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On Estimating Initial Conditions in Unstructured Models

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Abstract—Estimation of structured models is an important problem in system identification. Some methods, as an intermediate step to obtain the model of interest, estimate the impulse response parameters of the system. This approach dates back to the beginning of subspace identification and is still used in recently proposed methods. A limitation of this procedure is that, when obtaining these parameters from a high-order unstructured model, the initial conditions of the system are typically unknown, which imposes a truncation of the measured output data for the estimation. For finite sample sizes, discarding part of the data limits the performance of the method. To deal with this issue, we propose an approach that uses all the available data, and estimates also the initial conditions of the system. Then, as examples, we show how this approach can be applied to two methods in a beneficial manner. Finally, we use a simulation study to exemplify the potential of the approach.

I. INTRODUCTION

System identification deals with obtaining mathematical models for dynamical systems from experimental data. Different methods exist to obtain such models, e.g., the prediction error method (PEM; [1], [2]), subspace identification methods ([3], [4]), instrumental variable methods ([5], [6]), and the Steiglitz-McBride method [7]. Some methods estimate an unstructured high-order model as an intermediate step to obtain the model of interest. Examples are the methods in [8] and [9], and also some subspace identification methods, such as SSARX [10].

An inherent limitation of estimating unstructured high-order models is that, since the initial conditions of the system are typically unknown, the transient part of the data must be truncated for the estimation. The higher the order required to capture the dynamics of the system accurately, the longer the transient part is, and the more data must be discarded to estimate the unstructured parameters. Although this limitation does not change the asymptotic properties of the estimate, it affects performance for finite sample size.

The problem of dealing with unknown initial conditions is not new. The principle of backforecasting, for example, is one such approach ([11], [12]). In this contribution, we deal with this problem when estimating high-order unstructured models as an intermediate step to obtain the model of interest. We propose that, when estimating the unstructured model, the initial conditions be viewed as unknowns to be estimated. Although this allows the inclusion of the

previously truncated data, the extra equations are in the same number as the extra unknowns. Thus, this procedure does not yield improved estimates of the high-order model parameters. Nevertheless, for some methods, the new estimated variables can be useful in a posterior step to obtain the low-order model of interest.

As examples, we incorporate this procedure into two methods. First, we use SSARX to point out that the unstructured estimate we propose contains the same information as the original data set. To do so, we discard the data and use only the estimated parameters after these have been obtained. Then, for the method in [8], which we will denote by weighted null-space fitting (WNSF), we notice that the extra estimated variables are themselves estimates of the impulse response of a system with the same poles, and can be used to improve the structured model estimate. This is exemplified in the case of output-error models.

The paper is structured as follows. In Section II, the procedure for including initial conditions in the unstructured model estimate is delineated. Then, Sections III and IV concern the application to SSARX and WNSF, respectively. Numerical simulations are performed in Section V to illustrate how these methods perform with and without estimating initial conditions, followed by a discussion in Section VI.

II. ESTIMATING INITIAL CONDITIONS IN A HIGH-ORDER UNSTRUCTURED MODEL

Consider a single-input single-output (SISO) output-error (OE) model, given by

$$y_t = \frac{L(q)}{F(q)} u_t + e_t, \quad (1)$$

where $F(q)$ and $L(q)$ are rational functions of the delay operator q^{-1} , as

$$\begin{aligned} F(q) &= 1 + f_1 q^{-1} + \dots + f_{n_f} q^{-n_f}, \\ L(q) &= l_1 q^{-1} + \dots + l_{n_l} q^{-n_l}, \end{aligned}$$

and e_t is Gaussian white noise with variance λ_e . We consider that an output sequence $\{y_t\}$ has been obtained using such a model and a known input sequence $\{u_t\}$. The objective is to estimate the coefficients of the polynomials $L(q)$ and $F(q)$,

$$\theta := [l_1 \quad \dots \quad l_{n_l} \quad f_1 \quad \dots \quad f_{n_f}]^\top. \quad (2)$$

The prediction error method (PEM) is a well known approach to estimate these coefficients ([1], [2]), which minimizes the cost function (if a quadratic error criterion is used)

$$V(\theta) = \frac{1}{N} \sum_{t=1}^N \left(y_t - \frac{L(q, \theta)}{F(q, \theta)} u_t \right)^2, \quad (3)$$

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where N is the sample size.

