On the statistical tests over Fennoscandian GNSS/levelling networks

Sedigheh Zoghi

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School of Architecture and the Built Environment
Royal Institute of Technology (KTH)
Stockholm, Sweden

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Abstract

In Geodesy and Surveying we work with a large amount of observations which always contains different types of errors. The errors decrease the quality of the observations and propagate to the results. Therefore, detection and removing the gross errors are of vital importance. The Global Navigation Satellite Systems (GNSS) can be used to measure the ellipsoidal height and by subtracting an existing geoid height from that, the orthometric height can be determined. There is a simple linear relation amongst these triple heights, which cannot be fulfilled due to the presence of different types of error. One of the important sort of these errors is gross errors.

This study concerns about investigation and detection of blunders or gross errors on the 4346 GNSS/leveling points over Fennoscandia. Each country has its own data set with specific precision. The well-known gravity model EGM08 is used to compute the geoid heights with respect to WGS84 reference ellipsoid. We have a large amount of data and we expect that their errors follow the normal distribution. The main aim of this thesis is to apply some data screening methods both before and after adjustment process in such a way that the normal distribution of the data set is achieved by eliminating the erroneous data. This will be done by performing the pre- and post-adjustment data screening. For the pre-adjustment we performed data filtering, test of normality of observations and test of their variances for the GNSS/leveling data over Sweden, Denmark, Norway and Finland. We used the 4-, 5- and 7-parameter corrective surfaces for modelling the systematic trends of the differences between the EGM08 geoid model and the ellipsoidal and orthometric height differences. The test of normality of residuals, global test of variance, Baarda’s data snooping and Tau test will be performed after the removal of the trends. Numerical studies show that the GNSS/leveling data of Sweden, Denmark and Finland are of good qualities, but the claimed errors for the data are rather optimistic. The situation was complicated for the data of Norway and we could not see the normality of the data and even the claimed accuracies seem to be optimistic.
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1. Introduction

In Geodesy, measuring quantitates directly is not always possible. Therefore, we should find some mathematical relations between them and another set of quantities, which are measurable. In addition, we always try to measure more data than what is needed in practice to increase the precision of our measurements and the corresponding quantities being estimated. Least-squares (LS) is a method for solving such problems, but the main issue is that the errors of data should be of random type; otherwise, LS solution provides incorrect results. Therefore, it is important that before any LS computation we make sure that the errors of our data are random and they follow the normal distribution.

Global Navigation Satellite System (GNSS)/levelling network is a set of points with known geographical coordinates with three types of heights at these points; a) orthometric heights, b) ellipsoidal heights and c) geoidal heights. The orthometric height ($H$) is the distance between any point at the surface of the Earth and the geoid along the plumb line, ellipsoidal or geodetic height ($h$) is the distance between any point and the reference ellipsoid, and the geoidal height ($N$) which is geoid-ellipsoid separation. There is a fundamental linear equation amongst these heights (Hofmann-Wellenhof and Moritz 2005):

$$h - H = N$$ (1.1)

Figure 1.1 illustrates the measurements of the orthometric height, which is the aim of the GNSS/levelling. It means that when the geoid height ($N$) is available, we can simply use a GNSS receiver for measuring the ellipsoidal height ($h$) and by subtracting $N$ from $h$ the orthometric heights ($H$) is derived. So the lengthy, costly and time consuming processes of the traditional methods of levelling can be avoided if $N$ is known over a territory.
Figure 1.1. Relation between triple hights: orthometric ($H$), ellipsoidal ($h$) and geoid ($N$)
(http://www.anobanini.net/forum/showthread.php)

However, there are errors in all of these data, which causes that Eq. (1.1) is not satisfied. Therefore, it is necessary to investigate these data prior to using them for analysis of $N$ and the other types of heights.

The main important difference between GNSS/levelling networks with respect to the other geodetic networks is related to its simple mathematical model and very large number of data. There is not a rigorous mathematical model to estimate some quantities. In fact, the unknowns are directly measured by GNSS receiver and levelling methods and the geoid is already available. The condition (1.1) is not satisfied in practice due to existence of some systematic effects in the data and it is not easy to say what they are. In fact, we should be able to do condition adjustment if systematic trends do not exist, but in practice we use some mathematical models to remove them.

When a local geoid model is gravimetrically determined for a specific area, it needs to be evaluated by external and independent sources of data. GNSS/levelling data can be used for such a purpose as by subtracting the geodetic heights from the orthometric heights another geoid model is derived, which should be consistent with the gravimetrically-determined one. The differences between these two geoid models can give ideas about the quality of the gravimetric geoid model. Therefore, it is of vital importance to check the quality of the GNSS/levelling data as they play the role of a reference for the geoid model and remove the erroneous ones. The cleaned GNSS/levelling data are useful for evaluation of the next generations of the geoid models over Fennoscandia.
This thesis consists of 4 chapters, in Chapter 1, the theoretical background about the pre-adjustment data cleaning including the data filtering based on confidence intervals, goodness of fit test of the data, testing the normality of the misclosures, and test of variance are explained and discussed. In addition, different LS adjustment models are generally introduced and it is shown how they are implemented for the GNSS/levelling data. Finally, the post-adjustment data screening consisting of test of normality of the residuals, global test of the a posteriori variance factor, data snooping of Baarda and τ-test are introduced. In Chapter 3 all of the presented theories in Chapter 2 are applied for the GNSS/levelling data over Fennoscandian countries of Sweden, Denmark, Norway and Finland. Finally, our conclusions and recommendations will be presented in Chapter 4.
2. Theory

This chapter will discuss about the theories for analysis and process of the GNSS/levelling data with a special emphasis on the statistical tests and adjustment. The numerical studies about these theories will be presented in Chapter 3. This chapter is divided into 4 parts, the first one is an overview on errors and the second one is the pre-adjustment data cleaning which consists of goodness of fit test, data filtering, and test on the variance of the data. The next section deals with different adjustment models and their connections to the GNSS/levelling data. Later on the post-adjustment data screening will be presented and discussed.

2. 1. An overview on errors

Error is the deviations of an observation from its true value. In Geodesy, we often work with a large amount of observations which all contain errors of different types. Some of these errors can be removed from the observations and some cannot. However, the most important and may be the most serious one is the gross errors, which will destroy the observations and deviate them from their true values significantly. In the following part, we will discuss and explain the nature and sources of different type of errors.

2. 1.1 Gross errors

Gross errors or blunders occur through the human carelessness and/or the instrument malfunctioning. These errors are rather big and recognisable based on the surveyor's experiences. However, when a large number of data are measured detection of such errors becomes complicated. Some human source mistakes, like recording wrong values, misreading, or observing the wrong mark are classified in the category of gross errors. We
cannot treat with this error by the statistical or mathematical methods and the only way is to pay more attention in the data collection procedures and checking the data prior to their use.

2. 1.2. Systematic errors
The systematic errors are due to the nature or the instrument. They are removable, in three different ways, and we can handle this type of error by a) mathematical modelling, b) calibration of the instrument and c) using an observation technique so that our results will not be affected by the systematic errors. Some phenomenae like extension or contraction of tape due to the changes in the weather temperature, the effect of Earth curvature in levelling, atmospheric refractions and so on can be mathematically modelled. It will be enough to compute the effect of the errors and remove them from the observations. Calibration is mainly related to the instrument which is used. If the instrument does not work properly we can take it in the calibration office and perform some measurements and compare them with the corresponding ones which we already have them in the office. If the instrument cannot deliver the measurement with the acceptable level of errors the instrument will be mechanically adjusted by especial tools. After the calibration of the instrument we can safely work with it without worrying about the systematic errors due to the instrument. The last technique is more practical and may be more economical and it depends on the technique of observation. Collimation error is a well-known systematic error in the surveying instruments. This error can also be modelled mathematically or removed by calibrating the instrument. However, neither of these methods is economical as we have to spend time and energy for doing such processes. In levelling we are always recommended putting the levelling instrument in the middle of the two levelling rods, because the height difference is a differential type of measurement which is obtained by subtracting the foresight from the backsight. If the instrument has a collimation error and its line of sight is not exactly horizontal, the same error occurs in both readings and by subtracting them from each other this error is cancelled out. In measuring the angle using theodolites, we are asked to read the direction in two different faces of left- and right circles, in this case, if the theodolite contains the collimation error, when e.g. we read the direction in the left-circle face we read smaller / larger value for the direction and when we do it in the right-circle face we read it larger / smaller. By taking the average of the readings the effect of the collimation error will be removed because the effect appears with positive sign in one reading and negative in the other so that they cancel each
other. Another well-known example is the use of double frequencies in the GNSS so that the results are least-affected by the Ionospheric refraction. Also, using star pairs is another issue to remove the effect of the atmospheric refraction from angle observations in geodetic astronomy.

Based on what have been explained so far about the systematic errors, we can summarise the characteristic of these errors:

1. They are due to the nature and/or instrument
2. The systematic error is a one-way error, which means that it is either positive or negative.
3. This error is removable by three techniques of formulation, calibration and observation techniques.

2. 1.3 Random errors
Once we have made sure that the gross and the systematic errors have been successfully removed from the observations, the remaining error is nothing else than the random error. This error can be considered due to the human weakness for example the human eye cannot recognise the exact value of the reading. However, this error always exits even if the contribution of the human is reduced in the measurements. It is accidental, sometimes larger and sometimes smaller than a specific value without any regulation. Random errors are small in nature. We can reduce the effects of these errors by the statistical procedures. The characteristics of this error can be summarised as:

1. It has no specific source, but it could be limitation of the human's ability in the measurements.
2. It is a two-way error, positive and negative.
3. It cannot be removed, but since it is stochastic in nature, we can reduce it by repeating the observations.
2.2. Pre-adjustment data screening

All processes that should be done for cleaning the data before using them in the mathematical models for computations are called pre-adjustment data screening. Here, we mention some methods for this purpose such as: a) visualisation, b) filtering the measurements, c) finding external mathematical models, d) test of the misclosures, e) test on the normality of the observations and f) test on the variance of the data. Below we will describe these methods briefly.

