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Prediction of Apparent Trabecular Bone Stiffness through Fourth-Order Fabric Tensors

Rodrigo Moreno · Örjan Smedby · Dieter H. Pahr

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Abstract The apparent stiffness tensor is an important mechanical parameter for characterizing trabecular bone. Previous studies have modeled this parameter as a function of mechanical properties of the tissue, bone density and a second-order fabric tensor, which encodes both anisotropy and orientation of trabecular bone. Although these models yield strong correlations between observed and predicted stiffness tensors, there is still space for reducing accuracy errors. In this paper we propose a model that uses fourth-order instead of second-order fabric tensors. First, the totally symmetric part of the stiffness tensor is assumed proportional to the fourth-order fabric tensor in the logarithmic scale. Second, the asymmetric part of the stiffness tensor is derived from relationships among components of the harmonic tensor decomposition of the stiffness tensor. The mean intercept length (MIL), generalized MIL (GMIL) and global structure tensor fourth-order were computed from images acquired through micro computed tomography of 264 specimens of the femur. The predicted tensors were compared to the stiffness tensors computed by using the micro finite element method (μ FE), which was considered as the gold stan-

dard, yielding strong correlations (R^2 above 0.962). The GMIL tensor yielded the best results among the tested fabric tensors. The Frobenius error, geodesic error and the error of the norm were reduced by applying the proposed model by 3.75%, 0.07% and 3.16%, respectively compared to the model by Zysset and Curnier (1995) with the second-order MIL tensor. From the results, fourth-order fabric tensors are a good alternative to the more expensive μ FE stiffness predictions.

Keywords Fabric tensors, stiffness tensor, mechanical properties, trabecular bone, mean intercept length

1 Introduction

The stiffness (or elasticity) tensor is a mathematical entity that describes the elastic behavior of a material. In the specific field of trabecular bone research, the estimation of the homogenized (especially apparent) stiffness tensor is crucial for describing the mechanical competence of the cancellous structure, with relevant implications in the field of medical diagnosis, surgical planning, and interventions, especially related to osteoporosis (Kim et al 2014; Tjhia et al 2011).

There are mainly two standard approaches for estimating the apparent stiffness tensor of an inhomogeneous material. On the one hand, the results of mechanical tests on samples can be performed for measuring at least some entries of the stiffness tensor. On the other hand, finite element (FE) modeling can be used for the same purpose but allow the computation of the full stiffness tensor. Like in the experiments, also computational results depend on the employed boundary conditions (Pahr and Zysset 2008).

A fast alternative way for estimating the stiffness tensor computationally is to assume that the morphol-

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ogy and elasticity of the tissue are closely related. In his seminal paper, Cowin (1985) hypothesized that the apparent stiffness tensor of a representative elementary volume (RVE) of trabecular bone should be a function of both its geometry and the mechanical properties of the material matrix. To a great extent, this hypothesis has been shown reasonable for trabecular bone (Zysset 2003; Gross et al 2013). This approach is appealing since it can be used as an alternative to the more computationally expensive FE-based methods.

Assuming a linear elastic anisotropic material, Cowin proposed that the apparent stiffness tensor $\mathbb{C}$, which is a fourth-order tensor, could be estimated as a function f of the Lamé parameters, λ and μ of the tissue, the bone volume fraction, ρ , and a second-order tensor that encodes the orientation and anisotropy of trabecular bone, $\mathbf{M}$. The latter two can be estimated from a volume of interest. Thus:

$$\mathbb{C} = f(\lambda, \mu, \rho, \mathbf{M}). \quad (1)$$

Tensor $\mathbf{M}$ is usually referred to as fabric tensor. The main assumption in (1) is that ρ and $\mathbf{M}$ are the main geometric features that can influence the apparent stiffness tensor. Although other morphometric descriptors can be considered for describing the geometry of trabecular bone, those have been reported unable to improve the predictions of the stiffness tensor (Zysset 2003).

Different authors have proposed different functions f to estimate $\mathbb{C}$. A comparison of five of them was performed by Zysset (2003), including one isotropic model (Gibson 1985), one anisotropic model (Yang et al 1999), and three orthotropic models (Boehler 1987; Cowin 1985; Zysset and Curnier 1995). In this comparison, the best predictions were attained with the model proposed by Zysset and Curnier (1995). In a recent study, Gross et al (2013) also showed very strong correlations between the Zysset and Curnier (1995) model and stiffness tensors computed through micro-FE (μ FE) simulations (Pahr and Zysset 2008). Despite these strong correlations, the prediction error measured through the Frobenius error is, depending on the skeletal site, up to 20% (Gross et al 2013). Thus, there is still opportunities for further improvements aiming at reducing such error.

Although it is possible to estimate fabric tensors of order higher than two (Kanatani 1984), the use of second-order fabric tensors has been customary in trabecular bone research. The main reason of this is that trabecular bone is usually approximated as an orthotropic material. Thanks to the symmetries of orthotropic materials, second-order fabric tensors are usually sufficient for modeling orientation and anisotropy for orthotropic materials. Notwithstanding,

using higher-order fabric tensors can potentially be used to increase the accuracy of the predictions.

In this line, the aim of this paper is to propose a new model for reducing the error of the stiffness tensor predicted from fabric tensors. The main difference with previous approaches is that we use fourth-order fabric tensors instead of second-order ones. This approach has mainly three advantages. First, since fourth-order tensors make a better approximation of the geometry than second-order ones, it is expected that its use can lead to reduced prediction errors. Second, we argue that it is natural to use fourth-order fabric tensors for predicting stiffness tensors, given the fourth-order nature of the latter. Actually, the main hypothesis of the proposed method is that fabric tensors are proportional to the totally symmetric component of the stiffness tensor in the logarithmic scale. In contrast, methods based on second-order fabric tensors use different tensor products for augmenting the order of them (or their components) in order to make them of fourth order. Such procedures are not necessary with the proposed model, since the tensors are already of the same order. Finally, the proposed method does not require any symmetry assumption of the material. In comparison, most of the previously proposed methods assume orthotropy of the material. To the best of our knowledge, this is the first attempt of using fourth-order fabric tensors for performing predictions of the stiffness tensor.

In the last few years, many different alternatives for computing fabric tensors have been proposed (cf. Moreno et al (2014) for a comprehensive review of them). Although any type of fabric tensor can theoretically be used in (1), the methods listed above have mainly used the mean intercept length (MIL) fabric tensor (Whitehouse 1974) in their experiments. Thus, it is interesting to determine whether or not the prediction of the stiffness tensor can be improved by applying other types of fabric tensors. Thus, besides the MIL tensor, we include in the experiments the generalized MIL (GMIL) (Moreno et al 2012) and the global structure tensors (GST) (Tabor and Rokita 2007).