Other methods use the following idea. Consider a finite impulse response (FIR) model,

$$y_t = G(q, g)u_t + e_t, \quad (4)$$

where

$$G(q, g) = \sum_{k=1}^m g_k q^{-k}. \quad (5)$$

Using PEM, g_k can be obtained by minimizing the cost function

$$V(g) = \frac{1}{N} \sum_{t=1}^N (y_t - G(q, g)u_t)^2,$$

which is a linear regression problem, since (5) is linear in the model parameters. It can thus be solved using least-squares (LS) as follows. First, (4) can be written in vector form as

$$y = \Phi_u g + e, \quad (6)$$

with y , g , and e given by

$$\begin{aligned} y &= [y_1 \ y_2 \ \cdots \ y_N]^\top, \\ g &= [g_1 \ g_2 \ \cdots \ g_m]^\top, \\ e &= [e_1 \ e_2 \ \cdots \ e_N]^\top, \end{aligned}$$

and where Φ_u is a Toeplitz matrix with input values,

$$\Phi_u = \begin{bmatrix} u_0 & u_{-1} & \cdots & u_{-m+1} \\ u_1 & u_0 & \cdots & u_{-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N-1} & u_{N-2} & \cdots & u_{N-m} \end{bmatrix}. \quad (7)$$

Then, it is possible to solve for g using LS, obtaining

$$\hat{g} = \left(\Phi_u^\top \Phi_u \right)^{-1} \Phi_u^\top y, \quad (8)$$

where, for m large enough, the statistics of $\hat{g}$ are well approximated by

$$\hat{g} \sim \mathcal{N} \left(g, \lambda_e \left(\Phi_u^\top \Phi_u \right)^{-1} \right), \quad (9)$$

with $\mathcal{N}$ representing the normal distribution.

If the order m is chosen large enough, (5) can model (1) with good accuracy. However, the drawback is that the unstructured estimate (8) may have high variance, especially when m has to be chosen large. Nevertheless, some methods use this estimate to obtain the coefficients (2) of the structured model, which will have lower variance.

A limitation of this approach is that the first m rows in (7) normally have to be truncated. The reason is that a batch of data $\{u_t, y_t\}$ is typically obtained from an experiment for certain time instants $t = \{1, 2, \dots, N\}$. Therefore, u_t is not available for $t \leq 0$, which makes the first m rows of Φ_u unknown, requiring these equations to be removed. This implies that also y is truncated when estimating $\hat{g}$ in (8), and not all the available data is used in the estimation. This situation is particularly limiting for poorly damped systems, when the order m has to be chosen large.

A straightforward approach to estimate these missing data is as follows. Let

$$\tau := \begin{bmatrix} u_0 & u_{-1} & \cdots & u_{-m+1} \\ 0 & u_0 & \cdots & u_{-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_0 \end{bmatrix} g.$$

Then, if τ is treated as a standalone variable, it can be estimated simultaneously with g from

$$\begin{aligned} y &= \Phi_{\bar{u}} g + \begin{bmatrix} I_m \\ \mathbf{0}_{N-m \times m} \end{bmatrix} \tau + e \\ &= \Phi_{\bar{u}} g + \Phi_{\delta} \tau + e \\ &= \bar{\Phi} \bar{g} + e, \end{aligned} \quad (10)$$

where $\Phi_{\bar{u}}$ is constructed as Φ_u in (7), but using

$$\bar{u}_t \equiv \begin{cases} 0, & t \leq 0 \\ u_t, & t > 0 \end{cases},$$

and with

$$\begin{aligned} \bar{g} &= [g^\top \ \tau^\top]^\top, \\ \Phi_{\delta} &= \begin{bmatrix} I_m \\ \mathbf{0}_{N-m \times m} \end{bmatrix}, \\ \bar{\Phi} &= [\Phi_{\bar{u}} \ \Phi_{\delta}]. \end{aligned}$$

Then, LS can be applied to (10) as before, now using all N measurements. This approach can be seen as equivalent to estimating the initial states in a state-space model for an FIR system, and is thus in line with standard approaches for estimating initial states.