2.2.1. Visualisation

Blunder detection based on visualisation is highly dependent on the surveyor experiences working with data. This means that the surveyor will look at the numbers and guess which one of them may contain blunder. Also, he/she can make plots or maps of the data to see that if they are consistent. For example, the map of gravity data can simply show if they contain blunders, having a priori knowledge about the regime of the gravity in the study area is of vital importance which depends on the experiences of the surveyor by working on this type of data in the study area. Today, according to the advances in technology and instrument automation the risk of occurrence of blunders in the recording of the observation is rather low. In short, the efficiency and success of this method is highly related to the level of experience of the surveyor working in the area.

2.2.2. Data filtering

If we are sure that all systematic effects have been successfully removed from the observations we can use the statistical methods for filtering the data. It should be mentioned that in the first step we have to try to remove the blunder by the visualisation technique at the first step. If the magnitude of the erroneous observation is large it can simply be detected by visualisation.

All random errors are of stochastic nature and should have normal distribution which means that the histogram and polygon plot of the errors are bell-shaped. By increasing the number of repetition and correspondingly the number of data this plot will be smoother and smoother so that one can consider a mathematical model for that when this number goes to infinity. This model is called density function which means that probability of occurrence of the true value
of the measurements should be around the mean value of them and by increasing the number of repetition to infinity the mean value moves towards the true value. Considering a special distance from the mean value, which is called the confident interval, can be used as a method to filterise the data. The following steps should be done for this purpose:

1. The mean value of the measurements is computed
2. The standard deviation of the measurement should be derived
3. Confidence regions should be organised to test the data.

Here, we emphasise that this method is suitable when one quantity is repeatedly measured and it cannot be used when we have more than one repeatedly-measured quantity.

The area under the normal distribution function is 1 which corresponds to the probability of 100%. This means that a normal observation has the probability of 100% to be between $-\infty$ and $\infty$. However, if we limit the confidence interval to be equally placed around the mean value by $k$, the percentage of occurrence of the data is reduced. When $k = 3$ it means that the probability of having a normal observation is 99.7% which is very close to 1. This probability is useful to find different confidence intervals for example when $k = 1$ and 2 it corresponds to the probability of 68.3% and 99.7%. It can also be possible to find the confidence interval based on the probability, for example, the probability of 95% is achieved when $k = 1.96$.

Equation (2.1a) shows the mathematical model for the probabilities for $k = 1, 2$ and 3 (cf. Fan 2003):

$$ P(\bar{T} - ks < l_i < \bar{T} + ks) = \begin{cases} 68.3\% & k = 1 \\ 95.4\% & k = 2 \\ 99.7\% & k = 3 \end{cases} $$

(2.1a)

This equation can be used to detect possible gross errors in the observations. $l_i$ is an observation value, $\bar{T}$ and $s$ are mean value and standard deviation of the data set respectively. Assume that the quantity $l$ has been measured $n$ times, in this case, the mean value $\bar{T}$ and standard deviation $s$ of the data set are, respectively, computed by:

$$ \bar{T} = \frac{1}{n} \sum_{i=1}^{n} l_i $$

(2.1b)
and

$$s = \pm \sqrt{\frac{\sum_{i=1}^{n} (\bar{t} - t_i)^2}{n-1}}.$$  \hfill (2.1c)

2.2.3. Test of misclosures

This method can be done depending on the type of the network and data that we are working with, but not always possible. For example, in a triangulation network we know that the sum of angles of each triangle should be 180°. This is an external mathematical model for checking the angles. If this condition does not hold, therefore, we will have misclosure error for the sum of the angles and when it is in at acceptable level, which can be checked using statistical test. If it is large and unacceptable, therefore, one, two or three angles can be erroneous. In order to find which one we have to use other conditions with the other angles so that we can organise another condition having common angles. This method does not work for one quantity, as we have to find mathematical models amongst the measurement and it is hard to find such geometric condition for one single observation or may be even impossible.

2.2.4. Test of normality of observations

The observations should be normally distributed priori to being used in a LS adjustment process. One method to check the normality is to plot the histogram of the observations and check if it is symmetric and approximately bell-shaped. In practice, it is not easy to make sure that the observations have normal distribution even if they have such histograms. However, we can always use the $\chi^2$ goodness of fit test for checking the normality of the observations. In order to do that we use the following statistic (Vanicek and Krakiwsky 1987):

$$y = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} < \chi^2_{df, 1-\alpha} \hfill (2.2a)$$

where $O_i$ and $E_i$ are the observations and its expected values in each bin of histogram, respectively and $n$ stands for the number of bins and $df$ is the degree of freedom and $\alpha$ stands for the significant level of the test. Here, the null hypothesis is that the observations have normally distribution and the alternative hypothesis is that they have not. $y$ should be smaller than $\chi^2_{df, 1-\alpha}$, if the $y$ value computed from the histogram of the data and the expected values
derived from the normal distribution function based on the observation mean and standard deviations. If this condition fulfills then the null hypothesis cannot be rejected and the observation set has probably normal distribution. Today, there is lots of statistical software which are able to test the normality of the data based on Eq. (2.2a).

### 2.2.5. Test of variance

In some cases a set of data is given with a priori specified error. This means that the deviation of the observations from the mean value of the set should be consistent with the claimed error for the observations. If the test passed it means that the claimed variance, the squared error, is consistent with the estimated variance of the data, and when the test is failed it means that either there is blunder amongst the data or the claimed variance is not consistent with the set. The following confidence interval is considered as the acceptable levels for the claimed variance:

\[
\frac{(N-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(N-1)s^2}{\chi^2_{1-\alpha/2}}
\]

(2.2b)

where \(s^2\) is the variance that has been estimated from the data and \(N\) stands for the number of the data, \(\sigma^2\) is the claimed variance which is given for the data.

### 2.3. Least-squares adjustment models

After removing all possible blunders we can use the cleaned measurements for computing our desired quantities. To do so, we need mathematical models, because our unknowns which are usually coordinates cannot be measured directly. Therefore, we have to measure other quantities so that we can connect them to the unknowns by known mathematical models (Vanicek and Krakiwsky 1987). On the other hand, we know that for reducing the influence of the random errors we have to repeat the measurements and try to estimate the unknown parameters by using more than necessary observations. In such a case, the probability of reaching to the true value of the unknowns is higher.

Generally, the mathematical formulae leads to systems of equations of three types:
1. They have unique solution meaning that the number of observations and unknowns are equal.

2. They have less number of observations than the unknowns in this case the model is called under-determined and there are infinite numbers of solutions for such a model.

3. When the number of observations is larger than the unknowns and in this case the model is called over-determined and there is no solution to that. The unknowns can be estimated by the statistical method for such model.

According to our explanations above, we can simply conclude that the third types of the models is suitable for our purposes because we always try to have more observations than the unknowns to increase probability of obtaining some values for the unknowns which are close to their true values. In Surveying and Geodesy, three well-known mathematical models are used. a) Adjustment by element model (Bjerhammar 1973), which is also named as Gauss-Markov model (Koch 1999), b) condition adjustment model and c) combined model (Bjerhammar 1973), which is also named mixed model by Koch (1999) or Gauss-Helmert model (cf. Koch 1999, Sjöberg 1993). In the following we briefly explain these models.

2.3.1. Gauss-Markov model

Let the following mathematical model:

\[
Ax = L - \varepsilon, \quad E\{e\} = 0 \text{ and } E\{\varepsilon\varepsilon^T\} = \sigma_0^2 Q, \tag{2.3a}
\]

where \(A\) is an \(n \times m\) coefficient matrix, \(x\) is an \(m \times 1\) vector of unknowns, \(L\) is the \(n \times 1\) vector of observations and \(\varepsilon\) is the vector of observation errors, \(Q\) stands for the co-factor matrix of the observations and \(\sigma_0^2\) is the a priori variance factor, or the variance of unit weight. \(E\{.\}\) stands for the statistical expectation operator.

The Gauss-Markov model (2.3a) is well-known for solving different LS problems. The LS solution of this model is (Cooper 1987):

\[
\hat{x} = (A^TQ^{-1}A)^{-1}A^TQ^{-1}L \tag{2.3b}
\]

and the Variance-Covariance Matrix (VCM) of the estimated parameters is:

\[
C_{\hat{x}} = \sigma_0^2 (A^TQ^{-1}A)^{-1}. \tag{2.3c}
\]
The *a posteriori* variance factor can be estimated by:

\[ \hat{\sigma}_0^2 = \frac{\hat{\mathbf{e}}^T \mathbf{Q}^{-1} \hat{\mathbf{e}}}{n-m} \]  

(2.3d)

where \( \hat{\mathbf{e}} \) stands for the estimated residuals or the estimated errors with the following formula:

\[ \hat{\mathbf{e}} = \mathbf{L} - (\mathbf{A}^T \mathbf{Q}^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Q}^{-1} \mathbf{L} \]  

(2.3e)

We know that the choice of a priori variance factor has no influence in the estimated parameters, but the VCM of these parameters has a direct relation with it and should be scaled by the estimated *a posteriori* variance factor. The scaled VCM of the estimated parameters will be:

\[ \hat{\mathbf{C}}_x = \hat{\sigma}_0^2 (\mathbf{A}^T \mathbf{Q}^{-1} \mathbf{A})^{-1} \]  

(2.3f)

We normally consider that \( \sigma_0^2 \) is equal to 1. This means that we assume that the presented errors of the observations are properly scaled prior to the adjustment procedure. \( \hat{\sigma}_0^2 \) is expected to be close to \( \sigma_0^2 \) if the errors of the observations are consistent with the residuals derived from the adjustment process contributed to \( \hat{\sigma}_0^2 \), otherwise \( \hat{\sigma}_0^2 \) should be multiplied to the co-factor matrix of the estimated unknown parameters so that the VCM matrix is scaled properly with the adjustment process. It should be mentioned that the choice of \( \sigma_0^2 \) has no influence on the estimated unknowns, but, it is important for estimating their errors. In fact, the test statistic presented in Eq. (2.2b) is a special case of this test in which \( \sigma^2 \) is the same with \( \sigma_0^2 \) and \( \hat{\sigma}_0^2 \) with \( s^2 \) and the Gauss-Markov model is the same with the formulae (2.1b) which is nothing else that the mean value operator. Equation (2.3d) is in fact the generalised form of Eq. (2.1c).

