Element-Removal Games on Acyclic Graphs and Posets

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Abstract

This paper is about element-removal games on acyclic graphs and posets. Two players alternate in turn by removing one element at a time according to the rules. If a player on her turn cannot make a move she loses and the game ends.

Here we give formulas for the game value of games on trees where the players are only allowed to remove leaves. We also show how to compute the game value of games on some posets whose Hasse diagram is cycle-free and the players must remove maximal and/or minimal elements.

Sammanfattning

Denna uppsats handlar om elementborttagningsspel på acykliska grafer och pomängder. Två spelare turas om att ta bort element enligt reglerna. Spelet är slut när en spelare under sin tur inte kan göra ett drag och därmed förlorar.

Här ger vi formler för värden för spel på träd där spelarna enbart får ta bort löv. Vi visar även hur man kan beräkna värden för spel på pomängder vars Hassediagram är fritt från cykler och spelarna bara får ta bort maximala och/eller minimala element.
Acknowledgements

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1 Introduction

This is a master thesis that has been done at the Department of Mathematics at the Royal Institute of Technology on the topic of combinatorial game theory. This first chapter will give an introduction to element-removal games with background, content of this thesis and a summary of the main results.

An element-removal game is a 2-player game where the players remove one element at a time until one of the players is unable to do so and loses. These games are interesting because of their rare property of being integer-valued. This thesis will contain four different games. Three of the games, the max-removal game, min-removal game and min-max-removal game, are played on posets where a move is to remove a maximal, minimal and max- or minimal element respectively. The fourth game is the leaf-removal game which is played on a tree and the players are only allowed to remove leaves. Here are some examples on how the leaf- and min-max-removal game is played.

Example 1 (Min-max-removal game)

A min-max-removal game is played on a colored poset. Below is a poset where the minimal and maximal elements are marked in green.

![Figure 1: A poset in a min-max-removal game](image)

A gameplay can go as follows where Black’s and White’s moves are marked in blue and red respectively.

![Figure 2: An example of a gameplay of a min-max-removal game](image)
Example 2 (Leaf-removal game)
Here is a leaf-removal game played on a colored tree where the leaves are marked in green.

A gameplay could look like this where Black’s and White’s moves are marked in blue and red respectively.

Figure 3: A tree in a leaf-removal game

A gameplay could look like this where Black’s and White’s moves are marked in blue and red respectively.

Figure 4: An example of a gameplay of a leaf-removal game

1.1 Background

A poset game is played on a poset where a player chooses an element and removes it together with all greater elements. Nim is a well-known game played on piles of sticks. On a player’s turn, he chooses a pile and a number of sticks to remove from the pile. The player who removes the last stick in the end wins. Nim can be seen as a poset game played on vertical chains. A player makes a move by removing an element and all elements above. A gameplay of Nim can be seen in example 3.

Nim is an impartial game meaning that both players got the same moves in all positions of the game while a partizan game is a game where the options differ between the players. In partizan poset games the same rules are applied but each elements in the poset is colored black or white and the players must choose an element of their own color.
Example 3 (A gameplay of Nim)
An example of a gameplay of Nim on 3 heaps of sizes 4, 6 and 7. Right’s and Left’s moves are marked in red and blue.

Right wins

Figure 5: An example of a gameplay of Nim

Poset games on Young diagrams has been studied in F. Wahlberg [6]. He shows that these games are valued between 0 and 1 and he also provides a formula for chess-colored Young diagrams with 3 rows or less. Figure 6 shows an example of a Young diagram and how it can be seen as a poset.

Figure 6: A poset game created from a Young diagram with 3 rows.

In this thesis we will focus on element-removal games (the players removes only one element at the time) on colored graphs and posets. This has been done on Young diagram by R. Staffas [7]. She presents a linear time algorithm to calculate any Young diagram with only two rows. She also studied what integers could occur as value of a max-removal game on some Boolean lattice. Here we will continue with the work by E. Järleberg and J. Sjöstrand, [4] and [5]. In their work they prove that all element-removal games are integer-valued and that it is PSPACE-complete to determine the winner of a general max-removal game. They introduces a concept called blocking triple which turns out to be an important part in max-removal games and that some posets that do not contain a blocking triple or are chess-colored have a very simple formula for various element-removal games which is just the difference between the number of black and white elements. Also they show how to calculate the value of the max-removal game on any tree poset which is a poset where every element covers exactly one element except for the root which covers no elements.
1.2 Content

In this paper we will study three different acyclic posets: tree posets, double tree posets and zigzag posets, see Figure 7. By acyclic we mean that the Hasse diagram of the poset is cycle-free as an undirected graph. A double tree poset is two tree posets that share the same root but one of them is turned upside down. In a zigzag poset all elements have exactly two cover relations except for two elements that have only one each.

We will show formulas for the game value of the min-, max- and min-max-removal game on tree posets and double tree posets and also the max-removal game value on zigzag posets. The result can also be applied on similar acyclic posets. We will also prove a formula for the leaf-removal game. Lastly we show that it is PSPACE-complete to determine the winner of a min-max-removal game on general posets.

Figure 7: From left to right we have a tree poset, a double tree poset and a zigzag poset
2 Preliminaries

This section will give a brief introduction to game theory with definitions, theorems and notation that will be used in the rest of the thesis. All theory in this section is taken from the books [1], [2] and [3]. We will begin with the definition of posets and games and then we will show that all element-removal games are integers.

2.1 Posets, partially ordered sets

The games in this paper consists of a colored graphs or posets together with a set of rules.

**Definition 1 (Partially ordered sets)**
A partially ordered set $P$ (or poset, for short) is a set together with a binary relation denoted $\leq$ satisfying three axioms:

1) Reflexivity: $t \leq t \quad \forall t \in P$
2) Antisymmetry: If $s \leq t$ and $t \leq s$ then $t = s$
3) Transitivity: If $s \leq t$ and $t \leq u$ then $s \leq u$

We use the obvious notation that:

- $t \geq s$ means $s \leq t$
- $s < t$ means $s \leq t$ and $s \neq t$
- $t > s$ means $s < t$

Two elements $s$ and $t$ in a poset are comparable if $s \leq t$ or $t \leq s$; otherwise $s$ and $t$ are incomparable, denoted $s \nmid t$. We say that $t$ covers $s$ or $s$ is covered by $t$ denoted $s < t$ or $t > s$ if $s < t$ and no element $u$ in the poset satisfies $s < u < t$.

A Hasse diagram of a poset can be seen as a directed graph where the elements and cover relations in the posets are vertices and directed edges respectively. But we have to be careful to not write all relations in a chain of linear relations, only the cover relations. For example we write

since it is already known that the top and bottom element is related by transitivity.

The disjoint union (or direct sum) of two posets $P$ and $Q$ is a poset $P + Q$ on the union $P \cup Q$ such that $s \leq t$ in $P + Q$ if either $s, t \in P$ and $s \leq t$ in $P$ or $s, t \in Q$ and $s \leq t$ in $Q$.

A colored poset is a poset where each element also has a color which in this thesis will be black and white (one color to each player). In a chess-colored poset no adjacent elements have the same color.
Example 4 (A colored poset)

In the introduction we saw an example of a colored poset (the min-max-removal game) namely the following.

![Figure 8: A colored poset](image_url)

2.2 Combinatorial game theory

In combinatorial games two players alternate in turn. Both players know all the details of the game position at all times i.e. there is nothing hidden and there is no chance involved like drawing cards or rolling a dice. The game continues until the player in turn cannot make a move and loses. Examples of combinatorial games that have been popular are Connect 4, Checkers, Tic-Tac-Toe, Gomoku (5 in a Row), Othello, Chess and Nim. We assume that the players are always playing optimally. The players are often called Left and Right, Blue and Red or in this case Black and White. He and she refer to Black and White, respectively.

**Definition 2 (Definition of a game)**

A game (position) \( G \) is defined by its options, \( G = \{ G^L | G^R \} \), where \( G^L \) and \( G^R \) are the set of Black’s and White’s options respectively.

We usually omit the braces around the sets \( G^L \) and \( G^R \) for brevity. A typical element of \( G^L \) and \( G^R \) are often denoted by \( G_L \) and \( G_R \).
Example 5
The leaf-removal game in figure 2 can be written as

\[
\begin{cases}
    , & \text{Left player options} \\
    , & \text{Right player options}
\end{cases}
\]

Figure 9: Example of the game positions in a leaf-removal game

Definition 3
Let \( G \) be a game. Then

\[
\begin{align*}
G > 0 & \quad (G \text{ is positive}) \quad \text{Black will win} \\
G < 0 & \quad (G \text{ is negative}) \quad \text{White will win} \\
G = 0 & \quad (G \text{ is zero}) \quad \text{Second player in turn will win} \\
G \parallel 0 & \quad (G \text{ is fuzzy to } 0) \quad \text{First player in turn will win}
\end{align*}
\]

If we combine these we get an extended definition.

