A weighted least-squares method for parameter estimation in structured models

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Abstract—Parameter estimation in structured models is generally considered a difficult problem. For example, the prediction error method (PEM) typically gives a non-convex optimization problem, while it is difficult to incorporate structural information in subspace identification. In this contribution, we revisit the idea of iteratively using the weighted least-squares method to cope with the problem of non-convex optimization. The method is, essentially, a three-step method. First, a high order least-squares estimate is computed. Next, this model is reduced to a structured estimate using the least-squares method. Finally, the structured estimate is re-estimated, using weighted least-squares, with weights obtained from the first structured estimate. This methodology has a long history, and has been applied to a range of signal processing problems. In particular, it forms the basis of iterative quadratic maximum likelihood (IQML) and the Steiglitz-McBride method. Our contributions are as follows. Firstly, for output-error models, we provide statistically optimal weights. We conjecture that the method is asymptotically efficient under mild assumptions and support this claim by simulations. Secondly, we point to a wide range of structured estimation problems where this technique can be applied. Finally, we relate this type of technique to classical prediction error and subspace methods by showing that it can be interpreted as a link between the two, sharing favorable properties with both domains.

I. INTRODUCTION

The goal of system identification is to obtain mathematical models for dynamical systems, using input and output data. Such models can be derived using different methods. The prediction error method (PEM) has been studied in detail ([1], [2]) and is a benchmark in the field, yielding asymptotically efficient estimates if the model orders are correct, but has to solve a non-convex optimization problem, with the risk of converging to local minima only. Subspace identification methods (SIMs) are attractive methods due to their computational advantages ([3], [4]), but are not as accurate as PEM [5], and do not incorporate structural information. Many other methods to overcome the non-convexity of PEM have also been proposed, e.g. the instrumental variable method (IV) [6] and its variants [7]–[10], indirect PEM (IPEM) [11], and the Steiglitz-McBride (SM) method [12].

The SM method belongs to a family of methods that we will denote iterative quadratic maximum likelihood (IQML) algorithms. These algorithms appeared first for filter design, where the problem was to fit a rational frequency response to a desired response in the H_2 -norm [13]. This problem can be

re-written as a weighted least-squares (WLS) problem, where the weighting matrix depends on denominator coefficients of the to be designed rational function. By using an estimate of these coefficients from a previous step in the algorithm, a new estimate can therefore be obtained by solving a standard WLS problem. A major step in the developments, from an estimation view, was taken in [14]. For the problem of signal in noise estimation, the statistically optimal weighting was derived. It was also in this work that the term IQML was coined. IQML has found widespread application in the communications area, e.g. [15].

IQML-like algorithms have also been developed for identification of dynamical systems. Perhaps, the earliest work is the SM method, appearing in [12]. It was shown in [16] that IQML and SM are equivalent for an impulse-input case. Later work include [17], [18], and [19]. In none of these works is the used weighting determined by statistical considerations. In this perspective, the result in [20], showing that SM is not asymptotically efficient, is not surprising.

In [21], the statistics are taken into account when forming the weighting matrix. Here, the presented method is based on instrumental variables, and, while simulations indicate good performance, the asymptotic variance does not correspond to the Cramér-Rao lower bound.

A very recent contribution is the pre-filtered SM method, in [22]. Asymptotic efficiency is achieved for Box-Jenkins models by pre-filtering the data with a high order ARX-model before applying SM.

In this contribution, we continue this development. For output-error (OE) models, we propose a method where, first, an unstructured model – more precisely, a high order finite impulse response (FIR) model – is estimated. Then, this estimate is used as “data” in an IQML-like step to estimate an OE model. Here, the second order statistics of the FIR model are used to obtain the optimal weighting matrix. We conjecture that this leads to an asymptotically efficient estimate if the order of the FIR model is allowed to increase as the sample size increases, and this is backed up by a simulation study consisting of a thousand randomly drawn systems. The method, thus, differs from [22]: in this paper, an unstructured impulse response model is used as an intermediate, whereas [22] uses pre-filtered data.