Despite including m extra equations, (10) also includes m extra unknowns. Therefore, the accuracy of the estimate $\hat{g}$ is not expected to increase. Nevertheless, it can be useful in a later step for methods that use $\hat{g}$ to obtain an estimate of the coefficients (2). Two examples are SSARX and weighted null-space fitting (WNSF), which are discussed in the following sections.

III. SSARX

SSARX [10] is a method combining ARX modeling and subspace identification. The algorithm is similar to the subspace identification method CCA ([13]–[15]), but uses an ARX model to obtain unstructured estimates of the impulse response parameters, which are then used to restructure the data model for subspace identification.

In this section, we first review the SSARX method. Then, we observe that the measured output is still used in the algorithm after the impulse response has been estimated. We discuss why this is the case, noticing that an impulse response estimated with truncation of the initial conditions cannot fully replace the data. On the other hand, the unstructured coefficients obtained with the proposed procedure can, and thus contain no less information than the data.

Since our discussion only considers OE models, estimating a high-order FIR model is sufficient in SSARX, instead of an ARX.

A. Background

Consider the state-space model

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + e_t, \end{aligned} \quad (11)$$

which can be used to write the state as

$$x_t = \sum_{k=0}^{p-1} A^k Bu_{t-k-1} + A^p x_{t-p}.$$

Therefore, for stable A , and by choosing p large enough, the state can be estimated by a linear combination of past outputs,

$$\hat{x}_t = \mathcal{K}u_t^p, \quad (12)$$

where $\mathcal{K}$ is a matrix with unknown coefficients, and

$$u_t^p := [u_{t-1} \quad u_{t-2} \quad \dots \quad u_{t-p}]^\top.$$

Then, if an estimate of $\mathcal{K}$ is obtained, the state can be reconstructed, and the state-space matrices recovered. Moreover, since the output is a linear combination of the states, it can also be written as in (6).

Now, if we define

$$y_t^f := [y_t \quad y_{t+1} \quad \dots \quad y_{t+f-1}]^\top,$$

where f is a user-defined parameter whose influence has been analyzed in [16], and with analogous definitions for u_t^f and e_t^f , we can write the relation

$$y_t^f = \Gamma x_t + \Psi u_t^f + e_t^f, \quad (13)$$

with

$$\Gamma = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{f-1} \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 0 & \dots & 0 \\ CB & 0 & & \\ CAB & CB & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ CA^{f-2}B & & & CB & 0 \end{bmatrix}.$$

Replacing the state in (13) by its estimate (12) yields

$$y_t^f \approx \Gamma \mathcal{K} u_t^p + \Psi u_t^f + e_t^f.$$

Furthermore, define

$$z_t := y_t^f - \hat{\Psi} u_t^f \approx \Gamma \mathcal{K} u_t^p + e_t^f, \quad (14)$$

where $\hat{\Psi}$ is an estimate of Ψ . The approach of SSARX to obtain this estimate is to solve (6) for g in a LS sense, and use the fact that

$$g_k = CA^{k-1}B.$$

This allows constructing an estimate of Ψ and computing z_t . Then, (14) can be seen as a low rank linear regression problem in $\Gamma \mathcal{K}$. Recall that the objective is to estimate $\mathcal{K}$, so that we can reconstruct the state sequence. This is done by performing canonical correlation analysis (CCA) on z_t and u_t^p as follows. First, define

$$M = (\hat{R}_{zz})^{-1/2} (\hat{R}_{zu^p}) (\hat{R}_{u^p u^p})^{-1/2},$$

where

$$\hat{R}_{zu^p} = \frac{1}{N} \sum_{t=1}^N z_t u_t^p,$$

and analogously for $\hat{R}_{zz}$ and $\hat{R}_{u^p u^p}$. Then, compute a singular value decomposition

$$USV^T = M.$$

Finally, the CCA estimate of $\mathcal{K}$ is given by

$$\hat{\mathcal{K}} = V_n^\top (\hat{R}_{u^p u^p})^{-1/2}.$$

The state sequence is then estimated from (12), using $\hat{\mathcal{K}}$ as estimate of $\mathcal{K}$.

