NOISE FROM WIND TURBINES

OLIVIER FÉGEANT

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Abstract

A rapid growth of installed wind power capacity is expected in the next few years. However, the siting of wind turbines on a large scale raises concerns about their environmental impact, notably with respect to noise. To this end, variable speed wind turbines offer a promising solution for applications in densely populated areas like the European countries, as this design would enable an efficient utilisation of the masking effect due to ambient noise. In rural and recreational areas where wind turbines are sited, the ambient noise originates from the action of wind on the vegetation and about the listener’s ear (pseudo-noise). It shows a wind speed dependence similar to that of the noise from a variable speed wind turbine and can therefore mask the latter for a wide range of conditions. However, a problem inherent to the design of these machines is their proclivity to pure tone generation, because of the enhanced difficulty of avoiding structural resonances in the mechanical parts. Pure tones are deemed highly annoying and are severely regulated by most noise policies. In relation to this problem, the vibration transmission of structure-borne sound to the tower of the turbine is investigated, in particular when the tower is stiffened at its upper end. Furthermore, since noise annoyance due to wind turbine is mostly a masking issue, the wind-related sources of ambient noise are studied and their masking potentials assessed. With this aim, prediction models for wind-induced vegetation noise and pseudo-noise have been developed. Finally, closely related to the effect of masking, is the difficulty, regularly encountered by local authorities and wind farm developers, to measure noise immission from wind turbines. A new measurement technique has thus been developed in the course of this work. Through improving the signal-to-noise ratio between wind turbine noise and ambient noise, the new technique yields more accurate measurement results.
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A rapid growth of installed wind power capacity is expected in the next few years. However, the siting of wind turbines on a large scale raises concerns about their environmental impact, notably with respect to noise. To this end, variable speed wind turbines offer a promising solution for applications in densely populated areas like the European countries, as this design would enable an efficient utilisation of the masking effect due to ambient noise. In rural and recreational areas where wind turbines are sited, the ambient noise originates from the action of wind on the vegetation and about the listener’s ear (pseudo-noise). It shows a wind speed dependence similar to that of the noise from a variable speed wind turbine and can therefore mask the latter for a wide range of conditions. However, a problem inherent to the design of these machines is their proclivity to pure tone generation, because of the enhanced difficulty of avoiding structural resonances in the mechanical parts. Pure tones are deemed highly annoying and are severely regulated by most noise policies. In relation to this problem, the vibration transmission of structure-borne sound to the tower of the turbine is investigated, in particular when the tower is stiffened at its upper end. Furthermore, since noise annoyance due to wind turbine is mostly a masking issue, the wind-related sources of ambient noise are studied and their masking potentials assessed. With this aim, prediction models for wind-induced vegetation noise and pseudo-noise have been developed. Finally, closely related to the effect of masking, is the difficulty, regularly encountered by local authorities and wind farm developers, to measure noise emission from wind turbines. A new measurement technique has thus been developed in the course of this work. Through improving the signal-to-noise ratio between wind turbine noise and ambient noise, the new technique yields more accurate measurement results.

Keywords: Masking, vibration transmission, diffraction, ambient noise, pseudo-noise, cylindrical shell, perturbation methods, structural mobility, acoustic outdoor measurement.
à ma famille
Preface

This thesis is the result of my research work at the Division of Building Technology, Department of Building Sciences at the Royal Institute of Technology of Stockholm during the years 1995-2000. To this end, I would like to acknowledge the financial support of the Swedish National Energy Administration within the frame of the VKK program and of the European Commission (Project JOR3-CT95-0065 on Noise Immission from Wind Turbines).

I wish to express my gratitude to my supervisor, Professor Sten Ljunggren, for giving me the opportunity to carry out this research. On top of his guidance, the constant interest that he has shown in my work has been the best encouragement to carry this thesis through to its completion (Tack ska du ha, Sten!).

Finally, I would like to add that, although addressing an adverse effect of wind turbines, this thesis does not question the legitimacy of wind energy. In my mind, noise from wind turbines is an important issue in relation to public acceptance but it must be considered as a siting matter rather than as an insuperable problem. In this respect, it is primarily the concern of developers and local utilities, even if wind turbine manufacturers have much to gain by producing quieter machines.

Stockholm, in January 2001
The thesis comprises an introduction and the following papers:

**Paper A**  
O. Fégeant, “Vibratory power transmission to a stiffened cylindrical shell structure”.

**Paper B**  
O. Fégeant, “Closed-form solutions for the point mobilities of axisymmetrically excited cylindrical shells” *Journal of Sound and Vibration* (Accepted for publication).

**Paper C**  
O. Fégeant, “Structural mobilities for the edge-excited, semi-infinite cylindrical shell using a perturbation method” (Submitted for publication in the *Journal of Sound and Vibration*).

**Paper D**  

**Paper E**  

**Paper F**  

**Paper G**  
Introduction

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Papers A-G
Introduction
1. GENERAL INTRODUCTION

The beginning of the 21st century may well be a turning point in the history of wind energy. Because of concerns about climate changes, of an increasing awareness of the public and its representatives and of its technical maturity, wind energy, among other sources of renewable energies, is expected to be a significant part of the future energy production. For instance, the target set for wind energy in the European Commission’s 1997 white paper on renewable energy entails a 15-fold increase of the installed capacity by 2010.

In densely populated areas like Europe, noise aspects will thus become a matter of great concern. In the first place, it will set an upper bound to the number of possible sites since the environmental impact with respect to noise is strictly regulated in most countries. In the second place, it seems that, whereas people support wind energy as a renewable energy, they object to specific local projects because of the expected consequences concerning primarily noise and visual impact (Gipe, 1995). Consequently, wind farm developers will have to pay special attention to noise concerns not to tarnish the image of wind energy and thus jeopardise its development.

Noise annoyance is a complicated issue involving both a physical and a cognitive dimension (Fégeant, 2001). Thus, it is acknowledged that a positive attitude to the source of the noise reduces noise annoyance (Pedersen, 1994). Apart from the psychological aspects, a number of physical factors govern the perception of noise. In this respect, a most important concept is the one of masking. Masking is defined as ‘the interference with the perception of one sound (the signal) by another sound (the masker)’ (Buus, 1997). Due to masking, the intrusiveness of the signal is decreased, thereby leading to a decreased annoyance. As regards noise annoyance caused by wind turbines, the masking effect is a crucial issue. Indeed, partial or total masking of wind turbine noise is expected close to dwellings because of the high levels of ambient noise induced by the wind in the surrounding vegetation and about the listener’s ear. A related aspect is the regularly encountered difficulty of measuring noise immersion from wind turbines. Indeed, low signal-to-noise ratios between wind turbine noise and ambient noise often preclude accurate experimental assessment of the environmental impact.

In its present form, the Swedish noise guidelines for wind turbines are based upon existing rules for industrial plants and do not take the ambient noise into account: they only specify sound pressure levels ($L_{eq,T}$) which must not be exceeded at neighbouring dwellings (General Guideline, 1995). However, the above remarks show that a noise legislation that ignores the masking effect and its mechanisms cannot be a warrant for a low rate of annoyance caused by wind turbines (Fégeant, 1998). Knowledge of the characteristics of wind turbine noise and ambient noise is more important to assess the environmental impact than the level alone (Hayes, 1992). To this end, theoretical modellings of the noise characteristics, models accounting for outdoor propagation effects and accurate measurement techniques are most valuable tools.
2. OBJECTIVES OF THE WORK

The object of the work presented here is threefold: First, to develop tools to enable a reduction of the transmission of structure-borne sound from the machinery to a potential noise radiator: the wind turbine tower. Secondly, to develop prediction models for the noise generated by the wind in the vegetation and around the listener’s ear in relation to the masking issue. Thirdly, to develop a new measurement technique for wind turbine noise immission assessments.

Noise generated by modern wind turbines is mainly of aeroacoustic origin and strongly dependent on the rotor speed. Variable speed design appears as a promising solution to noise problems since it would enable a tailoring of the noise generation to the ambient noise level (Ljunggren et al., 1989), while bringing additionally some technical advantages such as reduced fatigue load and improved power quality. However, a problem of major importance concerning this design is the enhanced difficulty, in comparison with constant speed design, of designing the mechanical parts with respect to structural resonances and structure-borne sound propagation. Structural resonances might result in pure tone emission, the annoying character of which is accounted for in the Swedish noise legislation by a 5 dB penalty (General Guidelines, 1995). Because of its tubular steel design and its size, the tower of the wind turbine might become an effective noise radiator if excited via structure-borne paths. The fact that prominent pure tone emissions have been observed in relation to radiation by the tower of wind turbines (Lundmark, 1996) reflects the need to study the transmission of structure-borne sound from the nacelle to the tower. Thus, one objective of the present work is to describe the mechanical power transmission to circular, semi-infinite cylindrical thin shells, either non-stiffened or stiffened, by mechanical loads (Papers A, B and C).

A second objective of this thesis is to investigate the diverse wind-related sources of ambient noise and to evaluate their masking potential. Masking at immission points is an important issue because the levels of wind turbine noise and ambient noise are generally of the same order of magnitude. As a result, noise annoyance is not related to the absolute level of the wind turbine noise (Persson Waye, 1999) but to its intrusiveness. Assessing this later requires a determination of the amount of masking provided by the ambient noise. Wind turbines are generally sited in rural and recreational areas, where the wind is primarily responsible for the acoustical climate. Papers D and E describe a semi-empirical model for the prediction of ambient noise generated by wind in vegetation. Paper F presents original measurement results on the noise generation due to the wind about the listener’s ear. In addition, Paper F discusses the masking potentials of different types of wind-induced noise, in terms of frequency contents and wind speed dependence, and the influence of wind turbulence on masking.

Finally, the third issue addressed in this thesis is the measurement of wind turbine noise immission. Such measurements are usually motivated following complaints from neighbours or for control purposes. But, because of the regularly prevailing poor signal-to-noise ratio between wind turbine noise and ambient noise at remote immission points, measurement results are often difficult to interpret and may even be inaccurate. In relation to this problem, Paper G describes a new measurement technique for improving wind turbine noise immission assessments.
3. NOISE FROM WIND TURBINES

There is great design diversity in wind turbine technology but this section is focused on the most widespread design: the upwind horizontal axis wind turbine (that is, with the rotor facing the wind, see Figure 1).

3.1 NOISE GENERATION CHARACTERISTICS OF MODERN WIND TURBINES

Noise generated by wind turbines fits into two classes: mechanical noise and aeroacoustic noise. During the last decade, a considerable attention has been given to the design of the blades and of the mechanical parts of the machine, resulting in a significant noise reduction. For instance, a modern wind turbine of the megawatt class has a sound power level of about 100 dB(A), that is the level of a 150 kW wind turbine ten years ago (Klug, 1997).

![Diagram of wind turbine](image)

Figure 1. Sources of noise for an upwind, horizontal axis wind turbine.

3.1.1 Aerocoustic noise generation

Aeroacoustic noise generation, also referred to as rotor noise, is the main source of noise from modern, constant speed wind turbines. The noise results from the interaction of the rotating blades of the wind turbine with the surrounding atmosphere and is usually attributed to the following three sources of broadband noise: (a) interaction of atmospheric turbulence with the blade surface (inflow-turbulence noise), (b) interaction of the blade boundary-layer turbulence with the trailing edge (trailing-edge noise), and (c) vortex shedding caused by the bluntness of the trailing edge. Figure 2 illustrates the contributions of these three mechanisms to the total noise spectrum. An additional, but
usually less significant, source of noise is the so-called tip noise, arising from the flow separation at the blade tip. The horizontal and vertical directivity patterns of these noises are not very pronounced and therefore not discussed in what follows (see Sutherland, 1987, for a discussion of this subject).

As mentioned previously, a lot of attention has been, and is still, devoted to the blade design in the objective of reducing the noise generation. Thus, changes of the tip shapes (Andersen, 1993, Klug, 1997) and sharpening of the blade trailing edges, (Hubbard, 1991), have resulted in significant reductions of tip-noise and trailing edge bluntness noise. Accordingly, the inflow-turbulence and trailing-edge noises are considered as being the main sources of aeroacoustic noise for modern, well-designed wind turbines (Guidati et al., 1999). Their levels are proportional to the third and the fifth power, respectively, of the speed of the blade relative to the surrounding air. Inflow turbulence noise is directly related to wind speed and inflow turbulence. On the other hand, trailing-edge noise is greatly influenced by the blade design (Lowson, 1992) and studies show that noise reductions may be achieved by trailing edge serrations (Braun et al., 1999). Accordingly, for constant speed designs, trailing-edge noise is expected to dominate just above the cut-in wind speed of the wind turbine while inflow-turbulence noise becomes more prominent as the wind speed increases.

![Figure 2. Relative contributions of aeroacoustic noise sources to the total noise spectrum (from Grosveld, 1985).](image)

With respect to these considerations, the type of terrain on which the wind turbine is sited must be expected to influence the noise generation. Indeed, complex terrain is characterised by a high mechanical turbulence level (Panofsky, 1984) and should consequently lead to higher inflow-turbulence noise generation in comparison with a flat terrain. However, this hypothesis could not be confirmed by field measurement results (Bass, 1997) and this would tend to support the alternative hypothesis suggested by Glegg (1987), that the inflow turbulence at each blade is strongly influenced by the wake of the preceding blade. A further effect of potential significance, also in relation to complex terrain, is the effect of the wind incidence angle on the wind turbine rotor since
measurements carried out for a wind turbine sited at the edge of a steep hill have shown an unexpected strong noise generation (Fégeant, 1998).

3.1.2 Mechanical noise generation

Mechanical noise originates from different components of the machinery of the wind turbine. The principal sources of the noise emission are (Ljunggren, 1991): gearbox, generator, drive train and auxiliary equipment (hydraulic system, cooling units). The gearbox is usually the dominating component and generates primarily structure-borne sound (Pinder, 1992).

The generation of machinery noise is closely related to the vibration of the machine. It propagates directly through the nacelle openings or is radiated indirectly by the nacelle hub and blades or the tower excited via structure-borne paths (Andersen, 1993). Machinery noise will typically be both broadband and tonal due to the cyclical nature of the contacts between the rotating parts. Both the broadband noise level and the level and the frequency of pure tones depend on the rotational speed. Because of this tonal content, machinery noise is subjectively more annoying than the aerodynamic noise, even if, in terms of sound level, it is not dominating.

Structural dynamics analysis and noise control measures have figured increasingly in the design of the mechanical parts. As a result, tonal emission is no longer a problem for modern, constant speed wind turbines and the overall mechanical noise is generally masked by the rotor noise. However, as mentioned in the introduction, an increasing development of wind turbines with variable speed design is expected in Europe in a near future (European Commission, 1999). This design provides a simple and efficient mean to control the overall noise level but presents considerable drawbacks in relation to pure tone emission. Since the rotor speed is allowed to vary in a wide range, it is much more difficult to predict potential amplifications in the structural response of the mechanical parts. Thus, prominent pure tone emissions have been observed in relation to radiation by the tower of wind turbines (Lundmark, 1996).

Tubular steel tower is the most common design for large wind turbines. It is often built up from sections of 20-30 metres with flanges at either end bolted together. In Papers A, B and C, the transmission of structure-borne sound to the tower is investigated. The approach stems from the following hypothesis: (a) the tower has been modelled as a thin, circular cylindrical shell despite the fact that real towers are often slightly conical, (b) a low impedance excitation is assumed. The former assumption is supported by measurement results describing the structural response of the tower to impact test (Lundmark, 1996). Thus, Figure 3 shows that the measured resonance frequencies of the tower can be predicted using Flügge theory (Flügge, 1973) applied to a thin, cylindrical shell with a radius equal to that of the tower at the point of excitation. The non-acquainted reader might be introduced to thin-walled shell theory by referring to the first section of Paper A. The second hypothesis relies on the fact that vibration isolators are likely to be placed at the excitation points.

The design of the tower usually includes a thicker, cylindrical part welded to the tower end. The purpose of this stiffener is twofold: first to enable the mechanical assembling with the nacelle. Second to stiffen the structure with respect to the external loads. The
influence of such a stiffener on the transmission of structure-borne sound to the tower is studied in Paper A while Papers B and C shed some light on the physical mechanisms underlying this transmission. In those later papers, analytical formulae for the structural mobilities of the shell are derived using a perturbation approach, revealing the functional dependence of this transmission on the material and geometric parameters of the shell.

The main result outlined by the analysis presented in Paper A is that the influence of the stiffener is frequency dependent, as shown by Figure 4. Three frequency regions characterise the influence of the stiffener. At very low frequencies, the structure behaves as if the stiffener was only an added mass and the input power is similar to the one of the semi-infinite shell of thickness equal to the one of the tower (\( h_t \)). Conversely, at high frequencies, the stiffened shell behaves in a way similar to a semi-infinite shell of thickness equal to that of the stiffener (\( h_s \)). In the frequency region where the transition takes place, no resonant behaviour appears but only small undulations in the mobility curve. This result is of significant interest since it provides a simple mean to avoid structural resonances at the cut-on frequencies of the circumferential modes. Yet the benefit in terms of noise radiation is uncertain since these modes are inefficient radiators near their cut-on frequencies (Fahy, 1985). It is worth noting that at both low and high frequencies, the stiffened shell exhibit the circumferential resonance characteristics of the shell and the stiffener, respectively. Finally, it has been shown that the power transmission to the structure is similar to the power transmission to a shell having a thickness equal to the stiffener thickness when the following inequality is satisfied

\[
\alpha d \geq c_p \sqrt{\frac{h_s}{R}},
\]  

(1)
where $\omega$ is the circular frequency of the excitation, $R$ the radius of the shell, $L$ the length of the stiffener, and $c_p$ the velocity of quasi-longitudinal wave in a plate of the same material as the shell. Furthermore, the power transmission is in a frequency average sense reduced by $20\log(h_z/h_1)$ decibels compared to the non-stiffened shell if the above condition is satisfied.

![Figure 4. Real part of the radial input mobility for edge-excited semi-infinite shells, $(\Omega = \omega R/c_p)$. (-- --) shell of thickness $h_z$; (— --) shell of thickness $h_0$; (---) shell of thickness $h_1$ stiffened by a 10 cm long stiffener of thickness $h_z$.](image)

### 3.2 ENVIRONMENTAL ASPECTS OF WIND TURBINE NOISE

As outlined in the introduction, the environmental impact caused by wind turbine noise can usually be considered as a masking issue. Thus, its assessment requires a reasonable knowledge of potential maskers prevailing in rural and recreational environments, where most wind turbines are sited.

#### 3.2.1 Sources of masking

The acoustical climate of rural and recreational areas can be qualified as “natural” in the sense that it is not the result of human activities. As a matter of fact, the wind will likely to be the origin of the ambient noise when the wind turbine is in operation, that is for wind speeds higher than about 5-6 m/s at hub height.

The wind, as we experience it at ground level, is the result of large-scale flows set into motion by thermal contrasts in the atmosphere. It is convenient to regard it as having a steady component and a superimposed fluctuating component (called turbulence), the familiar gusts. In the region close to the earth’s surface, currently referred to as planetary boundary layer (PBL), the mean wind speed increases with height while turbulence
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depends on the ground roughness and topography and on the atmospheric stability (temperature gradient, etc.).

The sources of wind-induced masking noise are numerous and, most of them, of aeroacoustic origin. However, two of them can be expected to prevail in environments typical for rural and recreational areas. They are the noise due to the vegetation and the noise resulting from the action of the wind on the listening device. A third source of ambient noise would be the inherent turbulence of the PBL since vortices have been shown to radiate acoustical waves (Lighthill, 1954). However, at Mach numbers typical for wind, this generation should not amount to significant contributions (Sutherland, 1987).

Though few, there has been some attempts in the past to characterise the wind-induced vegetation noise (see Paper D for references). These studies report on measurement results obtained for different vegetation species and density and provide valuable information on such topics as typical noise levels, frequency distribution and dependence on wind speed. The study presented in Papers D and E was designed to model the mechanisms at the origin of the vegetation noise and to include this theoretical modelling in a semi-empirical analytical model for vegetation noise. Thus, in Paper D, it is shown that the two principal origins of vegetation noise are: (a) rustling of the leaves for deciduous species and (b) whistling of needles for coniferous ones. The former arises from the wind-induced flap of the leaves with each other. Its spectrum shape is wind speed independent and almost flat but with a broadband peak around 5000 Hz when analysed in 1/3 octave bands. The later arises from vortex shedding by the needles; an aeroacoustic mechanism which is also the origin of the whistling of telephone wires in a strong wind (Goldstein, 1978). Its spectrum shape is very wind speed dependent since it is characterised by a broad peak at the Strouhal frequency given by the needle diameter and by the wind speed.

The second source of ambient noise is the pseudo-noise. A microphone, or a human ear, placed in a turbulent air stream perceives the local aerodynamic pressure fluctuations as if they were real acoustic signals. The term “pseudo” refers to the non-acoustical nature of this noise (it does not propagate in contrast to acoustical waves), but, in practice, this term encompasses the aerodynamic noise as well as the aeroacoustic noise resulting from the introduction of the measurement system in the flow. However, microphone and human ear respond differently to wind turbulence since in the case of the ear, the resulting aerodynamic pressure fluctuations at the eardrum are strongly influenced by the system human head - ear canal. Consequently, a correction should be applied to measurement results gathered with a free microphone, equipped with a windscreen or not, in order to assess the amount of masking. This correction, referred to as the ear’s dynamic transfer function in Paper F, relates the pseudo-noise measured by a microphone at the eardrum to the one measured in the free field. The paper describes measurement results obtained for different wind speeds and several orientations of the head with respect to the wind direction. In addition, it has been found that human response to aerodynamic pseudo-noise can also be assessed by applying the A-weighting system, despite the non-acoustical nature of this noise. Thus, the pseudo-noise generated by a wind of speed $U$ is approximately given by

$$L_{Aeq, microphone} = 7 + 50 \log(U),$$

(2)
for a free microphone fitted with a 10 cm diameter wind screen, and

\[ L_{A_{eq, ear}} = 3 + 70 \log(U), \]  

(3)

at the eardrum for a position of the head such that the ear is facing the oncoming flow.

3.2.2 Measurement of wind turbine noise immission

As explained in the introduction, measurements of noise immission from wind turbines are uniquely difficult because of the regularly prevailing low signal-to-noise ratio between wind turbine noise and ambient noise (Fégeant, 1997). At low frequencies, the main source of this later is the pseudo-noise (Morgan, 1992), and will therefore depend on the choice of the measurement technique. Among others, three methods are in other cases used to achieve an abatement of pseudo-noise: (a) acquiring the data by dual microphone correlation technique which utilises the loss of coherence in the flow turbulence, (b) lowering the microphone elevation above the ground as the wind speed decreases with decreasing height, and (c) using a windscreen around the microphone which reduces the wind speed at the microphone diaphragm.

The first is not suitable as it causes a suppression of measured wind turbine noise of 1 dB or more (Kragh et al, 1998). Furthermore, it requires sophisticated instrumentation not commonly used by most wind turbine manufacturers or local government regulatory agencies. The second one cannot be used for immission measurements as existing recommendations specify measurements with a microphone at some distance above the ground in order to avoid shielding by obstacles (Ljunggren, 1997). Finally, the last method is the one commonly used in practice. The windscreen, used currently in outdoor measurement, is a 10 cm diameter open cell foam ball, but does not offer sufficient pseudo-noise reduction for wind speeds higher than approximately 4 m/s. An additional windscreen might nonetheless be used to provide a further abatement (Theofiloyiannakos et al, 1997).

An alternative technique, presented in Paper G of this thesis and in more detail in Fégeant (1997), uses a microphone mounted on a vertical board surface. It is shown that it improves the signal-to-noise ratio since the board acts as

- a source amplifier by doubling the turbine acoustical pressure at the microphone diaphragm because of reflection by the board surface,
- an acoustical shield which shelters the microphone from the sources of ambient noise situated behind the board,
- a wind shield as the wind speed, and thus the pseudo-noise, is reduced at the microphone due to the blocking effect of the board.

The board also shelters the microphone from possible contributions of the wind turbine noise by reflecting surfaces as building facades situated behind the microphone. However, diffraction occurs at the edges and corners of the board affecting the reflection and the acoustical shielding. Thus, the assessment of both the diffraction effects and the pseudo-noise reduction are important to evaluate the improvements achieved by the board measurement technique. Paper G shows that the signal-to-noise ratio is globally
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improved over the whole frequency range. This technique leads to satisfying results and is now recommended for certain immission measurements cases (Ljunggren, 1997 and 1998). A drawback of this technique is a reduced practical applicability since the board dimensions required in order to reduce the diffraction effects make its handling difficult in strong winds.

3.2.3 Masking of wind turbine noise by ambient noise

The concept of masking characterises the reduced ability of the auditory system to detect an intruding signal because of the presence of an ambient noise. The masking is said to be complete when the masker makes the signal inaudible while the adjective “partial” is used when the presence of the masker results in a decrease of the signal loudness or of the noticeability of a given change of the signal. In relation to masking is the description of the human auditory system as a set of filters, called critical bands, (Zwicker, 1999), which cover the audible frequency range [20 - 20000 Hz]. Thus, the detectability of a signal will depend on its signal-to-noise ratio with the masker in each critical band.

Consequently, masking is best achieved when signal and masker have similar frequency contents. In this case, and if both signal and masker are broadband and steady, the amount of masking can be assessed directly from the signal-to-noise ratio of the equivalent continuous A-weighted levels. It is accepted that the signal is barely perceptible when the $L_{Aeq}$ of the signal is the same as the $L_{Aeq}$ of the masker (Miaoa, 1992). However, in order to distinguish and segregate sounds emanating from different sources, the human auditory system resorts to an intelligent processing which focus on certain components of the acoustic signals. Hence, periodic time variations of the signal (amplitude modulation) and presence of pure tones make the masking more difficult by drawing the attention of the ear on the signal. In this respect, the amplitude modulation shown by the aerodynamic noise from some wind turbines (Arlinger, 1990) or the emission of pure tones might lead to a masking release, and thus induce an increase in annoyance.

In paper F, the masking-related characteristics of the ambient noise are reviewed. It is shown that, for constant speed wind turbines, masking increases gradually with wind speed. Consequently, annoyance, if any, is likely to occur at wind speeds just above the cut-in wind speed of the turbine. Variable speed wind turbines would afford to take advantage of this increasing masking. It is nonetheless worth to point out that reducing the rotor noise in light winds might make the mechanical noise more prominent and thus bring to the fore the annoying character of this later.

Figure 3 presents typical noise spectra of different sources of ambient noise and of a wind turbine. It is clearly evidenced that, despite similar $L_{Aeq}$, detectability of the wind turbine is high in the frequency range [200 - 2000Hz] if the ambient noise is generated by vegetation of deciduous type or by the pseudo-noise. On the other hand, coniferous species such as pine is potentially an efficient masker of wind turbine noise in this frequency range. Another aspect related to masking is the effect of wind turbulence. Random temporal variations of ambient noise levels will occur because of wind turbulence, thereby resulting in a fluctuating signal-to-noise ratio and maybe in the
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Audibility of the wind turbine during short time laps. This effect is detrimental to masking and is also discussed in Paper F.

Figure 3: Typical A-weighted spectra for the ambient noises and the wind turbine noise.

(—o—) wind turbine 38 dB(A), (—+—) ear 38 dB(A), (—) microphone 35 dB(A),
(— — —) birch 38 dB(A), (—) pine 38 dB(A).

4. CONCLUSION

Although psychological factors have to be taken seriously, noise annoyance caused by wind turbine is mostly a masking issue. This aspect is presently neglected in the Swedish noise legislation, which is believed to be unsatisfying both from community and industry standpoints. Indeed, this penalises wind turbines in strong winds while annoyance may occur in light winds. This is clearly a strong motive for implementing a noise policy related to ambient noise levels. However, consideration has also to be paid to the difficulties related to the implementation of such regulation, in contrast with the simplicity of the present one. It has therefore been a central issue in this work to provide a reasonable understanding of the ambient noise in typical wind farms environments. It should also be pointed out that an ambient noise-related policy would be an incentive for the development of variable speed wind turbines. This design provides a simple and efficient mean of tailoring the noise generation from the wind turbine to the masking potential of the sites. With respect to the enhanced risk of pure tone emission by such machines, the solution has to be found in careful design and advanced structure dynamic analysis of the mechanical parts.
5. SUMMARY OF THE APPENDED ARTICLES

Paper A: Vibratory power transmission to a stiffened cylindrical shell structure
By: O. Fégeant
Summary: This paper studies the vibratory power transmission to a stiffened, thin cylindrical shell in the case of a mechanical excitation positioned on the stiffened part. The structure under consideration is built up from a semi-infinite shell coupled at its edge to a thicker shell of finite length (the stiffened part). Theoretical predictions based upon the Flügge theory are shown to agree nicely with measurement results. Results from a parametric study are also presented, showing the influence of stiffener length and thickness and excitation position. It is revealed that appropriate design of the stiffener can yield a significant reduction of input power.

Paper B Closed-form solutions for the point mobilities of axisymmetrically excited cylindrical shells
By: O. Fégeant
Summary: In this paper, closed-form formulae are reported describing point mobilities for thin cylindrical shells in axisymmetric motion. Two cases are studied: (i) the infinite shell and (ii) the edge-excited semi-infinite shell and, for both, forces and moment excitation are considered. The solutions of these problems are obtained analytically by resorting to perturbation methods and presented in terms of Green’s functions and point mobilities. These results are further used to derive approximate expressions for the reflection coefficients of the shell-borne waves at a free end and for the Green’s functions of a finite free-free shell undergoing axisymmetric vibrations.

Paper C Structural mobilities for the edge-excited, semi-infinite cylindrical shell using a perturbation method
By: O. Fégeant
Summary: Approximate closed-form solutions for all the elements of the mobility matrix for a thin-walled cylindrical, semi-infinite shell are presented here. Excitation is applied at the shell edge and consists of four load types, namely an axial force, a circumferential force, a radial force and a bending moment. The aim of the research is to determine structural mobilities for the edge-excited, semi-infinite cylindrical shell using a perturbation method. The problem is formulated using Donnell-Mushtari shell equations where in-plane inertial forces are neglected. The derivation of the solution is based upon the method of the matched asymptotic
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Expansions, yielding approximate expressions for both the input and cross-mobilities for a given circumferential mode of vibration. Finally, the accuracy of the expressions derived is discussed in terms of individual modal mobilities and point mobilities in comparison with numerical results calculated using Flügge theory. It is shown that vibrational power transmission by mechanical point excitation is predicted with an acceptable level of agreement for a frequency below half the ring frequency.

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**Paper D**

*Wind-induced vegetation noise, part I: a prediction model*

**By:** O. Fégeant

**Summary:** This paper describes a semi-empirical analytical model developed to predict ambient noise generated by vegetative sources during a windy day. The model is based on measurements carried out with vegetation samples placed in a flow. Typical noise spectra and the dependence of radiated noise on wind speed have been obtained for different tree species. Then, by using simplified representations of the interaction of the wind flow with the foliage, analytical expressions giving the acoustic power generated by different configurations (single tree, forest, forest edges, shelterbelt) have been derived. The influence of the vegetation cover and the ground reflection on the sound propagation is taken into account so that the sound pressure level can be predicted at an arbitrary point. The model shows that the generating mechanisms are different for coniferous trees and deciduous ones. In a subsequent paper, a series of measurements carried out in the field to calibrate and validate the model are presented.

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**Paper E**

*Wind-induced vegetation noise, part II: field measurements*

**By:** O. Fégeant

**Summary:** The present paper is devoted to the calibration and the validation of this model developed in Paper F by field measurements. It is shown that the model gives reliable predictions, both in terms of frequency distribution and A-weighted sound pressure levels, near the vegetative sources, while some discrepancies between theory and measurements appear when the propagation distance increases. The validation is based on 41 measurement results obtained at seven sites for different propagation distances and wind speeds. It is seen that the model predicts the A-weighted level with a mean error of 0.25 dB and a standard deviation of 3.7 dB. Nevertheless, this standard deviation is reduced to 3.0 dB if some suspicious data are removed from the validation.
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Paper F  
*On the masking of wind turbine noise by ambient noise*

By:  
O. Fégeant

Summary:  
A prerequisite to discussing the environmental impact of wind turbine noise is a reasonable understanding of what is being impacted. This paper reviews the current knowledge in matter of natural wind-induced ambient noise and presents original measurement results on the pseudo-noise generation at the human ear. Finally, the masking of WT noise by ambient noise is discussed, especially its relationship to the temporal random noise level fluctuations induced by the turbulence in the wind. Indeed, though the ambient noise has a strong masking potential in terms of frequency contents and generated sound levels, the influence of these fluctuations on masking is difficult to assess. It is shown how the deterministic models predicting the ambient noise can be coupled to the statistical analysis of the wind turbulence to yield reliable assessment of the amount of masking.

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Paper G  
*On the use of a vertical microphone board to improve low signal-to-noise ratios during outdoor measurements*

By:  
O. Fégeant

Summary:  
In this paper, the wind noise generation and the scattering of acoustical waves by a vertical board mounted above the ground are studied. A theoretical diffraction model, collated by measurements, has been derived to assess the effects of diffraction, and it provides a simple way to locate the minima and maxima of deviation from pressure doubling. It is seen that the board shielding effect allows the measurements to be carried out in the presence of reflecting surfaces such as building facades. Furthermore, a set of measurements has been achieved both in a laboratory and in the field to assess the wind noise generation when the measurement system is placed in a flow. Dimensionless laws giving the wind noise as a function of the Strouhal number for both the board and the free microphone have been obtained. It is shown that the signal-to-noise ratio is globally improved by the presence of the board though reduced at high frequency by the wind turbulence.

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REFERENCES


Introduction


Introduction


VIBRATORY POWER TRANSMISSION TO A STIFFENED CYLINDRICAL SHELL STRUCTURE

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The vibratory power transmission to a stiffened, thin cylindrical shell is investigated theoretically and experimentally in the case of a mechanical excitation positioned on the stiffened part. The structure under consideration is built up from a semi-infinite shell coupled at its edge to a thicker shell of finite length (the stiffened part). Theoretical predictions based upon the Flügge theory are shown to agree nicely with measurement results. Results from a parametric study are also presented, showing the influence of stiffener length and thickness and excitation position. It is revealed that a significant reduction of input power can be obtained under certain conditions.

1. INTRODUCTION

A great number of engineering structures may accurately be modelled as thin-walled, cylindrical shells, thereby justifying an interest in the characterisation of their structure borne sound properties. However, as the diameter of a shell becomes large, use is often made of stiffeners to increase its rigidity. The presence of stiffeners results in structural discontinuities, which influence the free wave propagation [1, 2] and the structural response to dynamical excitation [3]. The cited studies show that appropriate design of stiffener characteristics and spacing yields a considerable reduction of vibration transmission in case of excitation on the unstiffened part.

However, a common engineering practice is to position the excitation at or close to stiffeners, thereby emphasising the need to study the forced response of reinforced structures excited on their stiffer part. This need has been partially answered by the interest paid to the coupled system consisting of a localised force driving a beam, which is attached to a thin plate [4-7]. On the other hand, the corresponding problem of the reinforced shell has received scant attention. As a matter of fact, to the author’s knowledge, Skudrzyk [6], in a tutorial paper presenting an approximate method to predict the mean-value response of complex vibrators, is the only author who has ever considered it. He reported on measurement results obtained for a ring-stiffened shell and provided a formula to estimate the impedance of the structure. This formula is based upon approximate expressions of the ring and the shell mobilities and is applicable when the ring is much stiffer than the shell.

The object of the present study is to evaluate the effects of an axisymmetric discontinuity of finite length in the thickness of a shell wall on vibratory power transmission, when the excitation is located on the discontinuity. The structure under consideration is built up from a semi-infinite, thin cylindrical shell with a discontinuity in thickness at its free edge. In contrast to the case treated in [6], the length of the stiffened part is allowed to vary beyond the ring assumption and focus is made on moderate discontinuities with a discontinuity-to-shell thickness ratio lower than 10.
The plan of the analysis is as follows: Section 2 and 3 are devoted to the derivation of the equations of motion as arrived at by Flügge and by Donnell and to the propagation of free waves in thin-walled, cylindrical shells. Similar material can be found in many reference books and articles, e.g. [1, 8-10], but the derivation is presented here to provide the non-acquainted reader with the basis of thin shell theory. In Section 4, the formalism to determine the dynamic response of the structure is described according to the Flügge theory. Vibration measurement results are reported in section 5 and compared with the theoretical predictions. Finally, section 6 presents the result of a parametric study, showing the effects of the geometry of the discontinuity on vibration transmission.

2. THEORY OF CIRCULAR CYLINDRICAL SHELLS

In the problems addressed here, the shell is of uniform thickness and thin, in the sense that the thickness-to-radius ratio satisfies the inequality $h/R \leq 1/10$, with $h$ being the shell thickness and $R$ the radius of the shell. There is no unanimous agreement in the literature on the form of the equations of motion governing shell deformations even under this restriction [9]. The differences among the existing theories generally stem from slightly different simplifying assumptions made about the taking into account of small terms in the formulation.

Like most thin shell theories, Flügge and Donnell-Mushtari theories have Love’s first approximations as a common basis. Use of these approximations permits the deformations at any point of the shell to be expressed relatively simply as a function of the deformations of the middle surface, thereby reducing the original three-dimensional problem to a two-dimensional one. Love’s first approximations can be expressed in the following way:

(i) The thickness of the shell is small compared to a characteristic dimension
(ii) The displacements are small in comparison with the shell thickness
(iii) The transverse normal stress acting on planes parallel to the shell middle surface is negligible in comparison with the other normal stress components
(iv) Normals to middle surface are conserved as such during deformation and the shell thickness remains unchanged

The characteristic dimension referred to in postulate (i) is generally the radius of curvature of the shell, or alternatively the wavelength of vibrations. The assumption of small deformations made in postulate (ii) is necessary in order to ensure linearity of the resulting theory. Postulate (iii) is a consequence of postulate (i). Finally postulate (iv) is known as Kirchhoff’s hypothesis, according to which the shear and normal strains in the transverse direction are zero.

2.1 THE FLÜGGE SHELL THEORY

In this section, the shell equations as derived by Flügge [8] are presented. To that end, the equations of dynamic equilibrium of a shell element are first evolved and expressed in terms of stress resultants and inertia forces using Newton’s laws. Then the components of stress acting on the sections of the shell elements are related to the components of strains with the help of Hooke’s law and the strain-displacement relationships are written in accordance with Love’s first approximations. At last are the expressions of the stress
resultants as function of the displacement of the middle surface inferred and the equations of motion of the shell given.

2.1.1 Equations of dynamic equilibrium

The first step when deriving the equations of motion is to formulate the equations of equilibrium for an element of the shell. This shell element is delineated by planes perpendicular to the middle surface, passing through two pairs of adjacent co-ordinate lines as shown in Figure 1.

![Figure 1. Geometry of the shell, co-ordinate system and direction of displacements.](image)

The displacement components of the middle surface in the longitudinal ($u$), tangential ($v$) and radial ($w$) directions are defined positive according to Figure 1. The variable $s$ is a non-dimensional axial length defined as

$$ s = x/R. \quad (1) $$

The shell element is acted upon by forces and moments as shown by Figure 2. These forces and moments are resolved into the following components: two normal forces ($N_s$, $N_\varphi$), two shearing forces ($N_{s\varphi}$, $N_{\varphi\varphi}$), two transverse forces ($Q_s$, $Q_\varphi$), two bending moments ($M_s$, $M_\varphi$), two twisting moments ($M_{s\varphi}$, $M_{\varphi\varphi}$), and three surface forces ($q_s$, $q_\varphi$, $q_z$). The surface forces are referred to unit middle area while the others forces and moments are referred to a unit length of section. For a shell vibrating in vacuum with no load applied on its surfaces, the surface forces reduce to the forces of inertia and read

$$ q_s = -\rho \ddot{u}, \quad q_\varphi = -\rho \ddot{v}, \quad q_z = -\rho \ddot{w}, \quad (2) $$

where the second derivative with respect to time is denoted by

$$ \frac{\partial^2 \cdot }{\partial t^2} = (\cdot \cdot). $$

It is worth noting that the rotational inertia of the shell element is not considered. The equations of equilibrium are obtained by applying Newton’s laws to the shell element. Writing the equations of equilibrium for the forces yields


\[ N_s' + N_{\varphi}' + Rq_s = 0, \quad N_{\varphi}' + N_s^* + Q_\varphi + Rq_\varphi = 0, \quad Q_s' + Q_{\varphi}^* - N_s + Rq_z = 0, \] (3)

where primes and dots denote differentiation with respect to the non-dimensional axial length \( s \) and the angular variable \( \varphi \), respectively. Similarly, the equilibrium conditions of the moments with respect to the axis \( s, \varphi \) and \( z \) read

\[ M_s' + M_{\varphi}' - RO_s = 0, \quad M_{\varphi}' + M_s^* - RO_\varphi = 0, \quad R(N_{s\varphi} - N_{\varphi s}) - M_{\varphi s} = 0. \] (4)

Figure 2. Forces and moments acting on the shell element.

The forces and moments involved in the six equations of equilibrium are stress resultants obtained by integrating the components of stress shown in Figure 3 over the thickness of the shell. The stress resultants are given by

\[ N_s = \frac{h}{2} \int \sigma_s (1 + z/R) dz, \quad N_{\varphi} = \frac{h}{2} \int \sigma_{\varphi} dz, \]
\[ N_{s\varphi} = \frac{h}{2} \int \tau_{s\varphi} (1 + z/R) dz, \quad N_{\varphi s} = \frac{h}{2} \int \tau_{\varphi s} dz, \]
\[ M_s = \frac{h}{2} \int \sigma_s (1 + z/R) zdz, \quad M_{\varphi} = \frac{h}{2} \int \sigma_{\varphi} zdz, \]
\[ M_{s\varphi} = \frac{h}{2} \int \tau_{s\varphi} (1 + z/R) zdz, \quad M_{\varphi s} = \frac{h}{2} \int \tau_{\varphi s} zdz, \] (5)

where the term \((1 + z/R)\) in the integrands arises from the curvature of the shell as the width of a shell section \((ABCD)\) depends upon \( z \) as shown in Figure 3. It will be seen later that as a result of postulate (iv) and of Hooke’s law, transverse shear stresses are supposed to be zero, thereby implying that their resultants is zero. In fact Love’s approximation has to be interpreted in the following manner: transverse forces exist but the deformations they generate are neglected. Instead, the expressions of \( Q_s \) and \( Q_\varphi \) will be deduced from the conditions of equilibrium (4a) and (4b).
Figure 3. Notation and positive direction of the stresses acting on a shell element.

The theory of elasticity provides six relations between components of stress and components of strain, known as Hooke’s law. For a homogeneous and linearly elastic material they read:

\[
\begin{align*}
\varepsilon_s &= \frac{1}{E}(\sigma_s - \mu(\sigma_\varphi + \sigma_z)), \\
\varepsilon_\varphi &= \frac{1}{E}(\sigma_\varphi - \mu(\sigma_s + \sigma_z)), \\
\varepsilon_z &= \frac{1}{E}(\sigma_z - \mu(\sigma_s + \sigma_\varphi)), \\
\gamma_{s\varphi} &= \frac{2(1+\mu)}{E}\tau_{s\varphi}, \\
\gamma_{s\zeta} &= \frac{2(1+\mu)}{E}\tau_{s\zeta}, \\
\gamma_{\varphi\zeta} &= \frac{2(1+\mu)}{E}\tau_{\varphi\zeta},
\end{align*}
\]

(6)

where \(E\) is the Young’s modulus and \(\mu\) the Poisson’s ratio of the shell material, and where \(\varepsilon\) and \(\gamma\) denote normal and shear strains, respectively.

As mentioned previously, the fourth of Love’s approximations, i.e. the Kirchhoff hypothesis, introduces an inconsistency by assuming that the normal of the shell section remains normal after deformation. That would imply that the shear strains \(\gamma_{s\zeta}\) and \(\gamma_{s\varphi}\), and consequently the transverse shear stresses (see equations (6e) and (6f)) are zero which is incompatible with the existence of transverse forces. The second inconsistency originates from the assumption that the shell does not suffer extension in the vertical direction (postulate (iii)), thereby yielding \(\varepsilon_z = 0\). It ensues that \(\sigma_z = \mu(\sigma_s + \sigma_\varphi)\) which is in contradiction with the third of Love’s approximations (postulate (iii)). However, as pointed out by Flügge, “whatever happens in the \(z\) direction, strain or stress, is without significance” as long as the shell is thin. Thus, these contradictions do not affect the validity of the theory.

Adoption of Love’s first approximations consequently leads to a problem of plane stress distribution and the form of Hooke’s law given by equations (6) reduces to

\[
\begin{align*}
\varepsilon_s &= \frac{1}{E}(\sigma_s - \mu\sigma_\varphi), \\
\varepsilon_\varphi &= \frac{1}{E}(\sigma_\varphi - \mu\sigma_s), \\
\gamma_{s\varphi} &= \frac{2(1+\mu)}{E}\tau_{s\varphi}.
\end{align*}
\]

(7)

From these equations, the expressions for the stresses as function of the strains are readily obtained
\[ \sigma_z = \frac{E}{1-\mu^2}(\varepsilon_z + \mu\varepsilon_\varphi), \quad \sigma_\varphi = \frac{E}{1-\mu^2}(\varepsilon_\varphi + \mu\varepsilon_z), \quad \tau_{z\varphi} = \frac{E}{2(1+\mu)}\gamma_{z\varphi}. \] (8)

2.1.2 Kinematics of deformation

The next stage in the derivation of the equations governing the shell motions is to relate the components of strain to the displacements of the shell middle surface. This is achieved by studying the deformations of a shell element with the help of the simplifying assumptions of Love.

Let \( M \) be an arbitrary point of the shell and \( M_0 \) its image by normal projection on the shell middle surface. Further let \( U = [U \ V \ W]^T \) and \( u = [u \ v \ w]^T \) be the displacement vectors of \( M \) and \( M_0 \), respectively. From postulate (iv), it can be shown that the displacements of \( M \) and \( M_0 \) are related in the following way

\[ U = u - \frac{z}{R} w', \quad V = \frac{R+z}{R} v - \frac{z}{R} w^*, \quad W = w. \] (9)

In a similar way, the normal strains, \( \varepsilon_z \) and \( \varepsilon_\varphi \) and the shear strain \( \gamma_{z\varphi} \) at point \( M \) can be expressed as function of the deformations at point \( M_0 \) by

\[ \varepsilon_z = \varepsilon_z^0 + z\kappa_z, \quad \varepsilon_\varphi = \frac{1}{1+z/R}(\varepsilon_\varphi^0 + z\kappa_\varphi), \quad \gamma_{z\varphi} = \frac{1}{1+z/R}(\gamma_{z\varphi}^0 + z(1+\frac{z}{2R})\tau), \] (10)

where \( \varepsilon_z^0, \varepsilon_\varphi^0, \gamma_{z\varphi}^0 \) are the normal and shear strain in the middle surface and \( \kappa_z, \kappa_\varphi, \tau \) are the middle surface changes in curvature and twist. On the middle surface, the strain-displacement relations for a circular cylindrical shell are given by

\[ \varepsilon_z^0 = \frac{1}{R} u', \quad \varepsilon_\varphi^0 = \frac{1}{R} w^* + \frac{w}{R}, \quad \gamma_{z\varphi}^0 = \frac{1}{R} (u^* + v'), \]
\[ \kappa_z = \frac{1}{R^2} w^*, \quad \kappa_\varphi = \frac{1}{R^2} (v^* - w^*), \quad \tau = \frac{2}{R^2} (v' - w^*). \] (11)

2.1.3 Stress resultants and equations of motion

Using equations (8), (10), (11) and (5) and performing the integration yields the force and moment resultants expressed as a function of the displacements of the middle surface:

\[ N_z = \frac{B}{R}(u' + \mu v' + \mu w') - \frac{D}{R^3}w^*, \quad N_\varphi = \frac{B}{R}(v' + w + \mu u') + \frac{D}{R^3}(w + w^*), \]
\[ N_{z\varphi} = \frac{B}{R}(u' + v') + \frac{D}{R^3}(v' - w^*), \quad N_{\varphi x} = \frac{B}{R}(u' + v') + \frac{D}{R^3}(v' - w^*), \]
\[ M_z = -\frac{D}{R^2}(w^* + \mu v'^* - u' - \mu u'^*), \quad M_\varphi = -\frac{D}{R^2}(w + w^* + \mu v'^*), \]
\[ M_{z\varphi} = -\frac{D}{R^2}(1-\mu)(w^* - v'), \quad M_{\varphi x} = -\frac{D}{R^2}(1-\mu)(w^* + \frac{1}{2} u' - \frac{1}{2} v'). \] (12)
Finally, the equations of motion are evolved by eliminating the transverse shears $Q_r$ and $Q_v$ from equations (3b) and (3c) using equations (4a) and (4b). Then the expressions of forces and moments (12) are introduced in the three equations of equilibrium (3), yielding the following eighth order system of differential equations

$$\begin{align*}
\frac{u''}{2} + \frac{1-\mu}{2} u^{**} + \frac{1+\mu}{2} v'' + \mu w' + \beta\left(\frac{1-\mu}{2} u^{**} - w'' + \frac{1-\mu}{2} w^{**}\right) &= \frac{R^2}{c_p^2} \ddot{u}, \\
\frac{1+\mu}{2} u^{**} + \frac{1-\mu}{2} v'' + v^{**} + w^* + \beta\left(\frac{3(1-\mu)}{2} v'' - \frac{3-\mu}{2} w^{**}\right) &= \frac{R^2}{c_p^2} \ddot{v}, \\
\mu u' + v^* + w + \beta(4w + 2w^{**} + \frac{1-\mu}{2} u^{***} - u^{**} - \frac{3-\mu}{2} v^{**}) &= -\frac{R^2}{c_p^2} \ddot{w},
\end{align*}$$

where $\beta = h^2/12 R^2$ and is commonly termed thickness parameter and $V^2 = (\cdot')' + (\cdot')''$ is a Laplacian type operator. The factor $c_p$ is the phase velocity of longitudinal waves in a flat plate ($c_p = \sqrt{E/\rho (1-\mu^2)}$).