Definition 4
Let \( G \) be a game. Then

\[
\begin{align*}
G \geq 0 & \quad \text{Black will win as second player} \\
G \leq 0 & \quad \text{White will win as second player} \\
G < 0 & \quad \text{Black will win as first player} \\
G > 0 & \quad \text{White will win as first player}
\end{align*}
\]

Games can be added together to create a larger game. A sum of games \( H = G_1 + \cdots + G_n \) is to place them side by side. Each \( G_i \) is called a component of the larger game \( H \) and a player makes a move in \( H \) by first choosing a component \( G_i \) and then a move in that component. The negative of a game
$G$ is denoted $-G$ and for element-removal games it is defined to be the same game but the players have swapped colors and hence the options. This gives that $G + (-G) = 0$ (a second player win) since the second player can mimic each move the first player makes. We also write $G - H$ instead of $G + -H$.

**Example 6**
The sum of the min-max-removal games $G$ in figure 8 and its negation $-G$, i.e. $G + (-G)$ would then look as follows,

$G + (-G) =$

![Figure 10: Examples of colored trees](image)

We see that if Black starts and makes a move in the component $G$ then White can do the same move in the component $-G$ and vice versa. The same case if White starts then Black can mimic White’s moves. This shows that this is a second player win and hence the game value is equal to 0.

Now we are ready define a sum of games and the negative of a game.

**Definition 5** (The sum and negatives of games)

Let $G$ and $H$ be games. Then,

$$-G = \{-G^R, -G^L\}$$

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$

This is useful when we want to prove a relation between two games. Most often we want to prove that two games, $G$ and $H$, are equal. This can be done by checking if $G - H = 0$ i.e. a second player win.
**Definition 6 (Game relations)**

Let $G$ and $H$ be games. Then

\[ G = H \iff G - H = 0 \]
\[ G > H \iff G - H > 0 \]
\[ G < H \iff G - H < 0 \]
\[ G \geq H \iff G - H \geq 0 \]
\[ G \leq H \iff G - H \leq 0 \]

In fact the equivalence classes of games defined this way form an abelian group which makes it easier to calculate the outcome of a complicated sum of games.

**Theorem 1 (The abelian group of games)**

Games form an abelian group. Let $G$ be the group of games and let $G, H, J \in G$ then,

1) **Closure:** $G + H \in G$

2) **Identity:** $G + 0 = G$ ($0$ is the identity game)

3) **Inverse:** $G + (-G) = 0$ (the inverse of $G$ is $-G$)

4) **Abelian:** $G + H = H + G$

5) **Associativity:** $(G + H) + J = G + (H + J)$

We end this section with one last useful proposition.

**Proposition 1 (Combinations of negative games)**

Let $G$ and $H$ be games, then

\[-(-G) = G\]
\[-(G + H) = (-G) + (-H)\]

### 2.3 Integer games

Some games are equal to integers, in a sense that will be made precise later. It is very good for Black (respectively White) if a game is equal to a integer that is very large in absolute value and positive (negative). We will later see that if we have an element-removal game $G = n > 0$, where $n$ is an positive integer, then Black got $n$ more moves in $G$ than White. For example when the game is over (and the players have made equally many moves) Black can make $n$ more
moves without giving White a move at all, or Black could start the game by making \( n \) moves in a row and then the remaining game would be a zero game (a second player win).

We start by determining the game value of some simple element-removal games.

**Example 7** (Some simple leaf-removal games)

Games where the second player wins:

The most trivial game is the empty game where both players got no options.

\[
G = \{ | \} = 0
\]

This is clearly a second player win since the first player cannot move and loses immediately.

Examples of graphs in the leaf-removal game where the second player wins are cycles, since there are no leaves in a cycle, and chains with an even number of elements that alternate in black and white.

\[
G_1 = \text{cycle} = 0 \quad \quad G_2 = \text{chain} = 0
\]

Games where Black or White wins:

A very good game for Black is the leaf-removal game on a tree where every element is black. If a tree contains \( n \) black elements and no white elements then Black got \( n \) moves in that tree. That game would then be written as

\[
G = n > 0
\]

The converse is true for White, i.e. all elements are white. Then we write

\[
G = -n < 0
\]

\[
G_3 = \text{tree with black elements} = 7 \quad \quad G_4 = \text{tree with white elements} = -13
\]

**Figure 11:** \( G_3 \) is a win for Left and the \( G_4 \) is a win for Right in the leaf-removal game

Games where the first player wins:

We will later prove that all element-removal games are integer-valued and hence there exists no element-removal game where the first player wins.

For instance if we add the four games \( G_1, G_2, G_3 \) and \( G_4 \) in example 7 we get

\[
G_1 + G_2 + G_3 + G_4 = 0 + 0 + 7 - 13 = -6
\]

which means that White will win since it is negative. Now it is a good opportunity to define integer games.
Definition 7 (Integer games)

Integer games are defined as follows

\[ 0 = \{ \quad \} \]
\[ 1 = \{ 0 \} \]
\[ n = 1 + 1 + \cdots + 1 \]

From this definition it follows that,

\[ 0 = \{ \quad \} \]
\[ n = \{ n - 1 \} \quad n > 0 \]
\[ n = \{ \quad \} \quad n < 0 \]

It also follows that integer games form a subgroup of games. Some of the easiest colored posets to investigate are chains.

Example 8

Consider a min-max-removal game on a chain with \( b \) black elements at the top followed by \( w \) white elements at the bottom. Since the players alternate in turn the winner of this game will be the player who got the most elements in the chain. The game is equal to

\[
G = b - w = \begin{cases} 
    \{ b - w - 1 \} & \text{if } b - w > 0 \\
    \{ |b - w| + 1 \} & \text{if } b - w < 0 \\
    0 & \text{if } b - w = 0
\end{cases}
\]
3 Previous work

Here we first look at what we already know about element-removal games and state some definitions before we continue with the main results in the next section. All results in this section are from Järleberg and Sjöstrand [4].

3.1 Proof procedures

Most of the theorems in this thesis simplify a game \( G \) to an easier game \( H \) that has the same outcome. We write \( G = H \iff G - H = 0 \), a second player win which we then have to prove. The games \( G \) and \( H \) will have a very similar structure. Often \( G \) and \( H \) look the same except that some part is removed in \( H \). For some elements we will have a very clear bijective mapping between \( G \) and \( H \) which we call the mirror mapping. For those elements \( s \) in \( G \) we say that they have a mirror image \( \bar{s} \) in \( H \) and vice versa. In the same way if we have a set \( S \) of elements then \( \bar{S} \) is the mirror image of \( S \) and vice versa. During play a poset \( P \) will change and become smaller so for the rest of the thesis let \( P' \) denote the remaining part of \( P \).

Example 9

In the max-removal game the players can only remove maximal elements. Figure 12 shows two posets, \( T \) and \( S \), which together create a zero-game in the max-removal game (this will later be shown by Theorem 3). \( S \) looks exactly like \( T \) except that the colors are swapped and everything below the element \( \bar{f} \) is removed. The elements \( a, b, ..., f \) in \( T \) has a very clear bijective mapping to the elements \( \bar{a}, \bar{b}, ..., \bar{f} \) in \( S \).

![Figure 12: An example to show the mirror image of some elements between two game components](image_url)

To make it easy to follow we will try to include as many figures as possible. These figures will contain general graphs where the important part is in detail and the rest something general. General graphs and subgraphs will be illustrated with ellipses or other shapes filled with different shades of gray. When the poset is a tree poset (or contains tree posets as subposets) we sometimes put a black or white element at the bottom of the gray shape to mark the color of the root. If an element got a black and white striped pattern then the color could be either...
black or white.

**Figure 13:** From left to right we have a poset (or a graph), a tree poset with a black root, a tree poset with a white root and a node which could be black or white.

### 3.2 Different element-removal games

An element-removal game consists of a finite set $X$ of black and white elements and a removability function $\rho$. Each element in $X$ is colored black or white and in each move the player (Black or White) removes an element of his own color. However, all elements are not necessarily removable at any stage but once an element becomes removable it remains so until it is removed. The removable elements are determined by the removability function $\rho : 2^X \to 2^X$ with the property

$$\rho(B) \cap A \subseteq \rho(A) \subseteq A$$

for any $A \subseteq B \subseteq X$. In example 2 and 1 in the first chapter, $\rho$ maps to the elements marked in green.

