory, geometrical assumptions, variational basis, element configuration or integration scheme. Even in a simple case like a beam element, the manual derivation of finite element stiffness matrix is a boring and error prone process. Symbolic computational softwares, e. g. Maple [39] and Mathematica [40], have been found very helpful in explicitly deriving element stiffness matrix. It is also believed that the use of explicit element stiffness matrix can largely save CPU time, [15–17, 21]. On the other hand, symbolic computational techniques are also powerful tools in finite element parametric study. By the aid of symbolic computational techniques, the effects of a factor, e.g. the assumed stress modes in hybrid stress model (Paper B) or the assumed strain modes in assumed strain method (Paper C), on the performance of an element can be investigated with less effort. Another potential application of symbolic computational techniques in FEM is to prove or show some equivalences between different finite element formulations. For the mentioned reasons, it is very difficult to express a finite element relation in a mathematical theorem with all the involved factors considered. Conclusions drawn from several numerical examples are even less convincing. Symbolic computational techniques provide an intermediate way between analytical mathematics and approximate numerical methods. The conclusions obtained from a symbolic derivation might not be so general as a mathematical theorem, but much more convincing than those drawn from even a large number of numerical examples. It must be realized that the strategy, comparing element stiffness matrices from different formulations, adopted for relationship investigation in this thesis is possible only with a symbolic software. In a numerical software, e. g. Matlab, it is very difficult to judge the relation between two element stiffness matrices for computer precision and rounding error.

On the other hand, although symbolic computational techniques are very powerful, they are only tools; they can not find a new element or an equivalence relationship. The researcher must have a sound understanding about structural theories and finite element formulations. The use of symbolic computational techniques can indeed relieve the researcher from boring and error-prone manual derivation, but the idea about a new element can only come from the researcher. Only after an investigator somehow spot a possible equivalence relation, a symbolic computational software can be used to verify or to check such a relation.

The existing symbolic computational softwares still have their limits in application. One may encounter memory trouble when large expressions are involved. Some operations such as an integration with a complicated integrand cannot be carried out. A symbolic computational software has very limited capability in finding general solutions to differential equations, especially those to a set of coupled partial differential equations with non-constant coefficients. Nevertheless, it can be predicted that in the near future symbolic computational softwares will play a more and more important role in developing new finite elements and improving existing elements with their power and performance more and more upgraded.
References