The paper is structured as follows. Section 2 lists some relevant properties of the stiffness and fabric tensors that are used by the proposed model. Section 3 presents the material and the proposed model for estimating stiffness tensors. Section 4 shows experimental results on 264 samples of trabecular bone. Finally, Section 5 discusses the results and makes some concluding remarks.

The following conventions are used throughout the paper. Scalars, vectors, second-order tensors and fourth-order tensors are depicted in italics, lowercase bold, uppercase bold, and blackboard bold fonts respec-

tively (e.g., t , $\mathbf{v}$, $\mathbf{A}$, $\mathbb{C}$). Also, the Einstein summation convention is used for sub indices, i.e., summation is implied over repeated sub indices.

2 Background

The following subsections introduce the properties of both stiffness and fabric tensors that are relevant for the proposed model presented in the next section.

2.1 Stiffness Tensors

A common simplification is to assume trabecular bone to be a linear elastic material (Pahr and Zysset 2008). Thus, stiffness of trabecular bone can be characterized through the apparent stiffness (or elasticity) tensor $\mathbb{C}$, which relates the stress and the strain tensors as:

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}, \quad (2)$$

where σ and ϵ are the stress and strain tensors, respectively. Notice that σ and ϵ are of second order, while $\mathbb{C}$ is of fourth order. This equation corresponds to the Hooke's law in 3D.

Both σ and ϵ are symmetric tensors, i.e., $\sigma_{ij} = \sigma_{ji}$. These symmetries induce the so-called minor symmetries in the stiffness tensor, which means that $C_{ijkl} = C_{jikl} = C_{ijlk}$. Moreover, by assuming that the stress-strain relationship stems from a strain energy density function, the stiffness tensor also exhibits major symmetry, that is, $C_{ijkl} = C_{klij}$. These two types of symmetries reduce the number of independent entries of the stiffness tensor from $3^4 = 81$ to 21 in case of a fully anisotropic material.

Notice that, in general, $\mathbb{C}$ is not totally symmetric. In addition to minor and major symmetries, a totally symmetric fourth-order tensor would show the relations $C_{ijkl} = C_{ikjl}$. These additional requirements are known as Cauchy relations (Campanella and Tonon 1994). With these extra requirements, the number of independent entries of such a tensor is further reduced to 15.

An important property of a stiffness tensor $\mathbb{C}$ is that it can unambiguously be decomposed into its totally symmetric, $\mathbb{C}^s$, and its asymmetric component, $\mathbb{C}^a$, which are given by (Backus 1970):

$$\mathbb{C}_{ijkl}^s = \frac{1}{3}(C_{ijkl} + C_{ikjl} + C_{iljk}), \quad (3)$$

$$\mathbb{C}_{ijkl}^a = \frac{1}{3}(2C_{ijkl} - C_{ikjl} - C_{iljk}) \quad (4)$$

An important property of this decomposition is that both components are orthogonal (Moakher 2008), that is:

$$\langle \mathbb{C}^s, \mathbb{C}^a \rangle := C_{ijkl}^s C_{ijkl}^a = 0. \quad (5)$$

Consequently,

$$\|\mathbb{C}\|_F^2 = \|\mathbb{C}^s\|_F^2 + \|\mathbb{C}^a\|_F^2 \quad (6)$$

where $\|\mathbb{T}\|_F = \sqrt{T_{ijkl} T_{ijkl}}$ is the Frobenius norm. Notice that this decomposition is not related to the symmetries of the material. As an example, both the totally symmetric and asymmetric components of an isotropic stiffness tensor are in general not null.

Stiffness tensors can also be decomposed into harmonic tensors (Backus 1970). A tensor of order $n > 0$ is harmonic if and only if it is totally symmetric and traceless (Auffray et al 2014). In addition, scalars are considered harmonic tensors of order zero. It can be shown that $\mathbb{C}^s$ can be decomposed into three harmonic tensors α , $\mathbf{A}$ and $\mathbb{H}$ of orders zero, two and four respectively, while $\mathbb{C}^a$ can be decomposed into two harmonic tensors β and $\mathbf{B}$ of degrees zero and two respectively. The two components $\mathbb{C}^s$ and $\mathbb{C}^a$ can be reconstructed through the tensorial functions $h_s(\mathbb{H}, \mathbf{A}, \alpha)$ and $h_a(\mathbf{B}, \beta)$ respectively, which are given by:

$$\begin{aligned} C_{ijkl}^s = & H_{ijkl} + [\delta_{ij} A_{kl} + \delta_{kl} A_{ij} + \delta_{ik} A_{jl} + \\ & \delta_{jl} A_{ik} + \delta_{il} A_{jk} + \delta_{jk} A_{il}] + \\ & \alpha [\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}], \end{aligned} \quad (7)$$

$$\begin{aligned} \mathbb{C}_{ijkl}^a = & [\delta_{ij} B_{kl} + \delta_{kl} B_{ij} - \frac{1}{2}(\delta_{ik} B_{jl} + \\ & \delta_{jl} B_{ik} + \delta_{il} B_{jk} + \delta_{jk} B_{il})] + \\ & \beta [\delta_{ij} \delta_{kl} - \frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})], \end{aligned} \quad (8)$$

where δ_{ij} is the Kronecker's delta. From (7) and (8), a general stiffness tensor, $\mathbb{C}$, can be decomposed into five harmonic tensors: α and β of order zero, $\mathbf{A}$ and $\mathbf{B}$ of order two, and $\mathbb{H}$ of order four. Due to orthogonality, it is easy to show that α , $\mathbf{A}$ and $\mathbb{H}$ are zero for asymmetric stiffness tensors and β and $\mathbf{B}$ are zero for totally symmetric stiffness tensors. It is important to remark that, unlike the totally symmetric component, the harmonic decomposition of the asymmetric component is only possible for dimensions below four (Backus 1970).