**Figure 14:** Venn diagram of the removability function $\rho$

So far we have shown examples of three different element-removal games namely the min-max-removal game, the leaf-removal game and the max-removal game where $\rho$ maps to the min- and maximal elements, the leafs and the maximal elements respectively. One similar game is the min-removal game played on a poset where a move is made by removing a minimal element.
Definition 8 (Pomax, Pomin, Pominmax and Leaf)
If $P$ is a colored poset and $T$ is a colored tree we define

- $\text{Pomax}(P)$ to be the max-removal game on $P$,
- $\text{Pomin}(P)$ to be the min-removal game on $P$,
- $\text{Pominmax}(P)$ to be the min-max-removal game on $P$,
- $\text{Leaf}(T)$ to be the leaf-removal game on $T$

3.3 Element-removal games are integer-valued

Recall that an integer-valued game is on the form $n = 1 + 1 + \cdots + 1$ or its negation. The following lemma identifies some integer-valued games.

Lemma 1
A game is integer-valued if its options are integer-valued and the difference between any left and right options is at least 2. In that case, the value of the game is the integer closest to zero that is strictly larger than any left option and strictly smaller than any right option.

From the theory section we know that all games form an abelian group and when the games are integers they work like the group of integers under addition. Integers are very comfortable to work with which makes the next theorem valuable.

Theorem 2
Any element-removal game is integer-valued.

Proof. We want to show that for any element-removal game $G$ and any left option $G^L$, we have that

$$G - G^L \geq 1 \quad (1)$$

If this holds, then by symmetry,

$$G - G^R \leq -1 \quad (2)$$

for any right option $G^R$. Adding equation (1) and (2) we get

$$G^R - G^L \geq 2$$

which together with Lemma 1 and induction gives that $G$ is integer-valued. To prove equation (1) we need to show that $G - G^L - 1$ is a win for Black as a second player. Let $x$ be the element that is removed from $G$ to obtain $G^L$. We get three cases.
White removes an element, \( y \) in \( G \):
Black replies by removing \( \bar{y} \) from \( -G^L \) which is removable since \( y \) was removable in \( G \). This results in \( G^R - G^{RL} - 1 \) where \( G^R \) is obtained by removing \( y \) from \( G \) and \( G^{RL} \) is obtained by removing \( x \) from \( G^R \) (which is removable from \( G^R \) since it was removable from \( G \)). By Conway induction, this game is non-negative.

White removes an element \( \bar{y} \) in \( -G^L \):
Black replies by removing \( x \) in \( G \) which results in the game \( G^L - G^{LL} - 1 \) where \( G^{LL} \) is obtained by removing \( y \) from \( G^L \). By induction the game is non-negative.

White makes his only move in \( -1 \):
Black replies by removing \( x \) in \( G \) which results in the position \( G^L - G^L = 0 \) and Black wins.

\[ \square \]

### 3.4 Balanced Games

One class of element-removal games are balanced games which has a very simple formula for the game value which is just the number of black elements minus the number of white elements. For the rest of the thesis let \( b_S \) and \( w_S \) be the number of black and white elements in a set \( S \)

<table>
<thead>
<tr>
<th>Definition 9 (Balanced Game)</th>
</tr>
</thead>
<tbody>
<tr>
<td>An element-removal game is balanced if it has the following two properties:</td>
</tr>
<tr>
<td>1) All options are balanced</td>
</tr>
<tr>
<td>2) If all removable elements are of the same color, then at least half of the total set of elements have that color</td>
</tr>
</tbody>
</table>

When playing a balanced game the outcome will always be the same no matter which strategies the players are using.

<table>
<thead>
<tr>
<th>Proposition 2 (Value of a balanced game)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The value of a balanced game ( G ) is</td>
</tr>
<tr>
<td>( G = b_G - w_G )</td>
</tr>
<tr>
<td>and the outcome of the game is independent of the players strategies</td>
</tr>
</tbody>
</table>

**Proof.** Since all options of \( G \) are balanced, by induction, the value of any left option is \( G^L = b_G - w_G - 1 \) and the value of any right option is \( G^R = b_G - w_G + 1 \). If \( G \) has at least one right and left option then the value of \( G \) is

\[
G = \{ b_G - w_G - 1 | b_G - w_G + 1 \} = b_G - w_G.
\]
If $G$ has no right option then since $G$ is balanced we got $b_G \geq w_G$ which gives that

$$G = \{b_G - w_G - 1\} = b_G - w_G.$$  

The case where $G$ has no left options is completely analogous.  

All options of $G$ has the same value since they are balanced and the fact that the players remove one element at the time. Hence the outcome does not depend on the strategies since it does not matter which options the players choose.  

To create a balanced game we could use a chess-coloring on a tree poset in the max-removal game or a chess-coloring on a tree in the leaf-removal game. In a chess-coloring no adjacent elements have the same color. In a poset two element $s$ and $t$ are adjacent if $s \preceq t$ or $t \preceq s$.

**Definition 10 (Tree poset)**

A non-empty tree poset is a poset where each element covers exactly one element except the root which covers zero elements. An empty tree poset is an empty poset.

Two examples of tree posets can be seen in example 1 and 9.

**Proposition 3 (Pomax on chess-colored tree posets)**

The pomax game on a chess-colored tree-poset $T$ is balanced.

$$\text{Pomax}(T) = b_T - w_T$$

**Example 10**

The value of a leaf-removal game or max-removal game on this chess-colored tree $T$ is $b_T - w_T = 5 - 5 = 0$ which means that the player who starts will lose.

Proof of proposition 3. Suppose that all removable elements are black. Then each white element can be paired with one of the black elements that covers it. This pairing shows that there are at least as many black elements as white elements and hence the game is balanced and by Proposition 2 we get the desired formula.  

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Proposition 4
The leaf-removal game on any chess-colored tree $T$ is balanced.

\[ \text{Leaf}(T) = b_T - w_T \]

Proof. Choose any element in the tree to be a root (unique minimal element) in a tree poset and let all edges (cover relations) be directed from the root. Then the proof from Proposition 3 applies.

The class of pomax games on chess-colored tree posets can be extended to tree posets without blocking triples.

Definition 11 (Blocking triple)
A blocking triple in a colored poset is a triple of elements $x \lessdot y \lessdot z$ such that $x$ and $y$ are of the same color and $z$ is of a different color. If $x$ and $y$ are black (white) then we say that $x \lessdot y \lessdot z$ is a black (white) blocking triple.

It turns out that blocking triples form an important part when determine the game value of pomax (and also pominmax) games. A black blocking triple can be seen in figure 15.

Proposition 5
The pomax game on any colored tree poset $T$ without blocking triples is balanced.

\[ \text{Pomax}(T) = b_T - w_T \]

Proof. Identical to the proof of proposition 3.
Example 11
Tree posets without blocking triples are often chess-colored at the bottom but not necessarily completely chess-colored at the top. The following tree $T$ contains no blocking triples and is a win for White in the pomax game since $b_T - w_T = 6 - 8 = -2$.

![Tree poset without blocking triples](image)

Figure 16: Tree poset without blocking triples

3.5 Max-removal games on tree posets

Consider the pomax game on the chain in figure 15. The game is clearly a zero-game. If Black starts he loses immediately and if White starts she loses after the 4 topmost elements have been removed. The two elements at the bottom will not affect the value at all. This suggests the following definition.

Definition 12 (Essential and non-essential part)
For any colored poset $P$, its essential part, denoted by $\text{ess } P$, is the (unique) maximal upper set that does not contain any blocking triple. The rest of the poset is the non-essential part of $P$, denoted $\text{non-ess } P$.

The elements in the essential part will be called the essential elements and those in the non-essential part will be called the non-essential elements. We will soon see that we can remove the non-essential part from a pomax game without affecting the game value. The rest of the game is then balanced and can easily be computed with Proposition 5. To state and prove a formula for the pomax value on any tree poset we first need the following lemma.

Lemma 2
Let $T$ be a black-rooted colored tree poset with at least one white element but no blocking triple. Let $m$ be the (integer) game value of $\text{Pomax}(T)$. Then, in the game $\text{Pomax}(T) - m$, if Black starts White can win before Black gets an opportunity to remove the root of $T$.