2.1.4 Boundary conditions for edge-loaded shells

When considering the boundary-value problem of a semi-infinite shell acted upon by a set of edge loads, five stress resultants, three forces ($N_r$, $N_{s\theta}$, $Q_r$) and two moments ($M_r$, $M_{s\theta}$), have to be considered at the edge. Specifying them arbitrarily will yield five boundary conditions which is obviously incompatible with a system of differential equations of the eighth order. However, the number of boundary conditions can be reduced to four by replacing the twisting moment distribution acting on the edge by a statically equivalent distribution of tangential and transverse forces. This force distribution is then combined with the tangential and transverse forces, $N_{s\theta}$ and $Q_r$, to form an equivalent shear force defined by

$$T_s = N_{s\theta} + \frac{M_{s\theta}}{R},$$

and an equivalent transverse force

$$S_s = Q_r + \frac{M_{s\theta}}{R}.$$  

Hence, the boundary conditions read with the Flügge theory

$$\begin{align*}
N_r &= \frac{B}{R} (u' + \mu v' + \mu w') - \frac{D}{R^3} w', \\
T_s &= \frac{B}{R} \frac{1-\mu}{2} (u' + v') + \frac{3(1-\mu)}{2} \frac{D}{R^3} (v' - w^{**}), \\
S_s &= -\frac{D}{R^3} (w'' + (2-\mu) w^{**} - u' + \frac{1-\mu}{2} u^{**} - \frac{3-\mu}{2} v^{**}), \\
M_s &= -\frac{D}{R^2} (w' + \mu w^{**} - u' - \mu v').
\end{align*}$$
2.2 THE DONNELL-MUSHTARI THEORY

The Flügge shell theory is considered as one of the most accurate thin shell theories and is therefore often used in engineering practice when computational results are aimed at. However, its complexity makes it unsuitable for analytical analysis of shell problems. Fortunately, as pointed out by Flügge himself, some terms due to bending in his theory can generally be disregarded without altering neither the order of the differential system nor the accuracy of the results.

The simplest form of eighth order thin shell theories is attributed to Donnell and to Mushtari who independently developed it in 1938 for static problems. The so-called Donnell-Mushtari equations can be inferred from the Flügge equations by making the following simplifications:

- the term containing the transverse forces \( Q_\phi \) in equation (3b) is neglected as in membrane theory. Thus, the following equation is used instead of equation (3b):
  \[
  N_{\varphi\varphi} + R q_\varphi = 0, \tag{17}
  \]

- the trapezoidal shape of the shell section at \( z = \text{const} \) is neglected, that is \( 1 + z/R \) are taken as 1 in the expressions of stress resultants (5) and of strains (10).

- in-plane displacements and their derivatives in the equations giving the middle surface changes in curvature and twist are neglected, which gives
  \[
  \kappa_\varphi = -\frac{1}{R^2} w^{**}, \quad \tau = \frac{2}{R^2} w^{**}, \tag{18}
  \]
  instead of equations (11e f).

It can be shown that these three simplifications amount to a systematic neglect of the factor \( z/R \) in comparison to 1 wherever the term \( (1 + z/R) \) occurs. With these simplifications, the equations of motion become

\[
\begin{align*}
  &u'^* + \frac{1 + \mu}{2} u''* + \frac{1 + \mu}{2} v''* + \mu v'^* = R^2 \frac{\varepsilon}{c_p^2}, \\
  &\frac{1 + \mu}{2} u''* + \frac{1 + \mu}{2} v''* + w'^* = R^2 \frac{v}{c_p^2}, \\
  &\mu u'^* + v'^* + w + \beta N^* w = -R^2 \frac{w}{c_p^2}, \tag{19}
\end{align*}
\]

and the boundary conditions,

\[
\begin{align*}
  &N_s = \frac{B}{R} (u' + \mu v' + \mu w'), \quad T_s = \frac{B}{R} \left( 1 - \frac{1 - \mu}{2} (u' + v') - (1 - \mu) \frac{D}{R^2} w'^* - \right), \\
  &S_s = \frac{D}{R^3} (w'' + (2 - \mu) w'^*'), \quad M_s = -\frac{D}{R^2} (w'' + \mu w'^*'). \tag{20}
\end{align*}
\]

The use of Donnell-Mushtari theory is justified for very thin shells by considering the fact that, for the shell to be considered thin, it is required that the wavelengths of the deformation in both the axial and circumferential directions are much greater than its
thickness, which is equivalent to assume that $\beta \left| (\ldots)'' \right| << 1$ and $\beta \left| (\ldots)' \right| << 1$. When applied to the Flügge equations these inequalities lead to the Donnell-Mushtari equations. Inspecting equation (19), it appears that bending stresses are represented in the Donnell-Mushtari shell equations by the term $\beta \nabla^4 w$. Disregarding this term yields the well-known membrane shell theory developed by Love [11]. This theory, also known as the extensional approximation, is very appealing by its simplicity and can even be used to reveal the essential feature of shell behaviour [12]. The term $\beta \nabla^4 w$ arises from the derivatives of transverse shear forces in equation (3c) and cannot be neglected in many cases. Indeed, though the shear forces are relatively small as it has been mentioned previously, their distributions can exhibit very big gradients, as for instance close to discontinuities (external force, edge, etc.), and $\beta \nabla^4 w$ may become predominant.

The Donnell-Mushtari shell equations can be simplified further for vibration problems involving predominantly transverse motions. When this is the case, the in-plane inertia forces may be disregarded, i.e. the right-hand terms are set to zero in equations (19a) and (19b). Introducing then a stress function $\psi$ defined by

$$
N_x = \Psi'' / R^2, \quad N_\phi = \Psi'/R^2, \quad N_{x\phi} = -\Psi''/R^2,
$$

(21)

it can be shown that the shell motions are described by [13]

$$
\beta \nabla^4 w + \frac{1 - \mu^2}{EhR} \Psi'' = -R^2 \ddot{w} / c_p^2 \quad \text{and} \quad w'' = \frac{1}{EhR} \nabla^4 \Psi = 0.
$$

(22)

Uncoupling of these equations is readily achieved by defining a function $\Phi$ such that

$$
w = \nabla^4 \Phi \quad \text{and} \quad \Psi = EhR \Phi' ,
$$

(23)

which leads to the differential equation

$$
\beta \nabla^4 \Phi + (1 - \mu^2) \Phi''' = -R^2 \nabla^4 \ddot{\Phi} / c_p^2 .
$$

(24)

The term $\Phi'''$ in the equation above arises from the normal force $N_\phi$, in the equation of equilibrium, owing to the shell curvature. The shell displacements are deduced from $\Phi$ using the relationships

$$
u = -\Phi''' - (2 + \mu) \Phi'' , \quad w = \nabla^4 \Phi .
$$

(25)

3. FREE VIBRATIONS IN CYLINDRICAL SHELLS

Before addressing the forced response of cylindrical shells to mechanical loads, it is illuminating to consider the free wave propagation in a semi-infinite shell. By virtue of the cylindrical symmetry, the displacements are conveniently expanded in terms of circumferential modes and read
\[ u = \sum_{n=0}^{\infty} \sum_{p=1}^{4} A_{np}^{p} e^{i\alpha + \kappa_{np}s} \cos(n\varphi), \quad v = \sum_{n=0}^{\infty} \sum_{p=1}^{4} B_{np}^{p} e^{i\alpha + \kappa_{np}s} \sin(n\varphi), \]

\[ w = \sum_{n=0}^{\infty} \sum_{p=1}^{4} C_{np}^{p} e^{i\alpha + \kappa_{np}s} \cos(n\varphi). \]  

(26)

The integer \( n \) is referred to as the circumferential modal order and represents one half of the number of nodes around the circumference (Figure 4). For a given mode \( n \) the displacement field in the half-space \( s > 0 \) is the sum of four waves, each of them being described by a generally complex propagation constant \( \kappa_{np} \) and complex amplitudes \( A_{np}^{p}, B_{np}^{p}, C_{np}^{p} \) in the axial, tangential and radial directions, respectively.

\[ n = 0 \quad n = 1 \quad n = 2 \quad n = 3 \]

Figure 4. Modal shape of the circumferential modes \( n = 0, 1, 2 \) and 3.

Substitution of the solutions (26) in the differential system (13) yields for each circumferential modal order three linear equations in the amplitudes \( A_{np}^{p}, B_{np}^{p}, C_{np}^{p} \). Using the Flügge theory, the equations read [9]

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
A_{np}^{p} \\
B_{np}^{p} \\
C_{np}^{p}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]  

(27)

\[ L_{11} = \kappa_{np}^2 \left( -\frac{1-\mu}{2} n^2 (1+\beta) + \Omega^2 \right), \quad L_{12} = \frac{1+\mu}{2} n\kappa_{np}, \quad L_{13} = \kappa_{np} \left( \mu - \beta (\kappa_{np}^2 + \frac{1-\mu}{2} n^2) \right), \]

\[ L_{21} = -\frac{1+\mu}{2} n\kappa_{np}, \quad L_{22} = \frac{1-\mu}{2} \kappa_{np}^2 (1+3\beta) - n^2 + \Omega^2, \quad L_{23} = -n(1-\beta^2 - \frac{3-\mu}{2} \kappa_{np}^2), \]

\[ L_{31} = \kappa_{np} \left( \mu - \beta \left( \frac{1-\mu}{2} n^2 + \kappa_{np}^2 \right) \right), \quad L_{32} = n(1-\beta^2 - \frac{3-\mu}{2} \kappa_{np}^2), \]

\[ L_{33} = 1 - \Omega^2 + \beta \left( \kappa_{np}^2 - n^2 \right)^2 + 1 - 2n^2. \]

where \( \Omega \) is the non-dimensional frequency given by \( \Omega = \omega R / c_p \).

Non-trivial solutions for \( (A_{np}^{p}, B_{np}^{p}, C_{np}^{p}) \) are obtained only if the determinant of the coefficient matrix \([L]\) is zero. Hence, for a given \( \Omega \), the propagation constants \( \kappa_{np} \) must satisfy the so-called characteristic equation

\[ G(\kappa_{np}) = \kappa_{np}^8 + a_6 \kappa_{np}^6 + a_4 \kappa_{np}^4 + a_2 \kappa_{np}^2 + a_0 = 0. \]  

(28)
The coefficients $a_i$ associated with the Flügge theory are given in Appendix A.

3.1 NATURE OF THE FREE WAVES

Since the dispersion relation is an even function of $\kappa_{np}$, every root has an image through the point $|\kappa| = 0$. Owing to the adopted time dependence, the four roots located in the half plane $\pi/2 < \arg(\kappa_{np}) \leq 3\pi/2$ are associated with waves propagating in the half space $x \geq 0$ while their images correspond to waves travelling in the opposite direction. Regarding the nature of the roots, Figure 5 shows that, for given values of $\mu$ and $\beta$, the $\Omega$-$\nu$ plane can be divided into eight domains within which the roots have the form:

- **Domain I:** $\pm(a \pm ib), \pm(c \pm id)$
- **Domain II:** $\pm a, \pm b, \pm(c \pm id)$
- **Domain III:** $\pm a, \pm ib, \pm(c \pm id)$
- **Domain IV:** $\pm a, \pm ib, \pm(c \pm id)$
- **Domain V:** $\pm a, \pm b, \pm c, \pm d$
- **Domain VI:** $\pm a, \pm ib, \pm c, \pm d$
- **Domain VII:** $\pm a, \pm ib, \pm ic, \pm td$
- **Domain VIII:** $\pm a, \pm ib, \pm ic, \pm id$

where $a$, $b$, $c$, and $d$ are real and positive. Pure imaginary propagation constants correspond to propagating waves transporting energy while real propagation constants are associated with non-propagating waves decaying exponentially. Complex constants appearing in conjugate pair describe a standing decaying field.

Further information about the nature of a given wave, that is if the associated displacements are primarily axial, circumferential or radial, is obtained by substituting its propagation constant into the two first of equation (27) to assess the respective wave amplitude ratios $\tau_{np}^a$ and $\tau_{np}^b$ given by

$$
\tau_{np}^a = \frac{A_p^n}{C_p^n} = \frac{L_{12}L_{23} - L_{13}L_{22}}{L_{11}L_{22} - L_{12}L_{21}} \quad \text{and} \quad \tau_{np}^b = \frac{B_p^n}{C_p^n} = \frac{L_{21}L_{13} - L_{11}L_{23}}{L_{11}L_{22} - L_{12}L_{21}}.
$$

(29)

For a pre-assigned value of $n$ ($n \geq 2$), the vibration field is at low frequencies totally evanescent (Domain I). Then, as the non-dimensional frequency increases, three forms of waves may propagate along the shell. The frequency at which a wave becomes propagating is known as the cut-on frequency and is characterised by a zero propagation constant, or in other words an infinite phase velocity. The cut-on frequencies characterizing the emergence of propagating waves are depicted in Figure 5 by the thick solid line, the dashed line and the dotted line. The thin solid lines denote a change from a pair of complex conjugate constants to two purely real ones.

<table>
<thead>
<tr>
<th>Physical properties of the shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $\rho$ (kg/m$^3$)</td>
</tr>
<tr>
<td>Young's modulus, $E$ (Pa)</td>
</tr>
<tr>
<td>Poisson's ratio, $\mu$</td>
</tr>
<tr>
<td>Radius, $R$ (m)</td>
</tr>
<tr>
<td>Thickness-to-radius ratio, $h/R$</td>
</tr>
</tbody>
</table>

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3.2 CHARACTERISTICS OF THE PROPAGATING WAVES

3.2.1 Phase velocity and cut-on frequency

Concerning the propagating waves, Figure 6 shows their axial phase velocities $c_x = \alpha R / \text{Im}(\kappa)$ normalised by the quasi-longitudinal wave velocity of a plate ($c_p$). The characteristics of these waves in terms of velocities, shapes and natures, have been discussed by Smith [10] and Fuller [1]. Waves having their cut-on frequencies given by the solid line in Figure 5, which is given by

$$\Omega^2 = \beta \frac{n^2(n^2 - 1)^2}{n^2 + 1}, \tag{30}$$

are referred to as flexural waves since they are characterised by a displacement field dominated by radial motion for circumferential modes $n \geq 2$. Their axial phase velocities tend towards the phase velocity of a bending wave in a flat plate when $\Omega > 1$. It can be noted that when the Donnell-Mushtari theory is used, a finite cut-on frequency for the beam mode ($n = 1$) is predicted, thereby resulting in big errors below this frequency.

A wave of another nature becomes propagating as the non-dimensional frequency is increased. Its cut-on frequency, shown by the dashed line in Figure 5, is given by

$$\Omega^2 = n^2 \frac{1 - \mu}{2}. \tag{31}$$

As shown in Figure 6, its axial phase velocity approaches the shear wave velocity. Finally, as the non-dimensional frequency passes the dotted line in Figure 5, still another type of propagating wave emerges at
\[ \Omega^2 = 1 + n^2. \]  

(32)

Its phase velocity tends rapidly towards the phase velocity of quasi-longitudinal wave in a flat plate and this wave is essentially extensional in nature.

![Dispersion diagram for free cylindrical shell-waves calculated with the Donnell-Mushtari theory.](image)

Figure 6. Dispersion diagram for free cylindrical shell-waves calculated with the Donnell-Mushtari theory.

### 3.2.2 Wavenumber

Figure 7 shows the wavenumbers associated with the modes \( n = 0, 1 \ldots 6 \). Of particular interest is the envelop of these wavenumbers since it corresponds to the axial wavenumber of the highest \( n \) mode which can propagate along the shell at a given frequency and which is expected to govern the input power. The form of the envelop can be approximated by

\[
kR = \frac{2^{1/4} \sqrt{\Omega}}{(1 - \mu^2)^{1/4}} \text{ for } \Omega \leq 2.66\sqrt{\beta}, \quad kR = \frac{7.85\Omega}{\beta^{1/4} (9 + \Omega)} \text{ for } 2.66\sqrt{\beta} < \Omega < 2, \quad kR = \frac{\sqrt{\Omega}}{\beta^{1/4} (1 - \mu^2)^{1/4}} \text{ for } \Omega \geq 2.
\]  

(33)

Equations (33a) and (33c) express nothing but the wavenumbers for a beam and for a plate expressed in the shell parameters \( \Omega \) and \( \beta \), respectively.

### 3.2.3 Strain energy

Finally, it is interesting to consider the apportionment of energy due to stretching and bending actions associated with those wave types which can propagate freely along the shell.
Figure 7. Wavenumbers times radius versus non-dimensional frequency for the shell described in Table 1. (---) plate, (—) modes \( n = 0 \ldots 6 \), (— —) envelop of the wavenumbers, (+) equation (33b).

From Love's first approximations, the strain energy contained in a shell element is given by [9]

\[
U_s = \frac{R^2}{2} \int \frac{h}{2\varphi - h/2} \left( \sigma_x e_x^* + \sigma_y e_y^* + \tau_{xy} \gamma_{xy}^* \right) (1 + \frac{z^*}{R}) d\varphi ds,
\]

(34)

where \( z^* \) indicates the complex conjugate of \( z \). Using equations (8) and (10) and performing the integration over the shell thickness, the strain energy reads for the Flügge theory

\[
U_s = \frac{EhR^2}{2(1-\mu^2)} \int \left[ (I_x + \frac{h^2}{12} I_b) d\varphi ds, \right.
\]

(35)

where

\[
I_x = e_x^0 (e_x^*)^* + e_y^0 (e_y^*)^* + \mu(e_x^0 (e_y^*)^* + e_y^0 (e_x^*)^*) + \frac{1-\mu}{2} \gamma_{xy}^0 (\gamma_{xy}^0)^*,
\]

(36)

\[
I_b = \kappa_x \kappa_x^* + \kappa_y \kappa_y^* + \mu(\kappa_x \kappa_y^* + \kappa_y \kappa_x^*) + \frac{1}{R^2} (\epsilon_x^0 (\epsilon_y^0)^* + \epsilon_y^0 (\epsilon_x^0)^*) + \frac{1}{R} (\epsilon_x^0 \kappa_x^* + \epsilon_y^0 \kappa_y^*) + \frac{1}{R} (\kappa_x^0 \epsilon_x^0 + \kappa_y^0 \epsilon_y^0) - (\epsilon_x^0)^* \kappa_y^* - (\epsilon_y^0)^* \kappa_x^*),
\]

(37)

\( I_x \) and \( I_b \) are the components of the strain energy due to stretching and bending stresses, respectively. Equations (36) and (37) have been obtained by expanding the quantity \( \ln(1+h/2R) \) in Taylor series up to the third order in accordance with the original derivation by Flügge. Figure 8 shows a contour plot of ratio \( I_b/I_x \) associated with flexural waves in the \( \Omega-n \) plane calculated for the shell described in Table 1. It is seen that stretching energy
dominates for non-dimensional frequencies higher than the cut-on frequency but below the ring frequency for low circumferential modes. The shell behaves consequently as a membrane [12] for this mode and the vibration field is then characterised by a strong coupling between in-plane and out-of-plane motions due to the shell curvature. For higher modes, the wave, when propagating, carries mainly bending energy and the shell can be accurately modelled as a plate strip [14]. Furthermore, for frequencies higher than the ring frequency, the influence of the shell curvature decreases rapidly whatever the circumferential modal order and the shell response can be studied using the in-plane and out-of-plane governing equations for a plate strip of similar thickness [15]. Similar plots for the extensional and torsional type waves show that the strain energy associated to these waves is due to stretching.

![Graph](image)

**Figure 8.** Ratio of bending energy to stretching energy associated with the flexural waves in the $\Omega-n$ plane. The dashed line indicates the cut-on frequency.

4. VIBRATORY POWER TRANSMISSION TO A STIFFENED CYLINDRICAL SHELL

Consider the stiffened, thin-walled circular semi-infinite shell sketched in Figure 9. It is acted upon by a set of harmonic loads applied at $x = x_0$ at a constant circular frequency $\omega$. It is assumed that the loads can be expressed as Fourier series expansions in the form

$$N_e = (N_0^\varepsilon + \sum_{n=1}^{\infty} N_n^\varepsilon \cos(n\varphi) + \sum_{n=1}^{\infty} N_n^\varepsilon \sin(n\varphi)) e^{i\omega t},$$

$$T_e = (\sum_{n=1}^{\infty} T_n^\varepsilon \cos(n\varphi) - \sum_{n=1}^{\infty} T_n^\varepsilon \sin(n\varphi)) e^{i\omega t},$$

$$S_e = (S_0^\varepsilon + \sum_{n=1}^{\infty} S_n^\varepsilon \cos(n\varphi) + \sum_{n=1}^{\infty} S_n^\varepsilon \sin(n\varphi)) e^{i\omega t},$$

$$M_e = (M_0^\varepsilon + \sum_{n=1}^{\infty} M_n^\varepsilon \cos(n\varphi) + \sum_{n=1}^{\infty} M_n^\varepsilon \sin(n\varphi)) e^{i\omega t}.$$
The positive convention adopted for the components of the excitation vector
\( \mathbf{F} = [N^e T^e S^e M^e ]^T \) is shown in Figure 9. It is worth noting that the Fourier expansion of the equivalent tangential shear force has been supposed to have no axisymmetric component \((n = 0)\) due to the inability of such a distribution to induce radial motions.

![Figure 9. The stiffened, semi-infinite cylindrical shell.](image)

### 4.1 GENERAL FORMULATION

As the shell and the stiffener possess the same axial symmetry, the components of the velocity vector \( \mathbf{v} = [\dot{u} \quad \dot{\varphi} \quad \dot{\omega} \quad \dot{\theta}]^T \) produced by the loads take the form

\[
\begin{align*}
\dot{u}(x, \varphi) &= \dot{u}_0 + \sum_{n=1}^{\infty} \dot{u}_n \cos(n \varphi) + \sum_{n=1}^{\infty} \dot{u}_n \sin(n \varphi), \\
\dot{\varphi}(x, \varphi) &= \dot{\varphi}_0 + \sum_{n=1}^{\infty} \dot{\varphi}_n \cos(n \varphi) - \sum_{n=1}^{\infty} \dot{\varphi}_n \sin(n \varphi), \\
\dot{\omega}(x, \varphi) &= \dot{\omega}_0 + \sum_{n=1}^{\infty} \dot{\omega}_n \cos(n \varphi) + \sum_{n=1}^{\infty} \dot{\omega}_n \sin(n \varphi), \\
\dot{\theta}(x, \varphi) &= \dot{\theta}_0 + \sum_{n=1}^{\infty} \dot{\theta}_n \cos(n \varphi) + \sum_{n=1}^{\infty} \dot{\theta}_n \sin(n \varphi),
\end{align*}
\]

in which time dependence has been omitted for brevity. The dot denotes time derivation and \( \theta \) represents the rotation of the normal to the middle surface about the \( \varphi \)-axis, \( \dot{\theta} = \partial \omega / \partial x \). The set of functions \((\cos(n \varphi), \sin(n \varphi))\) with \( n = 0, 1, 2 \ldots \) being orthogonal,
the velocities \( \dot{u}_n, \dot{v}_n \) and \( \dot{w}_n \) and the angular velocity \( \dot{\theta}_n \) taken at \( x = x_0 \) can be expressed as a linear combination of the Fourier components \( N_n^e, T_n^e, S_n^e \) and \( M_n^e \),

\[
V_n \bigg|_{x=x_0} = Y^n F_n,
\]

(40)

with \( V_n = [\dot{u}_n, \dot{v}_n, \dot{w}_n, \dot{\theta}_n]^T \) and \( F_n = [N_n^e, T_n^e, S_n^e, M_n^e]^T \).

The coefficients of \( Y_n \), that is the complex ratios of velocity to force taken at the same point, are known as direct mobilities. Diagonal and off-diagonal terms are referred to as the input mobilities and cross-mobilities, respectively, the latter entering in the calculation of the input power only in case of joint excitation. As the tangential excitation is assumed to have no axially symmetric component in equations (38), no circumferential velocity can be induced and \( Y_n \) is reduced to \( 3 \times 3 \) matrix for the axisymmetric mode \((n = 0)\).

By a similar reasoning a mobility matrix \( \bar{Y}_n \) can also be defined for the Fourier components \( \dot{u}_n, \dot{v}_n, \dot{w}_n \) and \( \dot{\theta}_n \) and the loads \( N_n^e, T_n^e, S_n^e \) and \( M_n^e \). Due to the forms adopted for the loads and the displacements in equations (38) and (39), it can be shown that

\[
\bar{Y}_n = Y_n.
\]

(41)

For simple harmonic time dependence, the time-averaged power transmission to the structure by the excitation is given by

\[
P_{\text{input}} = \frac{1}{2} \int_0^{2\pi} \text{Re} \{ (F^T)^* V_n \} R d\varphi,
\]

(42)

where \( R \) is the mean radius of the shell on which is applied the load and \((F^T)^*\) indicates the complex conjugate of \( F^T \). Using equations (38), (39), (40) and the orthogonal properties of the functions \((\cos(n \varphi), \sin(n \varphi))\), the input power can be expressed as function of the mobility matrices of the circumferential harmonics \( n \),

\[
P_{\text{input}} = \frac{\pi R}{2} \text{Re} \left\{ \sum_{n=0}^{\infty} \varepsilon_n \left( (F_n^T)^* Y_n F_n + (F_n^T)^* Y_n F_n \right) \right\},
\]

(43)

with

\[
\varepsilon_n = \begin{cases} 2 & \text{when } n = 0 \\ 1 & \text{when } n > 0 \end{cases}
\]

If the structure is assumed undamped, it can be shown that only the circumferential modes \( n \) that are propagating contribute to the input power. This facilitates substantially the calculation procedure as the summation in equation (43) has to be carried out over these propagating modes only and not over an infinite number of terms. For point source excitation applied at \( x = x_0, \varphi = 0 \), the Fourier components of the load \( X \) are given by
\[
\begin{cases}
  x_n = \frac{X}{\pi R \varepsilon_n} & \text{for } X = N_n, S_n \text{ or } M_n \text{ and } \quad x_n = 0 & \text{for } X = T_n.
\end{cases}
\] (44)

4.2 DETERMINATION OF THE MOBILITY MATRIX \( Y_n \)

The structure is modelled as an assembly of three thin-walled cylindrical shells which are referred to as shell \( a \), shell \( b \) and shell \( c \), respectively, as shown in Figure 9. The shells modelling the stiffener, i.e., shells \( a \) and \( b \), are supposed to have the same mean radius and thickness. A possible eccentricity between them and shell \( c \) is taken into account and is denoted \( \varepsilon \). However, this parameter should have almost no influence since all the shells are assumed thin. The shells are supposed to vibrate in their \( n \)th circumferential mode subsequently to load of type \( (N^e_n \cos(n\varphi), T^e_n \sin(n\varphi), S^e_n \cos(n\varphi), M^e_n \cos(n\varphi)) \) applied at the interface between the shells \( a \) and \( b \). The forced vibrations in the shells can be described by

\[
\begin{align*}
  u_n^d &= \sum_{p=1}^{m^d} \tau_{np}^{nd} \tilde{C}_p^{nd} \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \cos(n\varphi), \\
  v_n^d &= \sum_{p=1}^{m^d} \tau_{np}^{nd} \tilde{C}_p^{nd} \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \sin(n\varphi), \\
  w_n^d &= \sum_{p=1}^{m^d} \tilde{C}_p^{nd} \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \cos(n\varphi),
\end{align*}
\] (45)

where the superscript \( d \) is either \( a, b \) or \( c \) depending on whether reference is made to the shells sections defined by \( 0 \leq x < x_0, x_0 \leq x < L \) or \( x \geq L \), respectively. \( \tau_{np}^{nd} \) and \( \tau_{np}^{nd} \) are the wave amplitude ratios defined by equations (29). The parameter \( m^d \) indicates the number of waves describing the vibration field. To each wave corresponds a propagation constant which is a solution of the dispersion relation \( G(\kappa_{np}^{nd}) = 0 \) given by equation (28). The shell \( c \) being semi-infinite, only the four waves associated with propagation in the half space \( x \geq 0 \) have to be considered, i.e. \( m^c = 4 \). They correspond to waves either propagating or exponentially decaying toward positive \( x \) and their propagation constants are the roots of the dispersion relation satisfying the condition \( \pi / 2 < \arg(\kappa_{np}^{nd}) \leq 3\pi / 2 \). As shells \( a \) and \( b \) are of finite length, wave propagation in both directions must be considered and thus \( m^a = m^b = 8 \).

The internal stress resultants produced by the shell deformations are taken in the form

\[
\begin{align*}
  N_n^d &= \sum_{p=1}^{m^d} N_{np}^d \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \cos(n\varphi), \\
  T_n^d &= \sum_{p=1}^{m^d} T_{np}^d \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \sin(n\varphi), \\
  S_n^d &= \sum_{p=1}^{m^d} S_{np}^d \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \cos(n\varphi), \\
  M_n^d &= \sum_{p=1}^{m^d} M_{np}^d \exp \left( \frac{\kappa_{np}^{nd} x}{R_d} \right) \cos(n\varphi).
\end{align*}
\] (46)

From equation (16), it is straightforward to evolve, for a given wave \( p \), relations between the amplitudes of the internal forces and moment and the amplitude of the radial displacement. They can be expressed as
\[ N_{np}^d = \frac{B_d}{R_d} (\kappa_{np}^d \tau_{np}^{nd} + \mu_d \tau_{np}^{nd} + \mu_d - \beta_d (\kappa_{np}^d)^2) C_p^{nd}, \]
\[ T_{np}^d = \frac{1}{2} \frac{D_d}{R_d} (\kappa_{np}^d \tau_{np}^{nd} - n \tau_{up}^{nd} + 3 \beta_d \kappa_{np}^d (n + \tau_{up}^{nd}) C_p^{nd}, \]
\[ S_{np}^d = \frac{D_d}{R_d} (-\kappa_{np}^d)^2 + (2 - \mu_d) n^2 \kappa_{np}^d + \kappa_{np}^d (\kappa_{np}^d)^2 + \frac{1}{2} \frac{D_d}{R_d} (\kappa_{np}^d)^2 + \frac{3 - \mu_d}{2} n \kappa_{np}^d \tau_{np}^{nd} C_p^{nd}, \]
\[ M_{np}^d = \frac{D_d}{R_d} (-\kappa_{np}^d)^2 + \mu_d n^2 + \kappa_{np}^d \tau_{np}^{nd} + \mu_d n \tau_{np}^{nd} C_p^{nd}, \] (47)

where \( B_d = E h_d / (1 - \mu_d^2) \), \( D_d = E h_d^3 / 12 (1 - \mu_d^2) \) and \( \beta_d = h_d^2 / 12 R_d^2 \).

The displacement equations (44) involve 20 wave amplitudes \( C_p^{nd} \). These are the unknowns of the problem and they are determined from the following boundary conditions:

(i) four equations arising from the free edge condition, i.e. the vanishing of internal forces and moments at \( x = 0 \),
(ii) four equations from the continuity of the displacements \( u, v, w \) and \( \theta \) at \( x = x_0 \),
(iii) four equations from the discontinuity in the stress resultants of amplitude \( N_n^{e}, T_n^{e}, S_n^{e} \) and \( M_n^{e} \) at \( x = x_0 \),
(iv) eight equations from the continuity of internal stress resultants and displacements at \( x = L \).

The boundary conditions can be written in matrix form as
\[ A_n C_n = K_n, \] (48)

where
\[ C_n = \begin{bmatrix} C_1^{na} & \cdots & C_8^{na} \\ C_1^{nb} & \cdots & C_8^{nb} \\ C_1^{nc} & \cdots & C_4^{nc} \end{bmatrix}^T, \]
\[ K_n = \begin{bmatrix} 0 & \cdots & 0 & N_n^{e} & T_n^{e} & S_n^{e} & M_n^{e} & 0 & \cdots & 0 \end{bmatrix}^T. \]

As regards the condition of displacement continuity between shells \( b \) and \( c \), the influence of the eccentricity defined as \( e = R_b - R_c \) has to be taken into account. Hence, for the \( n^{th} \) mode, the four equations to be satisfied at \( M (x = L, r = R_c) \) read
\[ u_n^b + \frac{e \partial w_n^b}{R_b} = u_n^{e}, \quad v_n^b (1 + \frac{e}{R_b}) + n \frac{e \partial w_n^b}{R_b} = v_n^{e}, \quad w_n^b = w_n^{e}, \quad \frac{\partial w_n^b}{\partial x} = \frac{\partial w_n^{e}}{\partial x}, \] (49)

where \( u, v, w \) are the displacements of the shell middle surfaces in the axial, circumferential and radial direction.

This forms a system of twenty independent equations. Solving for \( C_p^{nd} \), in other words computing \( A_n^{-1} K_n \), yields the displacements at the excitation point as a function of the load amplitudes. The structural mobilities of the stiffened shell with respect to a given load \( X_n^{e} \),
$X_n$ being either $N_n$, $T_n$, $S_n$ or $M_n$, are determined by assuming $X_n = 1$ and the other load components equal to zero when evaluating the amplitudes $C_{np}$.

For the axisymmetric mode ($n = 0$), the radial-axial vibrations are described by three waves. Indeed the fourth wave associated with the propagation constant $\kappa_{op} = -i\Omega / \sqrt{(1-\mu)(1+3\beta)}$ yields purely torsional motion and should therefore be removed from equation (40), i.e. $m^a = m^b = 6$ and $m^c = 3$. This is consistent with the fact that no tangential load has been considered, thereby yielding a 16 x 16 matrix $A_n$ for this mode.

4.3 THE CASE OF THE FINITE STRUCTURE

As it will be described in section 5, measurements have been carried out to establish the validity of the Flügge theory for the present application. Obviously, the model of the semi-infinite structure does not lend itself to such validation since it implies a realisation of an anechoic termination of a finite structure. Thus, it was easier to make the measurements on a finite structure and to adapt the model in accordance. The structure chosen for the measurements was built up from two cylindrical shells of different thickness coupled at one edge and free at the other. This finite character is simply accounted for in the theoretical model presented previously by increasing the number of existing waves in shell $c$ from 4 to 8. Accordingly, four boundary conditions corresponding to the free edge condition (that is the vanishing of forces and moments) are introduced in the system described by equation (48). The response of the structure at any points is then obtained by solving this system for the wave amplitudes $C_n$.

5. EXPERIMENTAL VERIFICATION OF THE VIBRATION TRANSMISSION MODEL

5.1 EXPERIMENTAL SET-UP

In order to verify the theoretical predictions, vibration measurements were conducted on a built-up thin cylindrical shell structure. This structure was composed of two PVC shells having the same outer diameter but different thickness and joined at their edges as sketched in Figure 10 below. The joint between the shells was achieved by machining the thicker one over a length of 5mm in order to fit the second shell as shown in Figure 10(b). A cohesive joint was then obtained by solvent-welding the surfaces in contact using ethylacetate as a solvent.

The geometric properties of the two shells are reported in Table 2; the short one is referred to as the stiffener and the long one as the shell from now on.

<table>
<thead>
<tr>
<th>TABLE 2</th>
<th>Geometric properties</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Shell</td>
</tr>
<tr>
<td>Radius, $R$ (mm)</td>
<td>98.8</td>
</tr>
<tr>
<td>Thickness, $h$ (mm)</td>
<td>2.5</td>
</tr>
<tr>
<td>Length (mm)</td>
<td>4220</td>
</tr>
</tbody>
</table>
Figure 10. (a) Sketch of the structure; (b) detail of the joint between shell and stiffener.

Figure 11. The measurement set-up.

The structure was suspended by elastic strings as shown by Figure 10(a) at two locations (sections marked by points E and F in Table 3), thereby ensuring free-free end conditions. It was excited by a shaker, the driving force of which was monitored by a force transducer.
(mass 3g) mounted on the structure at the excitation point. Two positions of the excitation have been used. The first one with the shaker attached at point B (see Figures 10 and 11), thereby yielding a radial force excitation. The second configuration with the shaker mounted at point A (see Figure 10), positioned to excite the structure axially. A swept sine excitation was used over the frequency range [0-5000 Hz]. The response of the structure was measured at different points (A, B, C or D, see Figure 10) by a lightweight accelerometer (0.65g) with a transverse sensitivity of 1.4% at 30 Hz. The positions of these points in the cylindrical co-ordinate system \((x, \varphi)\) are reported in Table 3. The acceleration and force signals were conditioned in charge amplifiers and then processed by a Tektronic 2630 FFT analyser to determine the corresponding transfer functions. Table 4 lists the instruments used for the measurements.

**TABLE 3**

<table>
<thead>
<tr>
<th>Points</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, \varphi)) in [m, rad]</td>
<td>0, 0</td>
<td>0.025, 0</td>
<td>1.05, 0</td>
<td>2.67, 0</td>
<td>0.87, 0</td>
<td>3.6, 0</td>
</tr>
</tbody>
</table>

**TABLE 4**

<table>
<thead>
<tr>
<th>Measurement instruments</th>
<th>B&amp;K 4809</th>
<th>NAD electronics 912</th>
<th>Aquila PC486</th>
<th>Tektronic 2630</th>
<th>B&amp;K type 4374</th>
<th>B&amp;K 2635</th>
<th>B&amp;K type 8200</th>
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</thead>
<tbody>
<tr>
<td>Electro-dynamic vibration generator</td>
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<td>Power amplifier</td>
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<tr>
<td>Computer</td>
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<tr>
<td>FFT analyser</td>
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<tr>
<td>Accelerometer</td>
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<tr>
<td>Charge amplifier</td>
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<tr>
<td>Force transducer</td>
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</tbody>
</table>

5.2 MECHANICAL PROPERTIES OF THE PLASTIC MATERIALS

To enable comparison between measurement results and theoretical predictions, both Young’s moduli and loss factors of the PVC materials composing the structure had to be determined. This was achieved by conducting impact test measurements on rings made of in the same PVC materials as the shells. The rings were elastically suspended and impacted radially by a hammer (PCB piezotronics 086C03). By monitoring the impulse force exerted on the rings and the accelerations at the excitation point, the accelerances of the two rings were determined, that is the ratio of the acceleration to force taken at the same point.

The response of the rings to such excitation are governed by in-plane flexure modes as shown by Figures 12 and 13. The resonance frequencies \(f_R\) of these modes are given by the following formula [16]

\[
\frac{f_R^2}{\beta^2} = \frac{E}{\rho} \frac{n^2 (n^2 - 1)^2}{(2\pi R)^2 (1 + n^2)},
\]

where \(n\) is the mode number. Knowing all the parameters but the Young’s modulus, an estimate of the latter is easily obtained from the measured resonance frequencies.
The measurement results show that the Young's moduli were slightly frequency dependent at room temperature. However, the dependence was judged sufficiently weak (an increase by 3% and 7%, respectively, for the two rings in the frequency range [0-2000 Hz]), so that constant values were adopted for the calculations. Figures 12 and 13 show calculated results for the accelerances of the two rings obtained using the geometric and material properties given in Tables 2 and 5 (except for the length).

Furthermore, the measured accelerance curves allowed the loss factors of the two materials to be estimated. These loss factors are related to the half-value bandwidth of the resonance curve by the relationship [14]

\[ \eta = (f_2 - f_1)/f_R, \]  

(51)

where \((f_2 - f_1)\) is the half-value bandwidth. Figure 14 shows the measured loss factors for the two materials versus frequency. As again the frequency dependence is weak, it has been
disregarded and constant values adopted for the calculations. The properties of the two materials determined from the measurement results are listed in Table 5. It is seen that the Poisson’s ratio has been assumed to be 0.4, which is a typical value for PVC [17].

**TABLE 5**  
Material properties of the shells

<table>
<thead>
<tr>
<th>Material</th>
<th>Shell</th>
<th>Stiffener</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $\rho$ (kg/m$^3$)</td>
<td>PVC</td>
<td>PVC</td>
</tr>
<tr>
<td>Young’s modulus, $E$ (GN/m$^2$)</td>
<td>4.0</td>
<td>4.6</td>
</tr>
<tr>
<td>Poisson’s ratio, $\mu$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Damping loss factor, $\eta$</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The effect of damping has been accounted for in the calculations by introducing a complex Young’s modulus defined by

$$E' = E (1 + i\eta).$$  \hspace{1cm} (52)

Using data from Tables 2 and 5 yields ring frequencies for the stiffener and the shell at 3080 Hz and 2852 Hz, respectively.

![Figure 14. Measured loss factors for the two materials; (---) shell (---*) stiffener.](image)

**5.3 MEASUREMENTS RESULTS VERSUS THEORETICAL PREDICTIONS**

As mentioned in section 5.1, a swept sine excitation has been used in the frequency range [0-5000 Hz] and the response of the structure has been monitored at several locations. The response of the shell is calculated according to the method described in section 4. Measurements and theoretical results have been collected for the five following quantities:
(a) input radial mobility at \( B \), \( Y_{BB}^w = \dot{w}_B / F_B \); the excitation is applied radially at \( B \) and the radial velocity is measured at the same point,
(b) transfer radial mobility between \( B \) and \( C \), \( Y_{BC}^w = \dot{w}_C / F_B \); the excitation is applied radially at \( B \) and the radial velocity is measured at \( C \),
(c) transfer radial mobility between \( B \) and \( D \), \( Y_{BD}^w = \dot{w}_D / F_B \); the excitation is applied radially at \( B \) and the radial velocity is measured at \( D \),
(d) transfer cross-mobility between \( B \) and \( A \), \( Y_{BA}^x = \dot{u}_A / F_B \); the excitation is applied radially at \( B \) and the axial velocity is measured at \( A \),
(e) transfer cross-mobility between \( A \) and \( B \), \( Y_{AB}^x = \dot{w}_B / F_A \); the excitation is applied axially at \( A \) and the radial velocity is measured at \( B \).

Figures 15 to 19 show measured and predicted values for these five mobilities. Both magnitude and phase are compared. It is seen that for radial excitation the agreement is very satisfying over the whole frequency range though some non negligible discrepancies appear above 2000 Hz. However, for these frequencies the predicted values still yield acceptable estimates of the vibration levels. It is seen anyway that in case of axial excitation (Figure 19), large discrepancies occurs at low frequencies in the range [10 - 30 Hz]. They may be attributed to the difficulty to excite the structure axially in this frequency range. Finally, Figure 20 shows that measurement results for \( Y_{AB}^w \) and \( Y_{BA}^x \) are very close to each other, thereby evidencing the symmetry of the mobility matrix. Thus, the good agreement obtained between experimental results and theoretical predictions validates the modelling of the stiffened shell structure as described in section 4.

![Figure 15. Magnitude and phase of the radial input mobility at B, Y^w_{BB}.](image)

(----) measured (---) predicted.
Figure 16. Magnitude and unwrapped phase of the transfer mobility at point C, $Y_{BC}^w$.

(—) measured (—) predicted.

Figure 17. Magnitude and unwrapped phase of the transfer mobility at point D, $Y_{BD}^w$.

(—) measured (—) predicted.
Figure 18. Magnitude and phase of the cross-mobility, $Y_{AB}^w$.

( ) measured (—) predicted.

Figure 19. Magnitude and phase of the cross-mobility, $Y_{BA}^w$.

(—) measured (----) predicted.
6. PARAMETRIC STUDY

In what follows, the stiffener and the shell are assumed to be made of the same material (see Table 6). The study is confined to the case of input power by radial point excitation. The influence of various geometrical parameters on the real part of the input radial mobility has been investigated. These parameters are the following: length of the stiffener \( L \), stiffener-to-shell thickness ratio \( (h_s/h_t) \), eccentricity between stiffener and the shell \( (e) \) and excitation position on the stiffener \( (x_0) \), see also Figure 21.

<table>
<thead>
<tr>
<th>TABLE 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material properties of the stiffener and the shell</td>
</tr>
<tr>
<td>Density, ( \rho_s = \rho_r = \rho ) (kg/m(^3))</td>
</tr>
<tr>
<td>Young’s modulus, ( E_s = E_r = E ) (Pa)</td>
</tr>
<tr>
<td>Poisson’s ratio, ( \mu_s = \mu_r = \mu )</td>
</tr>
</tbody>
</table>
6.1 THE SEMI-INFINITE SHELL

Before addressing the problem of vibration transmission to a stiffened shell, it is illuminating to study the functional dependence of the input power into a non-stiffened, semi-infinite shell upon parameters like shell thickness and position of the excitation. This is achieved by assuming the thickness and mean radius of the stiffener and shell equal, that is \( h_s = h_r = h \) and \( R_s = R_r = R \), and by following the method presented in section 4.2 to determine the input power.

6.1.1 Influence of thickness

Consider a semi-infinite, cylindrical thin shell acted upon by a radial point force at its edge. For a given excitation frequency, the influence of the thickness on the input power is twofold. It governs both the number of circumferential modes that contribute to the power injection at this frequency as well as the individual mobilities of these modes [18]. Figure 22(a) shows calculation results for the real part of the radial input mobility for two shells with different thickness. Figure 22(b) shows the corresponding difference of power input versus non-dimensional frequency. It appears that this difference oscillates around a mean value dependent upon the thickness ratio of the two shells, with big deviations at low frequencies owing to the resonance of the lower circumferential modes. Thus, a thicker shell might be responsible for a higher power input if excited close to one of its resonance frequencies. In a frequency average sense, the input power is inversely proportional to the square of the shell thickness for point excitation [14], that is

\[
P_{\text{input}} \propto -20 \log(h).
\]
Figure 22. Input power into edge excited, semi-infinite shells \((R = 0.5 \text{ m})\).  
(a) Real part of mobility: \((\cdots)\) thickness \(h_1 = R/25\); \((\cdots)\) thickness \(h_2 = R/50\).  
(b) Input power difference: \((\cdots)\) calculated; \((\cdots)\) \(-20\log(h_1 / h_2)\).

6.1.2 Position of the excitation

Infinite and semi-infinite structures provide valuable estimates of the power transmission to similar, but finite, structures in a frequency-average sense [14]. The type of model to adopt, that is, infinite or semi-infinite, depends on the location of the excitation with respect to the boundaries of the finite structure. For semi-infinite structures excited transversally by point force at their free boundary, previous studies conducted on structural elements like beams, plates and shells \([14, 18-21]\) have shown the power transmission to be about 6dB higher in comparison with the case of the infinite structure. It is therefore advantageous for noise and vibration applications to locate the excitation at some distance from the free boundary in order to get ride of this excess of transmission. However, this distance is frequency dependent (the lower the frequency the greater the distance needed), and so it is of significant interest to be able to predict the frequency at which this excess transmission disappears, that is the frequency at which a finite structure might be considered as infinite.