### 2.3.2. Condition adjustment model

Sometimes we can find a mathematical relation amongst observations, which are called the condition models. Such models usually follow a geometric condition, but due to the existence of different types of errors such a condition is not satisfied. The problem is to adjust the
observations in such a way that these geometrical conditions are fulfilled. The corrections
which should be added to the observations for correcting them should satisfy the LS property.
A condition adjustment model can be mathematically presented in the following matrix form:

\[ \mathbf{B}(\mathbf{L}-\mathbf{e}) = \mathbf{c} \quad , \quad \mathbb{E}\{\mathbf{e}\} = 0 \quad ; \quad \mathbb{E}\{\mathbf{e}\mathbf{e}^T\} = \sigma_0^2 \mathbf{Q} \quad (2.4a) \]

where \( \mathbf{B} \) is the coefficient matrix of the observations, \( \mathbf{c} \) is the constant vector and \( \mathbf{e} \) stands for
the random errors.

Eq. (2.4a) is very different from Eq. (2.3a) which was the adjustment by elements model as,
in Eq. (2.3a) there are some unknown parameters to be estimated first and after that by using
the estimated parameters the observations are adjusted. However, there is no unknown
parameter to be estimated in the condition adjustment model (2.4a). In fact, here, the
observations are adjusted prior to estimating any unknown by them. The LS solution of Eq.
(2.4a) is:

\[ \hat{\mathbf{e}} = \mathbf{Q}^{\mathbf{B}}(\mathbf{BQ}^{\mathbf{B}})^{-1}\mathbf{w} \quad \text{and} \quad \mathbf{w} = \mathbf{c} - \mathbf{B}\mathbf{L} \quad (2.4b) \]

the adjusted observation vector is:

\[ \hat{\mathbf{L}} = \mathbf{L} - \hat{\mathbf{e}} \quad (2.4c) \]

The VCM of the estimated residuals and the \textit{a posteriori} variance factor \( \hat{\sigma}_0^2 \) are computed by:

\[ \hat{\mathbf{C}}_e = \sigma_0^2 \mathbf{Q}^{\mathbf{B}}(\mathbf{BQ}^{\mathbf{B}})^{-1}\mathbf{BQ} \quad \text{and} \quad \hat{\sigma}_0^2 = \frac{\mathbf{w}^T(\mathbf{BQ}^{\mathbf{B}})^{-1}\mathbf{w}}{df} \quad (2.4d) \]

where \( df \) stands for the degree of freedom which in the condition adjustment model it is the
same as the number of condition equations.

2.3.3. Gauss-Helmert model

Gauss-Helmert model is the combined model of Gauss-Markov and condition model. In other
words, there are unknown parameters in the model to estimate as condition models amongst
the observations. In the matrix form, we can write this model by:

\[ \mathbf{A} \mathbf{x} + \mathbf{B}(\mathbf{L}-\mathbf{e}) = \mathbf{c} \quad \text{or} \quad \mathbb{E}\{\mathbf{e}\} = 0 \quad ; \quad \mathbb{E}\{\mathbf{e}\mathbf{e}^T\} = \sigma_0^2 \mathbf{Q} \quad (2.5a) \]
where A is the coefficient matrix of the unknown parameters and B that of the observations, they are also called the first and second design matrices, x the vector of unknowns and L that of observations and finally c is the constant vector. The LS solution of the unknown parameters is:

\[ \hat{x} = (A^T C^{-1} A)^{-1} A^T C^{-1} w \]  

(2.5b)

Once the unknown parameters x are estimated the residual vector for adjusting the observations will be:

\[ \hat{\varepsilon} = Q B^T C^{-1} w \]  

(2.5c)

the VCM of the estimated parameters will be:

\[ C_\chi = \sigma_0^2 (A^T C^{-1} A)^{-1} \].  

(2.5d)

The scaled VCM and the a posteriori variance factor in this model become:

\[ \hat{C}_\chi = \hat{\sigma}_0^2 (A^T C^{-1} A)^{-1} \]  

and

\[ \hat{\sigma}_0^2 = \frac{(w - A\hat{x})^T (BQB^T)^{-1} (w - A\hat{x})}{df} \].  

(2.5e)

2.4. Post-adjustment data cleaning

After pre-adjustment data cleaning and LS adjustment which eliminate large magnitude of gross errors there are still some small gross errors in our data set. The purpose of post-adjustment data cleaning is to determine these data set or results which we have obtained are reliable or not.

2.4.1 Goodness of fit test of residuals

After performing the LS adjustment process and estimating the unknown parameters, we can estimate the residual vector \( \hat{\varepsilon} \), which should have normal distribution. \( \hat{\varepsilon} \) should be tested again to check if they have the normal distribution as well. The mathematical formulae for the goodness of fit test of \( \hat{\varepsilon} \) are the same with those presented for the pre-adjustment data screening. Therefore we do not repeat them here.
2.4.2. Global test of a posteriori variance factor

\( \hat{\epsilon} \) can be used to compute \( \hat{\sigma}_0^2 \) which is also a general form of the definition of variance. The mathematical model of this variance has been already presented for different adjustment models. \( \hat{\sigma}_0^2 \) should be in the following interval:

\[
\frac{\chi^2_{df, 1 - \frac{\alpha}{2}} \sigma_0^2}{(n - m)} < \hat{\sigma}_0^2 < \frac{\chi^2_{df, \frac{\alpha}{2}} \sigma_0^2}{(n - m)} \tag{2.6a}
\]

where \( n \) and \( m \) are respectively the number of observations and unknown parameters, \( \sigma_0^2 \) and \( \hat{\sigma}_0^2 \) are the a priori and a posteriori variance factors. There are normally three reasons if the test of variance is not accepted (Sjöberg 1993):

1. Improper choice of the weight of observations
2. Existence of blunder amongst observations
3. Improper mathematical model

In ordinary geodetic networks when we have the strict mathematical models for the observations, the first two reasons are more likely to consider. However, when a voluntary mathematical model is fitted to the data to model the systematic effects the third reasons should also be considered. In the case, of combined adjustment of the GNSS/levelling networks all three can be the reasons for the failing the global test of variance.

2.4.3. Data snooping

Baarda (1968) proposed a method for post-adjustment data screening which is well-known as data snooping. He assumed that only one erroneous observation exists amongst the observation set and tried to detect it by analysing the standardised residual vector. These normalised residuals should have the normal distribution with 0 mean and variance of 1 (cf. Kavouras, 1982):
\[ |w_i| = \left| \frac{\hat{e}_i}{\hat{\sigma}_i} \right| > \sqrt{F_{1-\alpha_0,1,\infty}} \]  

(2.6b)

where \( \hat{e} \) and \( \sigma_{\hat{e}} \) are respectively the residual and its standard error and \( F_{1-\alpha_0,1,\infty} \) means the value of the Fisher distribution function, with the significant level of \( \alpha_0 \) and degree of freedoms of 1 and infinity. The null hypothesis is that the \( i^{th} \) observation is free of any gross error and if the condition (2.6b) is fulfilled, the null hypothesis is rejected and the \( i^{th} \) observation is flagged for rejection. Baarda (1968) recommended the significant level of \( \alpha_0 = 0.001 \) for data snooping which corresponds to the value \( \sqrt{F_{1-0.001,1,\infty}} = 3.29 \) and this means that \( \hat{e}_i \) which have larger magnitude than 3.29 \( \sigma_{\hat{e}_i} \) are flagged for rejection. The main assumption of the data snooping test is to have only one erroneous observation; therefore, this process should be repeated if there are more than one gross error in the data. If the \( i^{th} \) observation is erroneous it will be removed from the observation set and the adjustment process is repeated without that. Again the standardised \( \hat{e}_i \) are tested based on the condition (2.6b) and if any other observation is flagged for rejection it will be removed from the set and the adjustment is repeated until no erroneous observation can be found in the data set.

2.4.4. Tau Test

In data snooping the variance of the data is assumed to be known, which means that the variances of the observations are properly selected and we can rely on them. If the variances are not very reliable we have to use the \( \hat{\sigma}_0^2 \) instead. This test has been proposed by Pope (1976) which is very similar to the one presented by Baarda’s (1968) but it uses \( \hat{\sigma}_0^2 \) for scaling the standard error of the residuals:

\[ \tau_i = \left| \frac{\hat{e}_i}{\hat{\sigma}_i} \right| > c_\tau \]  

(2.6c)

where \( \hat{\sigma}_i \) is the standard error of \( \hat{e}_i \), scaled by \( \hat{\sigma}_0^2 \). \( c_\tau \) is the value that is extracted from the Tau distribution.

Since the erroneous residual is also used for computing \( \hat{\sigma}_0^2 \) we have to use another strategy to select the significant level. The test will be defined on the maximum value of \( \tau \), as if there is
a gross error in \(i\)th observation its standardised residual should be larger than the rest of \(\tau\). Therefore the significant level of \(\alpha\) is a function of \(n\) independent one-dimensional tests at significant level of \(\alpha_0\) (Pope 1976):

\[
\alpha = 1 - (1 - \alpha_0)^n
\]  

(2.6d)

The solution of Eq. (2.6d) for \(\alpha_0\) will be:

\[
\alpha_0 = 1 - \sqrt[n]{1 - \alpha}
\]  

(2.6e)

Then the hypothesis test is done based on this significant level for each standardised residual. Tau-distribution is rarely mentioned in the text books, but it can be defined by (Pope 1976):

\[
c_{\tau} = \frac{\sqrt{r \times t_{r-1}}}{\sqrt{r - 1 + t_{r-1}^2}}
\]  

(2.6f)

where \(r\) is the degree of freedom. \(t_{r-1}\) is the value obtained from the t-distribution with the significant level of \(\alpha_0\) and degree of freedom of \(r - 1\). It should be mentioned that when the degree of freedom is too large the normal, t- and tau-distribution are more or less the same. For example, for a degree of freedom of 100 the t, Tau and normal-distribution values for the significant level of 0.05 is respectively are 1.984, 1.956 and 1.96 (Kavouras 1982).