We reinforce the discussion in [21] that this class of methods lies at the crossroad between PEM and SIM, and that it shares favorable properties with both domains, namely asymptotic efficiency and flexibility in the parametrization with PEM, and guaranteed convergence with SIM.

For convenience only, we use an OE model setting in the

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paper. However, we point to the wide applicability this family of methods has, when it comes to parameter estimation of structured systems.

II. PROBLEM STATEMENT

To explain the proposed method, we consider an OE model, given by

$$y(t) = \frac{B(q)}{F(q)}u(t) + e(t), \quad (1)$$

where $F(q)$ and $B(q)$ are rational functions of the delay operator q^{-1} , as

$$\begin{aligned} F(q) &= 1 + f_1 q^{-1} + \dots + f_{n_f} q^{-n_f} \\ B(q) &= b_1 q^{-1} + \dots + b_{n_b} q^{-n_b}, \end{aligned} \quad (2)$$

and $e(t)$ is Gaussian white noise with variance λ_e . The system identification problem is to estimate the coefficients of the polynomials $F(q)$ and $B(q)$,

$$\theta = [b_1 \ \dots \ b_{n_b-1} \ b_{n_b} \ f_1 \ \dots \ f_{n_f-1} \ f_{n_f}]^T, \quad (3)$$

using the known input sequence $\{u(t)\}$ and the observed output sequence $\{y(t)\}$, for $t = 1, 2, \dots, N$.

III. ESTIMATING RATIONAL MODELS

In this section, we recapitulate different methods to estimate rational models as (1), and then present our proposed method, which uses ideas from these.

A. Unstructured Models

Consider the FIR model

$$y(t) = G(q)u(t) + e(t), \quad (4)$$

where

$$G(q) = \sum_{k=1}^m g_k q^{-k}. \quad (5)$$

The coefficients $\{g_k\}$ can be estimated using a least-squares (LS) approach. First, (4) can be written in vector form as

$$y = \Phi g + e, \quad (6)$$

with y , g , and e given by

$$\begin{cases} y = [y(1) \ y(2) \ \dots \ y(N)]^T \\ g = [g_1 \ g_2 \ \dots \ g_m]^T \\ e = [e(1) \ e(2) \ \dots \ e(N)]^T \end{cases}, \quad (7)$$

and where Φ is a Toeplitz lower triangular matrix of the input,

$$\Phi = \begin{bmatrix} u(0) & 0 & \dots & 0 \\ u(1) & u(0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u(m-1) & u(m-2) & \dots & u(0) \\ \vdots & \vdots & & \vdots \\ u(N-1) & u(N-2) & \dots & u(N-m) \end{bmatrix}. \quad (8)$$

Then, it is possible to solve for g using LS, obtaining

$$\hat{g} = (\Phi^T \Phi)^{-1} \Phi^T y, \quad (9)$$

where, for m large enough, the statistics of $\hat{g}$ are well approximated by

$$\hat{g} \sim \mathcal{N}(g, \lambda_e (\Phi^T \Phi)^{-1}), \quad (10)$$

with $\mathcal{N}$ being the normal distribution. Although $(\Phi^T \Phi)^{-1}$ is known, λ_e usually is not. However, an unbiased estimate is given by

$$\lambda_e \approx \frac{(y - \Phi \hat{g})^T (y - \Phi \hat{g})}{N - m}. \quad (11)$$

If the order m is chosen large enough, (5) can model (1) with good accuracy. However, the drawback is that the unstructured estimate (9) may have high variance, especially when m has to be chosen large.