To that end, calculations have been carried out for a semi-infinite shell acted upon by a radial point force applied at a distance \(x_0\) from its free edge. This section reports on calculation results showing the influence of the distance \(x_0\) on the input power compared to the input power to a similar, but infinite, shell. The input power for a radial point force excitation of amplitude \(S_e\) is given by

\[
P_{\text{input}} = \frac{1}{2} \Re (Y'_{wo} |S_e|^2), \tag{54}
\]
where $Y_{w, s}^{x_0}$ is the radial input mobility and is calculated according to section 4.2. Figure 23 shows numerical results of the input power for the infinite shell and for the semi-infinite shell excited at its edge as well as at the distance $x_0 = 0.2R$. For convenience, the calculation results have then been normalised by the input power to an infinite plate of thickness $h$ excited transversally by a point force, which is given by

$$P_{\text{input}}^\text{plate} = \frac{1}{2} \Re(Y_{w, s, \text{plate}}^{\infty} \|S_e\|^2),$$

(55)

where $Y_{w, s, \text{plate}}^{\infty} = \frac{\sqrt{3} \sqrt{1-\mu^2}}{4h^2 \sqrt{E\rho}}$ (from [14]).

The figure clearly evidences the plate behaviour of the shell for frequencies above the ring frequency. Furthermore, the input power resulting from an excitation applied at the edge is 6 dB higher compared to the infinite shell at low frequencies and 5.7 dB higher at high frequencies. These values are the same as for a beam and a plate, respectively. Finally, concerning the excitation at $x_0 = 0.2R$, it appears that the input power is at low frequencies similar to the one of the edge-excited shell, but as the frequency increases, it tends toward the one of the infinite shell as expected.

Figure 23. Normalised input power by a radial point force $(R = 0.5 \text{ m and } h = R/50)$.

(—) infinite shell, (--- —) edge excited semi-infinite shell $(x_0 = 0)$,

(-----) semi-infinite shell excited at $x_0/R = 0.2$.

For beams and plates [22], the dependence of this ‘excess transmission’ upon frequency and $x_0$ due to the free boundary is conveniently evidenced by plotting the mobility ratio $\Re(Y_{w, s}^{x_0})/\Re(Y_{w, s}^{\infty})$ versus $kx_0$, $k$ being the wavenumber of the propagating wave. For cylindrical shells, the input power results from the contributions of a finite number of
circumferential modes which are characterised by different wavenumbers, thereby precluding such a representation. However, if the normalised power is plotted as a function of \( kx_0 \), where \( k \) is the value of the wavenumber value obtained from the envelop shown in Figure 7, the most important patterns of this dependence are retained. Figure 24 shows the mobility ratio \( \text{Re}(Y_{w3}) / \text{Re}(Y_{w5}) \) for a beam, a plate and a shell and for the latter, three curves corresponding to three non-dimensional frequencies are shown. Likewise, Figure 25 shows the average of the excess transmission versus \( kx_0 \) for non-dimensional frequency in the range \([0.01, 2]\). At a given \( kx_0 \), the average value is obtained from the “best fit” in the least squares sense of the normalised power versus frequency by a straight horizontal line. Considering Figures 24 and 25, it is possible to draw the conclusion that the excess transmission for shells is rather similar to the one of the plate and the beam. When \( kx_0 \) is greater than unity, the power transmission to a semi-infinite shell oscillates around the one for the infinite shell. Consequently, \( x_0 \) being constant and \( k \) almost proportional to the frequency (equation 33b), a frequency average would yield similar results as for the infinite shell in this range.

These results make it possible to rapidly appreciate the validity of the analogy with the infinite structure for point excitation. Indeed, Figure 25 clearly shows that this assumption is acceptable when the condition \( kx_0 \geq 1 \) is satisfied. For the sake of illustration, assume that the shell described in Table 1 is acted upon by a force located at \( x_0 = 0.2R \) as calculated for Figure 23. Using equation (33b), it is seen that the condition \( kx_0 \geq 1 \) is met when \( \Omega \geq 0.43 \), that is when \( f \geq 744 \text{ Hz} \), a value which agrees with the numerical results depicted in Figure 23.

![Figure 24. Mobility ratio \( \text{Re}(Y_{w3}) / \text{Re}(Y_{w5}) \) versus \( kx_0 \).](image)

\( \text{--- shell: } \Omega = 0.01, \; \text{--- shell: } \Omega = 0.5, \; \text{--- shell: } \Omega = 2, \)
\( \text{--- beam, } \; \text{--- plate.} \)
Figure 25. Average in the least squares sense of excess transmission versus $k \alpha_0$
for $\Omega$ varying in the range [0.01, 2].

6.2 THE STIFFENED, SEMI-INFINITE SHELL

6.2.1 Application example

Figure 26 illustrates the influence of a stiffener on the real part of the input radial mobility
of a semi-infinite shell. The excitation is located at the free edge of the stiffener. The length
of the stiffener is $L = 0.1 \text{m}$, its mean radius $R_r = 0.5 \text{m}$ and its thickness $h_r = R_r / 25$; the
geometric parameters of the shell are $R_s = R$, and $h_s = h_r / 2$. For comparison are also
shown the mobilities corresponding to edge-excited, semi-infinite shells of thickness $h_s$
and $h_r$, respectively.

Three frequency regions characterise the influence of the stiffener. At very low
frequencies, the structure behaves as if the stiffener was only an added mass and the input
power is similar to the one of the semi-infinite shell of thickness ($h_s$). Conversely, at high
frequencies, the stiffened shell behaves in a way similar to a semi-infinite shell of thickness
equal to that of the stiffener ($h_r$). In the frequency region where the transition takes place,
no resonant behaviour appears but only small undulations in the mobility curve. It is worth
noting that at both low and high frequencies, the stiffened shell exhibits the circumferential
resonance characteristics of the shell and the stiffener, respectively.
Figure 26. Real part of the radial input mobility for edge excitation.

(---) semi-infinite shell of thickness $h_r$; (---) semi-infinite shell of thickness $h_s$;

(—) stiffened, semi-infinite shell $L = 0.1$ m.

6.2.2 Influence of eccentricity

Figure 27 shows that eccentricity between stiffener and shell plays a minor role on the
displacement of the structure. A significant influence of eccentricity would require the
stiffener to be much thicker, thereby invalidating the use of thin shell theory. Consequently,
no eccentricity is assumed in what follows and the results can thus be presented using a
common non-dimensional frequency for stiffener and shell.

6.2.3 Influence of stiffener length

The application example of section 6.2.1 has shown that the structural mobility of a
stiffened shell departs from the mobility of the non-stiffened shell as the frequency is
increasing, tending asymptotically toward the mobility of a shell of thickness equal to the
stiffener thickness at high frequency. Increasing the length of the stiffener results in a
reduction of the frequency at which the structure mobility reaches the high frequency
asymptotic behaviour. The objective of the results presented here is to relate this frequency
to the stiffener length. To that end, calculations have been carried out with the geometry
described in section 6.2.1 except for the length of the stiffener, $L$. This latter has been
assumed frequency dependent, the dependence being chosen such as to keep the product $kL$
constant at every frequency. The wavenumber $k$ is associated to the propagating wave with
the lowest wavelength of the stiffener at the frequency under consideration and is obtained
from the envelope shown in Figure 7.
Figure 27. Influence of eccentricity \((R_e = 0.5m, h_e = R_e / 25, h_s = h_e / 2)\).

(—) same inner radius: \(R_s = R_e - e\) (eccentricity \(e = (h_e - h_s) / 2)\);

(—) same mean radius: \(R_s = R_e\) (no eccentricity).

Figure 28(a) shows the real part of the mobility for both semi-infinite shells and the stiffened, semi-infinite shell with \(kL = 0.2\) for edge excitation. Figure 28(b) shows the difference of input power in decibels between the stiffened shell structure and the semi-infinite shell of thickness \(h_s\). It appears that this difference is approximately constant for a frequency average and equal, in the least squares sense, to \(-2.6\) dB for the present value of \(kL\). Supported by this result, calculations have been done with \(kL\) varying in the range \([0.01, 10]\). The frequency average difference with \(\Omega \in [0.01, 2]\), that is the “best fit” in the least squares sense of the difference versus frequency by a straight horizontal line, has then be evaluated for each \(kL\) as displayed by Figure 29. The quantity \(20\log(h_s / h_e)\), where \(h_s / h_e\) is the shell-to-stiffener thickness ratio, is referred to as the nominal difference in what follows. It corresponds to the difference between the input power in decibels of two shells of thickness \(h_s\) and \(h_e\), respectively. As can be seen in Figure 29, this difference is reached at about \(kL = 2\), but the transition between low and high frequency asymptotic behaviours takes place for \(kL \leq 1\). From equation (33b) an estimation of the parameter \(kL\) is obtained, given in the frequency range \([0.42 \sqrt{\beta c_p / R, c_p / \pi R}]\), by

\[
kL \approx 1.2 \frac{aL}{c_p} \sqrt{\frac{R}{h_e}}.
\]  

(56)

This approximation together with Figure 29 enables a rapid appreciation of the influence of the stiffener for a given frequency. However, it should be kept in mind that the conclusions drawn are valid only in a frequency-average sense.
Figure 28. (a) Real part of the radial input mobility for edge excitation; (—) semi-infinite shell: \( h = h_r \), (— —) semi-infinite shell: \( h = h_s \), (— — —) stiffened shell: \( kL = 0.2 \).
(b) Input power difference; (—) calculated; (— — —) average value.

Figure 29. Difference in input power between non-stiffened and stiffened shells versus \( kL \) for edge excitation. (—) calculated, (— — —) nominal difference.
6.2.4 Influence of shell-to-stiffener thickness ratio

For beam-plate systems, previous studies have shown that the mobility of the coupled system is only slightly different from the mobility of the beam alone, provided that the beam thickness is three or more times the thickness of the plate [4, 5]. Thus, the shell-to-stiffener thickness ratio should also influence the vibration transmission to the structure in a significant way. Calculations similar to those described in 6.2.2 have therefore been performed with different thickness ratios. This was achieved by keeping the thickness of the stiffener constant while varying the thickness of the shell. The results are presented in Figure 30 in terms of averages of the input power difference and reveal that the high frequency asymptotic mobility is reached at about $kL = 2$; this value is almost independent of the thickness ratio.

However, evaluation of the ratio of the calculated difference to the nominal value indicates that the shell-to-stiffener thickness plays an important role in the dynamics of the transition process, as shown in Figure 31. Taking for instance $kL = 0.25$, the ratios are 0.5 and 0.8, respectively, for shell-to-stiffener thickness ratios of 2 and 10.

![Figure 30. Influence of shell-to-stiffener thickness ratio on the input power for edge excitation. $h_r/h_s = [1.5 \ 2 \ 4 \ 6 \ 8 \ 10 \ 12]$. (---) nominal difference: $-20\log(h_r/h_s)$ and (—) averages in the least squares sense for $\Omega \in [0.01, 2]$.](image)
Figure 31. The ratios of the calculated input power differences to the nominal values for thickness ratios $h_r/h_s = [1.5, 2, 4, 6, 8, 10, 12]$. (---) averages in the least squares sense for $\Omega \in [0.01, 2]$.

6.2.5 Influence of excitation position

To conclude this parametric study, the effects of the position of the excitation on the stiffener are evaluated by moving the radial force from the free edge to $x_0 = L/2$ and $x_0 = L$. Figure 32 shows calculation results for the two shell-to-stiffener thickness ratios: $h_r/h_s = 2$ and 8. As can be seen from the figure, the effects of excess transmission due to the free edge, see section 6.1.2, vanish for $kL \geq 2$ for an excitation located at the centre of the stiffener and the input power difference amounts to approximately $20\log(h_r/h_s) - 6\text{dB}$. Moreover, when the excitation is located at the interface between the shell and the stiffener ($x_0 = L$), Figure 32 reveals that the shell influences the input power even at high frequency when the stiffener thickness is twice the shell thickness. However, for high thickness ratios, the mobility of the structure is only slightly different from the mobility of the stiffener alone. Finally, Figure 32 indicates that the position of the excitation does not influence the input power below approximately $kL \leq 0.3$. Consequently, in this range the stiffener can be considered as a ring.
Figure 32. Influence of the position of the excitation on the stiffener. The straight lines (---) and (---) indicate the nominal differences and the nominal differences minus the excess transmission of 6dB, respectively. (---) $x_0 = 0$; (---) $x_0 = L/2$; (---) $x_0 = L$.

Upper curves: $h_r/h_s = 2$; lower curves: $h_r/h_s = 8$.

7. CONCLUSION

The objective of this study was to evaluate the vibratory power transmission to a stiffened thin cylindrical shell by an excitation located on the stiffened part. The stiffening under consideration consists of a finite length axisymmetric discontinuity in the thickness of the shell wall. In the first place, the validity of the Flügge shell theory for the present problem is established from measurement results for a structure built up from two shells of different thickness. In the second place, the stiffener thickness and length are shown to influence the power transmission in a significant way. The main finding is the following: the power transmission to the structure is similar to the power transmission to a shell having a thickness equal to the stiffener thickness when the following inequality is satisfied

$$\omega L \geq c_p \sqrt{\frac{h_r}{R}}.$$

and the power transmission is in a frequency average sense reduced by $20 \log (h_s/h_r)$ decibels compared to the non-stiffened shell. Furthermore, it has also been found that, when this inequality is satisfied, it is advantageous to position the excitation at the centre of the stiffener as regards noise and vibration control.
ACKNOWLEDGMENTS

The author wishes to thank Professor S. Ljunggren for suggesting this investigation and for many profitable discussions. The staff at Marcus Wallenberg Laboratory for Sound and Vibration Research at KTH and Ingemansson Technology AB, are gratefully acknowledged for enabling the measurements. This work was supported by the Swedish National Energy Administration.

REFERENCES

APPENDIX A. DISPERSION RELATION FROM THE FLÜGGE THEORY

Disregarding terms containing $\beta^2$ and $\beta^3$ when expanding the determinant of the coefficient matrix, the characteristic equation derived from the Flügge shell operator (equation (27)) reads

$$\beta (\kappa^8 + a_6 \kappa^6 + a_4 \kappa^4 + a_2 \kappa^2 + a_0) + b_4 \kappa^4 + b_2 \kappa^2 + b_0 = 0, \quad (A1)$$

where the coefficient $a_i$ and $b_i$ are given by

$$b_0 = \Omega^2 (1 - \Omega^2 + n^2)(\Omega^2 \frac{2}{1 - \mu} - n^2),$$

$$b_2 = \Omega^2 (3 + 2\mu + 2n^2 - \frac{3}{1 - \mu} \Omega^2),$$

$$b_4 = 1 - \mu^2 - \Omega^2,$$

$$a_0 = (\Omega^2 - n^2)(\Omega^2 \frac{2}{1 - \mu} - n^2)(1 - n^2)^2 - n^2 \Omega^2 (1 + n^2 - \Omega^2),$$

$$a_2 = (2\frac{3 - 2\mu}{1 - \mu} - 3\Omega^2)\Omega^2 - 2n^2(2 - \mu + \Omega^2 \frac{5 - \mu}{1 - \mu} + 2\frac{\Omega^4}{1 - \mu}) + n^4 (2(4 - \mu) + 3\Omega^2 \frac{3 - \mu}{1 - \mu}) - 4n^6,$$

$$a_4 = 6n^4 - 3n^2 \left( \frac{3 - \mu}{1 - \mu} \Omega^2 + 2 \right) + 4 - 3\mu^2 - \frac{3 - 7\mu}{1 - \mu} \Omega^2 + \frac{2}{1 - \mu} \Omega^4,$$

$$a_6 = \frac{3 - \mu}{1 - \mu} \Omega^2 - 4n^2 + 2\mu .$$

APPENDIX B: LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>$B$</td>
<td>$= Eh/(1-\mu^2)$, extensional rigidity</td>
</tr>
<tr>
<td>$D$</td>
<td>$= \beta BR^2$, flexural rigidity</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus</td>
</tr>
<tr>
<td>$L$</td>
<td>stiffener length</td>
</tr>
<tr>
<td>$R$</td>
<td>shell radius</td>
</tr>
<tr>
<td>$Y$</td>
<td>mobility matrix</td>
</tr>
<tr>
<td>$c_p$</td>
<td>$= \sqrt{E/\rho(1-\mu^2)}$</td>
</tr>
<tr>
<td>$h$</td>
<td>shell thickness</td>
</tr>
<tr>
<td>$i$</td>
<td>$= \sqrt{-1}$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>n</td>
<td>Circumferential mode order</td>
</tr>
<tr>
<td>r</td>
<td>cylindrical co-ordinate</td>
</tr>
<tr>
<td>s</td>
<td>= $x/R$, non-dimensional axial length</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>$u$</td>
<td>midsurface displacement in the $x$ direction</td>
</tr>
<tr>
<td>$v$</td>
<td>midsurface displacement in the $\varphi$ direction</td>
</tr>
<tr>
<td>$w$</td>
<td>midsurface displacement in the $r$ direction</td>
</tr>
<tr>
<td>$x$</td>
<td>cylindrical co-ordinate</td>
</tr>
<tr>
<td>$\beta$</td>
<td>= $h^2/(12R^2)$, thickness parameter</td>
</tr>
<tr>
<td>$\eta$</td>
<td>shell loss factor</td>
</tr>
<tr>
<td>$\theta$</td>
<td>= $\partial w / \partial x$, rotation of the normal to the middle surface about the $\varphi$-axe</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>cylindrical co-ordinate</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>non-dimensional propagation constant</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of the shell material</td>
</tr>
<tr>
<td>$\omega$</td>
<td>circular frequency of the excitation</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>= $\omega R\sqrt{\rho(1-\mu^2)/E}$, non-dimensional frequency</td>
</tr>
</tbody>
</table>
Paper B
CLOSED-FORM SOLUTIONS FOR THE POINT MOBILITIES OF
AXISYMMETRICALLY EXCITED CYLINDRICAL SHELLS

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Closed-form formulae are reported describing point mobilities for thin cylindrical shells in axisymmetric motion. Two cases are studied: (i) the infinite shell and (ii) the edge-excited semi-infinite shell and, for both, forces and moment excitation are considered. The solutions of these problems are obtained analytically by resorting to perturbation methods and presented in terms of Green’s functions and point mobilities. These results are further used to derive approximate expressions for the reflection coefficients of the shell-borne waves at a free end and for the Green’s functions of a finite free-free shell undergoing axisymmetric vibrations.

1. INTRODUCTION

Point mobility for a mechanical system is defined as the complex ratio of velocity to force during harmonic motion. In mechanical engineering this concept is widely used for studying the vibration transmission to structures by force and moment. Formulae describing mobilities of infinite and semi-infinite simple structural elements such as beams, plates and shells are desirable from the theoretical point of view. Indeed analytical solutions display explicitly the functional dependence of the solutions on the problem parameters, thus yielding a physical insight into the underlying mechanisms which govern the structure response. Their benefit is, however, not limited to theoretical considerations and they also find application in practice. For instance, design methods based upon substructuring techniques, e.g. [1, 2], resort to such formulae to study the vibration transmission in built-up structures. A further application is that they provide estimates for the vibration levels of similar but finite structures [3].

Mobilities of beams and plates having infinite or semi-infinite extent have been the focus of a vast number of past, as well as recent, studies, e.g. [3-6], yielding closed-form solutions for a number of problems. For shells, the results reported by the literature are fewer and have been obtained mainly numerically, e.g. [7-10]. In fact, scant attention has been devoted to the derivation of analytical solutions, perhaps due to the complexity of the differential equations governing shell motions. The first to attempt to fill this void was Franken in 1960 [11]. Neglecting the flexural stiffness of the shell, he derived an analytical expression for the mobility of an infinite thin-walled shell with respect to a radial point force. Addressing the same problem two years later, Heckl [12] obtained approximate solutions including the flexural stiffness. It can be pointed out that, by taking advantage of the infinite character of the structure, both studies resort to the method of residues. Regarding the case of semi-infinite shell
subject to loads applied at its end, the static case was solved explicitly by Simmonds in 1966 [13]. Recently, Ming et al. [10] have presented numerical results for the dynamical version of this problem. However, no attempt has been made to derive closed-form solutions of the mobilities of semi-infinite shells. This may be due to the fact that the method of residues cannot be used with equal success to deal with semi-infinite structures as shown by [5, 6].

Considering only the axisymmetric mode of vibration, the present paper reports on closed-form formulae describing point mobilities for thin-walled cylindrical shells of infinite and semi-infinite extent. The formulae are given for three types of load, viz. an axial force, a radial force and a bending moment. Regarding the infinite shell problem, the present results complete those obtained by Franken [11] and Heckl [12] in the case of radial excitation. Further, since being a stage in the derivation of the mobilities, approximate expressions for the shell Green’s functions are also presented. The validity of the present results is bounded by the limitations inherent in the following assumptions: (i) Love’s first approximation [14], (ii) vibrations in vacuo and (iii) point excitation in the axial direction, i.e. the size of the excitation in this direction is larger than the shell thickness but small compared to the axial wavelength.

The theoretical approach adopted in this study is based upon perturbation methods. This strategy was used successfully by Wong et al. [15] to derive the normal frequencies of a clamped cylindrical shell vibrating axisymmetrically. The problems addressed here are formulated using the theory by Flügge [16] and solved analytically by applying the method of the matched asymptotic expansions [17]. Solutions are obtained in the form of power series expansions with the thickness-to-radius ratio of the shell taken as the expansion parameter. Retaining the leading terms of these expansions yields approximate expressions of the shell Green’s functions with respect to the different loads. Finally, the analysis is pursued to higher order terms in order to obtain expressions for both the real and the imaginary parts of the point mobilities.

The plan of this paper is as follows. Section 2 is concerned with the problem formulation and presents the methodology for the computerisation of the results. In section 3, the equations are re-written in line with perturbation theory and solved for the leading term to exemplify the method. It appears that the derived expressions exhibit a singularity slightly below the first ring frequency. In section 4, the Green’s functions and the mobilities of the semi-infinite and infinite shells with respect to the different loads are given in closed form and are shown to compare well with exact calculations. Section 5 deals with the singularities appearing in the previous results. They are tackled by resorting again to perturbation theory and the derived results are represented by formulae containing tabulated functions. Finally, section 6 provides approximate expressions for the reflection coefficients of the shell-borne waves at a free end and for the response of a finite free-free cylindrical shell.

2. PROBLEM FORMULATION

The mechanical response of a cylindrical shell to an axisymmetric load is addressed for the three following problems: the case of the semi-infinite shell excited at its end (Problem I); the case of the infinite shell excited in its middle (Problem II) and the case of the free-free finite shell (Problem III). Problems I and II can be tackled by studying a semi-infinite shell with certain sets of boundary conditions while solutions for Problem III are derived from results of the two former problems.
2.1 GOVERNING EQUATIONS

Let a thin circular semi-infinite shell made of a homogeneous isotropic material vibrate in vacuo in its axisymmetric mode. The shell occupies the half-space \( x > 0 \) (see Figure 1) and, according to Flügge shell theory [16], its displacements in the axial and radial directions, respectively denoted \( u \) and \( w \), are governed by the two differential equations

\begin{equation}
\begin{align*}
u'' + \Omega^2 u + \mu w' - \beta w''' = 0, \\
\mu u' - \beta u'' + (1 - \Omega^2)w + \beta(w + w^{(4)}) = 0,
\end{align*}
\end{equation}

where \( \mu \) is the Poisson’s ratio, \( \beta \) the thickness parameter and \( \Omega \) the non-dimensional frequency. The reader is referred to the list of symbols given in Appendix B for the definition of parameters not explicitly given in the text. Primes denote differentiation with respect to the non-dimensional axial length \( s = x/R \) and derivatives of order \( n \) higher than 3 are denoted \( \partial^n (\cdot) / \partial s^n = (\cdot)^{(n)} \) to lighten the equations.

Figure 1. Geometry of the semi-infinite shell, co-ordinate system and positive convention of force and moment resultants.

In order to make the present analysis more convenient, equations (1) are rearranged in the following form

\begin{equation}
\begin{align*}
\beta(1 - \beta)w^{(6)} + \beta(\Omega^2 + 2\mu)w^{(4)} + (\beta + 4m^4)w'' + \Omega^2(1 - \Omega^2 + \beta)w &= 0, \\
(\mu + \Omega^2 \beta)u'' &= -(1 - \Omega^2 + \beta)w + \mu \beta w' + \beta(1 - \beta)w^{(4)} ,
\end{align*}
\end{equation}

where the notation

\begin{equation}
4m^4 = 1 - \mu^2 - \Omega^2 ,
\end{equation}

has been introduced.

Free axisymmetric vibrations of semi-infinite shells can be expressed in terms of three waves of propagating or near-field types. Omitting the time dependence \( e^{i\omega t} \), its general form may be taken as

\begin{equation}
\begin{align*}
\begin{bmatrix}
    w \\
    u
\end{bmatrix} = \begin{bmatrix}
    C_1 \\
    A_1
\end{bmatrix} e^{\kappa_1 s} + \begin{bmatrix}
    C_2 \\
    A_2
\end{bmatrix} e^{\kappa_2 s} + \begin{bmatrix}
    C_3 \\
    A_3
\end{bmatrix} e^{\kappa_3 s} ,
\end{align*}
\end{equation}
where \( \kappa_p, C_p \) and \( A_p \) are the non-dimensional propagation constant, the amplitude in the radial and the axial directions of the wave \( p (p = 1, 2, 3) \), respectively.

Substituting the form of solutions (4) into equation (2a) leads to a sixth order polynomial in \( \kappa_p \), also termed dispersion relation. The three propagation constants are the roots of the dispersion relation which yield solutions satisfying the Sommerfeld’s condition in the half space \( x > 0 \). Accordingly, and owing to the time dependence adopted, these constants are the roots located in the complex half-plane described by \( \pi/2 < \arg(\kappa) \leq 3\pi/2 \). Further, to each wave \( p (p = 1, 2, 3) \) is associated a wave amplitude ratio, denoted \( T_p = A_p/C_p \), and which expression is easily obtained by introducing equation (4) in equation (2b).

In both Problems I and II, the shell is submitted to an axisymmetric excitation at \( x = 0 \). The excitation consists of an axial force, a radial force and a bending moment. The distributions of these loads per unit length are \( (N_0, Q_0, M_0) \) and the convention which is employed regarding their positive directions is shown in Figure 1. Problem II (infinite shell) is transformed into a boundary-value problem by considering the response of a semi-infinite shell and a set of boundary conditions at \( x = 0 \) in accord with the symmetry of the infinite shell. For problem I, i.e. the semi-infinite shell excited at the edge, the set of boundary conditions reads [16]

\[
(4m^4w + \beta(w + w^{(4)} + \mu w^\mu - u^\mu))/\mu \bigg|_{x=0} = n_0, \\
(w^\mu - u^\mu)\bigg|_{x=0} = q_0, \\
(w^\mu - u^\mu)\bigg|_{x=0} = m_0. 
\]

The right-hand members of equations (5) are defined by

\[
n_0 = RN_0/B, \quad q_0 = R^3Q_0/D, \quad m_0 = -R^2M_0/D, 
\]

where \( B = Eh/(1-\mu^2) \) and \( D = Eh^3/12(1-\mu^2) \) are the extensional and the flexural rigidities, respectively. In Flügge’s theory, the boundary condition with respect to the axial force is usually given in the form

\[
(u^\mu + \mu w - \beta w^\mu)\bigg|_{x=0} = -n_0, 
\]

instead of equation (5a). Equation (5a) is obtained by re-arranging equation (7) together with equation (1b) and was derived in order to save some algebra in the future analysis.

For Problem II, i.e. the case of the infinite shell excited in its ‘middle’, symmetry implies that half of the exciting load acts on each half of the shell. Furthermore, two sets of boundary conditions have to be considered. One deals with the excitation by a radial force and reads

\[
u\bigg|_{x=0} = 0, \quad w\bigg|_{x=0} = 0, \quad (w^\mu - u^\mu)\bigg|_{x=0} = q_0/2, 
\]

while the second deals with axial force excitation and bending moment excitation:

\[
(4m^4w + \beta(w + w^{(4)} + \mu w^\mu - u^\mu))/\mu \bigg|_{x=0} = n_0/2, 
\]
\[(w^* - u^*)|_{s=0} = m_0 / 2, \quad w|_{s=0} = 0. \quad (9)\]

### 2.2 SOLUTIONS TO THE BOUNDARY-VALUE PROBLEMS

To determine the response of the shell, the dispersion relation is solved to find the three propagation constants satisfying the Sommerfeld's condition and the associated wave amplitude ratios \(T_p\) are calculated. Then, substituting the displacement equations (4) into the boundary conditions given by either equations (5), equations (8) or equations (9) and using the previously calculated \(T_p\) and \(\kappa_p\) yield a system of three equations with the wave amplitudes \(C_1, C_2\) and \(C_3\) as unknowns. Solving this system to find these constants gives the receptances of each wave to the different loads. Written in matrix form, the amplitudes of the waves are given by

\[C = \alpha^c X, \quad (10)\]

where \(C = [C_1, C_2, C_3]^T\) and \(X = [N_0, Q_0, M_0]^T\). The matrix \(\alpha^c\) is referred to as the wave receptance matrix. The element \(\alpha_{p,X}^c\) is the receptance of the wave \(p\) with respect to the load \(X\) \((X = N_0, Q_0, M_0)\). The superscript \(c\) is either \(c = \infty / 2\) or \(c = \infty\) depending on whether reference is made to problem I (semi-infinite shell) or II (infinite shell).

Once the receptances of the waves are known, the direct mobilities, which are defined as the complex ratio of velocity to force taken at the same point, are easily derived. For the sake of convenience, they are gathered in matrix form as follows:

\[
\begin{bmatrix}
iu & \dot{w} & \dot{\theta}
\end{bmatrix}^T = Y^c X.
\quad (11)
\]

The dot denotes time derivation and \(\theta\) represents the rotation of the normal to the middle surface about the \(\varphi\)-axe, \(\theta = \partial w / \partial x\). The mobility matrix is symmetric and reads explicitly

\[
Y^c = 
\begin{bmatrix}
y^c_{u,N_0} & y^c_{u,Q_0} & y^c_{u,M_0} \\
y^c_{w,N_0} & y^c_{w,Q_0} & y^c_{w,M_0} \\
y^c_{\theta,N_0} & y^c_{\theta,Q_0} & y^c_{\theta,M_0}
\end{bmatrix}.
\quad (12)
\]

Diagonal and off-diagonal terms are referred to as input and cross-mobilities, respectively. They are related to the wave receptances by the relationships

\[
Y^c_{u,X} = i \omega \sum_{p=1}^{3} T_p \alpha_{p,X}^c, \quad Y^c_{w,X} = i \omega \sum_{p=1}^{3} \alpha_{p,X}^c, \quad Y^c_{\theta,X} = i \omega (\sum_{p=1}^{3} \alpha_{p,X}^c \kappa_p) / R.
\quad (13a-c)
\]

For single force or moment excitation, the power injected into the shell is proportional to the real part of the input mobility. In case of joint excitation, i.e. excitation consisting of several components (forces, moments) correlated in time, cross-mobilities should be taken into consideration when calculating the injected power.
Unlike input mobilities, the sign of the real part of the cross-mobilities may change, reflecting the possible reduction of power input by a certain combination of loads at the end of the structure.

3. THE PERTURBATION APPROACH

For thin shells the thickness parameter, which is proportional to the square of the thickness-to-radius ratio, is very small ($\beta \ll 1$). Equation (2a) shows that this small parameter multiplies the highest derivative of the differential equation. This form of differential equation is typical of edge-layer problems for which the variable of the problem undergoes rapid changes across a narrow region [17]; the term edge-layer refers to the fact that these regions frequently adjoin the boundary of the system. However, such layers should also be expected in the narrow regions adjoining load application points or discontinuities in the structure.

Setting $\beta = 0$ in equation (2) yields the well-known membrane equation for shells. Using membrane theory for studying the axisymmetric vibrations of shells is valid as long as the edge-layer can be neglected, i.e. at some distance from the boundary and at low frequencies. The membrane equation being of second order, the vibration field is modelled by only one wave and fails to satisfy the three boundary conditions at the edge. The two boundary conditions involving bending stresses, i.e. the ones on the radial force and the bending moment, should be dropped. However, for infinite shell membranes, transverse shear stresses are balanced by extensional stresses and the membrane theory can be used to assess the receptance of the shell to a radial force at low frequencies [11]. Conversely, the membrane theory is inadequate to predict the response of a semi-infinite shell to a radial load located at its edge as the extensional stresses vanish there.

Among the perturbation methods, a most effective technique for treating edge-layer problems is the method of the matched asymptotic expansions [17]. This method divides the structure into an inner region adjoining the boundary and an outer region including the rest of the structure. The sharp changes taking place in the inner region are described using a magnified scale. The governing equations and the boundary conditions are expressed in the new co-ordinate system and expanded as a function of the thickness parameter. The displacement vector expansion, the solution of these equations, is referred to as the inner solution. In the outer region, the equations of motion are expanded using the original co-ordinate system. The solution of these equations is called the outer solution. The basic idea underlying the method is that the domains of validity of the two expansions overlap and hence their matching provides the additional equations which allow all the constants in the expansions to be determined. Finally, the inner and outer expansions are combined to form a composite expansion valid both in the inner and outer regions. In the following section, the governing equations and the boundary conditions for Problem I (semi-infinite shell) are re-arranged according to the method. Thereafter the derivation of the wave receptances is exemplified by determining the solutions at the leading order. However, it proved necessary to extend the analysis to higher orders in order to obtain approximate expressions for all the wave receptances. The results of this further analysis are presented for both the semi-infinite and infinite shells in section 4.
3.1 GOVERNING EQUATIONS IN THE INNER REGION

Following [15] and [17], the inner solutions, denoted \( u' \) and \( w' \), are sought in the form

\[
    u' = \sum_{k=0}^{\infty} \lambda^k \tilde{u}_k, \quad w' = \sum_{k=0}^{\infty} \lambda^k \tilde{w}_k,
\]

where the expansion parameter is chosen to be:

\[
    \lambda = \beta^{1/4}
\]

Accordingly, modified sets of governing equations and boundary conditions are derived by applying the stretching transformation \( \xi = s/\lambda \) to equations (2) and (5). Introducing equations (14) in the modified equations of motion and collecting by terms of equal power of \( \lambda \) yields

\[
    \tilde{w}_k^{(6)} + 4m^4 \tilde{w}_k^* = -((\Omega^2 + 2\mu)\tilde{w}_k^{(4)} + \Omega^2 (1-\Omega^2)\tilde{w}_k^{(4)} + \tilde{w}_k^{(6)} + \Omega^2 \tilde{w}_k^{(6)}),
\]

\[
    \tilde{u}_k = -\frac{1}{\mu}((1-\Omega^2)\tilde{w}_k^{(4)} + \tilde{w}_k^{(4)} + \mu \tilde{w}_k^{(4)} + \Omega^2 \tilde{w}_k^{(4)} + \tilde{u}_k^{(4)}).
\]

where the convention that \( \tilde{u}_k = \tilde{w}_k = 0 \) for \( k < 0 \) is adopted. Acting similarly with the modified boundary conditions for problem (1) yields

\[
    (4m^4 \tilde{w}_k + \tilde{w}_k^{(4)} + \mu \tilde{w}_k^{(4)} + \tilde{w}_k^{(4)} - \tilde{u}_k^{(4)}) \mu^{-1} |_{\xi = 0} = \bar{n}_0 \delta(k,0),
\]

\[
    (\tilde{w}_k^{(4)} - \tilde{u}_k^{(4)}) |_{\xi = 0} = \bar{q}_0 \delta(k,0), \quad (\tilde{w}_k^{(4)} - \tilde{u}_k^{(4)}) |_{\xi = 0} = \bar{m}_0 \delta(k,0),
\]

where \( \delta \) is Kronecker's delta function (\( \delta(k,0) = 1 \) if \( k = 0, 0 \) otherwise) and

\[
    \bar{n}_0 = n_0, \quad \bar{q}_0 = \lambda^2 q_0, \quad \bar{m}_0 = \lambda^2 m_0.
\]

3.2 GOVERNING EQUATIONS IN THE OUTER REGION

In the outer region, the solutions for the longitudinal and the radial displacements are denoted \( u^0 \) and \( w^0 \). The solutions are sought in the form of

\[
    u^0 = \sum_{k=0}^{\infty} \lambda^k \hat{u}_k, \quad w^0 = \sum_{k=0}^{\infty} \lambda^k \hat{w}_k.
\]

The equations satisfied by the functions \( \hat{u}_k \) and \( \hat{w}_k \) are derived by inserting equations (19) into equations (2) and collecting by terms of equal power of \( \lambda \). This yields

\[
    4m^4 \hat{w}_k^* + \Omega^2 (1-\Omega^2)\hat{w}_k = -(\Omega^2 \hat{w}_k^{(4)} + \hat{w}_k^{(4)} + \hat{w}_k^{(6)} + (\Omega^2 + 2\mu)\hat{w}_k^{(4)} - \hat{w}_k^{(6)}),
\]

\[
    \hat{u}_k' = -((1-\Omega^2)\hat{w}_k + \Omega^2 \hat{w}_k^{(4)} + \hat{w}_k^{(4)} + \mu \hat{w}_k^{(4)} + \hat{w}_k^{(4)} - \hat{w}_k^{(4)}) / \mu,
\]

where the convention that \( \hat{u}_k = \hat{w}_k = 0 \) for \( k < 0 \) is adopted. It appears that up to and including the 3rd order, these equations reduce to the membrane equation. For higher order, a non-homogeneous term appears, leading to particular solutions.
3.3 LEADING ORDER SOLUTIONS

The derivation of the wave receptances is exemplified for the leading order \( k = 0 \), i.e. the leading terms of the inner and outer expansions are sought which satisfy both the boundary conditions and the matching equations. These solutions are then combined to form a composite expansion valid everywhere, thereby yielding the first approximation for the wave receptances.

At the leading order, i.e. \( k = 0 \), the differential equations governing the vibrations in the inner region and given by equations (16) read

\[
\ddot{\bar{\omega}}^{(0)}_0 + 4m^4 \dot{\bar{\omega}}_0 = 0, \quad \ddot{\bar{u}}_0 = 0,
\]

and their solutions are

\[
\bar{\omega} = C^1_{0,0} + C^1_{0,1} \xi + C^2_{0,0} e^{-m(1+i)\xi} + C^3_{0,0} e^{-m(1-i)\xi}, \quad \bar{u}_0 = A^1_{0,0}.
\]

Likewise at the leading order, equations (20) become

\[
4m^4 \dot{\bar{\omega}}^{(0)} + \Omega^2 (1 - \Omega^2) \bar{\omega}_0 = 0, \quad \dot{\bar{u}}_0 = -((1 - \Omega^2) \dot{\bar{\omega}}_0) / \mu
\]

and the outer solutions read

\[
\dot{\bar{\omega}}_0 = \mu \bar{\omega}_0 e^{-\Omega \sqrt{\Omega^2 - 1} \xi}, \quad \bar{u}_0 = -\frac{2m^2 \sqrt{\Omega^2 - 1}}{\mu \Omega} \bar{\omega}_0 e^{-\Omega \sqrt{\Omega^2 - 1} \xi} + b_0.
\]

It is seen that the propagation constants not satisfying \( \pi/2 < \arg(\kappa_p) \leq 3\pi/2 \) (\( p = 1, 2, 3 \)) are disregarded in the solutions. According to the method, the outer solution is valid everywhere outside the inner region. Assuming damping, the solutions should yield vanishing displacements when the axial variable \( s \) becomes infinite. This implies that the constant \( b_0 \) must be set to zero. The six remaining constants appearing in the expressions of the inner and outer solutions have to be determined partly from the boundary conditions and partly from the matching conditions.

From equations (17), the boundary conditions satisfied by the leading term of the inner solutions are

\[
(4m^4 \bar{\omega}_0 + \bar{\omega}_0^{(4)}) \mu^{-1} \bigg|_{\xi = 0} = \bar{n}_0, \quad \bar{\omega}_0^{(0)} |_{\xi = 0} = \bar{\eta}_0, \quad \bar{\omega}_0^{(1)} |_{\xi = 0} = \bar{m}_0.
\]

By solving the system obtained when equations (22) are introduced into equations (25), the following results ensue:

\[
C^1_{0,0} = \mu \bar{n}_0 / 4m^4, \quad C^2_{0,0} = \bar{\eta}_0 / 2m^2 (1 + i) + \bar{\eta}_0 / 4m^3, \quad C^3_{0,0} = i \bar{\eta}_0 / 2m^2 (1 + i) + \bar{\eta}_0 / 4m^3.
\]

The matching of the outer and inner solutions is performed in the overlapping region, where the inner region variable \( \xi \) tends towards infinity and \( s \) tends towards
zero. According to [17], the matching principle used to provide the missing equations is

\[
\lim_{\xi \to \infty} \tilde{w}_0(s, \lambda) = \lim_{\xi \to 0} \tilde{w}_0(s, \lambda), \quad \lim_{\xi \to \infty} \tilde{u}_0(s, \lambda) = \lim_{\xi \to 0} \tilde{u}_0(s, \lambda).
\]  
(27)

Since the exponential terms in the inner solution describe a standing decaying wave, they might be taken as zero in this region. Furthermore, expressing the limits of the inner solutions in terms of the axial variable \(s\), equations (27) become

\[
C_{0,0} + C_{0,1} \frac{s}{\lambda} = D_0, \quad A_{0,0} = -\frac{2m^2 \sqrt{\Omega^2 - 1}}{\mu \Omega} d_0.
\]  
(28)

Inasmuch as the limit of the outer expansion does not contain a term proportional to \(\lambda^{-1}\), the term \(\lambda^{-1}C_{0,1}\) in equation (28a) cannot be matched and the constant \(C_{0,1}\) should be taken to zero.

Finally, the analysis of the leading order is completed by deriving composite solutions which are valid both in the inner and the outer regions. According to Nayfeh [17], these composite expansions, denoted \(w^c\) and \(u^c\), are defined by

\[
w^c = w^o + w^i - \lim_{s \to 0} \tilde{w}_0(s; \lambda), \quad u^c = u^o + u^i - \lim_{s \to 0} \tilde{u}_0(s; \lambda)
\]  
(29)

Thus, using equations (18), (22), (24), (26), (28), (29) and (6) and the relationship \(D = \beta BR^2\), the composite solutions, expressed as a function of the original load distributions, read at the leading order

\[
w^c = \frac{\mu R N_0}{4m^2 B} e^{\xi_1 s} + \frac{R Q_0}{4m^2 \beta^{1/4} B} \left(e^{\xi_2 s} + e^{\xi_3 s}\right) - \frac{M_0}{4m^2 \beta^{1/2} B} ((1 - i) e^{\xi_2 s} + (1 + i) e^{\xi_3 s}),
\]  
(30)

where the propagation constants are given by

\[
\kappa_1 = -\frac{\Omega \sqrt{\Omega^2 - 1}}{2m^2}, \quad \kappa_{2,3} = -\frac{m(1 \pm i)}{\beta^{1/4}}.
\]  
(31a-b)

3.4 DISCUSSION

The first propagation constant given by equation (31a) describes a wave with a phase velocity close to the extensional phase speed in a beam at low frequencies and in a plate at high frequencies. This wave is referred to as the extensional wave in what follows. Likewise, the two remaining propagation constants given by equation (31b) tend toward the propagation constants of flexural wave in a flat plate at high frequencies. These waves are referred to as the flexural waves. From the expressions
of the propagation constants, it appears that the vibration field can be described by
two natural length scales, namely the \( \ell_1 = s/\beta^{1/4} \) and \( \ell_2 = s \).

Considering equations (30), it is seen that the analysis of the leading order does not
yield a complete wave receptance matrix. Indeed the receptances of the extensional
wave with respect to \( Q_0 \) and \( M_0 \) and the receptances of the flexural waves with respect
to \( N_0 \) do not appear at this stage. To derive the complete matrix, the analysis should be
pursued to higher orders.

Finally, a second conclusion that can be drawn from equations (30) is that the
approximate expressions become singular when the parameter \( m \) given by equation (3)
vanishes, i.e. when the non-dimensional frequency \( \Omega \) tends towards \( \Omega_0 = \sqrt{1 - \mu^2} \).
These singularities do not appear in the real solutions and the problem of their
treatment is tackled in section 5 by resorting again to perturbation theory.

4. THEORETICAL RESULTS

In this section, results issuing from the analysis of higher order solutions are given.
The details of the calculations are not presented since the algebra, though becoming
more cumbersome, is the same as in the preceding section. Yet a novelty appears for
orders higher than 2 owing to the fact that equations (16) and (20) are no longer
homogenous. The particular solutions of these non-homogeneous equations yield
terms of the form \( s^p e^{\kappa_p} \) in the composite solutions. For propagating waves, these
terms, referred to as secular terms in perturbation theory [19], become unbounded as
\( s \to \infty \). Since the exact solution is bounded, the non-uniformities introduced by these
terms should be removed. This can be achieved by using perturbation methods, e.g.
the Lindstedt-Poincaré technique [19]. This method is based on the fact that these
terms are nothing but the power series expansion of the propagation constants.
Realising that, it is then rather straightforward to make the solutions uniformly valid.