2.5. GNSS/levelling networks and their adjustments

The observations in GNSS/levelling networks are ellipsoidal, orthometric and geoidal heights and the following relation exists between these heights (Hofmann-Wellenhof and Moritz 2005):

\[
h - H = N
\]  

(2.7a)

where \(h\), \(H\) and \(N\) are respectively the ellipsoidal, orthometric and geoidal heights. \(h\) is measured using a GNSS receiver, \(H\) is related to the national height system of each country and measured normally by levelling methods from the mean sea level. \(N\) is the geoidal height, or the geoid which is determined using the gravimetric data.

2.5.1 Condition adjustment in GNSS/levelling networks
Equation (2.7a) can be considered as a condition adjustment model:

\[ h - H - N = 0 \]  

(2.7b)

This means that \( h - H \) should be equal to \( N \), but this condition cannot be fulfilled in practice due to the presence of different source of errors in the heights. In a matrix form we can rewrite Eq. (2.7b):

\[ \mathbf{B}\mathbf{e} = \mathbf{w} \] where \( \mathbf{B} = [1 \quad -1 \quad -1] \), \( \mathbf{w} = 0 - \mathbf{B}[h \quad H \quad N]^T \)  

(2.7c)

Equation (2.7c) is only valid for one point having these three heights. For a large network of points the matrix \( \mathbf{B} \) and the vector of \( \mathbf{w} \) become:

\[ \mathbf{B} = [\mathbf{I}_n \quad -\mathbf{I}_n \quad -\mathbf{I}_n] \] and \( \mathbf{w} = 0 - \mathbf{B}[h \quad H \quad N]^T \)  

(2.7d)

where \( \mathbf{I}_n \) is the identity matrix of dimension \( n \) which \( n \) stands for number of GNSS/levelling points. \( \mathbf{0} \) is a \( n \times 1 \) column vector of zeros. \( \mathbf{h} \), \( \mathbf{H} \) and \( \mathbf{N} \) are \( n \times 1 \) the vector of ellipsoidal, orthometric and geoid heights, respectively.

Now, assume that we have the co-factor matrix of the heights:

\[
\mathbf{Q} = \begin{bmatrix}
\mathbf{Q}_h & 0 & 0 \\
0 & \mathbf{Q}_H & 0 \\
0 & 0 & \mathbf{Q}_N
\end{bmatrix}.
\]  

(2.7e)

According to Eqs. (2.7e), (2.7d) and (2.4b) and after some simplifications according to diagonality of the co-factor matrix the estimated errors for the heights will be:

\[
\begin{bmatrix}
\hat{\mathbf{e}}_h \\
\hat{\mathbf{e}}_H \\
\hat{\mathbf{e}}_N
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}_h \\
\mathbf{Q}_H \\
\mathbf{Q}_N
\end{bmatrix} (\mathbf{Q}_h + \mathbf{Q}_H + \mathbf{Q}_N)^{-1} \mathbf{w}.
\]  

(2.7f)

Accordingly \( \hat{\sigma}_0^2 \) will be:

\[
\hat{\sigma}_0^2 = \frac{1}{n} \mathbf{w}^T (\mathbf{Q}_h + \mathbf{Q}_H + \mathbf{Q}_N)^{-1} \mathbf{w}.
\]  

(2.7g)

Based on (2.4d) the scaled VCM of the \( \mathbf{\hat{e}} \) will be:
2.5.2. Gauss-Markov adjustment model in GNSS/levelling networks

The main important issue for using a condition adjustment model in a GNSS/levelling network is that the misclosure vector $\mathbf{w}$ should be normally distributed. However, this is not practically possible due to systematic effects in all heights. Normally, some mathematical models are considered to absorb the systematic trends of the misclosures, which are well-known as corrective surfaces in the adjustment of GNSS/levelling networks. Since the mean value of $\mathbf{w}$ is not zero, so simply we can compute it and subtract it from the misclosures. This is theoretically fine as the relative accuracy of the geoid models and its agreement with $h - H$ is desired rather than its absolute accuracy. This process can also be called a corrective surface with one parameter. However, we can select a better surface to model these discrepancies or the systematic trends, normally, 4-, 5- and 7-parameter corrective surfaces are used to model them. The mathematical models of these surfaces are (see Fotopoulos 2005, Kiamher and Eshagh 2009, Eshagh 2010):

$$f_4(\phi, \lambda) = x_0 + x_1 \cos \phi \cos \lambda + x_2 \cos \phi \sin \lambda + x_3 \sin \phi$$

$$f_5(\phi, \lambda) = x_0 + x_1 \cos \phi \cos \lambda + x_2 \cos \phi \sin \lambda + x_3 \sin \phi + x_4 \sin^2 \phi$$

$$f_7(\phi, \lambda) = x_0 + x_1 \cos \phi \cos \lambda + x_2 \cos \phi \sin \lambda + x_3 \sin \phi + x_4 \frac{\cos \phi \sin \phi \cos \lambda}{k} + x_5 \frac{\cos \phi \sin \phi \sin \lambda}{k} + x_6 \frac{\sin^2 \phi}{k}$$

where $k = (1 - e^2 \sin^2 \phi)^{1/2}$ and $e^2$ is the first eccentricity of the reference ellipsoid. These three equations are used to model the systematic trends of the differences between $N$ and $h - H$. In fact, the trend is absorbed in the $x$ coefficients, in other words, $x_i$, i=0,1,2,… is estimated by fitting these mathematical models to the discrepancies. After that by removing these trend from the discrepancies we can expect that the residuals of this fitting process have more or less random behaviour. However, the normality of $\hat{\varepsilon}$ should be also checked by comparing their histogram and corresponding normal distribution function and the goodness of fit test.

The matrix form of Eqs. (2.8a)-(2.8c) are:

$$\begin{bmatrix} Q_{xx} \\ Q_{xy} \\ Q_{yx} \\ Q_{yy} \end{bmatrix} = \sigma_0^2 \begin{bmatrix} Q_h \\ Q_H \\ Q_N \end{bmatrix} (Q_h + Q_H + Q_N)^{-1} \begin{bmatrix} Q_h \\ Q_H \\ Q_N \end{bmatrix}^T \quad (2.7h)$$
Our unknown parameters are 4, 5 or 7 in the adjustment of the GNSS/levelling networks and only for absorbing the systematic trends. \( \hat{\epsilon}, \hat{x} \), VCM of the unknowns and \( \hat{\sigma}_0^2 \) can be simply computed using Eq. (2.3b)-(2.3f).

2.5.3. Gauss-Helmert adjustment model in GNSS/levelling networks

The Gauss-Helmert model is the combination of Gauss-Markov and condition adjustment models. Therefore, if we use the matrix \( A \) from one of Eqs. (2.8d)-(2.8f), depending on corrective surface which is used, and according to Eqs. (2.7e) and (2.7d), the matrix \( C \) and \( w \) to have the following structures:

\[
C = Q_h + Q_H + Q_N \quad \text{(2.9a)}
\]

\[
w = 0 - h + H + N \quad \text{(2.9b)}
\]
\( \hat{x}, \hat{\epsilon}, \hat{\sigma}_0^2, C_\hat{x} \) and \( C_\hat{\epsilon} \) can be computed using Eqs. (2.5b)-(2.5e). This method removes the systematic trends from the misclosure vector \( \mathbf{w} \) meanwhile performing the condition adjustment, but based on the residuals not the misclosures. We can also perform a LS adjustment using Gauss-Markov model and remove the trend and later on apply the condition adjustment based on the trend-removed misclosures.
Chapter 3

3. Numerical studies

3.1 Data description

GNSS is used to determine the earth surface heights above the ellipsoid. The main idea of GNSS/levelling is to determine the orthometric height of any point by subtracting the geoid height from the ellipsoidal height which is derived from GNSS receivers. So having a precise geoid model, determined independently or in the other word gravimetrically, is of vital importance in GNSS/ levelling. The difference between the ellipsoidal and orthometric heights should be consistent with a gravimetrically-determined geoidal height. That is the principle of testing geoid models and/or gravity field model regionally with the GNSS/levelling data over a specific area. However, we mention that due to different sources of errors there will be discrepancies between the geoidal heights and the difference between the ellipsoidal and orthometric heights in practice. The important issue is that the geoid model and the heights should be relatively in agreements as there are always large biases between the geoidal heights and the height differences of ellipsoidal and orthometric heights. Furthermore, the surface that is obtained from subtraction of these heights will not be an equipotential surface so as the gravimetric geoid model.

In this study, Fennoscandia, limited between latitudes of 55° N and 70° N and longitudes of 0° and 30° E comprising Sweden, Denmark, Norway and Finland is considered as the test area. This area contains 4346 GNSS/ levelling points which 1570 belongs to Sweden, 50 to Denmark, 43 to Finland and 2683 to Norway. The well-known Earth Gravity model, EGM08 (Pavlis et al. 2012) has been used to compute the geoidal height at the GNSS/ levelling points.
This model contains spherical harmonics coefficients of the gravity field and their errors to the degree and order 2160 and partially to 2190.

EGM08 was used by Eshagh (2013) to compute geoid model over Fennoscandian countries and he mentioned that it is of vital importance to compute the geoid model based on the ellipsoidal reference surface and the spherical approximation of the Earth’s shape is not appropriate for such a purpose. The statistics of the difference between the GNSS/levelling data of the area and the computed geoidal heights are presented in Table 3.1.