B. Prediction Error Method

The essential idea of PEM is to minimize a cost function of the prediction errors. If a quadratic error criterion is used, the coefficients in (3) are obtained by the minimization

$$\hat{\theta} = \arg \min_{\theta} \sum_{t=1}^N \varepsilon^2(t, \theta), \quad (12)$$

where $\varepsilon(t, \theta)$ are the prediction errors

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t|t-1; \theta), \quad (13)$$

with $\hat{y}(t|t-1; \theta)$ the predictor of $y(t)$, given $y(t-1)$.

This is typically a non-convex optimization problem, which can converge to local minima. An advantage with PEM is that it is known to be asymptotically efficient [1].

C. Subspace Identification Methods

1) *Range space estimation*: SIMs are characterized by a rank reduction step based on a singular value decomposition (SVD) of a weighted matrix [5]. To present the main idea behind these methods, consider that the model in (1) has a state-space realization

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + e(t) \end{cases}, \quad (14)$$

and define the extended observability and controllability matrices as

$$\begin{cases} \mathcal{O}_e = [C \ CA \ \dots \ CA^{f-1}]^T \\ \mathcal{C}_e = [B \ AB \ \dots \ A^{p-1}B] \end{cases}, \quad (15)$$

respectively, where f and p are user-defined parameters, whose influence has been analyzed in [23]. The corresponding impulse response coefficients $\{g_k\}$ can be used to construct the Hankel matrix

$$H(g) = \begin{bmatrix} g_1 & g_2 & \dots & g_p \\ g_2 & g_3 & \dots & g_{p+1} \\ \vdots & \vdots & & \vdots \\ g_f & g_{f+1} & \dots & g_{f+p-1} \end{bmatrix}. \quad (16)$$

A key observation is that $H(g) = \mathcal{O}_e \mathcal{C}_e$, and, thus, has rank at most equal to the order n_f of the system. In an early version of SIM, the unstructured FIR estimate $\hat{g}$ is used to form an estimate $H(\hat{g})$ of $H(g)$. Due to the estimation error,

this estimate will have full rank. The key step is then to use the SVD of $H(\hat{g})$,

$$H(\hat{g}) = USV^T, \quad (17)$$

to form low rank estimates of $\mathcal{O}_e$ and $\mathcal{C}_e$. With $f = p$,

$$H(\hat{g}) = \sum_{k=1}^p s_k u_k v_k^T \approx \sum_{k=1}^{n_f} s_k u_k v_k^T = \hat{\mathcal{O}}_e \hat{\mathcal{C}}_e, \quad (18)$$

where s_k are the diagonal entries of S , u_k and v_k the column vectors of U and V , respectively, and

$$\begin{cases} \hat{\mathcal{O}}_e = [\sqrt{s_1}u_1 & \cdots & \sqrt{s_{n_f}}u_{n_f}] \\ \hat{\mathcal{C}}_e = [\sqrt{s_1}v_1 & \cdots & \sqrt{s_{n_f}}v_{n_f}]^T \end{cases}. \quad (19)$$

Then, the extended observability and controllability matrices can be estimated, for example, as in (19), from which the A , B and C matrices in (14) can be estimated using LS.

Subspace methods do not employ local non-linear optimization, and thus do not suffer from problems with local minima. This family of methods is also consistent under general conditions. Statistical (and numerical) properties can be improved by pre- and post-multiplying $H(\hat{g})$ with weighting matrices before the SVD [23]: $W_1 H(\hat{g}) W_2$. Different weighting matrices have been suggested in the literature, e.g. CVA [24], MOESP [25], and N4SID [4]. A drawback is that it is not easy to incorporate structural information in the A , B , and C estimates, e.g. that the numerator $B(q)$ in (1) has a different number of parameters than $F(q)$.