4.1 GREEN’S FUNCTIONS FOR THE RADIAL DISPLACEMENT

In order to obtain solutions for all the elements of the wave receptance matrices with
respect to the radial displacement, the calculations are performed up to the 4\(^{th}\) and to
the 2\( n \)th order in the case of the semi-infinite and infinite shells, respectively. Table 1
provides the first non-zero term of the wave receptances for both problems as well as
the order of magnitude of the first following term. From the knowledge of the wave
receptances and by retaining the leading term of the propagation constants, it is
straightforward to derive an approximate expression of the Green’s functions of the
shell. These functions, given in its general form by

\[
G^r_X(x,0) = \alpha^{\kappa_1}_p X e^{\kappa_1 x/R} + \alpha^{\kappa_2}_p X e^{\kappa_2 x/R} + \alpha^{\kappa_3}_p X e^{\kappa_3 x/R},
\]  

describe the radial displacement of semi-infinite or infinite shells with respect to an
axisymmetric load \( X \). The wave receptances \( \alpha^{\kappa_p}_p \) and the wave propagation constants
\( \kappa_p \) are given in Table 1 and by equations (31), respectively.
<table>
<thead>
<tr>
<th>Excitation</th>
<th>Wave type</th>
<th>Semi-infinite shell (Problem I)</th>
<th>Infinite shell (Problem II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-extensional wave $p=1$</td>
<td>$\alpha_{1,N_0}^{\omega/2} = \frac{\mu R}{4m^4 B} + O\left(\frac{\beta R}{B}\right)$</td>
<td>$\alpha_{1,N_0}^{\omega} = \frac{\mu R}{8m^4 B} + O\left(\frac{\beta R}{B}\right)$</td>
<td></td>
</tr>
<tr>
<td>$N_0$</td>
<td>Flexural waves</td>
<td>$\alpha_{2,N_0}^{\omega/2} = -\beta^{1/2} \frac{(1-i)\Theta \Psi R}{16m^6 B} + O\left(\beta^{3/4} \frac{R}{B}\right)$</td>
<td>$\alpha_{2,N_0}^{\omega} = -\frac{\mu R}{16m^6 B} + O\left(\beta^{1/2} \frac{R}{B}\right)$</td>
</tr>
<tr>
<td>$p=2$</td>
<td></td>
<td>$\alpha_{3,N_0}^{\omega/2} = -\beta^{1/2} \frac{(1+i)\Theta \Psi R}{16m^6 B} + O\left(\beta^{3/4} \frac{R}{B}\right)$</td>
<td>$\alpha_{3,N_0}^{\omega} = -\frac{\mu R}{16m^6 B} + O\left(\beta^{1/2} \frac{R}{B}\right)$</td>
</tr>
<tr>
<td>$p=3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>extensional wave $p=1$</td>
<td>$\alpha_{1,Q_0}^{\omega/2} = -\beta^{3/4} \frac{\mu \Theta \Psi R}{32m^{11} B} + O\left(\beta^{7/4} \frac{R}{B}\right)$</td>
<td>$\alpha_{1,Q_0}^{\omega} = \frac{-i \mu^2 \Omega R}{16m^6 B \sqrt{\Theta}} + O\left(\frac{\beta R}{B}\right)$</td>
<td></td>
</tr>
<tr>
<td>$Q_0$</td>
<td>Flexural waves</td>
<td>$\alpha_{2,Q_0}^{\omega/2} = \frac{R}{4m^3 \beta^{1/4} B} + O\left(\beta^{1/4} \frac{R}{B}\right)$</td>
<td>$\alpha_{2,Q_0}^{\omega} = \frac{(1+i)R}{16m^3 \beta^{1/4} B} + O\left(\beta^{1/4} \frac{R}{B}\right)$</td>
</tr>
<tr>
<td>$p=2$</td>
<td></td>
<td>$\alpha_{3,Q_0}^{\omega/2} = \frac{R}{4m^3 \beta^{1/4} B} + O\left(\beta^{1/4} \frac{R}{B}\right)$</td>
<td>$\alpha_{3,Q_0}^{\omega} = \frac{(1-i)R}{16m^3 \beta^{1/4} B} + O\left(\beta^{1/4} \frac{R}{B}\right)$</td>
</tr>
<tr>
<td>$p=3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>extensional wave $p=1$</td>
<td>$\alpha_{1,M_0}^{\omega/2} = \frac{\mu^2 \Theta^2}{16m^8 B} + O\left(\beta^{1/2} \frac{R}{B}\right)$</td>
<td>$\alpha_{1,M_0}^{\omega} = \frac{\mu^2 \Theta^2}{32m^8 B} + O\left(\frac{\beta R}{B}\right)$</td>
<td></td>
</tr>
<tr>
<td>$M_0$</td>
<td>Flexural waves</td>
<td>$\alpha_{2,M_0}^{\omega/2} = \frac{(1-i)}{4m^2 \beta^{1/2} B} + O\left(\frac{R}{B}\right)$</td>
<td>$\alpha_{2,M_0}^{\omega} = \frac{i}{8m^2 \beta^{1/2} B} + O\left(\frac{R}{B}\right)$</td>
</tr>
<tr>
<td>$p=2$</td>
<td></td>
<td>$\alpha_{3,M_0}^{\omega/2} = \frac{(1+i)}{4m^2 \beta^{1/2} B} + O\left(\frac{R}{B}\right)$</td>
<td>$\alpha_{3,M_0}^{\omega} = \frac{-i}{8m^2 \beta^{1/2} B} + O\left(\frac{R}{B}\right)$</td>
</tr>
<tr>
<td>$p=3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Excitation</td>
<td>Response</td>
<td>Semi-infinite shell (Problem I)</td>
<td>Infinite shell (Problem II)</td>
</tr>
<tr>
<td>------------</td>
<td>----------</td>
<td>---------------------------------</td>
<td>----------------------------</td>
</tr>
<tr>
<td>(\bar{N}_0)</td>
<td>(\rightarrow u)</td>
<td>(Y_{u,N_0}^{\infty} = \frac{\omega R}{B} \left( \frac{\sqrt{\Theta}}{2m^2 \Omega} + \frac{i \mu^2}{16m^9} \right))</td>
<td>(Y_{u,N_0}^{\infty} = \frac{\omega R}{B} \left( \frac{\sqrt{\Theta}}{4m^2 \Omega} - \frac{i \mu^2 \beta^{1/4}}{16m^5} \right))</td>
</tr>
<tr>
<td>(\bar{w})</td>
<td>(\rightarrow u)</td>
<td>(Y_{w,N_0}^{\infty} = \frac{\omega R}{B} \left( \frac{i \mu}{4m^4} - \beta^{1/2} \frac{\Theta \psi}{8m^6} (1 + \beta^{1/4} \frac{\Omega \sqrt{\Theta}}{2m^3}) \right))</td>
<td>(Y_{w,N_0}^{\infty} = 0)</td>
</tr>
<tr>
<td>(\bar{\theta})</td>
<td>(\rightarrow u)</td>
<td>(Y_{\theta,N_0}^{\infty} = \frac{\omega}{B} \left( \frac{\mu \Omega \sqrt{\Theta}}{8m^6} + \frac{i \beta^{1/4} \Theta \psi}{4m^3} \right))</td>
<td>(Y_{\theta,N_0}^{\infty} = \frac{\omega}{B} \left( \frac{\mu \beta \sqrt{\Theta}}{16m^5} + \frac{i \mu}{8m^3 \beta^{1/4}} \right))</td>
</tr>
<tr>
<td>(Q_0)</td>
<td>(\rightarrow u)</td>
<td>(Y_{w,Q_0}^{\infty} = Y_{w,N_0}^{\infty})</td>
<td>(Y_{u,Q_0}^{\infty} = 0)</td>
</tr>
<tr>
<td></td>
<td>(\rightarrow u)</td>
<td>(Y_{w,Q_0}^{\infty} = \frac{\omega R}{B} \left( \beta^{3/2} \frac{\Omega \sqrt{\Theta}^{3/2} \psi}{256m^{16}} + \frac{i}{2m^3 \beta^{1/4}} \right))</td>
<td>(Y_{w,Q_0}^{\infty} = \frac{\omega R}{B} \left( \frac{\mu^2 \Omega}{16m^6 \sqrt{\Theta}} + \frac{i}{8m^3 \beta^{1/4}} \right))</td>
</tr>
<tr>
<td></td>
<td>(\rightarrow u)</td>
<td>(Y_{\theta,Q_0}^{\infty} = -\frac{\omega}{B} \left( \beta^{3/4} \frac{\mu \Omega \sqrt{\Theta}^{3/2} \psi}{64m^{13}} + \frac{i}{2m^2 \beta^{1/4} - \frac{i \mu}{4m^4}} \right))</td>
<td>(Y_{\theta,Q_0}^{\infty} = 0)</td>
</tr>
<tr>
<td>(M_0)</td>
<td>(\rightarrow u)</td>
<td>(Y_{w,M_0}^{\infty} = Y_{w,N_0}^{\infty})</td>
<td>(Y_{w,M_0}^{\infty} = Y_{w,N_0}^{\infty})</td>
</tr>
<tr>
<td></td>
<td>(\rightarrow u)</td>
<td>(Y_{w,M_0}^{\infty} = Y_{w,Q_0}^{\infty})</td>
<td>(Y_{w,M_0}^{\infty} = 0)</td>
</tr>
<tr>
<td></td>
<td>(\rightarrow u)</td>
<td>(Y_{\theta,M_0}^{\infty} = \frac{\omega R}{B} \left( \frac{\mu^2 \Omega \sqrt{\Theta}}{32m^{10}} + \frac{i}{m \beta^{3/4}} \right))</td>
<td>(Y_{\theta,M_0}^{\infty} = \frac{\omega R}{B} \left( \frac{\mu^2 \Omega \sqrt{\Theta}}{64m^{10}} + \frac{i}{4m \beta^{3/4}} \right))</td>
</tr>
</tbody>
</table>

*The underlined terms represent the contributions of the extensional wave.*
In view of equation (31), three frequency domains can be distinguished depending on the nature of waves associated with the propagation constants. In domain I, \( \Omega < \sqrt{1-\mu^2} \), only the extensional wave is propagating while the flexural waves exhibit complex conjugate propagation constants characterising a standing decaying field. In domain II \( (\sqrt{1-\mu^2} < \Omega \leq 1) \), the extensional wave and one of the flexural waves are purely evanescent while the second flexural wave is propagating. Finally, in domain III \( (\Omega > 1) \), i.e. above the ring frequency, the extensional wave becomes propagating again and the flexural waves are both of the propagating and near-field types.

4.2 DIRECT MOBILITIES

Once the wave receptances are determined, the elements of the mobility matrix given by equation (12) are derived in accordance with equations (13) for both problems. The results are given in Table 2 where only the first non-zero terms for both the real and the imaginary part of the mobilities are presented. In order to be able to trace the role played by the different waves, the contributions of the extensional wave to the mobilities have been underlined; the remaining terms are due to the flexural waves. For the semi-infinite shell, it proved necessary to carry on the analysis up to the 7th order in order to obtain the real part of the input mobility \( Y_{w_o b}^{m/2} \) for low frequencies.

4.3 COMPARISON WITH "EXACT" RESULTS

Numerical solutions to Problems I and II have been calculated according to the methodology described in section 2 by means of MATLAB® over the non-dimensional frequency range \( \Omega = 0.1 \) to \( 1.5 \). The physical properties of the shell used in the calculations are listed in Table 3. These "exact" solutions will serve in what follows as references with which the results from the approximate expressions will be compared.

<table>
<thead>
<tr>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical properties of the shell</td>
</tr>
<tr>
<td>Density, ( \rho ) (kg/m(^3))</td>
</tr>
<tr>
<td>Young's modulus, ( E ) (N/m(^2))</td>
</tr>
<tr>
<td>Poisson's ratio, ( \mu )</td>
</tr>
<tr>
<td>Radius, ( R ) (m)</td>
</tr>
<tr>
<td>Thickness-to-radius ratio, ( h/R )</td>
</tr>
</tbody>
</table>

Figure 2 shows the real part of the input mobilities for a semi-infinite shell. The anti-resonance exhibited by the real part of \( Y_{w_o b}^{m/2} \) at \( \Omega = \sqrt{1-\mu} \) is due to the zero of the function \( \Psi \). The agreement between "exact" and approximate solutions is very good for the whole frequency range, apart from at the resonance which occurs at
\[ \Omega_0 = \sqrt{1 - \mu^2} \]. As mentioned in section 3, this is the result of the singularities contained in the approximate solutions of Tables 1 and 2 when the parameter \( m \) vanishes. As shown later in section 5, the singularities can be removed and it is seen in Figure 2 that the new forms of solution given in Table 4, together with the values from Table 5, are in good agreement with the exact calculations.

![Figure 2. Real parts of the input mobilities for the semi-infinite shell.](image)

The lines indicate results obtained numerically with Flügge's theory while the symbols represent the values predicted by the expressions given in Table 2. 

- (o), \( \text{Re}(Y_{u,K_0}) \)
- (*) \( \text{Re}(Y_{w,0}) \)
- (×) \( \text{Re}(Y_{\theta,Y_0}) \)

The "×" symbols at \( \Omega_0 = \sqrt{1 - \mu^2} \) are obtained using results from section 5.

The error on the input radial force mobility introduced by the approximate expression has been assessed for the semi-infinite shell. Figure 3 shows the error versus frequency curves for three thickness-to-radius ratios. The ratio \( h/R \) was changed by keeping the radius \( R \) equal to \( R = 0.5 \) m and varying accordingly the shell thickness. It is seen that the error is very small outside the resonance. Indeed, the error is lower than 1% outside the non-dimensional frequency region [0.92 - 0.97] for ratio \( h/R \) in the range [0.1 - 0.01]. Concerning the peak values at resonance, the simplified expressions derived in section 5 and given in Tables 4 and 5 yield a satisfying, even if poorer, agreement. These error values correspond to the modulus of \( Y_{w,\Omega_0} \) and can vary somehow between the results of Tables 1 and 2. A good estimate of the errors can be obtained by determining the first following term of the expansions for each quantities but this would involve very cumbersome algebra. As expected, it appears that the higher the thickness-to-radius ratio, the wider is the non-uniformity region where the solutions derived in section 3 break down. However, even for ratios \( h/R \) as high as 1/10, this region remains very narrow.
Figure 3. Error on the modulus of the input radial force mobility for the semi-infinite shell, \( |\mathbf{Y}_{\varphi z} / \mathbf{Y}_{0\varphi z}| \). \( \cdots \cdots \) \( \bar{h}/R = 1/10 \), \( \cdots \cdots \) \( \bar{h}/R = 1/50 \), \( \cdots \cdots \) \( \bar{h}/R = 1/100 \). The "•" symbols at \( \Omega_0 = \sqrt{1 - \mu^2} \) are obtained using results from section 5.

4.4 DISCUSSION

Considering the mobilities of the semi-infinite (Problem I) and the infinite shell (Problem II), respectively, it can be noticed that there is a factor 2 and 4 for the mobility parts due to the extensional wave and the flexural waves, respectively. Therefore the power flow injected by a bending moment, which in domain I is governed by the extensional wave, is doubled between Problems I and II whereas the actual velocity at the excitation point, governed by the flexural waves, is four times greater. It also appears that for infinite structures, the real part of the mobilities is always related to the receptance of the propagating waves only and that these receptances are purely imaginary. For semi-infinite structures, waves of near-field type can contribute to the real part of the mobilities as seen for instance in the case of the radial excitation. Indeed, for problem I, the receptances of the flexural waves are complex conjugate only for the first six orders but a difference occurs at \( k = 6 \). The contribution of the extensional wave to the injected power is due to the imaginary part of its receptance which arises at the order \( k = 11 \). Hence the input power is injected via the flexural waves, the contribution of the extensional wave being negligible. Nevertheless, the great difference between the real parts of the input radial force mobility in problems I and II reveals that this process of injecting power via the non-propagating wave is not efficient.

Another point of interest that has arisen in the course of this study is the discrepancies in the analytical solutions resulting from the use of different shell theories. Considering the boundary condition on the bending moment, the moment per unit length is obtained by integrating the moments generated by the normal stresses \( \sigma_z \) shown in Figure 4 about the \( \varphi \)-line and dividing by \( Rd\varphi \). This yields
\[ M_0 = \int_{-h/2}^{h/2} \sigma_z (1 + \frac{z}{R}) dz. \]  \[ (33) \]

Figure 4. Normal stresses and shell section.

Unlike in the case of the theory by Flügge [16], the term \( z/R \) is neglected in comparison to unity in the integrand in many theories, e.g. those by Love, Timoshenko, Donnell [14]. This simplification results in disregarding the contribution of the axial strain to moment resultant with respect to the effect of change of curvature of the mid-surface. Therefore, the boundary conditions are given for these theories by

\[ w\bigg|_{z=0} = m_0, \quad w\bigg|_{z=0} = q_0, \]  \[ (34) \]

wherein the axial displacement is absent. Using these boundary conditions yields the same approximate expressions for the Green’s functions and the mobilities but with the function \( \Psi \) defined as \( \Psi = -\frac{\mu \Omega^2}{4m^4} \) instead of \( \Psi = 1 - \frac{\mu \Omega^2}{4m^4} \). The effect of this change is significant below the ring frequency for Problem I since \( \Psi \) appears in the expression of the real part of some mobilities. Solutions to the infinite shell problem are unaffected by such considerations.

Finally, considering the asymptotic behaviour of the approximate expressions at low or high frequencies yields some well-known results. For instance, the input impedance to the total axial force \( N \), where \( N = 2\pi RN_0 \), acting on the shell extremity may be approximated at low frequencies (\( \Omega \ll 1 \)) by

\[ Z = 1 \frac{1}{Y_{u,2\pi R N_0}} = \frac{\Omega B \sqrt{1 - \frac{\mu^2}{\omega R}}}{2\pi R} = \rho S C_L, \]  \[ (35) \]

where \( S = 2\pi R h \) and \( C_L \) is the phase speed of extensional wave in a beam, \( C_L = \sqrt{E/\rho} \).

This result is similar to the input impedance of a beam of circular cross-section of radius \( R \) and thickness \( h \) excited at its end [3]. At high frequencies, the shell responds as a plate of thickness \( h \) submitted to a uniform axial load distribution \( N_0 \),

\[ Z = 1 \frac{1}{Y_{u, N_0}} = \frac{\Omega B}{\omega R} = \rho h C_{LJ}, \]  \[ (36) \]

where \( C_{LJ} \) is the phase speed of extensional wave in a plate, \( C_{LJ} = \sqrt{E/(\rho(1-\mu^2))} \).
Concerning the response to radial excitation at high frequencies \((\Omega \gg 1)\), the input impedance is given by

\[
Z = 1/Y_{w,Q_0}^{1/2} = \frac{2B(-1)^{3/4} \Omega^{3/2} \beta^{1/4}}{4^{3/4} \alpha R} = 1/2 \rho c_B (1 + i),
\]

where \(c_B\) is the phase speed of flexural wave in a plate, \(c_B = \sqrt{\omega^2/2\rho (1-\mu^2)}\).

One can recognize the impedance of a flat plate to a uniform load distribution \(Q_0\) located at its edge. In fact this impedance is the same as for a semi-infinite beam of unit width \([3]\), with a minor correction on the bending wave speed to take into account the obstruction of the cross-sectional contraction in the circumferential direction. Finally the fading of the leading term of the cross mobility \(Y_{w,Q_0}^{1/2}\) reflects the uncoupling occurring at high frequencies between axial and radial displacements.

5. TREATMENT OF THE SINGULARITIES AT \(\Omega_0 = \sqrt{1-\mu^2}\)

As mentioned in section 3, the previous solutions become singular when \(m = 0\), i.e. when \(\Omega = \Omega_0\), and it can be seen in Tables 1 and 2 that the higher the order of the expansions terms, the greater the singularities. However, the singularities are not part of the actual solutions since exact calculations show that the resonance peaks exhibited by the shell vibrations in the vicinity of \(\Omega_0\) are of finite magnitude. The frequency region where the approximate solutions break down is referred to as the non-uniformity region. In spite of the narrowness of this region, the treatment of the singularities is of major importance due to the high vibration level undergone by the structure in this frequency range. Furthermore, for infinite shells excited by radial point force, Borgiotti et al. \([8]\) have pointed out that the axisymmetric mode carries most of the power in this frequency region, the higher order circumferential modes being dominant at lower and higher frequencies.

5.1 WAVE RECEPTANCES FOR THE RADIAL DISPLACEMENT AND DIRECT MOBILITIES

To briefly outline the method used, the problem is re-formulated using a row of variable changes. Then a perturbation analysis is performed with the thickness parameter as expansion parameter. Retaining only the leading term of the expansions yields forms of solutions which are considerably simplified with regard to the initial problem. For the detailed analysis of the treatment of the singularities, the reader is referred to Appendix A. The present section is devoted to the presentation of the results and their discussion.

As shown in Appendix A, the wave receptances and the direct mobilities can be represented by the formulae given in Table 4. These formulae contain the functions \(\tilde{\alpha}_{\tilde{p},\tilde{X}}^c, \tilde{\tilde{y}}_{\tilde{u},\tilde{X}}^c, \tilde{\tilde{y}}_{\tilde{u},\tilde{X}}^c\) and \(\tilde{\tilde{y}}_{\tilde{q},\tilde{X}}^c\) (with \(\tilde{X} = \tilde{n}_0, \tilde{q}_0, \tilde{m}_0\) and \(p = 1, 2, 3\)) which are solely dependent upon the variable \(\gamma\) defined by
\[ \gamma = \frac{\Omega^2 - \Omega_0^2}{\beta^{1/3} \mu^{4/3} \Omega_0^{4/3}}. \] (38)

These functions should be computed, as they have no simple analytical expressions. For instance, the variations of the function \( \tilde{\gamma}_{\tilde{u}, \tilde{a}_0}^{m/2} \) versus the variable \( \gamma \) are plotted in Figure 5. This curve represents the input axial force mobility made dimensionless by normalisation with respect to \( \mu^{1/3} R \omega / (\beta^{1/3} \Omega_0^{5/3} B) \). In Figure 5 are also shown mobilities values after identical normalisation for the shell described in Table 3 and calculated with both Flügge’s theory and the expression obtained in section 4 (see Table 2). It appears that \( \tilde{\gamma}_{\tilde{u}, \tilde{a}_0}^{m/2} \) agrees nicely with the Flügge solution while the mobility given by Table 2 becomes singular at \( \gamma = 0 \).

![Figure 5. Input axial force mobility normalised with respect to \( \mu^{1/3} R \omega / (\beta^{1/3} \Omega_0^{5/3} B) \). The “o” symbols and the dash-dot line, (---), represent the modulus of the “exact” solution from Flügge’s theory and of the predicted solution given in Table 2, respectively. (---) \( |\tilde{\gamma}_{\tilde{u}, \tilde{a}_0}^{m/2}| \), (---) Re(\( \tilde{\gamma}_{\tilde{u}, \tilde{a}_0}^{m/2} \)), (---) Im(\( \tilde{\gamma}_{\tilde{u}, \tilde{a}_0}^{m/2} \)).](image)

Fortunately, owing to the sharpness of the resonance peaks and to their dependence upon \( \gamma \) only, there is no real need for an analytical description of \( \tilde{\omega}_{\tilde{p}, \tilde{x}}^e \), \( \tilde{\gamma}_{\tilde{u}, \tilde{x}}^e \), \( \tilde{\gamma}_{\tilde{w}, \tilde{x}}^e \) and \( \tilde{\gamma}_{\tilde{\theta}, \tilde{x}}^e \). Instead, they can be described in terms of their most interesting characteristics. These characteristics, illustrated in Figure 5 for the function \( |\tilde{\gamma}_{\tilde{u}, \tilde{a}_0}^{m/2}| \), are denoted \( \gamma_M \), \( \gamma_d \) and \( \Delta \gamma \). They correspond to the peak values, the \( \gamma \)-values at which these peaks occur and the bandwidth of the resonance curve at \( \gamma_M / \sqrt{2} \), respectively. They have been computed for all the functions and tabulated in Table 5 for the semi-infinite shell and in Table 6 for the infinite shell.
TABLE 4

Wave receptances and direct mobilities for semi-infinite and infinite shells at $\Omega_0 = \sqrt{1 - \mu^2}$

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Response</th>
<th>Wave receptance ($p=1, 2, 3$)</th>
<th>Point mobility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0$</td>
<td>$u$</td>
<td>$\alpha_{p,N0} = \frac{R}{\beta^{1/3} \mu^{1/3} (1 - \mu^2)^{1/3} B} \bar{\alpha}_{p,\theta0}$</td>
<td>$Y_{u,N0} = \frac{\omega R \mu^{1/3}}{\beta^{1/6} (1 - \mu^2)^{5/6} B} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
<tr>
<td></td>
<td>$w$</td>
<td>$\bar{\alpha}<em>{p,N0} = \frac{R}{\beta^{-1/3} \mu^{-1/3} (1 - \mu^2)^{-1/3} B} \bar{\alpha}</em>{p,\theta0}$</td>
<td>$Y_{w,N0} = \frac{\omega}{\beta^{1/3} \mu^{1/3} (1 - \mu^2)^{-1/3} B} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>$\bar{\alpha}_{p,N0}$</td>
<td>$Y_{\theta,N0} = \frac{\omega}{\beta^{1/2} (1 - \mu^2)^{1/2} B} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>$u$</td>
<td>$\alpha_{p,Q0} = \frac{R}{\mu^{1/2} (1 - \mu^2)^{1/2} B} \bar{\alpha}_{p,\theta0}$</td>
<td>$Y_{w,Q0} = \frac{\omega R}{\mu \beta^{1/2} (1 - \mu^2)^{1/2} B} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
<tr>
<td></td>
<td>$w$</td>
<td>$\bar{\alpha}_{p,Q0}$</td>
<td>$Y_{w,Q0} = \frac{\omega}{\mu B^{1/2} (1 - \mu^2)^{1/2} B} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>$\bar{\alpha}_{p,Q0}$</td>
<td>$Y_{\theta,Q0} = \frac{\omega}{\mu^{2/3} B^{2/3} (1 - \mu^2)^{1/3} B} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
<tr>
<td>$M_0$</td>
<td>$u$</td>
<td>$\alpha_{p,M0} = \frac{1}{\beta^{2/3} \mu^{2/3} (1 - \mu^2)^{1/3} B} \bar{\alpha}_{p,\theta0}$</td>
<td>$Y_{u,M0} = Y_{\theta,M0}$</td>
</tr>
<tr>
<td></td>
<td>$w$</td>
<td>$\bar{\alpha}_{p,M0}$</td>
<td>$Y_{w,M0} = Y_{\theta,M0}$</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>$\bar{\alpha}_{p,M0}$</td>
<td>$Y_{\theta,M0} = \frac{\omega}{\beta^{5/6} \mu^{1/3} (1 - \mu^2)^{1/6} B R} \bar{\gamma}_{\theta,\theta0}$</td>
</tr>
</tbody>
</table>

19
Regarding the mobilities, characteristics are given for both the modulus and the real part of the functions since they are needed to assess the maximum displacement and the power input, respectively. On the other hand, for the wave receptances, only the characteristics concerning the modulus of the receptance associated with the propagating wave are reported since they are sufficient to assess the vibration level in the far-field.

**TABLE 5**  
*Characteristics of the resonance curves for the semi-infinite shell*  
(a) Modulus

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{y}^c_{\theta,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-0.2</td>
<td>-0.3</td>
<td>-0.3</td>
<td>-0.3</td>
<td>-0.2</td>
<td>-0.2</td>
<td>-0.4</td>
<td>-0.2</td>
</tr>
<tr>
<td>$\gamma_{\infty}$</td>
<td>2.2</td>
<td>2.3</td>
<td>1.2</td>
<td>3.4</td>
<td>2.2</td>
<td>2.1</td>
<td>0.8</td>
<td>1.0</td>
</tr>
<tr>
<td>$\Delta \gamma$</td>
<td>1.0</td>
<td>1.0</td>
<td>0.9</td>
<td>1.0</td>
<td>1.0</td>
<td>1.6</td>
<td>1.8</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(b) Real parts

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{y}^c_{\theta,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-0.4</td>
<td>-0.3</td>
<td>-0.1</td>
<td>-0.1</td>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>$\gamma_{\infty}$</td>
<td>1.9</td>
<td>2.3</td>
<td>-1.1</td>
<td>3.2</td>
<td>-1.7</td>
<td>1.1</td>
</tr>
<tr>
<td>$\Delta \gamma$</td>
<td>0.8</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
<td>0.9</td>
<td>2.9</td>
</tr>
</tbody>
</table>

**TABLE 6**  
*Characteristics of the resonance curves for the infinite shell.*  
(a) Modulus

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{y}^c_{\theta,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-0.9</td>
<td>/</td>
<td>-1.0</td>
<td>-1.0</td>
<td>/</td>
<td>-0.8</td>
<td>-1.2</td>
<td>-0.6</td>
</tr>
<tr>
<td>$\gamma_{\infty}$</td>
<td>0.4</td>
<td>0</td>
<td>0.2</td>
<td>0.2</td>
<td>0</td>
<td>0.3</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$\Delta \gamma$</td>
<td>6.0</td>
<td>/</td>
<td>4.8</td>
<td>4.7</td>
<td>/</td>
<td>9.0</td>
<td>3.4</td>
<td>3.4</td>
</tr>
</tbody>
</table>

(b) Real parts

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{y}^c_{\theta,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
<th>$\tilde{y}^c_{\phi,\phi_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>-1.9</td>
<td>/</td>
<td>-0.6</td>
<td>-0.6</td>
<td>/</td>
</tr>
<tr>
<td>$\gamma_{\infty}$</td>
<td>0.28</td>
<td>0</td>
<td>0.2</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta \gamma$</td>
<td>5.7</td>
<td>/</td>
<td>3.4</td>
<td>3.4</td>
<td>/</td>
</tr>
</tbody>
</table>

The analysis carried out in Appendix A also yields simplified forms for the propagation constants. They read

20
\[
\kappa_p = \frac{\mu^{1/3} \Omega_0^{1/3}}{\beta^{1/6}} \bar{\kappa}_p,
\]

where the constants \( \bar{\kappa}_p \) are solutions of the dispersion relation

\[
\bar{\kappa}_p^6 - \gamma \bar{\kappa}_p^2 + 1 = 0.
\]

Figure 6 shows computed results for the propagation constants obtained with equation (40) (marked with the symbol "+"). The solid lines indicate the solutions of the "exact" dispersion relation from Flügge theory after normalisation with respect to \( \mu^{1/3} \Omega_0^{1/3} / \beta^{1/6} \). It is seen that the roots are either purely real or arise in complex conjugate pairs, the latter becoming real at \( \gamma = 3/\sqrt{4} \).

![Real and imaginary parts of \( \kappa_p \) normalised with respect to \( \mu^{1/3} \Omega_0^{1/3} / \beta^{1/6} \).](image)

Figure 6. Real and imaginary parts of the propagation constants normalised with respect to \( \mu^{1/3} \Omega_0^{1/3} / \beta^{1/6} \). (—) exact solutions calculated with Flügge theory, (+) solutions of equation (40).

As shown in Figure 5, the non-dimensional frequency associated to the peak value is slightly shifted from \( \Omega_0 \). Using the values given in Tables 5 and 6 for \( \gamma_0 \) together with equation (38), it is straightforward to obtain a better approximation of the "exact" resonance frequency \( \Omega_e \). This yields

\[
\Omega_e = \Omega_0 + \gamma_0 \beta^{1/3} \mu^{4/3} \Omega_0^{1/3} / 2.
\]

Finally, sharpness of resonance peaks can also be estimated using the values for \( \Delta \gamma \) provided by Tables 5 and 6. The current parameter used as a measure of sharpness is the quality factor \( Q \) defined by
\[ Q = \frac{\Omega_e}{\Delta \Omega_{1/\sqrt{2}}} , \]  

(42)

where \( \Delta \Omega_{1/\sqrt{2}} \) is the \( 1/\sqrt{2} \) -value bandwidth. Using equations (38), (41) and (42), the quality factor can be approximated by

\[ Q = 2(1-\mu^2)^{1/3} / (\beta^{1/3} \mu^{4/3} \Delta Y) . \]  

(43)

As expected, equation (43) shows that the lower the shell thickness-to-radius ratio, the higher is the quality factor, viz. the sharper is the resonance peak.

5.2 DISCUSSION

The analysis of the non-uniformity region reveals that, at the resonance, the vibration field is described by only one length scale \( (\ell_3 = s / \beta^{1/6}) \), as shown by equation (39). Thus, the axisymmetric radial-axial vibrations of shells involve three waves of the same nature in the vicinity of \( \Omega_0 = \sqrt{1-\mu^2} \). Furthermore, by comparing the values given in Tables 5 and 6, it appears that the magnitude and the quality factor of the resonance peaks for the semi-infinite shell are higher by a factor of 5 or more compared with those of the infinite shell.

Assuming damping in the shell material, its effect can be represented by introducing a complex modulus of elasticity, i.e. a complex non-dimensional frequency given by

\[ \Omega^{*} = (1 - \frac{\eta}{2})\Omega , \]  

(44)

where \( \eta \) is the loss factor. Replacing \( \Omega \) by \( \Omega^{*} \) in the expression of the input mobility with respect to the axial force given in Table 2, i.e. \( Y_{\omega, N_0} \), the quality factor associated with the loss factor is approximately given by

\[ Q = \frac{1}{\sqrt{3} \eta} . \]  

(45)

Equating equation (45) to (43), it appears that, if the loss factor is greater than the limit value given by

\[ \eta_a = \frac{\beta^{1/3} \mu^{4/3} \Delta Y}{2\sqrt{3}(1-\mu^2)^{1/3}} , \]  

(46)

the resonance is damping-controlled. Below this value, it is the flexural rigidity which bounds the peak values. In practice, assuming a Poisson’s ratio of 0.3 and by taking \( \Delta Y = 2 \) in view of Tables 5 and 6, the values of \( \eta_a \) are contained in the interval \( 0.0025 \leq \eta_a \leq 0.012 \) for values of \( h/R \) varying in the range 1/10 \( \leq h/R \leq 1/100 \).
6. DERIVED RESULTS

6.1 REFLECTION COEFFICIENTS OF THE SHELL-BORNE WAVES AT A FREE END

Consider a semi-infinite shell lying in the half-space \( x \geq 0 \) and a set of three negative-going waves of amplitude vector \( \mathbf{C}^- = \begin{bmatrix} C_1^- & C_2^- & C_3^- \end{bmatrix}^T \) which is incident upon the free end of the shell at \( x = 0 \). These waves will give rise to reflected waves of amplitude vector \( \mathbf{C}^+ = \begin{bmatrix} C_1^+ & C_2^+ & C_3^+ \end{bmatrix}^T \) and the radial displacement \( w \) of the shell is given by

\[
w = C_1^+ e^{k_1 x} + C_2^+ e^{k_2 x} + C_3^+ e^{k_3 x} + C_1^- e^{-k_1 x} + C_2^- e^{-k_2 x} + C_3^- e^{-k_3 x}.
\] (47)

The incident and reflected amplitude vectors are related by the reflection matrix \( \mathbf{r} \) for free termination defined by

\[
\mathbf{C}^+ = \mathbf{r} \cdot \mathbf{C}^-.
\] (48)

The coefficient of the reflection matrix for the different types of waves can be derived from the knowledge of the wave receptances for the semi-infinite case. Consider a negative-going wave of type \( p \) incident upon the free end of the shell. This wave generates forces \( (N_0^*, Q_0^*, M_0^*) \) at \( x = 0 \). The amplitude of the reflected waves can then be assessed by using the Green’s function of the semi-infinite shell together with a load \( (-N_0^*, -Q_0^*, -M_0^*) \). Hence the vanishing of the loads at the extremity, i.e. the free end condition, is satisfied.

Applying this method to both the extensional and flexural waves yields the complete reflection matrix which reads at the leading order

\[
\mathbf{r} = \begin{bmatrix}
-1 & -(1-i)\beta \frac{\mu \Omega^2 \Theta \Psi}{8m^8} & -(1+i)\beta \frac{\mu \Omega^2 \Theta \Psi}{8m^8} \\
-i\beta^{3/4} \frac{\Omega \Theta^{3/2} \Psi}{4\mu m^2} & -i & 1+i \\
-i\beta^{3/4} \frac{\Omega \Theta^{3/2} \Psi}{4\mu m^2} & 1-i & i
\end{bmatrix}.
\] (49)

where \( 4m^4 = 1 - \mu^2 - \Omega^2 \), \( \Theta = 1 - \Omega^2 \) and \( \Psi = 1 - \mu \Omega^2 / 4m^4 \).

It appears that the coupling between the extensional and flexural waves, given by the element \( r_{12}, r_{13}, r_{24} \) and \( r_{34} \), is very weak. Determined for the axial displacement, the element \( r_{11} \) is equal to 1, i.e. similar to the reflection coefficients for longitudinal wave in a rod [3]. The results derived here show that an extensional wave incident upon the free end of a shell does not yield radial displacement. Considering the reflection of flexural waves and their coupling, the results are the same as for the flexural waves in a beam with free termination [20].
6.2 PROBLEM III: THE FINITE FREE-FREE SHELL

Consider a free-free cylindrical shell of length $L$ excited at $x = x_0$ by an axisymmetric load $X$, as shown in Figure 7. The mechanical response of the shell at frequencies below $\Omega_0 = \sqrt{1 - \mu^2}$, referred to as domain I, is deduced from the values of the wave receptances derived in Problems I and II (see Table 1) and from the reflection matrix (see equation (49)). In domain I, only the extensional wave is propagating and the standing decaying field associated with the flexural waves vanishes much faster than the near-field of bending wave in a beam. It can be noted that the lower the frequency, the greater is the decay. If the excitation is located outside the edge-layer described in section 3, only the extensional wave reaches the edge. Furthermore, since it has been shown that the coupling between extensional and flexural waves is very small, the influence of the flexural waves can be assumed to be limited to a narrow region containing the excitation point and thus be disregarded at the shell extremities.

![Figure 7. The finite free-free shell](image)

Following Cremer and Heckl [3], it is first assumed that the shell is infinite. The radial displacement produced by the load $X$ and associated with the extensional wave is given by equation (32) and read

$$w(x) = \begin{cases} 
\alpha_{1,\infty} e^{\kappa_1 (x-x_0)/R} X & \text{for } x \geq x_0 \\
\alpha_{1,\infty} e^{-\kappa_1 (x-x_0)/R} X & \text{for } x < x_0
\end{cases},$$

(50)

where $\kappa_1$ is given by equation (31a) and the wave receptances $\alpha_{1,\infty}$ by Table 1. Owing to the finite length of the shell, these waves will undergo an infinite series of reflections at the shell extremities characterised by a reflection coefficient of $r_{11} = -1$. It might be shown that the total vibration field generated by the extensional wave can be expressed as a sum of a geometric series, the characteristics of which are

$$U_0 = e^{\kappa_1 x/R} - e^{\kappa_1 (2L-x)/R}, \quad q = e^{2\kappa_1 L/R}, \quad S(x) = \frac{e^{\kappa_1 x/R} - e^{\kappa_1 (2L-x)/R}}{1-e^{2\kappa_1 L/R}},$$

(51)

where $U_0$, $q$ and $S(x)$ are the first term, the common ratio and the sum of the geometric series, respectively. Finally, it might be shown that the radial vibrations of the shell are described by
\[ w(x) = 2i \alpha_{1,x} X \left[ \frac{\sin(i \xi_1 x_0 / R)}{1 - e^{2 \xi_1 L / R}} e^{\xi_1 x / R} - e^{\xi_1 (2L-x) / R} \right] + H(x_0 - x) \sin(i \xi_1 (x - x_0) / R) + \alpha_{2,x} X e^{\kappa_2 |x-x_0| / R} + \alpha_{3,x} X e^{\kappa_3 |x-x_0| / R} . \] (52)

where \( H(x) \) is the Heaviside step function, i.e. is 0 for \( x < 0 \) and 1 otherwise. The propagation constants and the wave receptances can be approximated by the expressions given in equations (31) and in Table 2, respectively. Finally, if the excitation is located at the edge, i.e. \( x_0 = 0 \), the radial vibrations are given by

\[ w(x) = \alpha_{1,x} X \frac{e^{\xi_1 x / R} - e^{\xi_1 (2L-x) / R}}{1 - e^{2 \xi_1 L / R}} + \alpha_{2,x} X e^{\kappa_2 x / R} + \alpha_{3,x} X e^{\kappa_3 x / R} . \] (53)

Equations (52) and (53) shows that the shell response becomes infinite at the frequencies for which the denominator of the function \( S(x) \) vanishes. These frequencies correspond to the principle of wave cycle closure described in [3], i.e. the extensional wave closes on itself with the same phase after being reflected at both ends. Using equation (31a), these “natural” frequencies are the solutions of the following equation:

\[ \frac{\Omega \sqrt{1 - \Omega^2}}{\sqrt{1 - \mu^2 - \Omega^2}} = \frac{n \pi R}{L} \quad \text{with} \quad n = 1, 2, 3... \] (54)

Finally, the derived Green’s functions derived for the finite free-free cylindrical shell are compared to computed results obtained with the spectral finite element formulation developed by Finnveden [21] for straight fluid filled pipes. The comparison is made with a shell of length \( L = 20 \) meters excited by a harmonic radial load of magnitude \( Q_0 = 1 \) N/m located at \( x = L/4 \). The others physical properties of the shell are listed in Table 3 and the non-dimensional frequency of the excitation is \( \Omega = 0.5 \). As shown in Figure 8, the radial displacement predicted by equation (52) is in good agreement with the numerical results obtained with the spectral finite element formulation.

7. CONCLUSIONS

Perturbation techniques are applied to the analysis of the axisymmetric mode of vibration of cylindrical shells. It is shown that they yield accurate approximate expressions for the Green’s functions of the semi-infinite or infinite shells with respect to force and moment excitation. Pursuing the analysis, it was also possible to derive closed-form formulae for the mobilities of the shell with respect to axial and radial forces and a bending moment. The analysis reveals that the response of the shell at the excitation point could be determined by the flexural waves while the power flow injected within the structure is governed by the extensional wave.
Figure 8. Shell response to a harmonic radial load applied at $x = 5$ m. The non-dimensional frequency of the excitation is $\Omega = 0.5$ and the load magnitude is $Q_0 = 1$ N/m. The solid line represents the modulus of the radial displacement calculated numerically with the spectral finite element formulation [21]. The values predicted by equation (52) are marked with "o" symbols.

The present study also shows that differences in the closed-form expressions arise in the case of the semi-infinite shell when different thin-walled shell theories are used to derive the real part of the input radial force mobility below the ring frequency. Regarding the infinite shell, the approximate expressions remain the same, whatever the thin shell theory used for the derivation. Finally, the derived Green’s functions are used to determine approximate expressions of the reflection coefficients of the shellborne waves at the free end and to study the axisymmetric vibrations of a finite free-free shell.

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REFERENCES


**APPENDIX A: FORM OF THE SOLUTIONS IN THE NON-UNIFORMITY REGION**

The singularities are dealt with by searching for approximate solutions valid in the region of non-uniformity. In perturbation theory [18], this problem is tackled by introducing a detuning parameter in the governing equations. In the present case, this parameter is taken as

\[ \sigma = 4m^4 / \beta^v . \]  

(A1)
where the constant $\nu$ is a measure of the extent of the non-uniformity region. Additionally to equation (A1), a stretching transformation defined by

$$\xi = s / \beta^\varepsilon,$$

is introduced to derive a new variant of equation (2a). This yields

$$(1 - \beta)\beta^{-6\varepsilon}w^{(6)} + \beta^{1-4\varepsilon}(\Omega^2 + 2\mu)w^{(4)} + \sigma\beta^{-2\varepsilon+\nu}w^{\nu} + \Omega^2(1 - \Omega^2)w + O(\beta) = 0.$$  \hspace{1cm} \text{(A3)}$$

The constants $\varepsilon$ and $\nu$ are determined by requiring that the first, third and fourth terms of equation (A3) are now of the same order of magnitude. It ensues

$$\varepsilon = 1/6, \quad \nu = 1/3.$$  \hspace{1cm} \text{(A4)}$$

The following notation is then defined

$$\tau = \beta^{1/6}, \quad \sigma = 4m^4 / \beta^{1/3},$$  \hspace{1cm} \text{(A5)}$$

together with the variables changes

$$\tilde{u} = -\Omega_0^{1/3}u / (\mu \tau^{2/3}), \quad \tilde{w} = w, \quad \xi = (\mu \Omega_0)^{1/3}s / \tau.$$  \hspace{1cm} \text{(A6)}$$

Introducing equations (A5) and (A6) into the governing equations (2) and omitting terms of order of magnitude $O(\tau^2)$ or lower in the writing yields

$$w^{(6)} - \gamma w^{\nu} + \tilde{w} + O(\tau^2) = 0, \quad \tilde{w}^{(4)} - \gamma w^{(4)} + O(\tau^2) = 0.$$  \hspace{1cm} \text{(A7)}$$

where the parameter $\gamma$ is defined by

$$\gamma = -\sigma / (\mu \Omega_0)^{4/3}.$$  \hspace{1cm} \text{(A8)}$$

Similarly, when re-arranged with respect to equations (A5), (A6) and (A8), the boundary conditions (5) read,

$$\left(\tilde{w}^{(4)} - \gamma w^{(4)} + O(\tau^2)\right)_{\xi=0} = \tilde{n}_0, \quad \left(\tilde{w}^{\nu} + O(\tau^2)\right)_{\xi=0} = \tilde{q}_0,$$

$$\left(\tilde{w}^{(4)\nu} + O(\tau^2)\right)_{\xi=0} = \tilde{m}_0,$$  \hspace{1cm} \text{(A9)}$$

where

$$\tilde{n}_0 = \frac{n_0}{\tau^2 \mu \Omega_0^{2/3}}, \quad \tilde{q}_0 = \frac{\tau^2 q_0}{\mu \Omega_0}, \quad \tilde{m}_0 = \frac{\tau^2 m_0}{\mu \Omega_0^{2/3}}.$$  \hspace{1cm} \text{(A10)}$$

Performing a perturbation analysis, solutions are sought in the form of straightforward expansions given by
\[
\begin{bmatrix}
\tilde{u} \\
\tilde{w}
\end{bmatrix} = \sum_{k=0}^{\infty} \tilde{w}_k
\]
(A11)

Inserting equation (A11) into equation (A7) and collecting by terms of equal power of \( \tau \), the leading term of the solution can be taken as

\[
\tilde{w}_0 = \sum_{p=1}^{3} \tilde{C}_p e^{\tilde{\tau} \varphi} , \quad \tilde{u}_0 = \sum_{p=1}^{3} \tilde{C}_p e^{\tilde{\tau} \varphi}
\]
(A12)

where the constants \( \tilde{\kappa}_p \) are solutions of the dispersion relation

\[
\tilde{\kappa}_p^6 - \gamma \tilde{\kappa}_p^2 + 1 = 0.
\]
(A13)

Thus these constants have no simple analytical expressions except when \( \gamma = 0 \) where they read \((\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3) = (-i, e^{5\pi i/6}, e^{-5\pi i/6})\).

The amplitudes of the waves \( \tilde{C}_p \) in equation (A12) are related to the loads \( \tilde{X} \) given by equation (A10) by the wave receptance matrix \( \tilde{\alpha}^c \) defined by

\[
\begin{bmatrix}
\tilde{C}_1 \\
\tilde{C}_2 \\
\tilde{C}_3
\end{bmatrix} = \tilde{\alpha}^c \begin{bmatrix}
\tilde{n}_0 \\
\tilde{q}_0 \\
\tilde{m}_0
\end{bmatrix},
\]
(A14)

where the superscript \( c \) indicates if reference is made to Problem I or II. The elements of this matrix, \( \tilde{\alpha}^c_{p,\varphi} \), are determined by inverting the system obtained when equations (A12) are introduced in the boundary condition (A9). The expressions of these coefficients are very cumbersome since they contain the roots of equation (A13). Using equations (A12), (A11), (A10), (A6), (A14) and (6), the receptances of the waves are expressed in terms of the original variables and read

\[
\begin{align*}
\tilde{\alpha}^c_{p,N_0} &= \frac{R}{\varphi^{1/3} \mu \Omega_0^{4/3} B} \tilde{\alpha}^c_{p,N_0}, \\
\tilde{\alpha}^c_{p,Q_0} &= \frac{R}{\varphi^{1/2} \mu \Omega_0 B} \tilde{\alpha}^c_{p,Q_0}, \\
\tilde{\alpha}^c_{p,M_0} &= \frac{1}{\varphi^{2/3} \mu \Omega_0^2 B} \tilde{\alpha}^c_{p,M_0}.
\end{align*}
\]
(A15)

Likewise, using equations (13) and (A15) for the derivation of the mobilities yields the following expressions

\[
\begin{align*}
Y^c_{w,N_0} &= \frac{\omega R \mu^{1/3}}{\beta \varphi^{1/3} \Omega_0^{5/3} B} \tilde{Y}^c_{w\varphi,0}, \\
Y^c_{w,Q_0} &= \frac{\omega R}{\varphi^{1/3} \mu \Omega_0^{4/3} B} \tilde{Y}^c_{w\varphi,0}, \\
Y^c_{w,M_0} &= \frac{\omega}{\varphi^{1/2} \Omega_0 B} \tilde{Y}^c_{w\varphi,0}, \\
Y^c_{\theta,N_0} &= \frac{\omega R}{\mu \beta^{1/2} \Omega_0 B} \tilde{Y}^c_{\theta\varphi,0}, \\
Y^c_{\theta,Q_0} &= \frac{\omega}{\mu \varphi^{1/3} \beta^{2/3} \Omega_0^{2/3} B} \tilde{Y}^c_{\theta\varphi,0}, \\
Y^c_{\theta,M_0} &= \frac{\omega}{\varphi^{5/6} \mu^{1/3} \Omega_0^{1/3} B R} \tilde{Y}^c_{\theta\varphi,0}
\end{align*}
\]
(A15)

where the functions \( \tilde{Y}^c_{w,\varphi,\dot{\varphi}}, \tilde{Y}^c_{w,\dot{\varphi}}, \tilde{Y}^c_{\theta,\varphi,\dot{\varphi}} \) and \( \tilde{Y}^c_{\theta,\dot{\varphi}} \) are given by
\[
\tilde{Y}_c \tilde{u}_\tilde{x} = i \sum_{p=1}^{3} \tilde{X}_p \tilde{X}_p, \\
\tilde{Y}_c \tilde{w}_\tilde{x} = i \sum_{p=1}^{3} \tilde{X}_p \tilde{X}_p, \\
\tilde{Y}_c \tilde{\theta}_\tilde{x} = i \sum_{p=1}^{3} \tilde{X}_p \tilde{X}_p \tilde{\theta}_p.
\]  
(A16)

Considering equations (A13), (A9) (A12) and (A16), it is seen that the coefficients \( \tilde{X}_p, \tilde{X}_p \tilde{X}_p, \tilde{X}_p \tilde{X}_p, \) and \( \tilde{X}_p \tilde{X}_p \tilde{\theta}_p \) are solely function of the parameter \( \gamma \) defined in equation (A8).