It is customary to represent $\mathbb{C}$ by the following 6×6 symmetric second-order tensor with 21 unknowns:

$$\hat{\mathbb{C}} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1113} & \sqrt{2}C_{1112} \\ & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2213} & \sqrt{2}C_{2212} \\ & & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3313} & \sqrt{2}C_{3312} \\ & \text{sym} & & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ & & & & 2C_{1313} & 2C_{1312} \\ & & & & & 2C_{1212} \end{pmatrix}. \quad (9)$$

As shown by Moakher (2008), there is an isometry between the vector spaces of $\mathbb{C}$ and $\hat{\mathbb{C}}$. Thus, $\|\mathbb{C}\|_F = \|\hat{\mathbb{C}}\|_F$. This fact eases the manipulation of

stiffness tensors. The associated second-order tensor of $\mathbb{C}^s$, $\hat{\mathbb{C}}^s$, has the following structure:

$$\hat{\mathbb{C}}^s = \begin{pmatrix} \hat{C}_{11}^s & \hat{C}_{12}^s & \hat{C}_{13}^s & \hat{C}_{14}^s & \hat{C}_{15}^s & \hat{C}_{16}^s \\ & \hat{C}_{22}^s & \hat{C}_{23}^s & \hat{C}_{24}^s & \hat{C}_{25}^s & \hat{C}_{26}^s \\ & & \hat{C}_{33}^s & \hat{C}_{34}^s & \hat{C}_{35}^s & \hat{C}_{36}^s \\ & \text{sym} & & 2\hat{C}_{23}^s & \sqrt{2}\hat{C}_{36}^s & \sqrt{2}\hat{C}_{25}^s \\ & & & & 2\hat{C}_{13}^s & \sqrt{2}\hat{C}_{14}^s \\ & & & & & 2\hat{C}_{12}^s \end{pmatrix}. \quad (10)$$

Notice that the highlighted entries in blue and pink are related. For example, the entry (6,6) is twice the entry (1,2), the entry (5,6) is $\sqrt{2}$ times the entry (1,4)

and so on and so forth. From these relationships, it is clear that $\hat{\mathbb{C}}^s$ has 15 independent entries.

Correspondingly, the associated second-order tensor of $\mathbb{C}^a$, $\hat{\mathbb{C}}^a$ has the structure:

$$\hat{\mathbb{C}}^a = \begin{pmatrix} 0 & \hat{C}_{12}^a & \hat{C}_{13}^a & \hat{C}_{14}^a & 0 & 0 \\ & 0 & \hat{C}_{23}^a & 0 & \hat{C}_{25}^a & 0 \\ & & 0 & 0 & 0 & \hat{C}_{36}^a \\ & \text{sym} & & -\hat{C}_{23}^a & -\frac{\sqrt{2}}{2}\hat{C}_{36}^a & -\frac{\sqrt{2}}{2}\hat{C}_{25}^a \\ & & & & -\hat{C}_{13}^a & -\frac{\sqrt{2}}{2}\hat{C}_{14}^a \\ & & & & & -\hat{C}_{12}^a \end{pmatrix}, \quad (11)$$

which also has related the highlighted entries. For example, the entry (6,6) is minus the entry (1,2) and the entry (5,6) is $-\sqrt{2}/2$ times the entry (1,4). Thus, $\hat{\mathbb{C}}^a$ only has 6 independent entries. Thanks to the aforementioned isometry between fourth-order tensors and second-order tensors of six dimensions, it can be shown that $\langle \hat{\mathbb{C}}^s, \hat{\mathbb{C}}^a \rangle := \hat{C}_{ij}^s \hat{C}_{ij}^a = 0$. In terms of the harmonic decomposition of $\mathbb{C}^a$, the diagonal values of $\mathbf{B}$ and β are responsible for the entries highlighted in blue in (11), while the values off the diagonal of $\mathbf{B}$ are responsible for the entries highlighted in pink. Thus, $\hat{\mathbb{C}}^a$ can also be decomposed into two orthogonal components

$\hat{\mathbb{C}}^d$ and $\hat{\mathbb{C}}^n$ with the entries highlighted in blue and pink respectively.

2.2 Fabric Tensors

The aim of fabric tensors is to encode in a tensor the orientation and anisotropy of porous media. Such a tensor can be computed in many different ways (cf. Moreno et al (2014) for a comprehensive review of methods). Most of the fabric tensors proposed in the literature are usually computed by following two steps. In a first

step, an orientation distribution function (ODF) is estimated from an orientation-dependent feature of interest. In the second step, the fabric tensor is computed as an approximation of such an ODF.

As an example, the MIL can be used as an orientation-dependent feature to create, first an ODF and then a fabric tensor. The MIL for a specific orientation is proportional to the inverse of the number of crossings between the interface between bone and marrow and a set of lines traced parallel to the orientation of interest. An ODF is obtained by performing the same procedure for different orientations. Once the ODF is computed, either an ellipse fitting or the covariance matrix is computed in order to obtain a second-order fabric tensor.

As already mentioned, the following fabric tensors are tested in this paper: the MIL tensor (Whitehouse 1974), the GMIL tensor (Moreno et al 2012) and the GST (Tabor and Rokita 2007). It has been shown that the ODFs of these three methods are closely related, since they can be obtained by performing an angular convolution of the extended Gaussian image (EGI) (Horn 1984) with different kernels. Basically, the EGI can be seen as the ODF of the gradient (Moreno et al 2012) and it can be computed as:

$$E(\mathbf{v}) = \int_I \delta \left(\frac{\nabla I}{\|\nabla I\|} - \mathbf{v} \right) \|\nabla I\| dI \quad (12)$$

with $E(\mathbf{v})$ being the EGI at the orientation $\mathbf{v}$, δ being the unit impulse function and I the whole image. The ODFs of the three aforementioned fabric tensors are computed as (Moreno et al 2012):

$$g_{\text{MIL}} = \frac{1}{E * H}, \quad (13)$$

$$g_{\text{GMIL}} = \frac{1}{E * K}, \quad (14)$$

$$g_{\text{GST}} = E, \quad (15)$$

where “*” is the angular convolution (usually performed in the spherical harmonics domain), H is the half-cosine kernel, and K is any convolution kernel provided that it is positive and rotationally symmetric with respect to the north pole in S^2 . H is given by:

$$H = \begin{cases} \cos \phi & \phi \leq \pi/2 \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

where ϕ is the colatitude in the surface of the unitary sphere S^2 . GMIL generalizes the MIL by applying any

kernel K . Notice that GMIL and MIL are equivalent when $K = H$. Also notice that g_{GST} is related to $\frac{1}{g_{\text{GMIL}}}$ by using the unit impulse function as a kernel, that is, $K = \delta$.

Whereas MIL and GST use a fixed kernel, GMIL can use kernels that can be adjusted to the needs of a specific application. For example, the bandwidth of the von Mises-Fisher kernel, which is given by:

$$K_{\text{vmf}} = \exp(\kappa \cos \phi) \quad (17)$$

can be controlled by the parameter κ . This kernel has shown a better performance than the half-cosine and δ kernels for estimations in vivo (Moreno et al 2015). Figure 1 shows three different kernels K .