2) *Null-space estimation*: The method in the preceding section estimates the range space of $H(g)$, represented by $\hat{\mathcal{O}}_e$. However, the null-space of $H(g)$ contains even more detailed information about the structure of the system. To see this, notice first that

$$\frac{B(q)}{F(q)} = G(q) \Leftrightarrow F(q)G(q) - B(q) = 0. \quad (20)$$

To keep matters simple, we consider that

$$F(q) = 1 + f_1 q^{-1} + f_2 q^{-2}, \quad B(q) = b_1 q^{-1}. \quad (21)$$

In that case,

$$(1 + f_1 q^{-1} + f_2 q^{-2})(g_1 q^{-1} + \cdots + g_m q^{-m}) = b_1 q^{-1}. \quad (22)$$

This can be expressed using $H(g)$ in the following way. First, write (22) in matrix form, as

$$\begin{bmatrix} 0 & 0 & g_1 \\ 0 & g_1 & g_2 \\ g_1 & g_2 & g_3 \\ g_2 & g_3 & g_4 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ \vdots \end{bmatrix}. \quad (23)$$

Then, extend it by adding columns:

$$\begin{bmatrix} 0 & 0 & g_1 & g_2 & \cdots \\ 0 & g_1 & g_2 & g_3 & \cdots \\ g_1 & g_2 & g_3 & g_4 & \cdots \\ g_2 & g_3 & g_4 & g_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} f_2 & 0 & \cdots \\ f_1 & f_2 & \cdots \\ 1 & f_1 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} b_1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}. \quad (24)$$

We recognize that the lower block of the g -dependent matrix is $H(g)$, and thus we see that the null-space of $H(g)$ determines f_1 and f_2 . Therefore, using $H(\hat{g})$ as starting point, one can, in a first step, estimate this null-space, and then obtain estimates of f_1 and f_2 . This method has been outlined in [26]. As in the range space based methods, the SVD of $H(\hat{g})$ is used to obtain an estimate of the space of interest, and, also here, the weighting influences the statistical properties. An advantage of this procedure is that it is flexible in parametrization, where one can specify structures such as $n_f \neq n_b$, for example, which is not possible with the standard approach to SIM. To see this, notice that the first equation in (23) relates to b_1 . If there are more b -parameters, there will be additional equations that determine these parameters.

While there have been significant contributions to the statistical analysis of SIMs ([23], [27]–[31]), a complete analysis is still lacking. However, it is generally believed that such methods can be asymptotically efficient only in special cases. We believe that the problem can be traced to the weighting that is applied to the noisy matrix for which the SVD is computed (the Hankel matrix, in our simplified exposition). Since it is a matrix weighting, it cannot be tailored to the statistical properties of the individual elements of the matrix in question.

In the next section, we will start afresh from (23), but will not add any more columns. This will allow us to tailor the weighting exactly to the statistics of $\hat{g}$, paving the way for an asymptotically efficient estimate.

D. Connecting PEM and SIM

Rewriting (22) as

$$\underbrace{(g_1 - b_1)}_{\triangleq z_1} q^{-1} + \underbrace{(f_1 g_1 + g_2)}_{\triangleq z_2} q^{-2} + \cdots = 0, \quad (25)$$

and defining vector z by

$$z \triangleq [z_1 \quad z_2 \quad \cdots \quad z_m]^T = 0, \quad (26)$$

we continue with the example (21), reformulating (23) as

$$z = \begin{bmatrix} -1 & g_1 & 0 & 0 \\ 0 & g_2 & g_1 & 0 \\ 0 & g_3 & g_2 & g_1 \\ 0 & g_4 & g_3 & g_2 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} b_1 \\ 1 \\ f_1 \\ f_2 \end{bmatrix} = 0 \quad (27a)$$

$$= \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & g_2 & g_1 \\ 0 & g_3 & g_2 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\triangleq Q(g)} \underbrace{\begin{bmatrix} b_1 \\ f_1 \\ f_2 \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix}}_g. \quad (27b)$$

From (27a), it is clear that the subspace we are looking at is the null-space determined by the system's impulse response. Since it is a null-space, the statistical considerations can be better tailored than in the range space approach.