**Table 3.1. Statistics of differences of EGM08-based geoid model and GNSS/levelling data over Fennoscandia, NP stands for number of points. Unit: 1 cm**

<table>
<thead>
<tr>
<th>Country</th>
<th>NP</th>
<th>max</th>
<th>mean</th>
<th>min</th>
<th>STD</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sweden</td>
<td>1570</td>
<td>-12.7</td>
<td>-21.9</td>
<td>-42.5</td>
<td>2.9</td>
<td>22.1</td>
</tr>
<tr>
<td>Denmark</td>
<td>50</td>
<td>-24.2</td>
<td>-29.7</td>
<td>-34.5</td>
<td>2.6</td>
<td>29.8</td>
</tr>
<tr>
<td>Norway</td>
<td>2683</td>
<td>-9.1</td>
<td>-26.5</td>
<td>-51.3</td>
<td>6.5</td>
<td>27.2</td>
</tr>
<tr>
<td>Finland</td>
<td>43</td>
<td>7.2</td>
<td>-21.5</td>
<td>-31.6</td>
<td>5.8</td>
<td>22.3</td>
</tr>
<tr>
<td>Fennoscandia</td>
<td>4346</td>
<td>6.5</td>
<td>-24.3</td>
<td>-51.1</td>
<td>5.7</td>
<td>25.0</td>
</tr>
</tbody>
</table>

Table 3.1 shows that the standard deviations (STD) of the differences is about 2.9 cm and 2.6 cm respectively in Sweden and Denmark. The largest STD is related to Norway and it is about 6.5 cm which can be expected as Norway is very mountainous. Finland has a STD of 5.8 cm which seems to be large for such a flat country. The largest mean value of the differences is related to that of Denmark which is 29.7 cm and after that Norway with 26.5 cm. The large and non-zero magnitude of mean values cause that the root mean square error (RMS) of the differences become large. Generally, the STD of the differences is 5.7 cm with RMS of 25.0 cm in whole area of Fennoscandia.

Our personal contacts with experts working on the GNSS/levelling data revealed that the GNSS/levelling data of Sweden are of three types, the first order one which is the most precise and the reliable one contains 25 points which accuracies between 10-20 mm, the second-order one consist of 180 point with accuracies of 20-30 mm and the rest of them has accuracies of 30 mm. Accuracies of the normal heights and levelling have been computed based on the motherised levelling of about 5 mm.. Therefore, based on the error propagation
law, the accuracy of the computed geoid heights from the heights should be about 14 mm for the first-order network, 24 mm for the second-order one and 34 for the rest of the points. No errors have been provided to us for the levelling network of Denmark but the accuracies of the ellipsoidal heights vary with the points and they are mostly below millimeter level, which seems to be optimistic when we compare them with the geoid model. Situation is very similar for Norway and Finland and even we did not have the heights directly and instead only their differences were provided to us. Generally, the accuracy of the GNSS ellipsoidal heights for Finland was about 25 mm for all points. Norway’s data are classified into two groups the first order network contain about 345 points with accuracies of about 5 mm and accuracies the rest of the data is 15 mm.

Figure 3.1a shows the distribution of the GNSS/levelling data over the study area. One can see that Norway and Sweden have relatively dense distribution of the data, but for Denmark and Finland the data are sparser. This could be understandable as the geoid models are rather smooth over these two countries and for purpose of GNSS/levelling the discrepancies between the geoid and the heights can simply be modelled.

![Figure 3.1. a) Distribution of GNSS/levelling data over Fennoscandia, b) misclosures, geoid error and height difference h-H errors. Unit: 1 metre](image)

Figure 3.1b shows the plot of the errors of the geoidal heights computed from the errors of the spherical harmonic coefficients of EGM08, and the claimed error of the data derived from personal communication with experts from each country and the differences between the
geoidal heights and the ellipsoidal and orthometric heights difference, or in other words, the misclosure errors. The plot illustrates the misclosures are much larger than the claimed errors for the GNSS/levelling heights. The errors of the geoidal heights are also large but still less than the magnitudes of the misclosures. Even if we compute the errors of the misclosures, they will be still smaller than the magnitudes of the misclosures. This means that there are systematic effects amongst them.

3.2. Pre-adjustment data cleaning

Since the basic assumption in the LS adjustment is that the data are free of any systematic and/or gross error and have random behaviour, therefore, the most important step after data collection is to detect and eliminate these types of errors as much as possible. To do so, we statistically test the data to check their normality prior to using them in the LS process. One simple test is to compare the histogram of the data with their corresponding normal distribution. In this study, we use the misclosures, or in other words, the difference between the geoid heights derived from EGM08 and the one obtained by subtracting the geodetic heights from the orthometric heights. We assume that the all height systems over Fennoscandia are of orthometric type. This assumption is not very far from the reality as this area has not very rough topography. However, we can always derive the normal heights from orthometric heights and the opposite in practice. Here, we perform three types of test on the misclosures, test of normality of the misclosures and goodness of fit test of the data to the normal distribution, finally the test of variance for testing the claimed error, or the variance of the data. All of these tests are performed in two cases of filtered data and the original one. In data filtering we have removed those data which are outside of the confidence interval of 95%. This means that the misclosures, which are far from the mean values by 1.96 STD are filtered out from the data set.

3.2.1. Test of the normality of the misclosures

Here we test the normality of the misclosures. The statistic of the differences of the computed geoid model and the GNSS/levelling data heights can be obtained by:

\[ h-H=N \]
Since the observations contain errors, the above statistics does not fulfill in reality, so we have:

\[ h-H=W+N \quad \text{or} \quad W=(h-H)-N \]

where \( h, H, N \) and \( W \) are geodetic, orthometric, geoid heights and misclosure respectively. In order to check the normality of \( W \) we plot the histograms and the area under that for the misclosures and the corresponding normal distribution curve derived based on the mean and the standard deviation of the data. The agreement of the histograms with the normal distribution function gives some ideas about the normality of \( W \). Figure 3.2a shows a good agreement between the histogram and the normal distribution function, but on the left hand side of the plot some parts of the histogram is not under the normal distribution function, so we can suspect that there should be some blunders in the data set. Figure 3.b shows a similar plot for Denmark’s misclosures and all bins of the histogram are close to the normal function. On the left side of Figure 3.2c shows some parts of the histogram outside of the normal function, so there should be some blunders in the Norwegian data set. Figure 3.2d illustrates that there should be outliers in the Finnish data set as well because one part of the histogram is outside of the range of the corresponding normal function. Similar situation is seen in Figure 3.2e which belongs to \( W \) of Fennoscandia.

Figures 3.3a, b, c, d and e are the plots of the histograms and the corresponding normal distribution functions for Sweden, Denmark, Norway, Finland and Fennoscandia, respectively. After filtering the data by a confidence interval of 95% we can see in the figures that the histograms are more in agreement with the corresponding normal distribution functions and those possible erroneous data have been removed by filtering them. Figure 3.3b is more or less the same with Figure 3.2b which means that the Danish data have had a better quality even before the filtering process. Another issue is still the non-symmetricity of the histograms in Fennoscandia visible in Figure 3.3e.

Table 3.2 represents the statistics of differences between the ellipsoidal and orthometric heights \( (h-H) \) and the geoid heights \( (N) \) before and after filtering the data with the confidence level of 95%. In the table, \( NP \) stands for the number of points, as we want to show the number of the acceptable data after data filtering process. By comparing this table with Table 4.1 we can see that the STD of the misclosures is reduced by 5 mm in Sweden after removing 70
points from the data set. No change is seen in the mean value of the differences, but the minimum value changed from -42.5 cm to -27.6 cm. It is not easy to say that those removed 70 points contain blunders but at least we can say that they are not in the confidence interval of 95%. Also, by removing these data the STD is reduced only by 5 mm this means that the quality of the data are relatively fine and if there are blunders amongst the misclosures they are small and very well suppressed by the large number of the data over the territory. The filtering process of Danish data leads to removing only one data and the STD of the misclosures reduced by 1 mm. Therefore, these data are of good quality. The removal of one data from the set does not mean that it necessarily contains blunder, but it is not in the confidence level of 95%. By this filtering process 117 points are removed from the data of Norway and the STD of the data is reduced by 7 mm. However, removal of 1 point in Finish data leads to a reduction of 17 mm in the STD, which probably means that that misclosure contains gross error.

Table 3.2. Statistics of the differences between \( h-H \) and \( N \) before and after data filtering with the confidence interval of 95%. Unit: 1 cm

<table>
<thead>
<tr>
<th></th>
<th>Before data filtering</th>
<th></th>
<th>After data filtering</th>
<th></th>
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<tbody>
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<td>NP</td>
<td>max</td>
<td>mean</td>
<td>min</td>
</tr>
<tr>
<td>Sweden</td>
<td>1570</td>
<td>-12.7</td>
<td>-21.9</td>
<td>-42.5</td>
</tr>
<tr>
<td>Denmark</td>
<td>50</td>
<td>-24.2</td>
<td>-29.7</td>
<td>-34.5</td>
</tr>
<tr>
<td>Norway</td>
<td>2683</td>
<td>-9.1</td>
<td>-26.5</td>
<td>-51.3</td>
</tr>
<tr>
<td>Finland</td>
<td>43</td>
<td>7.2</td>
<td>-21.5</td>
<td>-31.6</td>
</tr>
<tr>
<td>Fennosc.</td>
<td>4346</td>
<td>6.5</td>
<td>-24.3</td>
<td>-51.1</td>
</tr>
</tbody>
</table>
Figure 3.2. Histograms and normal distribution functions before data filtering over a) Sweden, b) Denmark, c) Norway, d) Finland and e) Fennoscandia
Figure 3.3. Histograms and normal distribution functions after data filtering over a) Sweden, b) Denmark, c) Norway, d) Finland and e) Fennoscandia
3.2.1. Goodness of fit test on the misclosures

Judging about the normality of the data based on the agreements of the histogram and distribution functions needs more investigation. As mentioned before, the test of normality of the data can be done by the goodness of fit test of the misclosures. The theory of the goodness of fit test presented in Section 2.2.4 and now we will apply this theory for W of height differences over Fennoscandia and the mentioned countries. Today, different software exists for statistical analysis including MATLAB. Here, we use the $\chi^2$ goodness of fit test of this software and it is enough to introduce the type of distribution, which in our case, is the normal distribution, the vector of the data which are going to be tested and the significant level of 5% in this study. The results of the test are presented in Table 3.3. This table shows the values of the goodness of fit test before and after data filtering and our goal is to see if we can somehow check the normality of the data by this test. This can be computed using Eq. (2.2a) and $x$ is the value of the $\chi^2$ distribution function with the significant level of 5% and the specified degree of freedom. We have done this test on each country and its different networks.