Once the ODF is computed, a second-order fabric tensor can be computed as:

$$M_{ij} = \int_{\mathbf{v} \in \Omega} g(\mathbf{v}) v_i v_j d\Omega, \quad (18)$$

where Ω is the surface of S^2 and $g(\mathbf{v})$ is the ODF computed at the orientation $\mathbf{v}$. Generalizing this equation to N -th-order fabric tensors is straightforward. The particular case of fourth-order is given by (Advani and Tucker 1987; Moakher 2008):

$$M_{ijkl} = \int_{\mathbf{v} \in \Omega} g(\mathbf{v}) v_i v_j v_k v_l d\Omega. \quad (19)$$

Notice that $\mathbb{M}$ in (19) is totally symmetric by construction.

An important point to remark is that fabric tensors are dimensionless. Actually, it is customary to normalize them before being used for prediction purposes, since any bias generated by numerical computations that are not related to the properties of the material must be disregarded (e.g. Zysset (2003); Moreno et al (2012)). The main consequence of this fact is that fabric tensors can potentially be used to predict orientation and anisotropy of the stiffness tensor, but they must be combined with other mechanical and geometrical descriptors for predicting the size (or strength) of the predicted stiffness tensor.

3 Material and Methods

3.1 Material

Segmented and gray-level trabecular bone cubes are taken from the study of (Gross et al 2013). Femur sections are scanned with CT (μCT 40, SCANCO Medical AG, Brüttisellen, Switzerland) at $18\mu\text{m}$ isotropic spatial resolution. After applying a Gaussian filter ($\sigma = 1.2$,

Fig. 1 Graphical representation of three angular convolution kernels. Left: Half-cosine kernel. Middle: von Mises-Fisher kernel ($\kappa = 10$). Right: unit impulse function (a single dot at the north pole).

Table 1 Comparison of the Frobenius norm of different components of the stiffness tensor. Statistics correspond to 264 samples of the femur.

	mean	std
$\ \mathbb{C}^s\ _F / \ \mathbb{C}\ _F$	0.9993	0.0007
$\ \mathbb{C}^a\ _F / \ \mathbb{C}\ _F$	0.0342	0.0152

support=2), 264 cubic subregions with a side length of 5.3mm were extracted from the μ CT scans. In case of segmented images the single-level threshold of IPL (SCANCO Medical AG, Brüttisellen, Switzerland) was applied, unconnected bone regions were removed from the scans, bone volume fraction was computed via voxel counting, and fabric tensors were measured using (18) and (19) for ODFs computed through (13)-(15). μ FE models of the segmented trabecular bone cubes were created by converting image voxels into linear isotropic eight-node hexahedral finite elements. Each element was assigned a Young's modulus of 12GPa and a Poisson's ratio of 0.3. The apparent elastic properties of the μ FE models were evaluated by performing FE simulations of six independent load cases under kinematic boundary conditions. The full elastic anisotropic stiffness tensor of each bone cube was computed by means of stress and strain averages of the FE results.

3.2 Prediction Model

Before proposing a new prediction model, it is important to determine the importance of each component of the stiffness tensor. Table 1 compares the Frobenius norms of different components of the stiffness tensor for 264 samples from the femur. As shown, the totally symmetric part of the stiffness tensor is largely dominant. Thus, the accuracy of the prediction will be closely related to the totally symmetric component of the predicted stiffness tensor $\mathbb{C}_p^s$. Despite this, it is also necessary to predict the asymmetric part of the stiffness tensor in order to improve the predictions.

As already mentioned, the proposed model uses fourth-order fabric tensors instead of second-order ones, that is, (19) is used instead of (18). It is important to recall that $\mathbb{M}$ in (19) is totally symmetric. This fact has two direct consequences. On the one hand, $\mathbb{M}$ cannot be used alone for predicting $\mathbb{C}^a$. Thus, $\mathbb{M}$ must be combined with other mechanical features for predicting $\mathbb{C}^a$. On the other hand, by following the same hypothesis of Cowin (1985), the anisotropy and orientation of $\mathbb{C}^s$ should be related to the ones of $\mathbb{M}$. This means that ideally, $\mathbb{C}^s \propto \mathbb{M}$, or at least a power law should relate both tensors. Thus, we propose to predict the totally symmetric component of the stiffness tensor as:

$$\mathbb{C}_p^s := \gamma \frac{s(\mathbb{M}^w)}{\|s(\mathbb{M}^w)\|_F} \quad (20)$$

where γ and w are parameters and $s(\cdot)$ extracts the totally symmetric part of the tensor. Notice that the exponential in (20) does not refer to the component-wise exponential but to the matrix exponential operation, which can be computed by rising the eigenvalues of the associated tensor $\hat{\mathbb{M}}$ to the power w . It is worthwhile to mention that the function $s(\cdot)$ in (20) is necessary since, in general, the matrix exponential operation does not preserve the total symmetry of fourth-order tensors. Notice that, since fabric tensors cannot be used for modeling the strength of the stiffness tensor due to their non-dimensionality nature, the exponential in (20) has been normalized with the Frobenius norm. In this way, the strength of the stiffness tensor is only modeled through γ .

Basically, w measures the deviation between both tensors in the logarithmic scale. In turn, as already mentioned, γ is used to appropriately scale the dimensionless tensor $s(\mathbb{M}^w)$ after normalization to the size of the stiffness tensor. Inspired by previous models (especially Zysset (2003)), γ is computed as:

$$\gamma = \Gamma_0 \rho^k, \quad (21)$$

where Γ_0 and k depend on the mechanical parameters of the tissue.

Parameters Γ_0 , k and w can be estimated from linear regression using observed data $\mathbb{C}_o$. On the one hand, Γ_0 and k are estimated from the regression between $\log(\rho)$ and $\log(\|\hat{\mathbf{C}}_o^s\|_F)$. In turn, w can be estimated through linear regression between the 15 independent entries of $\text{Log}(\hat{\mathbf{M}})$ and their corresponding entries of $\text{Log}(\hat{\mathbf{C}}_o^s)$, where Log refers to the logarithm of matrices.

As for $\mathbb{C}^a$, Campanella and Tonon (1994) have hypothesized that it is connected with a self-equilibrated stress field. This could mean that $\mathbb{C}^a$ is not completely independent from $\mathbb{C}^s$. Thus, certain relationships can be found between both components, which likely depend on the mechanical properties of the tissue. Let us recall the harmonic tensor decomposition of $\mathbb{C}^s$ and $\mathbb{C}^a$: $(\mathbb{H}, \mathbf{A}, \alpha)$ and $(\mathbf{B}, \beta)$ respectively. Since $\mathbb{C}^s$ can be estimated through (20), and consequently $(\mathbb{H}, \mathbf{A}, \alpha)$ are available, our strategy for estimating $\mathbb{C}^a$ is to find relationships between α and β as well as between $\mathbf{A}$ and $\mathbf{B}$ in order to predict $\mathbf{B}$ and β . Such relationships can be derived from the observed data and depend on the mechanical properties of the tissue. Once these two components are predicted, $\mathbb{C}^a$ can be obtained through (8).