When estimating $Q(\hat{g})$, the unstructured FIR estimates $\hat{g}$ are noisy, given by $\hat{g} = g + v$, where

$$v \sim \mathcal{N}(0, \lambda_e (\Phi^T \Phi)^{-1}). \quad (28)$$

Therefore, the estimate $\hat{z} \triangleq Q(\hat{g})\theta + \hat{g}$ can be written as

$$\hat{z} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ f_1 & 1 & 0 & & & \vdots \\ f_2 & f_1 & 1 & & & \vdots \\ 0 & f_2 & f_1 & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_1 & 1 \end{bmatrix} v \triangleq T v, \quad (29)$$

which is thus distributed as

$$\hat{z} \sim \mathcal{N}(0, \underbrace{\lambda_e T (\Phi^T \Phi)^{-1} T^T}_{\triangleq P_z}). \quad (30)$$

Knowing the statistics of $\hat{z}$ allows solving for θ using a maximum likelihood (ML) approach. Firstly, we observe that, by taking the order m of the unstructured FIR model $\hat{g}$ large enough, $\hat{g}$ is almost a sufficient statistic for our problem. The error only lies in the truncation of the impulse response. Thus, $\hat{g}$ can be used to obtain an estimate of θ that is, in some sense, almost asymptotically efficient. Secondly, we observe that the map from $\hat{z}$ to $\hat{g}$ is unique. Therefore, we can use $\hat{z}$ to obtain an estimate of θ that is almost asymptotically efficient. This, we can achieve by considering the maximum likelihood estimate of θ based on $\hat{z}$. The likelihood function of $\hat{z}$ is given by

$$f(\theta; \hat{z}) = \frac{C}{\sqrt{2\pi \det(P_z)}} e^{-\frac{1}{2\lambda_e} \hat{z}^T (T(\Phi^T \Phi)^{-1} T^T)^{-1} \hat{z}}, \quad (31)$$

where C is a constant. Thus, the ML estimate is obtained by maximizing (31), or, equivalently, its logarithm:

$$\hat{\theta} = \arg \max_{\theta} \left\{ \log(C) - \frac{1}{2} \log \left[2\pi \det \left(\lambda_e T (\Phi^T \Phi)^{-1} T^T \right) \right] - \frac{1}{2\lambda_e} \hat{z}^T T^{-T} (\Phi^T \Phi) T^{-1} \hat{z} \right\}. \quad (32)$$

This is a non-convex problem, as both $\hat{z}$ and T depend on θ , so it cannot be solved in a straightforward way. However, the parameter-dependent part in the second term of (32) is constant, namely $\det(T) = 1$, and thus the first two terms are not parameter-dependent. Therefore, we only need to maximize the last term, which can be done by solving for $\hat{z}$ iteratively, considering that T is constant at each iteration step. In that case, the maximum can be found by

$$\frac{\partial}{\partial \hat{z}} [\hat{z}^T W \hat{z}] = 0, \quad (33)$$

where

$$W = T^{-T} (\Phi^T \Phi) T^{-1}. \quad (34)$$

Taking the derivative in (33) and solving for $\hat{\theta}$ leads to the WLS problem

$$\hat{\theta} = -(Q^T W Q)^{-1} Q^T W g. \quad (35)$$

At the next step, T is updated using the obtained $\hat{\theta}$, and the new estimation is performed with updated weighting.

Notice that, for white inputs, $\Phi^T \Phi = kI$, where I is the identity matrix and k a constant, and can thus be discarded in the suggested optimization procedure. However, if the input is not white, it has to be considered. This is a feature that distinguishes our method from SM and IQML, which do not take the input statistics into account, and therefore yield non-optimal results when colored input is used, as we show in the numerical simulation.

In summary, the proposed method consists of three steps:

- 1) a high order unstructured estimate of the model is obtained using (9);
- 2) model reduction to a structured estimate is performed using LS, i.e. (35) with $W = I$;
- 3) re-estimate the structured model using WLS, according to (35), with weighting W (34) obtained from the previous iteration.