**APPENDIX B: LIST OF SYMBOLS**

- **A**: wave amplitude in the axial direction
- **B**: \( = E h (1 - \mu^2) \), extensional rigidity
- **C**: wave amplitude in the radial direction
- **D**: \( = \beta B R^2 \), flexural rigidity
- **E**: Young’s modulus
- **G**: Green’s function for the radial displacement
- **H**: Heaviside step function
- **L**: shell length
- **M_0**: bending moment per unit length in the \( r,x \) plane
- **N_0**: axial force per unit length
- **Q**: resonance quality factor
- **Q_0**: radial force per unit length
- **R**: shell radius
- **T**: \( = A/C \), wave amplitude ratio
- **Y**: direct mobility
- **Y^c**: mobility matrix
- **Z**: impedance
- **c**: superscript; \( \infty/2 \) and \( \infty \) for the semi-infinite and infinite shell, respectively
- **h**: shell thickness
- **i**: \( = \sqrt{-1} \)
- **m**: parameter defined by \( 4m^4 = 1 - \mu^2 - \Omega^2 \)
- **r**: cylindrical co-ordinate
- **r**: reflection matrix
- **s**: \( = x/R \), non-dimensional axial length
- **t**: time
- **u**: midsurface displacement in the \( x \) direction
- **w**: midsurface displacement in the \( r \) direction
- **x**: cylindrical co-ordinate
- **a\_c**: wave receptance matrix
- **a\_c \_p \_x**: receptance of the wave \( p \) with respect to the load \( X \)
- **\beta**: \( = k \beta R^2 \), thickness parameter
- **\delta**: Kronecker’s delta function \( (\delta(k,0) = 1 \text{ if } k = 0, 0 \text{ otherwise}) \)
- **\eta**: shell loss factor
- **\theta**: \( = \partial w/\partial \_x \), rotation of the normal to the middle surface about the \( \varphi \)-axe
$\phi$  
cylindrical co-ordinate  

$\kappa$  
non-dimensional propagation constant  

$\lambda$  
extension parameter  

$\mu$  
Poisson's ratio  

$\xi$  
ininner region axial length  

$\rho$  
density of the shell material  

$\omega$  
circular frequency of the excitation  

$\Omega = \omega R \sqrt{\rho (1 - \mu^2) / E}$, non-dimensional frequency  

$\cdot$  
differential with respect to time
Paper C
STRUCTURAL MOBILITIES FOR THE EDGE-EXCITED, SEMI-INFINITE CYLINDRICAL SHELL USING A PERTURBATION METHOD

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Approximate closed-form solutions for all the elements of the mobility matrix for a thin-walled cylindrical, semi-infinite shell are presented here. Excitation is applied at the shell edge and consists of four load types, namely an axial force, a circumferential force, a radial force and a bending moment. The aim of the research is to determine structural mobilities for the edge-excited, semi-infinite cylindrical shell using a perturbation method. The problem is formulated using Donnell-Mushtari shell equations where in-plane inertial forces are neglected. The derivation of the solution is based upon the method of the matched asymptotic expansions, yielding approximate expressions for both the input and cross-mobilities for a given circumferential mode of vibration. Finally, the accuracy of the expressions derived is discussed in terms of individual modal mobilities and point mobilities in comparison with numerical results calculated using Flügge theory. It is shown that vibrational power transmission by mechanical point excitation is predicted with an acceptable level of agreement for a frequency below half the ring frequency.

1. INTRODUCTION

In a noise and vibration control context, it is important to understand and quantify the vibration transmission to cylindrical shells by dynamical excitations such as forces and moments. To this end, a most convenient concept is structural mobility, because of the simple relationship existing between the input power into an elastic structure and the real part of its mobilities. However in the case of cylindrical shells, the mobility representation is considerably complicated by the coupling occurring between in-plane and out-of-plane vibrations. Indeed, cross-mobility, or the complex ratio of velocity in one direction to force or moment in another direction, can play a decisive role in the vibration transmission to shells, such as in the case of joint excitation.

By virtue of the cylindrical symmetry of the shell, loads applied and displacement produced can be expanded in a sum of modes by means of Fourier series expansions in the circumferential direction. The structural mobilities of the shell can then be calculated for each mode in a relatively simple way. Furthermore, owing to the waveguide character of shells, only a finite number of circumferential modes are able to propagate freely at a given frequency, that is contributes to the vibrational power transmission. This waveguide behaviour simplifies the analysis of vibration transmission substantially [1], as only the modes that are propagating are needed for the calculation of the input mechanical power.

The mobilities of the cylindrical shell of infinite extent are of considerable interest in particular [1-6], as they provide estimates of the mobilities for a similar but finite shell in a frequency-averaged sense. However, in practice shells are often subject to forces and moments
located at their edges, thereby invalidating the aforementioned analogy and motivating the investigation of the edge-excited, semi-infinite shell. The boundary-value problem of the semi-infinite shell being acted upon by edge loads was first addressed and solved explicitly in the static case by Simmonds in 1966 [7]. With regard to dynamical excitation, both problem formulation and computed results for the input and cross-mobilities were presented recently by Ming et al. [8], while closed-form solutions for both the infinite and semi-infinite shell problems under axially-symmetrical loads have been derived by the author in a previous article [9].

The main focus of this paper is the derivation of approximate solutions for the input and cross-mobilities of the semi-infinite, cylindrical shell excited at its edge by forces and moments with other circumferential distributions. The study is confined to vacuo vibrations, and the shell is assumed to be sufficiently thin to satisfy Love's first approximations. Four load types are considered in relation to the excitation: an axial force, a circumferential force, a radial force and a bending moment. The size of the region of excitation in the axial direction is assumed to be larger than the shell thickness, but relatively small compared to the axial wavelength. The edge loads are assumed to have slow spatial variations in the circumferential direction and exerted at a frequency below the ring frequency of the shell (that is the frequency at which the wavelength of extensional waves in the shell wall coincides with the shell circumference). Consequently only the modes of low-circumferential order \( n = 0, 1, 2, 3 \ldots \) are excited. The underlying reason of these restrictions is twofold. Firstly, a high proportion of the noise and vibration problems in which shells are involved, arise at frequencies below the ring frequency. Secondly, the analysis is thus limited to these vibrations, which are characteristic of shells in the sense that only vibrations for which a strong coupling between radial, circumferential and axial motions occurs are studied. At higher frequency, or for modes of high-circumferential order, the vibrational behaviour of the shell tends to liken that of the plate strip and can thus be studied through the in-plane and out-of-plane governing equations for thin plates [10-11].

Well below the ring frequency, membrane theory [2, 12] is often used as a simplified shell theory to model the shell vibrations. This is especially so for the axisymmetric mode \( (n = 0) \), the beam mode \( (n = 1) \) and other lower modes excited to well above their cut-on frequencies. However this theory alone cannot satisfy all the boundary conditions associated with the edge loads. Furthermore it fails to model the cut-on phenomena experienced by the circumferential modes, which is largely responsible for the power input to the shells. In the analysis presented here, the boundary-value problem is handled by describing the shell vibrations and the boundary conditions using the Donnell-Mushtari shell equations, where tangential inertial forces are neglected. Perturbation solutions are then sought by applying the method of the matched asymptotic expansions [9, 13]. According to this method, the shell is modelled as a membrane at some distance from the edge but bending actions localised in the edge region can also be accounted for. In this regard, the method distinguishes between an inner region adjoining the edge and an outer region describing the main body of the shell. Simplified but different sets of governing equations are derived for both regions, which can be solved analytically in the case presented. Solutions are obtained in the form of power series expansions for which the expansion parameter is a function of the shell thickness-to-radius ratio multiplied by the circumferential modal order \( n \). Thus, the solutions of the inner region satisfy the boundary conditions associated with the present boundary-value problem, while in the outer region, the solutions are to some extent, solutions to the membrane shell equations. Furthermore, matching the inner solution to the outer one provides additional conditions, which lead to the complete determination of these solutions. The leading terms of these expansions then yield approximate expressions for the elements of the mobility matrix.
Finally, the expressions derived are corrected to account partly for the influence of the inertial force of the circumferential displacement, and compared with the results obtained using Flügge shell theory for the mobilities of both the individual modes and point excitations.

2. FORMULATION OF THE BOUNDARY-VALUE PROBLEM

Consider the circular, semi-infinite shell of mean radius $R$ and thickness $h$ sketched in Figure 1 below. The shell is assumed to be thin, made of homogeneous isotropic material and acted upon by a set of edge loads specified by the following set of equations:

$$
N_s = N_n \cos(n\phi)e^{i\alpha t},
$$

$$
T_s = T_n \sin(n\phi)e^{i\alpha t},
$$

$$
R_s = R_n \cos(n\phi)e^{i\alpha t},
$$

$$
M_s = M_n \cos(n\phi)e^{i\alpha t}
$$

(1)

where the integer $n$ is the circumferential modal number, $\phi$ the circular frequency of the excitation, and $t$ the time. In this paper it is assumed that $n > 0$ since the mode $n = 0$ has been treated in detail elsewhere [9]. $N_s$, $T_s$, $R_s$ are force components in the axial, tangential and radial directions, respectively, while $M_s$ is a bending moment. The loads refer to a unit length of section and are defined positive accordingly to Figure 1. It is worth noting that excitation by a twisting moment does not need to be considered, as such a load type can be replaced by statically equivalent distributions of $T_s$ and $R_s$ [14].

Figure 1. The semi-infinite shell: (a) co-ordinate system and positive convention of force and moment, (b) modal shape of the circumferential modes $n = 0, 1, 2$ and $3$.

Let us assume first that the vibration field produced in the shell body is predominantly radial, thereby allowing us to neglect in-plane inertial forces in the shell equations of motion. The vibrations of the shell in vacuum are then conveniently expressed in terms of a function $\Phi$ [15], obeying the eighth-order differential equation,
\[ \beta \nabla^8 \Phi + (1 - \mu^2) \Phi''' + \frac{\rho(1 - \mu^2)}{E} \nabla^4 \Phi = 0, \]  

where
\[ s = \frac{x}{R}, \quad (\quad)' = \frac{\partial (\quad)}{\partial s}, \quad (\quad)'' = \frac{\partial (\quad)}{\partial \phi}, \quad \nabla^2 (\quad) = (\quad)'' + (\quad)'''' \quad \text{and} \quad \beta = \frac{h^2}{12R^2}. \]

The parameters \( \mu, \rho \) and \( E \) are the Poisson's ratio, the density and the Young's modulus of the shell material respectively, and \( \beta \) is referred to as the thickness parameter of the shell.

The shell displacements in the axial (\( u \)), circumferential (\( v \)) and radial directions (\( w \)) are related to \( \Phi \) by the equations:
\[ u = \Phi''' - \mu \Phi'', \quad v = -\Phi''' - (2 + \mu) \Phi'' \quad \text{and} \quad w = \nabla^4 \Phi. \]

Considering the semi-infinite shell where \( 0 < s < \infty \), the general form of solutions to equation (2) is:
\[ \Phi(s, \phi, t) = \Phi_n \cos(n\phi)e^{i \alpha t}, \]
where
\[ \Phi_n = C_1e^{\kappa_1 s} + C_2e^{\kappa_2 s} + C_3e^{\kappa_3 s} + C_4e^{\kappa_4 s}. \]

The propagation constants \( \kappa_j \) are the roots of the dispersion relation associated with equation (2), which are located in the complex half-plane \( \pi/2 < \arg(\kappa) \leq 3\pi/2 \). The wave amplitudes \( C_j \) are obtained from the boundary conditions. The boundary conditions for the boundary-value problem under consideration here (where the forces and moments are prescribed at the edge) are derived by inserting the function \( \Phi \) into the stress resultants obtained with the Donnell-Mushtari theory [16] as follows:

\[ N_n = \frac{B}{R} n (1 - \mu^2) \Phi_n' |_{s=0}, \]
\[ T_n = -\frac{1 - \mu^2}{R} B (\Phi_n''' + \frac{\beta}{1 + \mu} \nabla^4 \Phi_n') |_{s=0}, \]
\[ R_n = \frac{D}{R^3} \nabla^4 (\Phi_n''' - (2 - \mu) n^2 \Phi_n') |_{s=0} \quad \text{and} \]
\[ M_n = -\frac{D}{R^2} \nabla^4 (\Phi_n''' - \mu n^2 \Phi_n') |_{s=0}, \]

where \( B = Eh/(1 - \mu^2) \) \quad and \quad \( D = Eh^3/12(1 - \mu^2) \).

Once equation (5) is introduced into the boundary conditions (6) and the system solved for \( C_j \), the shell velocities, and consequently the shell mobilities, may readily be obtained using equations (3). Presented in matrix form, the shell mobilities read explicitly:
\[ \begin{bmatrix} \tilde{u}_n \\ \tilde{v}_n \\ \tilde{w}_n \\ \tilde{\theta}_n \end{bmatrix} = \begin{bmatrix} Y_{uN} & Y_{uW} & Y_{uR} & Y_{uM} \\ Y_{vN} & Y_{vW} & Y_{vR} & Y_{vM} \\ Y_{wN} & Y_{wW} & Y_{wR} & Y_{wM} \\ Y_{\theta N} & Y_{\theta W} & Y_{\theta R} & Y_{\theta M} \end{bmatrix} \begin{bmatrix} N_n \\ T_n \\ R_n \\ M_n \end{bmatrix}. \]  

(7)
where \( \theta \) represents the rotation of the normal to the middle surface about the \( \varphi \)-axis, \( \theta = \partial w / \partial x \). The mobility matrix is symmetrical and its diagonal and off-diagonal coefficients are referred to as input and cross-mobilities, respectively. They are obtained from the constants \( C_j \) using the following equations:

\[
Y_{wX}^n = -i\omega \sum_{j=1}^{n} \kappa_j (\mu \kappa_j^2 + n^2) C_j , \quad Y_{uX}^n = i\omega \sum_{j=1}^{n} n((2+\mu)\kappa_j^2 - n^2) C_j ,
\]

\[
Y_{wX}^n = i\omega \sum_{j=1}^{n} (\kappa_j^2 - n^2)^2 C_j , \quad Y_{ux}^n = -\frac{i\omega}{R} \sum_{j=1}^{n} (\kappa_j^2 - n^2)^2 \kappa_j C_j ,
\]  

(8)

where \( X \) denotes one of the excitation components (either \( N_n, T_n, R_n \) or \( M_n \)). For the determination of the mobilities relative to \( X \), the wave amplitudes \( C_j \) are calculated assuming that \( X \) equals one and the other components equal zero.

Despite the rather simple formulation of the boundary-value problem being considered, closed-form solutions for the coefficients of the mobility matrix are precluded due to the cumbersome analytical expressions of the propagation constants. Accordingly, approximate solutions are sought using perturbation techniques. With this in mind, equation (2) is rewritten assuming a variable change \( \tilde{s} = ns \), yielding:

\[
\beta n^4 (\Phi^{'''''} + 4\Phi^{''''} + 5\Phi^{''''} - 2\Phi^{''''}) + 4m^4 \Phi^{''''} + 2\sigma^2 \Phi^{''} - n^2 \Phi = 0 ,
\]  

(9)

where primes now denote differentiation with respect to \( \tilde{s} \), and where the notation

\[
\Omega^2 = \frac{\rho(1-\mu^2)\omega^2 R^2}{E} , \quad \sigma^2 = \Omega^2 - \beta n^4 , \quad 4m^4 = 1-\mu^2 - \sigma^2 ,
\]  

(10)

has also been introduced.

3. THE PERTURBATION METHOD

In the differential equation (2), the highest derivative is multiplied by the parameter \( \beta n^4 \). Provided that \( \beta n^4 \ll 1 \), (which means that the shell vibrates in one of its lower circumferential modes) the boundary-value problem is seen as belonging to the class of the edge-layer problems [17]. In the case under consideration, it is translated into axial variations of the shell motions characterised by two distinct length scales. One scale is associated with motions remaining localised in the edge region, while the other scale is characteristic of motions penetrating farther into the shell body. This class of problems lends itself to analysis by perturbation techniques and particularly by the method of the matched asymptotic expansions.

As mentioned in the introduction, this method divides the body of the shell into an inner region adjoining the edge and an outer region including the rest of the structure. In the inner region, the governing equation and the boundary conditions are re-arranged using a magnified scale and expanded as a function of the parameter \( \beta^{1/4} n \). The solution to these equations is referred to as the inner solution. In the outer region, the equation of motion is expanded using the original co-ordinate system and the associated solution is termed outer solution. The basic
idea underlying the method is that the domains of validity for the two expansions overlap. This matching provides the additional equations that allow all of the constants in the expansions to be determined. Finally, combining the inner and outer expansions forms a composite expansion that is valid for both the inner and the outer regions.

3.1 THE INNER PROBLEM

In the inner region, a stretched co-ordinate system is used, reading \( \zeta = \frac{s}{r (\beta n^4)^e} \) in its general form. The parameter \( e \) is a strictly positive constant and has to be determined. This is achieved by searching for the value of \( e \) that yields the least degenerate limit of the differential equation (2) when \( \beta n^4 \to 0 \) in the new co-ordinate system \([17]\). It ensues here that \( e = \frac{1}{4} \) and thus the inner solution has to be sought in a power series in the form:

\[
\Phi' = \sum_{k=0}^{\infty} \lambda^k \Phi_k ,
\]

where the expansion parameter \( \lambda \) is defined by

\[
\lambda = \beta^{1/4} n .
\]

The governing equations and boundary conditions satisfied by the functions \( \Phi_k \) (the successive terms of the expansion), are derived by inserting equation (11) into equations (2) and (6) expressed in the stretched co-ordinate system \( \zeta = \frac{s}{\lambda} \). Finally, terms of equal power of \( \lambda \) are collected. Adopting the convention \( \Phi_k = 0 \) for \( k < 0 \), then the governing equations (13) and boundary conditions (14) read:

\[
\Phi_k^{(4)} + 4m^4 \Phi_k^{(4)} = 4\Phi_{k-2}^{(4)} - 2\sigma^2 \Phi_{k-2}^{(4)} - 5\Phi_{k-4}^{(4)} + \sigma^2 \Phi_{k-4} + 2\Phi_{k-6}^{(4)}
\]

and

\[
\begin{align*}
\bar{N}_n \delta_{k2} &= \Phi_k, \\
\bar{T}_n \delta_{k3} &= \Phi_k^{(4)} - \Phi_k^{(4)} + (2\Phi_k^{(4)} + 5\Phi_k^{(4)} - (4 - \mu) \Phi_k^{(4)})/(1 + \mu n^2), \\
\bar{R}_n \delta_{k7} &= \Phi_k^{(4)} - (2 + \mu) \Phi_k^{(4)} + (1 + 2\mu) \Phi_k^{(4)} - (2 + \mu) \Phi_k^{(4)} - (2 + \mu) \Phi_k^{(4)}, \\
\bar{M}_n \delta_{k6} &= \Phi_k^{(4)} - (2 + \mu) \Phi_k^{(4)} + (1 + 2\mu) \Phi_k^{(4)} - (2 + \mu) \Phi_k^{(4)} - (2 + \mu) \Phi_k^{(4)},
\end{align*}
\]

with the notation

\[
\bar{N}_n = \frac{R}{(1 - \mu^2)n^4B} N_n, \quad \bar{T}_n = -\frac{R}{(1 - \mu^2)n^4B} T_n, \quad \bar{R}_n = \frac{R^3}{Dn^7} R_n \quad \text{and} \quad \bar{M}_n = -\frac{R^2}{Dn^6} M_n ,
\]

and where \( \delta \) is the Kronecker delta function defined by

\[
\delta_{km} = \begin{cases} 
0 & \text{if } k \neq m \\
1 & \text{if } k = m 
\end{cases}
\]
3.2 THE OUTER PROBLEM

The outer solution is sought in a power series using the form

$$\Phi^o = \sum_{k=0}^{\infty} \lambda^k \Phi_k.$$  \hspace{1cm} (16)

The differential equations governing the functions $\Phi_k$ are derived by inserting equation (16) into (2) and collecting terms of equal power of $\lambda$. Adopting the convention $\Phi_k = 0$ for $k < 0$, they read:

$$4m^4 \Phi_k^{'''''} + 2\sigma^2 \Phi_k^{'''} - \sigma^2 \Phi_k = -\Phi_{k-4}^{'''''} + 4\Phi_{k-4}^{''''} - 5\Phi_{k-4}^{''''} + 2\Phi_{k-4}^{'''}.$$  \hspace{1cm} (17)

Assuming an excitation frequency well above the cut-on frequency for the investigated mode, that is $\sigma^2 \equiv \Omega^2$, equation (17) reduces to the governing equation for membrane shells for the orders $k = 0 \ldots 3$, when in-plane inertial forces are neglected.

3.3 THE MATCHING EQUATIONS

In the inner region, the general form of the solution to equation (13) is

$$\Phi_k = \sum_{p=0}^{\text{Int}(k/2)} (C^{(1)}_{k,p} \tilde{P} e^{\rho_1 \tilde{\sigma}} + C^{(2)}_{k,p} \tilde{P} e^{\rho_2 \tilde{\sigma}}) + \sum_{p=0}^{\text{Int}(k/4)} \sum_{r=0}^{3} C^{r}_{k,p} \tilde{P} e^{\kappa_r \tilde{\sigma}},$$  \hspace{1cm} (18)

while in the outer region, the solution to equation (17) is

$$\Phi_k = \sum_{p=0}^{\text{Int}(k/4)} (D^{(1)}_{k,p} \tilde{P} e^{\kappa_1 \tilde{\sigma}} + D^{(2)}_{k,p} \tilde{P} e^{\kappa_2 \tilde{\sigma}}),$$  \hspace{1cm} (19)

where $\text{Int}(x)$ denotes the integer part of $x$, and where the propagation constants are given by:

$$\rho_1 = -m(1+i), \quad \rho_2 = -m(1-i), \quad \kappa_1 = \frac{\sigma}{\sqrt{\sigma + \sqrt{1-\mu^2}}}, \quad \kappa_2 = \frac{\sigma}{\sqrt{\sigma - \sqrt{1-\mu^2}}}.$$  \hspace{1cm} (20)

The constants $C^{(1)}_{k,0}, C^{(2)}_{k,0}, C^{(3)}_{k,0}$ and $D^{(j)}_{k,0}$ describe the general solutions to equations (13) and (17) and have to be determined from both the boundary conditions and the matching equations. On the other hand, the constants $C^{(1)}_{k,p}, C^{(2)}_{k,p}$ and $D^{(j)}_{k,p}$, where $p > 0$, are associated with particular solutions appearing when these equations become non-homogeneous.

In accordance with the method presented in [17], the matching principle used to infer the matching equations is:

$$\lim_{\tilde{\xi} \to 0} \Phi^o(\tilde{\xi}; \lambda) = \lim_{\xi \to \infty} \Phi^i(\xi; \lambda).$$  \hspace{1cm} (21)

Consider the case where the inner solution is given by equation (18). The polynomial part can be re-expressed in terms of the original variable $\tilde{\xi}$ while the exponential functions describing
a standing decaying field are disregarded as \( \zeta \to \infty \). The matching then being performed as \( \bar{s} \to 0 \) for the outer solution, equation (19) can be expanded in a Taylor series as follows:

\[
\lim_{\bar{s} \to 0} \Phi^0 (\bar{s}; \lambda) = \sum_{k=0}^{\infty} \lambda^k \sum_{j=0}^{\frac{\ln(k/4)}{2}} \sum_{p=0}^{\frac{\ln(k/4)}{2}} \frac{D_{k,p}^{(1)} \kappa_1^j + D_{k,p}^{(2)} \kappa_2^j}{j!} \bar{s}^j + p.
\]  

(22)

Matching is achieved by identifying the terms of the limits of the inner and outer solutions that are of equal power of \( \lambda \) and of \( \bar{s} \). It can be shown that the two following conditions or matching equations ensue:

\[
C_{k,m}^{(3)} = 0 \quad \forall k \quad \text{and for } m > k.
\]  

(23)

and

\[
\sum_{p=0}^{N} \frac{D_{k,p}^{(1)} \kappa_1^{m-p} + D_{k,p}^{(2)} \kappa_2^{m-p}}{(m-p)!} = -C_{k+m,m}^{(3)} = 0 \quad \forall m, \forall k.
\]  

(24)

where \( N = \min(m, \text{Int}(k/4)) \).

3.4 THE COMPOSITE EXPANSION

Finally, the analysis is completed by deriving a composite solution that is valid in both the inner and the outer regions. According to Nayfeh [17], this composite expansion, denoted as \( \Phi^c \), is defined by:

\[
\Phi^c = \Phi^o + \Phi^i - \lim_{\bar{s} \to 0} \Phi^0 (\bar{s}; \lambda).
\]  

(25)

Thus, using equations (11), (16), (18) and (19), plus the matching equations (23) and (24), the expression for the composite solution is:

\[
\Phi^c = \sum_{k=0}^{\infty} \lambda^k \left \{ \sum_{p=0}^{\frac{\ln(k/4)}{2}} \left ( D_{k,p}^{(1)} \bar{s}^p e^{\kappa_1 \bar{s}} + D_{k,p}^{(2)} \bar{s}^p e^{\kappa_2 \bar{s}} \right ) + \sum_{p=0}^{\frac{\ln(k/2)}{2}} \left ( C_{k,0}^{(1)} \bar{s}^p e^{\rho_1 \bar{s}} + C_{k,0}^{(2)} \bar{s}^p e^{\rho_2 \bar{s}} \right ) \right \}.
\]  

(26)

Terms in the form \( \bar{s}^p e^{\kappa \bar{s}} \) appear in the composite solution (26) as a consequence of the emergence of particular solutions. When \( \kappa \) describes a propagating wave, they become unbounded as \( \bar{s} \to \infty \) and thereby make the solution non-uniform. However, it has been shown that these terms are nothing but the power series expansion of the propagation constants and consequently the non-uniformity can be removed by resorting once again to perturbation methods, such as the Lindstedt-Poincaré technique [19].

3.5 THE RADIAL EXCITATION CASE

The method is further exemplified here by looking at the case of radial excitation. As the other types of excitation can also be handled in a similar manner, only the final results are reported
in the next section. Since it is assumed that the shell is acted upon by a radial force only, $\vec{N}_r$, $\vec{T}_r$ and $\vec{M}_r$ are all assigned a value of zero in the boundary conditions (14) and the condition on the applied force has to be satisfied by the seventh order solution to the inner region. Applying these boundary conditions and the matching equations (23) and (24), it can be shown that the inner solution to the seventh order is

$$\Phi_7 = C_{7,0}^{(1)} e^{\mu \xi} + C_{7,0}^{(2)} e^{\rho \xi} + \sum_{p=0}^{3} C_{k \rho \xi}^{(p)} e^{\rho \xi}.$$  

(27)

where

$$\begin{bmatrix} C_{7,0}^{(1)} \\ C_{7,0}^{(2)} \end{bmatrix} = \frac{1}{4 m^4} \begin{bmatrix} \frac{1}{\rho_1^2 (\rho_1 - \rho_2)} \\ \frac{1}{\rho_2^2 (\rho_1 - \rho_2)} \end{bmatrix} \vec{R}_n,$$

and

$$\begin{bmatrix} C_{6,2}^{(3)} \\ C_{7,3}^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{R}_n / 24 m^4 \end{bmatrix}.$$  

(28 and 29)

It can be seen that the two constants $C_{6,0}^{(3)}$ and $C_{6,1}^{(3)}$ remain undetermined but this does not prevent deriving the composite expansion. With respect to the outer solution, the first non-zero solution arises at the fourth order, reading:

$$\Phi_4 = D_{4,0}^{(1)} e^{\kappa \xi} + D_{4,0}^{(2)} e^{\kappa_2 \xi}.$$  

(30)

Writing equation (24) for the doublets $(k, m) = (4,2)$ and $(k, m) = (4,3)$, then using equation (29) yields the following solutions for the constants $D_{4,0}^{(1)}$ and $D_{4,0}^{(2)}$:

$$\begin{bmatrix} D_{4,0}^{(1)} \\ D_{4,0}^{(2)} \end{bmatrix} = \frac{1}{4 m^4} \begin{bmatrix} \frac{1}{\kappa_1^2 (\kappa_1 - \kappa_2)} \\ \frac{1}{\kappa_2^2 (\kappa_1 - \kappa_2)} \end{bmatrix} \vec{R}_n.$$  

(31)

A first approximation of the composite expansion can thus be obtained and expressed with the original axial variable $s$ as follows:

$$\Phi^{s} = \lambda^4 (D_{4,0}^{(1)} e^{s \kappa s} + D_{4,0}^{(2)} e^{s \kappa_2 s}) + \lambda^7 (C_{7,0}^{(1)} e^{s \rho s} + C_{7,0}^{(2)} e^{s \rho_2 s}).$$  

(32)

From equations (32) and (3), it is straightforward to derive approximations for the semi-infinite shell mobilities with respect to radial force. For instance, the radial input mobility is given by:

$$Y_n^{\text{rad}} = \frac{i \omega m_1^{(1)}}{R_n} \equiv \frac{i \omega R_n^4 \Phi^{s}}{s_0},$$

(33)

which yields the following by retaining the leading term of the respective wave amplitude:
\[ Y_{nk} = \frac{i \omega \lambda^2}{R_n} \left( (\kappa_1^2 - 1)^2 n^4 D_4^{(1)} + (\kappa_2^2 - 1)^2 n^4 D_4^{(2)} - \frac{4m^2 \lambda^3}{\beta} (c_{1,0} + c_{1,0}^{(2)}) \right). \] (34)

4. THEORETICAL RESULTS

4.1 DISCUSSION

Approximate expressions for the coefficients of the mobility matrix defined in equation (7) are reported in Tables 1 to 4. They are lifted from the perturbation analysis conducted with the different components of the excitation. Furthermore, as they are of practical interest in the field of structure-borne sound, the Green’s functions associated with the different edge loads derived in the course of the analysis are given in Appendix A. For the sake of brevity, only results concerning the prediction of the radial displacement in the far-field are reported.

Before discussing these results, it is interesting to consider the nature of the free waves associated with the propagation constants given by equation (20). The two constants \( \rho_1 \) appearing in the inner solution occur as a complex conjugate pair when \( \Omega^2 < 1 - \mu^2 \), and thus describe a standing decaying wave (termed wave Type 1 below). Furthermore, the propagation constants of the outer solution \( \kappa_1 \) are also a complex conjugate pair when \( \Omega^2 < \beta n^4 \). Therefore the vibration field is totally of a near field type in this frequency range and all the mobilities are purely imaginary since no propagation of energy is possible. At \( \sigma = 0 \) (that is where \( \Omega^2 = \beta n^4 \)), the propagation constants of the outer solution become real and purely imaginary, yielding an evanescent and a propagating wave respectively. They are referred to as Type 2 waves in the rest of this article. The frequency corresponding to \( \Omega^2 = \beta n^4 \) is called the cut-on frequency since it indicates the emergence of a propagating wave. This value was obtained using a simplified set of governing equations, whereas more rigorous shell theories [18] show the cut-on phenomenon occurring at

\[ \Omega^2 = \beta n^2 \frac{(n^2 - 1)^2}{1 + n^2}. \] (35)

Consequently, the following equation:

\[ \sigma^2 = \Omega^2 - \beta n^2 \frac{(n^2 - 1)^2}{1 + n^2}, \] (36)

is used instead of equation (10) in the calculation of the mobilities reported in Tables 1 to 4. For an excitation frequency well above the cut-on frequency of the circumferential mode considered, it appears that the propagation constants of the Type 2 waves tend toward those predicted by the membrane shell theory when tangential inertia is disregarded. With respect to the very low frequency behaviour of the beam and the “ovaling” modes \( n = 1 \) and 2, respectively, corrections need to be made to account for the influence of tangential inertial forces. Indeed within this frequency range, the displacement ratios associated with the Type 2 waves are approximately \( u/w < 1 \) and \( v/w = -1/n \) [18].

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### TABLE 1

Direct mobilities for excitation by axial force (n > 0):

\[ \Delta = 1 + \frac{1 - \Omega^2}{n^2}, \quad \sigma^2 = \Omega^2 - \frac{\beta n^2 (n^2 - 1)^2}{1 + n^2} \text{ and } \varepsilon_n = \begin{cases} -1 & \text{if } \sigma^2 < 0 \\ +1 & \text{if } \sigma^2 \geq 0 \end{cases} \]

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Response</th>
<th>Mobility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>[ Y_{uN}^n = \frac{\omega R}{n(1 - \mu^2) B} \varepsilon_n \sqrt{1 - \mu^2 - \sigma + i \sqrt{1 - \mu^2 - \sigma}} ]</td>
</tr>
<tr>
<td>(N_n)</td>
<td></td>
<td>[ Y_{\theta N}^n = \frac{\omega R}{nB} \varepsilon_n \sqrt{1 - \mu^2 - \sigma - i \sqrt{1 - \mu^2 + \sigma}} ]</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td></td>
<td>[ Y_{uN}^n = \frac{\omega R}{B} \varepsilon_n \sqrt{1 - \mu^2 - \sigma + i \sqrt{1 - \mu^2 + \sigma}} ]</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td></td>
<td>[ Y_{\theta N}^n = -\frac{\omega R}{B} \varepsilon_n \sqrt{1 - \mu^2 - \sigma + i \sqrt{1 - \mu^2 + \sigma}} ]</td>
</tr>
</tbody>
</table>

### TABLE 2

Direct mobilities for excitation by circumferential force (n > 0):

\[ \Delta = 1 + \frac{1 - \Omega^2}{n^2}, \quad \sigma^2 = \Omega^2 - \frac{\beta n^2 (n^2 - 1)^2}{1 + n^2} \text{ and } \varepsilon_n = \begin{cases} -1 & \text{if } \sigma^2 < 0 \\ +1 & \text{if } \sigma^2 \geq 0 \end{cases} \]

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Response</th>
<th>Mobility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>[ Y_{uN}^n = Y_{uN}^n \text{ (see TABLE 1)} ]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[ Y_{\theta N}^n = \frac{\omega R}{n(1 - \mu^2) B} \sqrt{1 - \mu^2 - \sigma + i \sqrt{1 - \mu^2 + \sigma}} ]</td>
</tr>
<tr>
<td>(T_n)</td>
<td></td>
<td>[ Y_{uN}^n = \frac{\omega R}{B} \varepsilon_n \sqrt{1 - \mu^2 - \sigma + i \sqrt{1 - \mu^2 - \sigma}} ]</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td></td>
<td>[ Y_{\theta N}^n = \frac{n \omega R \varepsilon_n \sqrt{1 - \mu^2 - \sigma^2}}{B} \sqrt{1 - \mu^2 - \sigma^2} ]</td>
</tr>
</tbody>
</table>
TABLE 3
Direct mobilities for excitation by radial force \((n > 0)\):
\[
\Delta = 1 + \frac{1 - \Omega^2}{n^2}, \quad \sigma^2 = \Omega^2 - \frac{\beta n^2 (n^2 - 1)^2}{1 + n^2} \quad \text{and} \quad \varepsilon_n = \begin{cases} -1 & \text{if} \quad \sigma^2 < 0 \\ +1 & \text{if} \quad \sigma^2 \geq 0 \end{cases}
\]

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Response</th>
<th>Mobility</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( u )</td>
<td>( Y_{\text{ur}}^n = Y_{\text{ru}}^n ) (see TABLE 1)</td>
</tr>
<tr>
<td>( v )</td>
<td>( v )</td>
<td>( Y_{\text{vr}}^n = Y_{\text{rv}}^n ) (see TABLE 2)</td>
</tr>
<tr>
<td>( w )</td>
<td>( w )</td>
<td>( Y_{\text{wr}}^n = \frac{\omega R}{B} \cdot \frac{\frac{i \sqrt{2}}{\beta^{1/4}} + n(1 - \mu^2) \frac{\varepsilon_n \sqrt{1 - \mu^2} + \sigma - i \sqrt{1 - \mu^2} - \sigma}{\sigma^{3/2} \Delta^{1/4}}}{(1 - \mu^2 - \sigma^2)^{3/4}} + n^2(1 - \mu^2) \frac{2i \sigma + \varepsilon_n \sqrt{1 - \mu^2} - \sigma^2}{\sigma^{3/2} \Delta^{1/4}} )</td>
</tr>
<tr>
<td>( \vartheta )</td>
<td>( \vartheta )</td>
<td>( Y_{\text{dr}}^n = \frac{\omega B}{\beta^{1/4} \sqrt{1 - \mu^2 - \sigma^2} + n^2(1 - \mu^2) \frac{2i \sigma + \varepsilon_n \sqrt{1 - \mu^2} - \sigma^2}{\sigma^{3/2} \Delta^{1/4}} )</td>
</tr>
</tbody>
</table>

TABLE 4
Direct mobilities for excitation by bending moment \((n > 0)\):
\[
\Delta = 1 + \frac{1 - \Omega^2}{n^2}, \quad \sigma^2 = \Omega^2 - \frac{\beta n^2 (n^2 - 1)^2}{1 + n^2} \quad \text{and} \quad \varepsilon_n = \begin{cases} -1 & \text{if} \quad \sigma^2 < 0 \\ +1 & \text{if} \quad \sigma^2 \geq 0 \end{cases}
\]

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Response</th>
<th>Mobility</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( u )</td>
<td>( Y_{\text{ur}}^n = Y_{\text{ru}}^n ) (see TABLE 1)</td>
</tr>
<tr>
<td>( v )</td>
<td>( v )</td>
<td>( Y_{\text{vr}}^n = Y_{\text{rv}}^n ) (see TABLE 2)</td>
</tr>
<tr>
<td>( M_n )</td>
<td>( w )</td>
<td>( Y_{\text{wr}}^n = Y_{\text{rw}}^n ) (see TABLE 3)</td>
</tr>
<tr>
<td>( \vartheta )</td>
<td>( \vartheta )</td>
<td>( Y_{\text{dr}}^n = \frac{\omega \sqrt{2}}{B^{1/4} \beta^{1/4} \sqrt{1 - \mu^2 - \sigma^2} + n^2(1 - \mu^2) \frac{\varepsilon_n \sqrt{1 - \mu^2} + \sigma + i \sqrt{1 - \mu^2} + \sigma}{\sqrt{\sigma}(1 - \mu^2 - \sigma^2)^{3/4}} \Delta^{1/4}} )</td>
</tr>
</tbody>
</table>

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Obviously inertial force attributed to circumferential vibrations cannot be neglected with regard to this frequency range. Fortunately this effect can easily be accounted for by applying a correction to the propagation constants $\kappa_1$ and $\kappa_2$. Indeed, by retaining terms arising from the inertial force associated with circumferential motion in the dispersion relation of the membrane theory, it can be shown that the corrected propagation constants can be approximately given by $\Delta^{1/4} \kappa$ instead of $\kappa$, where $\Delta = 1 + (1 - \Omega^2) / n^2$. The results reported in Tables 1 to 4 have thus been obtained by replacing $\kappa_j$ with $\Delta^{1/4} \kappa_j$ in the derivation. It can be noted that the approximate expressions confirm the respective importance of the in-plane inertial forces. It is clearly be shown from the following ratios

\[
\frac{\text{Re}(Y_{\psi}^n)}{\text{Re}(Y_{w}^n)} = \frac{-1 - \mu^2 - \sigma^2}{n(1 - \mu^2)}
\]

\[
\frac{\text{Re}(Y_{w}^n)}{\text{Re}(Y_{\psi}^n)} = \frac{\Delta^{1/4}\sqrt{\sigma}}{\sqrt{1 - \mu^2 - \sigma^2}} \frac{\sqrt{1 - \mu^2}}{n(1 - \mu^2)},
\]

for a given load component $X$, and for frequencies $\sigma << 1$, that the radial and circumferential displacements generated are of the same order of magnitude when $n$ is small and much larger than the axial displacement.

Inspecting equation set (20), it appears that the Type 2 wave associated with the propagation constant $\kappa_2$ remains evanescent regardless of the frequency. In fact, when axial inertial force is considered, it can be shown that this wave cuts on at [18]

\[
\Omega^2 = (1 - \mu)n^2 / 2.
\]

Accordingly, this wave only propagates in the $n = 1$ mode below the ring frequency. The influence of this change in character manifests itself in the real part of the mobilities by a high axial input mobility, small peaks of amplitude for $Y_{\psi}^n$ and $Y_{\psi}^n$ and local dips in the frequency curves for the remaining mobilities. However, since axial inertial force was not considered in the formulation of the problem, these patterns will not be reflected by the approximated solutions.

Tables 1 to 4 reflect the membrane-type behaviour of the shell for frequencies well above the cut-on frequency of the mode but below the ring frequency. Indeed, the existence of the Type 1 wave shows itself only in the imaginary part of the mobilities $Y_{\psi}^n$, $Y_{\psi}^n$, and $Y_{\psi}^n$, thereby resulting in a high compliance of the shell with respect to out-of-plane excitation. The vibrational power that is proportional to the real part of the mobilities is thus governed by the Type 2 waves and proportional to $\beta^{-1/2}$. The strong coupling between the different shell displacements is reflected by the fact that the cross-mobilities are of the same order of magnitude as the input mobilities. This coupling is also evidenced by the equations (A2-A5) provided in Appendix A, where the radial displacement in the far-field is of the same order of magnitude, whatever the load type exerted on the shell. Finally, as regard the dependence of input mobilities upon $n$, it can be seen in Tables 1 to 4 that the circumferential modes become either harder or softer with respect to in-plane and out-of-plane excitations respectively, as their order $n$ increases.
4.2 EXCITATION OF SHELLS BY RADIAL FORCE

Of considerable interest is the asymptotic expression of $Y_{wR}^n$ at very low frequency ($\Omega^2 << 1$). In this frequency range, the contribution of the Type 1 wave can be neglected and the input radial mobility reads:

$$Y_{wR}^n = \frac{n\omega R}{B(1-\mu^2)^{1/4}} \frac{\varepsilon_n^{-1}}{(\Omega^2 - \beta n^2 (n^2-1)^2/1+n^2)^{3/4}(1+1/n^2)^{-3/4}}$$

for $\Omega^2 << 1$ \hspace{1cm} (39)

For the beam mode ($n = 1$), assuming that a total force of 1 Newton is applied on the shell in such a manner that only the $n = 1$ mode is excited, the distribution of radial force is then given by $R_n = 1/\pi R$ N/m and the velocity at the excitation point is

$$\omega = Y_{wR}^n R_n = \frac{\omega(1-1)}{B(1-\mu^2)^{1/4} \Omega^{3/2} 2^{3/4} \pi} = \frac{1-i}{\rho Sc_B}$$

where $c_B = \sqrt{\omega_E^2/\rho S}$, $S$ is the shell cross-section ($S = 2\pi Rh$) and $I$ its moment of inertia ($I = \pi hR^3$). It can be seen that this corresponds to the mobility of a semi-infinite Euler beam excited at its end \cite{10}. For modes $n \geq 2$, the radial displacement due to a force distribution given by $R_n = 1$ N/m is

$$w = \frac{n(1-\mu^2)^{3/4}}{\beta^{5/4} E\sqrt{6(n^2-1)^{3/2}}} \quad \text{when} \quad \Omega^2 << \beta n^2 (n^2-1)^2/1+n^2$$

which is the result obtained by Simmonds \cite{8} in the static case when $2 \leq n \leq (1-\mu^2/4\beta)^{1/4}$.

Another point of significant interest is the comparison of these results to those for infinite shells. In Appendix B, the problem of the infinite shell excited in its "middle" by a radial force distribution is formulated according to the matched asymptotic expansions method. However, as the mathematical procedure is very similar, only the expression derived for the radial input mobility is presented, being:

$$Y_{wR}^{n,\infty} = \frac{\omega R}{4B} \left( \frac{i\sqrt{2}}{\beta^{1/4}(1-\mu^2-\sigma^2)^{3/4}} \right) + n\sqrt{1-\mu^2} \frac{\varepsilon_n(\sqrt{1-\mu^2} + \sigma)^{3/2} - i(\sqrt{1-\mu^2} - \sigma)^{3/2}}{\sigma^{3/2} \Delta^{3/4}(1-\mu^2-\sigma^2)^{3/2}}$$

and where

$$\varepsilon_n = \begin{cases} -1 & \text{if } \sigma^2 < 0 \\ +1 & \text{if } \sigma^2 \geq 0 \end{cases}$$

Assuming $\Omega << 1$, equation (42) reduces to the formula provided by Heckl \cite{2}. Furthermore, from Table 3 and equation (42), it can be shown that the ratio of the real parts of the mobilities for the infinite and semi-infinite shells is given by:

$$\text{Re}(Y_{wR}^{n,\infty})/\text{Re}(Y_{wR}^{n,1/2}) = (1+\sqrt{\Omega^2 - \beta n^4}/\sqrt{1-\mu^2})/4 \quad \text{where} \quad \beta n^4 < \Omega^2 < 1-\mu^2$$ \hspace{1cm} (44)
and the superscripts \( \infty \) and \( \infty/2 \) refer to the infinite and the semi-infinite shell respectively. The ratio given by equation (44) is independent of the circumferential modal number \( n \) and tends towards \( 1/4 \) at very low frequencies, which is the same as in the beam case [10].

Finally, the case of radial excitation lends itself to a study of the result's validity. From the expression for \( Y_{wr}^n \) given in Table 3, it can be seen that infinite shell responses are predicted at frequencies given by \( \sigma = 0 \) and \( \sigma^2 = 1-\mu^2 \). The former is to the lower cut-on frequency of the mode while the latter corresponds to a singularity introduced by the perturbation method. In fact results obtained numerically do indeed show a higher shell response in the vicinity of \( \sigma^2 = 1-\mu^2 \) for the lower circumferential modes, but the peak of magnitude remains finite; with its amplitude decreasing rapidly as the order \( n \) increases. It is worth noting that this type of non-uniformity is related to out-of-plane motions only and does not appear in the mobility expressions for the in-plane displacements \( u \) and \( v \). It can be shown that the non-uniformity region around \( \sigma^2 = 1-\mu^2 \), where the approximate solutions cease to be valid, grows proportionally to the factor \( \sqrt{\beta n^2} \). Removing this singularity can be achieved by introducing a detuning parameter in a manner similar to the one described in [9]. Once this is done, an approximate expression for the real part of the input radial mobility can be derived for frequencies contained in the range \( \beta n^2 < \Omega^2 \leq 1-\mu^2 \), reading

\[
\text{Re}(Y_{wr}^n) = \frac{\sigma R}{B n(1-\mu^2)} \frac{\sqrt{1-\mu^2 + \sigma}}{\sigma^{3/2} \sqrt{1-\mu^2 - \sigma^2} (\sigma^{3/2} + 2 \sqrt{\beta n^2 \sigma^2} )},
\]

(45)

which yields the following at \( \Omega = \sqrt{1-\mu^2} \):

\[
\text{Re}(Y_{wr}^n) = \frac{\sigma R}{n B \sqrt{2\beta(1-\mu^2)}}.
\]

(46)

5. COMPARISON WITH NUMERICAL RESULTS

In this section, the accuracy of the approximate solutions is shown by comparing mobilities calculated using the approximate solutions given by Tables 1 to 4 to numerical results obtained using Flügge theory [14,16]. The parameters of the shell used for the comparison are described in Table 5.