Having an estimate of the state sequence, the state-space matrices can be estimated by applying LS to the state-space equations (11). In particular, C is obtained by regressing y_t on $\hat{x}_t$, and A and B by regressing $\hat{x}_{t+1}$ on $\hat{x}_t$ and u_t .

B. Using the Unstructured Estimate Only

The SSARX method, as other subspace identification methods, requires, at some point, estimates of the impulse response of the system. In particular for SSARX, these are obtained by estimating a high-order ARX model, or, in this simplified example, a high-order FIR model. This is done by regressing the measured data on the input, in (6). In a subsequent step, the measured data is again used in (14) to construct z_t , and later to obtain C by regressing y_t on x_t . However, since an impulse response contains all the information about a system, provided that we have a sufficient amount of coefficients m , it should be possible to discard the data entirely after the impulse response is estimated.

Let

$$\hat{y} := \Phi_u \hat{g}, \quad (15)$$

which, in terms of data, is a function only of the LS estimate $\hat{g}$. Then, as an alternative procedure, we can replace y_t^f by $\hat{y}_t^f$ in (14). This approach, however, performs worse, as will be seen in Section V. The problem can be traced back to the truncation of Φ_u and y , when the impulse response coefficients $\hat{g}$ are estimated. For this reason, the transient part of the data cannot be recovered from (15).

This limitation can be tackled by using, instead of (15),

$$\hat{\hat{y}} := \bar{\Phi} \hat{\hat{g}}, \quad (16)$$

which is based on all data records. Then, (16) can be used not only to construct z_t in (14), but also to regress $\hat{\hat{y}}$ on the state to estimate C .

Although this approach does not bring an advantage with respect to the original version of SSARX, the example prompts the importance of estimating initial conditions in methods that use unstructured models as an intermediate step to estimate the model of interest. Unlike the estimate obtained with truncation, this extended estimate allows the full data records to be reconstructed, containing no less information about the system than the data. Furthermore, as will be discussed in Section VI, it might be possible to exploit computational advantages from only using the unstructured estimate, and no data, once it has been estimated.

IV. WEIGHTED NULL-SPACE FITTING

A. Background

In [8], a weighted least-squares (WLS) method, to which we will refer as WNSF, for estimation of structured models such as (1) is presented. The procedure consists of three steps. In the first step, a high-order FIR model is estimated by LS, as described in Section II. In the second step, this unstructured estimate, $\hat{g}$, is used to obtain a structured estimate, $\hat{\theta}_{\text{LS}}$, by least-squares (LS). Then, the third and final step consists on re-estimating θ using WLS, where the weighting depends on $\hat{\theta}_{\text{LS}}$ and the statistical properties of $\hat{g}$.

The last two steps are detailed as follows. By letting m be large enough, we have that

$$\frac{L(q)}{F(q)} = G(q). \quad (17)$$

Expanding the polynomials $L(q)$, $F(q)$, and $G(q)$, and rearranging, (17) can be written as

$$\begin{aligned} & (g_1 - l_1)q^{-1} + (g_2 + f_1g_1 - l_2)q^{-2} + \\ & + (g_3 + f_1g_2 + f_2g_1)q^{-3} + (g_4 + f_1g_3 + f_2g_2)q^{-4} + \\ & + \dots + (g_m + f_1g_{m-1} + f_2g_{m-2})q^{-m} = 0, \end{aligned} \quad (18)$$

where, for ease of notation, we let $n_f = 2 = n_l$. In vector form, by defining

$$Q(g) := [Q_l \quad Q_f(g)],$$

where

$$Q_l := \begin{bmatrix} -I_{n_l} \\ \mathbf{0}_{m-n_l \times n_l} \end{bmatrix}, \quad Q_f(g) := \begin{bmatrix} 0 & 0 & \dots & 0 \\ g_1 & 0 & \dots & 0 \\ g_2 & g_1 & & 0 \\ \vdots & \vdots & & \\ g_{m-1} & g_{m-2} & \dots & g_{m-n_f} \end{bmatrix},$$