Table 3.3. $\chi^2$ goodness of fit test before and after data filtering with the confidence interval of 95% , $y$ is the estimated statistic based on the data set and $x$ is the acceptable threshold which is derived from $\chi^2$ distribution function with a specific significant level .

<table>
<thead>
<tr>
<th>Study Region</th>
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<th>After data filtering</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$y$</td>
<td>$X$</td>
</tr>
<tr>
<td>Sweden</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.426</td>
<td>7.815</td>
</tr>
<tr>
<td>2</td>
<td>4.813</td>
<td>7.815</td>
</tr>
<tr>
<td>3</td>
<td>5.757</td>
<td>7.815</td>
</tr>
<tr>
<td>Denmark</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>5.660</td>
<td>7.815</td>
</tr>
<tr>
<td>Norway</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>84.885</td>
<td>11.070</td>
</tr>
<tr>
<td>2</td>
<td>60.417</td>
<td>12.592</td>
</tr>
<tr>
<td>all</td>
<td>29.728</td>
<td>12.592</td>
</tr>
<tr>
<td>Finland</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.988</td>
<td>12.592</td>
</tr>
<tr>
<td>Fennoscandia</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>192.748</td>
<td>9.488</td>
</tr>
</tbody>
</table>
As we can see, the first, second and third-order networks of the Swedish data pass the goodness of fit test before data filtering but the third-order network fails the test after the filtering process. This means that the normality of data is less after this process or we have removed more data than what was necessary. All Danish W passed the test but none of the Norwegian data can pass the test. By removing one data from Finnish W the data can pass the goodness of fit test. When we consider all data of Fennoscandia the test fails, this could be expected as Figure 3.2e and 3.3e already showed the histogram of the W is not symmetric and in agreement with the normal distribution function. This could be due to involvement of a large contribution of the data of Norway.

3.2.3. Test of the variance of the data

Since the a priori variance factor ($\sigma^2_0$) or precision of the observations is known, we can test whether the a priori variance of the data is acceptable or not. Every country has some estimates of precision of the GNSS/levelling data. Table 3.4 shows the result of the goodness of fit test on the variance before and after data filtering. NOP is the number of measurements for each country, $\sigma^2$ is the error of the misclosures derived by the error propagation of the claimed error for the heights and that of the geoid model computed by EGM08. The boundary values of $\sigma^2$ have been calculated using $\chi^2$ table based on 95% confidence interval at 5% significant level and degree of freedom of corresponding data set.

As Table 3.4 shows by considering the errors of the heights and the geoid model the test is passed neither before nor after data filtering. This means that the errors are much larger than what should it be. If we look at Figure 3.1b we can see that the claimed errors of the height differences are optimistically small whilst those of the geoid models are large and increasing by latitude. Therefore the main reason of the rejection of the test is due to the large error of the geoid model. Table 3.4 shows that the geoid model and the height differences are in good agreements so that the STD deviation of the differences will not be that large. Therefore, the large error of the geoid model which is larger than 10 cm is not very realistic as all STDs are smaller than this value. This is due to the downward continuation of the error of EGM08 to the reference ellipsoid. This has been pointed out by Eshagh (2013) and he has explained that by increasing the degree harmonics in the spherical harmonic expression of the geoid the errors are largely amplified and not realistic.
Table 3.4. Test of variance of the data by considering the geoid model error

<table>
<thead>
<tr>
<th>Country</th>
<th>NOP</th>
<th>order</th>
<th>Before data filtering</th>
<th>After data filtering</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\sigma^2$</td>
<td>T NOD $\sigma^2$</td>
</tr>
<tr>
<td>Sweden</td>
<td>25</td>
<td>1</td>
<td>0.012$^2$ 0.115$^3$ 0.022$^4$</td>
<td>F 0 0.022$^2$ 0.115$^3$ 0.026$^4$</td>
</tr>
<tr>
<td></td>
<td>181</td>
<td>2</td>
<td>0.025$^2$ 0.119$^3$ 0.031$^4$</td>
<td>F 8 0.021$^2$ 0.116$^3$ 0.025$^4$</td>
</tr>
<tr>
<td></td>
<td>1364</td>
<td>3</td>
<td>0.028$^2$ 0.119$^3$ 0.030$^4$</td>
<td>F 62 0.023$^2$ 0.121$^3$ 0.025$^4$</td>
</tr>
<tr>
<td>Denmark</td>
<td>43</td>
<td>_</td>
<td>0.022$^2$ 0.071$^3$ 0.033$^4$</td>
<td>F 1 0.020$^2$ 0.090$^3$ 0.031$^4$</td>
</tr>
<tr>
<td>Norway</td>
<td>345</td>
<td>1</td>
<td>0.073$^2$ 0.133$^3$ 0.082$^4$</td>
<td>F 30 0.053$^2$ 0.123$^3$ 0.082$^4$</td>
</tr>
<tr>
<td></td>
<td>2338</td>
<td>2</td>
<td>0.058$^2$ 0.142$^3$ 0.062$^4$</td>
<td>F 87 0.051$^2$ 0.132$^3$ 0.055$^4$</td>
</tr>
<tr>
<td>Finland</td>
<td>50</td>
<td>_</td>
<td>0.048$^2$ 0.235$^3$ 0.072$^4$</td>
<td>F 1 0.033$^2$ 0.211$^3$ 0.049$^4$</td>
</tr>
</tbody>
</table>

If we trust the geoid model and consider that it is error-free we can test the claimed variance of the heights. Table 3.5 shows the results of this test. The table shows that the first-order network of Sweden passes the test and it means that the variance of the data is in agreement with W. After filtering the data the second-order network of Sweden will pass the test as well. However, the rest of data of the countries will not pass this which means that the claimed variances are not fully realistic and in agreement with the data deviations. Consequently, we can say that neither the error of the heights nor that of geoid model is in agreement with W in the significant level of 5%.

Table 3.5. Test of variance of misclosures by assuming an error-free geoid model

<table>
<thead>
<tr>
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<th>order</th>
<th>Before data filtering</th>
<th>After data filtering</th>
</tr>
</thead>
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<td></td>
<td></td>
<td>$\sigma^2$</td>
<td>T NOD $\sigma^2$</td>
</tr>
<tr>
<td>Sweden</td>
<td>25</td>
<td>1</td>
<td>0.012$^2$ 0.013$^3$ 0.022$^4$</td>
<td>P 0 0.022$^2$ 0.014$^3$ 0.026$^4$</td>
</tr>
<tr>
<td></td>
<td>181</td>
<td>2</td>
<td>0.025$^2$ 0.022$^3$ 0.031$^4$</td>
<td>F 8 0.021$^2$ 0.022$^3$ 0.025$^4$</td>
</tr>
<tr>
<td></td>
<td>1364</td>
<td>3</td>
<td>0.028$^2$ 0.032$^3$ 0.030$^4$</td>
<td>F 62 0.023$^2$ 0.032$^3$ 0.025$^4$</td>
</tr>
<tr>
<td>Denmark</td>
<td>43</td>
<td>_</td>
<td>0.022$^2$ 0.002$^3$ 0.033$^4$</td>
<td>F 1 0.020$^2$ 0.002$^3$ 0.031$^4$</td>
</tr>
<tr>
<td>Norway</td>
<td>345</td>
<td>1</td>
<td>0.073$^2$ 0.005$^3$ 0.082$^4$</td>
<td>F 30 0.053$^2$ 0.005$^3$ 0.082$^4$</td>
</tr>
<tr>
<td></td>
<td>2338</td>
<td>2</td>
<td>0.058$^2$ 0.015$^3$ 0.062$^4$</td>
<td>F 87 0.051$^2$ 0.015$^3$ 0.055$^4$</td>
</tr>
<tr>
<td>Finland</td>
<td>50</td>
<td>_</td>
<td>0.048$^2$ 0.025$^3$ 0.072$^4$</td>
<td>F 1 0.033$^2$ 0.025$^3$ 0.049$^4$</td>
</tr>
</tbody>
</table>
3.3. Least-Square Adjustment

In this section, we focus on the estimated parameters and their accuracies. The mathematical formulae for these quantities have been already presented in Chapter 2 and we do not repeat them here. First, we assume that $\sigma_0^2 = 1$ and later on we will compute $\hat{\sigma}_0^2$ and multiply it to the VCM of the estimated parameters and the residuals.

Table 3.6 shows the estimated parameters and their accuracies before and after scaling. Those errors which are given in parentheses are the estimated errors by assumption of $\sigma_0^2 = 1$. As we can see, in all cases the estimated errors before scaling are larger than those after. This means that $\sigma_0^2 = 1$, is not a proper choice according to the presented accuracies of the observations. However, in most cases, after scaling the VCM of the estimated errors become smaller and more reasonable. In the case of performing the adjustment before scaling the VCM, some of the errors are even larger than the estimated parameters, which cannot be meaningful. However, after scaling the VCM the errors are more acceptable. In Sweden and before filtering the data, the fifth parameter in the 5-parameter corrective surface has larger error than the fifth parameter itself. The same happens for most of parameters for the 7-parameter surface. Nevertheless, the error will be at acceptable level after multiplying $\hat{\sigma}_0^2$ to the VCM.

In the case of using filtered data in the adjustment process, errors of the estimated parameters are slightly smaller than those obtained from the non-filtered data. Similar situation is seen for the estimated parameters and their errors for other countries and Fennoscandia and we do not repeat explaining them here. The cases where the errors are larger than estimated parameters are also happened in Denmark and Finland and in both countries the 4th parameter of the 5-parameter surface is smaller than its errors.