It is also possible to continue to iterate.

The method is formally between PEM and SIM. As PEM, it allows a flexible parametrization, and we conjecture that it is asymptotically efficient, which we support by a numerical simulation in section V; with SIM, it shares guaranteed convergence. It is similar to the SM method and IQML, but we use the statistics of the high order model parameter estimation to obtain the ML estimate of the desired model coefficients, even for non-white inputs.

The order m of the unstructured model is important. To achieve asymptotic efficiency, it has to tend to infinity as the sample size grows to infinity at a suitable rate [32]. A practical way is to select the order m of the unstructured estimate that gives the structured model with lowest quadratic cost function (12) – recall that we are actually trying to minimize (12) using non-convex methods. In the examples that follow we optimize m over a grid of values.

Although we believe it is enough with one WLS-step to achieve asymptotic efficiency, small sample size properties may improve by further iteration steps. As with the order m , the number of iterations may be optimized using (12) as criterion.

IV. RELATION TO OTHER METHODS

A. Steiglitz-McBride and IQML

The Steiglitz-McBride method consists of three steps [1]. Consider an OE model as in (1). In step 1, LS is applied to the ARX model

$$F(q)y(t) = B(q)u(t) + e(t), \quad (36)$$

providing $\hat{B}(q)$ and $\hat{F}(q)$. In step 2, the data is filtered as

$$\begin{aligned} y_F(t) &= \frac{1}{\hat{F}(q)} y(t) \\ u_F(t) &= \frac{1}{\hat{F}(q)} u(t), \end{aligned} \quad (37)$$

In step 3, LS is applied to the ARX model

$$F(q)y_F(t) = B(q)u_F(t) + e(t), \quad (38)$$

with the new estimates $\hat{B}(q)$ and $\hat{F}(q)$. Finally, steps 2 and 3 are repeated until $\hat{B}(q)$ and $\hat{F}(q)$ converge. For more details and analysis on convergence and accuracy, see [20].

As in our method, SM applies LS to obtain a first estimate of the model. However, SM then refines these estimates by pre-filtering the input and output data, and applying LS repeatedly, while the proposed method performs a model reduction with WLS, choosing the appropriate weighting based on a ML argument.

Here, we presented SM applied to the raw data. Alternatively, SM can also be applied to an intermediate estimate of the impulse response instead of raw data, by setting, in (36), $y(t) = \hat{g}(t)$ and $u(t) = \delta(t)$, where $\delta(t) = 1$ for $t = 0$, and 0 otherwise. Notice that these two approaches have different asymptotic properties. In particular, the signal based approach is always suboptimal [20].

Another algorithm belonging to the IQML-family is presented in [18], and uses, as we do, an intermediate estimate of the impulse response. It is also based on (23), but the weighting does not take the statistics of the impulse response estimate into account. In the case of white input, and using the same length for the estimated impulse response, this method would be equivalent to the method we propose. Taking the input statistics into account, we maintain the ML properties, which is not the case for the method in [18].

The impulse-input approach to SM and IQML algorithms, such as [18], are equivalent [16], meaning that they compute the same poles at each iteration step.

B. Indirect PEM

The approach of IPEM [11] is to use PEM to estimate a higher order model, and then perform model order reduction to the desired order. This reduction uses the statistics of the first model's coefficients, providing a ML estimate of the latter model. This approach is similar to our proposal, but we avoid applying PEM by using LS to compute a high order unstructured model.