<table>
<thead>
<tr>
<th>Physical properties of the shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, ( \rho ) (kg/m(^3))</td>
</tr>
<tr>
<td>Young's modulus, ( E ) (N/m(^2))</td>
</tr>
<tr>
<td>Poisson's ratio, ( \mu )</td>
</tr>
<tr>
<td>Radius, ( R ) (m)</td>
</tr>
<tr>
<td>Thickness-to-radius ratio, ( h/R )</td>
</tr>
</tbody>
</table>
Consider a load type \( X \) applied at the edge of the semi-infinite shell and the resulting velocity \( \dot{\alpha} \) at a point in the shell section \( x = 0 \). \( X \) symbolises the excitation components \( N_x, T_x, R_x \) and \( M_x \), and \( \alpha \) symbolises one of the velocities \( u, v, w \) and \( \theta \), and are assumed to have Fourier series expansions of the form:

\[
X(x = 0, \phi) = \sum_{n=0}^{\infty} X_n \cos(n\phi - \frac{\pi}{2} \delta_{XT}) + \sum_{n=0}^{\infty} X_n \sin(n\phi - \frac{\pi}{2} \delta_{XT}) \quad \text{and} \quad \dot{\alpha}(x = 0, \phi) = \sum_{n=0}^{\infty} \dot{\alpha}_n \cos(n\phi - \frac{\pi}{2} \delta_{\alpha\alpha}) + \sum_{n=0}^{\infty} \dot{\alpha}_n \sin(n\phi - \frac{\pi}{2} \delta_{\alpha\alpha}),
\]

(47)

(48)

where

\[
\delta_{XT} = \begin{cases} 0 & \text{if } X \neq T \\ 1 & \text{if } X = T \end{cases} \quad \text{and} \quad \delta_{\alpha\alpha} = \begin{cases} 0 & \text{if } \dot{\alpha} \neq \dot{v} \\ 1 & \text{if } \dot{\alpha} = \dot{v} \end{cases}.
\]

In these expansions, the first sum is associated with the loads \( (N_x, T_x, R_x, M_x) \) and the solutions \( (u, v, w, \theta) \) having a dependence upon the azimuthal coordinate \( \phi \) of the form \( \cos(n\phi), \sin(n\phi), \cos(n\phi), \cos(n\phi) \) while the underlined terms are of the form \( (\sin(n\phi), \cos(n\phi), \sin(n\phi), \sin(n\phi)) \). In the problem formulation, the first kind of solution was assumed, yielding the mobility matrix \( [Y'_{\alpha}] \) as defined in equation (7). By a similar reasoning, a matrix \( [Y''_{\alpha}] \) can also be defined for the Fourier components \( \dot{\alpha}_n \) and \( X_n \). However, the form adopted in equations (46) and (47), ensues the equality \( [Y''_{\alpha}] = [Y''_{\alpha}] \), and from equations (48) and (7), it follows that

\[
\dot{\alpha}(0, \phi) = \sum_{n=0}^{\infty} Y''_{\alpha n} X_n \cos(n\phi - \frac{\pi}{2} \delta_{\alpha\alpha}) + \sum_{n=0}^{\infty} Y''_{\alpha n} X_n \sin(n\phi - \frac{\pi}{2} \delta_{\alpha\alpha}).
\]

(49)

The errors introduced by using the approximate solutions given in Tables 1 to 4 for the assessment of the real part of the input mobilities have been estimated in decibels. Figure 2 shows contour plots of the error in the \( \Omega-n \) plane for \( Y''_{\alpha}, Y''_{\alpha}, Y''_{\alpha}, Y''_{\alpha} \) and \( Y''_{\alpha} \) (isoline at 1.5 dB). The curves corresponding to the cut-off frequencies given by equations (35) and (38) are also shown. As it can be shown that the errors are approximately proportional to \( \beta^{1/4}n \) or to \( \beta^{1/2}n^2 \), these curves can then be extrapolated to any shell by keeping the \( \beta^{1/4}n \) product constant. However, in the case at hand, it must be realised that the plots given in Figure 2, though represented as continuous functions of \( n \), have a physical significance for integer values of \( n \) only. Figure 2 shows that the error increases proportionally to the circumferential order but is higher closer to the cut-off frequencies. Furthermore, the non-uniformity regions for \( Y''_{\alpha}, Y''_{\alpha}, Y''_{\alpha} \) around the singularity at \( \sigma^2 = 1 - \mu^2 \) are seen to grow larger as \( n \) increases.

Figure 2 (c) shows poorer agreement for \( Y''_{\alpha} \), even for modes of low circumferential order \( n \). This can be attributed to the fact that the following term in the expansion of the real part of \( Y''_{\alpha} \) is only an order of \( \beta^{1/4}n \) lower than the leading term, while for the other mobilities, there is a difference in the order of \( \beta^{1/2}n^2 \).

The approximate solutions were also used to estimate the shell response to point excitations applied at \( x = 0, \phi = 0 \). For point load, the Fourier components of the load \( X \) are given by:
\[
\begin{align*}
X_n &= \frac{X}{2\pi R} \tau_n \quad \text{for } X = N_s, R_s \text{ or } M_s \quad \text{and} \\
Y_n &= 0 \\
X_n &= 0 \quad \text{for } X = T_s,
\end{align*}
\]

where

\[
\begin{align*}
\tau_n &= 1 \text{ when } n = 0 \\
\tau_n &= 2 \text{ when } n > 0.
\end{align*}
\]

Figure 2. Contour plots of the error in \(Y^n_{\alpha \phi}\) in decibels in the \(\Omega-n\) plan: (a) \(Y^n_{\alpha N}\), (b) \(Y^n_{\alpha R}\), (c) \(Y^n_{\omega R}\), (d) \(Y^n_{\omega \ell}\); (white region) = error < 1.5 dB, (grey region) = error > 1.5 dB,

\(\quad\text{(-- --) = equation (35), (---) = equation (38).}\)

The direct mobilities are easily obtained by inserting equation (50) into equation (49):

\[
Y_{\alpha \phi} = \frac{\alpha(s = 0, \phi = 0)}{X} = \begin{cases} 
0, & \text{if } \alpha = \dot{\alpha} \text{ and } X \neq T \text{ or } \dot{\alpha} \neq \dot{\phi} \text{ and } X = T \\
\frac{1}{2\pi R} \sum_{n=0}^{\infty} \tau_n Y^n_{\omega \phi}, & \text{otherwise}
\end{cases}
\]

By way of illustration, Figure 3 shows the real part of the input radial mobility plotted against non-dimensional frequency and obtained using both Flügge theory and the approximate solution for \(Y^n_{\omega R}\) given in Table 3. The mobility is normalised by the corresponding mobility for a semi-infinite plate, \(Y^{\text{plate}} = 0.462/\sqrt{\rho D}\) [19]. The contributions of the circumferential
modes \( n = 1 \) and \( 5 \) to the total mobility are also shown. As expected the semi-infinite plate mobility is a good approximation of the frequency-averaged mobility of the semi-infinite shell when \( \Omega > 1 \). The maxima exhibited by the mobility correspond to the cut-on frequencies given by equation (40) while the beam mode \( (n = 1) \) exhibits a dip at the cut-on frequency given by equation (38). Finally, comparing approximate and numerical results shows acceptable agreement in the frequency region \( \Omega < 0.5 \), and the non-uniformity at \( \sigma^2 = 1 - \mu^2 \) clearly appears responsible for the increasing discrepancy as the frequency approaches the ring frequency.

![Figure 3](image-url)  

**Figure 3.** Real part of the input radial mobility for the cylindrical shell normalised by the semi-infinite plate one and contributions of different circumferential modes.  
\( (---) = \) Flügge theory, \( (----) = \) approximate solutions.

Figure 4 shows the real part of the direct mobilities defined in equation (51) plotted against non-dimensional frequency for the cylindrical shell described by Table 5. Equation (45) is used instead of the expression given in Table 3 for the input radial mobility. For the axisymmetric mode \( (n = 0) \), the approximate solutions derived in [9] were used together with the input circumferential mobility

\[
y_{\nu r}^0 = \frac{\omega R}{B} \frac{\sqrt{2}}{\Omega \sqrt{(1 - \mu)(1 + 3\beta)}}.
\]

(52)

Figure 4 shows that the approximate solution yields acceptable agreement for \( \Omega < 0.5 \). Above this frequency, agreement for the axial input mobility is strongly affected by the influence of inertial force in the axial direction of the \( n = 1 \) mode and for mobilities related to out-of-plane motions by the non-uniformity at \( \sigma^2 = 1 - \mu^2 \). Comparing Figures 3 and 4(c), it appears that the accuracy of the approximate solution is greatly improved when the singularity is removed as in equation (45).
Figure 4. The real part of the point mobilities: (a) $Y_{NN}^n$, (b) $Y_{VT}^n$, (c) $Y_{WR}^n$, (d) $Y_{NN}^n$.
(e) $Y_{NN}^n$, (f) $Y_{NN}^n$ and (g) $Y_{NN}^n$; (—) Flugge theory, (---) = Tables 1 to 4.

6. CONCLUSION

A perturbation technique has been applied to study the dynamic response of a semi-infinite, thin-walled shell to forces and moment applied at its edge. Considering a given circumferential mode of low order only ($n = 1, 2, 3...$), approximate closed-form solutions have thus been derived for the associated input and cross-mobilities. The perturbation analysis allows the bending effects localised in the edge region to be accounted for, while bringing to the fore the membrane behaviour of the shell for these modes of vibration. Solutions are obtained in the form of power series expansions, with the expansion parameter being a...
function of both the mode order $n$, and the shell thickness parameter. The analytical expressions derived display the functional dependence of the solutions on the problem parameters, revealing the strong coupling taking place between the in-plane and out-of plane motions of the shell below the ring frequency. Furthermore, owing to the waveguide behaviour of the shell, the expressions derived enable the prediction of the vibrational power transmission by point excitation. Finally, the accuracy of the expressions derived has been discussed in terms of both the individual modal mobilities and the point mobilities, showing that acceptable agreement is obtained for frequency below half the ring frequency.

ACKNOWLEDGEMENTS

The author would like to gratefully acknowledge the support for this work provided by the Swedish National Energy Administration, along with Professors S. Ljunggren and J-L. Guyader, and Dr. S. Finnveden for their valuable discussions.

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**APPENDIX A: GREEN'S FUNCTIONS FOR THE SEMI-INFINITE SHELL**

Being a stage in the derivation of the mobilities, Green's functions for the semi-infinite shell excited at the edge are reported here when far-field conditions are met, that is at a distance from the edge where the decaying standing wave and the evanescent wave can be disregarded in the expression of the solution.

In the frequency range \( \beta n^2 \ll \Omega^2 \ll 1 - \mu^2 \), the far-field criterion can be defined as

\[
\frac{n\sqrt{\Omega} x}{R} \gg 1,
\]

(A1)

and Green's functions from axial force, tangential force, radial force and bending moment to the radial displacement \( w_r \) read as follows respectively:

\[
G^N_w(x|0) = -\frac{R}{2B\sqrt{1-\mu^2}} \frac{1}{\sigma^{\Delta}} (1 + i \frac{\sqrt{1 - \mu^2 + \sigma}}{\sqrt{1 - \mu^2 - \sigma}}) e^{\frac{n\Delta^{1/4} \kappa_2 x}{R}},
\]

(A2)

\[
G^T_w(x|0) = \frac{R}{2B\sqrt{1-\mu^2}} \frac{\sqrt{1-\mu^2 + \sigma}}{\sigma^{3/2} \Delta^{3/4}} (1 + i \frac{\sqrt{1 - \mu^2 + \sigma}}{\sqrt{1 - \mu^2 - \sigma}}) e^{\frac{n\Delta^{1/4} \kappa_2 x}{R}},
\]

(A3)

\[
G^R_w(x|0) = -\frac{nR\sqrt{1-\mu^2}}{2B} \frac{1}{\sigma^{3/2} \Delta^{3/4}} (1 + i \frac{\sqrt{1 - \mu^2 + \sigma}}{\sqrt{1 - \mu^2 - \sigma}}) e^{\frac{n\Delta^{1/4} \kappa_2 x}{R}},
\]

(A4)

\[
G^M_w(x|0) = \frac{n^2 \sqrt{1-\mu^2}}{2B} \frac{1}{\sigma^{\Delta} (1 - \mu^2 - \sigma^2)} (1 + i \frac{\sqrt{1 - \mu^2 + \sigma}}{\sqrt{1 - \mu^2 - \sigma}}) e^{\frac{n\Delta^{1/4} \kappa_2 x}{R}},
\]

(A5)

where \( \Delta = 1 + \frac{1 - \Omega^2}{n^2} \), \( \sigma^2 = \Omega^2 - \frac{\beta n^2 (n^2 - 1)^2}{1 + n^2} \) and \( \kappa_2 = -i \frac{\sigma}{\sqrt{1 - \mu^2 - \sigma}} \).
APPENDIX B: INPUT RADIAL MOBILITY OF THE INFINITE CYLINDRICAL SHELL

Consider a shell of infinite extent vibrating in its \( n \)th circumferential mode as a result of a radial force distribution \( R_n \) applied in its “middle”. This problem can be transformed into a boundary-value problem by considering a semi-infinite shell and the following set of boundary conditions

\[
\begin{align*}
\left. u_n \right|_{x=0} &= 0, & \left. v_n \right|_{x=0} &= 0, & \left. w_n \right|_{x=0} &= 0, & \frac{R_n}{2} &= \frac{D}{R^3} (w'''' + (2 - \mu) w^{*''}) \bigg|_{x=0},
\end{align*}
\]  

(B1)

which are in accord with the symmetry of the infinite shell. Using equation (3), equations (B1) can then be expressed in terms of the function \( \Phi_n \). Adopting the form of the solution given by equation (11), and the inner axial variable \( \zeta \), the boundary conditions associated with the inner solutions are given by

\[
\begin{align*}
\Phi'_{k} &= 0, & \Phi'''_{k} + \frac{1}{\mu} \Phi'_{k-2} &= 0, & \Phi''''_{k} - \frac{1}{2 + \mu} \Phi''_{k-2} &= 0, \\
\Phi''''''_{k} - (4 - \mu)\Phi''''_{k-2} + (5 - 2\mu)\Phi''_{k-4} - (2 - \mu)\Phi_{k-6} &= \frac{R_n}{2} \delta_{k7}.
\end{align*}
\]  

(B2)

The matching conditions and the form of the composite solution are unaffected by the change of boundary conditions from equations (14) to equations (B2). Following a similar reasoning as the one adopted in Section 3.4, an approximate solution is finally obtained for the input radial mobility, being

\[
\begin{align*}
Y_{\omega R}^n &= \frac{\alpha R}{4B} \left( \frac{i\sqrt{2}}{\beta^{1/4}} \frac{1}{(1 - \mu^2 - \sigma^2)^{3/4}} + n\sqrt{1 - \mu^2} \right) \varepsilon_n \left( \sqrt{1 - \mu^2 + \sigma} \right)^{3/2} - i \left( \sqrt{1 - \mu^2 - \sigma} \right)^{3/2} \\
&\quad \times \frac{\sigma^{3/2} (1 - \mu^2 - \sigma)^{3/2}}{\sigma^{3/2} \Delta^{3/4}}.
\end{align*}
\]  

(B3)
Wind-Induced Vegetation Noise. Part I: A Prediction Model

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Summary
This paper describes a semi-empirical analytical model developed to predict ambient noise generated by vegetative sources during a windy day. The model is based on measurements carried out with vegetation samples placed in a flow. Typical noise spectra and the dependence of radiated noise on wind speed have been obtained for different tree species.

Then, by using simplified representations of the interaction of the wind flow with the foliage, analytical expressions giving the acoustic power generated by different configurations (single tree, forest, forest edges, shelterbelt) have been derived. The influence of the vegetation cover and the ground reflection on the sound propagation is taken into account so that the sound pressure level can be predicted at an arbitrary point. The model shows that the generating mechanisms are different for coniferous trees and deciduous ones. In a subsequent paper, a series of measurements carried out in the field to calibrate and validate the model are presented.

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1. Introduction
Discussing the environmental impact of industrial plants requires a reasonable understanding of what is being impacted. Indeed, it is widely reckoned that natural ambient noise gradually masks any non-indigenous sounds as the wind speed increases. Since the wind is a running condition for wind turbines, this masking is of prime importance in the assessment of their noise annoyance. Nevertheless, no consideration of this effect appears in the noise regulation actually in force in Sweden. The main reason for this situation is that very little is known about ambient noise, particularly in suburban and recreational areas where the vegetation is the most likely source of ambient noise during the windy days.

The wind-induced vegetation noise has received comparatively little attention in the literature. Jakobsen and Andersen [1] reported an extensive series of measurements on the generation of noise by the action of the wind on the vegetation and the measurement system. Ten years later, Sneddon et al. [2] and Boersma and Etienne [3] presented data from measurements in coniferous and deciduous forests, respectively. Concerning the existence of prediction models, Miller [4], and more recently van der Toorn [5], have proposed empirical methods. However, these models are fairly rough and lead to different estimates of the A-weighted sound pressure level. Thus, this paper aims at developing a semi-empirical analytical model for the sound pressure levels generated by some typical features of the rural landscape such as forests, forest edges, shelterbelts and single trees in order to predict their potential masking effect.

Two parts might be distinguished in the structure of the article. In a first part, the foundations of the model are discussed, supported by results from laboratory measurements made on some vegetation samples. The measurements have been carried out using typical Swedish tree species to provide insights into the noise generation in a situation where numerous parameters that are important outdoors can be disregarded (wind speed profile, distance from the source, ground form and ground impedance, etc.). Typical vegetation noise spectra and the dependence of the radiated power on the flow speed were thus obtained. The second part is devoted to the development of the analytical prediction model. The model considers both the noise generation and the noise propagation aspects. The first one is based on simplified representations of the interaction between the wind flow and the vegetation. The latter deals with the extended nature of the sources, the additional attenuation of the acoustic waves induced by the vegetation cover and the reflection by the ground.

2. Noise generation and propagation model
Deciduous and coniferous species are different with regard to the noise generation mechanisms. For deciduous species, the rustling of the leaves arises quite obviously from vibrations in the leaves induced by the unsteady contacts between the leaves and their neighbouring elements (other leaves, branches). This is easily shown by observing that a leaf alone in a flow generates almost no sound in contrast to two leaves between which contacts are allowed. Coniferous species generate noise in an aeroacoustical way similar to the whistling of telephone wires in the wind. This noise, referred to as aeolian tone in the literature, arises from the unsteady forces exerted by the needles or branches on the flow which are due to the von Kármán vortex shedding. It is known to be dipole-like and is easily recognisable as its spectrum exhibits a peak at the Strouhal frequency based on the needle or branch cross-section dimension and on the flow velocity. In the present case, such a sound is expected to be generated by needles and by deciduous tree branches in winter.

In the model, the whole canopy of volume $V$ is supposed to be composed of volume elements, itself containing a multitude of elementary sources, e.g. needles and leaves. Though
each of these elementary sources exhibits directivity patterns, dipole-like in the case of aeolian tones for instance, the directivity of a volume element is expected to be uniform due to the great number and the random orientation of the elementary sources it contains. Hence, a volume element might be modelled by a monopole with an acoustic power $\delta W$. Furthermore, it is expected that no correlation occurs between the volume elements. Then, the RMS pressure measured at the microphone is obtained by summing the contributions of all the volume elements over the total canopy, which yields

$$ p_{eff}^2 = \frac{pc}{4\pi} \int_{V} \int_{V} \delta W(r, f) \left[ \frac{e^{-jkr_1}}{r_1} + R \frac{e^{-jkr_2}}{r_2} \right] dV, \quad (1) $$

where $t$ is tacitly understood that the time factor is $e^{j\omega t}$. $\omega$ is the angular frequency, $i$ the imaginary unit, $t$ the time, $k$ the complex wave number, $\rho$ the air density, $c$ the sound speed, $R$ the ground reflection coefficient, $r_1$ and $r_2$ the geometrical path lengths of the direct and the reflected ray as described by Figure 1.

2.1. Propagation effects

For the sake of simplicity and because vegetative sources do not have a very high acoustic power, the propagation effects will be discussed for flat terrain only and limited to relative short ranges. Thus, it is reasonable to assume that the sound follows straight ray paths and to neglect the refraction of the acoustic waves due to the wind velocity and temperature gradients. Likewise, scattering of high frequency sound by wind turbulence is disregarded, since this effect affects mainly sources with high directivity, which is not the case here. Thus, only the ground effect and the attenuation due to the air and the vegetation remain to be assessed.

The attenuation coefficients, $\alpha_a$ and $\alpha_v$, for air and vegetation, can be found in the standards ISO/DIS 9613-1 and ISO/DIS 9613-2.2 [6, 7] and are introduced in the model by assuming a complex wave number $k$ ($\Imag[k] = -\alpha < 0$). The acoustic waves in a forest are submitted to multiple scattering from the trees and then attenuated by the various absorbing elements (ground, bark, leaves, etc.). Carlson et al. [8] and Emberton [9] have shown that the ground is responsible for the most part of the attenuation while the bark and the foliage have very little effect over the frequency range of interest. The attenuation by vegetation increases with frequency as the wavelength becomes of the same order of magnitude as the trunk and branch diameters. It should be noted that for frequencies higher than approximately 2000 Hz, multiple scattering by tree boles also affects the correlation between the direct sound from the source and its reflection by the ground.

The volume elements are monopole sources and thus generate spherical waves. Nevertheless, the use of the reflection coefficient for plane waves as adopted in the model, is justified for two reasons. The first is that grazing incidence is unlikely to occur here due to the height of the vegetative sources and the range of propagation distances considered. The second reason is that the cumulated height of the canopy and the microphone should be much larger than any wavelength in the frequency range of interest. Thus, the reflection coefficient, $R$, is given by

$$ R = \frac{Z \cos \theta - 1}{Z \cos \theta + 1}, \quad (2) $$

where $Z$ is the ground impedance normalized to $pc$. The ground is assumed to be a locally reacting surface and the Delany and Bazley model [10] is used to describe its impedance:

$$ Z = 1 + 0.0571 \left( \frac{\rho f}{\sigma} \right)^{-0.734} - j0.087 \left( \frac{\rho f}{\sigma} \right)^{-0.732}, \quad (3) $$

where $\sigma$ is the ground effective flow resistivity.

2.2. Acoustic power of vegetative sources

The acoustic power generated by the volume elements is assumed to be related to the local mean wind speed $u(r)$ and to the local leaf/needle area density $S(r)$ by the relationship

$$ \delta W(r, f) = AS(r)u^2\chi(r)\Gamma(r, f). \quad (4) $$

$A$ is a radiation constant depending on the tree species, $\chi$ is called the wind speed coefficient and reflects the dependence of the noise level on the wind speed. The function $\Gamma$ is the normalised noise spectrum of the sound generated by the canopy. The normalisation is achieved by setting $\int f_m = 1$ where $f_m$ is the frequency of the peak of noise emission. For instance, for conifers, the frequency $f_m$ is the Strouhal frequency.

2.2.1. Leaf Area Density $S$

The leaf area density (LAD), which is the total one-sided leaf (needle) area per unit of ground surface and unit of height ($m^2/m^3$), describes the foliage distribution. In the model, it will be supposed to have only a vertical variation. Typical LAD distributions are found in [11, 12, 13] for coniferous forests and in [14] for deciduous forests. From these references, it is seen that LADs are difficult to classify as they depend to a large extent on the growing conditions (sparse or dense stand, access to light, etc.). Anyway, concerning the present problem, it will be shown in section 5 that a high LAD is balanced by a low wind velocity in the canopy and
thus the LAD has little influence on the noise generation. As a matter of fact, the knowledge of the LAD will only be required for the deciduous vegetative obstacles case, inasmuch as the good access of light for this configuration should lead to a rather uniform LAD between the two levels $H_0$ and $H$, the LAD will be described by

$$S(z) = S_0 \left( U(z - H_0) - U(z - H) \right),$$

where $U$ is the Heaviside step function and $S_0$ the averaged leaf area density. The Leaf Area Index, LAI, $(m^2/m^2)$ is obtained by integrating the LAD in the vertical direction up to the canopy top [15].

2.2.2. Normalised frequency spectrum $\Gamma$

In Sweden, just a few species of trees predominate. The most common ones are spruces (47%), followed by pines (38%), with deciduous species accounting for the remaining 15%. Sound power measurements were carried out with different samples of tree species, namely birch, aspen, pine and spruce at the Marcus Wallenberg Laboratory for Sound and Vibration Research. The samples were placed in a silent flow delivered by an opening in a reverberant room (Figure 2) and the sound power was measured in narrow bands (bandwidth 6.25 Hz). The use of a reverberant room for narrow-band measurements is generally questionable but is justified here by the broadband character of the sources. Furthermore, the microphone was traversed at constant speed over a 7 m long circular path in order to reduce the influence of the normal modes of the room. Finally, the results presented later, e.g. spectra, wind dependence coefficient, frequencies of the emission peaks, were obtained after averaging the sound power levels over ten successive bands.

Table 1 gives the levels (dL) and the frequency, $f_m$, of the emission peaks at each speed for the different species. For coniferous species, the frequency $f_m$ corresponds to the Strouhal frequency $f_0$. The needle diameters $d_n$ reported in Table 1 have been calculated from the knowledge of $f_0$ and $u$ by using equation (7), together with a Strouhal number $St = 0.2$. Nonetheless, the experimental results obtained with the aspen sample should be considered with caution as the leaves were in a bad state at the time of the measurements.

a. Deciduous tree  Measured spectra at different velocities are presented in Figure 3 after conversion to 1/3 octave bands. As expected, the spectrum shape is very little changed by a velocity variation for deciduous species. Furthermore, the figure reveals clearly that the branches are responsible for the low frequency emission while the rustling of the leaves generates a high-frequency noise. In the future, it will be assumed that the aeroacoustic sources are negligible compared to the rustling-noise ones for deciduous trees in leaf. Hence, the function $\Gamma$ will be supposed totally velocity-independent. The following equation (6) gives the analytical expression of the narrow-band spectrum used for the future calculations with deciduous species:

$$\Gamma(f) = C_1 f^{-1} + C_2 e^{-C_3(f-f_m)^2}/f_m^2.$$  

The three constants $C_i$ are determined from the measurements and from the additional condition $C_1/f_m + C_2 = 1$. 

![Figure 2. Laboratory measurements with a deciduous species in the reverberant room.](image-url)
Table 1. Frequencies and levels at the peaks of the emission from narrow band measurements.

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<td>(L_W)</td>
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<td>(L_W)</td>
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<td>(L_W)</td>
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<td>[dB]</td>
<td>[Hz]</td>
<td>[dB]</td>
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<td>2841</td>
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</tbody>
</table>

which insures the normalization of the spectrum at the frequency of maximum of emission. Figure 4 shows that expression (6) fits the laboratory measurements quite well.

b. Coniferous species

Typical noise spectra in 1/3 octave bands are shown in Figure 5 at two velocities for both spruces and pines. It appears that conifer noise spectra are characterised by a strong emission at the Strouhal frequency \(f_0\) corresponding to the frequency of vortex shedding by the needles. This frequency depends on the Strouhal number \(St\) (typically 0.2), on the wind speed \(u\) and on the needle diameter \(d_n\),

\[
f_0 = St \frac{u}{d_n}.
\]

From Table I, it is seen that the characteristic needle diameter is about 1.3 mm for pine trees and 1 mm for spruces, which agrees well with the observed values. The narrow-band spectra of the conifers may, after normalisation, be described by a gaussian function (Figure 6) given by

\[
\Gamma(f) = e^{-\lambda \log^2(f/f_0)},
\]

where \(\lambda\) is a constant deduced from measurements (about 10 for pine trees and 15 for spruces). Though the measurement results have shown that the peak at the Strouhal frequency becomes slightly broader with increasing velocity, \(\lambda\) will be put constant in the model.

2.2.3. Wind speed coefficient \(\chi\)

The parameter \(\chi\) is obtained from the narrow band values of the emission peaks (cf. Table I) at four different velocities by using the relationship

\[
\chi = \frac{1}{6} \sum_{i=1}^{3} \sum_{j=i+1}^{4} \frac{L_W(u_i) - L_W(u_i)}{20 \log(u_i/u_j)}.
\]

The coefficients \(\chi\) estimated from the measurements are respectively 1.8, 2.7, 1.8 and 1.5 for the birch, the aspen, the pine and the spruce, respectively. The discrepancy between the experimental results obtained for coniferous species and the \(u^3\)-law derived by Curle [16] for aerodynamic noise in the presence of solid surfaces might be explained by the complex flow field in the needle structure of a branch.

2.2.4. Radiation constant \(A\)

The laboratory measurements did not lead to a reliable determination of \(A\) since the leaf area densities were not known for the different samples. Estimation of the LAD requires a knowledge of both the volume and the leaf area index of the samples, which could not be obtained without special instrumentation. But, even if the LAIs had been assessed by a Leaf Area Meter [14], small errors in the estimation of the sample volumes would have lead to rather big errors on the values of \(A\) due to the small size of the samples. To get a rough idea of the radiation constant for the different species, the leaf/needle
area density of the samples were supposed to be 1.0 m$^2$/m$^3$. By using the wind speed coefficients determined previously, the parameters $A$ estimated from the measurements are respectively $1.8 \cdot 10^{-13}$, $6 \cdot 10^{-16}$, $1.3 \cdot 10^{-13}$ and $6 \cdot 10^{-13}$ for the birch, the aspen, the pine and the spruce, respectively. It appears that the radiation constant for the aspen is much lower than for the others while the aspen is known to be the noisiest species. A possible explanation can be the bad state of the aspen sample during the measurements. The radiation constants $A$ for different tree species will be determined in a subsequent paper from field measurements.

3. Wind flow model

The treatment of the wind-induced vegetation noise rests on the description of the wind flow within the canopy. This section presents the analytical expressions used to describe the canopy flow in the model.

3.1. The infinite forest

In an infinite forest, the velocity in the canopy results from the downward flux of momentum through turbulent mixing at the canopy top and decreases rapidly downwards in the canopy [11, 17]. From the different models reviewed by Massman [18], Cionco's one is preferred because of its simplicity. It states that, for a homogeneous medium with an uniform LAD, the velocity profile follows an exponential decrease given by

$$u(z) = u_h e^{-(z/H-1)},$$

where $H$ is the height of the canopy, $u_h$ the wind speed at the canopy top and $\alpha$ the canopy attenuation coefficient for which typical values are presented in Cionco [19].

3.2. The vegetative obstacle

The term "vegetative obstacle" refers to upwind forest edges, shelterbelts and single trees. Compared to the open-field wind field, the velocity within the canopy results from a windward speed reduction as the wind flows towards the obstacle, followed by a momentum exchange with the foliage and an upward movement as the wind flows through it.

Some literature has been devoted to the blocking effect of the obstacle, responsible for the windward velocity reduction [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Nevertheless, in this paper, this effect is not of primary interest and the wind speed is assumed to be known at the obstacle upwind boundary.

Measurements from Raynor [31] and calculations from Wang and Takle [22] show that the windward velocity reduction might be assumed independent of the height for obstacles with a rather high porosity (50% or more). This assumption should be satisfied by common vegetative canopies and thus the velocity profile at the obstacle upwind boundary will be taken as the same as in the open field.

If the flow around a shelterbelt has received quite a lot of attention, little is known about the wind speed distribution inside the three-dimensional structure formed by the foliage. Looking at the momentum balance at a forest edge, it appears that the drag force is mainly balanced by the positive pressure gradient into the canopy and by the momentum exchange [13]. The upward air movement is then rather weak at the edge [28, 31, 22]. This might certainly be explained by the rather high porosity of vegetative obstacles. Smoke visualisations made by Miller [32] at a forest edge show that, for velocities higher than 2 m/s, the flow penetrates horizontally into the canopy and into the trunk space to be dissipated after a distance of approximately 2$H$. From [31], it is seen that the wind profile adopts an exponential form already at a distance of about 2$H$ or 4$H$ from the edge. Thus, two regions will be distinguished in an upwind forest edge: the edge region which stretches out over the region $z \in [0, 2H]$ and the inner region for which $z \in [2H, +\infty]$. Shelterbelts and single trees will only exhibit an edge region. In the edge region, the wind profile is assumed to be described by the relationship

$$u(x, z) = u_h \left(\frac{z}{H}\right)^p e^{-\left(S(z)/z_0\right)},$$

with $p = 1/\ln(H/Z_0)$,

where $H$ is the canopy height, $z_0$ the open field ground roughness, $S(z)$ the leaf area density function, $\zeta$ the horizontal attenuation coefficient and $u_h$ the velocity at the canopy top at the edge. The parameter $\zeta$ takes into consideration the pressure gradient, the momentum exchange with the foliage and any upward air movement and needs to be determined from measurements. It is seen that equation (11) is reduced to

$$u(z) = u_h \left(\frac{z}{H}\right)^p$$

at the edge, which is the approximation given by Panofsky and Dutton [33] of the logarithmic profile in open field. For the forest region, the wind profile given by equation (10) is well suited. By taking the LAD values used by Li et al. [13]
and $\zeta = 0.05$, equation (11) shows an acceptable agreement with the forest edge velocity profiles measured by Raynor [31], as depicted on Figure 7.

Concerning the single tree, the wind is allowed to flow through, around, over and even below it if the tree exhibits a high porosity near the ground (see [34]). This multiplication of flow paths results in a higher windward velocity reduction and leads to complicated three-dimensional flow patterns strongly dependent on the geometry of the obstacle. Nevertheless, Gross' numerical calculations have shown that the wind velocity is especially high on the frontal canopy surface and decreases rather quickly inside the canopy as for the forest edge. Thus it will assumed that equation (11) can be applied. A single tree will then be assimilated to a shelterbelt of length $L$ where $L$ is obtained from the canopy diameter $D$ by the relationship

$$L = \frac{\pi D}{2}. \quad (13)$$

4. Application to special cases

In this section, a simplification, called assumption (A), is first presented as it is an important step in the derivation of the analytical expressions. Afterwards the cases of the infinite forest, the downwind forest edge and the vegetative obstacle are treated.

4.1. Assumption (A)

For each case, derivation of analytical expressions is achieved by an uncoupling between the propagation effects and the noise-generation ones. This uncoupling, referred to as assumption (A) in the remaining of the article, rests on the fact that the generated noise, proportional to $u^{2x}$, originates first and foremost from the canopies outer surfaces since the wind speed decreases strongly within the foliage. A three dimensional canopy might then be modelled by equivalent sources located on its outer surface exposed to the wind; the calculation of the acoustic power of the equivalent sources taking into account the inner part of the foliage. Thus, the calculation of the propagation effects is greatly simplified since it only applies to the equivalent sources.

4.2. Noise in an infinite forest

The ambient noise in an infinite forest is given by equations (1) and (4) with the appropriate expressions for the wind profile and canopy density. The derivation of an analytical expression is possible if use is made of assumption (A) mentioned previously. In the present case, this assumption is justified by the rapid downward decrease of the wind speed in the canopy. This leads to model the forest by a distribution of equivalent sources distributed over the canopy top plane (plane $P$ in Figure 8). The strength of the equivalent sources is calculated by taking into account the inner canopy part but the propagation effects are only assessed for points of the plane $P$:

$$p_{\text{eff}}^2 = \frac{p_{\text{c}}^2}{4\pi} W_{\text{eq}}(f) J(f), \quad (14)$$

where

$$W_{\text{eq}}(f) = A \int_0^H S(z) u^{2x}(z) \Gamma(f,z) \, dz, \quad (15)$$

and

$$J(f) = \int_P \left[ \frac{e^{-jkr_1}}{r_1} + \frac{R e^{-jkr_2}}{r_2} \right]^2 \, dS. \quad (16)$$

Theoretically, equation (15) is evaluated from the knowledge of the wind velocity profile and the LAD. Nevertheless, this approach results in a non-physical sensitivity to the description of these two parameters in the high part of the canopy. The problem can be evaded by the introduction of a relationship linking velocity profile and LAD as described in the appendix.

Concerning the propagation effects, it can be guessed that the integration over the plane $P$ allows the effects of the coherence between the direct and reflected sound to be neglected. Indeed, the effects of the destructive interferences between direct and reflected path or the noise spectra are weak compared to the compact source case. The reason is that the forest is an extended source composed of monopoles not correlated with each other. Moreover, the vegetation noise is expected to be important at rather high frequencies, which strengthens the choice of the simplification.

Approximate expressions of equation (15) are derived in appendix for both deciduous and coniferous forests and only the results are presented here. The equivalent source power, $W_{\text{eq}} \, [\text{W/m}^2]$, is given by

$$W_{\text{eq}}(f) = \alpha_0 A C_f \frac{u^{2x}}{2C_d} F_{\text{f0}}(f), \quad (17)$$

where, for deciduous species,

$$F_{\text{f0}}(f) = \Gamma(f), \quad (18a)$$

while for coniferous species,

$$F_{\text{f0}}(f) = \sqrt{\pi} \frac{e^{k^2/r} + 2k(f)x}{\sqrt{r}} \text{erfc}(c(f)), \quad (18b)$$
with \( b(f) = \ln(f/f_0(H)) \), \( \tau = \lambda/\ln^2(10) \), \( c(f) = (\chi + b(f))\tau / \sqrt{\tau} \) and \( C_f = \tau_a \rho u^2 \). \( C_d \) is the drag coefficient of a volume element of canopy, \( \alpha_0 \) a constant of value 1.5, \( \tau_a \) the shear stress and \( C_f \) the stand drag coefficient, which represents the momentum exchange between the wind flow and the canopy. The details of the propagation integral calculations are not given here but can be found in Fégant [35].

For the propagation effects, it is assumed that \( H \gg h \). An approximate expression for equation (16) is therefore

\[
J = 4\pi \left( E_1(2\alpha \omega H) - 2m e^{2\alpha \omega n} \right) + \Re \{ e^{-2\alpha \omega n} E_1(2\alpha \omega (H + m - jn)) \},
\]

with

\[
m = \frac{\Re\{v\}}{|v|^2} H,
\]

\[
n = \frac{H}{|v|} \sqrt{1 - \left( \frac{\Re\{v\}}{|v|^2} \right)^2},
\]

where \( E_1 \) is the exponential integral function, \( \Re\{z\} \) and \( \Im\{z\} \) stand for the real and imaginary parts of the complex number \( z \), \( v \) the relative ground admittance \( (v = 1/Z) \) and \( \alpha_0 \) the attenuation coefficient in the forest.

4.3. Noise from a downwind forest edge

The slight downward expansion of the flow described by Li et al. [13] which occurs at the downwind edge is disregarded here. This is justified especially if the porosity of the canopy is low. As for the infinite forest, the canopy is represented by a distribution of equivalent sources of power \( W_{eq} \) distributed on the canopy top plane. Hence, it may be assumed that equations (17) and (18) which give the equivalent power are still valid and that only the propagation integral \( J \) must be re-assessed. The canopy top plane is now semi-infinite and, as shown by Figure 9, the propagation effects occur partly in the forest and partly in the open field. The propagation integral \( J \) is then written in the form

\[
J = \int_0^\infty \int_{L_1}^{L_2} \frac{e^{-\eta r_1} + Re^{-\eta r_2}}{r_1 + R e^{-\eta r_2}} \, dy \, dx,
\]

where \( L_1 = L/2 - y_0 \) and \( L_2 = L/2 + y_0 \).

Further assumptions are necessary in order to assess equation (22). The calculations take into consideration the change of medium from vegetation to air only with regard to the change of attenuation coefficient and not the possible reflection induced at the forest boundary. Again, the correlation between the direct and reflected field is disregarded in the calculations for the same reasons as those discussed previously in the forest case. Finally, the ground reflection coefficient is kept constant and equal to \( R_0 \) over the domain of integration. \( R_0 \) is determined from the angle \( \theta_0 \) (Figure 9) corresponding to the reflection angle experienced by the acoustic waves originating from the edge part the closest to the microphone. It is then assumed that the contributions from distant parts of the foliage should be sufficiently weak so that this assumption introduces only a negligible error. The analytical calculations [35] lead to

\[
J = \left( 1 + |R_0|^2 \right)^2 e^{2\alpha_0 - \alpha_0 \sqrt{d^2 + H^2}}
\]

\[
\sum_{i=1}^2 \Theta_i \left( 1 + \frac{(\alpha_0 - \alpha_0)\sqrt{d^2 + H^2}}{3} \right)
\]

\[
E_1 \left( 2\alpha_0 \sqrt{d^2 + H^2} \right) - \frac{\Theta_i^2}{6} e^{-2\alpha_0 \sqrt{d^2 + H^2}}
\]

where \( \Theta_1 = \arctan(L_i/\sqrt{d^2 + H^2}) \).

4.4. Noise from a vegetative obstacle

The geometry of the obstacle is defined by its length, \( L \), its height, \( H \), and its depth, \( D \). The receiver has the coordinate \( M = (x, y, z) \) in the coordinate system described in Figure 10 and should, in any case, be within the foliage. The sign (+), respectively (−), corresponds to observer located on the downwind, respectively upwind, side of the obstacle. Deciduous trees will be modeled with an uniform LAD of value \( S_0 \) between the heights \( H_0 \) and \( H \). Nevertheless it will be seen that these heights are only required in conifers case.

The obstacle is divided into strips of depth \( D \) and height \( H \) and length \( dy \). The upwind forest edges case seems at first more complicated as the theoretical depth is infinite.
Table II. Difference of noise generation between the edge and the inner regions.

<table>
<thead>
<tr>
<th>Distance $d$ [m]</th>
<th>10</th>
<th>25</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{\text{edge}} - L_{\text{forest}}$ [dB]</td>
<td>17</td>
<td>15</td>
<td>13</td>
<td>12</td>
</tr>
</tbody>
</table>

Nevertheless, it has been seen previously (cf. section 3.2) that these obstacles exhibit an edge and an inner region. The respective noise levels generated by these two regions have been calculated for several distances $d$ using Raynor's measured wind profiles (Figure 7 and [31]). The calculation results, listed in Table II, shows that the sound level generated by the edge region is always 10 dB greater than the inner region one for distances $d$ up to 100 m. Thus, the contribution of the inner region will be neglected in the future and upward forest edges will be considered as obstacles of depth $D = 2H$. Hence, the upward edge forest, the shelterbelt and the single tree might be the object of the same type of calculation. In accordance with assumption (A), each strip is represented by a monopole source located at $(0, y, \delta = (H_0 + H)/2)$ where $\delta$ is called the equivalent height. As a matter of fact, assumption (A) would have lead to a distribution of equivalent monopoles on the line $(0, y, z \leq [H_0, H])$. The additional simplification assumed here to have only one monopole at an equivalent height is not drastic and will introduce an error for low $d/H$ ratios only. Thus, the integral in equation (1) is again split into a noise generation and a noise propagation parts because of the rapid horizontal decrease of the wind velocity within the canopy. The RMS pressure at the receiver $M$ is then given by

$$P_{\text{eff}}^2(f) = \frac{pcW_{eq}(f)}{4\pi} J(f),$$

where

$$W_{eq}(f) = A \int_{y_0}^{H_0} \int_{0}^{D} S(z) u^2(x, z) \cdot \Gamma(x, z, f) \, dx \, dz,$$

and

$$J(f) = \int_{-L_2}^{L_1} \left[ e^{-jkr_1} \frac{1}{r_1} + e^{-jkr_2} \frac{1}{r_2} \right] dy.$$  

To perform the integration, the correlation between the direct and reflected waves is taken into account but the reflection coefficient is kept constant and equal to $R_0$ for all the canopy points. This coefficient is calculated from the incidence angle $\theta_0$ which corresponds to the closest edge point to the microphone (Figure 10). Furthermore, an equivalent attenuation coefficient $\alpha$ given by equation (27) may be derived which takes into account the propagation in both the air and the vegetation. As in the downwind forest edge case, reflections at the interface of the media are neglected.

$$\alpha = \begin{cases} \alpha_d \frac{d - D}{d} + \alpha_v \frac{D}{d} & \text{downwind case}, \\ \alpha_v & \text{upwind case}. \end{cases}$$

Approximate expressions for equation (25) are derived in appendix for both deciduous and coniferous species. The equivalent source power in [W/m] is given by

$$W_{eq}(f) = \frac{A u_2^2 H}{2\pi} F_{eq}(f),$$

where, for deciduous species,

$$F_{eq}(f) = \frac{\tau}{\chi(2\chi p + 1)} \left(1 - \frac{H_0}{H} e^{2\chi p + 1}\right) \left(1 - e^{-2\chi D_S}\right),$$

while for coniferous species,

$$F_{eq}(f) = \frac{\sqrt{\tau} e^{2\chi p + 1}}{\sqrt{\tau}} \left[ \text{erfc} \left[ c(f) \right] - \text{erf} \left[ c(f) \right] \right] - \frac{H_0}{H} \text{erf} \left[ c(f) - \frac{1}{2\sqrt{\tau}} \right] + e^{2\chi p + 1} \left( \text{erf} \left[ c(f) + \frac{1}{2\sqrt{\tau}} \right] - \text{erf} \left[ c(f) - \frac{1}{2\sqrt{\tau}} \right] \right).$$

The calculations of the propagation effects can be found in [35]. It is shown that an approximate expression of $J$ is

$$J(f) = \sum_{m=0}^{\infty} \left[ \frac{1 + |R_0|^2}{r_0} \Theta_0 e^{-2\alpha r_0} \cdot \frac{\Theta_1}{m(2m + 1)} \cdot \left\{ (-1)^m \Theta_2^m \cdot \frac{J_0(\epsilon r_0)}{m r_0} \sin(m\Theta_0) + 2 \sum_{m=1}^{\infty} J_m(\epsilon r_0) \cos(m\Theta_0) \cdot \cos(\beta_0 - m\pi/2) \right\} \right].$$

where $\epsilon(f) = (\chi + b(f)) \sqrt{\tau}$, $\Theta_i = \arctan(L_i/r_0)$, $r_0 = \sqrt{d^2 + \delta^2}$, $\epsilon = 2k(f) \delta h$ with $\delta > h$, $R_0 = |R_0|e^{ib_0}$, $R_0$ is the value of the ground reflection coefficient with respect to the nearest point of the obstacle (Figure 10) and $\beta_0$ its argument. $J_m(z)$ is the Bessel function of order $m$.

5. Discussion

According to Landsberg and James [11], the drag coefficient $C_d$ for spruces is close to 0.32. Additionally, numerical calculations of the stand drag coefficient $C_d$ as a function of the product $C_d LAI$ have been performed by Massman [18]. The results, depicted in Figure 11, show that $C_d$ is constant and about 0.2 for values of $C_d LAI$ greater than 0.6. Thus,
equations (17) and (18) state that the power generated is independent of the leaf area index if the product $C_d LAI$ is greater than 0.6, a condition which should be fulfilled in practice for both coniferous species and deciduous species in leaf. This result might first seem peculiar but is explained by the fact that a dense canopy will exhibit a greater velocity gradient at the top than a sparse one and this effect balances the LAD difference.

For a vegetative obstacle, this counter-effect is also present and equations (28) and (29) show that the LAD is a minor parameter for the determination of the acoustic power. Thus, it does not matter how dense the canopy is. For coniferous species, this effect is enhanced by the noise spectrum variation as the wind velocity rapidly decreases within the foliage. For leafless deciduous trees, the noise is generated in an aeroacoustic way as the wind blows through the branches. Nevertheless, equation (29) might not be valid as the typical branch area density is low.

From equations (28) and (29a) it appears that small values of the product $\zeta S_0 D$ lead to

$$W_{eq}(f) = \frac{\bar{A}(f)\mu A^2}{2\chi^2 \lambda H} D S_0 \left(1 - \left(\frac{H_0}{H}\right)^{2\chi+1}\right). \tag{31}$$

Equation (31) shows the power emitted is proportional to the depth $D$ of the obstacle only if the leaf area density $S_0$ or the horizontal attenuation coefficient $\zeta$ tends to zero. For deeper dense obstacles, an asymptotic value is reached as the exponential tends to zero and the noise generation is neither correlated to the LAI or the obstacle depth.

Furthermore, if the obstacle length is small, e.g. as for a single tree, it may be shown that equation (30) is reduced to

$$J(f) = \frac{e^{-2\sigma r_0^2}}{r_0^2} L \left(1 + |R_0|^2 \right) + 2|R_0| \cos(\epsilon/r_0 - \beta_0), \tag{32}$$

which shows that, for short obstacle lengths, the total expression for the squared pressure at the receiver is proportional to the product $HL$. This agrees well with the observation made by Miller [4], saying that the radiated noise is more correlated to the frontal area of the canopy than to its volume.

Nevertheless, he concluded that this is the result of a strong attenuation undergone by the acoustic waves generated by the deep canopy regions. It seems more probable that the correct explanation is the decrease of wind velocity into the canopy.

Computer simulations have been carried out to test the validity of the simplifications in the analytical expressions of the propagation integral $J$ in the three previous cases, namely the infinite forest, the downwind forest edge and the vegetative obstacle. Figure 12 shows that a good agreement has been obtained in the cases of the forest and of the vegetative obstacle. This supports the assumption to neglect the correlation between direct and reflected sounds in forests. Also, keeping the reflection constant for the vegetative obstacle does not introduce big discrepancy. Nonetheless, it is seen that the correlation is responsible for significant destructive interferences in the case of the downwind forest edge and the vegetative obstacle. But, since the correlation has been disregarded for downwind forest edges, the model fails to reproduce these dips at low frequencies.