(18) corresponds to

$$z(g, \theta) := Q(g)\theta + g = 0. \quad (19)$$

Replacing g by its estimate $\hat{g}$ obtained in the first step, (19) becomes a linear set of equations in θ , and $\hat{\theta}_{\text{LS}}$ can be obtained by

$$\hat{\theta}_{\text{LS}} = - \left(Q^\top(\hat{g})Q(\hat{g}) \right)^{-1} Q^\top(\hat{g})\hat{g}.$$

Then, the final step consists on re-estimating θ in a statistically sound way. This is done by noticing that the estimate of g can be written as

$$\hat{g} = g + v,$$

where v is normally distributed, with zero-mean and covariance P_g , which can be estimated as in (9). Then, we can write $z(\hat{g}, \theta)$ as function of the noise v , i.e.,

$$z(\hat{g}, \theta) := T(\theta)v,$$

where

$$T(\theta) := \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ f_1 & 1 & 0 & \dots & \dots & 0 \\ f_2 & f_1 & 1 & & \dots & 0 \\ 0 & f_2 & f_1 & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & f_1 & 1 \end{bmatrix}.$$

The key observation here is that $z(\hat{g}, \theta)$ is linear in the noise, and distributed as

$$z(\hat{g}, \theta) \sim \mathcal{N} \left(0, T(\theta)P_g^{-1}T^\top(\theta) \right).$$

Then, we can estimate θ using WLS, by

$$\hat{\theta}_{\text{WLS}} = - \left(Q^\top(\hat{g})WQ(\hat{g}) \right)^{-1} WQ^\top(\hat{g})\hat{g},$$

where the weighting that minimizes the variance of $\hat{\theta}_{\text{WLS}}$ is [17]

$$W = \left(T(\theta)P_g^{-1}T^\top(\theta) \right)^{-1} = T^{-\top}(\theta)P_gT^{-1}(\theta).$$

Since the true parameters θ are unknown, $T(\hat{\theta}_{\text{LS}})$ is used instead.

It is also possible to continue to iterate by repeating step three, where the previously obtained estimate is, in each iteration, used to construct the weighting.

B. Including the Transient

Now, we reformulate the WNSF method in order to include the estimates τ from (10). The OE model in (1) can be written in state-space form as (11), initialized with $x_0 = x^o$ and $u_0 = u^o$. This system is equivalent to

$$\begin{aligned} x_{t+1} &= Ax_t + [B \quad \bar{B}] \begin{bmatrix} u_t \\ \delta_t \end{bmatrix} \\ y_t &= Cx_t + e_t, \end{aligned} \quad (20)$$

initialized with $x_0 = 0 = u_0$, and where δ_t is the Dirac delta and $\bar{B} = Ax^o + Bu^o$ [18].

It is now clear that (10) corresponds to (20) if m is large enough, as $\Phi_{\bar{u}}g$ captures the contribution from u_t , with null initial conditions, filtered by the transfer function

$$C(qI - A)^{-1}B \equiv \frac{L(q)}{F(q)},$$

and $\Phi_{\delta}\tau$ corresponds to the output of the transfer function

$$C(qI - A)^{-1}\bar{B} =: \frac{\bar{L}(q)}{F(q)},$$

with input δ_t , where $\bar{L}(q)$ is, in general, of the same order as $L(q)$. In particular, g and τ are the impulse responses of $L(q)/F(q)$ and $\bar{L}(q)/F(q)$, respectively. The key observation here is that these two transfer functions share the same poles, and, thus, τ contributes to the estimation of θ .