Another important parameter which should be considered in the adjustment process is $\hat{\varepsilon}$ and its error $\sigma_\varepsilon$. Similar to what we did for the estimated parameters, we consider, at the first step, $\sigma_0^2 = 1$ and compute $\hat{\sigma}_0^2$ from the residuals. Later on $\hat{\sigma}_0^2$ is multiplied to the VCM of the residuals to scale them. The results of these investigations are presented in Figures 3.4 to 3.8. p above each panel of the figure shows the type of the corrective surface which was used in the adjustment process e.g. $p = 4$ means the 4-parameter corrective surface and so on.
Table 3.6. The estimated parameters and their errors before and after data filtering

<table>
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<th>After data filtering</th>
<th></th>
</tr>
</thead>
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<td>5</td>
<td>7</td>
<td>4</td>
</tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>-11.9±1.8(8.6)</td>
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</tr>
<tr>
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</tbody>
</table>
Figure 3.4. $\hat{\varepsilon}$, $\hat{\sigma}_\varepsilon$ and $\hat{\tilde{\sigma}}_\varepsilon$ in Sweden, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering. Unit: 1 metre.

Figure 3.5. $\hat{\varepsilon}$, $\sigma_\varepsilon$ and $\tilde{\sigma}_\varepsilon$ in Denmark, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering. Unit: 1 metre.
Figure 3.6. $\hat{\epsilon}$, $\hat{\sigma}_\epsilon$ and $\hat{\sigma}_\epsilon$ in Norway, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering. Unit: 1 metre

Figure 3.7. $\hat{\epsilon}$, $\sigma_\epsilon$ and $\hat{\sigma}_\epsilon$ in Finland, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering. Unit: 1 metre

45
The green curve shows $\hat{\sigma}_e$ before scaling and as we can see these errors is larger than $\hat{\epsilon}$ considerably. However, after multiplying $\hat{\sigma}_e^2$ they are more reasonable.

Figures 3.4a-3.4c are related to $\hat{\epsilon}$ and $\sigma_e$ before filtering the data and Figures 3.4d-3.4f are those after data filtering. As we can see, there are large $\hat{\epsilon}$ which are removed after the filtering process. Figure 3.4 is very similar to Figure 3.3 but for Denmark. Since we already mentioned only one GNSS/levelling point was removed by filtering the data. Also, that data was not considerably larger than the others so that it is not very visible in the plots. However, as we can see again the errors of $\hat{\epsilon}$ are larger than them before scaling the VCM of the $\hat{\epsilon}$, but after scaling they are in the order of them.

According to Figure 3.5 some of the residuals are considerably larger than the errors even before scaling the data. These data are suspected to be blunder or at least inconsistent with the rest of the residuals according to the confidence level of 95%. However, after filtering of the data most of them are smaller than the errors and at least in the order of them.

Figure 3.7 shows $\hat{\epsilon}$ and their errors in Finland. One of $\hat{\epsilon}$ has a considerably large magnitude than the errors, which is removed after the filtering process. The scaling process again brings the errors in the order of the residuals. Similar issues are visible in Figures 3.7 and 3.8 which belong to Norway and Fennoscandia. However, in all of these figures one should consider is that if we take the absolute value of the residual we can avoid having any negative value for them so that the errors and the magnitudes of the residuals are more comparable.
Figure 3.8. $\hat{\epsilon}$, $\hat{\sigma}_\epsilon$ and $\hat{\sigma}_\epsilon$ in Fennoscandia, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering. Unit: 1 metre

3.4. Post-adjustment data cleaning

After applying LS adjustment usually there are still some errors in our data set, so we need to use some methods to test the data set again and detect these errors. Selecting of testing techniques and the probability levels are of importance. One observation can be considered as an outlier or not depending to the chosen methods.

3.4.1. Goodness of fit test on the adjusted observations

In this test, we consider whether our null hypothesis is true or not. The null hypothesis is that the adjusted observations follow the normal distribution. We use goodness of fit test in order to see if this hypothesis is accepted or not. If the test is not passed, one reason might be the existence of the gross errors in the data set. We apply $\chi^2$ test with 95% confidence interval at 5% significant level to check the normality of the residuals.

Table 3.7 shows the result of applying goodness of fit test before and after data filtering at 95% confidence level and $\mu \pm 1.96\sigma$ interval of height differences. $Y$ and $x$ are the computed values of statistics and $\chi^2$ value of the adjusted observations respectively based on the 95% confidence interval and 5% significant level. The column p in the table means the number of
the parameters in the applied corrective surface. As the table shows only the Danish data can pass the goodness of fit test before data filtering. This means that these data are more or less normally distributed. However, as we can see in the table the test fails the normality hypothesis of the residuals in the adjustment on non-filtered data. In the case of using the filtered data in the adjustment procedure most of the residuals can pass the goodness of fit test of normality of the residuals except the Norwegian and the Swedish residuals when the 7-parameter model is used as an adjustment model. The residuals over Fennoscandia do not pass the test even after data filtering this could be due to the involvement of the residuals over Norway and partly in Sweden.

Table 3.7. $\chi^2$ statistics and its acceptable level of the goodness of fit test of residuals in Fennoscandian countries

<table>
<thead>
<tr>
<th>Country</th>
<th>Before data filtering</th>
<th>After data filtering</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P y x test</td>
<td>y x test</td>
</tr>
<tr>
<td>Sweden</td>
<td>4 16.977 7.815 F</td>
<td>7.670 12.592 P</td>
</tr>
<tr>
<td></td>
<td>5 19.386 7.815 F</td>
<td>9.920 11.070 P</td>
</tr>
<tr>
<td></td>
<td>7 17.236 7.815 F</td>
<td>13.804 11.070 F</td>
</tr>
<tr>
<td>Denmark</td>
<td>4 2.922 7.815 P</td>
<td>0.912 9.488 P</td>
</tr>
<tr>
<td></td>
<td>5 1.428 7.815 P</td>
<td>0.671 7.815 P</td>
</tr>
<tr>
<td></td>
<td>7 4.509 7.815 P</td>
<td>4.011 7.815 P</td>
</tr>
<tr>
<td>Norway</td>
<td>4 34.556 11.070 F</td>
<td>104.292 14.067 F</td>
</tr>
<tr>
<td></td>
<td>5 57.051 12.592 F</td>
<td>78.218 14.067 F</td>
</tr>
<tr>
<td></td>
<td>7 59.558 12.592 F</td>
<td>58.800 14.067 F</td>
</tr>
<tr>
<td>Finland</td>
<td>4 13.336 5.991 F</td>
<td>5.240 7.815 P</td>
</tr>
<tr>
<td></td>
<td>7 8.339 5.991 F</td>
<td>7.312 9.488 P</td>
</tr>
<tr>
<td>Fennoscandia</td>
<td>4 207.607 11.070 F</td>
<td>203.195 14.067 F</td>
</tr>
<tr>
<td></td>
<td>5 182.238 11.070 F</td>
<td>203.195 14.067 F</td>
</tr>
<tr>
<td></td>
<td>7 127.118 9.488 F</td>
<td>203.195 14.067 F</td>
</tr>
</tbody>
</table>

Figures 3.9-3.13 are the histograms and normal distribution functions of the residuals before and after data filtering of each country, and the whole study area; Fennoscandia. As we can see from these figures, Sweden has some gross errors in both side of the histograms and some of them remain in the data set even after data filtering. The histograms and normal distribution curves of the residuals of Denmark are similar before and after data filtering and it seems that there is no blunder in the data set. Situation in Norway is different since the
histograms are not symmetric and we can see obviously that there are still some blunders in the data set even after data filtering and that is the reason of failure of the goodness of fit test. The plots related to Finland shows that there is a gross error in the data set before data filtering which is removed from the data set after filtering process and as we can see the histograms and the normal distribution curves are in better agreements after data filtering and the test of the normality is also passed. The whole data set of the Fennoscandia could not pass the test and we can see that there are some visible blunders in the adjusted data set before and after data filtering process.

Figure 3.9. Histograms and normal distribution functions of the residuals in Sweden, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering.
Figure 3.10. Histograms and normal distribution functions of the residuals in Denmark. a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering.
Figure 3.11. Histograms and normal distribution functions of the residuals in Norway, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering.

Figure 3.12. Histograms and normal distribution of the residuals in Finland, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering.
Figure 3.13. Histograms and normal distribution functions of the residuals over Fennoscandia, a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces before data filtering, and d), e) and f) similar ones but after filtering.

3.4.2. Global test on the a posteriori variance factor

One test which is applied after adjustment is global test on $\hat{\sigma}_0^2$ calculated based on $\hat{e}$. This test gives some ideas about the quality of the adjustment procedure. This test can be applied only if there is a priori knowledge about the precision of the observation, i.e. when the a priori variance factor is assumed to be known; otherwise the test has no meaning (Kavouras 1982). There are possible reasons for the case that the test is not passed:

1. Existence of blunder in the data set
2. Improper choice of correcting surface
3. Improper choice of weight matrix

Table 3.8 shows the results of the global test on $\hat{\sigma}_0^2$, as we can see none of countries can pass the test even after data filtering. Using different correcting surfaces to model systematic effects gave us the same results. One can conclude that the improper choice of the corrective surface is not the main reason for the test failure. In addition, $\hat{\sigma}_0^2$ is not passed even after using the filtered data. So the existence of blunder cannot be the reason either. Consequently,
the reason is the improper choice of the a priori precision of the data. Since the weight matrix has a reverse relation with VCM therefore the presented variance or the errors are not realistic. In most of cases \( \sigma_0^2 \) is large and we can conclude that the claimed a priori precisions of observation are optimistic.