V. NUMERICAL SIMULATIONS

A Monte Carlo simulation is performed for a thousand randomly generated fourth order OE systems, for the set of sample sizes $N = \{300, 600, 1000, 3000, 6000, 10000\}$, using the colored input

$$u(t) = \frac{1}{1 - 0.008q^{-1} - 1.941q^{-2} - 0.006q^{-3} + 0.966q^{-4}} u_w(t), \quad (39)$$

where $\{u_w(t)\}$ is a Gaussian white noise sequence with unitary variance. The noise sequence $\{e(t)\}$ is white and Gaussian, with variance $\lambda_e = 20$. The following methods are compared:

- 1) the proposed method, with 10 iterations, and $m = \{0.1, 0.2, \dots, 0.9, 1\}N/3$ as lengths for the impulse response estimation (WLS);
- 2) the proposed method, with the true parameters as weighting, and $m = \{0.1, 0.2, \dots, 0.9, 1\}N/3$ as lengths for the impulse response estimation (I-WLS);
- 3) subspace algorithm, with CVA weighting (CVA);

- 4) prediction error method, estimating the initial condition, with maximum number of 50 iterations, and specified tolerance of 10^{-6} (PEM);
- 5) the Steiglitz-McBride method, applied to the estimated impulse response, under the same conditions as WLS (SM);

The last three methods are used according to the implementation in MATLAB 2013b.

The accuracy of each estimate is computed by measuring the FIT, given by

$$\text{FIT} = 1 - \frac{\|g - \bar{g}\|_2}{\|g - \tilde{g}\|_2}, \quad (40)$$

where g is a vector with the impulse response coefficients of the true system, $\bar{g}$ its mean, and $\tilde{g}$ the impulse response for the estimated model. A sufficiently long impulse response is taken to make sure that it has died out for the true system and all the estimated models. The FIT has a maximum value of 1, when both impulse responses coincide.

In Figure 1, the average FIT for each sample size and each method is presented. The upper bound to the proposed method, using the true weighting, provides results similar to PEM. The proposed algorithm does not achieve, for small sample sizes, its theoretical upper bound or PEM, but gets closer for increasing sample size. This supports our conjecture that the method is asymptotically efficient. Moreover, it performs better than the SIM used, with CVA weighting. Finally, the SM method is not competitive. The reason is the colored noise: notice that, if the input were white, WLS and SM would provide exactly the same result, since the input statistics could be discarded, and SM is being applied to an impulse response estimate.

In this figure, estimated unstable models were discarded. Table I presents the percentage of unstable cases, which is higher for WLS than for PEM or CVA, but lower than for SM. It is known that models obtained by LS can be unstable, even if the system is not [33], which can be verified here for the proposed method and SM, in particular cases. This could be avoided by, at each iteration, projecting the unstable poles inside the unit circle.

Fig. 1. FIT average obtained for each sample size, with 1000 Monte Carlo simulations of random fourth order systems, comparing several methods with colored input. Unstable cases were not considered.

TABLE I
PERCENTAGE (%) OF UNSTABLE CASES

Samples	WLS	I-WLS	CVA	PEM	SM
300	0.1	0	0	0.3	0.6
600	0.1	0	0	0	0.7
1000	0	0	0	0	0.5
3000	0.8	0.1	0	0	0.9
6000	0.7	0.4	0	0	0.7
10000	0.6	0.2	0	0.1	0.6
	0.38	0.12	0	0.07	0.67

VI. CONCLUSIONS

A system identification method has been proposed for estimating structured models. It consists of three steps: (1) a high order model is estimated, using least-squares; (2) least-squares is applied again to perform a model reduction; (3) the model is re-estimated using weighted least-squares, with the weights obtained from the estimate in the previous step.

The method connects ideas from PEM and subspace methods, sharing favorable properties with both. It is related to IQML algorithms, having similarities with the Steiglitz-McBride method. However, when compared to existing SM and IQML-based algorithms for parameter estimation in linear dynamical systems, it has the advantage of using the statistically optimal weighting, which becomes especially relevant for colored inputs.

Although the proposed method was presented for an OE model, it is applicable to a wider range of structured estimation problems, including other rational polynomial models (Box-Jenkins, ARMAX), Hammerstein models, and multi-input multi-output block structured generalizations of such structures.

We have conjectured that the method is asymptotically efficient, which has been backed up by a simulation study. A formal proof of this is on the agenda.

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