Proof. The game $\text{Pomax}(T) - m$ is a second player win by proposition 5 no matter what strategies they are using. Hence Black will lose if White starts. If White removes all elements in the $-m$ component before making a move in the $\text{Pomax}(T)$ component the root of $T$ will never be removed (since it is the last element that can be removed).
**Theorem 3** (Value of pomax games on tree posets)

Let $T$ be a colored tree poset, then

$$\text{Pomax}(T) = \text{Pomax}(\text{ess } T) = b_{\text{ess }} T - w_{\text{ess }} T$$

**Example 12**

Here is a tree poset $T$ with blocking triples in a pomax game. The border between the essential and non-essential part is marked in blue. White will win since $b_{\text{ess }} T - w_{\text{ess }} T = 5 - 9 = -4$.

---

**Proof of Theorem 3.** We want to show that

$$\text{Pomax}(T) - \text{Pomax}(\text{ess } T)$$

is a second player win. Figure 17 contains a tree with a black and a white blocking triple with forests (possibly empty or single trees) covering them. Also there is a striped element below a blocking triple that could be covered by a forest or tree.

**Strategy:**

For each element the first player removes from the component $\text{Pomax}(T)$, the second player can respond by removing the mirror image of that element in the component $-\text{Pomax}(\text{ess } T)$ and vice versa. But we also need a strategy at the border between the essential part and non-essential part.
Let $x \preceq y \preceq z$ be a black blocking triple then if White also applies the strategy from the proof of lemma 2, $y$ can act as a "root" in a subtree and Black will never get the opportunity to remove the $y$-element in any black blocking triple or elements below $y$. This gives that the players will never reach the non-essential part of $T$ in the component $\text{Pomax}(T)$, hence it can be removed and then the remaining game is clearly a zero game.

White starts:
This proof is completely analogous to when Black starts.

3.6 Max-removal games are PSPACE-complete

So far we have only been looking at trees. We will now show that it is very hard in general to find the winner of a pomax game even for very shallow tree posets. The height of a poset is the length of the longest chain of the poset.

**Theorem 4** [E. Järleberg, J. Sjöstrand, 2014]
The problem of deciding whether a given pomax game equals zero is NP-hard even if the height of the colored poset is restricted to two.

**Proof.** We will make a reduction from the NP-complete problem 3-SAT.

**3-satisfiability (3-SAT)**

**Input:** A 3CNF-formula

**Output:** "Yes" if and only if the formula is true for some assignment of the variables

A 3CNF-formula (CNF is short for *conjunctive normal form*) is a conjunction of classes where each clause is a disjunction of 3 literals, each literal being a
variable or a negation of a variable. An example is \((x_1 \lor \neg x_2 \lor x_4) \land (x_2 \lor \neg x_3 \lor \neg x_4)\). Given a 3CNF-formula we will construct a colored poset (in polynomial time) whose pomax value is zero precisely when the formula is true. The construction goes as follows.

1. For each variable \(x_i\) in the formula we put two white assignment elements in the poset, called \(x_i = 0\) and \(x_i = 1\) (where 0 and 1 should be interpreted as "false" and "true", respectively.

2. For each clause in the formula we put a black clause element \(c_j\) in the poset and we let it be covered by exactly those assignment elements that would make the clause false. For example the clause element corresponding to \((x_1 \lor \neg x_2 \lor x_4)\) would be covered by \(x_1 = 0, x_2 = 1\) and \(x_4 = 0\).

3. Put one black candy element for each \(x_i\) and let the corresponding \(x_i = 0\) and \(x_i = 1\) cover that element.

4. Put as many black isolated elements in the poset as there are Boolean variables.

We want the-removal of assignment element \(x_i = \alpha\) during a play to correspond to actually assigning the value \(\alpha\) to the variable \(x_i\). If White tries to "cheat" by removing both \(x_i = 0\) and \(x_i = 1\) then Black gets a candy element. We will now show that this is a second player win if and only if the 3CNF-formula is satisfiable.

**White starts**

White cannot win since Black has an isolated element for each pair of white assignment elements and if White "cheats" Black gets candy.

**Black starts**

White will win unless some of the black clause elements are uncovered during the game. Clearly, White can avoid uncovering a clause element precisely when the 3CNF-formula is satisfiable.

The restriction that the height is at least two can not be improved because all elements in a tree of height one is removable which gives a balanced game.
Theorem 5 [E. Järleberg, J. Sjöstrand, 2014]

The problem of deciding whether a given pomax game equals zero is PSPACE-complete even if the height of the colored poset is restricted to three.

The proof uses a reduction from the archetypical PSPACE-complete problem in a similar way as in the proof of Theorem 4.
4 Main results

The goal of the thesis is to find formulas for element-removal games on acyclic graphs and posets. The element-removal games will be pomax, pomin, pominmax and leaf. We will begin with some lemmas which will be useful in all element-removal games. Then we will show a formula for the leaf-removal game before we continue with pomax, pomin and pominmax on tree posets, double tree posets and zigzag posets. Last we will show a transformation from pomax to pominmax that will be used to show that the pominmax value is at least as hard to find as the pomax value for general posets.

4.1 Element-removal games in general

Here, two small lemmas will be presented that will be useful in the coming subsections.

Lemma 3

Let $G(A)$ be an element-removal game on the set $A$ with the removability function $\rho$. Let $v$ be a black element in $\rho(A)$ then

$$G(A \setminus \{v\}) < G(A).$$

Proof. Since $v \in \rho(A)$ i.e. $v$ is a removable element in $G(A)$, we have that $G(A \setminus v)$ is a left option in the game and since all element-removal games are integer-valued we must have that $G(A \setminus \{v\}) < G(A)$. □

Consider a poset in a pomax game where a black maximal element $l$ only covers a black element $v$ (see figure 19). The set of elements $S$ that are less than $v$ is also less than $l$. When $l$ is removed the elements in $S$ will still be unreachable since $v$ is still in the poset. Also since $l$ only covers $v$, the removal of $l$ will not help White for the moment. Therefore it seems reasonable for Black to remove $l$. The next lemma is a result of this observation.

Lemma 4

Let $G(A)$ be an element-removal game on the set $A$ with the removability function $\rho$. Let $v$ be a black element in $\rho(A)$ such that $\rho(B \setminus \{v\}) \setminus \rho(B)$ does not contain any white element for any $B \subseteq A$. Then

$$G(A) = G(A \setminus \{v\}) + 1.$$
Example 13
Lemma 4 can be applied in pomax, pomin, pominmax and leaf to disconnect removable elements which only have relations/edges to elements of the same color. Here is an example where $G(T)$ could be any of those games on a tree poset $T$ (tree in leaf). The elements that can be disconnected are marked in green.

![Figure 20: Example of how Lemma 4 can be used](image)

\[ G(T) \quad \text{=} \quad G(T') + 2 - 3 \]

Proof of Lemma 4. With Lemma 3 and Theorem 2 we get that
\[ G(A) \geq G(A \setminus \{v\}) + 1 \]
Now we want to prove that $G(A) \leq G(A \setminus \{v\}) + 1$ i.e. that White will win
\[ G(A) + G(\bar{A} \setminus \{\bar{v}\}) - 1 \]
if Black starts, where $G(\bar{A} \setminus \{\bar{v}\}) = -G(A \setminus \{v\})$.
Note that $\rho(B \setminus \{v\}) \setminus \rho(B)$ is the set of elements that becomes removable in $B$ when $v$ is removed.

![Figure 21: Illustration of the game $G(A) + G(\bar{A} \setminus \{\bar{v}\}) - 1$](image)

Black removes $v \in A$
White responds by removing $-1$ and the remaining game becomes a zero game.

Black removes $u \in A \setminus \{v\}$
White responds by removing $\bar{u} \in \bar{A}$. Now if we let $A' = A \setminus \{u\}$ and $A' = A \setminus \{\bar{u}\}$ the remaining game becomes $G(A') + G(A' \setminus \{v\}) - 1$.
Since $A' \subseteq A$ we still have that $\rho(A' \setminus \{v\}) \setminus \rho(A')$ does not contain any white elements and the induction principle applies.
Black removes \(\bar{a} \in \bar{A} \setminus \{v\}\) 
White responds by removing \(u \in A\) which is possible since
\[
\begin{align*}
&\left. u \notin \rho(A \setminus v) \setminus \rho(A) \right) \\
&\left. u \in \rho(A \setminus v) \right) \\
\implies & u \in \rho(A)
\end{align*}
\]
Again, if we let \(A' = A \setminus \{u\}\) and \(\bar{A}' = \bar{A} \setminus \{\bar{a}\}\) the remaining game becomes \(G(A') - G(A' \setminus \{v\}) - 1\). By the same reasoning as in the previous case the induction principle applies.