On the one hand, β can be estimated as $t\alpha$, where t is the mean of the ratio β/α in the observed data. On the other hand, let us consider the eigendecomposition of $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \mathbf{Q}_A \mathbf{\Lambda}_A \mathbf{Q}_A^T, \quad (22)$$

$$\mathbf{B} = \mathbf{Q}_B \mathbf{\Lambda}_B \mathbf{Q}_B^T, \quad (23)$$

where $\mathbf{Q}_A$ and $\mathbf{Q}_B \in SO(3)$ and $\mathbf{\Lambda}_A$ and $\mathbf{\Lambda}_B$ are diagonal matrices with the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ respectively, with λ_1 , λ_2 and λ_3 being the maximum, intermediate and minimum eigenvalue of these tensors. Without loss of generality, the eigendecomposition can be arranged in such a way that $\mathbf{Q}_A$ and $\mathbf{Q}_B$ are as close as possible to the matrix identity. That is, the observed stiffness tensor is rotated in such a way that the eigenvectors of its associated tensors $\mathbf{A}$ and $\mathbf{B}$ are aligned as close as possible to the main axes. Such a rotation can be performed by following the methodology by Mehrabadi and Cowin (1990).

Notice that $\mathbf{Q}_A$ can be used to describe the orientation of the stiffness tensor. For predicting $\mathbf{B}$ from $\mathbf{A}$, it is necessary to determine relationships between $\mathbf{Q}_A$ and $\mathbf{Q}_B$ and between $\mathbf{\Lambda}_A$ and $\mathbf{\Lambda}_B$ respectively. First consider the orientations. Following Huynh (2009), it is possible to measure the distance between $\mathbf{Q}_A$ and $\mathbf{Q}_B$

as:

$$D(\mathbf{Q}_A, \mathbf{Q}_B) = \arccos(|\mathbf{q}_A^T \mathbf{q}_B|) \quad (24)$$

where $\mathbf{q}_A$ and $\mathbf{q}_B$ are the quaternions corresponding to $\mathbf{Q}_A$ and $\mathbf{Q}_B$ respectively. This function ranges between 0 and $\pi/2$. This distance can also be computed as $\frac{1}{2} \arccos((\text{trace}(\mathbf{Q}_A^T \mathbf{Q}_B) - 1)/2)$ (Moakher 2002). Considering that the mean and standard deviation of the distance computed through (24) for our 264 samples is relatively small (12.19° and 6.99° respectively), we propose to use the approximation $\mathbf{Q}_B \approx \mathbf{Q}_A$.

Regarding the eigenvalues, Figure 2 shows correlation plots between $\lambda_1(\mathbf{A})$ and $\lambda_1(\mathbf{B})$ and between $\lambda_1(\mathbf{A})$ and $\lambda_3(\mathbf{B})$. As shown, $\lambda_3(\mathbf{B})$ is strongly correlated. Although the correlation for $\lambda_1(\mathbf{B})$ is less strong, it can still be used for predictions. Thus, λ_1 and λ_3 of $\mathbf{B}$ can be approximated from the eigenvalues of $\mathbf{A}$. Notice that the correlations are negative for $\lambda_3(\mathbf{B})$ as expected. In turn, we have experimentally found that $\lambda_2(\mathbf{B})$ is more difficult to predict through regression, mainly because their values are usually close to zero. However, since $\mathbf{B}$ is traceless, $\lambda_2(\mathbf{B})$ can easily be obtained from the predictions of the other two eigenvalues. Hence, tensor $\mathbf{B}$ can be approximated as:

$$\mathbf{B} \approx \mathbf{Q}_A \tilde{\mathbf{\Lambda}}_B \mathbf{Q}_A^T \quad (25)$$

where $\tilde{\mathbf{\Lambda}}_B$ is a diagonal matrix with the estimated eigenvalues of $\mathbf{B}$:

$$\lambda_1(\mathbf{B}) = r_1(\lambda_1(\mathbf{A})), \quad (26)$$

$$\lambda_2(\mathbf{B}) = -r_1(\lambda_1(\mathbf{A})) - r_3(\lambda_1(\mathbf{A})), \quad (27)$$

$$\lambda_3(\mathbf{B}) = r_3(\lambda_1(\mathbf{A})) \quad (28)$$

with r_1 and r_3 being the linear estimation functions obtained through regression of observed data. For example, r_1 is computed through linear regression between $\lambda_1(\mathbf{A})$ and $\lambda_1(\mathbf{B})$ extracted from the observed stiffness tensors. For implementation purposes, $\mathbf{B}$ can be computed by modifying the corresponding eigenvalues of $\mathbf{A}$ with the values computed through (26)-(28). In summary, the predicted asymmetric component of the stiffness tensor, $\mathbb{C}_p^a$, is estimated by:

$$\mathbb{C}_p^a := h_a(\mathbf{Q}_A \tilde{\mathbf{\Lambda}}_B \mathbf{Q}_A^T, t\alpha) \quad (29)$$

where h_a is given in (8). The orthogonality between $\mathbb{C}_p^s$ and $\mathbb{C}_p^a$ is guaranteed by the fact that h_a yields completely asymmetric fourth-order tensors.

Algorithm 1 summarizes the process for estimating the stiffness tensor. As aforementioned, the precomputed parameters are related to the mechanical properties of the tissue and can be obtained from regressions using observed data.

Fig. 2 Correlation plots between eigenvalues of tensors **A** and **B** for 264 specimens from the femur. (a) $\lambda_1(\mathbf{B})$ vs. $\lambda_1(\mathbf{A})$. (b) $\lambda_3(\mathbf{B})$ vs. $\lambda_1(\mathbf{A})$.

Algorithm 1 Prediction of stiffness tensor

```

1: function STIFFNESSPREDICTION( $\mathbb{M}$ ,  $\rho$ ) ▷ Uses
   precomputed parameters  $w$ ,  $k$ ,  $\Gamma_0$ ,  $t$ ,  $r_1$  and  $r_3$ 
2:   Generate  $\mathbb{C}_p^s$  through (20)
3:    $(\mathbb{H}, \mathbf{A}, \alpha) = \text{harm\_decomp}(\mathbb{C}_p^s)$  ▷ Harmonic
   decomposition (Backus 1970)
4:    $(\mathbf{Q}_A, \mathbf{\Lambda}_A) = \text{eig}(\mathbf{A})$  ▷ Eigendecomposition of  $\mathbf{A}$ 
5:   Estimate  $\mathbf{\Lambda}_B$  through (26)-(28)
6:   Estimate  $\mathbf{B}$  through (25)
7:   Estimate  $\mathbb{C}_p^a$  through (29)
8:    $\mathbb{C}_p = \mathbb{C}_p^s + \mathbb{C}_p^a$ 
9:   return  $\mathbb{C}_p$ 
10: end function

```

An additional requirement of the stiffness tensor is that it must be positive definite. Since $\mathbb{M}$ is positive definite by construction, the only component that can lead to non-positive definite stiffness tensors is $\mathbb{C}^a$. However, due to the relatively small contribution of this component to $\mathbb{C}$, the risk of losing positive definiteness is small. A stronger condition than positive definiteness of $\mathbb{C}$ is to require $\hat{\mathbb{C}}$ to be positive definite. An easy way to check this condition is to assess the positiveness of the six eigenvalues of $\hat{\mathbb{C}}$, which are also referred to as Kelvin moduli (Yang et al 1999). As a matter of fact, all predicted stiffness tensors of Section 4 complied with this stronger condition, regardless of the used fabric tensor. Consequently, all predicted stiffness tensors of Section 4 are positive definite.