It is interesting to compare the noise generation and propagation from both an upwind and a downwind forest edge. The noise generations for those two configurations are given by equations (17) and (28) for deciduous species. By assuming a rather dense foliage which is uniform from the ground to the canopy top, equation (28) for the upwind edge is simplified to

$$W_{eq,up}(f) = \frac{\bar{A}(f)\mu A^2}{2\chi^2 \lambda H} \cdot \frac{D S_0}{D_{canopy}}. \tag{33}$$

Thus the difference of noise generation with a strip of depth $D$ is given by

$$\Delta = 10 \log \left(\frac{W_{eq,up}}{W_{eq,down}}\right) \approx 10 \log \left(\frac{H C_d}{\alpha_0 C_f D \zeta (2\chi + 1)}\right), \tag{34}$$

and by taking typical values for the constants ($\alpha_0 = 1.5$, $\zeta = 0.05$, $\chi = 2.5$, $C_d = 0.32$, $C_f = 0.2$, $p = 0.2$)
and \( D = H \), the difference is \( \Delta \approx 10 \text{ dB} \). Thus, an upwind edge is much more noisier than a downwind one for the same velocity at the canopy top. Furthermore, it should be added that the wind speed is likely to be higher at the upwind edge than at the downwind one. Nevertheless, the noise level will decrease faster for increasing distances \( d \) at an upwind edge than at a downwind one as shown by Table II.

Deciduous species exhibit a strong noise emission at about 4000 Hz, so this frequency is of great interest for any consideration on the propagation effects. The difference in the propagation effects for the two configurations is calculated by using equation (23) and (30). At this frequency, the correlation term appearing in equation (30) may be disregarded. Furthermore, the values of the attenuation coefficient are mainly due to the variation in the air and are thus very low. This implies that the first sum in equation (30) may be reduced to its first term and

\[
\sum_{m=0}^{\infty} (-1)^m \frac{(ar_0 \Theta_f^2)^m}{m!(2m+1)} \approx 1.
\]

Figure 13 represents the expected decrease of the sound pressure level as the microphone is moved further from the forest edge for both the upwind and downwind cases. It appears that the attenuation increases faster for the upwind configuration as expected and the difference in attenuation between the two configurations is about 3 dB at 50 m.

Another point of interest is the influence of the vertical profile on the noise generation for a vegetative obstacle. It has been seen that the atmospheric turbulence and the ground roughness affect only very little the blocking effect. Nevertheless, the wind profile, and thus the noise power generation, depend on \( z_0 \). Furthermore, an important parameter of the model is the velocity at the canopy top which might be rather difficult to measure in practice. Instead, the experimenters might choose to measure the wind speed at an intermediate height \( H' \) and then extrapolate it to the top canopy using equation (12). To operate this way, the ground roughness is needed and might be, for instance, determined approximately by looking at the ground aspect [36].

The error made by choosing a wrong ground roughness has been estimated for deciduous trees by assuming an uniform foliage leaf area density. Figure 14 shows that the error depends on the height at which the velocity is measured and the lower the measurement point, the stronger the error. Nevertheless, as the ground roughness might be assessed in practice with a rather good accuracy, it is seen that the error introduced is fairly small.

6. Conclusion

A semi-empirical analytical model has been presented to predict the ambient noise due to vegetation. It allows not only the A-weighted sound level to be predicted but also the spectrum shape, which is important for, e.g., the assessment of the masking effect on WT noise by wind-induced vegetation noise. The model takes into account the wind field distribution into the canopy and has shown that dense foliage results in strong velocity decrease. It is shown that the canopy density affects the noise generation only slightly since a change in this density is counter-balanced by a change in the velocity. Furthermore, the model highlights the differences between deciduous and coniferous species. On one hand, deciduous species are characterised by an important noise emission at about 4000 Hz regardless of the wind speed. On the other hand, for coniferous species, the spectrum exhibits a peak at the Strouhal frequency, determined by both the velocity and the needle diameter, and is thus strongly speed dependent. Though the investigation has been limited to some typical features of the rural landscape, such as forests, forest edges, single trees and shelterbelts, it is believed that almost any practical vegetative configurations might be decomposed in such elementary features.

A series of measurements in the field carried out to calibrate and validate the model will be presented in a subsequent paper. The calibration will serve to determine the radiation constant for different species as well as to provide some typical horizontal attenuation coefficients for different vegetative obstacles.
Appendix

A1. Forest noise generation

In a coniferous forest, the noise spectrum is given by equation (8). Then the equivalent power per m² is

\[
W_{eq}(f) = A \int_{0}^{H} S(z) u^2(z) e^{-\lambda (\log(f) - \log(0.2u(z)/d_0))^2} \, dz.
\]  

(A1)

To evaluate equation (A1), it is imperative to take into account the coupling between the velocity profile and the leaf/needle area density, i.e. the momentum transfer within the canopy. A first-order closure model [18] is adopted, which states that

\[
\tau_s(z) = \rho K(z) \frac{du(z)}{dz},
\]

(A2)

and

\[
\frac{d\tau_s(z)}{dz} = \rho C_d S(z) u^2(z),
\]

(A3)

where \( \tau_s \) is the shear stress into the canopy, \( K \) the eddy diffusivity and \( C_d \) the drag coefficient of a volume element. Furthermore, it will be assumed that the drag coefficient does not depend on the velocity. Indeed, the turbulent diffusivity \( K \) is known to have an exponential variation with the height [37] if the wind speed dependence of \( C_d \) is not taken into account.

According to Massman,

\[
C_d = \text{constant} \quad \text{and} \quad K(z) = K_h \frac{u(z)}{u_h}.
\]

(A4)

After some manipulations, equation (A1) is reduced to

\[
W_{eq}(f) = \frac{A}{C_d} \frac{K_h a}{u_h H} \frac{e^{-\tau f(f)}}{\sqrt{\pi} u_h^2} \int_{0}^{\tau} e^{-\tau z^2} \frac{1}{\sqrt{uv}} e^{-uv} \, dv,
\]

(A5)

with \( c(f) = \left( \chi + b(f) e^r \right) \sqrt{\lambda} \), \( b(f) = \ln(f / f_0(H)) \) and \( r = \lambda / \ln^2(10) \).

As \( \tau^2 \) is rather big, the integral may be extended to infinity and is seen to be a well-known Laplace transform, which yields

\[
W_{eq}(f) = \frac{A}{C_d} \frac{K_h a}{u_h H} \frac{e^{-\tau f(f)}}{\sqrt{\pi} u_h^2} \frac{1}{\sqrt{\lambda}} e^{x^2 / \tau + 2x f} \operatorname{erfc}(c(f)).
\]

(A6)

The quantity \( K_h a (u_h H) \) is assessed by writing that the eddy diffusivity at the top of the canopy is given by

\[
K_h = \alpha_0 K u^* |_{z=H} (H - d_0),
\]

(A7)

where \( u^* \) is the friction velocity, \( \alpha_0 \) a constant between about 1 and 2, \( k \) is the von Kármán constant (\( k = 0.41 \)) and \( d_0 \) is the height of the zero plane displacement. The stand drag coefficient is defined by

\[
C_f = 2 \left( \frac{u^2}{u_h^2} \right) |_{z=H} = \frac{\tau_s(H)}{\frac{1}{2} \rho u_h^2}.
\]

(A8)

Then,

\[
K_h \frac{a}{u_h H} = \alpha_0 k \sqrt{\frac{C_f}{2 \frac{H}{H}}}. \quad \frac{H - d_0}{d_0}.
\]

(A9)

The velocity profile above the canopy is given by [33]

\[
u(z) = u_h \ln \left( \frac{z - d_0}{z_0} \right) - \ln \left( \frac{z}{z_0} \right),
\]

(A10)

where \( z_0 \) is the ground roughness. The attenuation coefficient \( \alpha \) may then be related to \( d_0 \) by equating the first derivatives of the two profiles (logarithmic profile equation (A10) and exponential profile equation 10) at \( z = H \), which results in

\[
\frac{H - d_0}{H} = \ln \left( \frac{(H - d_0)/z_0}{1} \right).
\]

(A11)

Finally, still according to Massman, the ground roughness is given by

\[
\frac{z_0}{H} = \left( 1 - \frac{d_0}{H} \right) e^{-k \sqrt{C_f / \lambda}}.
\]

(A12)

Using equations (A11) and (A12) into equation (A9) leads to

\[
K_h \frac{a}{u_h H} = \alpha_0 \frac{C_f}{C_d}.
\]

(A13)

Thus, the power generated by an element of volume \( H \, dx \, dy \) is given by

\[
W_{eq}(f) = \frac{\sqrt{\pi} \alpha_0 u_h^2 A_2 C_f}{C_d} \frac{e^{x^2 / \tau + 2x f}}{\sqrt{\tau}} \operatorname{erfc}(c(f)).
\]

(A14)

For deciduous trees, it has been seen that the noise spectrum does not depend on the wind velocity and from the foregoing equation, it is straightforward to derive the generated power by letting \( \lambda \) (i.e. \( \tau \)) tend towards zero. Then it becomes [38]

\[
\sqrt{\pi} c(f) e^{x^2 / \tau + 2x f} \operatorname{erfc}(c(f)) \rightarrow 1 \quad \text{when} \quad \tau \rightarrow 0.
\]

The power generated for deciduous tree is then given by

\[
W_{eq}(f) = \alpha_0 A \frac{C_f}{2 \frac{C_d}{2}} \frac{u_h^2}{H} \Gamma(f).
\]

(A15)
A2. Vegetative obstacle noise generation

By inserting equation (11), which gives the velocity profile within the canopy, into equation (25), the equivalent power of a strip, [W/m], is

\[ W_{eq}(f) = A u_h^2 x H \int_{0}^{H} \int_{0}^{D} S(z) \left( \frac{z}{H} \right)^{2xp} e^{-2xS(z)z} \Gamma(x, z, f) \, dz \, dx. \]  \hspace{1cm} (A16)

Deciduous trees

As the noise spectrum does not depend on the wind velocity for deciduous trees, the integration along the obstacle thickness is straightforward and leads to

\[ W_{eq}(f) = A \Gamma(f) u_h^2 \frac{H}{2zH} \int_{H/2z}^{1} w^{2xp} \left( 1 - e^{-2zS(Hw)^2} \right) \, dw. \]  \hspace{1cm} (A17)

By assuming an uniform LAD $S_0$ between the levels $H_0$ and $H$, the expression becomes

\[ W_{eq}(f) = \frac{A \Gamma(f) u_h^2 H}{2z(2z + 1)} \left( 1 - \left( \frac{H_0}{H} \right)^{2xp+1} \right) \left( 1 - e^{-2zS_0} \right). \]  \hspace{1cm} (A18)

Conifers

The noise spectrum is now space-dependent and the needle area density is assumed to be an unknown function $S(z)$ between the levels $H_0$ and $H$, and 0 otherwise. Thus,

\[ W_{eq}(f) = A u_h^2 e^{-z^2(f)} \int_{H_0}^{H} S(z) \left( \frac{z}{H} \right)^{2xp} e^{-z^2(S(z)z)^2} \Gamma(x, f) \, dz. \]  \hspace{1cm} (A19)

where

\[ b(f) = \ln \left( \frac{f}{f_0(0, H)} \right), \]

and

\[ I(z, f) = \int_{0}^{D} e^{-2zS(z)z} e^{(z + b(f))z - \tau \ln(z/H)} \left[ e^{-z(S(z)z)^2} \right] \, dz. \]

By using the same Laplace transform as in Appendix A1 and providing that the needle area density is not too small between the two levels $H_0$ and $H$, the result becomes

\[ I(z, f) = \frac{\sqrt{\pi}}{2\sqrt{\pi}zS(z)} e^{(x+b(f))z - \tau \ln(z/H)} \left[ e^{-z(S(z)z)^2} \right] \frac{1}{\tau} \, \text{erfc} \left[ \frac{x + b(f)z - \tau \ln(z/H)}{\tau} \right]. \]  \hspace{1cm} (A20)

Replacing this expression into equation (A19) and performing the integration gives the final expression

\[ W_{eq}(f) = A u_h^2 \frac{H}{2zH} \frac{\sqrt{\pi} e^{x^2/\tau^2} + 2H \left( \Gamma(x, f) \right) \frac{1}{\sqrt{\pi} \tau} \left[ \text{erfc} \left[ \frac{x + b(f)z - \tau \ln(z/H)}{\tau} \right] \right]}{2\sqrt{\pi}zS(z)} \]  \hspace{1cm} (A21)

where

\[ c(f) = \frac{x + b(f)z}{\sqrt{\tau}}. \]

Acknowledgement

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References


Wind-Induced Vegetation Noise. Part II: Field Measurements

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Summary
In a preceding paper [1] the development of a semi-empirical analytical model for the prediction of the wind-induced vegetation noise was presented. The present paper is devoted to the calibration and the validation of this model by field measurements. It is shown that the model gives reliable predictions, both in terms of frequency distribution and A-weighted sound pressure levels, near the vegetative sources, while some discrepancies between theory and measurements appear when the propagation distance increases. The validation is based on 41 measurement results obtained at seven sites for different propagation distances and wind speeds. It is seen that the model predicts the A-weighted level with a mean error of 0.25 dB and a standard deviation of 3.7 dB. Nevertheless, this standard deviation is reduced to 3.0 dB if some suspicious data are removed from the validation.

PACS no. 43.28.Fv, 43.28.Py

1. Introduction

The model for the prediction of noise from vegetative sources developed in [1] is based on laboratory measurements carried out with vegetation samples and on simplified expressions of the wind flow through foliage. Field measurements are now a necessary step for both the calibration and validation of this model.

The results from the literature devoted to the subject [2, 3, 4, 5] are not suitably reported to give to the model the input data needed for the present purpose. Thus, a series of measurements were carried out in summertime and their results, presented here, have been used to both calibrate and validate the model. Seven measurement sites were selected and are namely, a coniferous forest, a single tree and five upwind forest edges (both deciduous and coniferous). At these places, the noise level was recorded for different wind speeds and at several distances from the vegetation. Four of these places have served to calibrate the model by providing empirical parameters such as the radiation constant for different species, the horizontal attenuation coefficient and typical Strouhal numbers, while the three remaining sites have been used for testing the accuracy of the model predictions.

2. Experimental setup and instrumentation

The wind speed was measured by a cup anemometer with an accuracy of ±0.4 m/s at 4 m/s. The anemometer was mounted on a 10 m mast at a height \( H_a \) above the ground varying from site to site. Simultaneous wind speed measurements using a second anemometer located at some distance within the canopy have also been performed in order to evaluate the horizontal or vertical attenuation coefficients at some sites. The wind direction values presented later come from estimates.

Table I. Attenuation by 1/3 octave band for the Rycote windscreen [7].

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>&lt; 400</th>
<th>500</th>
<th>630</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attenuation (dB)</td>
<td>0.0</td>
<td>0.3</td>
<td>0.9</td>
<td>2.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>1000</th>
<th>1250</th>
<th>1600</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attenuation (dB)</td>
<td>2.7</td>
<td>2.5</td>
<td>0.0</td>
<td>0.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>2500</th>
<th>3150</th>
<th>4000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attenuation (dB)</td>
<td>1.4</td>
<td>0.7</td>
<td>2.4</td>
<td>2.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>6300</th>
<th>8000</th>
<th>10000</th>
<th>12000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attenuation (dB)</td>
<td>3.4</td>
<td>3.2</td>
<td>4.1</td>
<td>4.6</td>
</tr>
</tbody>
</table>

For the noise measurements, two different techniques, both using a 1/2" B&K microphone, were applied. In both methods, the microphone was connected to an analyzer meeting the requirements of IEC Publication 225 [6] and recording the 1/3 octave band spectra for frequencies ranging from 50 Hz up to 10000 Hz. Wind speeds and noise signals were recorded simultaneously with identical integration time. This time was fixed to 1 minute at sites 1 and 2 and 10 seconds at sites 3 to 7.

The first measurement method is the free microphone technique in which the microphone is mounted on a tripod and protected by a 10 cm foam windscreen. The height of the microphone above the ground was fixed to 1.4 m. At all the sites except the one in the forest, an additional Rycote windscreen was also used in order to reduce the pseudo-noise. In order to reproduce the measurement results, the level decreases caused by the Rycote windscreen reported in Table I were taken into account as the attenuation is not insignificant for middle and high frequencies.

The second method is the ground board measurement technique described by Ljunggren [8]. This method has been used in order to provide data free from the influence of the ground. The microphone was taped on a square board of 1.0 m edge length and protected by an hemispherical 10 cm foam ball.
Reflection by an infinite ground board on which is taped a microphone of infinitely small diameter will lead to pressure doubling that simply adds 6 dB to the free field signal. In practice, departures from pressure doubling occur at both low (up to 300 Hz) and high frequencies due to the finite sizes of the board and of the microphone, respectively. The low frequency effects will be neglected assuming that the ground is hard in this frequency range. At high frequencies, the pressure will be calculated by assuming that the reflection coefficient is 1 and that the microphone height is \( h = d_m/2 \) where \( d_m \) is the microphone diameter.

The measured 1/3 octave band spectra, the A-weighted sound pressure levels and the corresponding measured wind speeds, reported in the calibration and the validation sections, are the averages of 10 s (or 1 min.) values. The average, for the sound pressure level, was taken as

\[
\bar{L}_p = \frac{1}{n} \sum_{i=1}^{n} L_{pi},
\]

where \( L_{pi} \) is the \( i \)-th 10 s (or 1 min.) sound pressure level value. For the wind speeds, the average is obtained by

\[
\bar{u} = \sqrt[n]{\prod_{i=1}^{n} u_i},
\]

where \( u_i \) is the \( i \)-th wind speed value. For deciduous species, the averaging is performed with all the \( n \) obtained at wind speeds higher than 2 m/s. For conifer species, due to the wind speed dependence of the spectral shape, the averages are calculated over several wind speed intervals (2, 3 m/s; 3, 4 m/s, etc.).

3. Description of the measurement sites

Geometry, uniform canopy and reasonably flat terrain were the main criteria which have guided the selection of the measurement sites. A short presentation of each site is given below together with some illustrations showing their mapping for the purpose of calculation. The site parameters, the highest canopy height (\( H \)) and the lowest canopy hei
Table II. Measurement descriptions (the superscripts b and m above the propagation distances stand for the ground board and the free microphone, respectively).

<table>
<thead>
<tr>
<th>site</th>
<th>main type</th>
<th>species</th>
<th>$H$ [m]</th>
<th>$H_0$ [m]</th>
<th>$H_a$ [m]</th>
<th>propagation distance [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>forest</td>
<td>pine</td>
<td>14</td>
<td>5</td>
<td>5.4, 11.2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>single tree</td>
<td>birch</td>
<td>11</td>
<td>2</td>
<td>6, 4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>edge</td>
<td>aspen</td>
<td>18</td>
<td>2</td>
<td>10</td>
<td>$4_b^{8}m, 16_b^{7}m, 50_b^{5}m$</td>
</tr>
<tr>
<td>4</td>
<td>edge</td>
<td>spruce</td>
<td>15</td>
<td>0</td>
<td>10</td>
<td>$4_b^{8}, 27_b^{7}m$</td>
</tr>
<tr>
<td>5</td>
<td>edge</td>
<td>aspen</td>
<td>15</td>
<td>2</td>
<td>10</td>
<td>$4_b^{8}, 29_b^{7}m, 60_b^{6}m$</td>
</tr>
<tr>
<td>6</td>
<td>edge</td>
<td>pine</td>
<td>10</td>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>edge</td>
<td>spruce</td>
<td>10</td>
<td>2</td>
<td>7</td>
<td>$4_b^{8}, 25_b^{7}m, 40_b^{6}m$</td>
</tr>
</tbody>
</table>

($H_a$) are reported in Table II. This table also presents the measurement setup used at each site, such as the anemometer height ($H_a$), the propagation distance from the microphone to the vegetative source and the corresponding measurement technique.

Site 1: the forest  The forest is mainly of pines (95%) with some single birches. The tree height varies between 10 and 15 m with the foliage in the top third of the canopy. The soil is composed of a 10 cm thick layer of leaf mould over a bed of rocks. The ground is mainly plane around the measurement place except for a down-hill slope at a distance of 150 m in the north-east direction. Both measurement techniques have been used. The vertical attenuation coefficient has been assessed by placing two anemometers at different heights. The wind speed at the canopy top has been extrapolated from the data obtained at the two lower heights assuming an exponential velocity profile. The attenuation coefficient $a$ was found to be in average 2.6, which agrees rather well with the profiles given by Finnigan [9] for pine forests.

Site 2: single birch tree  At site 2, the noise was measured downwind a single birch with a canopy of about 7 m diameter. Though other trees and bushes were also present, they were far enough not to influence the sound pressure level measured close to the birch. The horizontal attenuation coefficient was measured by locating anemometers both upwind and downwind the birch.

Site 3: edge 1  This site consists of an open field covered with grass with an average height of 7 cm and the edge of a small forest (Figure 1a). The forest, mainly composed of pine trees, is lined with a row of aspens. The wind was blowing toward this 104 m long edge with an incidence angle with the edge normal contained in the range [0°, 60°].

Site 4: edge 2  This dense edge is composed of spruces (Figure 2) and of two single oaks (Figure 1b). The incidence angle of the wind was in the range [0°, 30°]. During the first measurements, the microphone was located rather far from the oaks in order to measure only the noise from the spruces. Later, the mast and the microphone were moved near to one of the oaks.

Site 5: edge 3  This site (Figure 1c) consists of an open field planted with wheat of about 10 cm height at the time of the measurements and a spruce forest bordered by a row of aspens. To estimate the horizontal attenuation coefficient, one mast was situated in the open field at 8 m from the edge while the second was at 7 m from the edge within the canopy.

Site 6: edge 4  This site is an upwind edge of a forest consisting of pines and spruces with some single aspens on the edge. The open field was covered with straw. The wind angle of incidence was in the range [0°, 30°]. The horizontal attenuation coefficient was measured for two different positions of the mast within the forest (17 m and 8 m from the edge) while the open field mast was 4 m in front of it (Figure 1d).

Site 7: edge 5  This forest edge is quite irregular and composed only of spruces and pines (Figure 1e). The ground in the open field is a clear-felled area. The wind was blowing in the direction normal to the edge.
4. Model calibration and validation

4.1. Model calibration

Calibration of the model is a necessary stage as laboratory measurements could not lead to the assessment of all the parameters of the model [1]. In the model described in [1], the acoustic power of a canopy volume element is related to the mean wind speed $u(r)$ and to the local leaf/needle area density $S(r)$ by the relationship

$$
\delta W(r, f) = A S(r) u^2 e(x) \Gamma(r, f),
$$

where $A$ is a radiation constant dependent on the tree species, $S(r)$ the leaf area density (LAD); $x$ the wind speed coefficient reflecting the dependence of the noise level on the wind speed and $\Gamma$ the normalised noise spectrum.

Furthermore, the wind speed decrease within a vegetative obstacle canopy might be described by an exponential law given by

$$
\frac{u(x_2, z)}{u(x_1, z)} = e^{-\zeta S(x)(x_2 - x_1)},
$$

where $u(x_1, z)$ and $u(x_2, z)$ are wind speeds measured at the height $z$ but $(x_2 - x_1)$ apart in the wind direction and $\zeta$ the horizontal attenuation coefficient.

In this section, measurement results are used to calibrate the parameters $\zeta$, $A$, $\chi$ and the function $\Gamma$ for different species.

4.1.1. Calibration of $\zeta$

The quantity $\zeta S$ was measured at sites 2, 5 and 6 at the heights where the canopies were the most dense. In the case of the single tree, the product $\zeta S$ was larger, which is certainly the consequence of the high LAD often exhibited by single trees due to the good access to light. The LADs have been estimated roughly at 2.5, 1.5 and 1.2 m²/m³ for the single birch (site 2), the edge of aspens (site 5) and the edge of pines and aspens (site 6). The horizontal attenuation coefficient, $\zeta$, derived with those LAD values, appears to be constant for the different species as seen in Figures 3. The value 0.08 for $\zeta$ will be used in the future calculations involving vegetative obstacles.

4.1.2. Calibration of $\chi$

The wind speed coefficient is rather difficult to evaluate from outdoor measurement results. Indeed, for low wind speed or at some distance from the vegetation, the wind velocity is often of the same order as the background noise. Thus, the sound pressure level, as shown by the measurement results (Figures 4 and 5), appears almost constant wind speeds below 2 m/s. For stronger winds, it increases approximately linearly with the logarithm of the wind speed. Furthermore, the presence of vegetative obstacles as well as the topography of the site influence the spatial wind pattern. Thus, the knowledge of the wind speed at a point is not relevant for the whole canopy. Finally, the turbulence adds a greater dispersion of the results, especially those obtained for extended sources like the shelterbelt.

Hence, in order to minimize the background noise, the wind speed coefficient is evaluated by selecting the data obtained with the ground board located close to

![Figure 3. Scatter plot of the horizontal attenuation coefficient $\zeta$, (x) measured at site 2 for the birch; (o) measured at site 5 for the aspen; (c) measured at site 6 for the pine and the aspen.](image)

![Figure 4. Scatter plot of the corrected emission peak levels. (+) data obtained from the measurements close to the edge of aspens at site 2; (o) data obtained from the measurements close to the single birch at site 6; (c) data obtained from the measurements close to the single birch at site 4; (---) regression line ($= 32 + 30 \log(u)$).](image)

![Figure 5. Scatter plot of the corrected emission peak levels. (+) data obtained from the measurements near the edge of spruces at site 1; (o) data obtained from the measurements in the pine forest at site 5; (---) regression line ($= 15 + 35 \log(u)$).](image)
Table III. Calibration parameters for coniferous species.

<table>
<thead>
<tr>
<th>Species</th>
<th>(\bar{d}n) [m]</th>
<th>(\lambda)</th>
<th>(A \times 10^{-13})</th>
</tr>
</thead>
<tbody>
<tr>
<td>pine</td>
<td>0.0013</td>
<td>10</td>
<td>6.0</td>
</tr>
<tr>
<td>spruce</td>
<td>0.0011</td>
<td>15</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Vegetative sources. Furthermore, the values of \(\chi\) are derived from the levels of the emission peaks measured in 1/3 octave band spectra at wind speeds higher than 2 m/s. Finally, as the ground board was placed near the mast foot during the measurements, the influence of the wind spatial patterns should have been reduced.

**a. Deciduous species**

Figure 4 shows the data obtained at sites 2, 3 and 4 after applying a correction to get in average a level of 50 dB at 4 m/s. Different values of \(\chi\) were expected between species but from Figure 4, it is hard to attribute this difference to the species themselves or to the site geometries. For the sake of simplicity, it was decided to use only one coefficient \(\chi\) for all the deciduous species. A relative error of \(\%\) on \(\chi\) will lead to a relative error of \(\%\) on the acoustic power. Nonetheless, in practice, the error remains small since the wind speed range of interest is reduced and since the calibration of the model is done at about 5 m/s.

This coefficient \(\chi\) was chosen as 1.5 since it appears to give the best agreement with the measurement results for wind speeds smaller than 6 m/s. This corresponds to a regression line having a speed dependence of \(30 \log(u)\) if dimensional considerations are evaded. As expected, the standard deviation of the data around this regression line decreases for increasing wind speeds and is 2.3, 2.0 and 1.7 dB for velocity greater than 2, 3 and 4 m/s, respectively.

**b. Coniferous species**

For coniferous trees, the determination of the coefficient \(\chi\) is even more difficult since the forest shape depends on the wind speed. Hence, the derived coefficient \(\chi\) depends on the type of analysis bands. Anyway, computer simulations, achieved with typical values for the needle diameter \(\bar{d}n\) and \(\lambda\) (see equations (5) and (6) and Table III), have shown that the 1/3 octave band sound pressure level has a speed dependence of \(35 \log(u)\) if \(\chi = 1.5\) for wind speeds up to 6 m/s. Thus, the apparent wind speed coefficient is higher than its real value. Figure 5 shows the wind speed dependence of the emission peak level. As for Figure 4, these data have been corrected to have the same level at 4 m/s. The standard deviation around the regression line are 3.0, 2.7 and 2.9 dB, respectively, for wind speeds greater than 2, 3 and 4 m/s, respectively.

4.1.3. Calibration of \(\Gamma\) and \(A\)

Five different tree species, namely birch, aspen, oak, pine and spruce, were present at sites 1 to 4. The calibration of the model was achieved by assessing the species radiation constants \(A\) and adjusting the spectrum shape from the measurement results. The calculations were performed by using the analytical expressions derived in the first part [1].

![Figure 6. Scatter plot of the Strouhal number \((f_d\bar{d}n/u)\) versus wind speed. (a) measured for spruces with \(\bar{d}n = 0.001\) at site 4; (a) measured for pines with \(\bar{d}n = 0.0013\) at site 1; (a) equation (7).](image)

![Figure 7. Measured and predicted 1/3 octave band sound pressure levels for conifers. (a) measurements with the board at 4 m from the spruce edge at site 4, \(u = 3.4\) m/s; (a) measurements in the pine forest (site 1) with the board, \(u = 6.3\) m/s; (a) calculations for the forest; (a) and (a) calculations for the spruce edge including and excluding the oak noise, respectively.](image)

In the calibration and validation simulations, the site parameters are those given in the field sites description. The depth of the vegetative obstacle for the forest edges was taken equal to the tree height \((D = 2H)\). The air attenuation coefficient was calculated for an air temperature of 20°C and a relative humidity of 40% according to the document ISO/DIS 9613-1 [10]. Values for the vegetation attenuation coefficient have been found in ISO/DIS 9613-2.2 [11]. The drag coefficient \(C_d\), the wind speed coefficient \(\chi\) and the horizontal attenuation coefficient \(\zeta\) were taken identical for all the species, deciduous and coniferous, and their respective values were 0.32, 1.5 and 0.08.

**a. Coniferous trees**

It has been shown that conifer noise narrow band spectra can be described by the expression

\[
\Gamma(r, f) = e^{-\lambda \log(\frac{f}{f_0(r)})},
\]

where

\[
f_0(r) = \frac{S_L}{\bar{d}n}.
\]
Table IV. Calibration parameters for deciduous species.

<table>
<thead>
<tr>
<th>Species</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$f_m$ [Hz]</th>
<th>$A \times 10^{-13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>birch</td>
<td>932</td>
<td>0.8</td>
<td>2</td>
<td>4500</td>
<td>1.6</td>
</tr>
<tr>
<td>aspen</td>
<td>120</td>
<td>1.0</td>
<td>1</td>
<td>3200</td>
<td>10.0</td>
</tr>
<tr>
<td>oak</td>
<td>1580</td>
<td>0.5</td>
<td>1</td>
<td>3000</td>
<td>0.44</td>
</tr>
</tbody>
</table>

The measured Strouhal frequency $f_0$ is taken to be the central frequency of the 1/3 octave band exhibiting the emission peak. From each measured spectrum, the Strouhal number $St$ is assessed by taking a needle diameter of 1 mm for spruces and 1.3 mm for pines. Figure 6 illustrates the tendency of the Strouhal number to decrease slightly as the wind speed increases. The hachured patterns of the plot in Figure 6 is due to the fact that the analysis has been done in 1/3 octave bands, and that the Strouhal frequency was chosen as the central frequency of the 1/3 octave band containing the peak of emission. A narrow band analysis would have lead to a more uniform scattering of the plot. The bigger scattering of the values observed at low wind speeds might reflect the influence of the non-uniform spatial distribution of the wind in the canopy. In the future calculations, the dependence of the Strouhal number on the wind speed will be taken into account by the expression

$$St(u) = 0.15 + e^{-0.75(u-0.6)}.$$  \hspace{1cm} (7)

Figure 7 shows the calibration results for coniferous trees achieved with the parameters values reported in Table III. It is seen that the model fails to predict the high and the low frequency levels in the case of the forest. The reason is that the noise is not generated by the vegetation at those frequencies, at least not by the needles, and the background noise needs to be added to the calculated results. The figure also shows measurement results obtained near one of the single oaks contained in the edge of spruces at site 4. Figure 7 shows that the taking into account of this oak in the simulations leads to a good description of the high frequency emission while the spruces are responsible for the broad band peak at 630 Hz. The prediction of the low frequency levels is better, though still barely passable, as the model does not introduce any consideration of the measurement system pseudo-noise.

b. Deciduous trees

Noise spectra from aspen, birch and oak have been measured at the different sites. These spectra are quite similar to those measured in the laboratory, i.e. their shape is independent on the wind speed and exhibits an emission peak around 4000–5000 Hz. In the preceding paper, it has been pointed out that narrow band spectra for deciduous species can be described by

$$\Gamma(f) = C_1f^{-1} + C_2e^{-C_3(f-f_m)12/f_m^2},$$  \hspace{1cm} (8)

where $f_m$ is the frequency of the emission peak. Table IV gives the values of the spectrum parameters and the radiation constants used for the calibration calculations. The values of the radiation constants reported here show clearly that aspen is the noisiest species. For the same source size, it generates 8 and 13 dB more than a birch and an oak, respectively. A good agreement is obtained between measured results and calculations (as Figure 8), except at low frequencies where some discrepancies appear. These discrepancies were expected as the model was not designed to predict frequency noise generation (<200 Hz), which anyway should be dominated by the pseudo-noise during outdoor measurements. Furthermore, by looking at both Figures 7 and 8, obvious that the slight difference between the measured and calculations appearing on Figure 8 at about 500 Hz site 3 might be imputed to the omission of the spruce forest behind the row of aspens.

4.2. Model validation

The model is validated by comparing calculated 1/3 octave band spectra and A-weighted sound pressure levels ($L_{Pa}$) measurements within the frequency range [50, 10000]. Nevertheless, the validation will essentially focus on model frequency range of application, e.g. frequencies higher than 200 Hz for deciduous species and the broad-band around the Strouhal frequency for coniferous ones. The error made in the assessment of the A-weighted sound pressure levels is defined as the difference between the measured and predicted values: a positive error indicates that the model underestimates the real level. The errors have been calculated for different wind speeds and propagation distance different field sites.

The experimental results used for the validation are obtained at sites 3–7, where those already used for model calibration have been removed. The calculations were made by assuming a specific ground resistivity 250000 ohms/m at all sites and the same values as used in calibration simulations for the other parameters.
4.2.1. Prediction of A-weighted sound pressure levels

The upper part of Figure 9 shows the deviation in dB between the measured and predicted A-weighted levels while the lower parts provide information on the corresponding wind speeds and propagation distances. A variation of the error with the wind speed reveals an inadequate choice of the wind speed coefficient. A negative error increasing, respectively decreasing, with the wind speed or a positive error decreasing, respectively increasing, with the wind speed indicates the choice of a too high, respectively too low, wind speed coefficient.

For deciduous species, no conclusion about the error velocity dependence can be drawn due the small ranges of wind speed recorded during the measurements. Concerning coniferous species, Figure 9 shows a strongly velocity-dependent error at edge 2 when the propagation distance is 27 m. And the error trend indicates a real \( \chi \) much smaller than the value of 1.5 adopted for the calculations. From measurement data, it appears that the wind speed coefficient is about 0.6 at 27 m distance while it was about 2 close to the vegetative source. The same tendency can be observed at edge 5, i.e., a bigger variation of the error for increasing propagation distances. It is unlikely that this is caused by the refraction of the acoustic waves due to the small propagation distances at stake. Instead, it should be noted that data obtained at low wind speeds (<3 m/s) exhibit often high background noise levels (see Figure 6) and this tends to decrease the wind speed coefficient value. Indeed, it can be observed that, at edge 4, the error is almost velocity independent if the low wind speed data are neglected.

Concerning the influence of the propagation distance, the results from coniferous edges 2, 4 and 5 show that the agreement between prediction and measurements decreases for increasing propagation distances. Anyway, this tendency is not confirmed by the validation results at the deciduous edges 1 and 3.

4.2.2. Prediction of spectral distributions

Though often only the A-weighted sound pressure level is required in practice, the spectrum shape of the wind-induced vegetation noise is of major importance for the assessment of the masking effect of any non-natural noise by wind-induced vegetation noise. Figures 10 to 14 show both measured and predicted spectra obtained with the two measurement methods. The couple \((d, u)\) given in the legends refers to the propagation distance and to the wind speed recorded during the corresponding measurement.

For vegetative obstacles, the model takes into account the correlation between the direct and ground reflected sounds and important dips appear in the low frequency part of the predicted spectrum when destructive interferences occur. Nevertheless, in practice, the pseudo-noise dominates the low frequency range even when the additional windscreen is used and the measured spectra do not exhibit these interference dips. This is the reason why in the next figures the noise levels at low frequencies do not decrease with increasing distances from the source and that the use of the free microphone technique results in higher noise levels at low frequencies compared to the ground board technique. Thus, it has been decided to neglect the correlation between the direct and the
ground reflected sound waves in the simulations with the free microphone.

For the deciduous edges 1 and 3 (Figures 10 and 12), the agreement at about 4000 Hz between measurement and calculation is still good when the propagation distance increases up to 50 m. The discrepancy observed around 500 Hz due to the omission of the coniferous forests behind the deciduous edges in the calculations increases for increasing propagation distances (see Figures 8, 10 and 12). For instance, at 50 m from the edge, the sound pressure level at 500 Hz is of the same order as the one at 4000 Hz. Nevertheless, the assumption that the forest contribution might be neglected compared to the edge one in the case of upwind forest edges is not invalidated by this observation. Indeed, the hypothesis was based on calculations made at the same frequency while two very distinct frequencies (500 and 4000 Hz) are at stake in the present case. This influences obviously the ground reflection, the air attenuation, the windscreen correction and the background noise level. In addition, it was shown that the decrease of the noise level due to the spreading effects is 3 dB smaller for the forest than the upwind edge at a distance 50 m. Thus, the combination of the foregoing effects shi compensates the initial difference in the emission levels explain the small difference observed at 50 m between peak levels at 500 and 4000 Hz.

Figures 12, 13 and 14 show that the good agreement tained at short propagation distances is somehow alt when the distance increases. Nevertheless, the discrep of 4 dB observed at 40 m from edge 4 (Figure 13) m be explained by the incomplete description of the edg the simulations. Indeed, aspens were present all along edge but only the three ones closest to the mast have l reported in the measurement report, and thus modelled ir calculations. This simplification, though reasonable for assessment of the noise level close to the source, should l affected the prediction accuracy for distant points.

At edge 5, the discrepancy between prediction and m surement grows up from almost 0 to 7 dB when the pagation distance increases from 4 m to 20 m (Figure 15). The difference in the propagation distance is not suffici
justify the decrease of 10 dB in the sound level. These measurements are each the averages of 14 spectra recorded with the ground board technique for wind speed ranging from 3 up to 4 m/s and no explanation has been found to support these results.

Finally, a discrepancy of 3 or 4 dB appears between measurement results and predictions at 27 m distance from edge 2 (Figure 11). But, again the experimental results are peculiar as the level recorded with the free microphone system (20 spectra) was on average 1 dB higher than the one measured by the ground board (8 spectra) for the same average wind speed (the spectrum numbers given within the parenthesis are the numbers of spectra on which the averages are built).

5. Discussion and conclusion

The validation may have suffered from an insufficient number of data, especially at high wind speeds. Nevertheless, it has shown that reliable predictions are obtained close to the vegetative sources while it seems that increasing the propagation distance leads to bigger discrepancies. If the peculiar results obtained at edges 2 and 5 for distant points are taken into account in the validation, the A-weighted level is predicted with a mean error of 0.25 dB and a standard deviation of 3.7 dB. If they are excluded, these values are reduced to 0.1 and 3.0 dB, respectively.

Some important points have emerged from the validation results. One is the need for an accurate description of extended vegetative obstacles as shelterbelts when the prediction concerns distant measurement points. A second interesting point is the variation of the wind speed coefficient $\chi$ with the distance from the source. It was first thought that that might be imputed to the turbulence and the spatial patterns of the wind along the extended source. Nevertheless, the study has also shown that the $\chi$ decrease is greater for coniferous than for deciduous species, this phenomenon might more likely be attributed to a lower signal-to-noise ratio prevailing for great propagation distances and decreasing wind speeds. The difference observed between the deciduous and coniferous species may be explained by the fact that the signal-to-noise ratio should be better at 4000 Hz than at 500 Hz which are the respective range of noise emission of deciduous and coniferous species.

In conclusion, the semi-empirical model developed in these two articles allows both the A-weighted sound pressure level and spectrum shape to be predicted. In the future, data recorded at higher wind speeds are needed to strengthen and test the model, especially with regard to the propagation effects. These measurements should be carried out simultaneously close to the source and at some distance from it to avoid changes in the atmospheric conditions. Furthermore, the model might also be improved by adding the contribution of the low frequency pseudo-noise.

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References

ON THE MASKING OF WIND TURBINE NOISE BY AMBIENT NOISE

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ABSTRACT: A prerequisite to discussing the environmental impact of wind turbine noise is a reasonable understanding of what is being impacted. This paper reviews the current knowledge in matter of natural wind-induced ambient noise and presents original measurement results on the pseudo-noise generation at the human ear. Finally, the masking of WT noise by ambient noise is discussed, especially its relationship to the temporal random noise level fluctuations induced by the turbulence in the wind. Indeed, though the ambient noise has a strong masking potential in terms of frequency contents and generated sound levels, the influence of these fluctuations on masking is difficult to assess. It is shown how the deterministic models predicting the ambient noise can be coupled to the statistical analysis of the wind turbulence to yield reliable assessment of the amount of masking.

Keywords: Environmental Aspects, Legislation, Noise

1. INTRODUCTION

Noise annoyance is a difficult issue involving both a physical and a cognitive dimension. In general, government policies deal with the environmental impact of industrial plants by referring to the physical exposure level. For example, in Sweden, the basis of the noise legislation is to bound the equivalent continuous A-weighted sound pressure level below certain limits, depending on the type of environment, the time of day and possible tonal components [1]. This over-simplified approach of the problem is not satisfying both from the community and the industry standpoint. Indeed, the community response cannot be predicted on the knowledge of the A-level alone and detectability of the low-level noises yields a better indicator to their intrusiveness [2].

Wind turbines are particularly penalised by the actual regulation. At remote emission points, the high wind-induced background noise is likely to mask partially, or completely, their presence, resulting in a decrease in loudness or even in the inaudibility of the turbine. The noise immunity levels of wind turbines can be predicted with a reasonable accuracy [3]. And, the major obstacle to the implementation of a background noise related regulation is that very little is known about this noise, especially in windy conditions. Measuring the background noise level at each given site would be too time-consuming and would request an overwhelming number of data considering the importance of variables like wind speed and wind direction. Thus, models predicting the sound pressure level by frequency bands, \( L_p \), are necessary steps in the characterisation of masking.

The first part of this paper reviews briefly the parameters involved in masking. Then the wind-induced vegetation noise is described and original measurement results on the pseudo-noise generated at the human ear are presented. Finally, the masking of WT noise is discussed, especially in regard to the influence of the random ambient noise level fluctuations due to the wind turbulence. It is shown how their detrimental effect on masking might be assessed by studying the statistical characteristics of the wind turbulence at ground level and the mechanisms of signal detection involved by the human auditory system.

2. PARAMETERS OF MASKING

Masking is defined as "the interference with the perception of one sound (the signal) by another sound (the masker)" [4]. The masking is said to be complete when the masker leads to the inaudibility of the signal. The term "partial masking" is used when the presence of the masker results in a decrease of the signal loudness or of the noticeability of a given change of the signal.

A concept of importance in masking is the notion of critical bands. The auditory system can be seen as a set of filters having frequency dependent bandwidths. The detectability of a signal is based on the signal-to-noise ratio \( S/N \) in each critical band, on the bandwidth of the critical band of interest, \( W \), and on the signal duration, \( T_s \). The detectability index is given by equation (1.a) for steady signals (from [2]) and by equation (1.b) for bursts duration in the range [100 ms - 1 s] (from [4]),

\[
d^2 = \left( \frac{0.4 \sqrt{W} 10^{5/N/10} \tau}{\sqrt{\frac{T_s}{\tau} 10^{5/N/10} + \frac{W \tau}{0.4 W \tau + 2 (1 + 10^{5/N/10})}} \right)^{1/2}.
\]

where \( \tau \) is an integration time of 100 ms.

Hence, masking is best achieved when signal and masker have similar frequency contents. In this case and if both signal and masker are broadband and steady, the amount of masking can be assessed directly from the \( S/N \) of the equivalent continuous A-weighted levels. Hence, it is accepted that the signal is barely perceptible when the \( L_{10dB} \) of the signal is the same as the \( L_{10dB} \) of the masker.

However, in order to distinguish and segregate sounds emanating from different sources, the human auditory system resorts to an intelligent processing which focus on certain components of the acoustic signals. Hence, periodic time variations of the signal (amplitude modulation) and presence of pure tones draw the attention of the ear on the signal and makes it easier to distinguish from the masker. For instance, the ear is highly sensitive to frequency modulation in the range [2-5Hz], the threshold for the detection of amplitude modulation of broadband signals.
being as low as 1.0 dB in this range. The detectability enhancement implies that such signals are likely to be perceived as annoying.

3. WIND-INDUCED VEGETATION NOISE

In [5], a semi-empirical analytical model predicting the $L_n$ generated by typical vegetation features (forest, hedge, single tree) and species (coniferous and deciduous) is presented. The model is based on a mathematical description of the noise generation mechanisms and the wind foliage interaction and yields the sound power level of each canopy point. Analytical expressions of the $L_n$ at a given measurement point have been derived. They take into account propagation parameters like distance, ground reflection, absorption by air and vegetation and the integration over the whole canopy. Both laboratory and in-the-field measurement results have been used to calibrate and validate the model. It has appeared that deciduous species have higher sound power levels than conifers and that they generate sound in a vibro-acoustic way. They exhibit wind speed independent spectrum shapes which, in 1/3 octave bands, are almost flat but with a broadband peak around 5000 Hz. On the other hand, coniferous species noise generation is of aerodynamic origin and similar to the whistling of telephone wires in the wind. Their spectrum shapes are very wind speed dependent since they are characterised by a broad peak at the Strouhal frequency given by the needle diameter and by the wind speed. In the model, the sound power levels of different tree species (spruce, pine, aspen, birch, oak) have been incorporated.

Interesting results emerge straightforwardly from the analytical model like the small influence of the foliage density on the resulting $L_{eq}$ due to the interaction between foliage density and wind speed. Another point of importance is the poor correlation existing between noise level and canopy volume, also a consequence of the previously quoted interaction. Noise levels are better correlated to the size of the windward surface of vegetative obstacles.

In the present article, dimension considerations are evaded and the $L_{eq}$ dependency on the wind speed $U$ for all types of wind-induced noises is described by the so-called wind speed coefficient $\alpha$ according to equation (2).

$\alpha \approx 30$ for deciduous species and 40 for coniferous ones.

$$L_{eq} = \alpha \log(U).$$

4. PSEUDO-NOISE AND HUMAN EAR

A microphone placed in a turbulent flow responds to the local aerodynamic pressure fluctuations as if they were real acoustic signals. This pseudo-noise is the main source of background noise in the low frequency range and windscreens are necessary devices when outdoor measurements are to be carried out on a windy day. Anyway, pseudo-noise data obtained with a free microphone, equipped or not with a windscreen, cannot directly provide information about masking because the ear response to wind turbulence differs from the microphone one. Furthermore, not being an "acoustic" noise, the A-weighting system cannot a priori be applied to assess human response to such a noise.

As previously said, the system human head - ear canal strongly influences the pseudo-noise generation. Thus, measurement results gathered with a free microphone should be corrected with respect to the ear's dynamic transfer function. This function relates the pseudo-noise measured by a microphone at the eardrum to the one measured in the free field. Since, from the knowledge of the author, this transfer function has never been the subject of previous studies, a series of laboratory measurements using an artificial plaster head placed in a flow was conducted to determine it. Sensitivity to parameters like velocity and angle of attack of the flow was also studying by carrying out the measurements at four velocities (3, 6, 8.5 and 10.5 m/s) and seven angles of attack $(0°, 30°, 60°, 90°, 120°, 150°$ and $180°$). As seen on Figure 1, the angle $0°$ refers to the position for which the ear containing the microphone is facing the oncoming flow.

![Figure 1: Coordinate system for the angle of attack](image)

A common ear canal is 22.5 mm in length and 7.5 mm in diameter. In the experiment, it was modeled by a rigid tube which was 25 mm in length and 12.5 mm in diameter. The acoustic transfer function free-field-eardrum, measured in an anechoic room, has shown an acceptable agreement with previous studies [6], INSURING that the present experimental set-up is representative of a real human ear. The dynamic transfer function, from now on referred to as $T_d$, is defined as the difference between the $L_n$ measured at the eardrum and the $L_n$ measured in the free field by a microphone equipped of a 95 mm diameter foam windscreens (type B&K 10237).