4.2 The leaf-removal game on trees

Once again consider the chain in figure 15. In the max-removal game the players can only remove maximal elements while in the leaf-removal game the players are allowed to remove elements of degree one or zero. A game on the chain will end differently the leaf removal game than in the pomax game. With this in mind we define a blocking path to help us calculate the value of a leaf-removal game.

\begin{definition}
A blocking path is a path \(v_1, \ldots, v_n\) of length \(n \geq 4\) such that
- \(v_2\) and \(v_3\) has a different color than \(v_1\)
- \(v_{n-2}\) and \(v_{n-1}\) has a different color than \(v_n\).
\end{definition}

With this definition we can find a way to determine all leaf-removal games on graphs. We begin with trees that are balanced in the leaf-removal game but first we need a lemma.

\begin{lemma}
Let \(T\) be a tree without blocking paths of length 5. If \(T\) contains 2 blocking paths \(u_1, u_2, u_3, u_4\) and \(v_1, v_2, v_3, v_4\) then \(\{u_2, u_3\} = \{v_2, v_3\}\).
\end{lemma}

\begin{proof}
Recall that every pair of elements in a tree has a unique path between them (this is because trees do not have cycles). Let \(U = \{u_2, v_3\}\) and \(V = \{v_2, v_3\}\). We will use proof by contradiction and split the proof into two cases.

\begin{itemize}
- \(U\) and \(V\) has nothing in common
\end{itemize}

Without loss of generality we can assume that the shortest path from \(U\) to \(V\) starts in \(u_3\) and ends in \(v_2\). With this setting we have that \(u_2\) and \(u_1\) is not a part of this path because then the path from \(u_2\) to \(v_2\) is shorter. By symmetry the same holds for \(v_3\) and \(v_4\). This gives a contradiction because \(u_1, u_2, u_3, \ldots, v_2, v_3, v_4\) is a blocking path of length 6 or longer.

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\[ U \text{ and } V \text{ got one element in common} \]

Without loss of generality we can assume that \( u_2 \neq v_2 \) and \( u_3 = v_3 = x \).

Again a contradiction because \( u_1, u_2, x, v_2, v_1 \) is a blocking path of length 5.

According to Lemma 5, if a tree without blocking paths of length 5 has a blocking path \( v_1, v_2, v_3, v_4 \) then the unicolored pair \((v_2, v_3)\) is unique in the sense that you cannot find another unicolored pair of elements \((u_2, u_3)\) in a blocking triple \( u_1, u_2, u_3, u_4 \) such that \( u_1 \) and \( u_2 \) are adjacent.

**Theorem 6**

If a tree \( T \) does not contain a blocking path of 5 elements or more then \( T \) is balanced in the leaf-removal game,

\[
\text{Leaf}(T) = b_T - w_T
\]

**Proof.** We split the proof into 2 cases:

**\( T \) does not contain a blocking path**

With Lemma 4 we can disconnect all leafs that are connected to an element of the same color. Since the remaining game cannot contain a blocking path it must be chess-colored which is balanced (Proposition 4).

**\( T \) does contain a blocking path**

Let \( v_1, v_2, v_3, v_4 \) be a blocking path in \( T \). Turn the tree into a tree poset by letting \( v_3 \) be the root and let all edges (cover relations) be directed from the \( v_3 \). By symmetry it is enough show this when \( v_2 \) and \( v_3 \) are black.

![Figure 22: Sketch of proof of the case in Theorem 6 when a tree \( T \) contains a blocking path of size 4.](image)

Assume that all removable elements are white. We want to show that there exist no black element \( y_1 \neq v_3 \) that is covered by another black element \( y_2 \). Assume the opposite. Let \( V = \{v_1, v_4\} \).

- If \( y_2 \) is covered by a white element \( y_3 \), it will create a black blocking triple and then the path \( y_3 \) to one of the elements in \( V \) must contain a blocking path (since the end elements are white and \( y_1 \) and \( y_2 \) are in between and black). This is impossible because of Lemma 5 and the fact that there exist no blocking path of 5 elements or more.
• $y_2$ cannot be maximal because all removable elements are maximal and white.

• If $y_2$ is covered by another black element $y_3$ then the induction principle is applied.

Assume that all removable elements are black. The same procedure as when all removable elements are white but interchange black and white and let $V = \{v_1\}$

\section*{Theorem 7}
Let $T$ be a tree containing a blocking path $v_1, \ldots, v_n$ of at least 5 elements. $T$ can be transformed to tree poset $P$ where the root is any of the elements $v_3, \ldots, v_{n-2}$ and all edges (cover relations) are directed from the root s.t.

\[
\text{Leaf}(T) = \text{Pomax}(\text{ess } P)
\]

\textbf{Proof.} First note that $\text{Leaf}(T) = \text{Leaf}(P)$. We will use the same strategy as in the proof of Theorem 3 on the game $\text{Leaf}(P) - \text{Pomax}(\text{ess } P)$. Since $v_1 \triangleright v_2 \triangleright v_3$ and $v_n \triangleright v_{n-1} \triangleright v_{n-2}$ are blocking triples the players will not reach any of the elements $v_3, \ldots, v_{n-2}$ in $\text{Leaf}(P)$. Hence they can all act as a root in a pomax game.

\hfill $\square$
Example 14
Here is a tree with a blocking path marked in green. With Theorem 7 we can transform the leaf game to a pomax game.

\[
\begin{align*}
\text{Leaf}(T) & \quad \text{Pomax}(P) \\
\text{ess } T & \quad \text{non-ess } T \\
\text{Pomax}(P) & \quad \text{Pomax(ess } P) \\
\end{align*}
\]

If we then use Theorem 3 we get that Black wins.

\[
\begin{align*}
\text{Leaf}(T) = \text{Pomax}(P) = \text{Pomax(ess } P) = 4 - 4 = 0
\end{align*}
\]

The leaf-removal game could also be played on all graphs (not just trees) and it is easy to deal with cycles. Since each cycle is closed the players will never be able to remove its elements. Therefore we can remove all cycles and let all elements that had an edge to a cycle become roots in a new forest of tree posets and let all edges be directed from the roots. Then we could play the pomax game on those poset trees.

4.3 The min-removal game on tree posets

We will now continue with the min-removal game on a tree poset $T$. Since the root $r$ of $T$ is the only minimal element there exists only one move in $T$. If $r$ is black then White has no moves i.e. the game has no left options and the game value cannot be less than zero. The opposite is true if $r$ is white. When $r$ has been removed we get a forest of tree posets $S_1, \ldots, S_n$.

Figure 23: A tree with subtrees $S_1, \ldots, S_n$. 

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Proposition 6

Let $T$ be a tree poset and $S_1, \ldots, S_n$ the tree posets obtained when the root $r$ is removed from $T$. Then $\text{Pomin}(T) =$

$$= \begin{cases} 
\max \{ \text{Pomin}(S_1) + \cdots + \text{Pomin}(S_n) + 1, 0 \} & \text{if } r \text{ is black} \\
\min \{ \text{Pomin}(S_1) + \cdots + \text{Pomin}(S_n) - 1, 0 \} & \text{if } r \text{ is white}
\end{cases}$$

Example 15 (Pomin value of a tree poset)

To calculate the pomin value of the following tree poset $T$

we first we need to calculate the value of the subtrees:

$-1 = \min(-1 + 1 - 1, 0) = \min(-1 + 1, 0) = \max(1 - 2 + 1, 0) = 0$

Then we get that $\text{Pomin}(T) = \min(-1 + 0 - 1, 0) = -2$ which means that White will win this game.

Proof of Proposition 6. If the root is black then there exists only one option which is Black's. This move results in the game $H = \text{Pomin}(S_1) + \cdots + \text{Pomin}(S_n)$.

If $H < 0$:

Then $\text{Pomin}(T)$ is a second player win since if White starts she loses immediately and if Black starts he must pick the negative game $H$ and loses. Note that

$$\max \{ \text{Pomin}(S_1) + \cdots + \text{Pomin}(S_n) + 1, 0 \} = \max \{ H + 1, 0 \} = 0$$

since $H + 1 \leq 0$.

If $H \geq 0$:

Then $\text{Pomin}(T) = \{H\} = H + 1 > 0$.

The proof when the root is white is completely analogous.

This recursion formula can be computed in time linear in the number of elements.
4.4 The min-max-removal game on tree posets

Before we proceed with the formula for the game value of the min-max-removal game we need the following lemma.

**Lemma 6**
The pomax and pominmax game on any colored tree poset $T$ without blocking triples is balanced. Hence

$$\text{Pomax}(T) = \text{Pominmax}(T) = b_T - w_T.$$

**Proof.** Identical to the proof of Proposition 3.

The value of a min-max-removal game on a poset tree $T$ can be computed from the pomax value of the essential part of $T$ and the pomin value of the non-essential part of $T$.