4 Experimental Results

The MIL, GST and GMIL fabric tensors of both second and fourth order were computed for the 264 samples of the femur. As for the GMIL, the von Mises-Fisher kernel has been used (eq. 17). In order to optimize κ for the von Mises-Fisher kernel, values ranging from 1.5 to 4.5 with intervals of 0.1 were tested. These tests showed that $\kappa = 1.8$ yielded the best results for specimens with ρ larger or equal than 0.2, while 2.8 was the most appropriate value for specimens with lower values of ρ . Thus, the following adaptive function has been used for κ in the experiments:

$$\kappa = 2.8 - \frac{1}{1 + \exp(-\tau(\rho - 0.2))}, \quad (30)$$

that is, a sigmoid function centered at $\rho = 0.2$ is used in order to make κ range between 1.8 and 2.8 for high and low values of ρ respectively. τ has been set to 30 in the experiments in order to have a rapid transition between low and high values of κ . Second- and fourth-order fabric tensors have been labeled with the -2 and -4 suffixes respectively for the three fabric tensors. The model presented in the previous section was used for the computed fourth-order fabric tensors. Regarding the computed second-order tensors, the model of Zysset and Curnier (1995) was applied in order to obtain predictions of the stiffness tensor, which is given by:

Table 2 Estimated parameters for the proposed method with fourth-order fabric tensors.

Parameter	Value
k	1.4820
Γ_0	39654.2196
r_1	$0.2651 \lambda_1(\mathbf{A}) + 4.2681$
r_3	$-0.3547 \lambda_1(\mathbf{A}) + 3.3776$
t	-0.0317
$w_{\text{MIL-4}}$	2.1040
$w_{\text{GST-4}}$	-0.5201
$w_{\text{GMIL-4}}$	2.1219

Table 3 Estimated parameters for the Zysset-Cournier method with the tested second-order fabric tensors.

Parameter	MIL-2	GST-2	GMIL-2
λ_0	7332.8657	6152.5107	6996.3621
λ'_0	5732.1110	4809.5264	5469.0657
μ_0	6092.7121	5112.0356	5813.1189
k	1.5992	1.5747	1.5749
l	1.0946	-0.4583	1.0730

$$\mathbb{C}_p = \rho^k \left(\sum_{i=1}^3 ((\lambda_0 + 2\mu_0) m_i^{2l} \mathbf{M}_i \otimes \mathbf{M}_i) + \sum_{i,j=1, i \neq j}^3 (\lambda'_0 m_i^{2l} \mathbf{M}_i \otimes \mathbf{M}_j) + \sum_{i,j=1, i \neq j}^3 (2\mu_0 m_i^l m_j^l \mathbf{M}_i \overline{\otimes} \mathbf{M}_j) \right), \quad (31)$$

where $\otimes$ and $\overline{\otimes}$ are the tensor and double tensor products, ρ is the bone volume fraction, $\mathbf{M}$ is the second-order fabric tensor, $\mathbf{m}_i$ and m_i are the i -th eigenvector and eigenvalue of $\mathbf{M}$ respectively, $\mathbf{M}_i = \mathbf{m}_i \otimes \mathbf{m}_i$, and $\lambda_0, \lambda'_0, \mu_0, k$ and l are parameters that can be estimated through linear regression after a logarithmic transformation of the observed data (cf., Zysset (2003) for details).¹ It is worthwhile to mention that the aim of the tensor products in (31) is to increase the order of the tensors from second to fourth in order to make them compatible with the order of the stiffness tensor.

Tables 2 and 3 show the estimated parameters for the proposed and Zysset and Curnier (1995) methods respectively for the 264 used specimens. Notice on Table 2 that only w depends on the type of used fabric tensor. It is worthwhile to remark that for GST, the optimal value of both l and w is close to -0.5, which is consistent with the findings in Larsson et al (2014).

Figure 3 and 4 show correlation plots between the 21 entries of the observed and predicted stiffness tensor, both in standard and logarithmic scale of matri-

¹ Notice that the Einstein summation convention is not used in (31) in order to be consistent with the formulation from Zysset (2003).

Table 4 Coefficients of determination R^2 for the prediction of the stiffness tensor and its totally symmetric and asymmetric components for the different fabric tensors.

	$\mathbb{C}_o$ vs. $\mathbb{C}_p$	$\mathbb{C}_o^s$ vs. $\mathbb{C}_p^s$	$\mathbb{C}_o^a$ vs. $\mathbb{C}_p^a$
MIL-4	0.96254	0.96299	0.65477
GST-4	0.97876	0.97904	0.67566
GMIL-4	0.99182	0.99210	0.69696
MIL-2	0.97562	0.97581	0.39149
GST-2	0.96732	0.96732	0.40859
GMIL-2	0.99288	0.99347	0.35040

ces respectively, for the six estimated fabric tensors. As shown, the three types of tensor have strong correlations with the observed stiffness tensor, with R^2 larger than 0.962 in all cases. Moreover, the correlations on Figure 3 for the fourth-order and second-order tensors are very similar. As shown, the best correlations in both figures are attained by the GMIL tensors.

Table 4 shows R^2 for the predicted totally symmetric and asymmetric component when they are considered independently. As shown both models yield strong correlations for the whole stiffness tensor and its totally symmetric component. In turn, correlations for the asymmetric component are largely improved by the proposed model, although they are still far away from those attained for the totally symmetric component. As shown, errors in the estimation of the asymmetric component have little impact in the estimation of the whole tensor due to the fact that its size is usually negligible with respect to the totally symmetric one.