Above 1000 Hz, the measurements with the head were spoilt due to the amplification of the rather high flow noise by resonance in the ear canal. Anyway, this should not affect the determination of $T_d$ since pseudo-noises are low-frequency noises. Dimensionless scaling laws could be derived in the present experiment for the free microphone with and without the windscreens and the head for $0°$ of angle of attack. For the others angles of attack, changes in the flow patterns as the velocities increases precluded dimensionless analysis. The good correlation obtained for the angle $0°$ is certainly due to the fact that the microphone was then at the head stagnation point where no drastic changes in the flow occur whatever the velocity. Fortunately, this position corresponds to the lowest noise generation in each frequency band and is thus the most interesting one. The dynamic transfer function for the $0°$ orientation was found to be only dependent of the variable $f/U$ and is described by
\[ T_d = \begin{cases} 
5 + 31 \log(f/f_1) - 17 \log^2(f/f_1) & \text{for } f/f_1 < 100 \\
0 & \text{otherwise}
\end{cases} \]

(3)

The standard deviation between the predicted and measured values of \(T_d\) is 4.0 dB. This rather high values is nonetheless reduced to 2.5 dB when the measurement results obtained at 3 m/s are disregarded.

To describe briefly the spectra obtained at the other angles of attack, it might be reported that the frequency range [100-1000 Hz] is dominated by the 120° orientation at all the velocities. The measured levels in this case are rather close to the ones obtained with a free microphone not equipped with a windscreen. In the low frequency range [10-100 Hz], the orientation 180° exhibited the highest pseudo-noise levels at the velocities 8.5 and 10.5 m/s, obviously generated by the vortex shedding taking place behind the head.

As seen previously, the respective \(L_{A_{eq}}\) of the signal and the masker provide information about the amount of masking for similar steady broadband noises. But the A-weighting system has been designed to simulate the human response to acoustic sounds, not to aerodynamic pressure fluctuations. Nevertheless, the acoustic transfer function of the ear canal is close to zero for frequencies below 1000 Hz [6], indicating that the great variations of sensitivity of the human auditory system below 1000 Hz are due to the ear drum and the inner ear responses. That means that human response to aerodynamic pseudo-noise can also be assessed by applying the A-weighting once the data obtained with a free microphone have been corrected with respect to the ear's dynamic transfer function. Above 1000 Hz, the pseudo-noise being of aeroacoustic origin, the A-weighting and free microphone measurement results provide directly reliable assessment of the human response.

Pseudo-noise \(L_{A_{eq}}\) values obtained with a 9 cm diameter windscreen were collected within the frame of a EU-project [6], one task of which was to develop an immersion measurement method for wind turbine noise. Equation (4) fits the data measured at different sites with a standard deviation of 2 dB(A) for wind speeds higher than 3 m/s. When spectra were available, they were corrected for the predicted function \(T_d\) and the \(L_{A_{eq}}\) re-assessed, leading to equation (5). The excess of masking generated at the human ear compared to the free microphone is readily obtained by subtraction and is given by equation (6).

\[
L_{A_{eq, microphone}} = 7 + 50 \log(U),
\]

(4)

\[
L_{A_{eq, ear}} = 3 + 70 \log(U),
\]

(5)

\[
T_{d, A} = -4 + 20 \log(U).
\]

(6)

The influence of the head orientation on the total \(L_{A_{eq}}\) was studied in a similar way. The measured \(T_d\) for the different velocities and angles of attack were added to typical outdoor pseudo-noise spectra and the \(L_{A_{eq}}\) re-assessed. Figure 2 shows the addition of \(L_{A_{eq}}\) due to the ear's dynamic transfer function as a function of both the angle of attack and the velocity. It is seen that the orientation 120° generated the highest pseudo-noise but that the other orientations are very sensitive to velocity increases. The reductions of \(L_{A_{eq}}\) due to the windscreen are also indicated on Figure 2.

![Figure 2: A-weighted dynamic transfer function \(T_{d, A}\) as function of the wind speed and the angle of attack. The marks (*) indicate the values of the \(T_{d, A}\) of the windscreen.](image)

5. DISCUSSION

Wind turbines noise emission fits into two classes: mechanical noise and aeroacoustic noise. The former originates in the machinery (gear box, drive train and generator) and the latter from the rotor. Considerable improvements in the design of the mechanical parts have achieved abatement of both the noise level and the pure tones generated by the machinery. Therefore, the noise emission of modern WTs is dominated by the broadband rotor noise. The wind speed coefficient lies typically in the range [10-20] for turbines with constant rotational speeds; the value 10 prevailing for wind speeds just above the cut-in. Wind turbine noise may also exhibit an amplitude modulation at the blade/tower passing frequency. Even weak, this modulation may lead to a masking release since the frequency of the modulation is typically 2-3 Hz and thus is easily perceived by the ear.

By comparing the coefficients \(\alpha\) of the relevant noise sources, the ambient noise is seen to increase faster with the wind speed than does the WT noise. Thus masking is minimum at low wind speeds and should then be studied for wind speeds just above the cut-in wind speed of the turbine. Figure 3 presents typical spectra from different sources at wind speeds corresponding to about 6 m/s at hub height (taken to be 40 m). The microphone and the ear spectra are given for a wind speed of 3.5 m/s at 1.5 m in height. The birch, pine, ear and WT spectra have an overall \(L_{A_{eq}}\) of 38 dB(A). The calculation of the pine spectrum has been performed at a velocity of 5 m/s at 10 m height. Such a wind speed configuration is defined to give a WT noise of 40 dB(A) at a wind speed of 8 m/s at 10 m height which corresponds to the immission limit given in [1].

In terms of \(L_{A_{eq}}\), masking is achieved by the ear pseudo-noise and the vegetation noise. Nonetheless, as seen on Figure 3, the detectability of the WT noise is still high in the frequency range [200, 2000 Hz] when the ear pseudo-noise or the birch noise are used as background noise. Masking is the best achieved by the pine species at this velocity. Nevertheless, noise spectra from conifers are strongly wind speed dependent and the broadband peak will move toward higher frequency at higher wind speeds. Thus, the S/N of the \(L_{A_{eq}}\) will become negative but the detectability may increase in some critical bands. Hence,
consequences of a velocity increase on the masking are
difficult to predict.

\[ L_\text{eq} = U \text{ln}^{-1} \left( \frac{L_\text{eq}}{U} \right) \text{,} \quad \sigma_L = \alpha \sigma_U \text{,} \quad \Delta = \frac{L_{\text{eq},w} - L_{\text{eq},b}}{U \text{ln} 10} \text{,} \quad \sigma^2 = \sigma_{L,w}^2 + \sigma_{L,b}^2 \text{.} \]

The indices \( b \) and \( w \) refer to the background noise and the wind turbine noise, respectively. Assuming a ground roughness typical of terrain with many hedges and some trees, \( z_0 = 0.1 \), a hub height at 40 m and a characteristic height of the vegetation of 3 m, this yields a standard deviation of 5 dB(A) for the ambient noise and 0.7 dB(A) for the wind turbine noise. The S/N standard deviation \( \sigma \) is then 5.05 dB(A) and is governed by the fluctuations of the background noise.

Knowing the S/N distribution, it is straightforward to calculate the probability for the S/N to be positive. But masking mechanisms bring also a temporal aspect since the S/N has to be positive during a certain laps of time for the WT to be perceived. Hence, very fast fluctuations of the S/N would be beneficial to the masking. Information about the temporal sequencing of the S/N can be obtained from the knowledge of its autocorrelation function in time. This function is rather complicated as two different time scales are involved, i.e. time scales of the turbulence at hub height and around the measurement position. Nevertheless, since the S/N fluctuations are mainly determined by the ambient noise fluctuations, the autocorrelation function of the S/N can be approximated by the autocorrelation function of the wind speed around the measurement place. Equation (10) gives the probability of the S/N to be positive during a period \( T \).

\[ \text{Prob}\left[ S/N > 0, S/N(t + T) > 0 \right] = \frac{1}{2} \int_{0}^{T} p(x,y) \text{d}x \text{d}y \text{.} \]

The joint normal density function \( p(x,y) \) introduces the correlation coefficient \( r \) which is primarily function of the lag \( T \), the wind speed, the turbulence intensity and the height,

\[ p(x,y) = \frac{1}{2\pi \sigma^2 \sqrt{1-r^2}} \exp \left\{ -\frac{1}{2\sigma^2(1-r^2)} \left[ \left( x - \Delta \right)^2 + \left( y - \Delta \right)^2 - 2r(x - \Delta)(y - \Delta) \right] \right\} \text{.} \]

Figure 4 shows the isoprobability of occurrence of a positive S/N during a period \( T \), here called \( P_{S/N>0,T} \), as a function of the average S/N. The calculation was made with a standard deviation of 5 dB. It is seen that the probability decreases as expected with increasing \( T \). Nevertheless, signal duration is determinant for its detection and thus the masking probability cannot be readily defined by the probability \( P_{S/N>0,T} \), for periods \( T \) shorter than 1 or 2 seconds. Indeed, the probability of masking involves two aspects. The first is that the probability of signal detectability increases with the signal duration for duration up to 1 or 2 seconds as shown by
equation (1.2). For longer duration, detectability is only function of the S/N and the bandwidth (equation 1.1). The second aspect is that the duration of the signal is a statistical events which probability decreases for increasing values.

Figure 4: Positive S/N isoprobability curves as function of the average S/N and the period T from eqns (10) and (11).

The time dependency of the masking may be studied through the statistical analysis of the detectability index. This index is statistically related to the wind turbulence by the relationship

\[
d'(t) = f\left(\frac{U_b}{u_b}\right)^4\frac{U_b}{u_b}
\]

(12)

where the function \(f\) is given by equation (1). The bar denotes average values, the indices b and w refer to the wind speed around the measurement position and at the WT hub. Statistically, the turbulence at the WT hub has a weak influence on \(d'\) and may be neglected. Thus the probability of detection is given by

\[
Prob\left[d'(t) > L, d'(t + T) > L\right] = Prob\left[\frac{U_b}{u_b}(t) > f^{-1}(L), \frac{U_b}{u_b}(t + T) > f^{-1}(L)\right]
\]

(13)

since \(f\) is a monotonic rising function in the range of interest. \(L\) is the value of the detectability index for which the signal is reliably reported (typically about 5). The masking probability, \(P_M\), can then be defined as

\[
P_M = 1 - Prob\left[d'(t) > L, d'(t + T) > L\right]
\]

(14)

From Figure 3, it is seen that detectability is likely to occur in the frequency range [200-2000 Hz], with a peak around 630 Hz. Figure 5 shows the result of masking probability calculations for different average S/N and different times T. W was taken to be 100 Hz which is the bandwidth of the filter tuned to 630 Hz and the wind turbulence intensity was fixed to 30%. The masking probability is minimum for 2 seconds duration and the 90%-value is then reached for an average S/N of 2.5 dB. Additional calculations were performed to study the influence on the bandwidth on the masking probability for a zero S/N and T = 2s. For a central frequency of 100 Hz, respectively 2000 Hz, i.e. for \(W = 35\) Hz, respectively 240 Hz, the probability of masking is 90%, respectively 62%.

Figure 5: Masking isoprobability curves at 630 Hz

6. CONCLUSIONS

In the article, the masking related characteristics of the ambient noise have been reviewed. It has been shown that masking should be assessed for rather low wind speeds situations. At such wind speeds, the excess of masking in terms of \(L_{	ext{eq}}\) provided by the pseudo-noise generation at the ear does not affect the detectability of the WT in the important frequency bands. Furthermore, although that deciduous tree species are the noisiest, conifers are the most favorable to the masking with regard to frequency contents and have the additional advantage to be season independent since evergreen. Finally, the paper shows that, due to the wind turbulence, masking should be assessed on the basis of a statistical analysis of the S/N or of the detectability index. It appears that wind turbulence is detrimental to masking and negative signal-to-noise ratios, which values depend on the wind turbulence and on the frequency, are needed in order to insure reasonable masking probabilities.

7. ACKNOWLEDGMENTS

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REFERENCES

Paper G
On the Use of a Vertical Microphone Board to Improve Low Signal-to-Noise Ratios During Outdoor Measurements

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ABSTRACT

In this paper, the wind noise generation and the scattering of acoustical waves by a vertical board mounted above the ground are studied. A theoretical diffraction model, collated by measurements, has been derived to assess the effects of diffraction, and it provides a simple way to locate the minima and maxima of deviation from pressure doubling. It is seen that the board shielding effect allows the measurements to be carried out in the presence of reflecting surfaces such as building facades. Furthermore, a set of measurements has been achieved both in a laboratory and in the field to assess the wind noise generation when the measurement system is placed in a flow. Dimensionless laws giving the wind noise as a function of the Strouhal number for both the board and the free microphone have been obtained. It is shown that the signal-to-noise ratio is globally improved by the presence of the board though reduced at high frequency by the wind turbulence. © 1997 Published by Elsevier Science Ltd

Keywords: Acoustic outdoor measurements, wind noise.

INTRODUCTION

The growing interest in wind energy has increased the need of accuracy in wind turbine noise immission measurements. But acoustical outdoor measurements are often made difficult by the action of the wind on the microphone system, the existence of ambient noise and the presence of scattering obstacles (building facades, etc.) around the measurement position. In the case of wind turbine noise immission measurements, this position corresponds to the locations of the nearest dwellings and is rather far from the turbine
(300 m). At this distance, the signal-to-noise ratio is low and it has been thought that mounting the microphone on the surface of a rectangular vertical board would improve it. The reason supporting this idea is that the incident pressure is increased by 6 dB due to the reflection on the rigid surface whereas the microphone is sheltered from the other sources located behind the plate. Furthermore, the wind velocity, viz the pseudo-noise generation, is reduced at the microphone due to the blocking effect of the board on the flow.

Unfortunately, the incident acoustical waves are not only reflected but also diffracted by the plate boundaries, affecting the signal-to-noise ratio and thus the efficiency of the board. This study was designed to evaluate the scattering effect of a board placed vertically above a ground as well as the wind noise generation (Fig. 1). Scattering of plane waves by plates has received considerable attention in the literature.\(^1\,^2\) However, as no ground is considered under the plate in those studies, the influence of the scattered field reflection cannot be assessed. Concerning the wind noise generation, previous research in this field\(^3\,^4\,^5\) has been limited to the case of a free microphone with or without a windscreen. The present paper provides a theoretical formula, expressed in a simple form and collated by measurements, to assess the scattered field on a rectangular board surface. Concerning the board wind noise generation, both indoor and outdoor experimental results are shown and dimensionless laws derived following Strasberg's method.\(^3\) The dimensions of the board were first fixed to (1.8 m × 1.5 m), but for practical reasons, laboratory measurements were carried out with scale models while in the field, the full scale model has been used.

**THEORY OF DIFFRACTION**

**Analytical model**

The board is oriented so that the microphone is mounted on the side of the board facing the turbine (Fig. 1) while the sources of ambient noise are randomly distributed around the plate. The plane of the board divides the space into two volumes \(V_1\) and \(V_2\).

For any source located in \(V_1\), the corresponding pressure at the microphone is the sum of the incident, the reflected and the diffracted fields. As the microphone is mounted on the rigid board, the reflected and incident pressure are equal, and this leads to a pressure doubling effect (+ 6 dB) while the diffraction effects are responsible for the deviation from pressure doubling. Only the diffraction effects of the sources located in \(V_2\) are perceived by the microphone, since the board acts as a shield.
On the use of a vertical microphone board to improve low signal-to-noise ratios

The board is assumed to be rigid and the angle of the incident waves to be in the range \([0^\circ, 50^\circ]\). The influence of the actions of the wind, of the temperature gradient and of the air absorption on the acoustical waves is neglected and the turbine is considered as a point radiator. In order to assess the diffracted field, the problem is written in its integral formulation which states that the pressure, \(p\), at any point in a volume \(V\) bounded by a surface \(S\) may be written in the following form:\(^6\)

\[
p(\vec{r}) = \int \int \int_V G(\vec{r}_0, \vec{r}) F(\vec{r}_0) d\nu(\vec{r}_0) + \oint_S (p(\vec{r}_0^i) \frac{\partial G(\vec{r}, \vec{r}_0^i)}{\partial n}) dS(\vec{r}_0^i) - G(\vec{r}, \vec{r}_0^i) \frac{\partial p(\vec{r}_0^i)}{\partial n} dS(\vec{r}_0^i)
\]  

(1)

where \(G\) is a Green's function and \(F\) is the source strength; the time dependence \(e^{-i\omega t}\) has been removed by a Fourier transform. For a point radiator with unit strength located at \(\vec{r}_0\), one writes \(F(\vec{r}_0) = \delta(\vec{r}_0 - \vec{r}_0)\) and eqn (1) becomes:

\[
p(\vec{r}) = G(\vec{r}, \vec{r}_0) + \oint_S (p(\vec{r}_0^i) \frac{\partial G(\vec{r}, \vec{r}_0^i)}{\partial n}) dS(\vec{r}_0^i) - G(\vec{r}, \vec{r}_0^i) \frac{\partial p(\vec{r}_0^i)}{\partial n} dS(\vec{r}_0^i)
\]  

(2)

The integration surface \(S\) is chosen as the following (Fig. 2):

- The pressure \(p(\vec{r})\) should satisfy the boundary conditions:
  \[
  \frac{\partial p}{\partial n} = 0 \text{ on the board on } B
  \]
  \[
  \frac{\partial p}{\partial n} - \frac{ik}{\bar{Z}_g} p = 0 \text{ on the ground where } \bar{Z}_g \text{ is the normalised impedance of the ground}
  \]
  \[
  \lim R(\frac{\partial p}{\partial R} - ikp) = 0 \text{ when } R \to \infty \text{ i.e. on } S3 \text{ (Sommerfeld condition)}
  \]
In order to simplify the integral equation given by eqn (2), a tailored Greens' function $G(\vec{r}, \vec{r}_0)$ may be constructed by the theory of the images such that $G$ satisfies also the boundary conditions previously quoted:

$$G(\vec{r}, \vec{r}_0) = \frac{-1}{4\pi} \left( \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} + R'(\vec{r}, \vec{r}_0) \frac{e^{ik|\vec{r}-\vec{r}_{11}|}}{|\vec{r}-\vec{r}_{11}|} + \frac{e^{ik|\vec{r}-\vec{r}_{21}|}}{|\vec{r}-\vec{r}_{21}|} \right)$$

where $R'$ is the reflection coefficient of the ground; the board is assumed to be hard and $\vec{r}_0$, $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ represent the source and its image positions. Replacing $G$ in eqn (2) and using the boundary conditions lead to:

$$p(\vec{r}) = G(\vec{r}, \vec{r}_0) - \int \int_{S_2+B} G(\vec{r}, \vec{r}_0^s) \frac{\partial p(\vec{r}_0^s)}{\partial n} dS(\vec{r}_0^s)$$

The rigorous solution is obtained by solving this integral equation but the task is considerably simplified if the Kirchhoff's assumptions, well-known in diffraction theory, are used. With this aim in view, the total pressure is expressed as the sum of the free field incident pressure and a scattered field:

$$p(\vec{r}_0^s) = p_i(\vec{r}_0^s) + p_s(\vec{r}_0^s)$$

The Kirchhoff's assumptions assert that the scattered pressure in the part of the plane not occupied by the scatterer is zero and that on the illuminated side of the scatterer, the pressure is doubled, which means that the diffraction field is zero in the scatterer plane.

By introducing eqn (5) in eqn (4) and after some simple transformations, eqn (4) becomes:
\[ p(\vec{r}) = p_i(\vec{r}) - \int \int_B G(\vec{r}, \vec{r}_0) \frac{\partial p(\vec{r}_0)}{\partial z} ds(\vec{r}_0) \]  

Due to the large distance between the turbine and the board, the amplitude of the incident wave may be assumed constant for all the points of the board. Then, the incident field becomes:

\[ p_i(\vec{r}) = \frac{1}{4\pi} \left( \frac{e^{ik|\vec{r} - \vec{r}_0|}}{|\vec{r} - \vec{r}_0|} + R'(\vec{r}, \vec{r}_0) \frac{e^{ik|\vec{r} - \vec{r}_1|}}{|\vec{r} - \vec{r}_1|} \right) \]

\[ \approx \frac{1}{4\pi} \left( \frac{e^{ik|\vec{r} - \vec{r}_0|}}{|\vec{r} - \vec{r}_0|} + R'(\vec{r}_c, \vec{r}_0) e^{ik|\vec{r} - \vec{r}_1|} \right) \]

where \( \vec{r}_c \) is the position of the board centre. Replacing the expressions of \( G \) and \( p_i \) in the expression of the total pressure gives:

\[ p(\vec{r}) = p_i(\vec{r}) - \frac{ik}{4\pi} \int \int_B \frac{\nabla (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|} \left( B_1 e^{ik|\vec{r}_0 - \vec{r}_0|} + B_2 e^{ik|\vec{r}_0 - \vec{r}_1|} \right) dS(\vec{r}_0) \]

\[ - \frac{ik}{2\pi} \int \int_B R'(\vec{r}, \vec{r}_0) \frac{\nabla (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|} \left( B_1 e^{ik|\vec{r}_0 - \vec{r}_0|} + B_2 e^{ik|\vec{r}_0 - \vec{r}_1|} \right) dS(\vec{r}_0) \]

with \( B_1 = \frac{z_0}{4\pi|\vec{r}_c - \vec{r}_0|^2} \) and \( B_2 = \frac{z_0 R(\vec{r}_c, \vec{r}_0)}{4\pi|\vec{r}_c - \vec{r}_0|^2} \)

Hence, the total pressure is the sum of the incident field, its scattered effect and the reflection of this scattering by the ground, i.e.:

\[ p(\vec{r}) = p_i(\vec{r}) + p^s(\vec{r}) + p^{rd}(\vec{r}) \]

The scattering fields may be written as a sum of integrals of the form:

\[ J = \int \int_B g(x, y) e^{ikf(x, y)} dx dy \]

These terms may be evaluated asymptotically by the method of stationary phase as described by Stamnes.\(^8\) This method uses a Taylor expansion of both the phase and the amplitude functions, \( kf(x, y) \) and \( g(x, y) \), respectively. The only requirement for the validity of the results is that the amplitude function varies slowly compared with the exponential function containing the phase (i.e. \( k \) must be large). This condition is satisfied when the Helmholtz's number, \( ka \), is greater than 1, that is for frequency above \( f_1 = c/2\pi a. \)
This integrand has the interesting property that only neighbourhoods of critical points of various kinds contribute to the integral $J$; the others cancel each other out by destructive interference. These critical points are the following:

- critical points of the first kind which are the points inside $B$ at which the phase is stationary,
- critical points of the second kind on the boundary of $B$ at which the tangential derivative of the phase vanishes,
- critical points of the third kind which are the boundary corners.

The contribution of the critical points of the first kind gives the influence of the reflected pressure by the plate. If the microphone is mounted on the plate, this leads to a pressure doubling. The deviation from pressure doubling is generated by the edges and the corners (critical points of the second and third kind) and is called diffraction effect.

**Theoretical results for a board in a free field**

By considering a plane wave impinging on the board (Fig. 3), the total pressure in absence of the ground is given by:

$$p(x_m, y_m) = p_i(x_m, y_m) - \frac{ik \cos \theta}{2\pi} \iint_B e^{ik(r(x, y)-\gamma \sin \theta \cos \psi - x \sin \theta \sin \psi)} \frac{1}{r(x, y)} \, dx \, dy$$

The expression of the total pressure on the board surface is given by the equation:
\[
\frac{p}{p_i} = 2 - \cos \theta e^{i\frac{\pi}{4}} \left[ \frac{C_1 e^{i\beta_1 (\lambda + \varsigma)}}{\sqrt{\beta \lambda^{1/4}(\lambda + \varsigma)}} + \frac{C_2 e^{i\beta_2 (\varepsilon - \chi)}}{\sqrt{\mu \varepsilon^{1/4}(\varepsilon - \chi)}} + \frac{C_3 e^{i\varphi(\lambda - \varsigma)}}{\sqrt{\nu \lambda^{1/4}(\lambda - \varsigma)}} \right. \\
\left. + \frac{C_4 e^{ik_3 (\varepsilon + \chi)}}{\sqrt{\alpha \varepsilon^{1/4}(\varepsilon + \chi)}} \right] + \frac{r_1 e^{i (k_{1a} (\lambda + \varsigma) + \alpha)}}{2\pi k} + \frac{r_2 e^{i (k_{2b} (\varepsilon - \chi) + \beta)}}{2\pi k} \\
\left. + \frac{r_3 e^{i (k_{3c} (\lambda - \varsigma) + \mu)}}{2\pi k} + \frac{r_4 e^{i (k_{4d} (\varepsilon + \chi) + \nu)}}{2\pi k} \right]
\]

\[
\alpha = a + y_m \quad \beta = b + x_m \quad \mu = a - y_m \quad \nu = b - x_m
\]

\(x_m\) and \(y_m\) are the microphone coordinates relative to the board centre.

\[
r_{c1} = \sqrt{\alpha^2 + \beta^2}, \quad r_{c2} = \sqrt{\mu^2 + \nu^2}, \quad r_{c3} = \sqrt{\lambda^2 + \alpha^2}, \quad r_{c4} = \sqrt{\lambda^2 + \beta^2}
\]

\[
\chi = \sin(\theta) \cos(\psi), \quad \varsigma = \sin(\theta) \sin(\psi), \quad \lambda = \sqrt{1 - \chi^2}, \quad \varepsilon = \sqrt{1 - \varsigma^2}
\]

\[
C_1 = 1 \text{ if } |ym + \chi \beta| < a, \ 0 \text{ otherwise},
\]

\[
C_2 = 1 \text{ if } |xm + \varsigma \mu| < b, \ 0 \text{ otherwise},
\]

\[
C_3 = 1 \text{ if } |ym + \chi \nu| < a, \ 0 \text{ otherwise},
\]

\[
C_4 = 1 \text{ if } |xm + \varsigma \alpha| < b, \ 0 \text{ otherwise}.
\]

The positions for the critical points of the second kind are given by a simple function of both the microphone location and the angle of incidence. With the present notation, the positions are the following: \((-b, y_m + \chi \beta / \lambda), (x_m + \varsigma \mu / \varepsilon, a), (b, y_m + \chi \nu / \lambda), (x_m + \varsigma \alpha / \varepsilon, -a)\). When a critical point is outside the board boundary, the contribution of the edge should vanish and the coefficient \(C_i (i = 1..4)\) in eqn (9) becomes zero.

The model suffers from some limitations. These occur when one or more of the critical points of the second kind comes close to a critical point of the third kind. That situation arises when the observation point is placed near an edge or for some critical angles of incidence. This case of diffraction needs special treatment which was considered not necessary for the present application, and so, is not developed here. Then, eqn (9) is valid only if the distance between the critical of second kind and the nearest corner is greater than about \(a/5\).

At the centre of a square plate of half edge length \(a\), the expression of the total field is reduced under normal incidence to:
\[
\frac{p}{p_i} = 2 - 2\sqrt{2}e^{ikx} \frac{e^{ika}}{\sqrt{\pi}} \text{ (edges)} + \frac{4i}{\pi \sqrt{2}} \frac{e^{ika\sqrt{2}}}{ka} \text{ (corners)}
\]

The quantity 20 \lg(p/p_i) is taken to be a measure of the diffraction effect and it appears that it depends only on the Helmholtz's number (\(ka\)). The bigger the Helmholtz's number, the less the diffraction effects. It appears also that the pressure due to the edge effect spreads as a cylindrical wave while the corner wave spreads as a spherical one, their amplitudes being proportional to \(1/\sqrt{k}\) and \(1/k\), respectively. Thus the corner effect will decrease more rapidly as a function of the distance and of the frequency. Furthermore, the frequencies at which the extreme of departure from pressure doubling occur may be roughly obtained by neglecting the pressure due to corner waves (which is normally rather low at those frequencies).

\[
ka = \left( n - \frac{1}{4} \right)\pi \Rightarrow f_{\text{extr}} = \frac{(n - \frac{1}{4})c_0}{2a}
\]

\(c_0\) is the sound velocity, \(a\) the board typical dimension and \(n\) the number of the extremum.

Due to the simplicity of the solution obtained by the method of stationary phase, it is quite easy to obtain a presentation of the diffraction field over the board surface. Indeed, by computing eqn (9) over the frequency range [100, 5000 Hz] sampled with a frequency resolution of 10 Hz, the standard deviation of the departure from pressure doubling may be calculated at each frequency and summed over all the frequencies. By repeating this operation for numerous board points, it appears that some positions minimise the diffraction effect and that, under normal incidence, the centre and the board axis of symmetry are maxima of diffraction (Fig. 4). The position of the best minimum depends of course on the angle of incidence, the size of the board and the frequency range of interest. For the sake of convenience, this point will be denoted \(M_{\text{opt}}(x_{\text{opt}},y_{\text{opt}})\) in the rest of the article where \(x_{\text{opt}}\) and \(y_{\text{opt}}\) will be given according to the co-ordinate system described in Fig. 3.

**Taking into account the presence of the ground**

The ground, as seen in eqn (8), induces reflection for both the acoustical waves launched by the turbine and the field diffracted by the board. Due to the large distance between the board and the turbine, it is reasonable to use the reflection coefficient for plane waves to deal with the incident wave and to assume it constant for the points of the board. Thus, the scattering effects of both the direct and reflected fields from the turbine may be dealt separately.
by considering two plane waves impinging on the board with different angles of incidence. Equation (9) may then be applied to each wave and the results added to obtain the total scattered field. It should be nevertheless noted that the long propagation distances associated to the high height of the turbine hub and the scattering of the acoustical wave by the turbulence in the wind may affect the correlation between the direct and the reflected fields, thus reducing the ground effect.

Nonetheless, this solution neglects the reflection of the diffracted field by the ground, a problem which rises more complications. Indeed, the first is due to the cylindrical and spherical nature of these waves which prevents from using the reflection coefficient for plane waves. The second difficulty is to deal with the second diffraction experienced by the diffracted field after reflection by the ground and which occurs under grazing incidence relative to the board. Nevertheless, due to the expected weakness of the reflected diffracted field contribution, the solution may be approximated by using the plane wave reflection coefficient and by neglecting the second diffraction.

Board shielding effect

The board acts as an acoustical shield for sources located in the volume $V_2$ (Fig. 1) as the microphone perceives only their diffracted field. Though all the sources will not satisfy the assumptions required by the model (plane wave, limited range of incidence ...), it is expected that it will give at least a
rough assessment of this shielding effect. As the board thickness is assumed to be zero, the diffracted fields are equal on the two board sides and then the shielding effect is maximum at $M_{opt}$.

The sheltering effect is especially important when the measurements location allows reflections from building facades behind the microphone. By considering a plane incident field impinging on one side of the board and its image by the board plane impinging on the other side, the shielding effect is defined as the difference between the sound pressure levels induced by both fields at the microphone. Figure 5 shows the result of a calculation made with an incident field composed of two non-correlated waves of incidences ($10^\circ$, $0^\circ$) and ($-10^\circ$, $0^\circ$), respectively. It is observed that a board of dimensions (1.8 m×1.5 m) gives a 10 dB sheltering already at about 150 Hz when the microphone is positioned at $M_{opt}$ (0.45, 0). It is also shown by the figure that the shielding effect assessed at the board centre is lower and more irregular.

**DIFFRACTION MEASUREMENTS**

**Free field measurements**

These measurements were performed in an anechoic chamber (7 m×5.6 m×4.6 m). The sound pressure on the board surface, $p$, and the corresponding free field value, $p_0$, were obtained by a 13 mm microphone (B&K) at four positions. The scatterer was made from plywood plate of 15 mm thickness with the dimensions (0.9 m×0.75 m). The incident sound field was produced by a loudspeaker located at about 5 m from the board at

![Image](image-url)  
**Fig. 5.** Shielding effect assessed at $M_{opt}$ (—) and at the board centre (---).
different positions in order to collate the model for several angles of incidence. The assumption of plane wave is doubtful at 5000 Hz for this configuration (board size, distance obstacle-source and frequency). Anyway, considering the size of the scatterer, the diffraction effects are low frequency phenomena and then these drawbacks are not important.

The different studied parameters, defined according to Fig. 3, were the following:

- observation points: centre, \((b/2,a/2), (-b/2,0)\) (\(x_{\text{opt}},y_{\text{opt}}\))
- incidence angles \((\theta,\psi)\): (0,0) (15,0) (25,0) (35,67)

For each case, a narrow band spectrum with a 10 Hz frequency resolution was obtained by averaging over 100 samples. The spectra have then been converted into third-octave bands and the diffraction effect plotted in the form of the ratio \(|p_{1/3}/p_{0,1/3}|\) in dB. The good agreement between theory and experiment, shown by Fig. 6 for normal incidence, has been obtained for all the configurations. The importance of the microphone position is also revealed in this figure by comparing the diffraction effect at the board centre and at \(M_{\text{opt}}\) (0.288, 0.06). A discrepancy (about 2 dB) appears in the middle frequency part of the spectra as expected due to the approximation used in the method of stationary phase. The error almost vanishes when the Helmholtz’s number becomes greater than 1.4.

Measurement of the influence of absorbing edges

As the diffracted field may be modelled by secondary sources on the edges and corners of the board, an apparently obvious way to reduce it was to set

![Graph](image_url)

**Fig. 6.** Third octave band scattering effect in dB for normal incidence. (——) calculated at the centre, (-----) measured at the centre, (——□□) calculated at \(M_{\text{opt}}\), (-×-) measured at \(M_{\text{opt}}\).
some absorbent foam on the edges in order to decrease the strength of these sources.\textsuperscript{10,11} Figure 7 shows the result of measurements carried out with strips of absorbent Stepisol (density 185 kg m\textsuperscript{-3}, thickness 0.1 m, breadth 0.25 m) on the board edges. Contrary to expectancies, it is seen that the foam increases the diffracted effect. The explanation may be the introduction of a second change of impedance, board-foam, closer to the microphone. Nevertheless, that does not mean that the presence of absorbent is always negative. For points behind the plate, only the impedance change foam-air has to be considered as a secondary source, which leads to reduction of the diffracted field.

**Measurement of the board shielding effect**

One measurement has been made by setting the microphone in the shadow zone of the board at about 20 cm from the surface centre, called point M, and the sheltering effect has been measured by comparing the pressures at the microphone with and without the board, $p_M$ and $p_0$, respectively (Fig. 8). For normal incidence, the shielding effect, given by $20 \log \left( \frac{p_0}{p_M} \right)$, appears to be negative up to 500 Hz at the centre while the shielding is already efficient at 250 Hz at the minimum of diffraction $M_{\text{opt}}$ (0.288, 0.06).

**Measurements of the reflected diffracted field**

In order to measure the amplitude of the reflected diffracted field, a large plate was placed on the room floor under the vertical board. The dimensions of this plate (2.4 m × 2 m) were small enough to avoid the reflection of the incident field but large enough to reduce the inevitable diffraction by the plate. The measurements indicate that the contribution of the reflected diffracted field for this configuration may increase or decrease the diffraction
Fig. 8. Diffraction in the shadow zone of the board under normal incidence. (- - -) calculated at $M_{opt}$, (——) measured at $M$, (-- -) calculated at $M$.

effect of 1 or 2 dB. Indeed, the phase of the reflected diffracted field, determined by both the board height and the ground impedance, may lead to either constructive or destructive interference with the free field pressure. Unfortunately, the effects of diffraction by the ground board were of about the same magnitude as the reflection of the diffracted field, which prevented us from deducing reliable conclusions.

MEASUREMENT SYSTEM AND WIND NOISE GENERATION

During a windy day and in an environment free from human and animal activity, the background noise is reduced to its wind component. It arises from the action of the wind through foliage and around obstacles near the measurement location, from the turbulence of the atmosphere and from the measurement system itself. Though the other contributions are also responsible for a low signal-to-noise ratio, only the pseudo-noise linked to the measurement system is studied in this article. It is composed of a non-radiated aerodynamic noise due to the internal flow velocity fluctuations and by the near field aeroacoustical noise of the measurement system. According to previous studies, the first component is dominant in the wind and the second in a low turbulence rate oncoming flow.

Flow description

The characteristics of a flow passing a sharp-edged body are nearly insensitive to the Reynolds number. Bands of vorticity are generated at the edges of the obstacle and, at some distance behind, they roll up and form the vortex
street also called the wake. The action of the board on the oncoming flow is called “blocking effect” which reduces the velocity as the stream approaches the board surface. On the side which faces the oncoming flow appears a stagnation point which characterises a division of the stream and where the mean velocity is zero.

A microphone mounted on the board is submitted to the velocity fluctuations of the board boundary layer. They are perceived by the microphone as a strong low frequency additional noise. The pseudo-noise is then expected to be very low at the stagnation point compared to any other point of the board, due to the zero mean velocity.

Measurements

Both indoor and outdoor measurements have been carried out to provide an estimation of the attenuation given by the use of a board measurement system instead of a free microphone. The pressure perceived by a microphone mounted on the board and sheltered by a 9.5 cm hemispherical windscreen is compared with the “free microphone” value given by a microphone embedded in a 9.5 cm spherical windscreen. On the board, the microphone was located at the centre and at the point giving the minimum of diffraction effect, \( M_{\text{opt}} \). The position of \( M_{\text{opt}} \) is fixed under all the wind noise measurements and is \((b/2, 0)\). This position has been calculated considering two non-correlated waves impinging on the board with the incidences \((10^\circ, 0^\circ)\) and \((-10^\circ, 0^\circ)\), respectively. The velocity is always given at the height centre of the board (1.5 m during the outdoor measurements).

Laboratory measurements

Four boards of different size, \((0.24 \text{ m} \times 0.2 \text{ m}), (0.33 \text{ m} \times 0.27 \text{ m}), (0.48 \text{ m} \times 0.4 \text{ m})\) and \((0.6 \text{ m} \times 0.5 \text{ m})\), have been placed successively at the output of a square rectangular duct of area 0.49 m² for three different flow velocities \((6.5, 10, 12 \text{ m s}^{-1})\).

Figure 9 shows third octave band spectra for two different board sizes, two velocities and two microphone positions. In Table 1 are given the sound pressure levels at the board \((0.48 \text{ m} \times 0.4 \text{ m})\) for the frequency range [20–5000 Hz] with and without the A-weighting for both the centre and the point

<table>
<thead>
<tr>
<th>Velocity ([\text{m s}^{-1}])</th>
<th>6.5</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_p \text{(centre)} ) dB/dB(A)</td>
<td>72/46</td>
<td>82/53</td>
<td>92/59</td>
</tr>
<tr>
<td>( L_p \text{(M_{opt})} ) dB/dB(A)</td>
<td>81/47</td>
<td>90/55</td>
<td>96/62</td>
</tr>
</tbody>
</table>
of minimum diffraction effect $M_{\text{opt}}$. It appears clearly that the gain is better when the microphone is placed at the board centre, i.e. at the stagnation point. The comparison between the sound pressure levels with and without the weighting also shows that only the low and middle frequencies (up to 300 Hz) are sensitive to the position (Table 1 and Fig. 9).

The influence of the turbulence is expected to be very important as the nature of the pseudo-noise changes from aeroacoustical to aerodynamical when the turbulent rate increases. The turbulence rate has been modified by setting a turbulence screen of solidity 49% in the duct, at a distance of 1.5 m in front of the board. The screen was a grid made of square rods characterised by a bar thickness of 3 cm and a spacing of 10 cm. The turbulence rates, measured with a hot wire probe, were 4 and 6% with and without the screen, respectively. The results obtained from those measurements may be partly distorted by the screen blocking effect which reduces the velocity at the microphone location. Unfortunately this effect has not been measured correctly but should be about $-1 \text{ m s}^{-1}$ at 10 m s$^{-1}$.

Table 2 and Fig. 10 show that, unlike for the free microphone, the board system pseudo-noise spectrum is strongly affected by the presence of the screen in the frequency range [200–3000 Hz]. This increase of pseudo-noise generation appears for each measurement with both the screen and a board, though with different amplitudes. One explanation could be that the interaction screen-flow induced noise and that the board measurement system, adding +6 dB to all the source strengths located in front of it, has been particularly affected by this noise. An other explanation would be that the
<table>
<thead>
<tr>
<th>Velocity [m s⁻¹]</th>
<th>6.5</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Microphone</td>
<td>Turbulence rate 4%</td>
<td>79/50</td>
<td>85/58</td>
</tr>
<tr>
<td>dB/dB(A)</td>
<td>Turbulence rate 6%</td>
<td>77/50</td>
<td>85/60</td>
</tr>
<tr>
<td>Board</td>
<td>Turbulence rate 4%</td>
<td>68/44</td>
<td>79/52</td>
</tr>
<tr>
<td>dB/dB(A)</td>
<td>Turbulence rate 6%</td>
<td>71/53</td>
<td>80/62</td>
</tr>
</tbody>
</table>

**TABLE 2**

Lp in dB and in dB(A) for a Board [0.6 m × 0.5 m] and the Free Microphone

![Graph](image)

**Fig. 10.** Influence of the turbulence at 10 m s⁻¹. (−○−) free microphone with 4% turbulence, (−−−) free microphone with 6% turbulence, (−−−−) board [0.6×0.5] with 4% turbulence, (−□−) board [0.6×0.5] with 6% turbulence.

board measurement system is highly sensitive to the turbulence, but this hypothesis is rather doubtful considering the slight increase in the turbulence rate during the experiment (from 4 to 6%). Nevertheless, the gain provided by the board is now reduced to the low frequency range (under 300 Hz) and, if the A-weighted sound pressure levels are compared, the use of the board is no longer an advantage.

The gain provided by the board is strongly dependent on the board size, on the flow velocity and on the frequency of interest. For a fixed frequency, the gain increases for increasing board sizes and decreasing flow velocities. If those two parameters are fixed, it appears that the gain may be negative in the very low frequencies, then increases as the frequency increases due to the different slopes of the spectra and begins to decrease to finally vanish at high frequency. Furthermore, the turbulence rate is also an important factor which tends to reduce this gain drastically.

By plotting the spectra under the dimensionless form adopted by Strasberg³ (Fig. 11), it appears that the low frequency part of the board pseudo-noise satisfies a similarity law with a very good correlation.
The law is expressed by:

\[
20 \log_{10} \left( \frac{p_{13}(f)}{\rho V^2} \right) = A + B \log(St)
\]

where \( p_{13} \) is the pressure in the third octave band \( f \) and \( \rho \) is the flow density. \( St \), the Strouhal number, is defined by \( St = fD/V \) where \( D \) is a typical dimension of the board or of the windscreen and \( V \) is the flow mean velocity. Table 3 gives the constants \( A \) and \( B \) determined experimentally and their Strouhal number range of validity.

The sound pressure level for the frequency \( f \), \( L_p(f) \), is then derived:

\[
L_p(f) = (A + 95.8) + B \log(D) + B \log(f) + (40 - B) \log V, \text{ dB}
\]

and the total \( L_p \) integrated over the whole range of interest is given by:

\[
L_p = (A + 95.8) + B \log(D) + (40 - B) \log(\nu) + B \log(f_0) - 10 \log(1 - 2^{B/30})
\]

where \( f_0 \) is the frequency of the first third octave band of interest. This representation, though valid for limited Strouhal number ranges, allows the pseudo-noise generated by the measurement system to be predicted simply, for either the free microphone or board.

**Outdoor measurements**

This set of measurements was performed at the top of a hill (73 m height) in the neighbourhood of Stockholm in July 1996. The records were made with
two board sizes, [1.8 m x 1.5 m], [0.9 m x 0.75 m], during the day and for wind velocity at 1.5 m height contained in the range [4 m s\(^{-1}\), 7 m s\(^{-1}\)]. The vegetation, mainly pines and spruces, absent at the top of the hill, was present on the hillside and around one part of the hill. Furthermore, a rather important road, with a car speed limited to 70 km h\(^{-1}\), was situated at 2 km from the measurement place.

According to the literature\(^\text{13}\), the turbulence rates are expected to be 20% for velocity components in the plane parallel to the ground and 10% for the vertical component. Outdoors, the mean wind velocity is characterised by a logarithmic profile in the vertical direction, the gradient of that profile depending mainly on the ground roughness. Nevertheless, the presence of the hill should influence strongly both the turbulence and the wind profile.

Typical wind-noise spectra are shown on Fig. 12 together with a common wind turbine noise spectrum. It appears that the board is effective only in the range [10–300 Hz] as, outside this range, the board system measures a higher background noise level. This increase had been already observed during the laboratory measurements with screen but occurs here in much lesser proportions. Nevertheless, it is still impossible to determine if this might be imputed to a greater wind noise generation by the board or to the reflection of the background noise arising from sources located in front of the board. However, for these frequencies, the pressure doubling of the incident noise should also be taken into account. By plotting typical wind turbine spectra (Micon 600 kW) (Fig. 12), it is shown that the use of a board leads to a global improvement over the whole frequency range. The solid curve represents the wind turbine spectrum corrected for a distance of 300 m while the dashed
Fig. 12. Outdoor wind noise spectra. (–O–) free microphone at 5.5 m s\(^{-1}\), (— —) board [1.8×1.5] at 5.5 m s\(^{-1}\), (— — ) wind turbine SPL at 300 m, (— ——) wind turbine SPL at 300 m + 6 dB.

curve shows it corrected for both the distance and the pressure doubling (for frequencies greater than 100 Hz). So, in the case of the board, the slight increase of background noise at high frequencies is largely counteracted by the pressure doubling of the incident signal. Figure 13 shows the signal-to-noise ratios corresponding to Fig. 12 for both the free microphone and the board, likewise the gain obtained by using the board. It is seen that this gain is greater than 5 dB over the frequency range [20, 1000 Hz].

The spectrum dimensionless forms for the free microphone and the board centre are still satisfied, but with a standard deviation of 3 dB around the regression line (Fig. 14). The influence of the microphone position is reduced compared to the indoor measurements due to the uncertainties linked to the mean wind direction and profile and to the turbulence. But the board centre still shows a better pseudo-noise reduction than the point \(M_{\text{opt}}\).

DISCUSSION AND CONCLUSIONS

Some practical drawbacks concerning the board have been noted during the measurements. One is its handling, especially when the wind is strong and another is the special care one needs to take to the disposition of the windscreen on the board. Indeed, a slight gap between the windscreen and the board surface leads to a drastic increase of the wind noise. Furthermore, if the sources of ambient noise are non-correlated, of equal strength and uniformly distributed around the board, the board introduces not only some
diffraction effects but also a correlation which adds $+3\,\text{dB}$ to the free field value.

Dimensionless laws giving the wind noise generation for both the board and the free microphone have been derived, allowing the results to be extrapolated to different board sizes or wind velocities. Nevertheless, these laws could have suffered from the influence of the short velocity range [$4-7\,\text{m/s}$]. Furthermore, it has been shown that the board should be placed normal to...
the wind direction in order to maximise its blocking effect. Then, a suitable microphone position on the board surface is the stagnation point, which may be located a bit over the board centre (about 10 cm) due to the logarithmic wind velocity profile. Both indoors and outdoors, it has been observed that the board measurement system might lead to a higher background level compared to the free microphone at middle and high frequency. It is difficult to conclude if it arises from the generation of pseudo-noise by the board, i.e. a high sensitivity to the flow turbulence, or if it is the consequence of the reflection of the background noise by the board. Anyway, the addition of 6 dB to the incident signal counteracts largely this negative effect and makes the board measurement system still interesting for improving the signal-to-noise ratio.

Concerning the diffraction effect, the model developed in this article has shown a good agreement with measurements results. The positions of minimum of diffraction have been localized on the board surface, leading to an optimization of the departure from pressure doubling and of the sheltering effect. The sheltering for a board of (1.8 m x 1.5 m) is greater than 10 dB above 150 Hz and allows the measurements to be achieved near reflecting obstacles like buildings facades. So, if the wind is not at the origin of the poor signal-to-noise ratio, it is better to place the microphone at the point which reduces the diffraction.

Thus, this study shows that the use of a board is advantageous for outdoor measurements when one is dealing with low signal-to-noise ratios or with reflecting surfaces around the measurement location. Further work could be devoted to repeating the measurements for higher wind velocities and with different sizes of hemispherical windscreen in order to know how the gain is improved by increasing the windscreen diameter.

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