To include the estimates of the initial conditions, the WNSF method becomes as follows. In the first step, compute an estimate of $\bar{g}$, $\hat{\bar{g}}$, from (10). In the second step, calculate

$$\hat{\theta}_{\text{LS}} = - \left(\bar{Q}^\top(\hat{\bar{g}})\bar{Q}(\hat{\bar{g}}) \right)^{-1} \bar{Q}^\top(\hat{\bar{g}})\hat{\bar{g}},$$

where

$$\bar{\theta} := [\theta^\top \quad \bar{l}_1 \quad \cdots \quad \bar{l}_{n_l}]^\top, \\ \bar{Q}(\bar{g}) := \begin{bmatrix} Q_l & Q_f(g) & \mathbf{0}_{m \times n_l} \\ \mathbf{0}_{m \times n_l} & Q_f(\tau) & Q_l \end{bmatrix}.$$

Finally, in the third step,

$$\hat{\theta}_{\text{WLS}} = - \left(\bar{Q}^\top(\hat{g}) \bar{W} \bar{Q}(\hat{g}) \right)^{-1} \bar{W} \bar{Q}^\top(\hat{g}) \hat{g},$$

where

$$\bar{W} := \bar{T}^{-\top} (\hat{\theta}_{\text{LS}}) P_{\bar{g}} \bar{T}^{-1} (\hat{\theta}_{\text{LS}}), \\ \bar{T}(\bar{\theta}) := \begin{bmatrix} T(\bar{\theta}) & \mathbf{0}_{m \times m} \\ \mathbf{0}_{m \times m} & T(\bar{\theta}) \end{bmatrix}.$$

Besides the already discussed advantage of including the estimate of τ to estimate θ , this approach also allows for a better choice of the order m of the unstructured model. Theoretically, the order m has to tend to infinity at a suitable rate as the sample size grows to infinity [19]. In practice, m can be optimized over a grid of values that is function of the sample size, and the model that minimizes (3) is chosen. Computing (3) without knowing the initial conditions of the system requires also truncation of the transient part. By estimating both $L(q)/F(q)$ and $\bar{L}(q)/F(q)$, the function

$$V(\bar{\theta}) = \frac{1}{N} \sum_{t=1}^N \left(y_t - \frac{L(q, \bar{\theta})}{F(q, \bar{\theta})} \bar{u}_t - \frac{\bar{L}(q, \bar{\theta})}{F(q, \bar{\theta})} \delta_t \right)^2.$$

can be computed for particular values of $\bar{\theta}$, as a substitute for (3), without need for truncation, since this setting was constructed with $\bar{u}_t \equiv 0$ for $t \leq 0$.

V. NUMERICAL SIMULATIONS

In this section, we perform two separate Monte Carlo simulations to evaluate the performance gain by introducing initial condition estimates. The first simulation concerns SSARX and the second the WNSF method. In both cases, the data is generated by

$$y_t = \frac{q^{-1} - 2q^{-2}}{1 - 1.7q^{-1} + 0.9q^{-2}} u_t + e_t, \quad (21)$$

where u_t and e_t are uncorrelated Gaussian white noise sequences. The transfer function in (21) has an impulse response which is well approximated with 100 coefficients. The input sequence has unitary variance, and the variance of the measurement noise, λ_e , is chosen to give a signal-to-noise ratio (SNR) of 10dB at the output. Performance is evaluated by computing the FIT, in percent, given by

$$\text{FIT} = 100 \left(1 - \frac{\|g - \bar{g}\|_2}{\|g - \bar{g}\|_2} \right),$$

where, in this case, g is the true impulse response, $\bar{g}$ its mean, and $\bar{g}$ is the impulse response of the estimated system. An impulse response long enough so that it has died out is taken for the true and estimated systems.

The first simulation consists of 200 Monte Carlo runs and sample sizes $N = \{100, 200, 300, 600, 1000\}$, comparing SSARX in the following configurations:

Fig. 1. Average FIT obtained for each sample size, with 200 Monte Carlo runs, comparing three versions of SSARX – original version, data reconstructed from impulse response with transient truncated (SSARX-IR-0) and transient estimated (SSARX-IR-i).