The following measurements were used for assessing the accuracy of the estimation of the stiffness tensor. First, the Frobenius error, which is given by:

$$E_F = \frac{\|\mathbb{C}_o - \mathbb{C}_p\|_F}{\|\mathbb{C}_o\|_F}, \quad (32)$$

where $\mathbb{C}_o$ and $\mathbb{C}_p$ are the observed and predicted stiffness tensors respectively. This type of error was also used in Zysset (2003) and Gross et al (2013). Taking advantage of the positive definiteness and symmetry of $\hat{\mathbb{C}}$, the geodesic distance between the tensors can also be used as a measurement of error. This error can be computed as (Moakher 2005; Moakher and Norris 2006):

$$E_G = \frac{\left\| \text{Log} \left(\hat{\mathbb{C}}_p^{-\frac{1}{2}} \hat{\mathbb{C}}_o \hat{\mathbb{C}}_p^{-\frac{1}{2}} \right) \right\|_F}{\|\text{Log}(\hat{\mathbb{C}}_o)\|_F}. \quad (33)$$

Moreover, the error of the norm of the tensor has also been considered:

$$E_N = \frac{\sqrt{(\|\mathbb{C}_o\|_F - \|\mathbb{C}_p\|_F)^2}}{\|\mathbb{C}_o\|_F}. \quad (34)$$

E_G and E_N are related to errors in the shape and size of the tensors respectively, while E_F considers both features.

Fig. 3 Correlation plots of the 21 entries of the observed and predicted stiffness tensor for the 264 specimens using the six tested fabric tensors.

Table 5 shows the errors of the tested fabric tensors for the 264 specimens of the femur. This table compares not only the prediction of the complete stiffness tensor (which is the most important from an engineering point of view) but also its totally symmetric and asymmetric components. As shown, stiffness tensors obtained from the GMIL second-order fabric tensor have the best performance. In turn, GST performs worse. The obtained errors for the asymmetric part are especially high for the three tensors. As for the fourth-order model, GMIL-4 is much better than MIL-4. As in the second-order case, GST also have the worst performance of the three tensors. As expected, the error of the norm is very similar for the three tensors. This is due to the fact that the three tensors share the same parameters k and I_0 which, at a large extent, determine the size of the predicted stiffness tensor.

Comparing the two prediction models, the table shows that the Frobenius error is improved with the use of the fourth-order model for the GMIL and MIL, while the geodesic error is slightly better with the second-order model. In addition, the error of the norm is markedly improved by the fourth-order model, regardless of the used fabric tensor. A noticeable improvement is obtained in the asymmetric component of the

tensor with the fourth-order model. Although the errors for this component are still high, the effect in the errors of the complete predicted stiffness tensor is limited due to the fact that the asymmetric component is small compared to the totally symmetric one. All the differences between errors for the complete stiffness tensor are statistically significant based on paired t -tests at the 1% significance level, except for E_F between MIL-2 and MIL-4 ($p = 0.07766$), E_G between GMIL-4 and MIL-2 ($p = 0.1038$), all E_N of the fourth-order model ($p \geq 0.90$) and E_N between GMIL-2 and MIL-2 ($p = 0.0164$).

In order to assess the performance of the methods with respect to ρ , Figure 5, shows the errors of the complete stiffness tensor computed independently for specimens with low (below 0.2) and high values of ρ (higher or equal to 0.2). As shown, all methods perform better for specimens with $\rho \geq 0.2$ under all error measurements. The best method in both scenarios is GMIL-4 with a mean E_F , E_G and E_N of 7.82%, 1.36% and 3.18% for high ρ and 17.85%, 4.78% and 7.23% for low ρ respectively. As a reference, MIL-2 yields a mean error of 12.92%, 1.69% and 3.87% in the first case and 20.71%, 4.68% and 11.99% in the second case respectively.

Fig. 4 Correlation plots of the 21 entries of the logarithms of both observed and predicted stiffness tensor for the 264 specimens using the six tested fabric tensors.

Fig. 5 Box plots of E_F , E_G and E_N computed for low (below 0.2) and high (higher or equal to 0.2) values of ρ for all tested methods. The two groups are indicated with (L) and (H) for low and high values of ρ respectively. The number of specimens with low and high values of ρ are 160 and 104 respectively.

Table 5 Mean and standard deviation (in parenthesis) of E_F , E_G and E_N computed on 264 specimens from the femur. E_G cannot be computed for the asymmetric component of the stiffness tensor since it can be non-positive definite.

	$\mathbb{C}_o$ vs. $\mathbb{C}_p$			$\mathbb{C}_o^s$ vs. $\mathbb{C}_p^s$			$\mathbb{C}_o^a$ vs. $\mathbb{C}_p^a$	
	E_F (%)	E_G (%)	E_N (%)	E_F (%)	E_G (%)	E_N (%)	E_F (%)	E_N (%)
MIL-4	17.13 (5.40)	3.99 (2.17)	5.63 (5.14)	16.92 (5.36)	3.89 (2.16)	5.62 (5.11)	79.84 (59.60)	37.25 (40.62)
GST-4	21.27 (9.38)	5.05 (3.16)	5.63 (5.14)	21.11 (9.35)	5.00 (3.16)	5.62 (5.11)	81.55 (60.30)	37.25 (40.62)
GMIL-4	13.90 (6.66)	3.43 (2.33)	5.63 (5.14)	13.67 (6.59)	3.37 (2.32)	5.62 (5.11)	77.65 (59.60)	37.25 (40.62)
MIL-2	17.65 (7.26)	3.50 (2.15)	8.79 (7.19)	17.31 (7.33)	3.45 (2.11)	8.77 (7.17)	89.69 (23.11)	49.74 (20.54)
GST-2	19.56 (8.66)	3.78 (2.27)	11.34 (8.97)	19.29 (8.70)	3.74 (2.22)	11.34 (8.99)	89.39 (22.25)	50.50 (19.15)
GMIL-2	15.09 (7.11)	3.29 (2.11)	8.29 (6.09)	14.70 (7.16)	3.19 (2.11)	8.24 (6.07)	90.81 (22.40)	49.97 (20.22)

Regarding the computational cost, computing fourth-order fabric tensors is marginally more expensive than computing second-order ones. Actually, the most expensive steps are the estimation of the gradient and the convolutions for obtaining the ODF through (13)-(15). As an example, the computational cost for computing the two types of fabric tensor on a specimen of $296 \times 296 \times 296$ voxels in a standard PC is 3.84 and 3.83 seconds respectively. If both types of tensors are computed, the total cost is 3.87 seconds, since the ODF is only computed once.

5 Discussion

In this paper we have proposed a model to compute stiffness tensors based on fourth-order fabric tensors and bone density. This model assumes that the totally symmetric component of the stiffness tensor and fabric tensors are connected, and that the asymmetric component of the stiffness tensor is linked to its totally symmetric one. The results of the previous section show that the Frobenius error and the error of the norm are reduced by using fourth-order tensors. This means that fourth-order fabric tensors can be a good alternative to use instead of the more expensive μ FE methods.