**Theorem 8** (Pominmax on a tree poset)

Let $T$ be a tree poset then

$$\text{Pominmax}(T) = b_{\text{ess } T} - w_{\text{ess } T} + \text{Pomin}(\text{non-ess } T)$$

**Proof.** From Theorem 2 we have that $b_{\text{ess } T} - w_{\text{ess } T} = \text{Pomax}(\text{ess } T)$. Thus we could equally show that

$$\text{Pominmax}(T) - \text{Pomax}(\text{ess } T) - \text{Pomin}(\text{non-ess } T)$$

is a second player win, see figure 24. We only need to check what is happening at the border between the essential part of $T$ and the non-essential part of $T$. By symmetry, it suffices to show that that White will win if Black starts.
Removing maximal elements
White should mirror Black’s moves and apply the same strategy as in the proof of Theorem 3 (the formula for pomax games). This gives that as long as the players are removing maximal elements, Black will not be able to reach the non-essential part of the tree $T$ in the component Pominmax($T$).

Removing minimal elements
If Black removes minimal elements then White can keep on mirroring. If $s$, $t$ or $i$ gets removed then the forest above that element becomes a component without blocking triples and the pominmax value of that component is the same as the negative of the pomax value of the mirror image of that part (Lemma 6).

\[ \square \]

4.5 The max-removal game on ordinal sums
We now change the structure of the poset to a combination of two posets called an ordinal sum.

\[
\text{Definition 14 (Ordinal sum of posets)}
\]

The ordinal sum of two posets $P$ and $Q$ is the poset $P \oplus Q$ on the union $P \cup Q$ such that $s \leq t$ in $P \oplus Q$ if

- $s, t \in P$ and $s \leq t$ in $P$
- $s, t \in Q$ and $s \leq t$ in $Q$
- $s \in P$ and $t \in Q$

The pomax value on an ordinal sum $P \oplus Q$ is easy to compute when $Q$ is a tree.
poset. The root of $Q$ will be an important element since it is greater than all elements in $P$ and at the same time is less than all other elements in $Q$.

### Proposition 7

**Let $T$ be a tree poset that contains a blocking triple and let $S$ be any poset. Then**

$$\text{Pomax}(S \oplus T) = b_{\text{ess} \, T} - w_{\text{ess} \, T}$$

**Proof.** By adopting the same strategy as in the proof of Theorem 3 the players will never reach the non-essential part of $T$ which contains the root of $T$ (which is greater than all elements in $S$). Hence the non-essential part can be removed and the remaining game will not contain any blocking triple (Theorem 3).

### Proposition 8

**Let $P$, $Q$ and $R$ be posets such that $\text{Pomax}(P) = \text{Pomax}(Q)$ then**

$$\text{Pomax}(P \oplus R) = \text{Pomax}(Q \oplus R)$$

**Proof.** By the definition of an ordinal sum the players cannot remove any elements in $P$ before all element of $R$ are removed which means that we can replace $P$ with any poset of the same pomax value.

If a poset $A$ contains an element $x$ which is comparable with all elements in the poset and whose removal disconnects the poset into two disjoint parts then by Proposition 8 the subposet $P = \{v : v \leq x\}$ can be replaced by a poset of the same pomax value since $A = P \oplus R$ where $R = \{v : v > x\}$. The next proposition is about when $R$ is a tree poset.

### Proposition 9

**Let $T$ be a tree poset with a black root $r$ and let $S$ be any poset. If $T$ is unicolored then**

$$\text{Pomax}(S \oplus T) = \text{Pomax}(S \oplus \{r\}) + b_T - 1$$

**Otherwise**

$$\text{Pomax}(S \oplus T) = \begin{cases} b_T - w_T & \text{if } \text{Pomax}(S) \geq 0 \\ b_T - w_T - 1 & \text{if } \text{Pomax}(S) < 0 \end{cases}$$

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Example 16

Here is an ordinal sum of a poset $S$ and a tree poset $T$ with root $r$. We have that

$\text{Pomax}(S) = \{2 - 1\} = 2$.

Thus Proposition 9 gives us that

$\text{Pomax}(S \oplus T) = 4 - 4 = 0$

A second player win.

Proof of Proposition 9. If $T$ is unicolored then Lemma 4 can be applied to disconnect all elements above the root which then gives the desired formula. For the rest of the proof assume that $T$ is not unicolored.

Case 1: $\text{Pomax}(S) \geq 0$

By proposition 8 we can replace $S$ with a black unicolored chain $C$ of the same value as $\text{Pomax}(S)$. If $T$ contains a blocking triple or $C$ only contains one black element then we use Theorem 3. Otherwise we get a blocking triple at the bottom such that ess $C \oplus T = \text{ess } T$ which gives the same result after using Theorem 3.

Case 2: $\text{Pomax}(S) < 0$

Since

$\text{Pomax}(S \oplus T) = \text{Pomax}((S \cup \{r\}) \oplus (T \setminus \{r\}))$

and

$\text{Pomax}(S \oplus \{r\}) = \{\text{Pomax}(S)\} = 0$

we can use Proposition 8 to remove $S \cup \{r\}$. Together with Theorem 3 we get that $\text{Pomax}(S \oplus T) = \text{Pomax}(\text{ess } T \setminus \{r\})$.

When the root is white we can invert the colors of the poset and take the negative of the game value (Proposition 1).

4.6 The min-max-removal game on double tree posets

In the pominmax game we will look at a structure called a double tree poset. See figure 25. A double tree is almost like an ordinal sum on a tree poset and a tree poset that is turned upside down.
Definition 15 (Double tree poset)
A double tree poset, $T = T_1 \cup T_2^*$, is a poset where two tree posets, $T_1$ and $T_2^*$, share the same root but $T_2$ is turned upside down, denoted $T_2^*$.

The root $r$ in a double tree poset $T = T_1 \cup T_2^*$ is a part of both $T_1$ and $T_2^*$, see figure 26. A poset $P$ that is upside down is marked with a star like this $P^*$.

Theorem 9
The value of a min-max-removal game on a double tree poset $T = T_1 \cup T_2^*$ is

\[
\begin{align*}
& b_T - w_T & \text{neither } T_1 \text{ nor } T_2^* \\
& b_{\text{ess}} T_1 - w_{\text{ess}} T_1 + \text{Pomax}((\text{non-ess } T)^*) & T_1 \\
& b_{\text{ess}} T_2 - w_{\text{ess}} T_2 + \text{Pomax}(\text{non-ess } T) & T_2 \\
& b_{\text{ess}} T_1 + b_{\text{ess}} T_2 - w_{\text{ess}} T_1 - w_{\text{ess}} T_2 & T_1 \text{ and } T_2
\end{align*}
\]
Example 17
Consider the pomax game on the double tree poset $T = T_1 \cup T_2^*$ from figure 25,

Since $T_1$ but not $T_2$ contains a blocking triple we have to use the formula in the second case. The border between the essential and non-essential part is marked in blue.

\[
Pomimmax(T) = \text{Pomax}((\text{non-ess } T)^*) + 6 - 8 = 0 - 2 = -2
\]

Proof of Theorem 9. We split the proof into four cases,

1) Neither $T_1$ nor $T_2$ contains a blocking triple.
2) $T_1$ contains a blocking triple.
3) $T_2$ contains a blocking triple.
4) Both $T_1$ and $T_2$ contains a blocking triple.

Neither $T_1$ nor $T_2$ contains a blocking triple
We will show that the game is balanced. Assume that all removable elements are black and pair each white element in $T_1$ and $T_2$ with one of the elements covering it. This pairing shows that the majority of the ele-
ments are black and hence the game is balanced. The reverse is true if all removable elements are white.

\textbf{T}_1 \textbf{contains a blocking triple.}
With use of Theorem 3 and the fact that Pomax((non-ess \(T\)^*) = Pomin(non-ess \(T\)) we could equally show that

\[ Pominmax(T) - \text{Pomax(ess } T) - \text{Pomin(non-ess } T) = 0 \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sketch.png}
\caption{Sketch for the proof of case 2 in Theorem 9}
\end{figure}

As a second player we can mirror each move from the first player.
When removing maximal elements we can use the same strategy as in the proof of Theorem 3 so that the players will never reach the non-essential part of \(T_1\) from above.
When removing minimal elements we can keep on mirroring until the shared root is accessible, at that point the remaining game consists of tree posets and then we can use Theorem 8.

\textbf{T}_2 \textbf{contains a blocking triple.}
Since Pomax(\(T\)) = Pomin(\(T^*\)), this is exactly the same as previous case but on \(T^*\) instead of \(T\).