**Table 3.8.** Global test of a posteriori variance factor (\( \sigma_0^2 \)) before and after data filtering

<table>
<thead>
<tr>
<th>Country</th>
<th>p</th>
<th>Before data filtering</th>
<th>After data filtering</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Min ( \sigma_0^2 )</td>
<td>Max ( \sigma_0^2 )</td>
</tr>
<tr>
<td>Sweden</td>
<td>4</td>
<td>0.9335</td>
<td>4.5277</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.9335</td>
<td>4.5305</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.9334</td>
<td>4.5363</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.6710</td>
<td>24.8136</td>
</tr>
<tr>
<td>Denmark</td>
<td>5</td>
<td>0.6679</td>
<td>25.4665</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.6613</td>
<td>26.8807</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.9485</td>
<td>5.4243</td>
</tr>
<tr>
<td>Norway</td>
<td>5</td>
<td>0.9485</td>
<td>5.4270</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.9485</td>
<td>5.4307</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.6905</td>
<td>2.8644</td>
</tr>
<tr>
<td>Finland</td>
<td>5</td>
<td>0.6880</td>
<td>2.9284</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.6826</td>
<td>3.0644</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.9592</td>
<td>4.3532</td>
</tr>
<tr>
<td>Fennoscandia</td>
<td>5</td>
<td>0.9592</td>
<td>4.3535</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.9592</td>
<td>4.3552</td>
</tr>
</tbody>
</table>

3.4.3. **Data snooping using Baarda and \( \tau \) methods**

In Section 3.1, we have shown that by filtering the data based on a confidence interval of 95% lots of data are flagged as outliers and removed from the data set; correspondingly from the LS adjustment process. However, removal of such a large amount of data is not very reasonable as it means that lots of the jobs which responsible organisations have done are
meaningless. Therefore, in this case we focus on the Baarda and τ-tests without using the filtered data.

The global test of $\hat{\sigma}_0^2$ can be used to detect the blunders in an iterative manner. To do so, one of the data points are removed from the adjustment process and the test is done. If the global test rejects $\hat{\sigma}_0^2$, therefore we can conclude that that removed data point was not problematic so that by its removal the results has not changed. Therefore, the removed data point is restored into the observations set and the second one is removed and the adjustment and the test are performed again. If the test does not pass $\hat{\sigma}_0^2$ therefore, that data does not contain blunder. We repeat this process until the test is passed.

In data snooping we assume that only one observation in the data set contains gross error. In order to detect this erroneous observation we statistically test the residuals against its standard error and we repeat the test for each observation in the data set.

The null hypothesis is that there is no gross error in the data set or the chosen observation is free of error and the alternative hypothesis is that the chosen observation is the erroneous observation in the data set. We perform one dimensional statistic test and examine all standardised residuals one by one. According to Baarda’s suggestion the significant level is chosen at 0.001 and the critical value is 3.29, it means that the residuals bigger than 3.29STD are flagged as gross errors. However, we perform the Baarda data snooping test by finding and testing the maximum standardised residuals and if it does not passed through the test we remove it and perform the adjustment and do the test again to detect the next possible blunder. We repeat this process so that the maximum standardised residual will pass through the test.

Figure 3.14 shows the standardised $\hat{\varepsilon}$ derived based on the Baarda theory which are shown by $w_i$ and those derived based on the $\tau_i$ test. The former test relies on the errors of the observations and the later does not and $\hat{\sigma}_0^2$ is used to scale the errors of $\hat{\varepsilon}$. We can see that all of the normalised $\hat{\varepsilon}$, $(w_i)$ have smaller values than 3.29 which means that the Baarda test cannot detect any blunder amongst the data. This is due to the fact that the error of the geoid model computed from EGM08 is considerably larger than the errors of the heights. When the errors of the $W$ which are the square root of summation of variances of the heights and the geoid are used for normalisation the magnitude of $w_i$ becomes considerably small and even smaller than 3.29. In the case of considering the scaled errors for $\hat{\varepsilon}$ the normalised $\hat{\varepsilon}$ are
larger than 3.29 and some erroneous data can be detected. It should be noted that the value of \( \tau \) for high degrees of freedom is very close to 3.29, but for the cases that the degree of freedom is not large the tau value differs from 3.29; see e.g. Kovarous 1982. In order to make sure that this reason of having smaller residuals than 3.29 in Baarda test we neglect the error of the geoid model and assume that it is free of any error. Therefore, the normalisation process will be done solely based on the errors of the heights. However, since the errors are considerably smaller than residuals, therefore, the normalised \( \hat{\epsilon} \) becomes very large so that if we want to consider 3.29 as a threshold we have to remove a large amount of data, which is not very logical and realistic.

Figures 3.14a and Figure 3.14b show the results of the tau and Baarda tests in Sweden. From these figures we can see that there are some \( \tau_i \) values which are larger than 3.29 and probably can be removed from the data set. In the case of considering 4-, 5- and 7-parameter corrective surfaces, we found 20, 21 and 22 observations which are flagged as outliers in Sweden.

Figures 3.14c and 3.14d are similar plots but for \( w_i \) and \( \tau_i \) in Denmark. As it can be seen both of \( w_i \) and \( \tau_i \) are smaller than 3.29 and no outlier is detected in the GNSS/levelling data of this country based on the \( w_i \) and \( \tau_i \) test. Figures 3.14e and 3.14f are two other similar plots in Norway for \( w_i \) and \( \tau_i \) in the case of ignoring the error of the geoid model about half of the data should be removed from the data. In the case of using the 4-parameter corrective surface 23 residuals are failed in the test, for 5-parameter surface only 5 residuals are flagged as outliers and in the case of using 7-parameter model 9 residuals. Figures 3.15g and 3.15h are those related to Finland and only 1 residual is flagged as outlier based on all corrective surfaces.
Figure 3.14. Standardised residuals in Sweden, Denmark, Norway and Finland respectively a), c), e), d) and g) concerning geoidal errors and b), d), f), h) without concerning the geoidal errors.

Figure 3.15. shows the histograms of the residuals and their corresponding normal distribution curves of each country and the whole region of the study. As it can be seen obviously from these figures after removing the blunders which have located with tau test the histograms are
in a better agreement with the normal distributions. Since the histograms of each country using different correcting surfaces looks very similar here we plot only the histograms with the normal distribution curve of the 7 parameters corrective surfaces.

Figure 3.15. Histograms and normal distribution functions Sweden, Denmark, Norway and Finland with 7-parameter corrective surfaces, after removing gross errors using tau test.

The results of applying the goodness of fit test on the cleaned data set shows that all countries except Norway can pass the test. The main reason of the failure of the test can be the existence of gross errors with very small magnitude in the data set of Norway which are not
detectable easily and we can conclude that large contribution of the data set of Norway causes the failure of the goodness of fit test on the whole area of the study.

Figure 3.16. Distribution of the detected gross errors over Fennoscandia a), b) and c) are related to the 4-, 5- and 7-parameter corrective surfaces respectively.

Figure 3.16 shows the distribution of detected gross errors over Fennoscandia according to the 4, 5 and 7-parameter corrective surfaces. As it can be seen in the figures, the 7-parameter corrective surface can detect more gross errors which mainly distributed in Norway as it is expected, since observations of Norway has a large contribution in the data set and due to its rough topography.
4. Conclusions

In this study, we analysed the GNSS/levelling data over Fennoscandia comprising Sweden, Denmark, Norway and Finland. In the case of filtering the data based on the confidence interval of 95% large number of them will be outside the interval and should be removed from the data set. Based on this process 70 and 117 points are placed outside the interval in Sweden and Norway and only 1 for Denmark and Finland. However, not always the filtering process of the data leads to have normally distributed data. The $\chi^2$ goodness of fit test of normality of the data showed that all of the three types of the networks over Sweden can pass this test before data filtering whilst the third-order one cannot after data filtering. The Danish data have normal distribution before and after filtering. Removal of one data leads to have normally distributed data in Finland. However, the Norwegian data can pass this test neither before nor after filtering.

The test of variance of the data shows that none of the claimed errors for the data whether or not consider the error of the geoid model in our computations. This means that claimed precision for the data might be very optimistic. However, in the case of considering an error-free geoid model the first-order network of Sweden can pass the test, meaning that the claimed errors are closer to reality. In addition, the second-order network can pass the test after filtering and removing 8 data points. Nevertheless, the data of the rest of the countries cannot pass this test. It should also be mentioned that we have had only the errors of the ellipsoidal heights for some country.
The errors of the estimated parameters after LS process in some cases are larger than the parameters themselves, which is not very meaningful. However, by considering the a posteriori variance factor; they are balanced with the quality of the data and the adjustment process. The plots of the residuals and their errors before and after considering the a posteriori variance factor shows that in most of the cases, the errors of the residuals are larger than the residuals, but after scaling the VCM of the residuals by the a posteriori variance factor the errors will have more logical values.

After the LS adjustment and removing the systematic trends from the misclosures, the residuals based on Swedish data do not have the normal distribution, but in the case of using the filtered data, the goodness of fit test is passed when the 4- and 5-parameter models are used in the LS solution. The Danish data can pass the test with all corrective surfaces, and in the opposite the Norwegian data cannot. Before data filtering the Finnish residuals cannot pass the test but after data filtering they pass.

The LS solutions cannot pass the global test of a posteriori variance factor. In order to find the reason, we considered all three corrective surfaces and perform the test before and after filtering the misclosures. The global test is not passed based on these different situations; therefore, the reason could not be due to the mathematical models as the situation does not change by their choice. This cannot be the presence of the blunders either as even after using the filtered data the test is not passed. Consequently, the reason should be due to improper choice of the a priori errors. Since the Baarda data snooping methods relies on the a priori errors of the data, therefore, this method was not very successful as by standardising the residuals by this optimistically small errors the statistics become very large and about 50% of the data are flagged as outliers, which cannot be true. In the case of including the error of the geoid model, the Baarda data snooping method passes all of the data, which is not logical either. However, the situation will be better in tau-test. Since the degree of freedom is too large some large values may be obtained for the tau. In this case, we consider the Baarda threshold of 3.29 for the Tau-test and find that in Sweden about 20 GNSS/levelling points are suspected to contain blunders. Based on the 4-parameter model, 23 points of Norwegian data are flagged as outliers. In the case of using 5 and 7-paraemter models, they are 5 and 9 respectively. Only one residual is flagged as outlier in Finland and the Danish data can pass the tau-test.
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Kavouras M. (1982) On the detection of outliers and the determination of reliability in geodetic networks, Geodesy and Geomatics Engineering, University of New Brunswick, Canada


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