- original version (SSARX);
- using reconstructed output from the impulse response estimate, with truncated transient (SSARX-IR-0);
- using reconstructed output from the impulse response estimate, with estimated transient (SSARX-IR-i).

Here, different high-orders m are tried from a grid of ten linearly spaced values between 20 and $N/2$. Then, for each m , different f and p are tried out from a two-dimensional grid with five linearly spaced values between 10 and $m/2$, along each dimension. The low-order model that minimizes a quadratic cost function of the prediction errors is picked.

Fig. 1 presents the average FIT as function of the sample size. Here, it can be observed that the obtained average FIT is approximately the same when using data (dotted curve) and when the data are reconstructed from an estimated impulse response and transient (full curve). If the reconstructed data are truncated due to the unknown transient part (dashed curve), a similar performance is achieved only for larger sample sizes. This is in accordance with the discussion in Section III-B. Moreover, the small difference observed between the dotted and full curves for the smallest sample size can be explained by the fact that the maximum order $m = N/2$ is not large enough to fully capture the dynamics of the system. Nevertheless, we still observe a great improvement with respect to the case where these initial conditions are not estimated.

Concerning the second simulation, it consists of 200 Monte Carlo runs and sample sizes $N = \{100, 200, 400, 900, 2000, 5000, 10000\}$. The following methods are compared:

- the prediction error method (PEM), starting at the true parameters, with a maximum of 50 iterations;
- the weighted null-space fitting method with truncated transient (WNSF-0), with a maximum of 10 iterations;
- the weighted null-space fitting method with transient estimated (WNSF-i), with a maximum of 10 iterations.

Fig. 2. Average FIT obtained for each sample size, with 200 Monte Carlo runs, comparing WNSF, with transient truncated (WNSF-0) and estimated (WNSF-i), and PEM.

For all the methods, a function tolerance of 10^{-5} is used as stopping criterion. For WNSF, the unstructured model order m is chosen as in the first simulation, according to which minimizes a quadratic cost function of the prediction errors for the low-order model. In the case of WNSF-0, the first m values of y and their predictions are not considered.

For the second simulation, Fig. 2 shows the average FIT as function of the sample size, where it can be observed that WNSF with estimation of initial conditions (full curve) has approximately the same average performance as PEM (dotted curve) for sample sizes $N \geq 200$. If WNSF is applied without estimating these initial conditions (dashed curve), a comparable performance is only observed for much larger values of N . Regarding the case $N = 100$, a performance difference to PEM is observed even if WNSF is used with estimated initial conditions. This can be explained as in the first simulation.

VI. DISCUSSION

In this paper, we discussed a procedure to estimate initial conditions in high-order unstructured models. Although the estimates of the system's impulse response coefficients are not improved in this step, some methods can benefit from the extra estimated variables in posterior steps to obtain a structured model. The procedure was presented for models with additive white noise, but a similar idea can be applied if the noise is colored, where a high-order ARX is estimated instead of an FIR.

Regarding SSARX, reconstructing the data using the parameters estimated with the proposed procedure cannot improve the accuracy of the original version, since no new information is included. However, as backed up by a simulation study, it serves to demonstrate how these parameters contain no less information than the data, unlike if the initial conditions are discarded. Also, the information contained in the data is now condensed in the estimates of the impulse response and the initial conditions, which, potentially, are of

smaller size than the data. Thus, reformulating a subspace algorithm to be applied to these estimated parameters directly instead of the data may yield computational advantages. How to tackle this problem is in the agenda for future work.

The WNSF method is an example of a method that benefits from using the proposed procedure. In the model reduction step to a structured estimate, the method uses the relation between the impulse response and the structured model parameters. Since, for OE models, the extra variables estimated in the unstructured model step are themselves estimates of an impulse response of a system with the same poles, they contain useful information for the estimation of the system of interest. A simulation showed that this procedure improves the method, in the sense that it performs similarly to PEM for much smaller sample sizes.

The delineated procedure can potentially be incorporated into other methods that use a high-order model as an intermediate step to obtain a low-order model. Although the asymptotic properties remain unchanged, significant improvements may be observed for finite sample sizes.

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