\textbf{Both \(T_1\) and \(T_2\) contains a blocking triple.}
With use of Theorem 3 we could equally show that

\[ Pominmax(T) - \text{Pomax}(T_1) - \text{Pomax}(T_2) = 0 \]
The second player can use the same strategy as in the proof of Theorem 3 between the components Pominmax(T) and Pomax(T₁) + Pomax(T₂). In this way the players will never touch the root in any of the trees T, T₁, or T₂ since they never reach the non-essential part. Hence it does not matter if the T₂ and T₁ share root or not.

4.7 The max-removal game on zigzag posets

**Definition 16 (Zigzag poset)**

A zigzag poset \( Z = \{v_1, ..., v_n\} \) is a poset where all elements have two cover relations except for \( v_1 \) and \( v_n \) who only got one each.

A zigzag poset is not the same as a chain since chains are totally ordered. The elements \( v_1, ..., v_n \) in a zigzag poset is drawn from left to right in consecutive order. A slope \( S_i \) is a maximal chain minus elements which covers 2 elements or is covered by 2 elements. The slopes are ordered from left to right. The zigzag poset has a very simple structure which can be found as subposets in other posets. We say that we *hook* a zigzag poset onto another poset with the following definition.
Figure 29: A zigzag poset with 4 slopes.

**Definition 17**

Let $P$ be a poset and $Z = \{v_1, \ldots, v_n\}$ be a zigzag poset. Then define $\text{Hook}(Z, m, P)$ to be the union of $Z$ and $P$ together with the relation $v_n \triangleright m$ where $m \in P$.

**Theorem 10**

Let $Z$ be a zigzag poset containing a blocking triple $x \lesssim y \lesssim z$ where $z$ covers exactly one element and let $Z$ be hooked onto any poset $P$. Then all elements $s < y$ can be removed and the pomax value stays the same.
**Example 18**

Note that $P$ in Theorem 10 could be an empty poset like in this example. The following zigzag poset contains a white blocking triple $x \preceq y \preceq z$ (marked in green). Theorem 10 gives that we can remove everything below $y$. Also Lemma 4 gives that we can disconnect the last two elements.

We get 2 white singleton elements and 2 new larger games, $S$ and $T$. Both $S$ and $T$ have only one option, each consisting of chains without blocking triples which are easy to calculate with Proposition 5.

\[
Pomax(Z) = Pomax(S) + Pomax(T) - 2
\]

\[
= \{ |b_S - w_S + 1| + |b_T - w_T - 1| \} - 2
\]

\[
= \{ |4 - 5 + 1| + |4 - 4 - 1| \} - 2
\]

\[
= \{ |0| + |-1| \} - 2
\]

\[
= -1 + 0 - 2
\]

\[
= -3
\]

In the end White will win this pomax game.

Proof of Theorem 10. Let $Q = \text{Hook}(Z, x, P)$ where $x$ is some element in $P$. We want to prove that $Pomax(Q) - Pomax(R)$ is a second player win where $R$ is the reduced poset. By symmetry it is sufficient to prove this when we have a black blocking triple. If Black starts, White should mirror all Black’s moves except when Black removes $\bar{z} \in R$ (see figure 30). In that case White should remove $\bar{y}$ instead of $z$ to prevent Black from getting below $z$ in $Q$. White can then mirror all Black’s remaining moves. When White starts Black has no trouble mirroring each move White makes.
A zigzag poset consists of several chains that are hooked together. The next theorem is about chains that are hooked onto a poset.

**Theorem 11**

Let $Q = \text{Hook}(C, m, P)$ where $C$ is the chain $x_1 \triangleright \ldots \triangleright x_n$, $P$ is a poset and $m \in P$ is a minimal element. If $C \cup \{m\}$ does not contain a blocking triple then

$$\text{Pomax}(Q) = b_C - w_C + \text{Pomax}(P)$$
Example 19
Below we have a zigzag poset where the first respectively last slope can be disconnected with Theorem 11. The remaining part in the middle \( Z' \) becomes a pomin game when turned upside down where Black has one option consisting of chains without blocking triples. Together with Proposition 5 and 6 we get

\[
Pomax(Z) = 4 - 1 + Pomax((Z')^*) + 2 - 2
\]
\[
= 3 + \min\{b_{Z'} - w_{Z'}, 1 - 1, 0\}
\]
\[
= 3 + \min\{3 - 5, 0\}
\]
\[
= 3 + \min\{-2, 0\}
\]
\[
= 3 - 2
\]
\[
= 1
\]

**Proof of Theorem 11.** We use Proposition 5 to get that \( (b_{\bar{C}} - w_{\bar{C}}) = Pomax(\bar{C}) \). Now we want to show that

\[
Pomax(Q) + Pomax(\bar{C}) + Pomax(\bar{P}) = 0
\]

Let \( \bar{S} \) be the set of elements in \( \bar{P} \) that covers \( \bar{m} \) and let \( C' \) and \( P' \) be the remaining part of \( C \) and \( P \) during gameplay. The second player can mirror each move from the first player and if all elements in \( \bar{S} \cup S \) is removed by mirroring then \( C' \cup \{m\} \) and \( \bar{m} \) are disconnected from \( P' \) and \( \bar{P}' \). With help of Proposition 5 we get that

\[
Pomax(C' \cup \{m\}) + Pomax(\bar{C}') + Pomax(\bar{m}) = 0
\]
\[
Pomax(P') + Pomax(\bar{P}') = 0
\]

and the second player wins. \( \square \)

Now we continue with a zigzag poset with 2 slopes that is hooked to some poset.
Theorem 12

Let $Q = \text{Hook}(H, m, P)$ where $H$ is the poset $x_1 \preceq \ldots \prec x_a \prec p \succ y_1 \succ \ldots \succ y_b$, $P$ is a poset and $m \in P$ is a minimal element in $P$. If $x_1 \preceq \ldots \prec x_a$ and $y_1 \succ \ldots \succ y_b \succ m$ do not contain a blocking triple and if $p$ is black then

$$ \text{Pomax}(Q) = \begin{cases} \text{Pomax}(P) + \text{Pomax}(H) & \text{if } b_H - w_H > 0 \\ \text{Pomax}(P \setminus \{m\}) & \text{if } b_H - w_H < 0 \\ \text{Pomax}(R) & \text{if } b_H - w_H = 0 \end{cases} $$

where

$$ R = \begin{cases} P & \text{if } m \text{ is black} \\ P \setminus \{m\} & \text{if } m \text{ is white} \end{cases} $$

Figure 31: Sketch of proof of Theorem 11.
Example 20
Here we have a poset \( Q = \text{Hook}(H_1, v_1, P_1) \) with equally many black and white elements in \( H_1 \). Since \( v_1 \) is black we can remove \( H_1 \) without affecting the game value (case 3 in Theorem 12 where \( R = P \)).

In the new poset \( P_1 = \text{Hook}(H_2, v_2, P_2) \) we invert the colors (because \( q \) is white) to make it possible to use Theorem 12 but we have to remember to take the negative of the game value (Proposition 1). Now since \( H_2 \) contains more black than white elements, case 1 in Theorem 12 can be used.

This results in two games with one option each where both options consists of chains without blocking triples (Proposition 5). In total we get

\[
Pomax(Q) = Pomax(P_1) = -Pomax(\bar{P}_1) \\
= -Pomax(H_2) - Pomax(\bar{P}_2) \\
= -\{4 - 4\} - \{|4 - 3|\} \\
= -\{0\} - \{|1|\} \\
= -1
\]

Proof of Theorem 12. In each case the second player will mirror the first player’s move. When it is not possible a new strategy will be presented in each case. Let \( S_1 = \{x_1, ..., x_a\} \) and \( S_2 = \{y_1, ..., y_b\} \). First note two things that is frequently used in the proof.
N1 Each time both $m$ and $\bar{m}$ are removable we will ignore the remaining part of $P$ and $\bar{P}$ since they cancel each other out.

N2 Each time $p$ is removed, Theorem 11 is used to disconnect $S_2$ which then becomes a chain without blocking triples (Proposition 5).

Case 1: $b_H - w_H > 0 \implies \text{Pomax}(Q) - \text{Pomax}(H) - \text{Pomax}(P) = 0$

If $p$ and $\bar{p}$ gets removed the game value clearly becomes 0 (N2 is used).
If White starts and manage to remove $\bar{m}$ then Black can respond by removing $p$ (N2 is used). Note that $m$ is black. Since

$$\text{Pomax}(H \setminus \{p\}) + \text{Pomax}(m) = b_H - w_H - 1 + 1 > 0$$

White is forced to remove $\bar{p}$. The remaining game consists of chains without blocking triples with one more black element which Black can remove and wins (Proposition 5 and N1 is used).
If Black starts and manage to remove $\bar{m}$ we use the same thinking. That is, White removes $\bar{p}$, Black is forced to remove $p$ since $\text{Pomax}(H \setminus \{\bar{p}\}) < 1$ and then the rest of the game equals -1 which White can remove to win (Proposition 5 and N1 is used).