From the results, the fourth-order GMIL fabric tensor had the best performance among the tested fabric tensors. By considering MIL-2 as a reference, GMIL-4 is able to reduce the errors E_F , E_G and E_N by 3.75%, 0.07% and 3.16% respectively. The reductions are consistent for both, specimens with high ρ (5.1%, 0.33% and 0.69% respectively) and low ρ (2.86%, -0.10%, 4.76% respectively). From these differences, only E_G for low ρ is not statistically significant ($p = 0.1552$).

Notice in Figure 3 that despite MIL-2 and GMIL-2 yield slightly better correlations than MIL-4 and GMIL-4 respectively, the fitting lines yielded by the fourth-order models are less biased. In turn, the second-order models on Figure 4 are slightly better than the fourth-order ones but their fitting lines of models are not biased in this case. We think these observations are re-

lated to the improvements of the proposed model with respect to E_F and E_N but not to E_G .

GMIL is able to yield better results thanks to the fact that this fabric tensor can be optimized by selecting the most appropriate kernel. In particular, the parameter of the von Mises-Fisher kernel was optimized to get better predictions. In turn, the bad results yielded by the GST might be attributed to the use of an excessively narrow kernel (the impulse function) which can make this tensor less stable compared to tensors that use broader kernels. For example, the half-cosine or the von Mises-Fisher kernel can do a better job smearing out gradient information coming from regions that do not significantly contribute to the apparent mechanical competence of the tissue.

It is necessary to state that the proposed model and the one from Zysset and Curnier (1995) are closely related. First, parameter k plays the same role in both models: to describe the influence of the bone volume fraction on the size of the predicted stiffness tensor. Second, both w in the proposed and l in Zysset and Curnier (1995)'s model deal with power laws between the fabric tensor and the stiffness tensor, although in a slightly different fashion. Finally, Γ_0 in the proposed model describes the size of the predicted stiffness tensor that is not related to the bone volume fraction. This means that Γ_0 can be seen as an intrinsic property of the material matrix. Correspondingly, the combination of λ_0 , λ'_0 and μ_0 determines the size of the stiffness tensor in the Zysset and Curnier (1995)' model.

However, there are also important differences between both models. First, the proposed model is strongly based on a pure proportionality relationship between the fabric and stiffness tensors, which is described through (20). In turn, by using three different parameters: λ_0 , λ'_0 and μ_0 , Zysset and Curnier (1995)'s model provides more freedom to deviate from such a proportionality relationship. While the proposed method yields consistently better results related to the size of the predicted stiffness tensor (see E_N on Table 5 and Fig. 5) which are independent of the used fabric tensor, the second-order-based model yields slightly smaller errors in the logarithmic scale than the pro-

posed one (see E_G on Table 5 and Fig. 5), likely because of this freedom. Unfortunately, a consequence of giving such a freedom is that, unlike Γ_0 ; λ_0 , λ'_0 and μ_0 do not depend only on the properties of the material matrix, but also on the used fabric tensor as shown on Table 3.

An advantage of the proposed model compared to the model by Zysset and Curnier (1995) is that it was not assumed any type of material symmetries. Thus, all entries of the stiffness tensor are available. This means that the difference in performance between the two tested models is expected to be larger in specimens (or other applications) where the orthotropic symmetry assumption is not valid. If necessary, approximations to different symmetries can be performed as a post-processing step, e.g., by following the methodology by Moakher and Norris (2006).

The proposed model requires 8 parameters: 3 for estimating the totally symmetric component of the stiffness tensor and 5 for estimating the asymmetric one. However, from the results, the asymmetric component is usually very small with respect to the totally symmetric one. This means that this component could safely be neglected, reducing in that way the number of parameters to 3: Γ_0 , k and w , from which only w depends on the used fabric tensor. As a comparison, Zysset and Curnier (1995)'s model requires 5 parameters.

As already mentioned, the estimation of the asymmetric component is based on the assumption that $\mathbb{C}^a$ and $\mathbb{C}^s$ are related. Although the errors yielded by the proposed method are in the range between 10% and 15% smaller than the ones obtained by Zysset and Curnier (1995)'s method, they are still very high. These errors are similar both in specimens with high and low ρ . We think that more research is necessary to get a better understanding of the relationship between the totally symmetric and asymmetric components. However, it is important to note that the errors on the estimation of the asymmetric component have little effect on the total error, due to the fact that the asymmetric component is very small compared to the totally symmetric one.

Although fabric tensors of order higher than four can be computed by generalizing (18) or using the methodology by Kanatani (1984), it is unlikely that this procedure can yield better results due to the fourth order nature of the stiffness tensor. For example, any model based on sixth-order fabric tensors will require tensor contractions to make the order compatible with the one of the stiffness tensors. In this manner, any possible advantage of using sixth order vanishes due to the tensor contraction operation.

One limitation of this study is that we only compared three fabric tensors. Thus, it is interesting to assess whether the reported errors can further be reduced by changing the fabric tensor. We think that the pure proportionality relationship between fabric tensors and the totally symmetric component of the stiffness tensor should be valid in most cases. However, the current state-of-the-art fabric tensors, including the GMIL tensor, might not be appropriate for all specimens, especially those with low values of ρ . We expected that the proposed model will yield better results by using more appropriate fabric tensors.

In order to show this, Figure 6 plots the shape of the observed and predicted stiffness tensors using the two models and the three fabric tensors for a sample with a high ρ (27.22%) and a sample with low ρ (6.05%). These two samples represent cases where all methods yield satisfactory and unsatisfactory results respectively. As shown, GMIL performs better than other fabric tensors in both cases. However, it also shows that while the methods are accurate for specimens with high ρ , their performance is reduced for low ρ . This shows the need for better fabric tensors for low ρ .

In this line, ongoing research includes assessing whether or not disregarding trabecular termini in the fabric tensor as we proposed in Moreno and Smedby (2014) can lead to more accurate results, while keeping the pure proportionality relationship between fabric tensors and the totally symmetric component of the stiffness tensor. Moreover, fabric tensors estimated through machine learning approaches will be considered (Lekadir et al 2015; Hazrati Marangalou et al 2013). In addition, we plan to apply the proposed model to trabecular bone from different skeletal sites and with different degrees of osteoporosis. Since the tested fabric tensors can be applied to images acquired in vivo, it is also planned to assess the accuracy of the proposed model under such conditions.

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Fig. 6 Shape of the observed (in green) and predicted (in red) stiffness tensors for the samples depicted at the left. The samples have a ρ of 27.22% (top) and 6.05% (bottom). E_F is shown in parenthesis.

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