Case 2: $b_H - w_H < 0 \implies \text{Pomax}(H \cup P) - \text{Pomax}(P \setminus \{\bar{m}\}) = 0$

If Black starts and removes $p$ then (N2 is used)

$$\text{Pomax}(H \setminus \{p\}) = b_H - w_H - 1 < -1$$

which means that White got at least two moves in that component, one to make now and (at least) one extra if $m$ is Black. White can then mirror Black’s remaining moves.
Case 3a: $b_H - w_H = 0$ and $m$ is black $\implies$ $\text{Pomax}(Q) - \text{Pomax}(P) = 0$

If Black starts and removes $p$ then (N2 is used)

$\text{Pomax}(H \setminus \{p\}) = b_H - w_H - 1 = -1$

which White can remove to win.

If White starts and manages to remove $\bar{m}$ then Black can respond by removing $p$. The rest of the game becomes (N1 is used)

$\text{Pomax}(H \setminus \{p\}) + \text{Pomax}(m) = (b_H - w_H - 1) + 1 = 0$

which is a second player win.

Case 3a: $b_H - w_H = 0$ and $m$ is white $\implies$ $\text{Pomax}(Q) - \text{Pomax}(P \setminus \{m\}) = 0$

If Black starts and removes $p$ then (N2 is used)

$\text{Pomax}(H \setminus \{p\}) = b_H - w_H - 1 = -1$

which White can remove. White can the mirror the remaining black moves.

\[ \square \]

Theorem 11 and 12 can be used to determine the value of any zigzag poset or to disconnect zigzag posets that are hooked to minimal elements by iteratively removing slopes.

4.8 The min-max-removal games are PSPACE-complete

In this section we will show that the min-max-removal game is PSPACE-complete.
Theorem 13

Any pomax game can be transformed to a pominmax game in polynomial time such that the gameplay and game value stays the same.

Proof. First we define a block poset $B$ as follows.

- One black element $s_1$ at the top.
- One black element $s_2$ at the bottom.
- A sufficient number of white elements $v_1, ..., v_n$ between $s_1$ and $s_2$.
- The cover relations $s_2 \preceq v_i \preceq s_1$, $1 \leq i \leq n$.

Figure 32: A block poset

The construction of $B$ can be seen in figure 32. Now we can take a poset $P$ from any pomax game and connect the block poset $B$ by letting the minimal elements in $P$ cover $s_2 \in B$ ($s_1 \in B$ also works but then the height of the poset becomes larger). An example is illustrated in figure 33.

In the pominmax game, Black will not remove $s_2$ (or $s_1$) unless he gets more black elements from the bottom of $P$ than the number of white elements in $B$. Therefore a sufficient number on $n$ is $b_P + 3$. The three extra elements are two cover up for $s_1$, $s_2$ and to prevent Black from making a move in $B$. Note that this is only an upper bound on $n$. In many cases it is enough with a smaller value on $n$. If there are more black elements than white elements in $P$ then we could invert the colors in $B$ and set $n = w_P + 3$. Since all minimal element in $P$ covers $s_2$ the players will choose to only remove maximal elements. Hence, the gameplay and game value stay the same. The creation of $B$ and connecting it to $P$ clearly takes polynomial time in the number of elements in $P$. \hfill $\Box$

Theorem 14

The problem of deciding if a given min-max-removal game equals zero is NP-hard even when the height of the colored poset is restricted to 3 and it is PSPACE-complete when the height is at least 4.

Proof. Use the transformation explained in the proof of Theorem 13 and invoke Theorem 4 and 5. An example is illustrated in figure 33. \hfill $\Box$
Figure 33: The colored poset (in the pomimmax game) constructed from the CNF3-formula \((\neg x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3)\).

We cannot get the height to be smaller than 3 because then all elements would have been removable from the beginning and we get a balanced game.
5 Summary and open problems

This thesis has been about element-removal games on acyclic graphs and posets. We have discovered that tree posets have a very simple formula in the pomax, pomin and pominmax game. For the leaf-removal game we are now able to determine the game value for all posets (even with cycles).

To extend the class of tree posets we have introduced two new structures, the double tree poset and the zigzag poset, where formulas for the game value have been found for the pomax, pomin and pominmax game. We have found tools on how to disconnect zigzag posets that are hooked onto a minimal element in another poset in the pomax game. If we could remove the condition that the last element in a zigzag poset must be connected to a minimal then we are very close of finding a way to determine the pomax value on all acyclic posets. Hopefully this is also possible for the min-max-removal game. For all games we have also seen that blocking triples (and blocking paths) is a fundamental part to determine the game value. All of this leads to the following open problems.

Open problem 1: Find a formula for $\text{Hook}(Z, m, P)$ where $Z$ is a zigzag poset, $P$ a poset and $m$ to be any element in $P$.

Open problem 2: Find a formula for the value of the pomax game on any acyclic poset.

Open problem 3: Find a formula for the value of the pominmax game on any acyclic poset.

We have also shown a transformation from the pomax game to pominmax in linear time such that the game value and strategies stays the same. With this and theory from E. Järleberg and J. Sjöstrand [4] we get that it is PSPACE-complete to determine the value of the pomax, pomin or pominmax games on general posets.
6 List of notation

**Posets**

Let $P$ be a poset, $a, b \in P$ and let $S$ be a set.

- $P$: A set together with a binary relation $\leq$ which is reflexive, antisymmetric and transitive.
- $P'$: The remaining part of $P$ during a gameplay.
- $a < b$: $a$ is less than $b$.
- $a \leq b$: $a$ is less than or equal to $b$.
- $a \bowtie b$: $a$ is covered by $b$ or $b$ covers $a$.
- $a \parallel b$: $a$ and $b$ are incomparable (not comparable).
- $\bar{v} \in \bar{S}$: The mirror image of $v \in S$.
- $\bar{S}$: The mirror image of the set $S$.
- $\text{ess } P$: The essential part of $P$, the maximal upper set without blocking triples.
- $\text{non-ess } P$: The non-essential part of $P$, $P \setminus \text{ess } P$.
- $b_S$: The number of black elements in $S$.
- $w_S$: The number of white elements in $S$.

**Zigzag and double tree posets.**

- $P^*$: A poset $P$ turned upside down.
- $T_1 \cup T_2^*$: A double tree poset where $T_1$ and $T_2$ are tree posets and share the same root but $T_2$ is turned upside down.
- $\text{slope, } S_i$: A maximal chain minus elements which do not cover only one element.
- Hook$(Z, m, P)$: The union of a zigzag poset $Z = \{v_1, ..., v_n\}$ and $P$ together with the relation $v_n \rightarrow m$ where $m \in P$.

**Games**

Let $G$ be a game and let $n$ be an integer.

- $G^L$: The set of Left’s/Black’s options in $G$.
- $G^R$: The set of Right’s/White’s options in $G$.
- $G^L$: A typical element of $G^L$.
- $G^R$: A typical element of $G^R$.
- $G > 0$: $G$ is positive, i.e. Black will win $G$.
- $G < 0$: $G$ is negative i.e. White will win $G$.
- $G = 0$: $G$ is zero i.e. second player in turn will win $G$.
- $G \parallel 0$: $G$ is fuzzy to 0 i.e. first player in turn will win $G$.
- $G \geq 0$: Left will win as second player in $G$.
- $G \leq 0$: Right will win as second player in $G$.
- $G < 0$: Left will win as first player in $G$.
- $G > 0$: Right will win as first player in $G$.
- $-G$: $\{ -G^R | -G^L \}$
- $G + H$: $\{ G^L + H, G + H^L | G^R + H, G + H^R \}$
\( G = 0 \) The empty game \{ | \}
\( G = n \)
\{ \( n - 1 \) | \} \( n > 0 \)
\{ \( n + 1 \) \} \( n < 0 \)

**Element-removal games**

\( \rho(S) \) The removability function \( \rho : 2^X \rightarrow 2^X \) in an element-removal game. Returns the removable elements in the set \( S \) of \( G \).

Pomax\((P)\) Max-removal game on a poset \( P \).

Pomin\((P)\) Min-removal game on a poset \( P \).

Pominmax\((P)\) Min-max-removal game on a poset \( P \).

Leaf\((T)\) Leaf-removal game on a tree \( T \).
References


